ROBOTICS INSIGHT II
ASSIGNMENT

This example problem will give you an idea on how to use Lagrangian to obtain an equation of motion.

PROBLEM:
We hang the pendulum of length ‘l’ from a cart of mass M and position x, acted upon by a force u in the direction of x and moving on frictionless rails.
Find equations of motion using the Euler-Lagrangian method.

SOLUTION:
The x position of the pendulum is x+ l*sin θ and the y position is l*cos θ, so the kinetic energy is First taking the time-derivatives, then squaring, then noting that $\cos^2\theta + \sin^2\theta = 1$ we obtain

$$K = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\frac{d}{dt}(x+\ell\sin\theta)\right)^2 + \frac{1}{2}m\left(\frac{d}{dt}(\ell\cos\theta)\right)^2$$

$$mg\ell(1-\cos\theta)$$

The potential energy is so:
Accordingly

$$L = K - U = \frac{1}{2}(M+m)\dot{x}^2 + m\ell\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(1-\cos\theta)$$

Clearly ∂L/∂x = 0 and

$$\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + m\ell\dot{\theta}\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (M+m)\ddot{x} + m\ell\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) = u$$

Next, we consider the θ direction and velocity, taking

$$\frac{\partial L}{\partial \theta} = -m\ell\dot{x}\dot{\theta}\sin\theta + mg\ell\dot{\theta}\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = m\ell\dot{x}\cos\theta + m\ell^2\dot{\theta}$$

Taking the time derivative yields

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = (m\ell\ddot{x}\cos\theta - m\ell\dot{x}\dot{\theta}\sin\theta) + m\ell^2\ddot{\theta}$$

The Lagrangian equation of motion is thus

$$m\ell(\ddot{x}\cos\theta + \ell\ddot{\theta} - g\sin\theta) = 0$$

We can write this all as:

$$\begin{bmatrix} M+m & m\ell\cos\theta \\ \cos\theta & \ell \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} m\ell\dot{\theta}^2\sin\theta + u \\ g\sin\theta \end{bmatrix}$$

After obtaining equations of motion, you can solve them on Matlab using solve function(Matlab will be taught in the next lecture of the same series)

In robotics, we do calculate the force required or torque required for any task through these kinds of equations as the opposing force/torque we need will make the system in equilibrium and adding up that force/torque will follow E-L equations. In this case 'u' was the required force.

Now there are some practice problems for you:

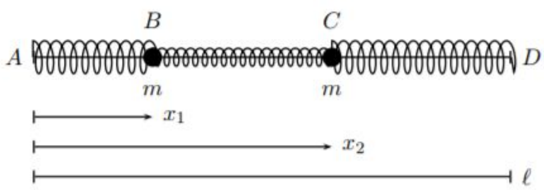
1. Consider a mass m on the end of a spring of natural length l and spring constant k. Let y be the vertical coordinate of the mass as measured from the top of the spring. Assume the mass can only move up and down in the vertical direction.

Show that

$$L = \frac{1}{2}m\dot{y}^2 - \frac{1}{2}k(y-\ell)^2 + mgy$$

Determine and solve the corresponding Euler-Lagrange equations of motion.

2. Three massless perfectly elastic springs AB, BC and CD are attached in a horizontal line as shown in the diagram. The ends at A and D are fixed. The objects of mass m are located where the springs join at B and C. The two outer springs AB and CD have natural lengths a and stiffness k, while BC has natural length a' and stiffness k'. The distance '*l*' between the fixed endpoints AD is greater than the total natural lengths of the springs: *l* > 2a + a'.



(a) Using x1 and x2, as shown in the diagram for the coordinates of the two masses, construct the Lagrangian.

(b) Determine the Lagrange equations of motion.

(c) Try to solve these equations.