Weakly Dominant Strategy

Definition: Given $\Gamma = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N} \rangle$ a strategy $s_i^* \in S_i$ is called a weekly dominant strategy if

(i) $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \forall s_{-i} \in S_{-i} \left(:= \times_{j\in N, j\neq i} S_j\right)$
$\forall s_i \in S_i$

(ii) $\forall s_i \in S_i \setminus \{s_i^*\}, \exists s_{-i} \in S_{-i}$ such that

$$u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i})$$

A strategy profile $(s_i^*)_{i\in N}$ is called a weakly dominant strategy equilibrium (WDSE) if s_i^* is a weakly dominant strategy for player i, for all $i \in N$

Very Weakly Dominant Strategy

A strategy $s_i^* \in S_i$ is called a very weakly dominant strategy if

$$u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}, \ \forall s_i \in S_i$$

A strategy profile $(s_i^*)_{i \in N}$ is called a very weakly dominant strategy equilibrium (VWDSE) if each s_i^* is a very weakly dominant strategy.

Theorem: Bidding Valuations is a weakly dominant strategy equilibrium.

Proof: To prove the result, we need to show that bidding valuation is a weakly dominant strategy for player i, $i \in N$.
Let v_i be the valuation of player i. Let $s_{-i} \in S_{-i}$ be any strategy profile of other players. $\theta_i = \min\limits_{j \in N, j \neq i} s_j$

Case I: Player i wins the auction. $\Rightarrow s_i^* \leq \theta_i$, $s_i^* = v_i \leq \theta_i$

$$u_i(s_i^*, s_{-i}) = \theta_i - v_i \geq 0 .$$

$$u_i(s_i', s_{-i}) \leq \theta_i - v_i \leq u_i(s_i^*, s_{-i})$$

Case II : (Player i does not win)

$u_i(s_i^*, s_{-i}) = 0$, $\Rightarrow \boxed{v_i \geq \theta_i}$

$s_i' \in S_i$ (any) if player i loses in (s_i', s_{-i}), then,

$u_i(s_i', s_{-i}) = 0 = u_i(s_i^*, s_{-i})$

if player i wins in (s_i', s_{-i}), then

$u_i(s_i', s_{-i}) = \theta_i - v_i \leq 0 = u_i(s_i^*, s_{-i})$

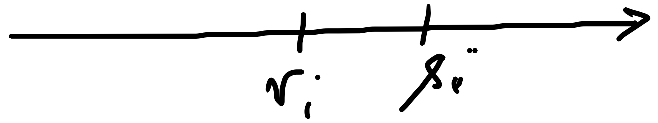
Hence bidding valuation maximizes utility of player i.

Let $s_i \in S_i$, $s_i \neq s_i^*$, need to show, $\exists\, s_{-i}$ s.t.

$u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i})$

If $\underline{s_i > v_i}$

Let $s_j = \frac{v+s_i}{2} \quad \forall\, j \in N,\ j \neq i$

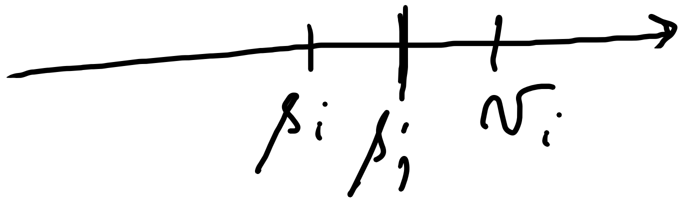


$$u_i(s_i, s_{-i}) = 0,$$

$$u_i(s_i^*, s_{-i}) = \frac{v+s_i}{2} - v_i = \frac{s_i - v_i}{2} > 0 = u_i(s_i, s_{-i})$$

If $\boxed{s_i < v_i}$

Let $s_j = \frac{v+s_i}{2} \quad \forall\, j \in N,\ j \neq i$



$$u_i(s_i, s_{-i}) = \frac{v+s_i}{2} - v_i = \frac{s_i - v_i}{2} < 0 = u_i(s_i^*, s_{-i}).$$