Lecture 8.1

Selfish Load Balancing Game

n jobs to be scheduled on m machines.

- Players : $[n]$ the set of jobs.
- Strategy set of each player : $[m]$
  the set of machines.
- Strategy profile : an assignment
  $$A : [n] \longrightarrow [m]$$
- Cost function : suppose i-th job is assigned to

j-th machine. Each job $i \in [n]$ has a length w_i and each machine j has a speed of s_j. Time taken to run job i in machine j is $\frac{w_i}{s_j}$. The load l_j of machine j in an assignment $A : [n] \to [m]$ is

$$\sum_{i \in [n] : A(i) = j} \frac{w_i}{s_j}.$$

The cost of job i in a strategy profile $A : [n] \longrightarrow [m]$ is l_j where $A(i) = j$.

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Theorem: Every selfish load balancing game has a PSNE.

Proof: Associate every assignment $A:[n] \to [m]$ with its sorted "load vector" $(\lambda_1, \dots, \lambda_m)$ $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$.

We say $(\lambda_1, \dots, \lambda_m) > (\lambda'_1, \dots, \lambda'_m)$ if there exists $j \in [m]$ such that $\lambda_k = \lambda'_k \ \forall k \in [j-1]$,

$$\lambda_j > \lambda'_j .$$

Let $(\lambda_1, \dots, \lambda_m)$ be the smallest sorted load vector, let the assignment A corresponds to $(\lambda_1, \dots, \lambda_m)$. We claim that A is a PSNE. Suppose <u>not</u>. Then, there exists a job $i \in [n]$ which benefits by moving from machine s to machine t.

$$\Big(\lambda_1, \dots, \underset{\substack{\uparrow \\ s}}{\lambda_a}, \dots, \underset{\substack{\uparrow \\ t}}{\lambda_b}, \dots, \lambda_m\Big)$$

$$> \big(\lambda_1, \dots \lambda_{a-1}, \dots \quad \dots \big)$$ new sorted load vector

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This contradicts our assumption that $(\lambda_1, \ldots, \lambda_m)$ is the smallest sorted load vector. $\boxtimes$

<u>Theorem:</u> The PoA of any selfish load balancing game with n jobs and m identical (same speed) machines is at most $2 - \frac{2}{m+1}$.

<u>Proof:</u> Since the machines are identical, wma wlog that $s_j = 1 \ \forall j \in [m]$,

Let A be a PSNE assignment. Social welfare function is the makespan i.e. $\max_{j \in [m]} l_j$. Let OPT be the minimum makespan possible.

Let $c(A) = l_{j^*}$

There must be at least 2 jobs assigned to j^*.

Otherwise, $c(A) = l_{j^*} = OPT$ since $OPT \geq \max_{i \in [n]} w_i$

and $PoA = 1$

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Let $i^* \in [n)$ be a job assigned to j^* in A.

wma wlog $\quad w_{i^*} \leq \frac{1}{2} c(A)$

$j \in [m) \setminus \{j^*\}$, $\quad l_j + w_{i^*} \geq l_{j^*}$

$$\Rightarrow \quad l_j \geq c(A) - w_{i^*}$$

$$\geq c(A) - \frac{1}{2} c(A)$$

$$= \frac{1}{2} c(A)$$

$$\mathrm{OPT} \geq \frac{\sum_{i=1}^{n} w_i}{m}$$

$$= \frac{\sum_{j=1}^{m} l_j}{m}$$

$$= \frac{l_{j^*} + \sum_{\substack{j=1 \\ j \neq j^*}}^{m} l_j}{m}$$

$$\geq \frac{c(A) + \frac{1}{2}(m-1)\, c(A)}{m}$$

$$= \frac{m+1}{2m}\, c(A)$$

$$\mathrm{PoA} = \frac{c(A)}{\mathrm{OPT}} \leq \frac{2m}{m+1} = 2 - \frac{2}{m+1}$$

Lecture 8.2

Other Forms of Games

Normal form game models "simultaneous move" "complete information" setting.

Relaxing "simultaneous move" $\rightsquigarrow$ extensive form game

Relaxing "complete information" $\rightsquigarrow$ Bayesian game.

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Bayesian Game

Definition: A Bayesian game $\Gamma = \langle N, (S_i)_{i\in N}, (\Theta_i)_{i\in N}, p, (u_i)_{i\in N} \rangle$

- N : set of players
- S_i : set of strategies for player $i \in N$
- Θ_i : set of types for player $i \in N$
- $p \in \Delta\left(\bigtimes_{i\in N} \Theta_i\right)$

- $u_i : \underset{i\in N}{\times} S_i \times \underset{i\in N}{\times} \Theta_i \longrightarrow \mathbb{R}$.

In a Bayesian game, Γ is common knowledge.

The prior distribution p is a common knowled and thus the posterior distribution $p(\cdot|\theta_i)$ is known to player i only.

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Example: (Sealed Bid Selling Auction)

- $N = \{1, \dots, n\}$ the set of players (buyers)
- $\Theta_i = [0, 1]$ the typeset (valuation) for player $i \in N$
- $S_i = [0, 1]$ the strategy set for player $i \in N$
- p is the product distribution $p = \times_{i=1}^{n} p_i$ where

$p_i \sim \mathcal{U}([0,1])$

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- Allocation function

$$a : \bigtimes_{i=1}^{n} S_i \longrightarrow \mathbb{R}^N \qquad a((s_i)_{i\in N}) = (a_i)_{i\in N}$$

$a_i = 1$ if and only if player i wins.

- payment function

$$q : \bigtimes_{i=1}^{n} S_i \longrightarrow \mathbb{R}^N \qquad q((s_i)_{i\in N}) = (q_i)_{i\in N}$$

where q_i is the money paid by player i.

- $u_i((\theta_i)_{i\in N}, (s_i)_{i\in N}) = a_i(\theta_i - q_i)$

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A Bayesian game can be equivalently represented by a normal form game. Such a normal form game is called a Selten game.

Selten game: Let $\Gamma = \langle N, (\Theta_i)_{i\in N}, (S_i)_{i\in N}, p, (u_i)_{i\in N} \rangle$ be any Bayesian game. The corresponding normal form game $\Gamma^S = \langle N^S, (S_{\theta_i})_{\theta_i\in\Theta_i, i\in N}, (U_{\theta_i})_{\theta_i\in\Theta_i, i\in N} \rangle$

- $N^s = \bigcup_{i \in N} \Theta_i$

- $S_{\theta_i} = S_i \quad \forall \; \theta_i \in \Theta_i$

- $U_{\theta_i} : \bigtimes_{\substack{\theta_i \in \Theta_i \\ i \in N}} S_{\theta_i} \longrightarrow \mathbb{R}$

$$U_{\theta_i}\left(\underline{(s_{\theta_i})}_{\theta_i \in \Theta_i,\, i \in N} \right) = \mathbb{E}_{\theta_{-i} \sim p(\cdot \mid \theta_i)} \left[u_i\left((\theta_i, \theta_{-i}), (s_{\theta_i})_{i \in N} \right) \right]$$

$$= \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i} \mid \theta_i) \; u_i\left((\theta_i, \theta_{-i}), (s_{\theta_i})_{i \in N} \right)$$

Pure Strategy Bayesian Nash Equilibrium

A pure strategy Bayesian Nash equilibrium of a Bayesian game $\Gamma = \langle N, (\Theta_i)_{i\in N}, (S_i)_{i\in N}, p, (u_i)_{i\in N} \rangle$ is a pure strategy profile $(s_1^*, \ldots, s_n^*)$, $s_i^* : \Theta_i \to S_i$ if for all $i \in N$ and all $s_i : \Theta_i \to S_i$, we have

$$U_{\theta_i}\left((s_i^*, s_{-i}^*) \right) \geq U_{\theta_i}\left(s_i, s_{-i}^* \right) \quad \forall \, \theta_i \in \Theta_i$$

Equivalently,

$$\mathbb{E}_{\theta_{-i}}\left[u_i(\theta_i, \theta_{-i}, s_i(\theta_i), s_{-i}(\theta_{-i})) \,\middle|\, \theta_i\right] \geqslant \mathbb{E}_{\theta_{-i}}\left[u_i(\theta_i, \theta_{-i}, a_i, s_{-i}(\theta_{-i})) \,\middle|\, \theta_i\right]$$

$$\forall i \in N \quad \forall a_i \in S_i \quad \forall \theta_i \in \Theta_i$$

<u>Theorem:</u> In the Bayesian game corresponding to the first price auction, each buyer bidding half of their valuation forms a pure strategy Bayesian Nash equilibrium under the following assumptions:

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(i) We have only 2 buyers.

(ii) Each buyer's valuation is distributed uniformly in $[0,1]$.

(iii) Each buyer is "risk neutral": bid $b_i(\theta_i)$ of player $i \in [2]$ is of the form $\alpha_i \theta_i$ for some $\alpha_i \in [0,1]$.

<u>Proof</u>: Utility of player 1:

$$u_1(\theta_1, \theta_2, b_1, b_2) = (\theta_1 - b_1(\theta_1)) \Pr[b_1(\theta_1) > b_2(\theta_2)]$$

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$$= (\theta_1 - b_1(\theta_1)) \, P_r[b_1(\theta_1) > \alpha_2 \theta_2]$$

$$= (\theta_1 - b_1(\theta_1)) \, P_r\left[\theta_2 \leq \frac{b_1(\theta_1)}{\alpha_2}\right]$$

$$u_1(\theta_1, \theta_2, b_1 b_2) = \begin{cases} \theta_1 - b_1(\theta_1) & \text{if } b_1(\theta_1) \geq \alpha_2 \\ (\theta_1 - b_1(\theta_1)) \cdot \dfrac{b_1(\theta_1)}{\alpha_2} & \text{if } b_1(\theta_1) < \alpha_2 \end{cases} \qquad [\because \theta_2 \sim U([0,1])]$$

$b_1^*(\theta_1)$ which maximizes $u_1(\theta_1, \theta_2, b_1, b_2)$ is

$$b_1^*(\theta_1) = \begin{cases} \dfrac{\theta_1}{2} & \text{if } \dfrac{\theta_1}{2} < \alpha_2 \\ \alpha_2 & \text{if } \dfrac{\theta_1}{2} \geq \alpha_2 \end{cases}$$

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Doing the same calculation we get that the following

$b_2^*(\theta_2)$ maximizes $u_2(\theta_1, \theta_2, b_1, b_2)$

$$b_2^*(\theta_2) = \begin{cases} \frac{\theta_2}{2} & \text{if } \frac{\theta_2}{2} < \alpha_1 \\ \alpha_1 & \text{if } \frac{\theta_2}{2} \geq \alpha_1 \end{cases}$$

To maximize $u_1(\theta_1, \theta_2, b_1, b_2)$ and $u_2(\theta_1, \theta_2, b_1, b_2)$ simultaneously, we chose/get,

$$\alpha_1 = \alpha_2 = \frac{1}{2}$$

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That is, $\left(b_1^*(\theta_1) = \frac{\theta_1}{2},\ b_2^*(\theta_2) = \frac{\theta_2}{2}\right)$ is a pure strategy Bayesian Nash equilibrium for the first price auction. $\blacksquare$

The above proof can be extended to show that with n risk-neutral buyers, the first price auction has $\left(b_i^*(\theta_i) = \frac{n-1}{n}\theta_i\right)_{i \in N}$ as its pure strategy Bayesian Nash equilibrium.

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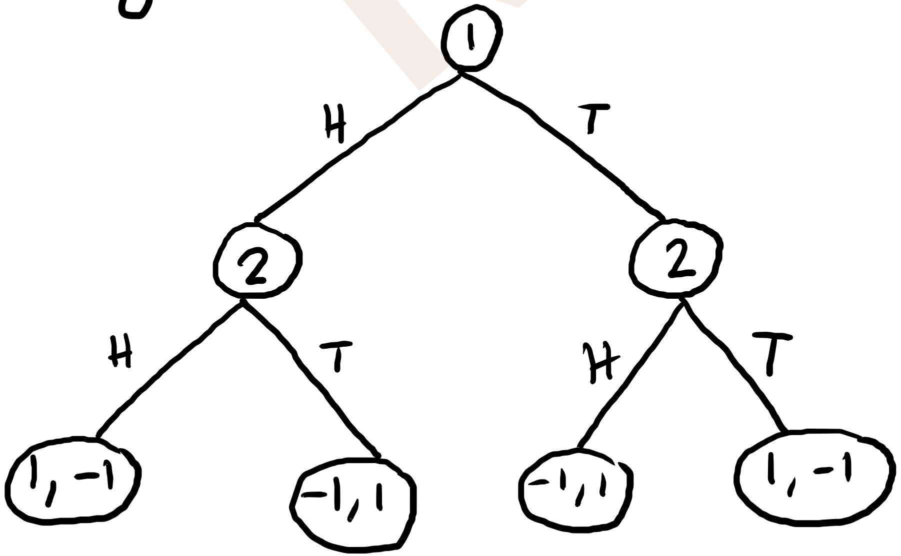
Extensive Form Game

Relaxes "simultaneous move" condition of normal form game.

"Game tree"

Example (Matching Pennies with Observation):



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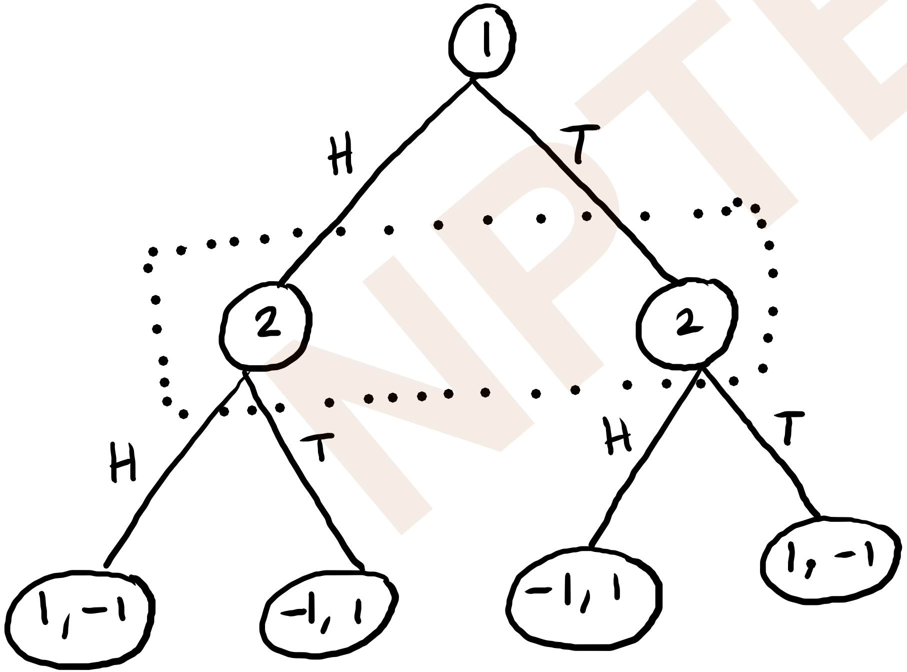
Example 2 (Matching Pennies without Observation)



Information set.

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Definition (Information set): An information set of a player is a subset of the player's decision nodes which are indistinguishable to him.

- In the matching pennies game with observation, player 1 has only one information set $\{\epsilon\}$ whereas player 2 has two information sets $\{H\}$ and $\{T\}$.
- In the matching pennies game without observation, player 2 has one information set $\{H, T\}$.

Definition (Extensive form game).

$\Gamma = \langle N, (S_i)_{i\in N}, \mathbb{H}, P, (\mathbb{I}_i)_{i\in N}, C, (u_i)_{i\in N} \rangle$ where

- N = set of players.
- S_i = set of strategies for player i.
- $\mathbb{H}$: set of all paths from root to leaf nodes. $S_{\mathbb{H}}$ is the set of all proper sub-histories.
- $P : S_{\mathbb{H}} \longrightarrow N$ maps nodes to player.
- $\mathbb{I}_i$: set of all information sets of player i.

- $C : \bigcup_{i \in N} \mathbb{I}_i \longrightarrow \bigcup_{i \in N} S_i$, $C(J) \subseteq S_i \;\; \forall \; J \in \mathbb{I}_i$

- $u_i : H \longrightarrow \mathbb{R}$ maps histories (leaf nodes) to the utility of player $i \in N$

- An extensive form game is called a perfect information game if all its information sets are singletons. Otherwise, the game is called an imperfect information game.

Representing Extensive form games as strategic form games.

Given an extensive form game $\Gamma = \langle N, (S_i)_{i \in N}, H, P, (\mathbb{I}_i)_{i \in N}, C, (u_i)_{i \in N} \rangle$, the corresponding normal form game

$\Gamma^s = \langle N^s, (S'_i)_{i \in N}, (u'_i)_{i \in N} \rangle$ is given by

- $N^s = N$
- $S'_i = \{ s_i : \mathbb{I}_i \to S_i \mid s_i(J) \in C(J) \ \forall J \in \mathbb{I}_i \}$
- $u'_i(s_1, \ldots, s_n)$ is the utility that player i

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receives if all the players play according to $(s_1, \dots, s_n)$.

Lecture 8.5

Mechanism Design

There is a social planner who wants to compute a social choice function $f : \times_{i \in N} \Theta_i \rightarrow X$.
But the inputs of the function f are held by n-strategic players. Mechanism designer designs a game with these n-players in such a way which enables the mechanism

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designer to implement the social choice function f.

We have n players. Each player $i \in [n]$ has a utility function $u_i : \mathcal{X} \times \Theta \longrightarrow \mathbb{R}$; $\Theta = \times_{i \in [n]} \Theta_i$.

We also assume that there is a prior distribution $p \in \Delta(\Theta)$.

Mechanism designer decides the set S_i of strategies of player i, and a function

$$g : \bigtimes_{i=1}^{n} S_i \longrightarrow \mathcal{X}.$$

$$u_i\left((s_i)_{i\in[n]}\right) = u_i\left(g\left((s_i)_{i\in[n]}\right), \theta\right). \quad \forall\, \theta \in \Theta$$

Indirect Mechanism. An indirect mechanism is a tuple $M = \langle N = [n], \mathcal{X}, (\Theta_i)_{i\in N}, (S_i)_{i\in N}, \mathbb{P} \in \Delta(\Theta),$

$g : \bigtimes_{i\in N} S_i \longrightarrow \mathcal{X}, \quad u_i : \mathcal{X} \times \Theta \longrightarrow \mathbb{R} \rangle$

- N : set of players
- $\mathcal{X}$: set of outcome

- Θ_i : type set of player i
- S_i : strategy set of player i
- $\mathbb{P}$: common prior distribution over Θ
- g : maps strategy profiles to outcomes
- u_i : utility function of player i.

If $N, \mathcal{X}, (\Theta_i)_{i \in N}, \mathbb{P}, u_i$ is clear from the context, we denote an indirect mechanism simply as

$$((S_i)_{i \in N}, g(\cdot)).$$

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A mechanism is called direct if $S_i = \Theta_i$ for all $i \in [n]$, and $g = f$. Clearly direct mechanism belongs to the set of indirect mechanisms. Typically $N, \mathcal{X}, (\Theta_i)_{i\in N}, \mathbb{P}, u_i$ are clear from the context and we denote a direct mechanism simply as $((\Theta_i)_{i\in N}, f(\cdot))$.

<u>Example</u> (Buying Auction): One buyer and n potential sellers. The mechanism is

- $N : \{0, 1, \ldots, n\}$ player 0 is the buyer and players $1, \ldots, n$ are the sellers.

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- $\mathcal{X} = \{(a_0, a_1, \dots, a_n, p_0, p_1, \dots, p_n) \in \mathbb{R}^{2n+2} : a_i \in \{0, \pm 1\}, i \in \{0, \dots, n\},$

$$\sum_{i=0}^{n} a_i = 0, \quad \sum_{i=0}^{n} p_i = 0 \}$$

$a_i = 1$ if player i receives the item

$= -1$ —— " —— i gives —— " ——

$= 0$ o/w.

p_i is the payment made by player i.

- Θ_i : set of all possible valuations of the item to player i

- S_i : the set of all possible bids.
- g : decides the outcome
- $u_i((a_0, \ldots, a_n, p_0, \ldots, p_n)) = a_i \theta_i - p_i$