**14.** (a) $2^{-7}[8^n \sin(8x + n\pi/2) - 2 . 6^n \sin(6x + n\pi/2)$

$$- 2 . 4^n \sin(4x + n\pi/2) + 6 . 2^n \sin(2x + n\pi/2)]$$

(b) $2^{-8}[9^n \sin(9 + n\pi/2) - 7^n \sin(7x + n\pi/2)$

$$- 4 . 5^n \sin(5x + n\pi/2) + 4 . 3^n \sin(3x + n\pi/2) + 6 \sin(x + n\pi/2)]$$

**15.** $(n!/2a)\{(a-x)^{-n-1} + (-1)^n (a+x)^{-n-1}\}$

**16.** $(-1)^n n!\left[\dfrac{1}{(x+2)^{n+1}} - \dfrac{1}{(x+3)^{n+1}}\right] + \dfrac{1}{2}\left\{\cos\left(x + \dfrac{n\pi}{2}\right) - 5^n \cos\left(5x + \dfrac{n\pi}{2}\right)\right\}$

**17.** $\dfrac{(-1)^n n!}{4a^3}\left[\dfrac{1}{(x-a)^{n+1}} - \dfrac{1}{(x+a)^{n+1}} - \dfrac{2}{a^{n+1}} \sin^{n+1}\theta \sin(n+1)\theta\right]$ [where $\theta = \tan^{-1}(a/x)$]

**18.** $\dfrac{(-1)^{n-1}(n-1)!}{2}\left\{\dfrac{1}{(x+a)^n} - \dfrac{1}{(x-a)^n}\right\}$

**19.** $\dfrac{(-1)^n . 2^{n+2} n!}{3(n+2)/2} \sin(n+1)\theta \sin^{n+1}\theta$ $\left[\text{where } x = \dfrac{\cos(\theta + \pi/6)}{\sin\theta}\right]$

**20.** $\dfrac{(-1)^n . 2^{n+1} n!}{3^{n+2/2}}\{\sin(n+1)\theta - \sin^{n+1}\theta - \sin n + \phi \sin^{n+1}\phi\}$

$$\left[\text{where } x = \frac{\cos(\theta - \pi/6)}{\sin\theta} = \frac{\cos(\phi + \pi/6)}{\sin\phi}\right]$$

**21.** If $z = (-1)^{n-1}(n-1)! \sin n\theta \sin^n \theta$, where $\theta = \cot^{-1} x$ then we have the following:

(a) $2z$ (b) $2z$ (c) $\frac{1}{2}z$.

## § 3.6. LEIBNITZ'S THEOREM

The $n$th differential coefficient of the product of two functions is conveniently evaluated by the use of this theorem. Its statement is :

**Theorem.** *If u and v are two functions of x, then*

$$D^n(uv) = D^n u . v + {}^nC_1 D^{n-1}u\, Dv + {}^nC_2 D^{n-2} u\, D^2 v + \ldots$$
$$+ {}^nC_r D^{n-r} u\, D^r v + \ldots + u\, D^n v.$$

**[Jiwaji 1990, 94; Vikram 92, 2001; Indore 92, 2001; Bilaspur 93, 98, 2001; Sagar 94; Bhoj 99; Jabalpur 99; Ravishankar 97, 99S, 2005S, 2007; Bhopal 96, 2000, 2005, 2006; Rewa 2004]**

**Proof.** We shall prove this theorem by mathematical induction. By actual differentiation, we have

$$D(uv) = Du . v + u . Dv$$
$$D^2(uv) = D^2u . v + 2Du . Dv + u . D^2v$$

or $$D^2(uv) = (D^2u)\, v + {}^2C_1 Du . Dv + {}^2C_2 u . D^2 v.$$

Thus the statement is true for $n = 1, 2$.

Now let the theorem be true for some particular value $m$ of $n$ *i.e.*,

$$D^m(uv) = D^m u . v + mC_1 . D^{m-1}u . Dv + {}^mC_2 . D^{m-2}u . D^2 v + \ldots$$
$$+ {}^mC_r . D^{m-r}u . D^r v + \ldots + u . D^m v \text{ is true.}$$

Now differentiating w.r.t. $x$ again, we have

$$D^{m+1}(uv) = \{D^{m+1}u \,.\, v + D^m u \,.\, Dv\} + {}^mC_1\{D^m u \,.\, Dv + D^{m-1} \,.\, uD^2 v\} + \ldots$$
$$+ {}^mC_r\{D^{m-r+1}u \,.\, D^r v + D^{m-r}u \,.\, D^{r+1}v\} + \ldots$$
$$+ \{Du \,.\, D^m v + u \,.\, D^{m+1}v\}.$$

Rearranging the terms as follows :

$$D^{m+1}(uv) = D^{m+1}u \,.\, v + (1 + {}^mC_1)\, D^m u \,.\, Dv + \ldots$$
$$+ ({}^mC_r + {}^mC_{r+1})\, D^{m-r}u \,.\, D^{r+1}v + \ldots + u\, D^{m+1}v.$$

But ${}^mC_r + {}^mC_{r+1} = {}^{m+1}C_{r+1}$. [from Algebra]

Hence $D^{m+1}u \,.\, v = D^{m+1}u \,.\, v + {}^{m+1}C_1\, D^m u \,.\, Dv \ldots$

$$+ {}^{m+1}C_{r+1} D^{m-r}u \,.\, D^{r+1}v + \ldots + u \,.\, D^{m+1}v. \quad \ldots(1)$$

The result (1) shows that the theorem is true for $n = m + 1$ if it is true for $n = m$. We have already proved that the theorem is true for $n = 1, 2$. Hence by mathematical induction it follows that the theorem is true for all positive integral values of $n$.

**Important Note I.** While applying Leibnitz's theorem if one of the two functions is such that its higher differential coefficients become zero then this function should be taken as $v$ and the remaining function $u$.

**II.** The formula of Leibnitz's theorem can be rewritten in the following form, [by taking successive integration of $D^n(u)$ and successive differentiation of $v$]

$$D^n(u \,.\, v) = D^n u \,.\, v + {}^nC_1\{\textstyle\int (D^n u)\, dx\}\, Dv$$
$$+ {}^nC_2 [\textstyle\int \{\int (D^n u)\, dx\}\, dx]\, D^2 v + \ldots + u \,.\, D^n v.$$

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Find the nth differential coefficients of the following :*

(i) $x^3 \cos x$, [Sagar 1993]

(ii) $x^2 \tan^{-1} x$, (iii) $e^x \log x$, (iv) $x^2 (ax + b)^m$, (v) $x^3 \log x$.

**Solution. (i)** Clearly the fourth and higher differential coefficients of $x^3$ are all zero, therefore, for the sake of convenience we shall take $x^3$ as $v$ and $\cos x$ as $u$. Applying Leibnitz's theorem, we get

$$D^n[(\cos x) \,.\, x^3] = D^n(\cos x) \,.\, x^3 + {}^nC_1 D^{n-1}(\cos x)\, D(x^3)$$
$$+ {}^nC_2 D^{n-2}(\cos x)\, D^2(x^3) + {}^nC_3 D^{n-3}(\cos x)\, D^3(x^3)$$
$$= x^3 \cos(x + \tfrac{1}{2}n\pi) + n\,[\cos\{x + \tfrac{1}{2}(n-1)\pi\}]\, 3x^2$$
$$+ \frac{n(n-1)}{1 \,.\, 2}[\cos\{x + \tfrac{1}{2}(n-2)\pi\}]\, 3 \,.\, 2 \,.\, x$$
$$+ \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3}[\cos\{x + \tfrac{1}{2}(n-3)\pi\} \,.\, 3 \,.\, 2 \,.\, 1]$$
$$= x^3 \cos(x + \tfrac{1}{2}n\pi) + 3nx^2 \sin(x + \tfrac{1}{2}n\pi) - 3n(n-1)\, x \cos(x + \tfrac{1}{2}n\pi)$$
$$- n(n-1)(n-2) \sin(x + \tfrac{1}{2}n\pi)$$

$$= x\,\{x^2 - 3n\,(n-1)\}\cos\left(x + \frac{1}{2}n\pi\right)$$
$$+ n\,\{3x^2 - (n-1)(n-2)\}\sin\left(x + \frac{1}{2}n\pi\right).$$

**(ii)** Taking $u = \tan^{-1} x,\ v = x^2$

$$D^n\,[(\tan^{-1} x)\,x^2] = D^n\,(\tan^{-1} x)\,.\,x^2 + {}^nC_1\,D^{n-1}\,(\tan^{-1} x)\,.\,D(x^2)$$
$$+ {}^nC_2 D^{n-2}\,(\tan^{-1} x)\,D^2(x^2)$$
$$[\because\ D^3\,(x^2) = 0 \text{ etc.}]$$

but $\qquad D^n\,(\tan^{-1} x) = (-1)^{n-1}\,(n-1)\,!\,\sin^n \phi\,\sin n\phi$ where $\phi = \tan^{-1}(1/x)$.

$$\therefore\quad D^n\,[(\tan^{-1} x)\,x^2] = \{(-1)^{n-1}\,(n-1)\,!\,\sin^n \phi\,\sin n\phi\}\,.\,x^2$$
$$+ \{n\,.\,(-1)^{n-2}\,(n-2)\,!$$
$$\sin^{n-1}\phi\,\sin\,(n-1)\,\phi\}\,2x + \left[\frac{n(n-1)}{1\,.\,2}\,(-1)^{n-3}\,(n-3)\,!\,\sin^{n-2}\phi\,\sin\,(n-2)\,\phi\right].\,2$$
$$= (-1)^{n-1}\,(n-3)\,!\,[(n-1)\,(n-2)\,x^2\,\sin^n \phi\,\sin n\phi$$
$$- 2nx\,(n-2)\,\sin^{n-1}\phi\,\sin\,(n-1)\,\phi + n(n-1)\,\sin^{n-2}\phi\,\sin\,(n-2)\,\phi]$$

**(iii)** $D^n\,(e^x \log x) = D^n\,(e^x)\,.\,\log x + {}^nC_1\,D^{n-1}\,(e^x)\,.\,D(\log x)$

$$+ {}^nC_2 D^{n-2}\,(e^x)\,.\,D^2\,(\log x) + \ldots + e^x\,D^n\,(\log x)$$
$$= e^x \log x + {}^nC_1\,e^x\,.\,\frac{1}{x} + {}^nC_1\,e^x\left(-\frac{1}{x^2}\right) + \ldots + \frac{e^x\,\{(-1)^{n-1}\,(n-1)\,!\}}{x^n}$$
$$= e^x\,[\log x + {}^nC_1\,x^{-1} - {}^nC_2\,x^{-2} + \ldots + (-1)^{n-1}\,(n-1)\,!\,.\,x^{-n}].$$

**(iv)** $D^n\,[(ax+b)^m\,x^2] = [D^n\,(ax+b)^m]\,.\,x^2 + {}^nC_1\,D^{n-1}\,(ax+b)^m\,D(x^2)$

$$+ {}^nC_2 D^{n-2}(ax+b)^m\,.\,D^2(x^2)$$
$$= m(m-1)\ldots(m-n+1)\,a^n\,(ax+b)^{m-n}\,.\,x^2$$
$$+ n\,.\,m(m-1)\ldots(m-n+2)\,a^{n-1}\,(ax+b)^{m-n+1}\,.\,2x$$
$$+ \frac{n\,.\,(n-1)}{1\,.\,2}\,m(m-1)\ldots(m-n+3)\,a^{n-2}\,(ax+b)^{m-n+2}\,.\,2$$
$$= m(m-1)\ldots(m-n+3)\,a^{n-2}\,(ax+b)^{m-n}$$
$$\times [(m-n+2)\,(m-n+1)\,a^2x^2$$
$$+ 2n\,(m-n+2)\,a(ax+b)\,x + n(n-1)\,(ax+b)^2].$$

**(v)** $D^n\,[\log x\,.\,x^3] = D^n\,(\log x)\,.\,x^3 + {}^nC_1 D^{n-1}\,(\log x)\,D(x^3)$

$$+ {}^nC_2 D^{n-2}\,(\log x)\,D^2(x^3) + {}^nC_3 D^{n-3}\,(\log x)\,D^3(x^3)$$
$$= \frac{(-1)^{n-1}\,(n-1)\,!}{x^n}\,x^3 + n\,\frac{(-1)^{n-2}\,(n-2)\,!}{x^{n-1}}\,.\,3x^2$$
$$+ \frac{n\,(n-1)}{1\,.\,2}\,.\,\frac{(-1)^{n-3}\,(n-3)\,!}{x^{n-2}}\,.\,6x + \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}$$
$$.\,\frac{(-1)^{n-4}\,(n-4)\,!}{x^{n-3}}\,.\,6$$
$$= \frac{(-1)^{n-1}\,(n-4)\,!}{x^{n-3}}\,[(n-1)\,(n-2)\,(n-3) - 3n(n-2)\,(n-3)$$
$$+ 3n(n-1)\,(n-3) - n(n-1)\,(n-2)]$$

$$= (-1)^{n-1}(n-4)!\,x^{3-n}\,[(n-1)(n-2)(n-3-n)$$
$$+ 3n(n-3)(n-1-n+2)]$$
$$= (-1)^{n-1}(n-4)!\,x^{3-n}\,[3(n^2-3n+2)+3(n^2-3n)]$$
$$= 6(-1)^n (n-4)!\,x^{3-n}.$$

**Example 2.** *If $y = x^2 e^x$ then prove that*

$$y_n = \frac{1}{2}n(n-1)\frac{d^2y}{dx^2} - n(n-2)\frac{dy}{dx} + \frac{1}{2}(n-1)(n-2)\,y. \quad \textbf{[Bhopal 1994, 97]}$$

**Solution.** $y = x^2 e^x$ ...(1)

$$\therefore \quad \frac{dy}{dx} = x^2e^x + 2xe^x = y + 2xe^x \quad ...(2)$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + 2\,[xe^x + e^x] = \frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) + 2e^x \quad \text{[from (1)]}$$

$$= \frac{2dy}{dx} - y + 2e^x. \quad ...(3)$$

Now $D^n(x^2e^x) = y_n = D^n(e^x)\,.\,x^2 + {}^nC_1\,D^{n-1}(e^x)\,.\,2x + {}^nC_2 D^{n-2}(e^x)\,.\,2$

$$= e^x\,.\,x^2 + 2nx\,e^x + n(n-1)\,e^x$$

$$= y + n\left(\frac{dy}{dx} - y\right) + n(n-1) \times \frac{1}{2}\left(\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y\right)$$

[using (1), (2) and (3)]

$$= \frac{1}{2}n(n-1)\frac{d^2y}{dx^2} - n(n-2)\frac{dy}{dx} + \frac{1}{2}(n-1)(n-2)\,y.$$

**Example 3.** *Prove that*

$$\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right) = \left[P\sin\left(x + \frac{n\pi}{2}\right) + Q\cos\left(x + \frac{n\pi}{2}\right)\right] \Big/ x^{n+1}$$

*where* $P = x^n - n(n-1)\,x^{n-2} + n(n-1)(n-2)(n-3)\,x^{n-4} - \ldots$

*and* $Q = nx^{n-1} - n(n-1)(n-2)\,x^{n-3} + \ldots.$

**Solution.** Let $u = \sin x$.

$$\therefore \quad u_n = \sin\left(x + \frac{n\pi}{2}\right) = \sin\alpha \text{ (say).}$$

$$\therefore \quad u_{n-1} = \sin\{x + (n-1)\tfrac{1}{2}\pi\} = \sin[\alpha - (\pi/2)] = -\cos\alpha,$$

$$u_{n-2} = \sin\{x + (n-2)\tfrac{1}{2}\pi\} = \sin(\alpha - \pi) = -\sin\alpha,$$

$$u_{n-3} = \sin\{x + (n-3)\tfrac{1}{2}\pi\} = \sin(\alpha - 3\pi/2) = \cos\alpha, \text{ etc.}$$

Again let $v = \frac{1}{x}$, $\quad \therefore \; v_n = \frac{(-1)^n\,n!}{x^{n+1}}.$

$$\therefore \quad v_1 = -\frac{1}{x^2}, \quad v_2 = \frac{2!}{x^3}, \quad v_3 = -\frac{3!}{x^4} \text{ etc.}$$

$$\therefore \quad \left(\frac{d^n}{dx^n}\right)\left(\sin\frac{x}{x}\right) = u_n\,.\,v + {}^nC_1 u_{n-1}\,.\,v_1$$

$$+ {}^nC_2 u_{n-2}\,.\,v_2 + {}^nC_3 u_{n-3}\,.\,v_3 + {}^nC_4 u_{n-4}\,.\,v_4 + \ldots$$

$$= \frac{1}{x}\sin\alpha + n(-\cos\alpha)\left(-\frac{1}{x^2}\right) + \frac{n(n-1)}{2!}(-\sin\alpha)\left(\frac{2!}{x^2}\right)$$

$$+ \frac{n(n-1)(n-2)}{3!}(\cos\alpha)\left(-\frac{3!}{x^4}\right)$$

$$+ \frac{n(n-1)(n-2)(n-3)}{4!}(\sin\alpha)\left(\frac{4!}{x^5}\right) + \ldots$$

$$= (1/x^{n+1})\,[\{x^n - n(n-1)x^{n-2}$$

$$+ n(n-1)(n-2)\,.\,(n-3)x^{n-4}\ldots\}\sin\alpha$$

$$+ \{nx^{n-1} - n(n-1)(n-2)x^{n-2} + \ldots\}\cos\alpha$$

$$= \left[P\sin\left(x + \frac{n\pi}{2}\right) + Q\cos\left(x + \frac{n\pi}{2}\right)\right] \Big/ x^{n+1}.$$

**Second Method.** See Note II of § 3.6,

$$D^n(\sin x\,.\,x^{-1}) = \sin(x + \tfrac{1}{2}n\pi)\,.\,x^{-1} + {}^nC_1\{-\cos(x + \tfrac{1}{2}n\pi)\}(-1\,.\,x^2)$$

$$+ {}^nC_2\{-\sin(x + \tfrac{1}{2}n\pi)\}\{(-1)(-2)x^{-3}\}$$

$$+ {}^nC_3\{\cos(x + \tfrac{1}{2}n\pi)\}\{(-1)(-2)(-3)x^{-4}\} + \ldots$$

Now proceed as above.

**Example 4.** *If* $y = x^{n-1}\log x$, *then prove that* $y_n = \dfrac{(n-1)!}{x}$.

**[Sagar 1998; Ravishankar 2002, 2006; Bilaspur 2008]**

**Solution.** Let $y = x^{n-1}\log x$

$$y_n = D^n(x^{n-1}\log x) = D\{D^{n-1}(x^{n-1}\log x)\}$$

$$= D\Big\{(n-1)!\,.\log x + {}^nC_1(n-1)!\,.\,\frac{x}{1}\,.\,\frac{1}{x}$$

$$+ {}^nC_2(n-2)!\left(\frac{x^2}{1\,.\,2}\right).\left(-\frac{1}{x^2}\right) + \ldots$$

$$= D\{(n-1)!\log x + \text{terms not containing } x\}$$

$$= (n-1)!\,.\,\frac{1}{x} + 0 = \frac{(n-1)!}{x}.$$

**Example 5.** *Prove that*

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{(-1)^n n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots \frac{1}{n}\right].$$

**Solution.** $y = \log x\,.\,x^{-1}$.

Differentiating $n$ times by Leibnitz's theorem taking $\log x$ at first function, have

$$y_n = \frac{(-1)^{n-1}(n-1)!}{x^n}\,.\,\frac{1}{x} + n\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}\left(-\frac{1}{x^2}\right)$$

$$+ \frac{n(n-1)(-1)^{n-3}(n-3)!}{1\,.\,2x^{n-2}}\,.\,\frac{(-1)(-2)}{x^3} + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}$$

$$.\,\frac{(-1)^{n-4}(n-4)}{x^{n-3}}\,.\,\frac{(-1)(-2)(-3)}{x^4} + \ldots + (\log x)\,.\,\frac{(-1)^n n!}{x^{n+1}}$$

$$= \frac{(-1)^n n!}{x^{n+1}}\left[-\frac{1}{n}-\frac{1}{n-1}-\frac{1}{n-2}-\frac{1}{n-3}-\ldots+\log x\right]$$

$$= \frac{(-1)^n n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{n}\right].$$

**Example 6.** *If* $I_n = \frac{d^n}{dx^n}(x^n \log x)$, *prove that* $I_n = nI_{n-1} + (n-1)$.

**[Jiwaji 1980; Ravishankar 918; Sagar 96, 2005]**

*Hence show that* $I_n = n!\,(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + 1/n)$.

**Solution.** $I_n = D^n (x^n \log x) = D^{n-1} [D (x^n \log x)]$

$= D^{n-1} [x^n (1/x) + nx^{n-1} \log x]$

$= n \,.\, D^{n-1} (x^{n-1} \log x) + D^{n-1} (x^{n-1})$

$= n\, I_{n-1} + (n-1)\,!.$ ...(1)

Now replacing $n$ by $n-1$ in the above relation (1), we have

$$I_{n-1} = (n-1)\, I_{n-2} + (n-2)\,!$$

$\therefore$ $I_n = n\,[(n-1)\, I_{n-2} + (n-2)\,!] + (n-1)\,!$

$= n(n-1)\, I_{n-2} + n(n-2)\,! + (n-1)\,!$

$= \frac{n!}{(n-2)!} I_{n-2} + \frac{n!}{(n-1)} + \frac{n!}{n}.$

Thus applying the reduction formula (1) $(n-2)$ times again, we have

$$I_n = \frac{n!}{0!} I_0 + \frac{n!}{1!} + \frac{n!}{2!} + \frac{n!}{3} + \ldots + \frac{n!}{n-1} + \frac{n!}{n}$$

$$= n!\,[\log x + 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \{1/(n-1)\} + (1/n)]$$

$[\because\ I_0 = D^0 (x^0 \log x) = \log x]$

**Example 7.** *By forming in two different ways the nth derivative of* $x^{2n}$, *prove that*

$$1 + \frac{n^2}{1^2} + \frac{n^2 (n-1)^2}{1^2 \,.\, 2^2} + \frac{n^2 (n-1)^2 (n-2)^2}{1^2 \,.\, 2^2 \,.\, 3^2} + \ldots = \frac{(2n)\,!}{(n\,!)^2}.$$

**Solution.** Let $y = x^{2n}$

$\therefore$ $y_n = \frac{(2n)!}{n!} x^{2n-n} = \frac{(2n)!}{n!} x^n$ ...(1)

Again $y = 2^{2n} = x^n \,.\, x^n.$

Differentiating $n$ times by Leibnitz's theorem, we have

$$y_n = n\,!\,x^n + n \,.\, \frac{n!}{1!} xnx^{n-1} + \frac{n(n-1)}{2!} \cdot \frac{n!}{2!} x^2$$

$$.\, n(n-1)\, x^{n-2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{n!}{3!} x^3$$

$$.\, n(n-1)(n-2)\, x^{n-3} + \ldots + x^n \,.\, n\,!$$

$$= n\,!\,x^n \left[1 + \frac{n^2}{1^2} + \frac{n^2 (n-1)^2}{1^2 \,.\, 2^2} + \frac{n^2 (n-1)^2 (n-2)^2}{1^2 \,.\, 2^2 \,.\, 3^2} + \ldots\right]. \quad \text{...(2)}$$

Equating the two values of $y_n$ obtained in (1) and (2), we get the required result.

**Example 8.** *If* $y^{1/m} + y^{-1/m} = 2x$, *then prove that*

$$(x^2 - 1)\, y_{n+2} + (2n+1)\, x \,.\, y_{n+1} + (n^2 - m^2)\, y_n = 0.$$

**[Ravishankar 1990, 96; Bhopal 2000, 2001; Rewa 98; Vikram 94, 99, 2002, 2007; Indore 98, 2004, 2007; Sagar 2004]**

**Solution.** $y^{1/m} + y^{-1/m} = 2x,$

$$y^{1/m} + 1/(y^{1/m}) = 2x \quad \text{or} \quad (y^{1/m})^2 - 2xy^{1/m} + 1 = 0.$$

$\therefore$ This is a quadratic equation in $y^{1/m}$

$\therefore$ $$y^{1/m} = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

or $$y = [x \pm \sqrt{x^2 - 1}]^m \qquad \ldots(1)$$

$\therefore$ $$y_1 = m\,[x \pm \sqrt{x^2 - 1}]^{m-1} \left\{1 \pm \frac{x}{\sqrt{x^2 - 1}}\right\}$$

$$\sqrt{x^2 - 1}\; y_1 = m\,[x \pm \sqrt{x^2 - 1}]^m = \pm\, my. \qquad \text{[from (1)]}$$

Squaring both sides, we have

$$y_1^2\,(x^2 - 1) = m^2 y^2.$$

Differentiating again we get,

$$2y_1 y_2\,(x^2 - 1) + y_1^2 \,.\, 2x = 2m^2\, yy_1.$$

Dividing by $2y_1$, we get $\quad y_2\,(x^2 - 1) + xy_1 - m^2 y = 0.$

Differentiating each term $n$ times by Leibnitz's theorem, we have

$$[(x^2 - 1)\, y_{n+2} + {}^nC_1\, y_{n+1} \,.\, (2x) + {}^nC_2\, y_n \,.\, 2] + [xy_{n+1} + {}^nC_1\, y_n \,.\, 1] - m^2 y_n = 0$$

or $$(x^2 - 1)\, y_{n+2} + (2n+1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0.$$

**Example 9.** *If* $y = a \cos(\log x) + b \sin(\log x)$, *then prove that*

$$x^2 y_2 + xy_1 + x = 0$$

*and* $$x^2 y_{n+2} + (2n+1)\, xy_{n+1} + (n^2 + 1)\, y_n = 0.$$

**[Vikram 1990; Jiwaji 93, 95; Ravishankar 94, 97, 97S, 99, 2008; Bhopal 98; Jabalpur 98; Indore 90, 91, 96]**

**Solution. First Part.** $y = a \cos(\log x) + b \sin(\log x)$

$\therefore$ $$y_1 = -\frac{a \sin(\log x)}{x} + \frac{b \cos(\log x)}{x}$$

or $$xy_1 = -a \sin(\log x) + b \cos(\log x).$$

Differentiating again w.r.t. $x$,

$$xy_2 + y_1 = -\frac{a \cos(\log x)}{x} - \frac{b \sin(\log x)}{x}$$

or $$x^2 y_2 + xy_1 = -[a \cos(\log x) + b \sin(\log x)] = -y \qquad \text{[use (1)]}$$

or $$x^2 y_2 + xy_1 + y = 0.$$

**Second Part.** Now differentiating each term $n$ times by Leibnitz's theorem, we get

$$[y_{n+2} \,.\, x^2 + {}^nC_1 \,.\, y_{n+1} \,.\, 2x + {}^nC_2 \,.\, y_n \,.\, 2] + [y_{n+1} \,.\, x + {}^nC_1 \,.\, y_n \,.\, 1] + y_n = 0$$

or $$x^2 y_{n+2} + (2n+1)\, xy_{n+1} + (n^2 + 1)\, y_n = 0.$$

**Example 10.** *If* $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, *then prove that*

$$x^2y_2 + xy_1 + n^2y = 0$$

*and* $x^2y_{n+2} + (2n+1)\,xy_{n+1} + 2n^2y_n = 0.$ **[Ravishankar 1990, 93, 94, 95, 99S, 2003, 2010; Indore 93, 94, 2006; Bilaspur 95; Rewa 95; Jabalpur 2000; Jiwaji 2006]**

**Solution.** $y = b\cos\log(x/n)^n$ ...(1)

$$y_1 = -b\sin\log(x/n)^n \,.\, n\left(\frac{1}{x/n}\right).\,1/n$$

or $$xy_1 = -bn\sin\log(x/n)^n.$$

Differentiating again, we get

$$xy_1 + 1\,.\,y_1 = -bn\cos\log(x/n)^n\,.\,n\left(\frac{1}{x/n}\right)n$$

or $$x^2y_2 + xy_1 = -n^2b\cos\log(x/n)^n = -n^2y$$

or $$x^2y_2 + xy_1 + n^2y = 0.$$

Differentiating each term $n$ times, we get

$$y_{n+2}x^2 + {}^nC_1\,.\,y_{n+1}\,.\,2x + {}^nC_2\,y_n\,.\,2 + y_{n+1}\,.\,x + {}^nC_1\,y_n + n^2y_n = 0$$

or $$x^2y_{n+2}\,(2n+1)\,xy_{n+1} + 2n^2y_n = 0.$$

**Example 11.** *If* $y = e^{\tan^{-1}x}$, *then prove that*

$$(1+x^2)\,y_2 + (2x-1)\,y_1 = 0$$

*and* $$(1+x^2)\,y_{n+2} + [2(n+1)\,x - 1]\,y_{n+1} + n(n+1)\,y_n = 0.$$

**[Bilaspur 1997; Ravishankar 2000; Rewa 2006]**

**Solution.** $y = e^{\tan^{-1}x}$.

$\therefore$ $$y_1 = e^{\tan^{-1}x}\frac{1}{1+x^2} \quad\text{or}\quad y_1 = \frac{y}{1+x^2} \quad\text{or}\quad (1+x^2)\,y_1 = y.$$

Differentiating again each term w.r.t. $x$, we get

$$(1+x^2)\,y_2 + 2xy_1 = y_1 \quad\text{or}\quad (1+x^2)\,y_2 + (2x-1)\,y_1 = 0.$$

Now differentiating each term $n$ times by Leibnitz's theorem, we have

$$y_{n+2}\,(1+x^2) + {}^nC_1\,y_{n+1}\,.\,2x + {}^nC_2\,y_n\,.\,2 + y_{n+1}\,(2x-1) + {}^nC_1\,y_n\,.\,2 = 0$$

or $$(1+x^2)\,y_{n+2} + [2\,(n+1)\,x - 1]\,y_{n+1} = n\,(n+1)\,y_n = 0.$$

**Example 12.** *If* $y = \sin(m\sin^{-1}x)$, *then show that*

$$(1-x^2)\,y_2 - xy_1 + m^2y = 0$$

**[Jabalpur 1995, 96; Indore 99, 2005; Ravishankar 94S, 99, 2007; Sagar 91; Bilaspur 95, 96, 99; rewa 2001; Vikram 2004]**

*and deduce from it that*

$$(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - (n^2-m^2)\,y_n = 0.$$

**[Jabalpur 1996, 2002, 2004; Ravishnkar 96; Bilaspur 99; Rewa 2000; Indore 2005]**

**Solution.** Here $y = \sin(m\sin^{-1}x)$. ...(1)

Differentiating (1) w.r. to $x$, we get

$$y_1 = \cos(m\sin^{-1}x)\,.\,\frac{m}{\sqrt{1-x^2}}$$

or $$y_1^2(1-x^2) = m^2\{1-\sin^2(m\sin^{-1}x)\} \qquad ...(2)$$

or $$(1-x^2)y_1^2 = m^2(1-y^2).$$

Again, differentiating w.r. to $x$, we get

$$(1-x^2)\,2y_1y_2 - 2xy_1^2 = -2m^2yy_1.$$

Dividing by $2y_1$, we get ...(3)

$$(1-x^2)y_2 - xy_1 + m^2y = 0.$$

Differentiating (3) $n$ times, we get

$$D^n\{(1-x^2)y_2\} + D^4\{-xy_1\} + D^n\{m^2y\} = D^n(0) = 0.$$

Now by Leibnitz's theorem, we get

$$D^n\{(1-x^2)y_n\} = y_{n+1}\cdot(1-x^2) + ny_{n+1}\cdot(-2x) + \frac{n(n-1)}{2!}y_n\cdot(-2)$$

$$D^n\{-xy_1\} = \qquad + y_{n+1}\cdot(-x) + ny_n\cdot(-1)$$

$$D^n\{m^2y\} = \qquad m^2y_n$$

Adding, $$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0.$$

**Example 13.** *If $x+y=1$, then prove that*

$$\frac{d^n}{dx^n}(x^ny^n) = n!\,\{y^n - ({}^nC_1)^2y^{n-1}x + ({}^nC_2)^2y^{n-2}x^2 + \ldots + (-1)^nx^n\}.$$

**[Jabalpur 1995; Jiwaji 98, 2004]**

**Solution.** Given $x+y=1 \Rightarrow y = 1-x.$

$$\therefore \frac{d^n}{dx^n}(x^ny^n) = D^n\{x^n(1-x)^n\}$$

$$= (D^nx^n)(1-x)^n + {}^nC_1(D^{n-1}x^n)\,.\,D(1-x)^n$$

$$+ {}^nC_2(D^{n-2}x^n)\,.\,D^2(1-x)^n + \ldots + x^nD^n(1-x)^n$$

$$= n!\,(1-x) + {}^nC_1\frac{n!}{2!}x\,.\,(-1)\,n(1-x)^{n-1}$$

$$+ {}^nC_1\frac{n!}{2!}x^2(-1)^2\,n(n-1)(1-x)^{n-2} + \ldots + x^n(-1)^nn!$$

$$= n!\left[(1-x)^n - n\,.\,{}^nC_1(1-x)^{n-1}x + \frac{n(n-1)}{2!}{}^nC_2(1-x)^{n-2}\,.\,x^2 + \ldots + (-1)^nx^n\right]$$

$$= n!\,[y^n - ({}^nC_1)^2y^{n-1}x + ({}^nC_2)^2y^{n-2}x^2 + \ldots + (-1)^nx^n].$$

Since $n = {}^nC_1,\ \dfrac{n(n-1)}{2!} = {}^nC_2$ etc.

## EXERCISE 3 (C)

**1.** Find the fourth differential-coefficients of $x^4e^{ax}$, $x^3\log x$, $x^2\sin 3x$, $xe^{ax}\sin bx$. Find $n$th differential-coefficient of the following.

**2.** $x^2e^{ax}$.

**3.** $x^2(ax+b)^m$.

**4.** $x^2\sin x$. **[Sagar 2001]**

**5.** $x^2\log_e x$.

**6.** $x^3\cos x$. **[Sagar 1993]**

**7.** $\sin x\log_e(ax+b)$.

8. $x^2 \tan^{-1} x$.

9. $x^2 . e^{3x} \cos 4x$.

10. $\frac{x^4}{1-x}$.

11. Differentiate $n$ times the equation $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + a^2y = 0$.

12. Differentiate $n$ times the equation $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + m^2y = 0$.

13. If $y = x^2 e^x$, then prove that

$$\frac{d^ny}{dx^n} = \frac{1}{2} n(n-1)\frac{d^2y}{dx^2} - n(n-2)\frac{dy}{dx} + \frac{1}{2}(n-1)(n-2)\,y.$$ **[Bhopal 1994]**

Hence prove that $\frac{d^8y}{dx^8} = 28\frac{d^2y}{dx^2} - 48\frac{dy}{dx} + 21y$.

14. If $I_n = \frac{d^n}{dx^n}\{x^n \log_e x\}$, then prove that $\left[\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \ldots + \log x\right].(n\,!)$.

15. If $y = \sin^{-1} x$, then prove that $(1-x^2)\,y_2 - xy_1 = 0$.

Hence prove that $(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - n^2y_n = 0$.

**[Sagar 2001; Rewa 2003; Bhopal 2004]**

16. If $y = \frac{\cos^{-1} x}{\sqrt{1-x^2}}$, then prove that $(1-x^2)\,y_{n+1} - (2n+1)\,xy_n - n^2y_{n-1} = 0$.

17. If $y = (\sin^{-1} bx)^2$, then prove that $(1+x^2)\,y_{n+2} + (2n+1)\,xy_{n+1} + n^2y_n = 0$.

18. If $y = [\log\{x + \sqrt{1+x^2}\}]^2$, then prove that $(1+x^2)\,y_{n+2} + (2n+1)\,xy_{n+1} + n^2y_n = 0$.

**[Ravishankar 1991S; Indore 2000]**

19. If $\cos^{-1}(y/b) = \log(x/m)^m$, then prove that $x^2y_{n+2} + (2n+1)\,xy_{n+1} + (n^2+m^2)\,y_n = 0$.

20. If $y = (1-x)^{-\alpha}e^{-\alpha x}$, then prove that $(1-x)\,y_{n+1} - (n+\alpha x)\,y_n - n\alpha y_{n-1} = 0$.

21. If $y = x(a^2+x^2)^{-1}$, then prove that $y_n = (-1)^n\, n\,!\, a^{-n-1} \sin^{n+1}\phi \cos(n+1)\phi$,

where $\phi = \tan^{-1}(a/x)$.

22. If $x = \cosh\left(\frac{1}{m}\log_e y\right)$, then prove that

$$(x^2-1)\,y_2 + xy_1 - m^2y = 0 \quad \text{and} \quad (x^2-1)\,y_{n+2} + (2n+1)\,xy_{n+1} + (n^2-m^2)\,y_n = 0.$$

23. Prove that $\frac{d^n}{dx^n}\left(\frac{\cos x}{x}\right) = \frac{P\cos\left(x+\frac{n\pi}{2}\right) - Q\sin\left(x+\frac{n\pi}{2}\right)}{x^{n-1}}$,

where $P = x^n - n(n-1)\,x^{n-2} + n(n-1)(n-2)(n-3)\,x^{n-4} + \ldots$

$Q = nx^{n-1} - n(n-1)(n-2)\,x^{n-3} + \ldots$.

24. Prove that $n$th differential-coefficient of $x^n(1-x)^n$ is equal to :

$$n\,!\,(1-x)^n\left(1 - \frac{n^2}{1^2}\cdot\frac{x}{1-x} + \frac{n^2(n-1)^2}{1^2\,.\,2^2}\cdot\frac{x^2}{(1-x)^2} - \ldots + (-1)^n\,.\,\frac{x^n}{(1-x)^n}\right).$$

25. Find $n$th differential coefficient of $x^{2n}$ by two distinct method, prove that

$$1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2\,.\,2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2\,.\,2^2\,.\,3^2} + \ldots = \frac{2n\,!}{(n\,!)^2}.$$

26. (a) State and prove that Leibnitz's theorem.

(b) Give statement of Leibnitz's theorem. **[Bhopal 2007]**

## ANSWERS

1. $e^{ax}(a^4x^4 + 12a^3x^3 + 54a^2x^2 + 72ax + 18)$; $6/x$; $3^3(3^2 - 4)\sin 3x - 6^3 x\cos 3x$;
$(a^2+b^2)^2 xe^{ax}\sin\{bx + 4\tan^{-1}(b/a)\} + 4(a^2+b^2)^{3/2} a^{ax}\sin\{bx + 3\tan^{-1}(b/a)\}$

2. $a^{n-2}e^{ax}\{a^2x^2 + 2nax + n(n-1)\}$

3. $m(m-1)(m-2)\ldots(m-n+2)a^{n-2}(ax+b)^{m-n}\{(m-n+2)$
$.(m-n+1)a^2x^2 + 2n(m-n+2)a(ax+b)x + n(n-1)(ax+b)^2\}$

4. $x^2\sin[x + n\pi/2] + 2nx\sin[x + (n-1)\pi/2] + n(n+1)\sin[x + (n-2)\pi/2]$

5. $2(-1)^{n-1}(n-3)!/x^{n-2}$

6. $x\{x^2 - 3n(n-1)\}\cos(x + n\pi/2) + n\{3x^2 - (n-1)(n-2)\}\sin(x + n\pi/2)$

7. $(-1)^{n-1}\{(n-1)!\,a^n(ax+b)^{-n}\sin x - {}^nC_1(n-2)!\,a^{n-1}$
$(ax+b)^{-n-1}\cos x - {}^nC_2(n-3)!\,a^{n-2}(ax+b)^{-n+2}$
$\sin x + \ldots + {}^nC_r(-1)^r(n-r-1)!\,a^{n-r}(ax+b)^{-n+r}$
$\times\sin(x + r\pi/2) + \ldots\log(ax+b)\sin(x + n\pi/2)\}$

8. $(-1)^{n-1}(n-3)!\,[(n-1)(n-2)x^2\sin^n\phi\sin n\phi - {}^nC_1\,2x(n-2)\sin^{n-1}\phi\sin(n-1)\phi$
$+ 2\,.\,{}^nC_2\sin^{n-2}\phi\sin(n-2)\phi\}$, where $x = \cot\phi$

9. $e^x x\{x^2\,.\,5^n\cos(4x + n\,.\,\alpha)\} + 2nx\,.\,5^{n-1}\cos\{4x + (n-1)\alpha\}$
$+ n(n-1)5^{n-2}\cos\{4x + (n-2)\alpha\}$, where $\alpha = \tan^{-1}(4/3)$

10. $n!/(x-1)^{n+1}$

11. $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} - (n^2 - a^2)y_n = 0$

12. $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0.$

## § 3.7. $n$th DIFFERENTIAL COEFFICIENT FOR SPECIAL VALUE OF $x$

Some times it is difficult to find general value of $y_n$, but its value can be found for special value of $x$. The value of $y_n$ at $x = 0$ is usually denoted by $(y_n)_0$.

## ILLUSTRATIVE EXAMPLES

**Example 1.** *If $y = \tan^{-1} x$, then prove that*

$$(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$$

**[Indore 198**

*hence find $(y_n)_0$.*

**Solution.**

$$y = \tan^{-1} x \qquad \ldots(1)$$

$$\therefore \quad y_1 = \frac{1}{1+x^2} \qquad \ldots(2)$$

or $$(1+x^2)y_1 = 1.$$

Differentiating again

$$(1+x^2)y_2 + y_1\,2x = 0 \quad \text{or} \quad y_2(1+x^2) + 2xy_1 = 0. \qquad \ldots(3)$$

Differentiating each term $n$ times by Leibnitz's theorem, we get

$$[y_{n+2}(1+x^2) + {}^nC_1\,.\,y_{n+1}\,.\,2x + {}^nC_2\,.\,y_n\,.\,2] + 2[y_{n+1}\,.\,x + {}^nC_1 y_n\,.\,1] = 0$$

or $$(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0.$$

Putting $x = 0$, we have

$$(y_{n+2})_0 = -n(n+1)(y_n)_0. \quad \text{...(4)}$$

But from (1), (2) and (3), we get

$$(y)_0 = \tan^{-1} 0 = 0, \ (y_1)_0 = \frac{1}{1+0} = 1, \ (y_2)_0 = 0.$$

Putting $n = 2, 4, 6, \ldots$ in equation (4), we get

$$(y_4)_0 = (y_6)_0 = (y_8)_0 = \ldots = 0.$$

$\therefore$ When $n$ is even $(y_n)_0 = 0$.

Again, $(y_1)_0 = 0$.

$\therefore$ From (4), $(y_3)_0 = -1.2(y_1)_0 = -1.2.1 = (-1).2\,!$

$$(y_5)_0 = -3.4(y_3)_0 = -3.4(-1)\,2\,! = (-1)^2\,4\,!$$

$\therefore$ When $n$ is odd, then we have

$$(y_n)_0 = (-1)^{(n-1)/2}\,(n-1)\,!.$$

**Example 2.** *If* $y = [\log\{x + \sqrt{(1+x^2)}\}]^2$, *then prove that* $(y_{n+2})_0 = -n^2\,(y_n)_0$ *hence find* $(y_n)_0$. **[Bilaspur 1991; Vikram 96, 2005]**

**Solution.** $y = [\log\{x + \sqrt{(1+x^2)}\}]^2$ ...(1)

$$\therefore \quad y_1 = 2\log[x + \sqrt{(1+x^2)}] \times \frac{1}{[x + \sqrt{(1+x^2)}]}\left[1 + \frac{1}{2\sqrt{(1-x^2)}}\,.\,2x\right]$$

or
$$y_1 = \frac{2\log[x+\sqrt{(1+x^2)}]}{\sqrt{[(1+x^2)]}}. \quad \text{...(2)}$$

Squaring both sides, we get

or $(1+x^2)\,y_1^{\,2} = 4\,[\log x + (1+x^2)]^2 = 4y$ or $(1+x^2)\,y_1^{\,2} = 4y$.

Differentiating again, we get

$$(1+x^2)\,2y_1 y_2 + y_1^{\,2}\,.\,2x = 4y_1$$

Dividing by $2y_1$, we get

$$y_2\,(1+x^2) + xy_1 = 2. \quad \text{...(3)}$$

Differentiating each term $n$ times, we get

$$[y_{n+2}\,(1+x^2) + {}^nC_1\,.\,y_{n+1}\,.\,2x + {}^nC_2\,.\,y_n\,.\,2] + [y_{n+1}\,.\,x + {}^nC_1\,.\,y_n\,.\,1] = 0$$

or $(1+x^2)\,y_{n+2} + (2n+1)\,xy_{n+1} + (n^2 - n + n)\,y_n = 0$

or $(1+x^2)\,y_{n+2} + (2n+1)\,xy_{n+1} + n^2 y_n = 0$. **[Indore 2000; Rewa 2003]**

Putting $x = 0$, we get

$$(y_{n+2})_0 = -n^2\,(y_n)_0. \quad \text{...(4)}$$

Again when $x = 0$, then from (1), (2) and (3), we have

$$y_1 = 0$$

$$y_2 = (1+0) + 0\,.\,0 = 2 \quad \text{or} \quad y_2 = 2.$$

From (4), $y_3 = -1^2\,.\,y_1 = 0$ (Taking $n = 1$)

$$y_5 = -3^2\,.\,y_3 = 0, \text{ etc.}$$

$\therefore$ **When n is odd integer** $(y_n)_0 = 0$.

Again $y_2 = 2,$

$$y_4 = -2^2\,.\,y_2 = -2^2\,.\,2$$

$$y_6 = -4^2 y_4 = -4^2(-2^2) \cdot 2 = (-1)^2 4^2 \cdot 2^2 \cdot 2$$

$$y_8 = -6^2 y_6 = -6^2 \cdot (-4^2)(-2^2) \cdot 2 = (-1)^3 6^2 \cdot 4^2 \cdot 1^2 \cdot 2.$$

$\therefore$ **When n is even integer, then**

$$(y_n)_0 = (-1)^{n/2-1}(n-2)^2(n-4)^2 \ldots 6^2 \cdot 4^2 \cdot 2^2 \cdot 2.$$

**Example 3.** *If* $y = (\sinh^{-1} x)^n$, *then prove that*

$$(1+x^2)\, y_{n+2} + (2n+1)\, x y_{n+1} + n^2 y_n = 0$$

*hence find* $(y_n)_0$.

**Solution.** $y = (\sinh^{-1} x)^2 = [\log \{x + \sqrt{(1+x^2)}\}]^2$.

Now see Ex. 2 above.

**Example 4.** *If* $y = \sin(a \sin^{-1} x)$, *then find* $(y_n)_0$.

**[Bhopal 1996; Indore 90, 94, 96, 99; Jiwaji 91; Vikram 82; Jabalpur 95, 96; Ravishankar 90S, 94S, 96, 99; Bilaspur 95, 96, 99, 2006; Rewa 2007]**

**Solution.**

$$y = \sin(a \sin^{-1} x). \quad \ldots(1)$$

$\therefore$

$$y_1 = \cos(a \sin^{-1} x)\frac{a}{\sqrt{(1-x^2)}}$$

or

$$\frac{y_1\sqrt{(1-x^2)}}{a} = \cos(a \sin^{-1} x). \quad \ldots(2)$$

Squaring (1) and (2) and adding, we get

$$y^2 + \frac{(1-x^2)\, y_1^2}{a^2} = 1 \quad \text{or} \quad a^2 y^2 + (1-x^2)\, y_1^2 = a^2.$$

Differentiating again

$$a^2 2yy_1 + (1-x^2)\, 2y_1y_2 + y_1^2(-2x) = 0$$

or

$$y_2(1-x^2) - xy_1 + a^2 x = 0. \quad \ldots(3)$$

Now differentiating each term $n$ times, we have

$$y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2) - [y_{n+1}x + {}^nC_1\, y_n \cdot 1] + a^2 y_n = 0$$

or

$$(1-x^2)\, y_{n+2} + (2n+1)\, x \cdot y_{n+1} - (n^2 - a^2)\, y_n = 0.$$

Putting $x = 0$, we get

$$(y_{n+2})_0 = (n^2 - a^2)(y_n)_0. \quad \ldots(4)$$

But from (1) and (3), we have

$$(y_1)_0 = a \quad \text{and} \quad (y_2)_0 = 0.$$

$\therefore$ From (4), $\quad (y_2)_0 = 0 = (y_4)_0 = (y_6)_0 = \ldots.$

$\therefore$ **When n is even integer** $(y_n)_0 = 0$.

Again

$$(y_1)_0 = a$$

$$(y_3)_0 = (1^2 - a^2)(y_1)_0 = (1^2 - a^2)\, a.$$

Similarly

$$(y_5)_0 = (3^2 - a^2)(1^2 - a^2)\, a.$$

$\therefore$ **When n is odd integer**

$$(y_n)_0 = [(n-2)^2 - a^2]\,[(n-4)^2 - a^2] \ldots (5^2 - a^2)(3^2 - a^2)(1^2 - a^2)\, a.$$

**Example 5.** *If* $y = \sin^{-1} x$, *then find* $(y_n)_0$. [Bilaspur 2005]

**Solution.** $y = \sin^{-1} x$ ...(1)

$$\therefore \quad y_1 = \frac{1}{\sqrt{(1-x^2)}} \qquad ...(2)$$

$$\Rightarrow \quad (1-x^2)\, y_1^2 = 1. \qquad \text{[squaring]}$$

Differentiating again, we have

$$(1-x^2)\, 2y_1 y_2 + y_1^2 (-2x) = 0 \quad \text{or} \quad y_2 (1-x^2) - xy_1 = 0. \qquad ...(3)$$

$\therefore$ Differentiating $n$ times, we get

$$y_{n+2} (1-x^2) + {}^nC_1\, y_{n+1} (-2x) + {}^nC_2\, y_n (-2) - [y_{n+1} \cdot x + {}^nC_1 \cdot y_n \cdot 1] = 0.$$

Putting $x = 0$, we have

$$(y_{n+2})_0 - \left[\frac{n(n-1)}{2} \cdot 2 + n\right] (y_n)_0 = 0$$

$$(y_{n+2})_0 = n^2\, (y_n)_0. \qquad ...(4)$$

But from (1), (2) and (3), we have

$$(y)_0 = 0,\ (y_1)_0 = 1,\ (y_2)_0 = 0.$$

$$\therefore \quad (y_4)_0 = 0 = (y_6)_0 = \ldots \qquad \text{[from (4)]}$$

$\therefore$ **When n is even, then** $(y_n)_0 = 0$.

Again $(y_1)_0 = 1$.

Again $(y_3)_0 = 1^2\, (y_1)_0 = 1^2 \cdot 1$ [from (4)]

Similarly, $(y_5)_0 = 3^2 \cdot 1^2 \cdot 1$.

$\therefore$ **When n is odd, then**

$$(y_n)_0 = (n-2)^2\, (n-4)^2 \ldots 3^2 \cdot 1.$$

**Example 6.** *If* $y = [x + \sqrt{1+x^2}]^m$, *then find* $(y_n)_0$.

[Vikram 1993, 2000; Indore 2001; Ravishankar 92S]

**Solution.** $y = [x + \sqrt{1+x^2}]^m$. ...(1)

$$\therefore \quad y_1 = m\, [x + \sqrt{1+x^2}]^{m-1} \left(1 + \frac{1}{2} \cdot \frac{2x}{\sqrt{1+x^2}}\right)$$

$$= m\, [x + \sqrt{1+x^2}]^{m-1} \left(\frac{x + \sqrt{1+x^2}}{\sqrt{1+x^2}}\right) \qquad ...(2)$$

$$\therefore \quad \sqrt{1+x^2}\, y_1 = my$$

$$(1+x^2)\, y_1^2 = m^2 y^2.$$

Differentiating again,

$$(1+x^2)\, 2y_1 y_2 + y_1^2 \cdot 2x = 2m^2 y y_1 \qquad ...(3)$$

$$(1+x^2)\, y_2 + xy_1 - m^2 y = 0. \qquad \text{[Ravishankar 1997; Bilaspur 2002]}$$

Now differentiating $n$ times, we have

$$[y_{n+2} (1+x^2) + {}^nC_1 \cdot y_{n+1} \cdot 2x + {}^nC_2 \cdot y_n \cdot 2] + [y_{n+1} \cdot x + {}^nC_1 \cdot y_n \cdot 1] - m^2 y_n = 0$$

$$(1+x^2)\, y_{n+2} + (2n+1)\, xy_{n+1} + (n^2 + m^2)\, y_n = 0. \qquad \text{[Bilaspur 2002]}$$

Putting $x = 0$, we get

$$(y_{n+2})_0 = (m^2 - n^2)(y_n)_0. \quad ...(4)$$

But from (1), (2) and (3), we have

$$(y)_0 = 1,\ (y_1)_0 = m \cdot 1 = m,\ (y_2)_0 = m^2 \cdot 1 = m^2.$$

From (4), $\quad (y_4)_0 = (m^2 - 2^2)(y_2)_0 = (m^2 - 2^2) \cdot m^2.$

Similarly, $\quad (y_6)_0 = (m^2 - 4^2)(m^2 - 2^2) \cdot m^2.$

$\therefore$ **When n is even, then**

$$(y_n)_0 = [m^2 - (n-2)^2][m^2 - (n-4)^2] \ldots (m^2 - 4^2)(m^2 - 2^2) m^2.$$

Also $\quad (y_1)_0 = m.$

Now from (4), $\quad (y_3)_0 = (m^2 - 1^2)(y_1)_0 = (m^2 - 1^2) m.$

Similarly, $\quad (y_5)_0 = (m^2 - 3^2)(m^2 - 1^2) m.$

$\therefore$ **When n is odd, then**

$$(y_n)_0 = [m^2 - (n-2)^2][m^2 - (n-4)^2] \ldots (m^2 - 3^2)(m^2 - 1^2) m.$$

**Example 7.** *If* $y = e^{a \cos^{-1} x}$, *prove that*

$$(1 - x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2 + a^2) y_n = 0. \quad \textbf{[Ravishankar 2009]}$$

*Also find the* $n^{th}$ *differential coefficient of* $e^{a \cos^{-1} x}$ *at* $x = 0$.

**[Indore 1992, 97; Ravishankar 92, 94S, 2003]**

**Solution.** $\quad y = e^{a \cos^{-1} x} \quad ...(1)$

$$\therefore \quad y_1 = \frac{-a}{\sqrt{(1-x^2)}} e^{a \cos^{-1} x}$$

or $\quad \sqrt{(1-x^2)}\, y_1 = -ay \quad ...(2)$

$\therefore \quad (1-x^2) y_1^2 = a^2 y^2.$ [squaring]

Differentiating again, we have

$$(1-x^2) 2y_1 y_2 + y_1^2(-2x) = 2a^2 y y_1 \quad \text{or} \quad y_2(1-x^2) - x y_1 = a^2 y. \quad ...(3$$

Differentiating each term $n$ times, we get

$$y_{n+2}(1-x^2) + {}^nC_1 \cdot y_{n+1}(-2x) + {}^nC_2 \cdot y_n(-2) - [y_{n+2} \cdot x + {}^nC_1 y_n \cdot 1] = a^2 y_n$$

$$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2 + a^2) y_n = 0. \quad \textbf{[Indore 199}$$

Putting $x = 0$, we get

$$(y_{n+2})_0 = (a^2 + n^2)(y_n)_0. \quad ...($$

But from (1), (2), (3), we get

$$(y_0) = e^{a \cos^{-1} 0} = e^{a\pi/2}$$

$$(y_1)_0 = -a e^{a\pi/2}$$

$$(y_2)_0 = -a^2 e^{a\pi/2}.$$

$\therefore$ From (4), $\quad (y_4)_0 = (a^2 + 2^2)(y_2)_0 = (a^2 + 2^2) a^2 e^{a\pi/2}.$

Similarly, $\quad (y_6)_0 = (a^2 + 4^2)(a^2 + 2^2) a^2 e^{n\pi/2}.$

$\therefore$ **When n is even, then**

$$(y_n)_0 = [a^2 + (n-2)^2]\,[a^2 + (n-4)^2] \ldots (a^2 + 4^2)\,(a^2 + 2^2)\,a^2 e^{a\pi/2}.$$

Again $\qquad (y_1)_0 = -\,ae^{a\pi/2}.$

Now from (4) $\qquad (y_3)_0 = -\,(a^2 + 1^2)\,(y_1)_0 = -\,(a^2 + 1^2)\,ae^{a\pi/2}.$

Similarly, $\qquad (y_5)_0 = -\,(a^2 + 3^2)\,(a^2 + 1^2)\,ae^{a\pi/2}.$

$\therefore$ **When n is odd, then**

$$(y_n)_0 = -\,[a^2 + (n-2)^2]\,[a^2 + (n-4)^2] \ldots (a^2 + 3^2)\,(a^2 + 1^2)\,ae^{a\pi/2}.$$

## EXERCISE 3 (D)

1. If $y = (\sin^{-1} x)^2$, then find $(y_n)_0$. **[Ravishankar 2000]**

2. If $x = \sin [(1/a) \log_e y]$, then prove that

$$(1 - x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - (n^2 + a^2)\,y_n = 0$$

and find also the value of $(y_n)_0$. **[Ravishankar 1993, 95; Indore 95, 2003; Bhopal 93, 95, 2003; Vikram 97; Bilaspur 2001; Jabalpur 97; Sagar 2005]**

[Hint : Transform into $y = e^{a \sin^{-1} x}$]

3. If $y = \tan^{-1} x$, then prove that

$$(1 + x^2)\,y_2 + 2xy_1 = 0$$

and find the value of $(y_n)_0$ and also show that when $n = 2p,\ 4p + 1,\ 4p + 3$, then their values are $0,\ (n-1)\,!,\ -\,(n-1)\,!$, respectively. **[Bilaspur 2003; Jabalpur 2007]**

## ANSWER

2. $a\,(n^2 + 1^2)\,(a^2 + 3^2) \ldots \{a^2 + (n-2)^2\}$, when $n$ is odd.
   $a^2\,(a^2 + 2^2)\,(a^2 + 4^2) \ldots \{a^2 + (n-2)^2\}$ when $n$ is even.

## MISCELLANEOUS EXERCISE 3 (E)

1. If $y = \sec x$, then prove that $\quad y\dfrac{d^2y}{dx^2} = y^4 + \left(\dfrac{dy}{dx}\right)^2.$

2. Show that

$$D^{n+1} xy = (n+1)D^n y + xD^{n+1} y.$$

Prove that

$$D^n\,(x^{n-1} e^{1/x}) = (-1)^n x^{-n-1} e^{1/x}.$$

3. Show that $y = \cos\,(m \sin^{-1} x)$ satisfies the following equations :

$$(1 - x^2)\,y'' - xy' + m^2 y = 0.$$

Using Leibnitz's theorem differentiate this equation $n$ times and find the value of $(y_n)_0$.

□■□

Chapter 4

# Expansion by Maclaurin's and Taylor's Series

## § 4.1. MACLAURIN'S THEOREM

**Statement.** *If $f(x)$ be a function of the variable $x$ such that it can be expanded in ascending powers of $x$ and this expansion be differentiable any number of times then*

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots$$

**[Bhopal 1991, 94, 95; Bilaspur 95, 97, 2001, 2007; Sagar 97; Jabalpur 2000; Vikram 2005; Ravishankar 2000, 2006S]**

**Proof.** Let $$f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots \quad \ldots(1)$$
where $A_0, A_1, A_2, \ldots$, are constants.

Now by successive differentiation of (1), w.r.t. $x$, we have

$$f'(x) = A_1 . 1 + 2A_2 x + 3A_3 x^2 + 4A_4 x^3 + \ldots$$
$$f''(x) = 2 . 1 A_2 + 3 . 2 A_3 x + 4 . 3 A_4 x^2 + \ldots$$
$$f'''(x) = 3 . 2 . 1 A_3 + 4 . 3 . 2 A_4 x + \ldots, \text{ etc.}$$

Substituting $x = 0$ in each of above relations, we get

$$f(x) = A_0, f'(0) = A_1, f''(0) = 2! . A_2,$$
$$f'''(0) = 3! . A_3, \ldots, f^{(n)}(0) = n! A_n \text{ etc.}$$

$\therefore$ $$A_0 = f(0), A_1 = f'(0), A_2 = f''(0)/2!, A_3 = f'''(0)/3! \ldots$$
$$A_n = f^n(0)/n! \text{ etc.}$$

Substituting the values of $A_0, A_1, A_2, \ldots$ in eqn. (1), we get

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots . \quad \ldots(2)$$

**Note.** If $f(x)$ be denoted by '$y$' then (2) may be written as :

$$y = (y)_0 + \frac{x}{1!} (y_1)_0 + \frac{x^2}{2!} (y_2)_0 + \ldots + \frac{x^n}{n!} (y_n)_0 + \ldots . \quad \ldots(3)$$

## § 4.2. TAYLOR'S THEOREM

**Statement.** *If $f(a+h)$, [where $a$ is independent of $h$] be a function of the variable $h$ such that it can be expanded in ascending powers of $h$ and this expansion be differentiable any number of times then*

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^n}{n!} f^n(a) + \ldots$$

**[Bilaspur 1990; avishankar 92, 2000; Indore 96; Vikram 90, 96, 2003; Bhopal 90, 2003, 2006, 2007]**

**Proof.** Let $$f(a+h) = A_0 + A_1 h + A_2 h^2 + A_3 h^3 + \ldots + A_n h^n + \ldots \quad \ldots(1)$$
where $A_0, A_1, A_2, \ldots, A_n$ are constants independent of $h$.

Now by successive differentiation of (1) w.r.t. '$h$', we have

$$f'(a+h) = A_1 + 2A_2h + 3A_3h^2 + 4A_4h^3 + \dots$$
$$f''(a+h) = 2 \,.\, 1 A_2 + 3 \,.\, 2A_3h + 4 \,.\, 3A_4h^2 + \dots$$
$$f'''(a+h) = 3 \,.\, 2 \,.\, 1A_3 + 4 \,.\, 3 \,.\, 2A_4h + \dots \text{ etc.}$$

Putting $h = 0$, in each of above relations, we have

$$f(a) = A_0, f'(a) = A_1, f''(a) = 2\,! \,.\, A_2,$$
$$f'''(a) = 3\,!\, A_3, \dots, f^n(a) = n\,!\, A_n \text{ . etc.}$$

Hence $\quad A_0 = f(a), A_1 = f'(a), A_2 = \dfrac{f''(a)}{2\,!}, A_3 = \dfrac{f'''(a)}{3\,!}, A_n = \dfrac{f^n(a)}{n\,!} \dots .$

Substituting, these values of $A_0, A_1, A_2, \dots$ in (1), we have

$$\mathbf{f(a+h) = f(a) + hf'(a) + \frac{h^2}{2\,!} f''(a) + \dots + \frac{h^n}{n\,!} f^{(n)}(a) + \dots .} \qquad \dots(2)$$

The power series given by (2) is called the **Taylor's infinite series** for the expansion of $f(a+h)$ in ascending powers of $h$.

**Cor. 1.** Putting $a = x$ in (2), we have

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2\,!} f''(x) + \dots + \frac{h^n}{n\,!} f^{(n)}(x) + \dots . \qquad \dots(3)$$

**Cor. 2.** Putting $h = x - a$ in (2), we have

$$f(x) = f(a) + (x-a)\,f'(a) + \frac{(x-a)^2}{2\,!} f''(a) + \dots + \frac{(x-a)^n}{n\,!} f^{(n)}(a) + \dots \qquad \dots(4)$$
$$= \sum_{k=0}^{\infty} \frac{(x-a)^k}{k\,!} f^k(a).$$

It is called Taylor's series of $f(x)$ around $x = a$.

**Note 1.** If we put $a = 0$, $h = x$ in (2) of § 4.2, we get Maclaurin's equation (2) of § 4.1.

**Note 2.** The power series in § 4.1 and § 4.2 should be convergent.

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Expand the following by Maclaurin's theorem :*

(*i*) $(a+x)^m$.

(*ii*) $e^x$. [Vikram 2006; Bilaspur 1988]

(*iii*) $\log(1+x)$. [Jabalpur 1993; Sagar 95, 2000; Indore 99]

(*iv*) $\sin x$. , cos x, tan x [Rewa 1990; Ravishankar 93S, 98, 99S; Bilaspur 96]

(*v*) $\sin^{-1} x$. [Jabalpur 1994]

(*vi*) $a^x$. [Sagar 1996]

(*vii*) $e^x . \cos x$. , e^x sin x [Ravishankar 1992, 93, 96S, 2001; Vikram 95, 2001; Rewa 95; Jabalpur 96, 98; Bhopal 93, 98, Indore 90, 92, 94]

**Solution (i)** Let $\quad f(x) = (a+x)^m, \qquad f(0) = a^m$

$$f'(x) = m\,(a+x)^{m-1}, \qquad f'(0) = ma^{m-1}$$
$$f''(x) = m(m-1)\,(a+x)^{m-2}, \quad f''(0) = m(m-1)\,a^{m-2}$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$f^{(n)}(x) = m(m-1)\,(m-2)\,\dots\,(m-n+1)\,(a+x)^{m-n}$$
$$f^{(n)}(0) = m(m-1)\,(m-2)\,\dots\,(m-n+1)\,a^{m-n} \text{ etc.}$$

Now by Maclaurin's theorem, we get

$$f(x) = f(0) + x f'(0) + (x^2/2!) f''(0) + (x^3/3!) f'''(0) + \dots$$

Hence $\quad (a+x)^m = a^m + x \cdot ma^{m-1} + (x^2/2!) m(m-1) a^{m-2} + \dots$

$$+ (x^n/n!) \cdot m(m-1)(m-2) \dots (m-n+1) a^{m-n} + \dots$$

**(ii)** Let

$$f(x) = e^x, \qquad f(0) = 1$$
$$f'(x) = e^x, \qquad f'(0) = 1$$
$$f''(x) = e^x, \qquad f''(0) = 1$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$f^{(n)}(x) = e^x \qquad f^{(n)}(0) = 1 \text{ etc.}$$

By Maclaurin's theorem, we get

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots .$$

**(iii)** Let

$$f(x) = \log(1+x), \qquad f(0) = 1$$
$$f'(x) = -(1+x)^{-1}, \qquad f'(0) = 1$$
$$f''(x) = -(1+x)^{-2}, \qquad f''(0) = -1$$
$$f'''(x) = 2(1+x)^{-3}, \qquad f'''(0) = 2(1)^{-3} = 2!$$
$$f^{(iv)}(x) = -2 \times 3 (1+x)^{-4}, \qquad f^{iv}(0) = -(3)!$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$f^{(n)}(x) = (-1)^{n-1}(n-1)!\,(1+x)^{-n},$$
$$f^{(n)}(0) = (-1)^{n-1}(n-1)! \text{ etc.}$$

$\therefore$ By Maclaurin's theorem, we have

$$\log(1+x) = x - (x^2/2) + (x^3/3) - (x^4/4) + \dots + (-1)^{n-1}(x^n/n) + \dots$$

**(iv)** Let

$$f(x) = \sin x, \qquad f(0) = \sin 0 = 0$$
$$f'(x) = \cos x, \qquad f'(0) = \cos 0 = 1$$
$$f''(x) = -\sin x, \qquad f''(0) = -\sin 0 = 0$$
$$f'''(x) = -\cos x, \qquad f'''(0) = -\cos 0 = -1$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$f^{(n)}(x) = \sin\left(x + n \cdot \frac{\pi}{2}\right), \qquad f^{(n)}(0) = \sin \frac{1}{2} n\pi.$$

When $n$ is even, then let $n = 2m$,

$$\therefore \quad f^{2m}(0) = 0$$

When $n$ is odd, then let $n = 2m+1$,

$$\therefore \quad f^{2m+1}(0) = (-1)^m.$$

$\therefore$ By Maclaurin's theorem, we get

$$\sin x = 0 + x \cdot 1 + 0 + \frac{x^3}{3!}(-1) + 0 + \dots + 0 + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + \dots$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + \dots$$

**(v)** Let $\quad f(x) = \sin^{-1} x, \qquad f(0) = 0$ **[Vikram 1982, 79]**

$$f(x) = y_1 = \frac{1}{\sqrt{1-x^2}}, \qquad (y_1)_0 = f'(0) = 1$$

$\therefore \quad y_1^2 (1 - x^2) = 1.$

Differentiating again, we have

$$2y_1 y_2 (1 - x^2) - 2xy_1^2 = 0 \quad \text{or} \quad (1 - x^2) y_2 - xy_1 = 0. \qquad ...(1)$$

Differentiating $n$ times w.r.t. '$x$', we have

$$(1 - x^2) y_{n+2} - (2n + 1) y_{n+1} - n^2 y_n = 0.$$

Putting $x = 0$, $\quad (y_{n+2})_0 = n^2 (y_n)_0.$ ...(2)

Putting $x = 0$ in (1), $\quad (y_2)_0 = 0 = f''(0).$

Putting $n = 1, 2, 3, \ldots$ in (2), we have

$$(y_3)_0 = 1^2 (y_1)_0 = 1^2 . 1$$

$$(y_4)_0 = 2^2 (y_2)_0 = 0$$

$$(y_5)_0 = 3^2 (y_3)_0 = 3 . 1^2$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$(y_n)_0 = (n - 2)^2 (y_{n-2})_0 = (n - 2)^2 (n - 4)^2 \ldots 5^2 . 3^2 . 1^2, \text{ if } n \text{ is odd.}$$

and $\qquad = 0$ if $n$ is even.

By Maclaurin's theorem, we have

$$f(x) = (y)_0 + x(y_1)_0 + (x^2/2!) (y_2)_0 + (x^3/3!) (y_3)_0 + \ldots + (x^n/n!) (y_n)_0 + \ldots$$

$$\therefore \sin^{-1} x = 0 + x . 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!} \cdot 1^2 + \frac{x^4}{4!} . 0 + \frac{x^5}{5!} . 3^2 . 1^2 + \ldots$$

$$+ \frac{x^n}{n!} (n - 2)^2 (n - 4)^2 \ldots 5^2 . 3^2 . 1^2 + \ldots$$

$$= x + 1^2 \frac{x^3}{3!} + 3^2 . 1^2 \frac{x^5}{5!} + \ldots + (n - 2)^2 (n - 4)^2 \ldots 5^2 . 3^2 . 1^2 . \frac{x^n}{n!} + \ldots$$

If $n$ is odd and the general term is 0 if $n$ is even.

(vi) Let $\quad f(x) = a^x, \qquad f(0) = 1$

$$f'(x) = a^x \log a, \qquad f'(0) = \log a$$

$$f''(x) = a^x (\log a)^2, \qquad f''(0) = (\log a)^2$$

$$\vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots$$

$$f^{(n)}(x) = a^x (\log a)^n, \qquad f^{(n)}(0) = (\log a)^n.$$

By Maclaurin's theorem, we get

$$a^x = 1 + x \log a + \frac{x^2}{2!} (\log a)^2 + \ldots + \frac{x^n}{n!} (\log a)^n + \ldots$$

(vii) Let $\quad y = e^x \cos x, \qquad (y)_0 = 1$

$$y_1 = e^x \cos x - e^x \sin x$$

or $\quad y_1 = y - e^x \sin x, \qquad (y_1)_0 = (y)_0 - 0 = 1$

$$y_2 = y_1 - y - e^x \sin x, \qquad (y_2)_0 = 1 - 1 = 0$$

or $\quad y_3 = y_2 - y_1 - y - e^x \sin x$

$$y_3 = y_2 - 2y, \qquad (y_3)_0 = -2.$$

**Example 4.** *Expand $e^{ax}\cos bx$ by Maclaurin's theorem. Hence prove that*

$$e^{x\cos\alpha}\cos(x\sin\alpha) = 1 + x\cos\alpha + \frac{x^2}{2!}\cos 2\alpha + \frac{x^3}{3!}\cos 3\alpha + \ldots$$

**Solution.** Let $y = e^{ax}\cos bx$

$\therefore$ $y_n = r^n e^{ax}\,.\cos(bx + n\phi)$

where $r = \sqrt{a^2+b^2},\ \alpha = \tan^{-1}\dfrac{b}{a}.$

$\therefore$ $(y_n)_0 = r^n \cos n\phi.$ ...(1)

Substituting $n = 1, 2, 3, \ldots$ in (1), we get

$$(y_1)_0 = r\cos\phi = (a^2+b^2)^{1/2}\,.\frac{a}{\sqrt{a^2+b^2}} = a$$

$$(y_2)_0 = r^2\cos 2\phi = (a^2+b^2)\,.\frac{1-\tan^2\phi}{1+\tan^2\phi} = \frac{(a^2+b^2)(a^2-b^2)}{a^2+b^2} = a^2-b^2,$$

$$(y_3)_0 = r^3\cos^3\phi = (a^2+b^2)^{3/2}(4\cos^3\phi - 3\cos\phi) = a(a^2-3b^2) \text{ etc.}$$

$$(y_n)_0 = (a^2+b^2)^{n/2}\cos\{n\tan^{-1}(b/a)\}.$$

Hence by Maclaurin's theorem, we have

$$e^{ax}\cos bx = 1 + ax + (a^2-b^2)\frac{x^2}{2!} + a(a^2-3b^2)\,.\frac{x^3}{3!} + \ldots.$$

**For second part,** putting $a = \cos\alpha$ and $b = \sin\alpha$, we get

$$e^{x\cos\alpha}\cos(x\sin\alpha) = 1 + x\cos\alpha + \frac{x^2}{2!}\cos 2\alpha + \frac{x^3}{3!}\cos 3\alpha + \ldots.$$

**Example 5.** *Expand $e^{a\sin^{-1}x}$ by Maclaurin's theorem (upto at least four non-zero terms). Hence show that*

$$e^{\theta} = 1 + \sin\theta + \frac{1}{2!}\sin^2\theta + \frac{2}{3!}\sin^3\theta + \ldots.$$

**[Jiwaji 2003, 2006; Rewa 2005; Jabalpur 2005; Bhopal 2006]**

**Solution.** Let $y = e^{a\sin^{-1}x}$, $(y)_0 = 1$

$\therefore$ $y = e^{a\sin^{-1}x}\,.\dfrac{a}{\sqrt{1-x^2}}$

$\Rightarrow$ $(1-x^2)\,y_1^{\,2} = a^2y^2$, $(y_1)_0 = a.$

Differentiating again, we get

$$(1-x^2)\,2y_1y_2 - 2xy_1^{\,2} = a^2\,.\,2yy_1$$

*i.e.,* $(1-x^2)\,y_2 - xy_1 - a^2y = 0$, $(y_2)_0 = a^2.$

Differentiating $n$ times by Leibnitz's theorem, we get

$$(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - (n^2+a^2)\,y_n = 0.$$

Putting $x = 0$, $(y_{n+2})_0 = (n^2+a^2)\,(y_n)_0.$ ...(1)

Now substituting $n = 1, 2, 3, 4, 5 \ldots$ in equation (1), we have

$$(y_3)_0 = a(1^2+a^2)$$

$$(y_4)_0 = a^2(2^2+a^2)$$

$$(y_5)_0 = a(1^2+a^2)(3^2+a^2).$$

Again $\quad y_4 = y_3 - 2y_1, \qquad (y_4)_0 = -2 - 2 = -2^2$

$y_5 = y_4 - 2y_2, \qquad (y_5)_0 = -2^2$

$y_6 = y_5 - 2y_3, \qquad (y_6)_0 = 0$

$y_7 = y_6 - 2y_4, \qquad (y_7)_0 = 2^3$, etc.

$\therefore$ By Maclaurin's theorem, we get

$$e^x \cos x = 1 + x - \frac{2x^3}{3\,!} - \frac{2^2 x^4}{4\,!} - \frac{2^2 x^5}{5\,!} + \frac{2^3 x^7}{7\,!} + \ldots .$$

**Example 2.** *Apply Maclaurin's theorem to prove that*

$$\log \sec x = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \ldots .$$

**[Vikram 1993, 2000, 2001, 2003; Indore 93, 2007; Ravishankar 95, 2003S, 2008; Jabalpur 2004; Jiwaji 93, 95, 2004; Bilaspur 93; Rewa 92; Sagar 2004; Bhopal 2004**

**Solution.** Let $f(x) = \log \sec x, \qquad f(0) = 0$

$$f'(x) = \frac{\sec x \tan x}{\sec x} = \tan x, \qquad f'(0) = 0$$

$$f''(x) = \sec^2 x, \qquad f''(0) = 1$$

$$f'''(x) = 2\sec^2 x \tan x, \qquad f'''(0) = 0$$

$$f^{(iv)}(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x, \qquad f^{(iv)}(0) = 2$$

$$f^{(v)}(x) = 8\sec^2 x \tan^3 x + 16\sec^4 x \tan x, \qquad f^{(v)}(0) = 0$$

$$f^{(vi)}(x) = 16\sec^2 x \tan^4 x + 88\sec^4 x \tan^2 x + 16\sec^6 x, \quad f^{(iv)}(0) = 16.$$

By Maclaurin's theorem, we have

$$\log \sec x = f(0) + x f'(0) + \frac{x^2}{2\,!} f''(0) + \frac{x^3}{3\,!} f'''(0) + \frac{x^4}{4\,!} f^{iv}(0) + \frac{x^5}{5\,!} f^{v}(0) + \frac{x^6}{6\,!} f^{vi}(0) + \ldots$$

$$= 0 + x \,.\, 0 + \frac{x^2}{2\,!} \,.\, 1 + \frac{x^3}{3\,!} \,.\, 0 + \frac{x^4}{4\,!} \,.\, 2 + \frac{x^5}{5\,!} \,.\, 0 + \frac{x^6}{6\,!} \,.\, 16 + \ldots$$

$$= \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \ldots .$$

**Example 3.** *Apply Maclaurin's theorem to prove that*

$$\frac{e^x}{\cos x} \quad \text{or} \quad e^x \sec x = 1 + x + \frac{2x^2}{2\,!} + \frac{4}{3\,!}x^3 + \ldots .$$

**[Indore 1995; Ravishankar 2005]**

**Solution.** Let $y = e^x \sec x, \qquad \therefore \quad (y_0) = 1$

$\therefore \quad y_1 = e^x \sec x + e^x \,.\, \sec x \tan x = (1 + \tan x)\, y, \qquad \therefore \quad (y_1)_0 = 1$

$y_2 = \sec^2 x \,.\, y + (1 + \tan x)\, y_1, \qquad \therefore \quad (y_2)_0 = 2$

$y_3 = \sec^2 x \,.\, y_1 + 2\sec x \sec x \tan x \,.\, y + \sec^2 x \,.\, y_1 + (1 + \tan x)\, y_2$

$= 2\sec^2 x \,.\, y_1 + 2\sec^2 x \tan xy + (1 + \tan x)\, y_2, \qquad \therefore \quad (y_3)_0 = 4.$

$\therefore$ By Maclaurin's theorem, we have

$$e^x \,.\, \sec x = 1 + x + \frac{2x^2}{2\,!} + \frac{4x^3}{3\,!} + \ldots .$$

$$y_4 = y_3(\cos x - x\sin x) + y_2(-2\sin x - x\cos x) - 2y_2(2\sin x + x\cos x)$$
$$-2y_1(3\cos x - x\sin x) - y_1(3\cos x - x\sin x) - y(-4\sin x - x\cos x)$$
$$= y_3(\cos x - x\sin x) - 3y_2(2\sin x + x\cos x)$$
$$-3y_1(3\cos x - x\sin x) + y(4\sin x + x\cos x)$$
$$y_5 = y_4(\cos x - x\sin x) - 4y_3(2\sin x + x\cos x) - 6y_2(3\cos x - x\sin x)$$
$$+4y_1(4\sin x + x\cos x) + y(5\cos x - x\sin x).$$

Putting $x = 0$ in above results, we get

$$(y)_0 = e^0 = 1;\ (y_1)_0 = (y_0)\,.\,1 = 1;$$
$$(y_2)_0 = (y_1)_0\,.\,1 = 1;$$
$$(y_3)_0 = (y_2)_0\,.\,1 - (y)_0\,.\,3 = -2;$$
$$(y_4)_0 = (y_3)_0 - 3(y_1)_0\,.\,3 = -11;$$
$$(y_5)_0 = (y)_4\,.\,1 = 0 - 6(y_2)_0\,3 + 0 + (y)_0\,.\,5 = -24 \text{ etc.}$$

Hence substituting these values in Maclaurin's series, we get

$$e^{x\cos x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{11x^4}{24} - \frac{x^5}{5} - \dots .$$

**Example 13.** *Prove that* $\quad \log\cosh x = \dfrac{x^2}{2} - \dfrac{x^4}{12} - \dfrac{x^6}{45} - \dots$

**[Jabalpur 2007; Jiwaji 2007]**

**Solution.** Let $\quad y = \log\cosh x.$ ...(1)

Therefore $\quad y_1 = \dfrac{1}{\cosh x}\sinh x = \tanh x$ ...(2)

$$y_2 = \text{sech}^2 x = 1 - \tanh^2 x = 1 - {y_1}^2, \quad \text{[from (2)]} \quad ...(3)$$
$$y_3 = -2y_1y_2 \quad ...(4)$$
$$y_4 = -2(y_1y_3 + 2{y_2}^2) \quad ...(5)$$
$$y_5 = -2(y_1y_4 + 3y_2y_3) \quad ...(6)$$
$$y_6 = -2(y_1y_5 + 4y_2y_4 + 3{y_3}^2). \quad ...(7)$$

Putting $x = 0$ in (1), (2), ..., (7), we have

$$(y)_0 = \log\cosh 0 = \log 1 = 0;$$
$$(y_1)_0 = \tanh 0 = 0$$
$$(y_2)_0 = 1 - (y_1)_0{}^2 = 1;$$
$$(y_3)_0 = -2(y_1)_0(y_2)_0 = 0$$
$$(y_4)_0 = -2[(y_1)_0(y_3)_0 + (y_2)_0{}^2] = -2(0+1) = -2$$
$$(y_5)_0 = -2[(y_1)_0(y_4)_0 + 3(y_2)_0(y_3)_0] = -2(0) = 0$$
$$(y_6)_0 = -2[(y_1)_0(y_5)_0 + 4(y_2)_0(y_4)_0 + 3(y_3)_0{}^2] = 16.$$

Hence substituting these values in Maclaurin's Series, we have

$$\log\cosh x = 0 + 0 + \frac{x^2}{2!}\cdot(1) + 0 + \frac{x^4}{4!}\cdot(-2) + 0 + \frac{x^6}{6!}\cdot(16) + \dots$$
$$= \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{45} - \dots .$$

Substituting these values in (1), we have

$$\log (x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} \dots .$$

**Example 27.** *Expand $\log \sin x$ in powers of $(x-2)$ by Taylor's theorem.*

**[Ravishankar 1993, 2005, 2008, 2010; Jiwaji 90; Bhopal 96]**

**Solution.** Let $f(x) = \log \sin x$.

By Taylor's theorem, we have

$$f(x) = f[2 + (x-2)] = f(2) + (x-2) f'(2) + \frac{(x-2)^2}{2!} f''(2) + \dots \qquad \dots(1)$$

Now $f(x) = \log \sin x, f'(x) = \cot x,$

$f''(x) = -\operatorname{cosec}^2 x, \; f'''(x) = 2 \operatorname{cosec}^2 x \, . \cot x.$

$\therefore$ $f(2) = \log \sin 2, \; f'(2) = \cot 2,$

$f''(2) = -\operatorname{cosec}^2 2, \; f'''(2) = 2 \operatorname{cosec}^2 2 \cot 2.$

Substituting these value in (1), we have

$$\log \sin x = \log \sin 2 + (x-2) \cot 2 - \frac{(x-2)^2}{2!} \operatorname{cosec}^2 2 + \frac{(x-2)^3}{3!} . \, 2 \operatorname{cosec}^2 2 \cot 2 + \dots .$$

**Example 28.** *Expand $\log \sin (x+h)$ in powers of $h$ by Taylor's theorem.*

**[Jabalpur 1988, 94; Ravishankar 93; Indore 91, 94; Jiwaji 95; Sagar 2007]**

**Solution.** We are to expand $\log \sin (x+h)$ in powers of $h$. So let

$f(x) = \log \sin x$, then $f(x+h) = \log \sin (x+h)$.

$\therefore$ By Taylor's theorem, we have

$$f(x+h) = f(x) + h f'(x) + (h^2/2!) f''(x) + \dots \qquad \dots(1)$$

Now $f(x) = \log \sin x, f'(x) = \cot x, \; f''(x) = -\operatorname{cosec}^2 x$, etc.

Substituting these values in (1), we have

$$\log \sin (x+h) = \log \sin x + h \cot x - (h^2/2!) \operatorname{cosec}^2 x + (2h^2/3!) \operatorname{cosec}^2 x \cot x + \dots .$$

**Example 29.** *Expand $\sin^{-1} (x+h)$ in powers of $x$ as far as the term in $x^3$.*

**Solution.** We are to expand $\sin^{-1}(x+h)$ in ascending powers of $x$. So let

$f(h) = \sin^{-1} h$ then $f(x+h) = \sin^{-1}(x+h)$.

By Taylor's theorem, we have

$$f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \dots . \qquad \dots(1)$$

Now $f(h) = \sin^{-1} h$, therefore $f'(h) = \dfrac{1}{\sqrt{1-h^2}} = (1-h^2)^{-1/2}$

$$f''(h) = -\tfrac{1}{2}(1-h^2)^{-3/2}(-2h) = h(1-h^2)^{-3/2}$$

$$f''(h) = h(-5/2)(1-h^2)^{-5/2}(-2h) + 1(1-h^2)^{-3/2}$$

$$= (1-h^2)^{-5/2}(1+2h^2) \text{ etc.}$$

Substituting these values in (1), we get

$$\sin^{-1}(x+h) = \sin^{-1} h + x(1-h^2)^{-1/2} + x^2 h(1-h^2)^{-3/2}/2! + x^3(1-h^2)^{-3/2}(1+2h^2)/3! + \dots$$

**Example 16.** *Expand* $\log\{1-\log(1-x)\}$ *in powers of* $x$ *by Maclaurin's theorem as far as the term* $x^3$. *By substituting* $x/(1+x)$ *for* $x$ *deduce the expansion of* $\log\{1+\log(1+x)\}$ *as far as the term in* $x^3$.

**Solution.** Let $\quad y=\log\{1-\log(1-x)\}, \qquad \therefore \quad e^y = 1-\log(1-x).$

Differentiating, we have

$$e^y \cdot y_1 = (1-x)^{-1}. \qquad \text{...(1)}$$

Differentiating (1), we get

$$e^y \cdot y_2 + e^y y_1^2 = (1-x)^{-2}. \qquad \text{...(2)}$$

Differentiating (2), we get

$$e^y \cdot y_3 + e^y \cdot y_1 y_2 + e^y \cdot y_1^3 + e^y \cdot 2y_1 y_2 = 2(1-x)^{-3}. \qquad \text{...(3)}$$

Putting $x=0$ in above results, we get

$$(y)_0 = \log\{1-\log 1\} = \log(1-0) = 0$$

$$(y_1)_0 = (1-0)^{-1} = 1,\ (y_2)_0 = 0\ \ (y_3)_0 = 1.$$

Hence from Maclaurin's theorem, we have

$$\log\{1-\log(1-x)\} = 0 + x \cdot 1 + 0 + \frac{x^3}{3\,!} \cdot 1 + \ldots = x + \frac{1}{6}x^3 + \ldots . \qquad \text{...(4)}$$

Now substituting $x/(1+x)$ for $x$ on both sides of (4), we have

$$\log\left\{1-\log\left(1-\frac{x}{1+x}\right)\right\} = \frac{x}{1+x} + \frac{1}{6}\left(\frac{x}{1+x}\right)^3 + \ldots$$

or $\quad \log\{1+\log(1+x)\} = x(1+x)^{-1} + \left(\dfrac{1}{6}\right)x^3(1+x)^{-3} + \ldots$

$$= x\{1-x+x^2 \ldots\} + \left(\frac{1}{6}\right)x^3\{1+(-3)+\ldots\}$$

$$= x - x^2 + x^3 + \ldots + (1/6)\,x^3 + \ldots$$

$$= x - x^2 + (7/6)\,x^3 + \ldots .$$

**Example 17.** *Expand* $\{x+\sqrt{1+x^2}\}^m$ *in ascending powers of* $x$ *and find the general term also.*

**Solution.** Let $\qquad y = \{x+\sqrt{1+x^2}\}^m.$

Proceeding as in Ex. 6 of Ch. 5, § 5.3,

$$(y)_0 = 1,\ (y_1)_0 = m,\ (y_2)_0 = m^2,$$

$$(y_3)_0 = (m^2-1^2)\,m,\ (y_4)_0 = (m^2-2^2)\,m^2,$$

$$(y_5)_0 = (m^2-3^2)(m^2-1^2)\,m, \ldots$$

**In general,**

$$\left.\begin{aligned} &\textbf{If n is even, } (y_n)_0 = \{m^2-(n-2^2)\} \ldots (m^2-4^2)(m^2-2^2)\,m^2 \\ &\textbf{If n is odd, } (y_n)_0 = \{m^2-(n-2)^2\} \ldots (m^2-3^2)(m^2-1^2)\,m. \end{aligned}\right\} \qquad \text{...(A)}$$

Now by Maclaurin's theorem, we get

$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2\,!}(y_2)_0 + \ldots + \frac{x^n}{n\,!}(y_n)_0 + \ldots$$

**Solution.** Here $y = (\sin^{-1} x)/\sqrt{1 - x^2}$. ...(1)

Squaring, we get $y^2 (1 - x^2) = (\sin^{-1} x)^2$.

Differentiating w.r.t. $x$, we get

$$(1 - x^2)\, 2yy_1 - 2xy^2 = 2(\sin^{-1} x)/\sqrt{1 - x^2} = 2y.$$

Dividing by $2y$ $\quad (1 - x^2)\, y_1 - xy = 1.$ ...(2)

Differentiating again,

$$(1 - x^2)\, y_2 - 3xy_1 - y = 0. \quad ...(3)$$

Differentiating (2) $n$ times by Leibnitz's theorem, we get

$$y_{n+1} (1 - x^2) + n \cdot y_n (-2x) + \frac{n(n-1)}{2!} y_{n-1} (-2) - xy_n - ny_{n-1} = 0$$

or $\quad (1 - x^2)\, y_{n+1} - (2n + 1)\, xy_n - n^2 y_{n-1} = 0.$ ...(4)

Now putting $x = 0$ in (1), (2), (3), (4), we get

$$(y)_0 = 0,\ (y_1)_0 = 1,\ (y_2)_0 = 0,$$

$$(y_{n+1})_0 = n^2 (y_{n-1})_0. \quad ...(5)$$

By Maclaurin's theorem, we get

$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \ldots + \frac{x^n}{n!}(y_n)_0 + \ldots. \quad ...(6)$$

Also we are given that

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots. \quad ...(7)$$

Comparing the coefficients of $x^n$ in the two expansions given by (6) and (7)

$$a_n = (y_n)_0 / n!.$$

$$\therefore \quad \frac{a_{n+1}}{a_{n-1}} = \frac{(y_{n+1})_0}{(n+1)!} \div \frac{(y_{n-1})_0}{n(n-1)!} = \frac{(y_{n+1})_0}{(y_{n-1})_0} \cdot \frac{(n-1)!}{(n+1)!}$$

$$= n^2 \left[\frac{1}{(n+1)\, n}\right] \quad \text{[from (5)]}$$

$$= n/(n+1).$$

$\therefore \quad (n + 1)\, a_{n+1} = na_{n-1}$ or $a_{n+1} = n/(n + 1)\, a_{n-1}$. ...(8)

Again from (6) and (7), we get

$$a_0 = (y)_0 = 0,\ a_1 = (y_1)_0 = 1,\ a_2 = (y_2)_0/2! = 0.$$

$\therefore$ Putting $n = 1, 3, 5, \ldots$ in (8), we have

$$a_2 = \frac{1}{2} a_0 = 0,\ a_4 = \frac{3}{4} \cdot a_2 = 0,\ a_6 = 0, \text{ etc.}$$

Thus $a_n = 0$, if $n$ is even *i.e.*, $a_{2m} = 0$.

Again putting $n = 2, 4, 6, \ldots$ in (8), we have

$$a_3 = \frac{2}{3} \cdot a_1 = \frac{2}{3},\ a_5 = \frac{4}{5} \cdot a_3 = \frac{4}{5} \cdot \frac{2}{3},\ a_7 = \frac{6}{7} \cdot \frac{4}{7} \cdot \frac{2}{7} \text{ etc.}$$

Thus if $n$ is odd let $n = 2m + 1$, then we get

$$a_{2m+1} = \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdots \frac{4}{5} \cdot \frac{2}{3}. \quad ...(9)$$

Hence the general term of expansion is $a_{2m+1}, x^{2m+1}$, where the value of $a_{2m+1}$ is given by (9).

$\therefore$ By Maclaurin's theorem, we have ...(2)

$$e^{a \sin^{-1} x} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a(1^2 + a^2)x^3}{3!} + \dots .$$

Putting $a = 1$, $x = \sin\theta$ in (2), we get

$$e^{\theta} = 1 + \sin\theta + \sin^2\theta/2! + 2\sin^3\theta/3! + \dots .$$

**Example 6.** *Find the first five terms in the expansion of $e^{\sin x}$ by Maclaurin's theorem.*

**Solution.** Let $y = e^{\sin x}$, $(y)_0 = 1$.

Differentiating successively, we get :

$y_1 = \cos x \, . \, e^{\sin x}$, $(y_1)_0 = 1$

i.e., $y_1 = y \cos x$, $(y_2)_0 = 1$

$y_2 = y_1 \cos x - y \sin x$,

$y_3 = y_2 \cos x - y_1 \sin x - y_1 \sin x - y \cos x = y_2 \cos x - 2y_1 \sin x - y \cos x$, $(y_3)_0 = 0$

$y_4 = y_3 \cos x - y_2 \sin x - 2y_2 \sin x - 2y_1 \cos x - y_1 \cos x + y \sin x$

$= y_3 \cos x - 3y_2 \sin x - 3y_1 \cos x + y \sin x$, $(y_4)_0 = -3$

$y_5 = y_4 \cos x - 4y_3 \sin x - 6y_2 \cos x + x + 4y_1 \sin x + y \cos x$, $(y_5)_0 = -8$

$\therefore$ By Maclaurin's theorem,

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15} + \dots .$$

**Example 7.** *Expand $\tan^{-1} x$ in ascending powers of $x$ by Maclaurin's theorem.*

**[Jiwaji 1991, 92; Jabalpur 93, 97; Indore 91, 2006; Bilaspur 2004]**

**Solution.** Let $y = \tan^{-1} x$, $(y)_0 = \tan^{-1} 0 = 0$

$\therefore$ $y_1 = \dfrac{1}{1+x^2}$, $(y_1)_0 = \dfrac{1}{1+0} = 1$

or $(1+x^2)\, y_1 = 0$ ...(1)

$\therefore$ $(1+x^2)\, y_2 + 2xy_1 = 0$, $(y_2)_0 = 0$.

Differentiating (1) $n$ times by Leibnitz's theorem, we get

$$(1+x^2)\, y_{n+1} + {}^nC_1\, y_n\, (2x) + {}^nC_2\, y_{n-1}\, (2 \, . \, 1) = 0$$

or $$(1+x^2)\, y_{n+1} + 2nxy_n + n(n-1)\, y_{n-1} = 0.$$

Putting $x = 0$, $(y_{n+1}) = -(n-1)\, n\, (y_{n-1})_0$. ...(2)

Putting $n = 2, 3, 4, \dots$ in (2),

$(y_3)_0 = -1 \, . \, 2\, (y_1)_0 = -1 \, . \, 2(1) = -2\,!$

$(y_4)_0 = -2 \, . \, 3\, (y_2)_0 = 0$ $[\because (y_2)_0 = 0]$

$(y_5)_0 = -3 \, . \, 4\, (y_3)_0 = 3 \, . \, 4 \, . \, 2\,! = 4\,!$, $(y_5)_0 = 0$

$(y_7)_0 = -5 \, . \, 6\, (y_5)_0 = -5 \, . \, 6 \, . \, 4\,! = -6\,!$, etc.

$\therefore$ By Maclaurin's theorem, we get

$$\tan^{-1} x = 0 + x \, . \, 1 + 0 + \frac{x^3}{3!}(-2!) + 0 + \frac{x^5}{5!}(4!) + 0 + \frac{x^7}{7!}(-6!) + \dots$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + \dots .$$

**[Bilaspur 1994]**

Putting $x = 0$ in (3)

$$1 \,.\, (y_3)_0 + 2(-1) \,.\, 1 - 0 = -1 \Rightarrow (y_3)_0 = 1.$$

Differentiating (3), we have

$$e^y y_4 + e^y y_1 y_3 + 2y_3 \cos x - 2y_2 \sin x - y_2 \sin x - y_1 \cos x = \sin x \quad \text{[from (1)]} \quad ...(4)$$

or $$e^y y_4 + 3y_3 \cos x - 3y_2 \sin x - y_1 \cos x = \sin x,$$

Putting $x = 0$ in (4), we get

$$e^0 (y_4)_0 + 3 \,.\, 1 \,.\, 1 - 0 - 1 \,.\, 1 = 0 \Rightarrow (y_4)_0 = -2.$$

Differentiating (4), again we get

$$e^y y_5 + e^y y_1 y_4 + 3y_4 \cos x - 3y_3 - \sin x - 3y_3 \sin x - 3y_2 \cos x - y_2 \cos x + y_1 \sin x = \cos x \quad ...(5)$$

Putting $x = 0$ in (5)

$$1 \,.\, (y_5)_0 + 1 \,.\, 1 \,.\, (-2) + 3(-2) \,.\, 1 - 0 - 3(-1) - (-1) + 0 = 1 \Rightarrow (y_5)_0 = 5.$$

Substituting the values in Maclaurin's theorem, we get

$$\log (1 + \sin x) = 0 + x \,.\, 1 \frac{x^2}{2!} \,.\, (-1) + \frac{x^3}{3!} \,.\, 1 + \frac{x^4}{4!}(-2) + \frac{x^5}{5!} \,.\, 5 + \ldots$$

$$= x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{2x^4}{4!} + \frac{5x^5}{5!} + \ldots .$$

**Example 10.** *Find the first four terms in the expansion of log (1 + tan x) by Maclaurin's theorem.* **[Vikram 1980]**

**Solution.** Let $y = \log(1 + \tan x)$, then $(y_0) = \log(1 + 0) = 0$.

Now $$e^y = 1 + \tan x.$$

By differentiating, we get $e^y \,.\, y_1 = \sec^2 x$ ...(1)

Putting $x = 0$ in (1), we get

$$e^0 \,.\, (y_1)_0 = 1 \Rightarrow (y_1)_0 = 1.$$

Differentiating (1), we have

$$e^y y_2 + e^y y_1^2 = 2 \sec^2 x \tan x$$

or $$e^y (y_2 + y_1^2) = 2 \sec^2 x \tan x. \quad ...(2)$$

Putting $x = 0$ in (2),

$$(y_2)_0 + 1 = 0 \Rightarrow (y_2)_0 = -1.$$

Differentiating (2), we get

$$e^y (y_3 + 2y_1 y_2) + e^y \,.\, y_1 (y_2 + y_1^2) = 2 \sec^4 x + 4 \sec^2 x \tan^2 x$$

or $$e^y (y_3 + 3y_1 y_2 + y_1^3) = 6 \sec^4 x - 4 \sec^2 x \quad ...(3)$$

$$[\because \tan^2 x = \sec^2 x - 1]$$

Putting $x = 0$ in (3), we have

$$(y_3)_0 + 3 \,.\, 1 \,.\, (-1) + 1 = 6 - 4 \Rightarrow (y_3)_0 = 4.$$

**Example 24.** *Expand* $\tan^{-1} x$ *in powers of* $(x - \frac{1}{4}\pi)$ *by Taylor's theorem.*

**[Sagar 1992, 93, 97, 2004; Ravishankar 94, 98, 99, 99S, 2001; Rewa 92; Bilaspur 90, 96; Indore 91, 97, 2004; Bhopal 91; Vikram 99]**

**Solution.** Let $\quad f(x) = \tan^{-1} x.$

$\therefore \quad f(x) = f[\frac{1}{4}\pi + (x - \frac{1}{4}\pi)].$

$\therefore$ By Taylor's theorem, we have

$$f(x) = f(\tfrac{1}{4}\pi) + (x - \tfrac{1}{4}\pi) f'(\tfrac{1}{4}\pi) + \frac{(x - \frac{1}{4}\pi)^2}{2\,!} f''(\tfrac{1}{4}\pi) + \dots \qquad \dots(1)$$

Now $\quad f(x) = \tan^{-1} x,\ f'(x) = \dfrac{1}{1+x^2} = (1+x^2)^{-1},\ f''(x) = -2x/(1+x^2)^2$, etc.

$\therefore \quad f(\frac{1}{4}\pi) = \tan^{-1}\frac{1}{4}\pi,\ f'(\frac{1}{4}\pi) = 1/(1 + 1/16\pi^2),$

$f''(\frac{1}{4}\pi) = -\pi/[2\{1 + (1/16)\pi^2\}^2]$, etc.

Putting these values in (1), the required expansion is given by

$$\tan^{-1} x = \tan^{-1}\left(\frac{\pi}{4}\right) + (x - \tfrac{1}{4}\pi)\frac{1}{(1 + 1/16\pi^2)} - \frac{(x - \frac{1}{4}\pi)^2}{2\,!} \cdot \frac{\pi}{2\,(1 + 1/16\pi^2)^2}.$$

**Example 25.** *Expand* $e^x$ *in powers of* $(x - 1)$ *by Taylor's theorem.*

**Solution.** Let $f(x) = e^x$. We can write $f(x)$ as

$$f(x) = f[1 + (x - 1)].$$

By Taylor's theorem, we have

$$e^x = f(x) = f[1 + (x - 1)]$$

$$= f(1) + (x - 1) f'(1) + \frac{(x-1)^2}{2\,!} f'(1) + \dots \qquad \dots(1)$$

Now $\quad f(x) = e^x,\ f'(x) = e^x, f''(x) = e^x$ etc.

Therefore $\quad f(1) = e, f'(1) = e, f''(1) = e$ etc.

Putting these values in (1), we have

$$e^x = e + (x - 1)\, e + \frac{(x-1)^2}{2\,!} e + \frac{(x-1)^3}{3\,!} e + \dots .$$

**Example 26.** *By Taylor's theorem, prove that*

$$\log(x + h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} + \dots .$$

**[Bhopal 1990, 98, 2001; Jabalpur 90; Ravishankar 2005S; Bilaspur 93; Indore 91; Sagar 98; Rewa 2000, 2003, 2004]**

**Solution.** We are to expand $\log(x + h)$ in ascending powers of $x$.

So let $f(h) = \log h$ then $f(x + h) = \log(x + h)$.

By Taylor's theorem, we get

$$\log(x + h) = f(x + h) = f(h) + x f'(h) + \frac{x^2}{2\,!} f''(h) + \frac{x^3}{3\,!} f'''(h) + \dots \qquad \dots(1)$$

Here

$\therefore \quad f(h) = \log h,$

$f'(h) = 1/h, f''(h) = -(1/h^2), f'''(h) = 2/h^3$, etc.

**Example 22.** *Expand $2x^3 + 7x^2 + x - 1$ in powers of $(x - 2)$ by Taylor's theorem.*
**[Jiwaji 1993, 99; Ravishankar 94, 2002, 2006S, 2007, 2009; Rewa 2000; Bhopal 92, 95, 2007; Jabalpur 91, 2000; Vikram 98, 2007; Bilaspur 2000, 2005; Sagar 2003; Indore 2005]**

**Solution.** Let $f(x) = 2x^3 + 7x^2 + x - 1.$ ...(1)

We can write $f(x)$ as follows :

$$f(x) = f\{2 + (x - 2)\}.$$

Now expanding $f\,[2 + (x - 2)]$ by Taylor's theorem in powers of $(x - 2)$, we have

$$f(x) = f\{2 + (x - 2)\}$$

$$= f(2) + (x - 2) f'(2) + \frac{(x-2)^2}{2\,!} f''(2) + \ldots . \quad ...(2)$$

Differentiating (1) successively, we have

$$f'(x) = 6x^2 + 14x + 1, f''(x) = 12x + 14, f'''(x) = 12,$$

$$f^{iv}(x) = 0, \text{ hence } f^{(n)}(x) = 0 \text{ when } n \geq 4.$$

Putting $x = 2$ in above, we have

$$f(2) = 2 \,.\, 2^3 + 7 \,.\, 2^2 + 2 - 1 = 45, f'(2) = 53$$

$$f''(2) = 38, f'''(2) = 12, f^{iv}(2) = 0, n \geq 14.$$

Now putting these values in (2), we have

$$f(x) = 45 + (x - 2) \,.\, 53 + \frac{(x-2)^2}{2\,!} \,.\, 38 + \frac{(x-2)^3}{3\,!} \,.\, 12$$

$$= 45 + 53\,(x - 2) + 19\,(x - 2)^2 + 2(x - 2)^3.$$

**Example 23.** *Expand $\sin x$ in powers of $(x - \frac{1}{2}\pi)$ by Taylor's theorem.*
**[Ravishankar 1991, 98; Vikram 92, 2004; Bilaspur 95, 99, 2006; Bhopal 97; Jablpur 98, 99, 2006; Indore 2000, 2003; Jiwaji 2000, 2001]**

**Solution.** Let $f(x) = \sin x$, we can write

$$f(x) = f\,[\tfrac{1}{2}\pi + (x - \tfrac{1}{2}\pi)].$$

By Taylor's theorem, we have

$$f(x) = f\,[\tfrac{1}{2}\pi + (x - \tfrac{1}{2}\pi)]$$

$$= f(\tfrac{1}{2}\pi) + (x - \tfrac{1}{2}\pi) f'(\tfrac{1}{2}\pi) + \frac{(x - \frac{1}{2}\pi)^2}{2\,!} f''(\tfrac{1}{2}\pi) + \ldots . \quad ...(1$$

Now $f(x) = \sin x,$ $f''(x) = -\sin x,$

$\therefore$ $f'(x) = \cos x,$ $f^{iv}(x) = \sin x.$

$f'''(x) = -\cos x,$

$\therefore$ $f(\tfrac{1}{2}\pi) = \sin \tfrac{1}{2}\pi = 1,$

$f'(\tfrac{1}{2}\pi) = 0,$ $f''(\tfrac{1}{2}\pi) = -1$

$f'''(\tfrac{1}{2}\pi) = 0,$ $f^{iv}(\tfrac{1}{2}\pi) = 1.$

Putting these values in (1), we have

$$\sin x = 1 + (x - \tfrac{1}{2}\pi) \,.\, 0 + \frac{(x - \frac{1}{2}\pi)^2}{2\,!} \,.\, (-1) + \frac{(x - \frac{1}{2}\pi)^3}{3\,!} \,.\, 0 + \frac{(x - \frac{1}{2}\pi)^4}{4\,!} \,.\, 1 +$$

$$= 1 - (1/2\,!)\,(x - \tfrac{1}{2}\pi)^2 - (1/4\,!)\,(x - \tfrac{1}{2}\pi)^4 \ldots .$$

**Example 14.** *Prove that* $(\sin^{-1} x)^2 = \frac{2}{2!}x^2 + \frac{2 \cdot 2^2}{4!}x^4 + \frac{2 \cdot 2^2 \cdot 4^2}{6!}x^6 + \ldots$

*and hence deduce* $\theta^2 = 2 \cdot \frac{\sin^2 \theta}{2!} + 2^2 \cdot \frac{2 \sin^4 \theta}{4!} + 2^2 \cdot 4^2 \cdot \frac{\sin^6 \theta}{6!} + \ldots$

[Vikram 1994; Ravishankar 2003]

**Solution.** Let $y = (\sin^{-1} x)^2, \qquad (y)_0 = (\sin^{-1} 0)^2 = 0.$

$\therefore \qquad y_1 = 2(\sin^{-1} x) \cdot \frac{1}{\sqrt{1-x^2}}, \qquad (y_1)_0 = 0$

or $\qquad (1-x^2) y_1^2 = 4y.$

Differentiating it, we get

$$(1-x^2) 2y_1 y_2 - 2xy_1^2 = 4y_1$$

or $$(1-x^2) y_2 - xy_1 = 2, \qquad (y_2)_0 = 2.$$

Differentiating $n$ times by Leibnitz's theorem, we get

$$(1-x^2) y_{n+2} - (2n+1) xy_{n+1} - n^2 y_n = 0.$$

Putting $x = 0$, $\qquad (y_{n+2})_0 = n^2 (y_n)_0.$ ...(1)

Putting $n = 1, 3, 5, \ldots$ in (1), we have

$$(y_3)_0 = (y_5)_0 = (y_7)_0 = \ldots = 0 \qquad [\because (y_1)_0 = 0]$$

Puting $n = 2, 4, 6$ in (1), again, we get

$$(y_4)_0 = 2^2 (y_2)_0 = 2^2 \cdot 2$$

$$(y_6)_0 = 4^2 \cdot (y_4)_0 = 4^2 \cdot 2^2 \cdot 2 \text{ etc.}$$

Hence substituting these values in Maclaurin's theorem, we get

$$(\sin^{-1} x)^2 = 0 + x \cdot 0 + \frac{x^2}{2!} \cdot 2 + \frac{x^4}{4!} \cdot 2^2 \cdot 2 + \frac{x^5}{5!} \cdot 0 + \frac{x^6}{6!} 4^2 \cdot 2^2 \cdot 2 + \ldots$$

$$= \frac{2x^2}{2!} + \frac{2 \cdot 2^2 \cdot x^4}{4!} + \frac{2 \cdot 2^2 \cdot 4^2 \cdot x^6}{6!} + \ldots \qquad \ldots(2)$$

Now putting $x = \sin \theta$ in (2), we get

$$\theta^2 = \frac{2 \sin^2 \theta}{2!} + 2^2 \cdot \frac{2 \sin^4 \theta}{4!} + 2^2 \cdot 4^2 \cdot \frac{2 \sin^6 \theta}{6!} + \ldots$$

**Example 15.** *Expand* $\sin (a \sin^{-1} x)$ *by Maclaurin's theorem as far as* $x^5$. *Hence expand* $\sin m\theta$ *in powers of* $\sin \theta$.

**Solution.** Let $\qquad y = \sin (a \sin^{-1} x).$

Proceeding as in Ex. 4 of Ch. 5, § 5.3.

We get $\qquad (y)_0 = 0, (y_1)_0 = a, (y_2)_0 = (y_4)_0 = (y_6)_0 = 0$

$$(y_3)_0 = (1^2 - a^2) a, (y_5)_0 = (3^2 - a^2)(1 - a^2) a.$$

Hence from Maclaurin's theorem,

$$\sin (a \sin^{-1} x) = ax + \frac{a(1^2 - a^2)}{3!} x^3 + \frac{a(1^2 - a^2)(3^2 - a^2) x^5}{5!} + \ldots .$$

Putting $a = m$ and $x = \sin \theta$ on both sides, we have

$$\sin (m\theta) = m \sin \theta + \frac{m(1^2 - m^2)}{3!} \sin^3 \theta + \frac{m(1^2 - m^2)(3^2 - m^2)}{5} \sin^5 \theta + \ldots .$$

**Example 30.** *Use Taylor's theorem to prove that*

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin z \,.\, \frac{\sin z}{1} - (h\sin z)^2\frac{\sin 2z}{2} + (h\sin z)^3\frac{\sin 3z}{3} - \ldots + (-1)^{n-1}(h\sin z)^n\frac{\sin nz}{n} + \ldots$$

*where* $z = \cot^{-1}x$.

**[Ravishankar 1990, 92, 93, 95; Rewa 90, 95, 97, 2003, 2005; Bilaspur 2001, 2003; Indore 90, 93, 98, 2003; Jabalpur 92, 95, 97; Bhopal 84; Vikram 79; Sagar 2005]**

**Solution.** First of all we observe that we are to expand $\tan^{-1}(x+h)$ in powers of $h$. So let $f(x) = \tan^{-1}x$, then $f(x+h) = \tan^{-1}(x+h)$.

By Taylor's theorem, we have

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2\,!}f''(x) + \ldots + \frac{h^n}{n\,!}f^n(x) + \ldots \quad \ldots(1)$$

Now $\quad f(x) = \tan^{-1}x.$

$$f^{(n)}(x) = D^n(\tan^{-1}x) = (-1)^{n-1}(n-1)\,!\,\sin^n z\sin nz$$

where $\quad z = \cot^{-1}x.$

For $n=1$, $\quad f'(x) = (-1)^0\,0\,!\,\sin z\sin z = \sin z\sin z$

For $n=2$, $\quad f''(x) = -(1)\,!\,\sin^2 z\sin 2z$

For $n=3$, $\quad f'''(x) = 2\,!\,\sin^3 z\sin 3z$ etc.

Substituting these values in (1), we have

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin z\sin z - (h^2/2\,!)\sin^2 z\sin 2z + \frac{h^3}{3\,!}2\,!\,\sin^3 z\sin 3z + \ldots + \frac{h^n}{n\,!}(-1)^{n-1}(n-1)\,!\,\sin^n z\sin nz + \ldots$$

$$= \tan^{-1}x + h\sin z\frac{\sin z}{1} - (h\sin z)^2\frac{\sin 2z}{2} + (h\sin z)^3\frac{\sin 3z}{3} - \ldots + (-1)^{n-1}(h\sin z)^n\frac{\sin nz}{n} + \ldots$$

or $$\tan^{-1}(x+h) = \tan^{-1}x + h\sin\cot^{-1}x\,.\,\frac{\sin\cot^{-1}x}{1} - (h\sin\cot^{-1}x)^2\,.\,\frac{\sin 2\cot^{-1}x}{2} + (h\sin\cot^{-1}x)^3\,.\,\frac{\sin 3\cot^{-1}x}{3} - \ldots.$$

**Example 31.** *Prove that*

(i) $f\left(\dfrac{x^2}{1+x}\right) = f(x) - \dfrac{x}{1+x}f'(x) + \dfrac{x^2}{(1+x)^2}\dfrac{f''(x)}{2\,!} - \ldots$

(ii) $f(mx) = f(x) + (m-1)\,x\,f'(x) + \dfrac{1}{2\,!}(m-1)^2x^2f''(x) + \ldots$

**[Bilaspur 1992; Jiwaji 98; Bhopal 2005]**

(iii) $f(x) = f(0) + x\,f'(x) - \dfrac{x^2}{2\,!}f''(x) + \dfrac{x^3}{3\,!}f'''(x) - \ldots$

**Solution.** (i) We are to expand $f\left(\dfrac{x^2}{1+x}\right)$ in powers of $\left(-\dfrac{x}{x+1}\right)$.

Putting the values of $a_0, a_1, a_2, \ldots$ in (7), we get

$$y = 0 + 1 \,.\, x + 0 \,.\, x^2 + \frac{2}{3} x^3 + 0 \,.\, x^4 + \frac{4}{5} \cdot \frac{2}{3} x^3 + \ldots$$

$$= x + (2/3)\, x^3 + (8/15)\, x^5 + \ldots .$$

**Example 21.** *If* $y = e^{m \tan^{-1} x} = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$ *prove that*

(*i*) $y = 1 + mx + \dfrac{m^2}{2\,!} x^2 + \dfrac{m(m^2 - 2)}{3\,!} x^3 + \dfrac{m^2 (m^2 - 8)}{4\,!} x^4 + \ldots .$ **[Jiwaji 2004]**

(*ii*) $(n + 1)\, a_{n+1} + (n - 1)\, a_{n-1} = m a_n.$ **[Jiwaji 2004]**

**Solution.** (i) Let $\quad y = e^{m \tan^{-1} x}$ ...(1)

$\therefore \quad y_1 = e^{m \tan^{-1} x} \dfrac{m}{1 + x^2}$ ...(2)

$$(1 + x^2)\, y_2 + 2xy_1 = my_1. \quad \text{...(3)}$$

Differentiating (3) $n$ times, we get

$$(1 + x^2)\, y_{n+2} + \{2(n + 1)\, x - m\}\, y_{n+1} + n(n + 1)\, y_n = 0. \quad \text{...(4)}$$

Putting $x = 0$ in (4), we get

$$(y_{n+2}) = m(y_{n+1})_0 - n(n + 1)\, (y_n)_0. \quad \text{...(5)}$$

Putting $x = 0$ in (1), (2), (3) and (4) and $n = 1, 2, 3, \ldots$ in (5), we get

$$(y)_0 = 1, \qquad (y_1)_0 = m,$$

$$(y_2)_0 = m^2, \qquad (y_3)_0 = m \,.\, m^2 - 2m = m(m^2 - 2),$$

$$(y_4)_0 = m^2\, (m^2 - 8) \text{ etc.}$$

Now by Maclaurin's theorem, we get

$$e^{m \tan^{-1} x} = 1 + mx + \frac{m^2}{2\,!} x^2 + \frac{m(m^2 - 2)}{3\,!} x^3 + \ldots .$$

(ii) We are given that

$$e^{m \tan^{-1} x} = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

$$= (y)_0 + x(y_1)_0 + \frac{x^2}{2\,!} (y_2)_0 + \ldots + \frac{x^n}{n\,!} (y_n)_0 + \ldots$$

Comparing the coefficients of $x^n$, we have

$$\therefore \quad a_n = \frac{(y_n)_0}{n\,!} \quad \text{or} \quad (y_n)_0 = a_n \,.\, n\,!. \quad \text{...(6)}$$

Putting $(n + 1)$ and $(n - 1)$ in place of $n$ in (6), we get

$$(y_{n+1})_0 = a_{n+1}\, (n + 1)\,! \quad \text{...(7)}$$

and $$(y_{n-1})_0 = a_{n-1}\, (n - 1)\,!. \quad \text{...(8)}$$

Putting $n - 1$ in place of $n$ in (5), we get

$$(y_{n+1})_0 = m(y_n)_0 - n(n - 1)\, (y_{n-1})_0.$$

Putting the values of $(y_n)_0, (y_{n+1})_0, (y_{n-1})_0$ from (6), (7), (8), we have

$$a_{n+1}\, (n + 1)\,! = m \,.\, a_n \,.\, n\,! - (n - 1)\, a_{n-1} \cdot (n - 1)\,!$$

or $$(n + 1)\, a_{n+1} = m a_n - (n - 1)\, a_{n-1}$$

or $$(n + 1)\, a_{n+1} = (n - 1)\, a_{n-1} = m a_n.$$

Differentiating (3), we get

$$e^y (y_4 + 3y_2^2 + 3y_1 y_3 + 3y_1^2 \cdot y_2) + e^y \cdot y_1 (y_3 + 3y_1 y_2 + y_1^3)$$
$$= 24 \sec^4 x \tan x - 8 \sec^2 x \tan x$$

or $$e^y (y_4 + 4y_1 y_2 + 3y_2^3 + 6y_1^2 y_2 + y_1^4) = (24 \sec^4 x - 8 \sec^2 x) \tan x.$$

Putting $x = 0$ in (4),

$$(y_4)_0 + 4 \cdot 1 \cdot 4 + 3(-1)^2 + 6 \cdot 1 (-1) + 1 = 0 \Rightarrow (y_4) = -14.$$

Substituting the values, by Maclaurin's theorem, we get

$$\log (1 + \tan x) = 0 + x \cdot 1 + \frac{x^2}{2!}(-1) + \frac{x^3}{3!} \cdot 4 + \frac{x^4}{4!}(-14) + \ldots$$
$$= x - \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{7}{12}x^4 + \ldots .$$

**Example 11.** *Expand sinh x cos x, to fifth powers of x.*

**Solution.** Here $y = \sinh x \cos x$, $\therefore\ (y_0) = \sinh 0 \cos 0 = 0 \cdot 1 = 0$

$$y_1 = \cosh x \cos x - \sinh x \sin x. \qquad \ldots(1)$$

Putting $x = 0$ in (1), we have

$$(y_1)_0 = \cosh 0 \cos 0 - \sinh 0 \sin 0$$

or $$(y_1)_0 = 1 \cdot 1 - 0 \cdot 0 \Rightarrow (y_1)_0 = 1.$$

Differentiating (1), we have

$$y_2 = \sinh x \cos x - \cosh x \sin x - \cosh x \sin x - \sinh x \cos x$$

or $$y_2 = -2 \cosh x \sin x, \qquad \therefore\ (y_2)_0 = 0$$

$$y_3 = -2 \sinh x \sin x - 2 \cosh x \cos x, \qquad \therefore\ (y_3)_0 = -2$$

$$y_4 = -2 \cosh x \sin x - 2 \sinh x \cos x + 2 \cosh x \sin x - 2 \sinh x \cos x$$

or $$y_4 = -4 \sinh x \cos x$$

or $$y_4 = -4y_1, \qquad \therefore\ (y_4)_0 = -4(y)_0 = 0$$

$$y_5 = -4y_1, \qquad \therefore\ (y_5)_0 = -4.$$

Substituting the values in Maclaurin's theorem, we have

$$\sinh x \cos x = 0 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!} \cdot (-2) + \frac{x^4}{4!} \cdot 0 + \frac{x^5}{5!} \cdot (-4) + \ldots$$
$$= x - (2/3!)\, x^3 - (4/5!)\, x^5 + \ldots .$$

**Example 12.** *Expand $e^{x \cos x}$ by Maclaurin's theorem.*

**[Ravishankar 2006; Indore 1990, 92, 94; Vikram 81; Jabalpur 91]**

**Solution.** Let $y = e^{x \cos x}$. Then

$$y_1 = e^{x \cos x} (1 \cdot \cos x - x \sin x) = (y \cos x - x \sin x)$$
$$y_2 = y_1 (\cos x - x \sin x) + y (-\sin x - 1 \cdot \sin x - x \cdot \cos x)$$
$$= y_1 (\cos x - x \sin x) - y (2 \sin x + x \cos x)$$
$$y_3 = y_2 (\cos x - x \sin x) + y_1 (-\sin x - 1 \cdot \sin x - x \cdot \cos x)$$
$$- y_1 (2 \sin x + x \cos x) - y (2 \cos x + 1 \cdot \cos x - x \sin x)$$
$$= y_2 (\cos x - \sin x) - 2y_1 (2 \sin x + x \cos x) - y (3 \cos x - x \sin x)$$

$$\therefore \quad [x+\sqrt{1+x^2}]^m = 1 + mx + \frac{m^2}{2!}x^2 + \frac{m(m^2-1^2)}{3!}x^3 + \frac{m^2(m^2-2^2)}{4!}x^4 + \dots .$$

The general term $= (x^n/n!)\,(y_n)_0$, where $(y_n)_0$ is given by (A).

**Example 18.** *Expand* $[\log\{x+\sqrt{1+x^2}\}]^2$ *in asscending powers of* $x$.

**[Jiwaji 1989]**

**Solution.** Let $\qquad y = [\log\{x+\sqrt{1+x^2}\}]^2.$

Proceeding as in Ex. 2 of Ch. 5 § 5.3, we have

$$(y)_0 = 0,\ (y_1)_0 = 0,\ (y_2)_0 = 2,$$

$$(y_3)_0 = 0,\ (y_4)_0 = -2^2 \cdot 2,\ (y_5)_0 = 0,\ (y_6)_0 = 4^2 \cdot 2^2 \cdot 2, \text{ etc.}$$

Now by Maclaurin's theorem, we get

$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \dots$$

$$[\log\{x+\sqrt{1+x^2}\}]^2 = 0 + x \cdot 0 + \frac{x^2}{2!}\cdot 2 + \frac{x^3}{3!}\cdot 0 + \frac{x^4}{4!}(-2^2 \cdot 2) + \dots$$

$$= \frac{2x^2}{2!} - \frac{2 \cdot 2^2 \cdot x^4}{4!} + 2 \cdot 2^2 \cdot 4^2 \cdot \frac{x^6}{6!} - \dots .$$

**Example 19.** *If* $y = e^{\tan^{-1}x}$, *prove that*

$$(1+x^2)\,y_{n+2} + \{2(n+1)x - 1\}\,y_{n+1} + (n+1)\,y_n = 0.$$

*Hence or otherwise find out the coefficient of* $x^5$ *if* $e^{\tan^{-1}x}$ *is expanded in powers of* $x$.

**Solution.** Here $\qquad y = e^{\tan^{-1}x}.$

Proceeding as in Ex. 11 of Ch. 5 § 5.3, we have

$$(y)_0 = e^0 = 1 \qquad [\because \tan^{-1} 0 = 0]$$

$$(y_1)_0 = \frac{1}{1+0} = 1,\ (y_2)_0 = (y_1)_0 = 1$$

and

$$(y_{n+2})_0 = (y_{n+1})_0 - n(n+1)\,(y_n)_0.$$

Putting $n = 1, 2, 3$

$$(y_3)_0 = (y_2)_0 - 1 \cdot 2\,(y_1)_0 = 1 - 2 \cdot 1 = -1$$

$$(y_4)_0 = (y_3)_0 - 2 \cdot 3\,(y_2)_0 = -1 - 6(1) = -7$$

$$(y_5)_0 = (y_4)_0 - 3 \cdot 4\,(y_3)_0 = -7 - 12(-1) = 5.$$

Coefficient of $x^5$ in the expansion of $e^{\tan^{-1}x} = \dfrac{(y_5)}{5!} = \dfrac{5}{5!} = \dfrac{1}{24}.$

**Example 20.** *If* $\quad y = (\sin^{-1}x)/\sqrt{1-x^2}$ *where* $\quad -1 < x < 1$ *and* $-\pi/2 < \sin^{-1}x < \pi/2$, *prove that*

$$(1-x^2)\,y_{n+1} - (2n+1)\,xy_n - n^2 y_{n-1} = 0.$$

*Also if* $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$

*Prove that* $\qquad (n-1)\,a_{n+1} = na_{n-1}$

*hence obtain the general term of the expansion. Find also the first three terms in the expansion.*

$Ho\,x \to \log \sec x$.



**Example 8.** *Expand by Maclaurins theorem* $\frac{e^x}{1+e^x}$ *as far as the term* $x^3$.

**[Bilaspur 1994]**

**Solution.** Here

$$y=\frac{e^x}{1+e^x}=\frac{1+e^x-1}{1+e^x}=1-\frac{1}{1+e^x}, \qquad \therefore\ (y)_0=\frac{e^0}{1+e^0}=\frac{1}{2}$$

$$y_1=0+\frac{e^x}{(1+e^x)^2}=\frac{e^x}{1+e^x}\cdot\frac{1}{1+e^x}=y(1-y)=y-y^2, \qquad \therefore\ (y_1)_0=\frac{1}{4}$$

$$\therefore \quad y_2=y_1-2yy_1, \qquad \therefore\ (y_2)_0=\tfrac{1}{4}-2\,.\tfrac{1}{2}\cdot\tfrac{1}{4}=0$$

$$y_3=y_2-2yy_2-2y_1^2, \qquad \therefore\ (y_3)_0=-1/8 \text{ etc.}$$

$\therefore$ By Maclaurin's theorem, we have

$$\frac{e^x}{1+e^x}=\frac{1}{2}+x\,.\frac{1}{4}+\frac{x^2}{2\,!}\,.\,0+\frac{x^3}{3\,!}\left(-\frac{1}{8}\right)+\ldots$$

$$=\frac{1}{4}+\frac{x}{4}-\frac{x^3}{8}+\ldots.$$

**Example 9.** *Find the first five terms in the expansion of* $\log(1+\sin x)$ *by Maclaurin's theorem.* **[Jabalpur 1992; Bilaspur 92, 98, 2001; Rewa 93; Vikram 90; Indore 96, 2001]**

**Or**

*Use Maclaurin's theorem to prove that*

$$\log(1+\sin x)=x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}+\frac{x^5}{24}+\ldots.$$

**[Bilaspur 2002; Jabalpur 2002; Rewa 2007]**

**Solution.** Let $y=\log(1+\sin x)$, then $(y)_0=\log(1+0)=0$.

Now $\qquad e^y=1+\sin x.$

Differentiating, we get

$$e^y\,.\,y_1=\cos x. \qquad \ldots(1)$$

Putting $x=0$ in (1), we get

$$e^0\,.\,(y_1)_0=1 \Rightarrow (y_1)_0=1.$$

Differentiating (1) again, we get

$$e^y\,.\,y_2+e^y\,.\,y_1^2=-\sin x$$

or $$e^y\,.\,y_2+y_1\cos x=-\sin x, \qquad \text{[from (1)]} \qquad \ldots(2)$$

Putting $x=0$ in (2),

$$e^0\,(y_2)_0+1\,.\,1=0 \Rightarrow (y_2)_0=-1.$$

Differentiating (2) again, we get

$$e^y\,y_3+e^y\,.\,y_1y_2+y_2\cos x-y_1\sin x=-\cos x$$

or $$e^y\,y_3+y_2\cos x+y_2\cos x-y_1\sin x=-\cos x \qquad \text{[from (1)]}$$

or $$e^y\,y_3+2y_2\cos x-y_1\sin x=-\cos x. \qquad \ldots(3)$$

We can write $f\left(\frac{x^2}{1+x}\right)=f\left[x+\left(\frac{-x}{1+x}\right)\right]$.

Now expanding by Taylor's theorem in powers of $\left(\frac{-x}{1+x}\right)$, we get

$$f\left(\frac{x^2}{1+x}\right)=f\left[x+\left(-\frac{x}{1+x}\right)\right]=f(x)+\left(\frac{-x}{1+x}\right)f'(x)+\frac{1}{2\,!}\left(\frac{-x}{1+x}\right)^2 f''(x)+\ldots$$

$$=f(x)-\frac{x}{1+x}f'(x)+\frac{x^2}{(1+x)^2}\cdot\frac{f''(x)}{2\,!}-\ldots .$$

**(ii)** We are to expand $f(mx)$ in powers of $(m-1)\,x$. Now we can write

$$f(mx)=f\,[x+(m-1)\,x].$$

Expanding by Taylor's theorem in powers of $(m-1)\,x$, we get

$$f(mx)=f\,[x+(m-1)\,x]$$

$$=f(x)+(m-1)\,x\,f'(x)+(1/2\,!)\,\{(m-1)\,x\}^2 f''(x)+\ldots$$

**(iii)** We can write $f(0)=f\,[x+(-x)]$.

Now expanding by Taylor's theorem in powers of $(-x)$, we get

$$f(0)=f\,[x+(-x)]=f(x)+(-x)\,f'(x)+\frac{(-x)^2}{2\,!}f''(x)+\frac{(-x)^3}{3\,!}f'''(x)+\ldots$$

or $$f(0)=f(x)-x\,f'(x)+\frac{x^2}{2\,!}f''(x)+\frac{x^3}{3\,!}f'''(x)\ldots$$

or $$f(x)=f(0)+x\,f'(x)-\frac{x^2}{2\,!}f''(x)+\frac{x^3}{3\,!}f'''(x)-\ldots .$$

## EXERCISE 4

**Expand the following functions by Maclaurin's theorem in ascending power of $x$.**

**1.** $\cos x$.

**2.** $\sec x$.

**3.** $\cos^{-1} x$. **[Bilaspur 2009]**

**4.** $\tan x$. **[Vikram 1995; Jabalpur 2003]**

**5.** $e^{\sin x}$.

**6.** $e^{x\cos x}$.

**7.** $e^{a\cos^{-1} x}$

**8.** $e^x \log_e(1+x)$.

**9.** Show that (Use of Taylor's series)

(a) $$e^{ax}\sin bx=bx+abx^2+\frac{3a^2b-b^3}{3\,!}x^3+\ldots+\frac{(a^2+b^2)^{n/2}}{n\,!}x^n\sin\left(n\tan^{-1}\frac{b}{a}\right)+\ldots$$

(b) $$e^{ax}\cos bx=1+ax+\frac{a^2-b}{2\,!}x^2+\frac{a^2(a^2-5b)^2}{3\,!}x^3+\ldots$$

$$+\frac{(a^2+b^2)^{n/2}}{n\,!}x^n\cos\left(n\tan^{-1}\frac{b}{a}\right)+\ldots$$

(c) $$\log(1+e^x)=\log 2+\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{192}x^4+\ldots$$ **[Sagar 1998; Vikram 97]**

(d) $$\log(x+h)=\log_e h+\frac{x}{h}-\frac{x^2}{2h^2}+\frac{x^3}{3h^3}-\ldots$$

**[Bilaspur 1993; Bhopal 90; Jabalpur 90; Indore 91; Sagar 98; Ravishankar 2003S]**

(e) $\log (x+h) = \log_e x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \dots$

(f) $(x+h)^{-1} = \frac{1}{x} - \frac{h}{x^2} + \frac{h^2}{x^3} - \frac{h^3}{x^4} + \dots$ **[Bhopal 1998; Sagar 91; Ravishankar 2006]**

(g) $\log (1+\sin^2 x) = x^2 - \frac{5}{6}x^4 + \dots$

(h) $e^x \sin x = x + x^2 + \frac{2}{3\,!}x^3 - \frac{2^2}{5\,!}x^5 - \dots + \sin\frac{n\pi}{4}\cdot\frac{2^{n/2}}{n\,!}x^n + \dots$ **[Sagar 2003]**

(i) $\cos x \sinh x = x - \frac{2}{3\,!}x^3 - \frac{4}{5\,!}x^5 + \dots$

(j) $\sin (m \sin^{-1} x) = mx + \frac{m(1^2 - m^2)}{3\,!}x^3 + \frac{m(1^2-m^2)(3^2-m^2)}{5\,!}x^5 + \dots$ **[Jiwaji 2005]**

10. (a) Prove that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

Hence expand $\sin^{-1}\left(\frac{2x^2}{1+x^4}\right)$ and find its general term. **[Jiwaji 1991]**

(b) Prove that $\tan^{-1}(x+h) = \tan^{-1} x + \frac{h}{1+x^2} - \frac{xh^2}{(1+x^2)^2} + \dots$ **[Rewa 2006]**

11. Use Maclaurin's theorem to prove that

$$[x+(1+x^2)]^m = 1 + mx + \frac{m^2x^2}{2\,!} + \frac{m\,(m^2-1)^2\,x^3}{3\,!} + \frac{m^2\,(m^2-2^2)}{4\,!}x^4 + \dots$$

and by using $a^m = 1 + m \log_e a + \frac{m^2}{2\,!}(\log_e a)^2 + \dots$ to deduce the expansions of

(i) $\log [x + \sqrt{1+x^2}]$; (ii) $[\log \{x+\sqrt{1+x^2}\}]^2$.

12. Expand $\sin^{-1}(x+h)$ in powers of $x$ by Taylor's theorem upto the terms of $x^3$.

13. Expand $\sin\left(\frac{\pi}{4}+\theta\right)$ in powers of $\theta$ by Taylor's theorem.

14. Expand $\tan^{-1} x$ in powers of $\left(x - \frac{\pi}{4}\right)$ by Taylor's theorem. **[Ravishankar 1991S]**

15. Expand $\tan x$ in powers of $\left(x - \frac{\pi}{4}\right)$. **[Jabalpur 2001]**

16. (a) State and prove Maclaurin's theorem. **[Bhopal 1991, 94, 95; Rewa 92; Bilaspur 95]**

(b) State and prove Taylor's theorem.

**[Vikram 1990; Bhopal 90, 2003; Ravishankar 91; Indore 96]**

(c) State Taylor's theorem. **[Bhopal 2007]**

## ANSWERS

1. $1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \frac{x^6}{6\,!} + \dots + (-1)^m \frac{x^{2m}}{(2m)\,!} + \dots$

2. $1 + \frac{x^2}{2\,!} + \frac{5x^4}{4\,!} + \frac{61x^6}{6\,!} + \dots$

3. $\frac{\pi}{2} - x - \frac{1^2 . x^3}{3\,!} - \frac{3^2 . 1^2 . x^5}{5\,!} - \dots - \frac{(n-2)^2 . (n-4)^2 \dots 3^2 . 1^2 . x^n}{n\,!} - \dots,$

when $n$ is even, $n^{th}$ term will be zero.

**4.** $x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots + (-1)^{n-1} \dfrac{x^{2n-1}}{2n-1} + \ldots$

**5.** $1 + x + \dfrac{x^2}{2} - \dfrac{x^4}{8} + \ldots$

**6.** $1 + x + \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{11x^4}{24} - \dfrac{x^5}{5} + \ldots$

**7.** $\dfrac{a\pi}{2} \left\{ 1 - 2x + \dfrac{a^2 x^2}{2\,!} - a\,(1 + a^2)\, \dfrac{x^3}{3\,!} + (2^2 + a^2)\, \dfrac{a^3 x^4}{5\,!} + \ldots \right\}$

**8.** $x + \dfrac{x^2}{2\,!} + \dfrac{2x^3}{3\,!} + \dfrac{9x^5}{5\,!} + \ldots$

**10.** $2 \left[ x^2 - \dfrac{x^6}{3} + \dfrac{x^{10}}{5} - \ldots + (-1)^{n-1} \dfrac{x^{4n-2}}{2^{n-1}} + \ldots \right]$

**11.** (i) $x - \dfrac{1^2}{3\,!} x^3 + \dfrac{1^2 . 3^2}{5\,!} x^5 - \ldots$ (ii) $2 \left[ \dfrac{x^2}{2\,!} - \dfrac{2^2}{4\,!} x^4 + \ldots \right]$

**12.** $\sin^{-1} h + x\,(1 - h^2)^{-1/2} + (x^2/2\,!)\, h\,(1 - h^2)^{-3/2} + (x^3/3\,!)\, \{(1 - h^2)^{-3/2}\, (1 + 2h^3)\} + \ldots$

**13.** $\dfrac{1}{\sqrt{2}} \left( 1 + \theta - \dfrac{\theta^2}{2\,!} - \dfrac{\theta^3}{3\,!} + \dfrac{\theta^4}{4\,!} + \dfrac{\theta^5}{5\,!} + \ldots \right)$

**14.** $\tan^{-1} \dfrac{\pi}{4} + \dfrac{x - \pi/4}{1 + \pi^2/16} - \dfrac{\pi\,(x - \pi/4)^2}{2\,(1 + \pi^2/16)} + \ldots$

**15.** $1 + 2\left(x - \dfrac{\pi}{4}\right) + 2\left(x - \dfrac{\pi}{4}\right)^2 + \dfrac{8}{3}\left(x - \dfrac{\pi}{4}\right)^3 + \dfrac{10}{3}\left(x - \dfrac{\pi}{4}\right)^4 + \ldots .$