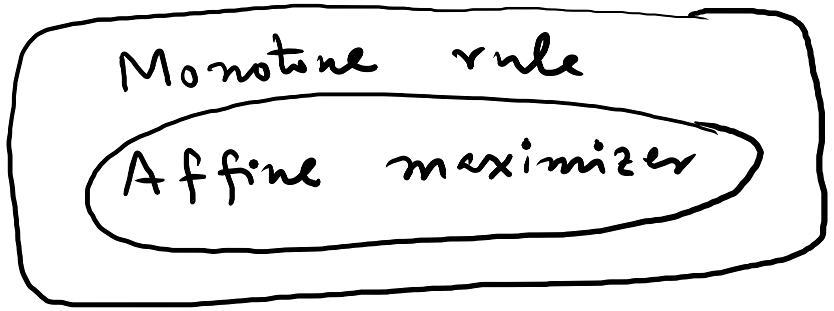
Lecture 12.1

Recall:

Quasi-linear enviroment: DSIC allocation rules are the affine maximizers (Grove's Theorem and Robert's Theorem)

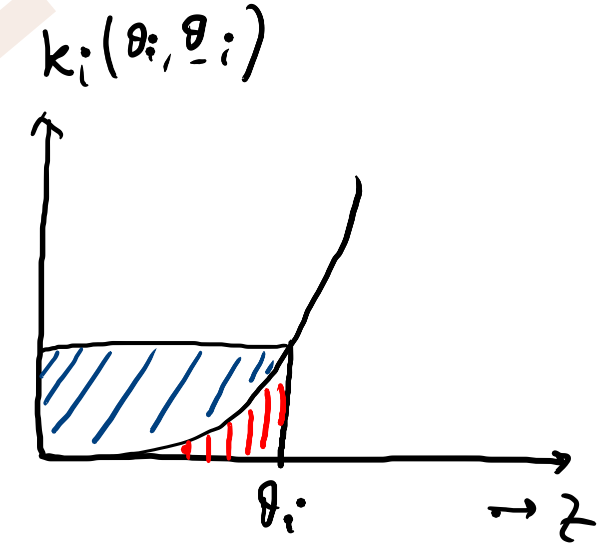
Single-Parameter domain: DSIC allocation rules are the monotone rules (Myerson's Lemma).

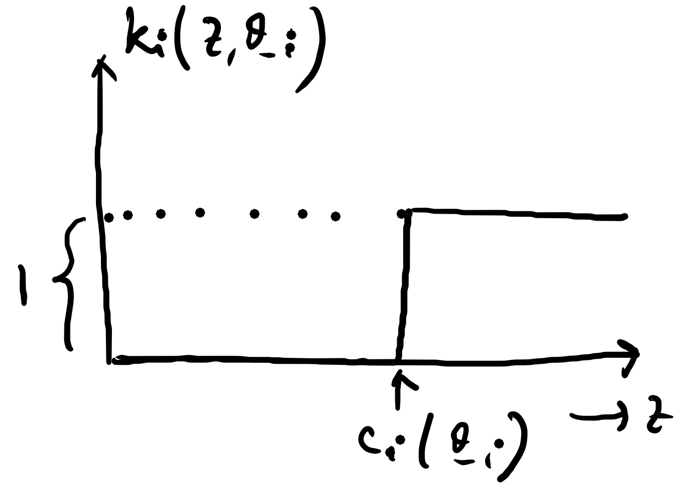


$$k(\cdot) = \left(k_1(\cdot), \ldots, k_n(\cdot)\right) \quad \underline{\text{monotone}}$$

$$t_i(\theta) = \int_0^{\theta_i} z \, \frac{d}{dz}\left(k_i(z, \theta_{-i})\right) \cdot dz$$

$$= \theta_i \cdot k_i(\theta_i, \theta_{-i}) - \int_0^{\theta_i} k_i(z, \theta_{-i}) \, dz$$





Implementability in Intermediate Domain

Convex domain: Each Θ_i is a convex set in some Euclidean space.

Weak monotonicity:- An allocation rule $k^*: \Theta \to \mathcal{K}$ is called weakly monotone if we have the following for $i \in [n]$, $\theta_i, \theta_i' \in \Theta_i$, $\theta_{-i} \in \Theta_{-i}$,

$$k^*(\theta_i, \theta_{-i}) = x \neq y = k^*(\theta_i', \theta_{-i})$$

$$v_i(x,\theta_i) - v_i(y,\theta_i) \geq v_i(x,\theta_i') - v_i(y,\theta_i')$$

Theorem: If a mechanism $(k^*, t_1, \dots, t_n)$ is DSIC, then k^* is weakly monotone. On the other hand, if Θ_i is a convex set for each $i \in [n]$, then, for every weakly monotone allocation rule $k^*(\cdot)$, there exist payment rules $t_1(\cdot), \dots, t_n(\cdot)$ such that the mechanism $(k^*, t_1, \dots, t_n)$ is DSIC.

<u>Proof</u>: (first part) Suppose a mechanism $(k^*, t_1, \ldots, t_n)$ is DSIC. Then we have the following: for every $i \in [n]$, $\theta_i, \theta_i' \in \Theta_i$, $\theta_{-i} \in \Theta_{-i}$ such that $k^*(\theta_i, \theta_{-i}), k^*(\theta_i', \theta_{-i}) \in K_i$, we have $t_i(\theta_i, \theta_{-i}) = t_i(\theta_i', \theta_{-i})$. Since the mechanism is DSIC, in the type profile (θ_i, θ_{-i}), player i does not benefit by reporting θ_i' instead of θ_i. So we have

$$u_i(x, \theta_i) \geq u_i(y, \theta_i)$$

$$\Rightarrow \quad v_i(x, \theta_i) + t_i(\theta_i, \underline{\theta}_{-i}) \geq v_i(y, \theta_i) + t_i(\theta_i', \underline{\theta}_{-i})$$

$$\Rightarrow \quad v_i(x, \theta_i) - v_i(y, \theta_i) \geq 0 \quad \text{--- (1)} \qquad \left[\because t_i(\theta_i, \underline{\theta}_{-i}) = t_i(\theta_i', \underline{\theta}_{-i})\right]$$

Similarly, since the mechanism is DSIC, player i does not benefit in the type profile $(\theta_i', \underline{\theta}_{-i})$ by reporting θ_i instead by θ_i'.

$$\Rightarrow \quad v_i(y, \theta_i') - v_i(x, \theta_i') \geq 0$$

$$\Rightarrow \quad v_i(x, \theta_i') - v_i(y, \theta_i') \leq 0$$

$$\Rightarrow \quad v_i(x, \theta_i') - v_i(y, \theta_i') \leq v_i(x, \theta_i) - v_i(y, \theta_i)$$

[from inequality (1)]

$\blacksquare$