Lecture 4.3

Potential Game

Best response dynamic :-

① – Pick any strategy profile s

② – If s is not a PSNE

ⓐ – pick a player i who has a beneficial unilateral deviation from s_i to s_i'.

ⓑ – replace s_i with s_i' in s.

ⓒ – Go to step ②

Best response dynamic always terminates with a PSNE for finite potential games.

For arbitrary games, the best response dynamic may not terminate.

Even for potential games, for example the network congestion games, what is the speed of convergence ??

If the potential function takes only polynomially many values, then the best response dynamic converges in polynomial time.

$\underline{\varepsilon\text{-PSNE}}$: Given a game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ a pure strategy profile $(s_i)_{i \in N} \in \bigtimes_{i \in N} S_i$ is called an ε-PSNE if

$$\forall i \in N, \forall s_i' \in S_i, \quad u_i(s_i', s_{-i}) \leq (1+\varepsilon)\, u_i(s_i, s_{-i})$$

or,

$$\forall i \in N, \forall s_i' \in S_i, \quad C_i(s_i', s_{-i}) \geq (1-\varepsilon)\, C_i(s_i, s_{-i})$$

ε-Best Response Dynamic:

1. Pick any strategy profile s.
2. If s is not an ε-PSNE,
   a. Pick a player i who has a move from s_i to s_i' which increases its utility by more than $(1+\varepsilon)$ times its current utility.
   b. Replace s_i with s_i'
3. Go to step 2.

Theorem (Fast convergence of ε-Best Response Dynamics)

In an atomic network congestion game, suppose the following holds.

- All the players have the same source and destination.
- cost function satisfies "α-bounded jump condition" $(\alpha > 1)$:

  $c_e(x+1) \in [c_e(x), \alpha c_e(x)]$ for all edge e and all positive real number x.
- The max gain version of ε-Best response dynamic is used.

among all players who have an ε-best response move,

pick a player and a move which achieves largest

absolute cost decrease, for that player.

Then an ε-PSNE will be reached in $O\left(\frac{n\alpha}{\varepsilon} \log \frac{\Phi(s^0)}{\Phi(s^{min})}\right)$

number of iterations, s^0 is the initial strategy profile,

$\Phi(s^{min})$ is the minimum value of the potential function.