Properties of Stable Matching

Lecture 12.5

Men proposing deferred acceptance algorithm outputs the men-optimal stable matching.

For a man m, let us define h(m) to be the woman whom m prefers most among all his partners in all stable matching.

Theorem: In the matching M output by the men-proposing deferred acceptance algorithm, every man m is matched with $h(m)$.

Proof: $R_i = \{ (a,b) :$ the woman b has rejected the man a in the first i iterations$\}$

For an instance, suppose the algorithm runs for k iterations.

Claim: $i \in \{0, 1, \ldots, k\}$, for every $(a,b) \in R_i$, there is no stable matching where man a is matched with the woman b.

Proof by induction on i.

Base case: $i = 0$, $R_0 = \phi$, the claimed statement is vacuously true.

Inductive step: Let us assume the statement for $R_0, R_1, \ldots, R_i$.

Suppose a man a proposes a woman b in the $(i+1)$-th iteration.

case I: b was unmatched and has accepted a.

Then $R_{i+1} = R_i$ and the proof follows from induction hypothesis.

case II: b has rejected some man a' who could be a also.

$$R_{i+1} = R_i \cup \{(a', b)\}$$

We only need to prove that there is no stable matching containing (a', b).

<u>sub-case I:</u> $a' \neq a$

Then we have $a >_b a'$

For the sake of finding a contradiction, let us assume that there exists a stable matching M' which contains (a', b)

$b >_a M'(a)$, $a >_b a'$

$a: \_\_\_\_ > b > \cdots$

$b: \cdots > a > \cdots > a' > \cdots$

(a,b) form a blocking pair for M' which contradicts our assumption that M' is stable.

$a'': \longrightarrow b > \cdots$ *

$b: \cdots > a'' > \cdots > a > \cdots$

$a: \cdots > b > \cdots$

subcase II: $a' = a$

If possible, suppose there exists a stable matching M' containing (a,b)

$b >_{a''} M'(a'')$ $\quad a'' >_b a = M'(b)$

(a'', b) form a blocking pair for the matching

M' which contradicts our assumption that M' is stable. $\blacksquare$

Green book of stable matching 1989

Matching Manlove 2007