Pure Strategy Bayesian Nash Equilibrium

Lecture 8.3

A pure strategy Bayesian Nash equilibrium of a Bayesian game $\Gamma = \langle N, (\Theta_i)_{i\in N}, (S_i)_{i\in N}, P, (u_i)_{i\in N} \rangle$ is a pure strategy profile $(s_1^*, \ldots, s_n^*)$, $s_i^* : \Theta_i \to S_i$ if for all $i \in N$ and all $s_i : \Theta_i \to S_i$, we have

$$U_{\theta_i}\left((s_i^*, s_{-i}^*)\right) \geq U_{\theta_i}(s_i, s_{-i}^*) \quad \forall \theta_i \in \Theta_i$$

Equivalently,

$$\mathbb{E}_{\theta_{-i}}\left[u_i\left(\theta_i, \theta_{-i}, s_i(\theta_i), s_{-i}(\theta_{-i})\right) \middle| \theta_i \right] \geq \mathbb{E}_{\theta_{-i}}\left[u_i\left(\theta_i, \theta_{-i}, a_i, s_{-i}(\theta_{-i})\right) \middle| \theta_i \right]$$

$$\forall i \in N \quad \forall a_i \in S_i \quad \forall \theta_i \in \Theta_i$$

<u>Theorem:</u> In the Bayesian game corresponding to the first price auction, each buyer bidding half of their valuation forms a pure strategy Bayesian Nash equilibrium under the following assumptions:

(i) We have only 2 buyers.

(ii) Each buyer's valuation is distributed uniformly in $[0,1]$.

(iii) Each buyer is "risk neutral": bid $b_i(\theta_i)$ of player $i \in [2]$ is of the form $\alpha_i \theta_i$ for some $\alpha_i \in [0,1]$.

<u>Proof</u>: Utility of player 1 :

$$u_1(\theta_1, \theta_2, b_1, b_2) = (\theta_1 - b_1(\theta_1)) \Pr[b_1(\theta_1) > b_2(\theta_2)]$$

$$= (\theta_1 - b_1(\theta_1)) \Pr[b_1(\theta_1) > \alpha_2 \theta_2]$$

$$= (\theta_1 - b_1(\theta_1)) \Pr\left[\theta_2 \leqslant \frac{b_1(\theta_1)}{\alpha_2}\right]$$

$$u_1(\theta_1, \theta_2, b_1, b_2) = \begin{cases} \theta_1 - b_1(\theta_1) & \text{if } b_1(\theta_1) \geqslant \alpha_2 \\ (\theta_1 - b_1(\theta_1)) \cdot \dfrac{b_1(\theta_1)}{\alpha_2} & \text{if } b_1(\theta_1) < \alpha_2 \end{cases} \qquad [\because \theta_2 \sim U([0,1])]$$

$b_1^*(\theta_1)$ which maximizes $u_1(\theta_1, \theta_2, b_1, b_2)$ is

$$b_1^*(\theta_1) = \begin{cases} \dfrac{\theta_1}{2} & \text{if } \dfrac{\theta_1}{2} < \alpha_2 \\ \alpha_2 & \text{if } \dfrac{\theta_1}{2} \geqslant \alpha_2 \end{cases}$$

Doing the same calculation we get that the following $b_2^*(\theta_2)$ maximizes $u_2(\theta_1, \theta_2, b_1, b_2)$

$$b_2^*(\theta_2) = \begin{cases} \frac{\theta_2}{2} & \text{if } \frac{\theta_2}{2} < \alpha_1 \\ \alpha_1 & \text{if } \frac{\theta_2}{2} \geq \alpha_1 \end{cases}$$

To maximize $u_1(\theta_1, \theta_2, b_1, b_2)$ and $u_2(\theta_1, \theta_2, b_1, b_2)$ simultaneously, we chose/get,

$$\alpha_1 = \alpha_2 = \frac{1}{2}$$

That is, $\left(b_1^*(\theta_1) = \frac{\theta_1}{2},\ b_2^*(\theta_2) = \frac{\theta_2}{2} \right)$ is a pure strategy Bayesian Nash equilibrium for the first price auction. $\blacksquare$

The above proof can be extended to show that with n risk-neutral buyers, the first price auction has $\left(b_i^*(\theta_i) = \frac{n-1}{n} \theta_i \right)_{i \in N}$ as its pure strategy Bayesian Nash equilibrium.