Lecture 3.1

Iterative Elimination of Dominated Strategies

Definition (Strongly Dominated Strategy): Given $\Gamma = \langle N, (S_i)_{i\in N}, (u_i)_{i \in N}\rangle$ a strategy $s_i \in S_i$ for a player $i \in N$ is called a strongly/weakly dominated strategy if there exists a mixed strategy $\sigma_i \in \Delta(S_i)$

s.t.

$$u_i(s_i, s_{-i}) \;<\; / \;\leq\; u_i(\sigma_i, s_{-i}) \qquad \forall s_{-i} \in S_{-i}$$

<u>Lemma</u>: In any game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ if a pure strategy $s_i \in S_i$ is strongly dominated, then, in every MSNE $(\sigma_i^*)_{i \in N}$ of Γ, we have $\sigma_i^*(s_i) = 0$.

<u>Proof</u>: by contradiction; Suppose there exists an MSNE $(\sigma_i^*)_{i \in N}$ of Γ such that $\sigma_i^*(s_i) \neq 0$ and s_i is strongly dominated by a mixed strategy $\sigma_i \in \Delta(S_i)$,

we have $\sigma_i \neq s_i$

Consider a mixed strategy $\underline{\pi \in \Delta(S_i)}$ as follows

$$\pi(s_i') = \sigma_i^*(s_i') + \sigma_i^*(s_i) \cdot \sigma_i(s_i') / (1 - \sigma_i(s_i)) \quad \forall s_i' \in S_i \setminus \{s_i\}$$

$$\pi(s_i) = 0$$

$$\pi(s_i') \geq 0 \quad \forall s_i' \in S_i ,$$

$$\sum_{s_i' \in S_i} \pi(s_i') = \sum_{\substack{s_i' \in S_i \\ s_i' \neq s_i}} \left(\sigma_i^*(s_i') + \sigma_i^*(s_i) \frac{\sigma_i(s_i')}{1 - \sigma_i(s_i)} \right)$$

$$= \sum_{\substack{s_i' \in S_i \\ s_i' \neq s_i}} \sigma_i^*(s_i') + \boxed{\frac{\sigma_i^*(s_i)}{1 - \sigma_i(s_i)} \boxed{\sum_{\substack{s_i' \in S_i \\ s_i' \neq s_i}} \sigma_i(s_i')}}$$

$$= \sum_{s_i' \in S_i} \sigma_i^*(s_i') = 1$$

We have

$$u_i(\sigma_i, \sigma_{-i}^*) > u_i(s_i, \sigma_{-i}^*) \quad \text{[by our assumption]}$$

$$u_i(\pi, \sigma_{-i}^*) = \sum_{s_i' \in S_i} u_i(s_i', \sigma_{-i}^*) \cdot \pi(s_i')$$

$$= \sum_{\substack{s_i' \in S_i \\ s_i' \neq s_i}} \sigma_i^*(s_i') \cdot u_i(s_i', \sigma_{-i}^*) + \sum_{\substack{s_i' \in S_i \\ s_i' \neq s_i}} u_i(s_i', \sigma_{-i}^*) \cdot \boxed{\frac{\sigma_i^*(s_i)\, \sigma_i(s_i')}{1 - \sigma_i(s_i)}}$$

$$= \;\text{''} \;+ \frac{\sigma_i^*(s_i)}{1 - \sigma_i(s_i)} \boxed{\sum_{\substack{s_i' \in S_i \\ s_i' \neq s_i}} \sigma_i(s_i')\, u_i(s_i', \sigma_{-i}^*)}$$

$$= \;\text{''} \;+ \frac{\sigma_i^*(s_i)}{1 - \sigma_i(s_i)} \cdot \Big[\underbrace{u_i(\sigma_i, \sigma_{-i}^*) - u_i(s_i, \sigma_{-i}^*)}_{} \Big]$$

$$= u_i(\sigma_i^*, \sigma_{-i}^*) - \underline{\sigma_i^*(s_i')\, u_i(s_i', \sigma_{-i}^*)} + \frac{\sigma_i^*(s_i)}{1-\underline{\underline{\sigma_i(s_i)}}}\boxed{\left[u_i(\sigma_i, \sigma_{-i}^*) - u_i(s_i, \sigma_{-i}^*)\right]}$$

$$= u_i(\sigma_i^*, \sigma_{-i}^*) + \frac{\cancel{u_i(s_i, \sigma_{-i}^*)}\, \sigma_i^*(s_i) - \sigma_i^*(s_i)\cancel{u_i(s_i, \sigma_{-i}^*)} + \sigma_i(s_i)\cdot\sigma_i^*(s_i')}{1-\sigma_i(s_i)}$$

$$u_i(s_i, \sigma_{-i}^*) \quad + \quad \sigma_i^*(s_i)\cdot u_i(\sigma_i, \sigma_{-i}^*)$$

$$= u_i(\sigma_i^*, \sigma_{-i}^*) + \frac{\sigma_i(s_i)\,\sigma_i^*(s_i')\,u_i(s_i, \sigma_{-i}^*) + \sigma_i^*(s_i)\,u_i(\sigma_i, \sigma_{-i}^*)}{1-\sigma_i(s_i)}$$

$$> u_i(\sigma_i^*, \sigma_{-i}^*)$$

This contradicts our assumption that $(\sigma_i^*)_{i \in N}$ is an MSNE.

Iterative Elimination of Dominated Strategies

Lecture 3.2

Example 1:

	A	B	C
→ A	2, $\underline{3}$	3, 0	0, $\underline{1}$
→ B	0, $\underline{0}$	1, $\underline{6}$	4, 2

$u_2(A,\sigma) = 1.5 > 1 = u_2(A,C)$

$u_2(B,\sigma) = 3 > 2 = u_2(B,C)$

Claim: $\sigma = \left(A:\frac{1}{2}, B:\frac{1}{2}\right)$ strongly dominates the pure strategy C for the column player. ✓

Reduced game:

	A	B
A	2, 3	3, 0
~~B~~	~~0, 0~~	~~1, 6~~

$$u_1(A,A) = 2 > 0 = u_1(B,A)$$

$$u_1(A,B) = 3 > 1 = u_1(B,B)$$

Claim: The strategy B is strongly dominated by the strategy A for the row player.

Reduced game:

	A	~~B~~
A	2, 3	~~3, 0~~

Claim: The strategy B is strongly dominated by the strategy A for the column player.

Reduced game:

	A
A	2,3

(A, A) is the unique MSNE for the given game. ∎

Lemma: Given a game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, if a pure strategy $s_i \in S_i$ is weakly dominated by some mixed strategy, then there exists an MSNE $(\sigma_i^*)_{i \in N}$ such that $\sigma_i^*(s_i) = 0$.

Proof: Analogous to the lemma for strongly dominated strategy. ∎

Example:

	A	B
A	2, 3	3, 3

A is a weakly dominated strategy for the column player.

$\sigma \in \Delta(\{A, B\})$, (A, σ) is an MSNE.

Example: Suppose there are 50 students in a class. Each student writes a number in $\{0, 1, 2, \ldots, 100\}$. Let $s_1, \ldots, s_{50}$ be the numbers written.

$$l = \frac{2}{3} \cdot \frac{s_1 + \ldots + s_{50}}{50}$$

. The winner is the student whose number is closest to l.

Find an MSNE of the game.

Iterative elimination of strongly dominated strategies.

The strategies $68, 69, \ldots, 100$ are weakly dominated by the strategy 67.

<u>Reduced game:</u> $S_i = \{0, 1, \ldots, 67\}$

$$l \leq \frac{2}{3} \cdot 67 \approx 44\ldots$$

<u>Reduced game:</u> $S_i = \{0, 1, \ldots, 45\}$

$\vdots$

$S_i = \{0\}$

$(0, 0, \ldots, 0)$ is an MSNE.

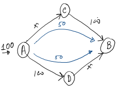
Lecture 3.3

Braess paradox:

Set of players: $\{1, \dots, 100\}$

Set of strategies:

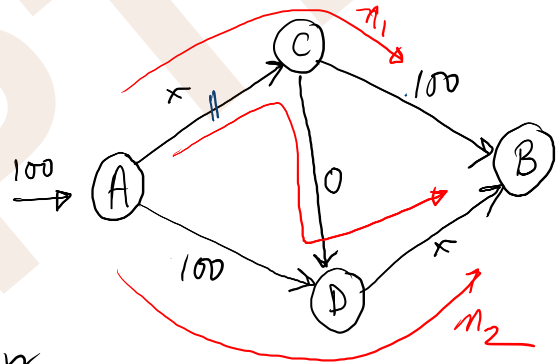
$\{A \to C \to B,\ A \to D \to B\}$



50 people using $A \to C \to B$ and the other 50 people using $A \to D \to B$ is a pure strategy Nash equilibrium.

Average delay of each player $= \underline{\underline{150}}$

Strategy set $= \{ A \to C \to B,\ A \to D \to B,\ A \to C \to D \to B \}$



<u>Claim:</u> $A \to C \to D \to B$ is a weakly dominated strategy

→ The delay of a player i using $A \to C \to D \to B$ is $(n - n_2) + (n - n_1)$

$= 2n - (n_1 + n_2) = 200 - (n_1 + n_2)$

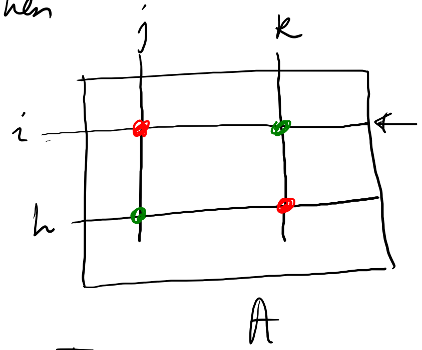
→ Suppose player i deviates to $A \to C \to D$. Then its utility is

$= (n - n_2) + 100 = 200 - n_2 \geq 200 - (n_1 + n_2)$

All players playing $A \to C \to D \to B$ is a WDSE.

Average delay $= 200$

<u>Lemma:</u> Let $A \in \mathbb{R}^{m \times n}$ be a matrix. Let (i,j) and (h,k) be two PSNEs of the matrix game A. Then (h,j) and (i,k) are also PSNEs.



<u>Proof:</u> $\underline{A[h,j]} \leq A[i,j] \leq A[i,k] \leq A[h,k] \leq \underline{A[h,j]}$

∎

Yao's Lemma.

Comparism based sorting algorithm.

We know that any comparison-based deterministic sorting algorithm must make $\Omega(n \log n)$ comparisons.

Proof by decision trees.

Can randomization help us break the $\Omega(n \log n)$ lower bound?

No.

A perspective of randomized algorithms:

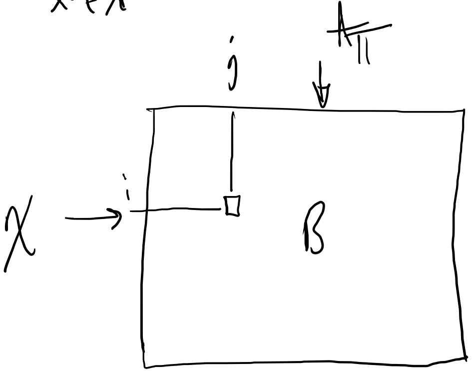
A randomized algorithm can be equivalently viewed as a probability distribution over the set $\{ \underline{\underline{A(s_i)}} \mid \underline{\underline{s_i}} \in \{0,1\}^{s} \}$ of deterministic algorithms.

<u>Theorem</u> (Yao's Lemma) Let A be a <u>randomized</u> algorithm for some problem Π. For an input x, let $T(A, x)$ be the random variable denoting the "cost" of A on x. Let

$\mathcal{X}$ be the set of all inputs to Π of length n, X be a random variable having distribution $\underline{p}$ on $\mathcal{X}$, A_Π be the set of all deterministic algorithms for Π. Then

$$\max_{x \in \mathcal{X}} \mathbb{E}\left[T(A, x)\right] \geq \min_{a \in A_\Pi} \mathbb{E}\left[T(a, X)\right]$$

<u>Proof:</u>



$$\min_{z \in \Delta(A_\Pi)} \boxed{\max_{x \in \mathcal{X}} e_x B z} \geq \max_{y \in \Delta(\mathcal{X})} \boxed{\min_{a \in A_\Pi} y B e_a}$$

$$\Rightarrow \max_{x \in \mathcal{X}} e_x B \sigma_A \geq \min_{a \in A_\Pi} X B e_a$$

i.e. $$\max_{x \in \mathcal{X}} \mathbb{E}\left[T(A, x)\right] \geq \min_{a \in A_\Pi} \mathbb{E}\left[T(a, X)\right]$$

$\blacksquare$

__Theorem:__ Any comparison-based randomized algorithm to sort n objects must make $\Omega(n \log n)$ comparisons.

__Proof:__ Let A be any randomized comparison-based sorting algorithm, $\mathcal{A}$ the set of all deterministic

" , $T(A, x)$ the random variable denoting the number of comparisons made by A on x.

__To show:__ $\max\limits_{x \in \mathcal{X}} \mathbb{E}[T(A, x)] = \Omega(n \log n)$

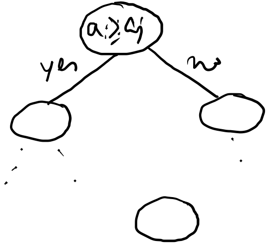
Let X be a random variable having <u>uniform distribution</u> on the set $\mathcal{X}$ of all inputs.

By Yao's lemma, it is enough to show that

$$\mathbb{E}[T(a,X)] = \Omega(n \log n)$$ for every

deterministic algorithm a.

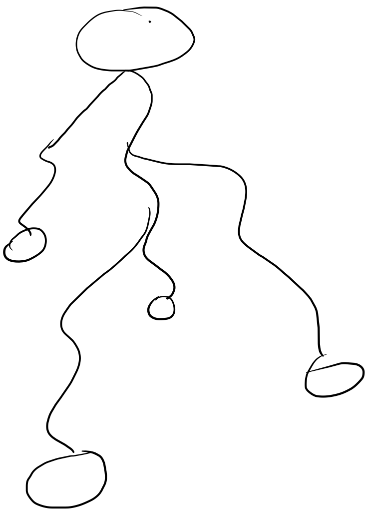
<u>To show:</u>



number of comparisons is the depth of the tree.

There are at least $n!$ leaf nodes.

The depth $\Omega(\log n!) = \Omega(n \log n)$



<u>To show:</u> Average depth of the leaf nodes of the decision tree for a is $\Omega(n \log n)$.

If the tree is balanced, then the average depth of leaf nodes is $\Omega(\log n!) = \underline{\underline{\Omega(n \log n)}}$. $\blacksquare$

Computing Equilibrium

Lecture 3.5

Computational task:

Input : A finite game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$

Output: An MSNE

Is it possible to have an MSNE where some probability values are irrational numbers even when input consists of only rational numbers?

YES if the number of players in T is at least 3.

If the number of players is 2, then the answer is NO.

$\varepsilon > 0$ ε-MSNE: Suppose all the utility values are in between 0 and 1. Then a mixed strategy profile $(\sigma_i^*)_{i \in N} \in \times_{i \in N} \Delta(S_i)$ is called an MSNE if unilateral deviation by any player can benefit it by at most ε.

ε-NASH:

Input: A normal form game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$

Output: An ε-MSNE.

Support Enumeration: Suppose we have only 2 players. Their utility matrices are $A, B \in \mathbb{R}^{m \times n}$. We are looking for an MSNE $(\sigma_1^*, \sigma_2^*) \in \Delta([m]) \times \Delta([n])$.

We guess the supports I and J of σ_1^* and σ_2^* respectively.

That is $\quad \mathcal{I} = \{ i \in [m] \mid \sigma_1^*(i) \neq 0 \}, \subseteq 2^{[m]} \setminus \{\emptyset\}$

$$\mathcal{J} = \{ j \in [n] \mid \sigma_2^*(j) \neq 0 \} \subseteq 2^{[n]} \setminus \{\emptyset\}$$

Let $\quad \sigma_1^* = (x_1, \dots, x_m), \quad \sigma_2^* = (y_1, \dots, y_n).$

$$\forall i \in \mathcal{I}, \quad \sum_{j=1}^{n} a_{ij} \cdot y_j = u, \qquad \Bigg| \qquad \forall j \in \mathcal{J}, \quad \sum_{i=1}^{m} b_{ij} \cdot x_i = v,$$

$$\forall i \in [m] \setminus \mathcal{I}, \quad \sum_{j=1}^{n} a_{ij} y_j \leqslant u, \qquad \Bigg| \qquad \forall j \in [n] \setminus \mathcal{J}, \quad \sum_{i=1}^{m} b_{ij} x_i \leqslant v,$$

$$y_1 + y_2 + \dots + y_n = 1, \qquad \Bigg| \qquad x_1 + \dots + x_m = 1,$$

$$\forall j \in [n] \quad y_j \geqslant 0, \quad \forall j \in [n] \setminus \mathcal{J}, \; y_j = 0 \qquad \Bigg| \qquad x_1, \dots, x_m \geqslant 0, \quad \forall i \in [m] \setminus \mathcal{I}, \; x_i = 0$$

Iterate over all possible $2^m - 1$ non-empty subsets I of $[m]$

and — — $2^n - 1$ — — J of $[n]$,

solve the linear program. If the LP is feasible for some I and J then, $((x_1^*, \ldots, x_m^*), (y_1^*, \ldots, y_n^*))$ is an MSNE where $(x_1^*, \ldots, x_m^*, y_1^*, \ldots, y_n^*, u^*, v^*)$ is a feasible solution.

Proof of correctness follows from indifference principal.

Run time: $O\left(2^m \cdot 2^n \cdot \text{poly}(\text{input size})\right)$.