Lecture 2.5

Maxmin and Minmax Values Coincide in Matrix Games

Theorem: $A \in \mathbb{R}^{m \times n}$.

$$\max_{x \in \Delta([m])} \min_{y \in \Delta([n])} xAy = \min_{y \in \Delta([n])} \max_{x \in \Delta([m])} xAy$$

Proof: We have already shown before,

$$\max_{x \in \Delta([m])} \min_{y \in \Delta([n])} xAy \leq \min_{y \in \Delta([n])} \max_{x \in \Delta([m])} xAy$$

To show: $\max\limits_{x \in \Delta([m])} \min\limits_{y \in \Delta([n])} xAy \geq \underline{\min\limits_{y \in \Delta([n])}} \underbrace{\max\limits_{x \in \Delta([m])} xAy}$. ✓

Let x^* be an optimal solution of LP1

y^* — — — LP2

$$\min_{y \in \Delta([n])} \max_{x \in \Delta([m])} xAy \geq \underline{\min_{y \in \Delta([n])} x^* A y} = \max_{x \in \Delta([m])} xAy^* \geq \max_{x \in \Delta([m])} \min_{y \in \Delta([n])} xAy$$

∎

Theorem: $A \in \mathbb{R}^{m \times n}$. Then the utility of both the players in any MSNE is exactly their values.

Proof: (x^*, y^*) be any MSNE.

$$v_r = \max_{x \in \Delta([m])} \min_{y \in \Delta([n])} xAy, \quad v_c = -\min_{y \in \Delta([n])} \max_{x \in \Delta([m])} xAy$$

From minmax theorem, we know that $v_r = -v_c$ for the matrix game A.

Let (x^*, y^*) be an MSNE

$$x^* A y^* \leq \max_{x \in \Delta([m])} x A y^* = \min_{y \in \Delta([n])} x^* A y \leq \max_{x \in \Delta([m])} \min_{y \in \Delta([n])} x A y = v_r$$

$$x^* A y^* \leq v_r \quad , \quad x^* A y^* \geq v_r \implies x^* A y^* = v_r$$

$$-x^* A y^* \leq -\min_{y \in \Delta([n])} x^* A y = -\max_{x \in \Delta([m])} x A y^* \leq -\min_{y \in \Delta([n])} \max_{x \in \Delta([m])} x A y = v_c$$

$$-x^* A y^* \leq v_c \quad , \quad -x^* A y^* \geq v_c \implies -x^* A y^* = -v_c .$$

$\blacksquare$

		A	B
$x_A \to$	A	1, −1	−1, 1
$x_B \to$	B	−1, −1	1, −1

Security level of row player.

$$\max_{\substack{x_A, x_B :\\ \boxed{x_A + x_B = 1,\ x_A, x_B \geq 0}}} \boxed{\min\{x_A - x_B,\ x_B - x_A\}}$$

Solution $x_A = \frac{1}{2}$, $x_B = \frac{1}{2}$

$(A : \frac{1}{2}, B : \frac{1}{2})$ value guaranteeing strategy for the row player.

Similarly, $(A : \frac{1}{2}, B : \frac{1}{2})$ column player.

$\left\{\left(A:\frac{1}{2}, B:\frac{1}{2}\right), \left(A:\frac{1}{2}, B:\frac{1}{2}\right)\right\}$ is an MSNE.

<u>Corollary:</u> $A \in \mathbb{R}^{m \times n}$. $x^* \in \Delta([m])$, $y^* \in \Delta([n])$. Then,

(x^*, y^*) is an MSNE if and only if x^* and y^* are value guaranteeing strategies for the row and column player respectively.

Q: Give an example of a matrix game with 10 MSNEs.

		Rock	Paper	Scissor
x_R	→ Rock	0, 0	−1, 1	1, −1
x_P	→ Paper	1, −1	0, 0	−1, 1
x_S	→ Scissor	−1, 1	1, −1	0, 0

$$\max_{x_R, x_P, x_S} \boxed{\min \left\{ x_P - x_S,\ x_S - x_R,\ x_R - x_P \right\}}$$

$$\boxed{\begin{array}{l} x_R + x_P + x_S = 1 \\ x_R, x_P, x_S \geq 0 \end{array}}$$

maximize if

$x_P = x_S = x_R = \frac{1}{3}$

$\left\{ \left(R:\frac{1}{3}, P:\frac{1}{3}, S:\frac{1}{3}\right), \left(R:\frac{1}{3}, P:\frac{1}{3}, S:\frac{1}{3}\right) \right\}$ unique MSNE for the rock-paper-scissor game.