A Glimpse of Algorithmic Mechanism Design : Knapsack Allocation

Lecture 12.2

Recall the Knapsack problem:

<u>Input</u>: n items with weights $w_1, \ldots, w_n$ and valuations $v_1, \ldots, v_n$; a knapsack of size W.

<u>Goal</u>: Find $S \subseteq [n]$ such that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} v_i$ is maximized.

This problem is weakly NP-complete.

We assume each item corresponds to a player and the valuation is the type (i.e. private information) of the player.

Crucial observation: Each player's type is a single real number.

Set of all allocations:

$$\mathcal{K} = \left\{ (x_1, \ldots, x_n) \in \{0,1\}^n : \sum_{i=1}^{n} x_i w_i \leq W \right\}$$

$i \in [n]$,

$$\mathcal{R}_i = \{ (x_1, \ldots, x_n) \in \mathcal{R} \mid x_i = 1 \}$$

This is a single parameter domain.

Is the Knapsack allocation rule monotone?

Of course it is !!

<u>Problem:</u> Computing knapsack allocation is NP-complete.

Real-world example of Knapsack problem:

W minutes are allocated for showing Ads.

n potential advertisements competing for slot.

w_i is the duration of Ad i.

v_i is the valuation/utility of Ad i (known only to player i).

Two step design rule for algorithmic mechanism design.

(i) Assume DSIC for free.

(ii) Design an allocation rule which (i) can be computed in polynomial time, (ii) approximates our optimal objective value, (iii) monotone.

Greedy allocation rule for Knapsack:

Delete all items having weight more than W.

Sort the items by valuation/weight. Let us assume w.l.o.g by renaming that

$$\frac{v_1}{w_1} \geq \frac{v_2}{w_2} \geq \frac{v_3}{w_3} \geq \cdots \geq \frac{v_n}{w_n}$$

Pick items till there is space. Let the set of items picked $S = \{1, 2, \ldots, i^*\}$ i.e. $\sum_{j=1}^{i^*} w_j \leq W$, $\sum_{j=1}^{i^*+1} w_j > W$.

Output S if $\sum_{j=1}^{i^*} v_j \geq v_{i^*+1}$; otherwise output i^*+1.

This allocation rule can be computed in poly-nomial time.

The allocation rule is clearly monotone.

Theorem: $k^{greedy}(\cdot)$ has an approximation factor of $\frac{1}{2}$.

proof:

$$ALG \geq \max\left\{\sum_{j=1}^{i^*} v_j, v_{i^*+1}\right\}$$

$$\geq \frac{\sum_{j=1}^{i^*+1} v_j}{2}$$

$$\geq \frac{OPT}{2}$$

$\blacksquare$