Lecture 10.5

Theorem: $f(\cdot) = \left(k^*(\cdot), t_1(\cdot), \dots, t_n(\cdot)\right)$.

$k^*(\cdot)$ is an affine maximizer.

$$\forall i \in [n], \quad t_i(\theta) = \sum_{\substack{j \in [n] \\ j \neq i}} \left[\frac{w_j}{w_i} v_j\left(k^*(\theta), \theta_j\right) + \frac{c_{k^*(\theta)}}{w_i}\right] + h_i(\underline{\theta}_{-i})$$

$$h_i : \underline{\Theta}_{-i} \longrightarrow \mathbb{R}.$$

Proof: Suppose $f(\cdot)$ is <u>not</u> DSIC.

Then there exist $i \in [n]$, $\theta_i \in \Theta_i$, $\theta_{-i} \in \Theta_{-i}$, $\theta_i' \in \Theta_i'$ such that

$$u_i(f(\theta_i', \theta_{-i}), \theta_i) > u_i(f(\theta_i, \theta_{-i}), \theta_i)$$

$$\Rightarrow v_i(k^*(\theta_i', \theta_{-i}), \theta_i) + p_i(\theta_i', \theta_{-i}) > v_i(k^*(\theta_i, \theta_{-i}), \theta_i) + p_i(\theta_i, \theta_{-i})$$

$$\Rightarrow v_i(k^*(\theta_i', \theta_{-i}), \theta_i) + \sum_{\substack{j \in [n] \\ j \neq i}} \left[\frac{w_j}{w_i} v_j(k^*(\theta_i', \theta_{-i}), \theta_j) + \frac{c_{k^*(\theta_i', \theta_{-i})}}{w_i} \right] >$$

$$v_i(k^*(\theta_i, \theta_{-i}), \theta_i) + \sum_{\substack{j \in [n] \\ j \neq i}} \left[\frac{w_j}{w_i} v_j(k^*(\theta_i', \theta_{-i}), \theta_j) + \frac{c_{k^*(\theta_i, \theta_{-i})}}{w_i} \right]$$

$$\Rightarrow \sum_{j \in [n]} w_j v_j\left(\underline{k^*(\theta_i', \theta_{-i})}, \theta_j\right) + C_{k^*(\theta_i', \theta_{-i})} > \sum_{j \in [n]} w_j v_j\left(\underline{k^*(\theta_i, \theta_{-i})}, \theta_j\right) + C_{k^*(\theta_i, \theta_{-i})}$$

$\Rightarrow$ In the type profile (θ_i, θ_{-i}), the allocation rule $k^*(\cdot)$ does <u>not</u> satisfy the condition of affine maximizer.

$\Rightarrow$ This contradicts our assumption that $k^*(\cdot)$ is an affine maximizer. $\blacksquare$

Roberts' Theorem: If $|\mathcal{K}| \geq 3$, the allocation rule $k^*(\cdot)$ is an onto function, the valuations are arbitrary, and the allocation rule $k^*(\cdot)$ is implementable in VWDSE using some payment rule, then $k^*(\cdot)$ is an affine maximizer.

Characterization of DSIC Mechanisms

A social choice function $f(\cdot) = (k(\cdot), t_1(\cdot), \dots, t_n(\cdot))$ is DSIC if and only if the following conditions hold for every $i \in [n]$ and $\underline{\theta}_{-i} \in \Theta_{-i}$

(i) The payment $t_i(\theta)$ depends on θ_i only via $k(\theta_i, \underline{\theta}_{-i})$. That is, for $\theta_i, \theta_i' \in \Theta_i$ and $\underline{\theta}_{-i} \in \Theta_{-i}$ such that $k(\theta_i, \underline{\theta}_{-i}) = k(\theta_i', \underline{\theta}_{-i})$, then $t_i(\theta_i, \underline{\theta}_{-i}) = t_i(\theta_i', \underline{\theta}_{-i})$

(ii) The allocation rule simultaneously optimizes for all the players. That is, for every $\theta_i \in \Theta_i$, $\underline{\theta}_{-i} \in \Theta_{-i}$,

$$k(\theta_i, \underline{\theta}_{-i}) \in \underset{K \in k(\cdot, \underline{\theta}_{-i})}{\arg\max} \left[v_i(K, \theta_i) + t_i(\theta_i, \underline{\theta}_{-i}) \right]$$

for every $i \in [n]$

(If part)

Proof: Let $\theta_i, \theta_i' \in \Theta_i$, $\theta_{-i} \in \Theta_{-i}$

$$u_i(\theta_i, \theta_{-i}) = v_i(k(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i})$$
$$= v_i(k(\theta_i, \theta_{-i}), \theta_i) + t_i(k(\theta_i, \theta_{-i}), \theta_{-i})$$
$$\geq v_i(k(\theta_i', \theta_{-i}), \theta_i) + t_i(k(\theta_i', \theta_{-i}), \theta_{-i})$$

(Only if part)

Let $f(\cdot)$ be DSIC.

$i \in [n]$, $\theta_i, \theta_i' \in \Theta_i$, $\theta_{-i} \in \Theta_{-i}$ s.t. $k(\theta_i, \theta_{-i}) = k(\theta_i', \theta_{-i})$

Suppose, $t_i(\theta_i, \theta_{-i}) > t_i(\theta_i', \theta_{-i})$

$$\Rightarrow \quad \underbrace{v_i(k(\theta_i,\theta_{-i}),\theta_i) + t_i(\theta_i,\theta_{-i})} > v_i(k(\theta_i,\theta_{-i}),\theta_i) + t_i(\theta_i',\theta_{-i})$$

$$= \underbrace{v_i(k(\theta_i',\theta_{-i}),\theta_i) + t_i(\theta_i',\theta_{-i})}$$

Contradicts DSIC of $f(\cdot)$ at (θ_i', θ_{-i}).

Contradicts DSIC property of $f(\cdot)$ at (θ_i, θ_{-i}).

$\exists\, \theta_i, \theta_i' \in \Theta_i,\ \theta_{-i} \in \Theta_{-i}$ such that

$$v_i(k(\theta_i,\theta_{-i}),\theta_i) + t_i(\theta_i,\theta_{-i}) < v_i(k(\theta_i',\theta_{-i}),\theta_i) + t_i(\theta_i',\theta_{-i})$$

Contradicts DSIC property of $f(\cdot)$ at (θ_i,θ_{-i}). ∎