1) Inclusion and exclusion principle states for finite sets that number of elements in them all combined together is sum of the number of elements in each set (inclusion), subtracting the number of elements that has counted more than once (exclusion), i.e, for two finite sets A and B :-

$$n(A \cup B) = \underbrace{n(A) + n(B)}_{\text{inclusion}} - \underbrace{n(A \cap B)}_{\text{exclusion}}$$

for 3 finite sets A, B, C :-

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - \{n(A \cap B) + n(A \cap C) + n(B \cap C)\} + n(A \cap B \cap C)$$

Application of inclusion and exclusion principle

(i) How many solutions does

$$x_1 + x_2 + x_3 = 11$$

where x_1, x_2 and x_3 are nonnegative integers with $x_1 \leq 3$, $x_2 \leq 4$ and $x_3 \leq 6$

(ii) count the number of onto functions

(iii) Derangements:

The principle of inclusion - exclusion will be used to count the permutations of n objects that leave no objects in their original positions. for example the Hatcheck problem

The number of derangements of a set with n elements is

$$D_n = n! \left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right]$$

(2.) Define Power sets of a set and find the power set of $S = \{1, 2, 3\}$

Power set :- for a given set S, there is a class of all subsets of S. This class is called the power set of S and will be denoted by P(S) if S is finite then so is P(S)

the number of elements in P(S) is 2 raised to the power n(S) That is

$$n(P(S)) = 2^{n(S)}$$

power set of S is also denoted as 2^S

find the power set of $S = \{1, 2, 3\}$

$P(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, S\}$

empty set ϕ is also a subset of S.

Hence P(S) has $2^3 = 8$ elements.

$3 = n(S) =$ no. of element in original set

(3.) Explain Principal of mathematical and by using mathematical induction method prove that $1 + 3 + 5 + \cdots (2n-1) = n^2$

Principal of mathematical induction :-

Suppose there is a given statement P(n) involving the natural number n such that

(i) Basis step: The statement is true for $n=1$
i.e. $p(1)$ is true
(ii) Inductive step: if the statement is true for $n=k$ (where k is some positive integer), then the statement is also true for $n=k+1$
i.e., truth of $p(k)$ implies the truth of $p(k+1)$
then $p(n)$ is true for all natural number n.

$1=1 \qquad 1+3=4 \qquad 1+3+5=9$

$1+3+5+7=16 \qquad 1+3+5+7+9=25$

$1+3+5+ \text{-------} (2n-1)=n^2 \quad \text{---(i)}$

Basis step: $p(1)$ states that the sum of the first one add positive integer is 1^2
This is true because the sum of the first add positive integer is 1

Inductive step:
To complete the inductive step we must show that the proposition $p(k) \rightarrow p(k+1)$ is true for every positive integer k.

The inductive hypothesis is the statement that $p(k)$ is true for an arbitrary positive integer k, that is

$1+3+5+ \text{------} - +(2k+1)=k^2$

is true we must known that if $P(k)$ is true (the inductive hypothesis) then $P(k+1)$ is true hence

$$1+3+5+\ldots\ldots(2k-1)+(2k+1) = k^2+2k+1$$
$$1+3+5+\ldots\ldots(2k-1)+(2k+1) = (k+1)^2$$
$$1+3+5+\ldots\ldots(2k+1)^2 = (k+1)^2$$

above equation is similar when we put $n = k+1$ in equation (i)

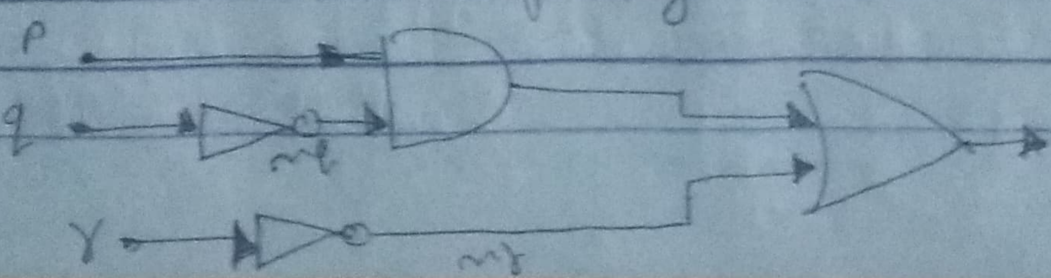
This shows that $P(k+1)$ follows from $P(k)$

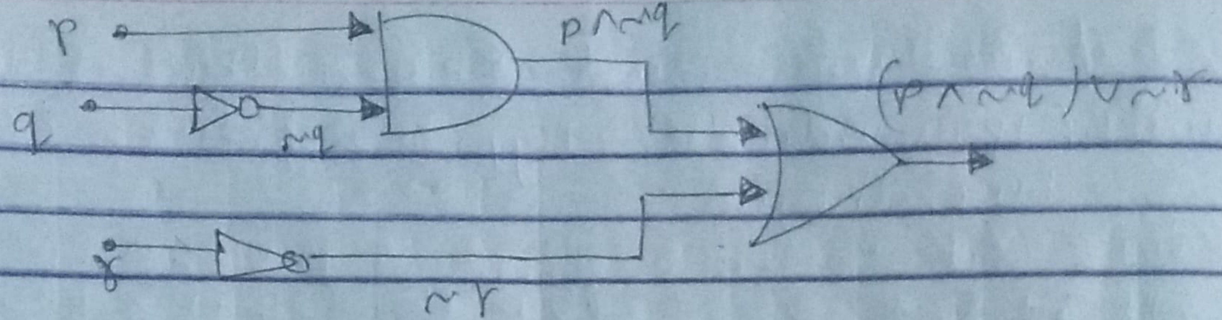
4. Construct the truth table for the compound proposition
$$(p \vee \sim q) \rightarrow (p \wedge q)$$

Truth Table

p	q	$\sim q$	$p \vee \sim q$	$p \wedge q$	$(p \vee \sim q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

5. Find the output for logic circuit. Construct the truth table of logic circuit.





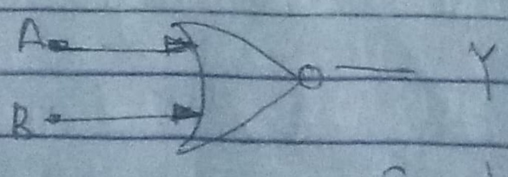
(6.) Define Basic logic gate and explain NOR gate and NAND gate. construct the truth table of all logic gates.

logic gate :-

- A gates is a digital circuit which operates on one or more signals and produce single output.
- Gates are digital circuits because the input and output signals are denoted by either 1 (high voltage) or 0 (low voltage)

NOR gate:

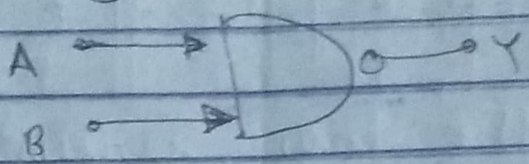
The NOR gate is one of the universal gates. A NOR gate combines an OR gate and NOT gate. So we can say it is an OR + NOT operation. it may have two or more inputs and an output.



Symbol

NAND gate :

The NAND gate or "NOT-AND" gate is the combination of two basic logic gate. the AND gate and the NOT gate connected in series. The NAND gate is also called a universal gate



The truth table for conjunction of two propositions.

p	q	p∧q
T	T	T
T	F	F
F	T	F
F	F	F

The truth Table for the disjunction of two propositions.

p	q	p∨q
T	T	T
T	F	T
F	T	T
F	F	F

truth table for exclusive OR

p	q	p ⊕ q
T	T	F
T	F	T
F	T	T
F	F	F

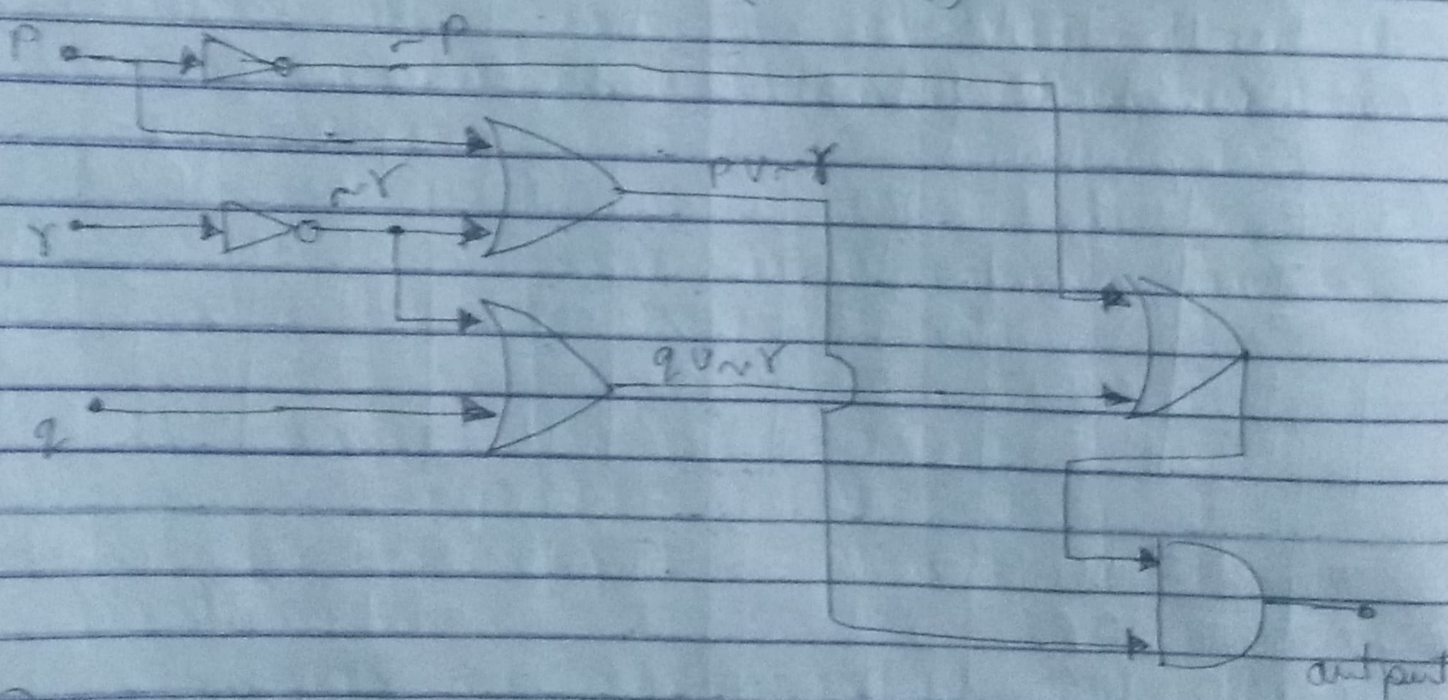
Truth table for conditional statements

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

Truth table for biconditional

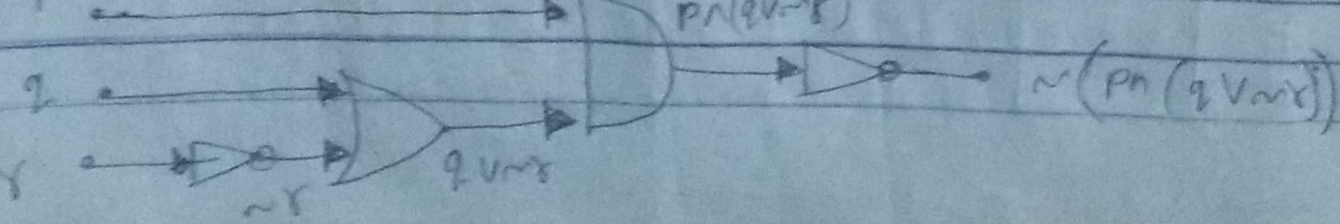
p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

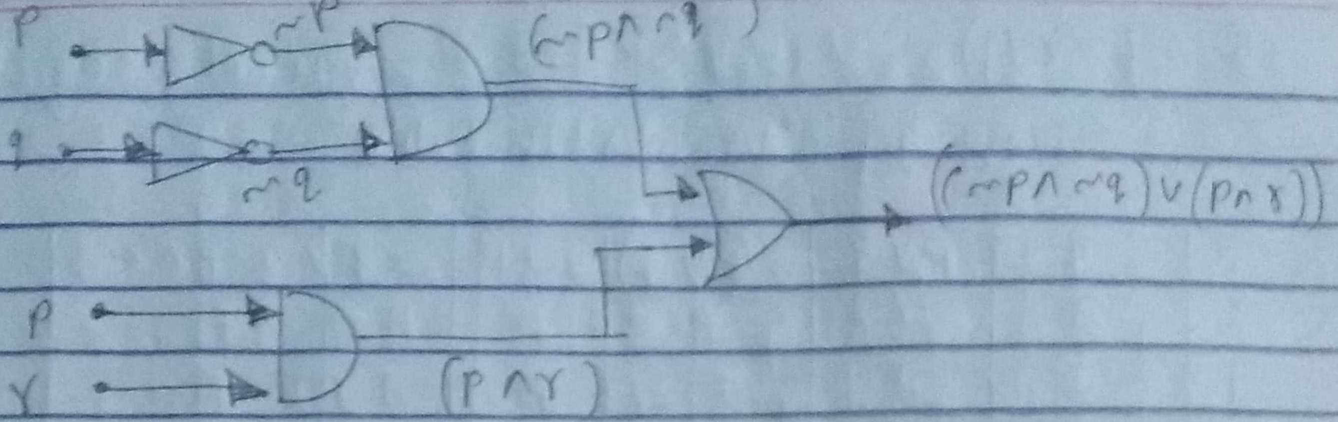
(7) Construct the logic circuit for
$(p \vee \sim r) \wedge (\sim p \vee (q \vee \sim r))$



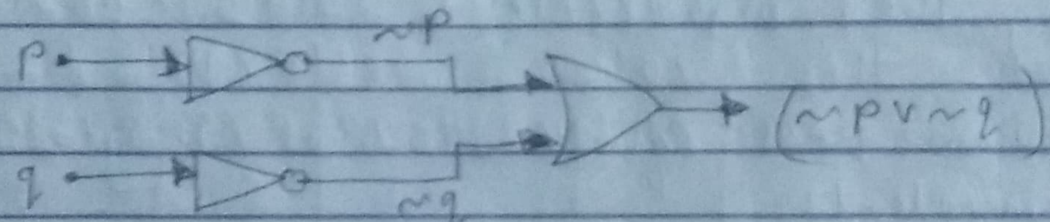
(8) Find the output of each of these combinational circuits.

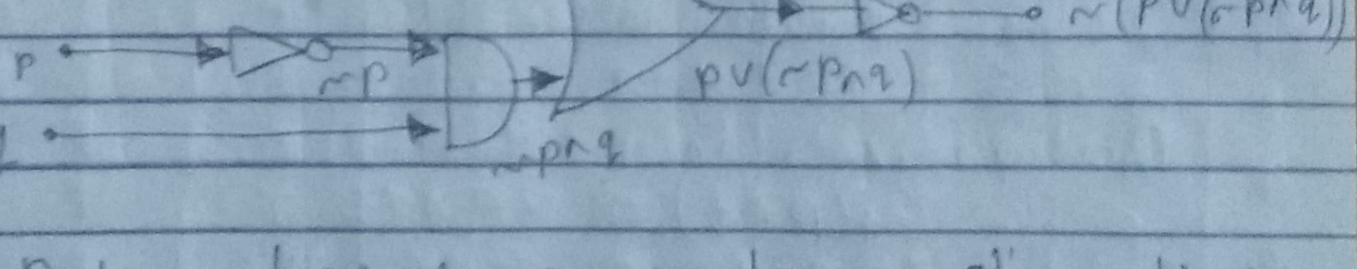
(a)



(b) 

9. Find the output of each of these combinational circuits

(a) 

(b) 

10. Define logical equivalences. Show that $\sim(p \vee q)$ and $\sim p \wedge \sim q$ are logically equivalent.

Logical equivalences:-
The compound propositions p and q are called logically equivalent if $p \leftrightarrow$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

$\sim(p \vee q)$ and $\sim p \wedge \sim q$ are logically equivalent
solved by the truth table

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

by the above table we can easily see that the values in $\sim(p \vee q)$ and $\sim p \wedge \sim q$ column are equal. hence they are logically equivalent

11. State and prove De Morgan's law by using truth table.

De morgan's law

$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$

$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$

first law:

Demorgan's theorem state that the negation of an logical AND operation of two statement gives the logical or operation on individual negation operation of given statement second law:

Demorgan's theorem that that the negation on logical OR operation of two statements gives the logical AND operation on individual negations operation of given statements.

Proof by table.

(i) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

(ii) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

(12) (a) show that $p \to q$ and $\sim p \vee q$ are logically equivalent.

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim p \vee q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

(b) show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

(c) show that $\sim (p \to q)$ and $p \wedge \sim q$ are logical equivalent

p	q	$p \to q$	$\sim (p \to q)$	$\sim q$	$p \wedge \sim q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

(e) show that $(p \to q) \wedge (p \to r)$ and $p \to (q \wedge r)$ are logically equivalent.

p	q	r	$p \to q$	$(p \to r)$	$(p \to q) \wedge (p \to r)$	$q \wedge r$	$p \to (q \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	F	F	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	F	T
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

(f.) show that $p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$ are logically equivalent.

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

(d) Show that $\sim(p \vee (\sim p \wedge q))$ and $\sim p \wedge \sim q$ are logically equivalent by developing a series of logical equivalences.

$$\sim(p \vee (\sim p \wedge q)) \equiv \sim p \wedge \sim q$$

in LHS $\quad \sim(p \vee (\sim p \wedge q))$

$$= (\sim p \wedge p) \vee (\sim p \wedge \sim q)$$

$$= \phi \vee (\sim p \wedge \sim q)$$

$$= (\sim p \wedge \sim q) = RHS$$

Hence proved.

(13.) Explain Quantifiers with example. What are the negations of the statements $\forall x (x^2 > x)$ and $\exists x (x^2 = 2)$

Quantifiers.

The universal quantification of $P(x)$ is the statement.

"$P(x)$ for all values of x in the domain."

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the universal quantifier. We read $\forall x P(x)$ as "for all x $P(x)$" or "for every x $P(x)$". An element for which $P(x)$ is false is called a counter example of $\forall x\, P(x)$

Quantifiers

Statement	When True?	When False?
$\forall x\, P(x)$	$P(x)$ is true for every x	There is an x for which $P(x)$ is false
$\exists x\, P(x)$	There is an x for which $P(x)$ is true	$P(x)$ is false for every x

finding the negation of $\forall x (x^2 > x)$

$= \sim(\forall x (x^2 > x))$

$= \exists x (x^2 \leq x)$

finding the negation of $\exists x (x^2 = 2)$

$= \sim(\exists x (x^2 = 2))$

$= \forall x (x^2 \neq 2)$

(15.) use mathematical induction to show that

$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$

for all nonnegative integers n

let $P(n)$ be the preposition that

$1 + 2 + 2^2 + \cdots 2^n = 2^{n+1} - 1$

for the integer n

Basis step: $P(0)$ is true because $2^0 = 1 = 2^1 - 1$

Inductive step: for inductive hypothesis, we assum that $P(k)$ is true for an arbitrary nonnegative integer k. that is we assume that.

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1 \quad \text{———} (i)$$

$P(k)$ is true then $P(k+1)$ is also true hence

$$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1 \quad \text{—} (ii)$$

by using equation (i) if we add 2^{k+1} in both side then we get the same result as equation (ii)

$$\begin{aligned} 1 + 2 + 2^2 + \cdots 2^k + 2^{k+1} &= (1 + 2 + 2^2 + \cdots + 2^k) + 2^{k+1} \\ &= (2^{k+1} - 1) + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+2} - 1 \end{aligned}$$

By using the basis and inductive step by ~~step~~ mathematical induction we know that $P(n)$ is true for all non negative integers n.

(16) What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$

$$x^2 - x - 2 = 0$$

$$x_1 = 2 \quad \text{and} \quad x_2 = -1$$

$$a_n = \alpha(x_1)^n + \beta(x_2)^n$$

$$a_n = \alpha(2)^n + \beta(-1)^n$$

$$a_0 = \alpha(2)^0 + \beta(-1)^0$$

$$a_0 = \alpha + \beta$$

$$a_1 = \alpha(2)^1 + \beta(-1)^1$$

$$a_1 = 2\alpha - \beta$$

but $a_0 = 2$ and $a_1 = 7$ given

Hence

$$a_0 = \alpha + \cancel{\beta} = 2$$

$$a_1 = 2\alpha - \cancel{\beta} = 7$$

$$3\alpha = 9$$

$$\alpha = 3$$

$$\alpha + \beta = 2 \Rightarrow 3 + \beta = 2 \Rightarrow \beta = -1$$

Hence $\alpha = 3$ and $\beta = -1$

(19) Explain Equivalence relation with example.

Equivalence Relation:
An equivalence relation R a set S is one that satisfies these three properties for all x, y, z ∈ S.

(i) Reflexive xRx, $\forall x \in R$
let $A = \{a, b, c, d\}$ and R be defined as follows
$R = \{(a,a), (a,c), (b,a), (b,b), (c,c), (d,c), (d,d)\}$
R is a reflexive relation

(ii) Symmetric: if xRy, then yRx, $\forall x, y \in R$
let $A = \{a, b, c, d\}$ and $R = \{(a,a), (b,c), (c,d), (d,d)\}$
show that R is symmetric

(iii) Transitive: if xRy and yRz then xRz, $\forall x, y, z \in R$.
let $A = \{a, b, c, d\}$ and R be defined as follows. $R = \{(a,b) (a,c), (b,d), (a,d), (b,c), (d,c)\}$.
then R is transitive relation on A

example For any non-empty set S, the equality relation = defined by the subset $\{(x, y) \mid x \in S\}$ of $S \times S$ is an equivalence relation

(18) What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with initial condition $a_0 = 1$ and $a_1 = 6$

$$x^2 - 6x + 9 = 0$$

$$x = 3, 3$$

$$a_n = \alpha(x_1)^n + \beta.n.(x_2)^n$$

$$= \alpha(3)^n + \beta.n.(3)^n$$

$$a_0 = \alpha(3)^0 + \beta.0.(3)^0$$

$$a_0 = \alpha$$

but $a_0 = 1$ given

Hence $a_0 = 1$

Now

$$a_1 = \alpha(3)^n + \beta.n.(3)^n$$

$$= 1.(3)^1 + \beta.1.(3)^1$$

$$a_1 = 3 + 3\beta$$

but $a_1 = 6$ given

Hence $3 + 3\beta = 6$

$$3\beta = 3$$

$$\beta = 1$$

Hence $\alpha = 1$ and $\beta = 1$

(17.) Show that $\sim\forall x(P(x)\to Q(x))$ and $\exists x(P(x)\wedge\sim Q(x))$ are logically equivalent.

$\Rightarrow$

p	q	$\sim p$	$\sim q$	$p(x)\to q(x)$	$p(x)\wedge\sim q(x)$
T	T	F	F	T	F
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	F

(17)

$\sim\forall x(P(x)\to Q(x))$

By universal law of quantifier

$\exists x\{\sim(P(x)\to Q(x))\}$

$\exists x\{P(x)\wedge\sim Q(x)\}$

also;

p	q	$\sim p$	$\sim q$	$p(x)\to q(x)$	$\sim[p(x)\to q(x)]$	$p(x)\wedge\sim q(x)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	F	F

$\therefore$ They are logically equivalent.

Q17. Explicit formula for fibonacci number.

Soln:-

Fibonacci sequence general term is represented by:-

$$T(n) = T(n-1) + T(n-2)$$

or $a_n = a_{n-1} + a_{n-2}$

Let $x^2 = a_n \quad a_{n-1} = x^1 \quad a_{n-2} = x^0$

$\therefore \quad x^2 = x + 1$

$\Rightarrow x^2 - x - 1 = 0$

$$x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Now both roots are distinct

$\therefore \quad a_n = \alpha r_1^n + \beta r_2^n$

$$a_n = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^n + \beta\left(\frac{1-\sqrt{5}}{2}\right)^n$$

for $n=0 \Rightarrow \quad 0 = \alpha + \beta \Rightarrow \underline{\underline{\alpha = -\beta}}$

for $n=1 \Rightarrow \quad 1 = \alpha\left(\frac{1+\sqrt{5}}{2}\right) + \beta\left(\frac{1-\sqrt{5}}{2}\right)$

$$1 = \left\{\frac{1+\sqrt{5} - 1 + \sqrt{5}}{2}\right\}\alpha$$

$$\alpha = \frac{1}{\sqrt{5}} \Rightarrow \beta = -\frac{1}{\sqrt{5}}$$

$$a_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$