Markowitz Portfolio Optimization and Capital Asset Pricing Model

What is Markowitz Portfolio Optimization?

Markowitz portfolio optimization is a strategy for constructing investment portfolios. It aims to maximize returns while minimizing risk. Below are the key steps involved:

- Selecting assets.
- Estimating their returns and risks.
- Analyzing correlations between assets.
- Constructing an efficient portfolio to balance risk and return.
- Diversifying across assets with different risk-return profiles is key.

1. Pick any 10 risky assets from the market. Use their 3-month closing price and obtain simple returns.

The 10 risky assets are picked from the market using their 3-month data from 1st October 2023 to 31st December 2023.

The assets selected are as follows:

Asset Data Table :

Asset No.	Stock	Expected Return/Mean	Risk/Variance
1	Infosys Pvt Ltd	0.00155378	0.00019820
2	Tata Motors	0.00341353	0.00021744
3	Sun Pharma	0.00174823	0.00008766
4	HDFC Bank	0.00212513	0.00008180
5	Reliance	0.00201997	0.00007556
6	Bharti Airtel	0.00197941	0.00010340
7	Hindustan Unilever	0.00112592	0.00005984
8	Tata Steel	0.00137406	0.00016177
9	Bajaj	0.00148609	0.00018778
10	NTPC	0.00458761	0.00022940

Mean Vector/Expected Return Vector (M) :

Asset No.	1	2	3	4	5	6	7	8	9	10
M	0.00155	0.00341	0.00174	0.00212	0.00201	0.00197	0.00112	0.00137	0.001486	0.00458

From the above M matrix, we can find the maximum possible expected return ($\mu_{\max}$): 0.00458.

Covariance Matrix (C) :

Asset No.	1	2	3	4	5	6	7	8	9	10
1	0.00020	0.00006	0.00001	0.00005	0.00005	0.00004	0.00002	0.00007	0.00007	0.00005
2	0.00006	0.00022	0.00005	0.00001	0.00003	0.00006	0.00002	0.00010	0.00005	0.00006
3	0.00001	0.00005	0.00009	0.00001	0.00003	0.00004	0.00000	0.00004	0.00002	0.00004
4	0.00005	0.00001	0.00001	0.00008	0.00005	0.00004	0.00001	0.00002	0.00003	0.00003
5	0.00005	0.00003	0.00003	0.00005	0.00008	0.00004	0.00002	0.00005	0.00005	0.00006
6	0.00004	0.00006	0.00004	0.00004	0.00004	0.00010	0.00003	0.00005	0.00003	0.00006
7	0.00002	0.00002	0.00000	0.00001	0.00002	0.00003	0.00006	0.00002	0.00001	-0.00000
8	0.00007	0.00010	0.00004	0.00002	0.00005	0.00005	0.00002	0.00016	0.00005	0.00010
9	0.00007	0.00005	0.00002	0.00003	0.00005	0.00003	0.00001	0.00005	0.00019	0.00005
10	0.00005	0.00006	0.00004	0.00003	0.00006	0.00006	-0.00000	0.00010	0.00005	0.00023

Now for Markowitz Portfolio Optimization, we need to find out the weight matrix W such that the risk or the variance is minimum, that is:

$$\text{Minimize, } \sigma^2 = W \cdot C \cdot W^T \text{ such that } O \cdot W^T = 1$$

$$W_{min} = \frac{O \cdot C^{-1}}{O \cdot C^{-1} \cdot O^T}$$

On solving, we get $W_{\min}$ to be:

Asset No.	1	2	3	4	5	6	7	8	9	10
$W_{\min}$	-0.0318	-0.1029	0.3334	0.1844	-0.0533	-0.0641	0.4597	0.1538	0.1684	-0.0475

So, the maximum possible expected return $\mu_{\max}$ is: 0.0006, and the minimum risk $\sigma^2_{\min}$ is: 0.000033539, corresponding to $W_{\min}$.

2. Use the mean-variance theory and build the Markowitz efficient frontier.

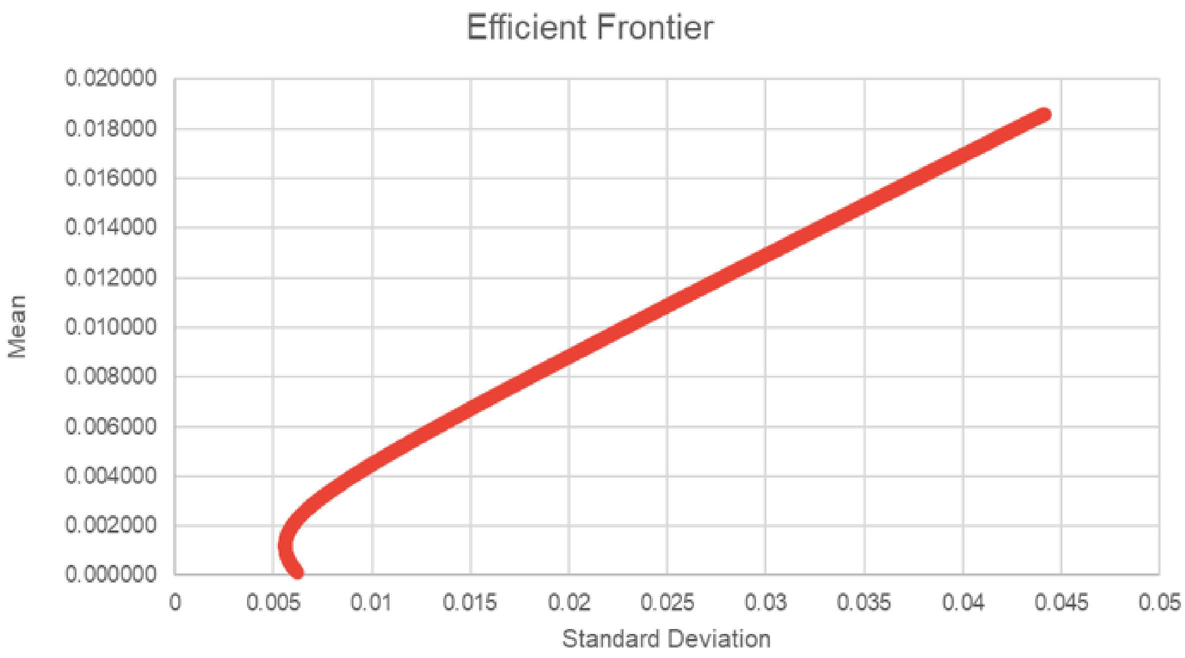
For Problem-2 of Markowitz Portfolio Optimization, we need to find the minimum risk and the corresponding value of weight matrices for a given value of expected return which lies between the minimum and maximum expected returns $[\mu_{\min}, \mu_{\max}]$.

$$\text{Minimize, } \sigma^2 = W \cdot C \cdot W^T \text{ such that } O \cdot W^T = 1 \text{ and } M \cdot W^T = \mu$$

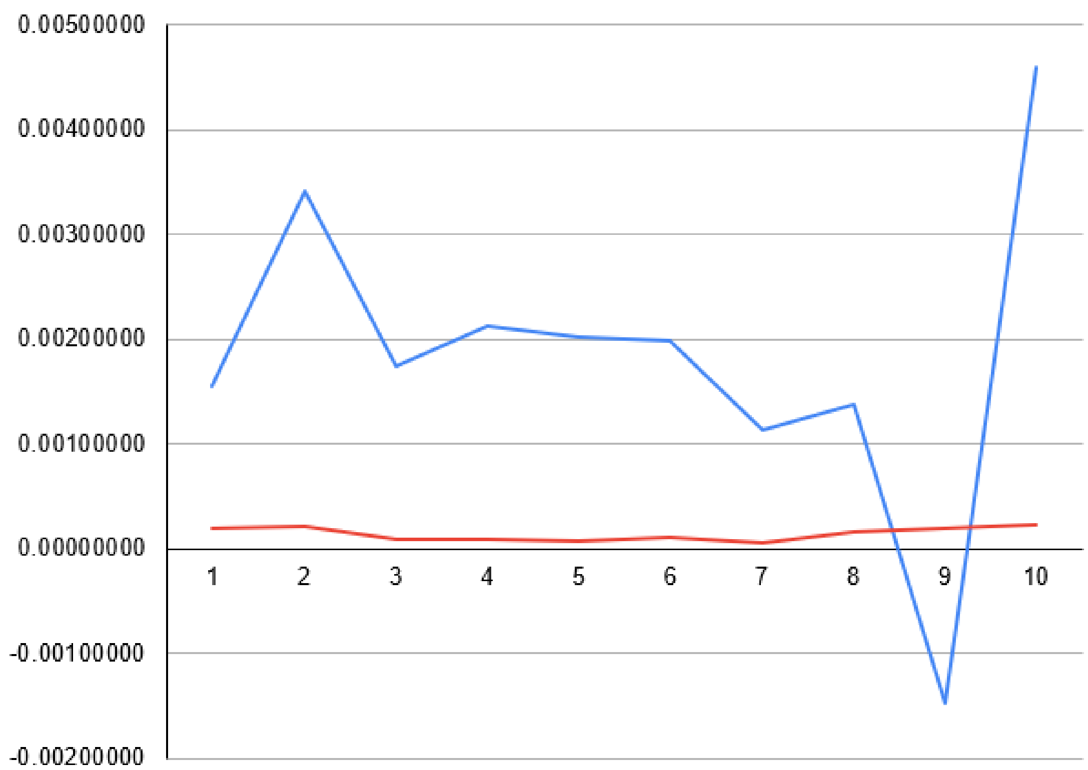
$$W_{min} = \frac{(B2 - B1 \cdot \boldsymbol{\mu}) \cdot O \cdot C^{-1} + (\boldsymbol{\mu} \cdot A1 - A2) \cdot M \cdot C^{-1}}{A1 \cdot B2 - A2 \cdot B1}$$

,where, $A1 = O \cdot C^{-1} \cdot O^T$, $A2 = M \cdot C^{-1} \cdot O^T$, $B1 = O \cdot C^{-1} \cdot M^T$, and $B2 = M \cdot C^{-1} \cdot M^T$.

The Efficient Frontier/Markowitz Curve obtained for these 10 risky assets is as follows: (The graph between Mean and Standard Deviation is as follows:)



The trade-off between risk and return is,



Limitations of Markowitz Optimization :

- Markowitz optimization assumes that asset returns are normally distributed, which may not hold true in real-world scenarios.
- The optimization is highly sensitive to input parameters such as expected returns, covariance matrix, and risk tolerance, which can lead to unstable and unreliable results.
- Estimating expected returns and covariance matrix accurately is challenging, especially with limited historical data, leading to potential errors in portfolio construction.
- Markowitz optimization may lead to highly concentrated portfolios, especially when dealing with a large number of assets, which can result in poor diversification and increased risk.

Application of Markowitz Portfolio Optimization :

- **Portfolio Management:** Maximizes returns for a given risk level.
- **Asset Allocation:** Determines optimal asset mix based on historical returns and correlations.
- **Risk Management:** Balances assets to reduce overall portfolio risk.
- **Wealth Management:** Tailors investment strategies to individual risk tolerance and return objectives.
- **Mutual Funds:** Constructs diversified portfolios for competitive returns.

What is Capital Asset Pricing Model (CAPM) ?

CAPM is a financial model used to calculate the expected return on an investment. It helps investors understand the relationship between risk and expected return. The model consists of three main components:

- **The Risk-Free Rate:** Represents the theoretical return on an investment with zero risk, typically based on government bond yields.
- **The Market Risk Premium:** The excess return that investors demand for bearing the risk of investing in the market compared to the risk-free asset.
- **Beta:** Measures the sensitivity of the investment's returns to the returns of the market as a whole.

3. Use a risk-free asset along with the 10 risky assets to obtain the Capital Asset Pricing Model (CAPM). Draw the straight line representing the Capital Market Line (CML) and show that it is tangent to the Efficient Frontier. Obtain the market portfolio.

We will now include a risk-free asset (Asset 11) with the 10 risky assets available. The variance/risk of a risk-free asset is zero. The mean/expected return for the risk-free asset is set to be:

$$\mu_{rf} = k \cdot \mu_{min}, \ k = 0.5$$

So, the mean/expected return vector **M** is changed to

Asset No.	1	2	3	4	5	6	7	8	9	10	11
μ	5.634%	6.420%	5.255%	5.399%	5.773%	2.731%	4.694%	6.231%	5.689%	4.915%	4.915%

The covariance matrix **C** is also changed as follows:

Asset No.	1	2	3	4	5	6	7	8	9	10	11
1	0.05065	0.01675	0.00359	0.01123	0.01361	0.01134	0.00499	0.01929	0.01829	0.01278	0
2	0.01675	0.05795	0.01257	0.00295	0.00834	0.01447	0.00739	0.02436	0.01329	0.01270	0
3	0.00359	0.01257	0.02086	0.00332	0.00745	0.00963	0.00215	0.00784	0.00384	0.00853	0
4	0.01123	0.00295	0.00332	0.01996	0.01207	0.01103	0.00299	0.00699	0.00956	0.00813	0
5	0.01361	0.00834	0.00745	0.01207	0.01941	0.01022	0.00442	0.01222	0.01128	0.01397	0
6	0.01134	0.01447	0.00963	0.01103	0.01022	0.02619	0.00689	0.01158	0.00785	0.00910	0
7	0.00499	0.00739	0.00215	0.00299	0.00442	0.00689	0.01461	0.00594	0.00423	0.00149	0
8	0.01929	0.02436	0.00784	0.00699	0.01222	0.01158	0.00594	0.03886	0.01114	0.02223	0
9	0.01829	0.01329	0.00384	0.00956	0.01128	0.00785	0.00423	0.01114	0.04710	0.01185	0
10	0.01278	0.01270	0.00853	0.00813	0.01397	0.00910	0.00149	0.02223	0.01185	0.05474	0
11	0	0	0	0	0	0	0	0	0	0	0

Now, the mean/expected return vector of risky assets only is represented as M_1, and the covariance matrix of risky assets only is represented by C_1.
So, now the weight vector is represented as

$$W = (W_1, W_2, \dots, W_{10}, W_0)^T, \text{ where } (\Sigma W_i + W_{rf}) = 1 \text{ and } \Sigma W_i = W_{risky}$$

Portfolio return:

$$\mu = M \cdot W^T$$

Portfolio risk:

$$\sigma^2 = W \cdot C \cdot W^T = \sigma^2_{risky}$$

Derived Portfolio:

$$W^* = \left(\frac{W_1}{W_{risky}}, \frac{W_2}{W_{risky}}, \ldots, \frac{W_{10}}{W_{risky}}\right) = (W_1^*, W_2^*, \ldots, W_{10}^*)$$

Derived Portfolio Return and Risk:

$$\mu_{der} = M_1 \cdot (W^*)^T$$

$$\sigma_{der}^2 = W^* \cdot C_1 \cdot (W^*)^T$$

From the derived portfolio equations we get:

$$\mu = \mu_{rf} + \left(\frac{\mu_{der} - \mu_{rf}}{\sigma_{der}}\right)\sigma \quad \text{and} \quad W_{risky} = \frac{\sigma}{\sigma_{der}}$$

W^*, the weight corresponding to the optimal point (μ_M, σ_M) is given by:

$$W^* = \frac{(M_1 - \mu_{rf} \cdot \mathbf{O}) \cdot C_1^{-1}}{(M_1 - \mu_{rf} \cdot \mathbf{O}) \cdot C_1^{-1} \cdot \mathbf{O}^T}$$

On solving we get W^* to be:

Asset No.	1	2	3	4	5	6	7	8	9	10
W^*	0.5073	0.3210	0.1700	0.1509	-0.0172	0.3601	-0.4007	-0.0912	0.3387	-0.3389

The optimum point (μ_M, σ_M) obtained is: $(0.0042, 0.0162)$.

Now, for a given value of μ, we can find W_{risky} from the equation:

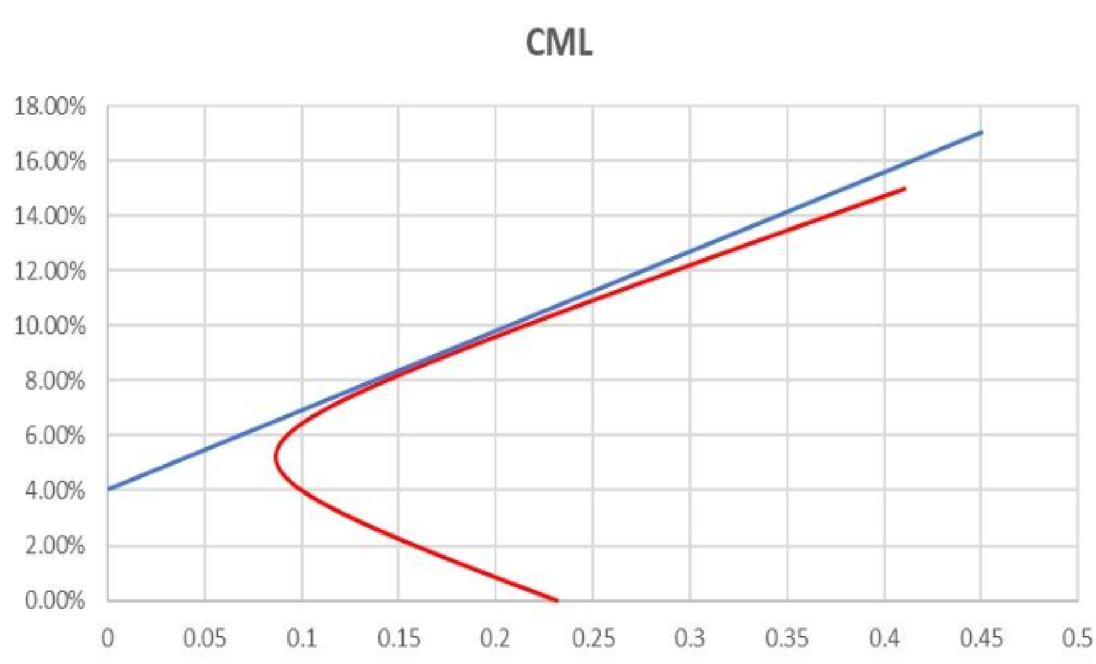
$$\mu = \mu_{rf} + (\mu_M - \mu_{rf}) W_{risky}$$

And the final weight vector (W):

$$W = (W_1, W_2, \ldots, W_{10}, W_r), \text{ where } W_i = W_i^* \cdot W_{risky} \text{ and } W_r = 1 - W_{risky}$$

And finally finding the portfolio risk corresponding to given μ. This helps us in making the Capital Asset Pricing Model Market Line which is a tangent to the efficient frontier of the risky assets.

Here is the graph between Mean and Standard Deviation:



It is clearly observed that the Capital Market Line is tangent to the efficient frontier of the risky assets and the tangent point is $(\mu_M, \sigma_M) : (0.0042, 0.0162)$.

The reaction of the Security Market Line (SML) is given as:

$$\mu_k = \mu_f + (\mu_M - \mu_f)\beta_k, \quad \text{where } \beta_k = \frac{\operatorname{cov}(R_M^r, R_k)}{\sigma_M^2}$$

The value of β_k is calculated for all the risky assets:

Asset No.	1	2	3	4	5	6	7	8	9	10
β_k	1.1948	0.1225	0.5317	0.0521	0.5667	0.0887	0.0068	0.3305	0.7945	0.0062

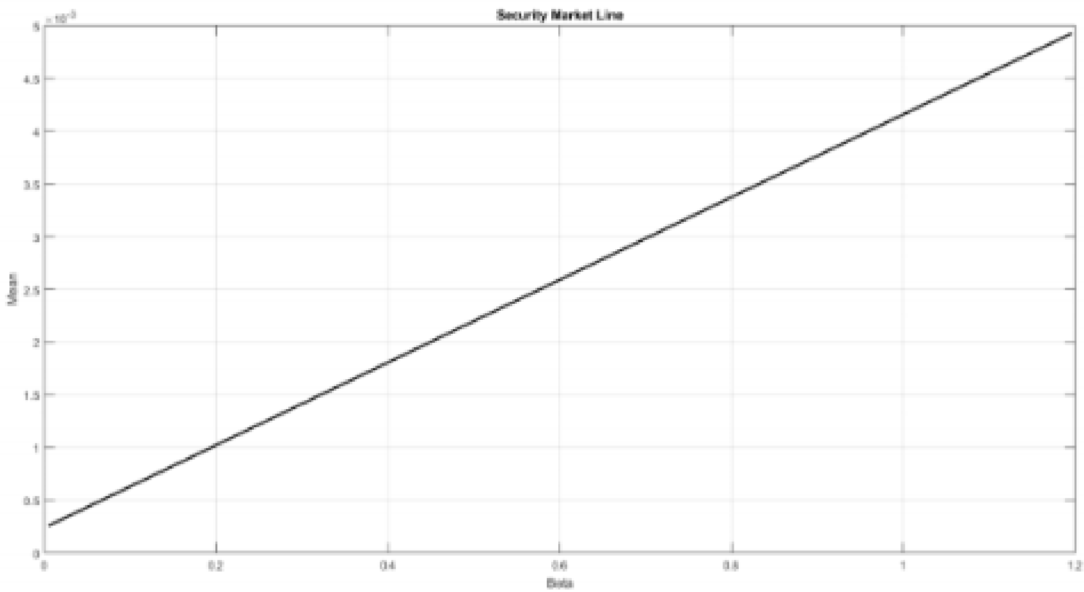
WHAT IS SHARPE RATIO?

- The Sharpe ratio compares the return of an investment with its risk.
- It's a mathematical expression of the insight that excess returns over a period of time may signify more volatility and risk, rather than investing skill.

$$S = \frac{R_p - R_f}{\sigma_p}$$

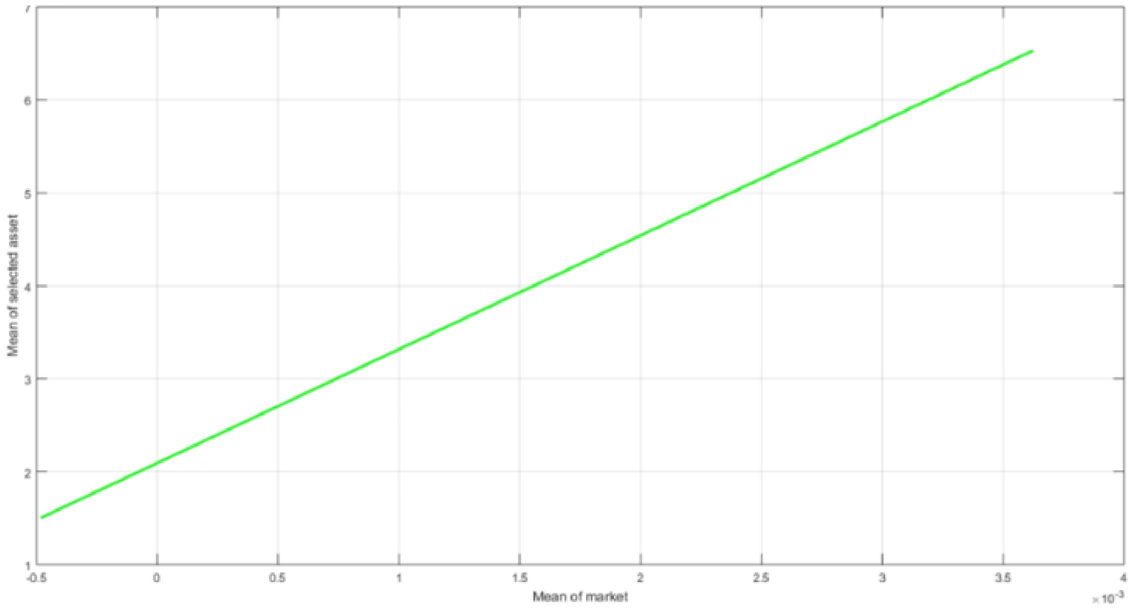
- **S** = Sharpe Ratio
- $\mathbf{R_p}$ = Return of portfolio
- $\mathbf{R_f}$ = Risk-free return
- σ_p = Standard deviation of the portfolio's excess return

For tangent point $(\mu_M, \sigma_M) : (0.0042, 0.0162)$ the relationship between the μ_k and β_k can be plotted as:



4. Use any two assets out of the 10 risky assets to get two different SMLs.

We take two assets out of the risky assets to get different SMLs: Tata Motors, Sun Pharma.
For Tata Motors, the security market line is:



For Sun Pharma, the security market line is: