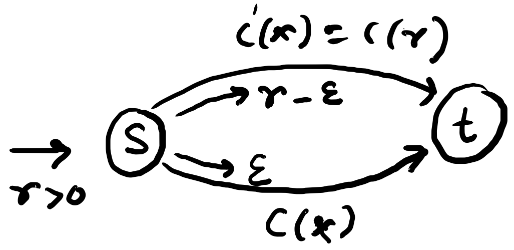
Lecture 7.5

PoA of Pigou's network:

$$\alpha(c) = \sup_{\varepsilon \in [0,r]} \left[\frac{r\, c(r)}{\varepsilon\, c(\varepsilon) + (r-\varepsilon)\, c(r)} \right]$$



Observe that, $\boxed{\varepsilon\, c(\varepsilon)} + (r-\varepsilon)\, c(r)$ is non-decreasing in $\varepsilon \in [r, \infty)$ since $c(\cdot)$ is a non-decreasing function.

$$\alpha(c) = \sup_{\varepsilon \geq 0} \frac{r\, c(r)}{\varepsilon\, c(\varepsilon) + (r-\varepsilon)\, c(r)}$$

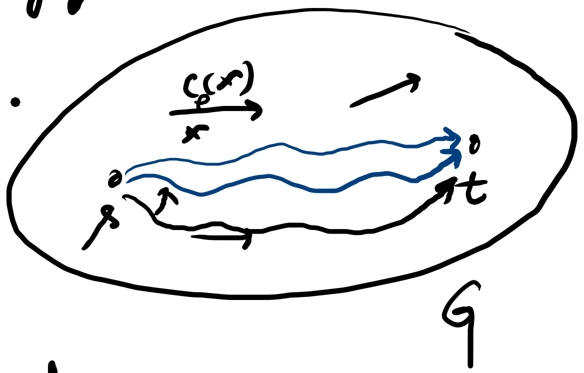
We let the cost function c belong to a class of function $\mathcal{C}$.

$$\alpha(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{r > 0} \alpha(c)$$

$$= \sup_{c \in \mathcal{C}} \sup_{r > 0} \sup_{\varepsilon \geq 0} \left[\frac{r\, c(r)}{\varepsilon\, c(\varepsilon) + (r - \varepsilon)\, c(r)} \right]$$

Theorem: For any class $\mathcal{C}$ of cost functions, the PoA of any network with one source and one destination where all the cost functions belong to $\mathcal{C}$, is at most $\alpha(\mathcal{C})$.

Proof: Let G be a graph, $f: E[G] \to \mathbb{R}_{\geq 0}$ be an s-t flow in G. Observe that every strategy profile corresponds to a flow in G.



Observe that a flow f corresponds to a PSNE if and only if only shortest paths carry any flow.

$$f_{\bar{P}} > 0 \Rightarrow \bar{P} \in \underset{P \in \mathcal{P}}{\operatorname{argmin}} \left\{ \sum_{e \in P} c_e(f_e) \right\}$$

Let f^* be a flow which minimizes the sum of cost of all the edges.

$$f^* \in \underset{\substack{f \text{ is an } s\text{-}t \text{ flow of} \\ \text{value } r}}{\operatorname{argmin}} \left\{ \sum_{e \in E} f_e \, c_e(f_e) \right\}$$

Since PSNE flow f^{eq} only uses shortest paths according to the cost function $c_e(f_e^{eq})$, $e \in E[G]$, all the s-t path carrying any flow in f^{eq} must have the same cost.

$$\sum_{p \in \mathcal{P}} f_p^{eq} \, \underline{c(f_p^{eq})} = rL$$, where L is the cost of the min-cost path. —(1)

$$\sum_{p \in \mathcal{P}} f_p^{*} \, \underbrace{c(f_p^{eq})}_{\geq L} \geq L \sum_{p \in \mathcal{P}} f_p^{*} = rL \quad \text{—(2)}$$

(1) − (2)

$$\sum_{p \in \mathcal{P}} (f_p^{eq} - f_p^{*}) \, c(f_p^{eq}) \leq 0 \quad \text{—(3)}$$

$\alpha(\mathcal{C})$ is supremum over all $c \in \mathcal{C}$, $r > 0$, $\varepsilon \geq 0$, we have the following for any edge $e \in E[G]$.

$$\alpha(\mathcal{C}) \geq \frac{f_e^{eq} \cdot c_e(f_e^{eq})}{f_e^{*} c_e(f_e^{*}) + (f_e^{eq} - f_e^{*}) c_e(f_e^{eq})}$$

$$\xrightarrow[e]{(r \Rightarrow) f_e^{eq}} \quad c_e(f_e^{eq})$$

$\varepsilon = f_e^{*}$

$r = f_e^{eq}$

$$\Rightarrow f_e^{*} c_e(f_e^{*}) \geq \frac{f_e^{eq} \cdot c_e(f_e^{eq})}{\alpha(\mathcal{C})} - (f_e^{eq} - f_e^{*}) c_e(f_e^{eq})$$

$$\Rightarrow \underbrace{\sum_{e\in E} f_e^* c_e(f_e^*)}\geq \frac{1}{\alpha(C)} \underbrace{\sum_{e\in E} f_e^{eq} c_e(f_e^{eq})} - \underbrace{\sum_{e\in E} (f_e^{eq} - f_e^*)\, c_e(f_e^{eq})}_{\leq 0}$$

$$\Rightarrow \quad c(f^*) \geq \frac{c(f^{eq})}{\alpha(C)}$$

$$\Rightarrow \quad \frac{c(f^{eq})}{c(f^*)} \leq \alpha(C)$$

$$\Rightarrow \quad PoA \leq \alpha(C)$$

$\blacksquare$