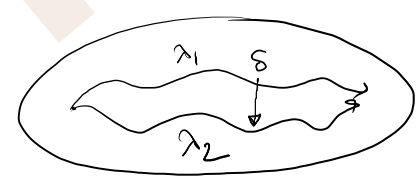
Lecture 4.4



Therem: In an atomic network congestion game suppose the following holds:

- The source and destination are the same for all the players.
- The cost functions satisfy α-bounded jump condition.
- Max-gain version of ε-Best response dynamic is used.

Then a ε-PSNE will be reached in $O\left(\frac{n\alpha}{\varepsilon} \log \frac{\Phi(s^0)}{\Phi(s^{min})}\right)$ iterations.

<u>Proof</u>: Let s be a strategy profile which is <u>not</u> an ε-PSNE. The proof has two parts. In the first part we show that there exists a player i whose cost in s is "high". In the second part, we show that if player j is chosen by the max-gain version of ε-best response dynamics, then the drop in potential is some significant fraction of the cost of player i in s.

Claim: In every strategy profile s, there exists a player $i^* \in N$ such that $C_{i^*}(s) \geq \frac{\Phi(s)}{n}$.

Proof:

$$C(s) = \sum_{i \in N} C_i(s) = \sum_{e \in E[G]} f_e \underline{c_e(f_e)}$$

$$\Phi(s) = \sum_{e \in E[G]} \sum_{i=1}^{f_e} c_e(i)$$

$$\Phi(s) \leq C(s) = \sum_{i \in N} C_i(s)$$

$\Rightarrow$ There exists a player $i^* \in N$ s.t. $C_{i^*}(s) \geq \frac{\Phi(s)}{n}$. ∎

Claim: Let j makes a move from s_j to s_j' in the current iteration of the max-gain version of ε-best response dynamics. Then,

$$c_j(s_j, s_{-j}) - c_j(s_j', s_{-j}) \geq \frac{\varepsilon}{\alpha} c_i(s) \quad \forall\ i \in N,\ i \neq j$$

Proof: Let fix a player $i \in N$, $i \neq j$.

Case I: player i has an ε-move.

$$\begin{aligned} c_j(s_j, s_{-j}) - c_j(s_j', s_{-j}) &\geq \max_{s_i' \in S_i} \left\{ c_i(s_i, s_{-i}) - c_i(s_i', s_{-i}) \right\} \\ &\geq \varepsilon \cdot c_i(s) \\ &\geq \frac{\varepsilon}{\alpha} \cdot c_i(s) \end{aligned}$$

<u>Case II</u>: Player i does not have an ε-move.

The strategy (path) s_j' is available to player i also.

$$C_i(s_j', s_{-i}) > (1-\varepsilon) C_i(s) \quad \text{——— (1)}$$

However, player j has an ε-move.

$$C_j(s_j', s_{-j}) \leq (1-\varepsilon) C_j(s) \quad \text{——— (2)}$$

Due to α-bounded jump property,

$$C_i(s_j', s_{-i}) \leq \alpha\, C_j(s_j', s_{-j}) \quad \text{—— (3)}$$

$$C_i(s) \underset{(1)}{<} \frac{C_i(s_j', s_{-i})}{1-\varepsilon} \underset{(3)}{\leq} \frac{\alpha\, C_j(s_j', s_{-j})}{1-\varepsilon} \underset{(2)}{\leq} \frac{\alpha\, \cancel{(1-\varepsilon)}\, C_j(s)}{\cancel{(1-\varepsilon)}} = \alpha \cdot C_j(s)$$

$$c_i(s) < \alpha \, c_j(s) \Rightarrow c_j(s) > \frac{c_i(s)}{\alpha}$$

from (2),

$$c_j(s) - c_j(s_j', s_{-j}) \geq \varepsilon \, c_j(s) > \frac{\varepsilon \, c_i(s)}{\alpha} \qquad \blacksquare$$

$$\Phi(s) - \Phi(s_j', s_{-j}) = c_j(s) - c_j(s_j', s_{-j})$$

$$\geq \frac{\varepsilon}{\alpha} \cdot \max_{\substack{i \in N \\ i \neq j}} c_i(s)$$

$$\geq \frac{\varepsilon \, \Phi(s)}{n\alpha}$$

$$\Phi(s_j', s_{-j}) \leq \left(1 - \frac{\varepsilon}{n\alpha}\right) \Phi(s)$$

The number of iteration needed $O\left(\frac{n\alpha}{\varepsilon} \log \frac{\Phi(s^0)}{\Phi(s^{min})}\right)$. $\blacksquare$