Lecture 11.1

Brief recall of what we have learnt so far.

(1) Game Theory: studied various kinds of games, for example, normal form game, extensive form game, and Bayesian game. We studied various game theoretic tools, for examples, equilibrium concepts, learning dynamics to predict the outcome of a game. Moreover, we studied various algorithms

and corresponding complexity theoretic framework of PLS, PPAD etc. to formalize hardness results. Last but not the least, we studied cost/price of anarchy for PSNEs.

(2) Mechanism Design: "Reverse game theory". Given a social choice function $f : \times_{i=1}^{n} \Theta_i \longrightarrow \mathcal{X}$, can we design a game to "implement" f. This question

is settled by the Revelation Theorem. What are the social choice functions implementable? This is answered by Gibbard-Satterwaite Theorem. Assume quasi-linear environment ——— (i) outcome has a special structure: (allocation, payment) (ii) utility function also has a special structure : utility is valuation + payment.

"Affine maximizers" are the social choice functions implementable in quasi-linear environment.

Characterization of DSIC Mechanisms:

(i) payment for player i depends on θ_i only through the allocation $k(\theta_i, \underline{\theta}_{-i})$

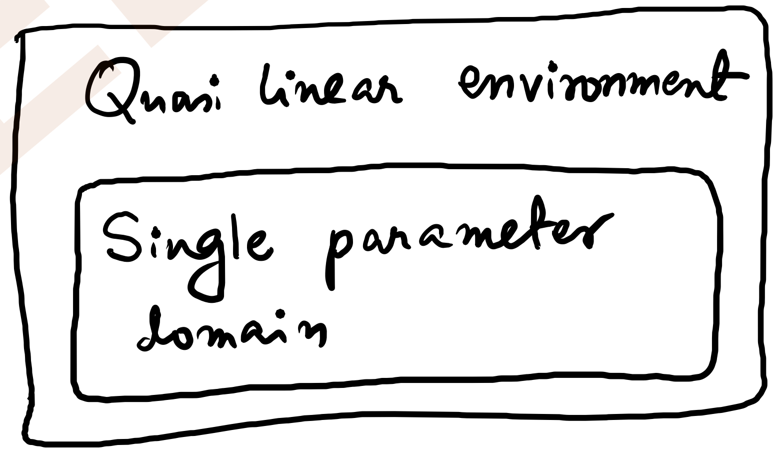
(ii) allocation function $k(\cdot)$ simultaneously optimizes for all the players.

$$k(\theta_i, \underline{\theta}_{-i}) \in \underset{K \in k(\cdot, \underline{\theta}_{-i})}{\arg\max} \left[v_i(k(\theta_i), \theta_i) + t_i(\theta_i, \underline{\theta}_{-i}) \right]$$

Single Parameter Domain

Quasi linear environment allows arbitrary valuation function

$$v_i : \mathcal{K} \times \Theta_i \longrightarrow \mathbb{R}$$



Define: A single parameter domain Θ_i is defined by $\mathcal{R}_i \subseteq \mathcal{K}$ and Θ_i is a real interval, $v_i(k, \theta_i) = \theta_i \ \forall k \in \mathcal{R}_i$, $v_i(k, \theta_i) = 0 \ \forall k \in \mathcal{K} \setminus \mathcal{R}_i$. Both $\mathcal{R}_i$ and

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H_i are common knowledge.

Lecture 11.2

"Social Choice Functions" Implementable in Single Parameter Domain

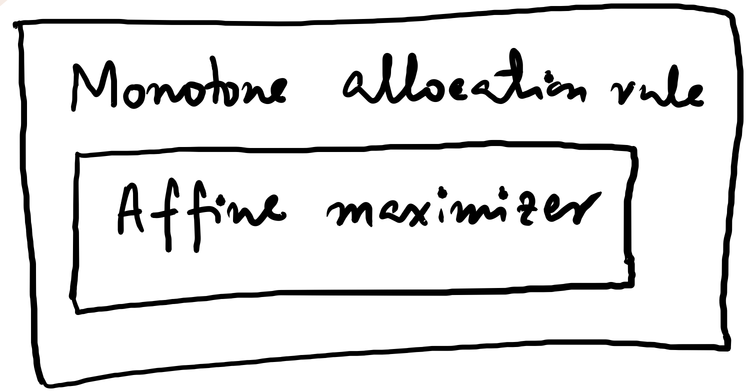
→ Allocation rules.

Monotone Allocation Rules in a Single Parameter Domain:

An allocation rule $k : \Theta \longrightarrow \mathcal{K}$ in a single parameter domain is called monotone in θ_i if, for every $\theta_{-i} \in \Theta_{-i}$, $\theta_i, \theta_i' \in \Theta_i$, $\theta_i \leq \theta_i'$, $k(\theta_i, \theta_{-i}) \in \mathcal{K}_i$, we have $k(\theta_i', \theta_{-i}) \in \mathcal{K}_i$. That is, for a fixed type profile θ_{-i}

of other players, if player i wins with type θ_i, then player i continues to win with increase of its type.

- We will show that monotone allocation rules form the set of allocation rules implementable in a single parameter domain.



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Critical Value Function :- Given an allocation rule in a single parameter domain, the critical value function $c_i(\theta_{-i})$ of player i is defined as

$$c_i(\theta_{-i}) = \sup \{ \theta_i \in \Theta_i \mid k(\theta_i, \theta_{-i}) \notin K_i \} \in \mathbb{R}$$

$$c_i : \Theta_{-i} \longrightarrow \mathbb{R}$$

If for some $\theta_{-i} \in \Theta_{-i}$, the set $\{ \theta_i \in \Theta_i \mid k(\theta_i, \theta_{-i}) \notin K_i \} = \phi$, then $c_i(\theta_{-i})$ is undefined.

Theorem: A social choice function $f(\cdot) = (k(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$ in a single parameter domain is dominant strategy incentive compatible and losers pay nothing if and only if the following conditions hold:

(i) The allocation rule $k(\cdot)$ is monotone.

(ii) Every winning player essentially pays its critical value.

That is,

$$t_i(\theta_i, \theta_{-i}) = \begin{cases} -c_i(\theta_{-i}) & \text{if } k(\theta_i, \theta_{-i}) \in K_i \\ 0 & \text{o/w.} \end{cases}$$

If $c_i(\theta_{-i})$ is undefined for some θ_{-i}, then we define $c_i(\theta_{-i}) = \lambda_i$ for any real number λ_i.

Proof: (If part) For a player $i \in [n]$, $\theta_i \in \Theta_i$, $\theta_{-i} \in \Theta_{-i}$, if player i wins, its utility is $\theta_i - c_i(\theta_{-i})$ and if player i loses then its utility is 0.

$$u_i(\theta_i, \theta_{-i}) = v_i(k(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i})$$

To show:

$$v_i(k(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i}) \geq v_i(k(\theta_i', \theta_{-i}), \theta_i) + t_i(\theta_i', \theta_{-i})$$

case I: $k(\theta_i, \theta_{-i}) \in K_i$

$$v_i(k(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i}) = \theta_i - c_i(\theta_{-i}) \geq 0$$

if $k(\theta_i', \theta_{-i}) \in K_i$ ✓ player prefers winning

if $k(\theta_i', \theta_{-i}) \notin K_i$ if and only if

$\theta_i \geq c_i(\theta_{-i})$

Lecture 11.3

Characterization of DSIC Social Choice Function in Single Parameter Domain

Theorem: A social choice function $f(\cdot) = (k^*(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$ in a single parameter domain is DSIC and losers do not pay anything if and only if all the following conditions hold.

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(i) The allocation $k^*(\cdot)$ is monotone in every θ_i

(ii) Every winning player pays its critical value.

$$t_i(\theta_i, \underline{\theta}_{-i}) = \begin{cases} -c_i(\underline{\theta}_{-i}) & \text{if } k^*(\theta_i, \underline{\theta}_{-i}) \in \mathcal{K}_i \\ 0 & \text{otherwise.} \end{cases}$$

For some $\underline{\theta}_{-i} \in \underline{\Theta}_{-i}$, if $c_i(\underline{\theta}_{-i})$ is not defined, then we set $c_i(\underline{\theta}_{-i}) = \lambda_i$ for any real number λ_i.

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<u>Proof:</u> (If part) ✓

(Only if part)

<u>proof for (i):</u> Since $k^*(\cdot)$ is <u>not</u> monotone in every θ_i, there exists a type profile $\underline{\theta}_{-i} \in \Theta_{-i}$, types $\theta_i, \theta_i' \in \Theta_i$, $\boxed{\theta_i < \theta_i'}$ and $k^*(\theta_i, \underline{\theta}_{-i}) \in R_i$ but $k^*(\theta_i', \underline{\theta}_{-i}) \notin R_i$.

Since $f(\cdot)$ is DSIC, player i prefers winning in type profile $(\theta_i, \underline{\theta}_{-i})$ compared to losing.

$$\theta_i - c_i(\underline{\theta}_{-i}) \geq 0 \quad \Rightarrow \theta_i \geq c_i(\underline{\theta}_{-i}) \qquad (1)$$

But player i prefers losing in type profile $(\theta_i', \underline{\theta}_{-i})$ compared to winning.

$$0 \geq \theta_i' - c_i(\underline{\theta}_{-i})$$

$$\Rightarrow \quad \theta_i' \leq c_i(\underline{\theta}_{-i}) \qquad (2)$$

From (1) and (2) $\theta_i' \leq \theta_i$ which contradicts our assumption that $\theta_i' > \theta_i$

Proof of part (ii) :- We know that, for every $i \in [n]$, $\theta_i, \theta_i' \in \Theta_i$, $\underline{\theta}_{-i} \in \Theta_{-i}$, $\underbrace{k^*(\theta_i, \underline{\theta}_{-i}) = k^*(\theta_i', \underline{\theta}_{-i})}_{\in K_i}$, we have $t_i(\theta_i, \underline{\theta}_{-i}) = t_i(\theta_i', \underline{\theta}_{-i})$.

Suppose $t_i(\theta_i, \underline{\theta}_{-i}) \neq -c_i(\underline{\theta}_{-i})$

(a) $t_i(\theta_i, \underline{\theta}_{-i}) < -c_i(\underline{\theta}_{-i})$ ✓

(b) $t_i(\theta_i, \underline{\theta}_{-i}) > -c_i(\underline{\theta}_{-i})$ ✓

Refuting possibility (a):

$t_i(\theta_i, \underline{\theta}_{-i}) < -c_i(\underline{\theta}_{-i})$

$\Rightarrow -t_i(\theta_i, \underline{\theta}_{-i}) > c_i(\underline{\theta}_{-i})$

$\Rightarrow \exists\, \theta_i'' \in \Theta_i$, $\underbrace{-t_i(\theta_i, \underline{\theta}_{-i}) > \theta_i''} \boxed{\theta_i'' > c_i(\underline{\theta}_{-i})}$

player i wins in $(\theta_i'', \underline{\theta}_{-i})$

In $(\theta_i'', \underline{\theta}_{-i})$, $u_i(\theta_i'', \underline{\theta}_{-i}) = \theta_i'' + t_i(\theta_i'', \underline{\theta}_{-i})$

$$= \theta_i'' + t_i(\theta_i, \underline{\theta}_{-i})$$

$$< 0$$

But then player i can report any very small type $(< c_i(\underline{\theta}_{-i}))$ and lose and thereby receive more utility.

This contradicts our assumption that $f(\cdot)$ is DSIC

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Single Parameter Domain

Lecture 11.4

The allocation rules implementable in DSE in the single parameter domain are monotone allocation rule.

<u>Recall:</u> The allocation rules implementable in DSE in quasi-linear environment (Groves Theorem and Robert's Theorem) are the affine maximizers.

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Example of affine maximizer:

$$k(\theta_1, \ldots, \theta_n) = \underset{k \in \mathcal{K}}{\operatorname{argmax}} \sum_{i=1}^{n} a_i\, v_i(k, \theta) + c$$

$$a_i \geqslant 0, \quad c \in \mathbb{R}$$

$$\boxed{\begin{array}{c}\text{Monotone} \\ \boxed{\text{Affine maximizer}}\end{array}}$$

Examples of monotone allocation rules which are not affine maximizer:–

$$k(\theta_1, \ldots, \theta_n) = \underset{k \in \mathcal{K}}{\operatorname{argmax}} \sum_{i=1}^{n} a_i\, v_i(k, \theta_i)^{\lambda_i} + c$$

$$a_i, \lambda_i \in \mathbb{R}_{\geqslant 0}, \quad c \in \mathbb{R}.$$

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Recall, we have worked with a restricted version of single parameter domain.

Mayerson's Lemma: We have the following in any single-parameter domain.

(i) An allocation rule $k : \Theta \to \mathcal{K}$ is DSIC if and only if $k(\cdot)$ is monotone in each θ_i.

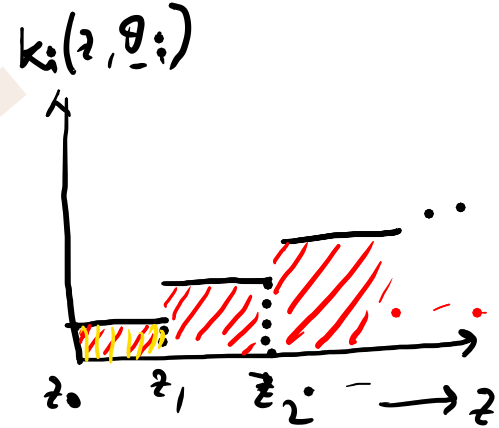
(ii) If $k(\cdot)$ is monotone, there there exist unique payment rules $t_1(\cdot), \ldots, t_n(\cdot)$ where players reporting 0 as their type do <u>not</u> pay anything such that the mechanism $f(\cdot) = (k(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$ is DSIC.

(iii) The payment rule of part (ii) is given by the following explicit formula.

$$t_i(\theta_i, \underline{\theta}_{-i}) = -\int_0^{\theta_i} z \cdot \frac{d}{dz} k_i(z, \theta_{-i})\, dz.$$

where $k(\cdot) = (k_1(\cdot), \ldots, k_n(\cdot))$, $k_i(\cdot)$ is differentiable in its domain.

If $k_i(\cdot)$ is a step function having jumps at $z_0, z_1, z_2, \dots$



$$t_i(\theta_i, \theta_{-i}) = k_i(z_0, \theta_{-i})\cdot(z_1 - z_0) + k_i(z_1, \theta_{-i})\cdot(z_2 - z_1) + \dots$$

Uniqueness of VCG Payment Rule: Assume Θ_i is connected in some Euclidean space for every $i \in [n]$. Let $f(\cdot) = (k(\cdot), t_1(\cdot), \dots, t_n(\cdot))$ be DSIC. If $f'(\cdot) = (k(\cdot), t_1'(\cdot), \dots, t_n'(\cdot))$ is also DSIC, then

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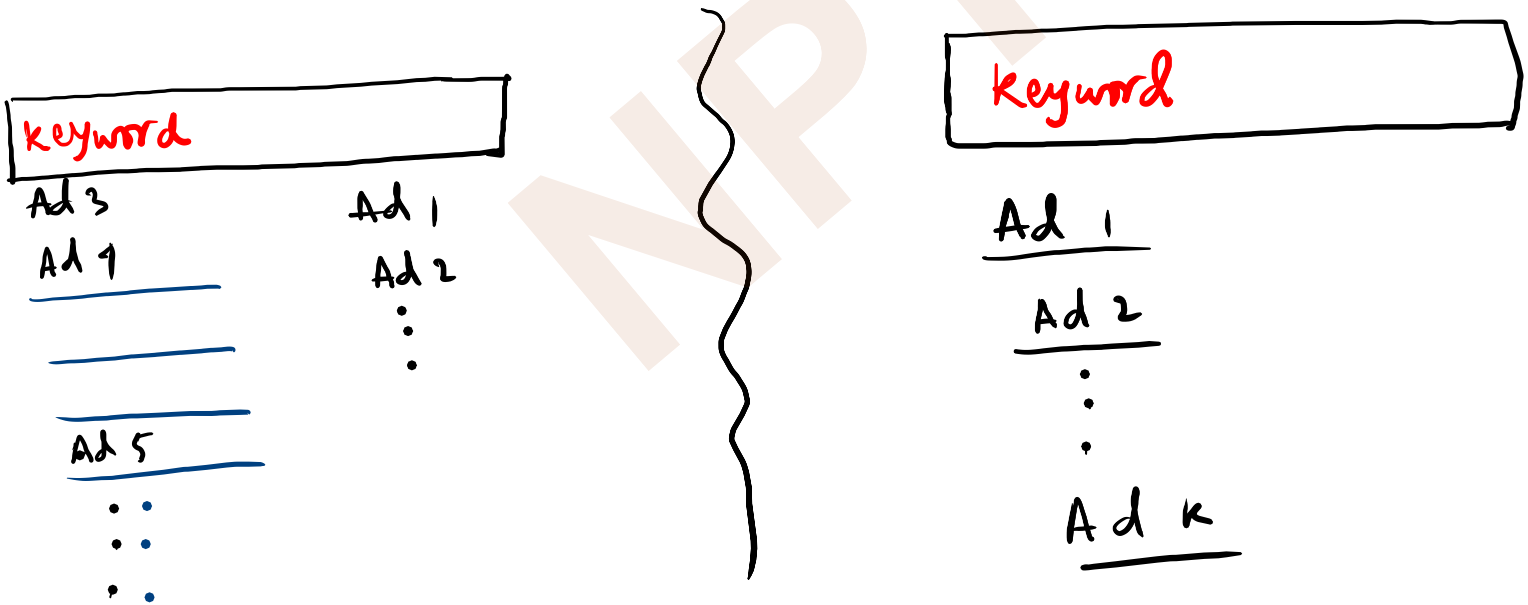
$\forall \theta \in \Theta$, $t_i'(\theta) = t_i(\theta) + h_i(\underline{\theta}_{-i})$

for some function $h_i : \underline{\Theta}_{-i} \longrightarrow \mathbb{R}$.

Lecture 11.5

Application of Myerson's Lemma:

Sponsored Search Auction



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Click-through-rate (CTR)

$\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_k$

α_1 ______ ←
α_2 ______
$\vdots$
α_k ______

Each Ad j has a quality score β_j

Assumption: The probability that an user clicks an Ad j shown at position i is $(\alpha_i \cdot \beta_j)$.

Each advertiser i has valuation v_i for its ad, and if its Ad is shown in the j-th slot, then

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the advertiser has valuation $v_i \alpha_j$.

Sponsored search auction belongs to single parameter domain.

Monotone allocation rule: CTRs: $\alpha_1, \dots, \alpha_k$

types: $v_1, \dots, v_n$ $k \leq n$

quality score: $\beta_1, \dots, \beta_n$.

$$\underset{\substack{k \in \mathcal{K} \\ \shortparallel \\ (k_1(\cdot), \dots, k_n(\cdot))}}{\arg\max} \left\{ \sum_{i=1}^{n} \mathbb{1}(k_i(\theta) \neq 0) \cdot \alpha_{k_i(\theta)} \cdot \beta_i \cdot v_i \right\}$$

is a monotone allocation rule.

The payment formula due to Myerson's lemma:

Fix player i: $(v_i, \underline{v}_{-i})$

If player i's ad is not chosen in $(v_i, \underline{v}_{-i})$ then $t_i(v_i, \underline{v}_{-i}) = 0$.

If player i's ad is shown in the l-th slot for some $l \in \{1, \dots, k\}$, then

$$t_i(v_i, \underline{v}_i) = \alpha_1 v_i \cdot (z_2 - z_1) + \alpha_2 v_i (z_3 - z_2) + \dots + (v_i - z_\ell) \cdot \alpha_\ell \cdot v_i$$