Computing Equilibrium

Lecture 4.1

Support enumeration algorithm for NASH for 2 players.

Computing a PSNE Given a game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ find a PSNE.

→ Observe that checking if a given pure strategy profile is a PSNE takes polynomial time.

Let all players have s strategies, and there are n players.

The number of strategy profiles is s^n.

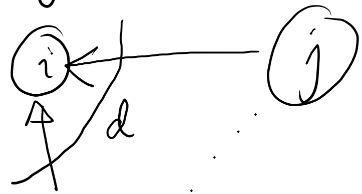
The input consists of $\underline{n \cdot s^n}$ utility values (numbers).

$O(n \cdot s)$ time is required to check if a pure strategy profile is a PSNE.

We can find a PSNE (if it exists) in $O(\underline{n \cdot s^{n+1}})$ time.

Important Classes of Succinct Games

1. Graphical games: We have a directed graph G on the set N of players. The utility of a player i depends only on the players who have a directed edge to i, including i. If the in-degree of G is at most d, then $(n \cdot s^{d+1})$ numbers are needed to represent the game.



2. <u>Sparse game</u>: In all but a few strategy profiles give non-zero utility.

3. <u>Symmetric game</u>: All players are identical. The utility of a player depends on the strategy played by the player and the number of players playing each strategy.

4. <u>Anonymous game</u>: The utility still depends **only** on the the number of players playing each strategy.

$$n \cdot s \cdot \binom{n-1+s-1}{s-1} = n \cdot s \cdot \binom{n+s-2}{s-1}$$ numbers in input.

For symmetric game: $s \cdot \binom{n+s-1}{s-1}$.

5. <u>Network Congestion game:</u> we have a graph G. Each player i has a source s_i and destination t_i. The load $l(e)$ of each edge is the number of players using that edge. Each edge e has a non-decreasing cost function $c_e : \mathbb{R} \to \mathbb{R}$. The strategy set of player i is the set of paths from s_i to t_i in G.

Let $(P_i)_{i \in N}$ be a strategy profile.

$$u_i((P_i)_{i \in N}) = -\sum_{e \in P_i} c_e(\ell(e))$$

Succinct Games

Graphical games, sparse games, symmetric games, anonymous games, network congestion games.

6. Congestion game: Generalization of network congestion game. We have a set $\mathcal{E}$ of resources. A strategy is a subset of $\mathcal{E}$. A strategy set of a player is a subset of power set of $\mathcal{E}$.

Load of a resource e is the number of players in a strategy profile using that resource.

Each resource has a non-decreasing cost function.

The utility of a player in a strategy profile is the sum of the cost of the resources it is using.

7. <u>Multi-matrix game:</u> For each $(i,j) \in [n] \times [n]$, $i \neq j$, we have

a $m \times m$ matrix A^{ij}

The utility of player i in a strategy profile $(s_j)_{j \in [n]}$,

is $\sum_{\substack{j=1 \\ j \neq i}}^{n} A^{ij}(s_i, s_j)$

Potential Game

A game is called a potential game if it has a "potential function".

<u>Theorem:</u> (Rosenthal) Every network congestion game has at least one PSNE.

<u>Proof:</u> Let f be a flow in G.

$f : E(G) \longrightarrow \mathbb{N}$ associated with strategy profiles.

"potential function maps strategy profiles to real numbers".

$$\Phi(f) = \sum_{e \in E[G]} \sum_{i=1}^{f(e)} c_e(i)$$

If there is a player i who can reduce its cost by changing its path from P_i to P_i', then the reduction in cost of player i is the same as the reduction in potential value.

f be the flow corresponding to (P_i, P_{-i})

$\hat{f}$ —— —— (P_i', P_{-i})

$$\Phi(\hat{f}) - \Phi(f) = \sum_{e \in P_i' \setminus P_i} c_e(\hat{f}_e) - \sum_{e \in P_i \setminus P_i'} c_e(f_e)$$

$$= \underbrace{\sum_{e \in P_i'} c_e(\hat{f}_e) - \sum_{e \in P_i} c_e(f_e)}$$

decrease in cost for player i by deviating unilaterally from P_i to P_i'.

$\overline{\notin P_i, P_i'}$ $\quad$ $\overline{\in P_i, \notin P_i'}$

$\overline{\in P_i', \notin P_i}$ $\quad$ $\overline{\in P_i, \in P_i'}$

G

The domain of Φ is a finite set. So it attains a minimum value. Clearly, the pure strategy profile corresponding to the minimum value is a PSNE.

$\square$

Potential Game: A game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is called a potential game if there exists a function Φ s.t.

$$\forall i \in N \; \forall s_i, s_i' \in S_i, s_{-i} \in S_{-i}, \quad \Phi(s_i, s_{-i}) - \Phi(s_i', s_{-i}) = u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}).$$

$\square$

Lecture 4.3

Potential Game

Best response dynamic :-

1 - Pick any strategy profile s

2 - If s is not a PSNE

a - pick a player i who has a beneficial unilateral deviation from s_i to s_i'.

b - replace s_i with s_i' in s.

c - Go to step 2

Best response dynamic always terminates with a PSNE for finite potential games.

For arbitrary games, the best response dynamic may not terminate.

Even for potential games, for example the network congestion games, what is the speed of convergence ??

If the potential function takes only polynomially many values, then the best response dynamic converges in polynomial time.

<u>ε-PSNE:</u> Given a game $\Gamma = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N} \rangle$ a pure strategy profile $(s_i)_{i\in N} \in \bigtimes_{i\in N} S_i$ is called an ε-PSNE if

$$\forall i \in N, \forall s_i' \in S_i, \quad u_i(s_i', s_{-i}) \leq (1+\varepsilon)\, u_i(s_i, s_{-i})$$

or, $$\forall i \in N, \forall s_i' \in S_i, \quad c_i(s_i', s_{-i}) \geq (1-\varepsilon)\, c_i(s_i, s_{-i})$$

ε-Best Response Dynamic:

1. Pick any strategy profile s.
2. If s is not an ε-PSNE,
   a. Pick a player i who has a move from s_i to s_i' which increases its utility by more than $(1+\varepsilon)$ times its current utility.
   b. Replace s_i with s_i'
3. Go to step 2.

Therem (Fast convergence of ε-Best Response Dynamics)

In an atomic network congestion game, suppose the following holds.

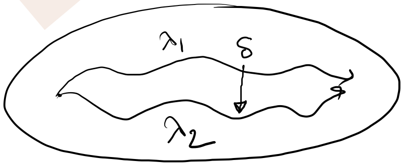
- All the players have the same source and destination.
- Cost function satisfies "α-bounded jump condition" $(\alpha > 1)$:

  $c_e(x+1) \in [c_e(x), \alpha c_e(x)]$ for all edge e and all positive real number x.
- The max gain version of ε-Best response dynamic is used.

among all players who have an ε-best response move, pick a player and a move which achieves largest absolute cost decrease, for that player.

Then an ε-PSNE will be reached in $O\left(\frac{n\alpha}{\varepsilon}\log\frac{\Phi(s^0)}{\Phi(s^{min})}\right)$ number of iterations, s^0 is the initial strategy profile, $\Phi(s^{min})$ is the minimum value of the potential function.

Theorem: In an atomic network congestion game suppose the following holds:



- The source and destination are the same for all the players.
- The cost functions satisfy α-bounded jump condition.
- Max-gain version of ε-Best response dynamic is used.

Then a ε-PSNE will be reached in $O\left(\frac{n\alpha}{\varepsilon} \log \frac{\Phi(s^0)}{\Phi(s^{min})}\right)$ iterations.

Proof: Let s be a strategy profile which is not an ε-PSNE. The proof has two parts. In the first part we show that there exists a player i whose cost in s is "high". In the second part, we show that if player j is chosen by the max-gain version of ε-best response dynamics, then the drop in potential is some significant fraction of the cost of player i in s.

Claim: In every strategy profile s, there exists a player $i^* \in N$ such that $C_{i^*}(s) \geq \frac{\Phi(s)}{n}$.

Proof:

$$C(s) = \sum_{i \in N} C_i(s) = \sum_{e \in E[G]} f_e \underline{c_e(f_e)}$$

$$\Phi(s) = \sum_{e \in E[G]} \sum_{i=1}^{f_e} c_e(i)$$

$$\Phi(s) \leq C(s) = \sum_{i \in N} C_i(s)$$

$\Rightarrow$ There exists a player $i^* \in N$ s.t. $C_{i^*}(s) \geq \frac{\Phi(s)}{n}$.

Claim: Let j makes a move from s_j to s_j' in the current iteration of the max-gain version of ε-best response dynamics. Then,

$$c_j(s_j, s_{-j}) - c_j(s_j', s_{-j}) \geq \frac{\varepsilon}{\alpha} c_i(s) \quad \forall\, i \in N,\ i \neq j$$

Proof: Let fix a player $i \in N$, $i \neq j$.

case I: player i has an ε-move.

$$c_j(s_j, s_{-j}) - c_j(s_j', s_{-j}) \geq \max_{s_i' \in S_i} \left\{ c_i(s_i, s_{-i}) - c_i(s_i', s_{-i}) \right\}$$

$$\geq \varepsilon \cdot c_i(s)$$

$$\geq \frac{\varepsilon}{\alpha} \cdot c_i(s)$$

Case II: Player i does not have an ε-move.

The strategy (path) s_j' is available to player i also.

$$c_i(s_j', s_{-i}) > (1-\varepsilon)\, c_i(s) \quad \text{——— (1)}$$

However, player j has an ε-move.

$$c_j(s_j', s_{-j}) \leq (1-\varepsilon)\, c_j(s) \quad \text{——— (2)}$$

Due to α-bounded jump property,

$$c_i(s_j', s_{-i}) \leq \alpha\, c_j(s_j', s_{-j}) \quad \text{—— (3)}$$

$$c_i(s) \underset{(1)}{<} \frac{c_i(s_j', s_{-i})}{1-\varepsilon} \underset{(3)}{\leq} \frac{\alpha\, c_j(s_j', s_{-j})}{1-\varepsilon} \underset{(2)}{\leq} \frac{\alpha\, \cancel{(1-\varepsilon)}\, c_j(s)}{\cancel{(1-\varepsilon)}} = \alpha \cdot c_j(s)$$

$$c_i(s) < \alpha\, c_j(s) \Rightarrow c_j(s) > \frac{c_i(s)}{\alpha}$$

from (2),

$$c_j(s) - c_j(s_j', s_{-j}) \geq \varepsilon\, c_j(s) > \frac{\varepsilon\, c_i(s)}{\alpha} \qquad \boxtimes$$

$$\Phi(s) - \Phi(s_j', s_{-j}) = c_j(s) - c_j(s_j', s_{-j})$$

$$\geq \frac{\varepsilon}{\alpha} \cdot \max_{\substack{i \in N \\ i \neq j}} c_i(s)$$

$$\geq \frac{\varepsilon\, \Phi(s)}{n\alpha} \longrightarrow \Phi(s_j', s_{-j}) \leq \left(1 - \frac{\varepsilon}{n\alpha}\right) \Phi(s)$$

The number of iteration needed $O\left(\frac{n\alpha}{\varepsilon} \log \frac{\Phi(s^0)}{\Phi(s^{min})}\right)$. $\boxtimes$

Lecture 4.5

Theorem: In an atomic network congestion game, suppose the following holds:

$\star$
i) All players have the same source and destination.

ii) Cost functions have α-bounded jump.

iii) Max-gain version of ε-best response dynamic is used.

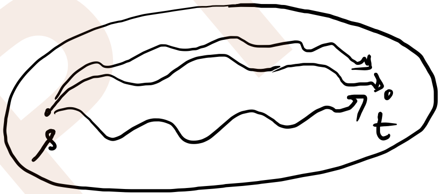
Then an ε-PSNE is reached in $O\left(\frac{n\alpha}{\varepsilon}\log\frac{\Phi(s^0)}{\Phi(s^{min})}\right)$ iterations.

Corollary: For an atomic network congestion game satisfying $(\star)$, an ε-PSNE can be computed in

$$O\left(\frac{n\alpha}{\varepsilon}\log\frac{\Phi(s^0)}{\Phi(s^{min})}\cdot \mathrm{poly}(N, m, n)\right)$$

Proof: Enough to show: each iterations of the max-gain version of ε-best response dynamics can be executed in $\mathrm{poly}(N, m, n)$ time.

- Let us fix a player i and a strategy profile s'.



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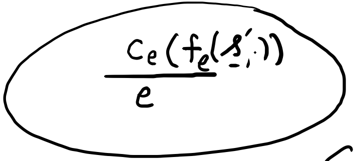
- Consider s'_{-i}. define weight of an edge e to be $c_e(f_e(s'_{-i}))$.



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Find the shortest s–t path p.

- Let the costs of s'_i and p be c'_i and c_p.
- If $c_p > (1-\varepsilon)\, c'_i$, then player i does not have

any ε-move.

- The absolute reduction in cost for player i is

$c_i' - c_p$.

- Since single-source shortest weight path can be computed in polynomial time, the result follows. $\square$

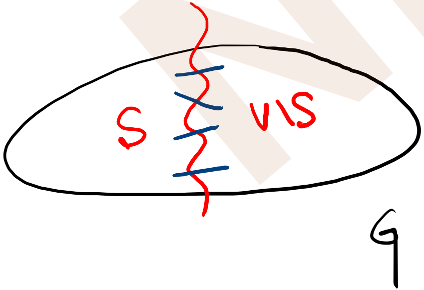
Qn: Can we relax the conditions in ☆?

Qn: Can the result be generalized to congestion games?

Local Search

A local search problem is to find a local minima for an optimization problem.

The canonical local search problem is weighted maximum-cut.

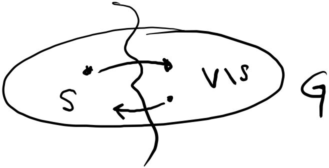


$S \neq \phi, \; S \neq V$

The weight of a cut is the sum of the weights of the <u>cut edges</u> (edges having one end point in S and another in $V \setminus S$).

Local maximum cut

A cut whose size can not be increased further by moving any "single" vertex from its current set.



i.e. no improvement possible by local moves.

Canonical NP problem is SAT.