Lecture 11.3

Characterization of DSIC Social Choice Function in Single Parameter Domain

Theorem: A social choice function $f(\cdot) = (k^*(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$ in a single parameter domain is DSIC and losers do not pay anything if and only if all the following conditions hold.

(i) The allocation $k^*(\cdot)$ is monotone in every θ_i

(ii) Every winning player pays its critical value.

$$t_i(\theta_i, \theta_{-i}) = \begin{cases} -c_i(\theta_{-i}) & \text{if } k^*(\theta_i, \theta_{-i}) \in \mathcal{K}_i \\ 0 & \text{otherwise.} \end{cases}$$

For some $\theta_{-i} \in \Theta_{-i}$, if $c_i(\theta_{-i})$ is not defined, then we set $c_i(\theta_{-i}) = \lambda_i$ for any real number λ_i.

Proof: (If part) ✓

(Only if part)

proof for (i): Since $k^*(\cdot)$ is not monotone in every θ_i, there exists a type profile $\underline{\theta}_{-i} \in \Theta_{-i}$, types $\theta_i, \theta_i' \in \Theta_i$, $\boxed{\theta_i < \theta_i'}$ and $k^*(\theta_i, \underline{\theta}_{-i}) \in \mathcal{K}_i$ but $k^*(\theta_i', \underline{\theta}_{-i}) \notin \mathcal{K}_i$.

Since $f(\cdot)$ is DSIC, player i prefers winning in type profile $(\theta_i, \underline{\theta}_{-i})$ compared to losing.

$$\theta_i - c_i(\underline{\theta}_{-i}) \geq 0 \quad \Rightarrow \theta_i \geq c_i(\underline{\theta}_{-i}) \text{ --- (1)}$$

But player i prefers losing in type profile $(\theta_i', \underline{\theta}_{-i})$ compared to winning.

$$0 \geq \theta_i' - c_i(\underline{\theta}_{-i})$$

$$\Rightarrow \theta_i' \leq c_i(\underline{\theta}_{-i}) \text{ --- (2)}$$

From (1) and (2) $\theta_i' \leq \theta_i$ which contradicts our assumption that $\theta_i' > \theta_i$

proof of part (ii) :- We know that, for every $i \in [n]$, $\theta_i, \theta_i' \in \Theta_i$, $\theta_{-i} \in \Theta_{-i}$, $\underbrace{k^*(\theta_i, \theta_{-i}) = k^*(\theta_i', \theta_{-i})}_{\in \mathcal{K}_i}$, we have $t_i(\theta_i, \theta_{-i}) = t_i(\theta_i', \theta_{-i})$.

Suppose $t_i(\theta_i, \theta_{-i}) \neq -c_i(\theta_{-i})$

(a) $t_i(\theta_i, \theta_{-i}) < -c_i(\theta_{-i})$

(b) $t_i(\theta_i, \theta_{-i}) > -c_i(\theta_{-i})$

Refuting possibility (a):

$$t_i(\theta_i, \theta_{-i}) < -c_i(\theta_{-i})$$

$$\Rightarrow -t_i(\theta_i, \theta_{-i}) > c_i(\theta_{-i})$$

$$\Rightarrow \exists\, \theta_i'' \in \Theta_i, \quad -t_i(\theta_i, \theta_{-i}) > \boxed{\theta_i'' > c_i(\theta_{-i})}$$

player i wins in $(\theta_i'', \theta_{-i})$

In $(\theta_i'', \theta_{-i})$, $\quad u_i(\theta_i'', \theta_{-i}) = \theta_i'' + t_i(\theta_i', \theta_{-i})$

$$= \theta_i'' + t_i(\theta_i, \theta_{-i})$$

$$< 0$$

But then player i can report any very small type $(< c_i(\theta_{-i}))$ and lose and thereby receive more utility.

This contradicts our assumption that $f(\cdot)$ is DSIC $\blacksquare$