Homework 7

Madilyn Simons

1. If $f(x) = x^7 - 7x + 20$, then

$$f(x+1) = x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 14$$

By Eisenstein's Criterion, for $p = 7$, $f(x+1)$ is irreducible in $\mathbb{Q}[x]$, and so $f(x)$ is irreducible in $\mathbb{Q}[x]$.

2. Let $f(x) = x^5 - 4x^4 + 4x^3 + 3x^2 - 26x + 21$. If $p = 2$, then $\overline{f}(x) = x^5 + x^2 + 1$ in $\mathbb{Z}_p[x]$. We know that $\overline{f}(x)$ is irreducible in $\mathbb{Z}_p[x]$ because since it has no roots, it is either irreducible or the product of an irreducible quadratic and an irreducible cubic. However, the only irreducible quadratic in $\mathbb{Z}_p[x]$ is $x^2 + x + 1$, which does not divide $\overline{f}(x)$. Therefore, $f(x)$ is irreducible in $\mathbb{Q}[x]$.

3. (a) Since $-1+3i$ is a root of $f(x)$, $-1-3i$ is also a root of $f(x)$. Therefore

$$f(x) = (x + 1 + 3i)(x + 1 - 3i)(h(x)) = (x^2 + 2x + 10)(h(x))$$

for some polynomial $h(x)$.

By division,

$$h(x) = f(x)/(x^2 + 2x + 10) = x^4 + x^2 - 6.$$

Thus,

$$f(x) = (x^2 + 2x + 10)(x^4 + x^2 - 6) = (x^2 + 2x + 10)(x^2 + 3)(x^2 - 2)$$

We know that $(x^2 + 2x + 10)$ and $(x^2 - 2)$ are irreducible in $\mathbb{Q}[x]$ by Eisenstein's Criterion for prime $p = 2$, and $(x^2 + 3)$ is irreducible in $\mathbb{Q}[x]$ by Eisenstein's Criterion for prime $p = 3$. Therefore, the irreducible factorization of $f(x)$ in $\mathbb{Q}[x]$ is

$$f(c) = (x^2 + 2x + 10)(x^2 + 3)(x^2 - 2).$$

(b) In $\mathbb{R}[x]$,

$$f(x) = (x^2+2x+10)(x^2+3)(x^2-2) = (x^2+2x+10)(x^2+3)(x+\sqrt{2})(x-\sqrt{2})$$

We know that $(x^2+2x+10)$ and (x^2+3) are irreducible $\mathbb{R}[x]$ because a polynomial of the form $f(x) = ax^2 + bx + c$ is irreducible in $\mathbb{R}[x]$ if

$b^2 - 4ac < 0$. Therefore, the irreducible factorization of $f(x)$ in $\mathbb{R}[x]$ is

$$(x^2 + 2x + 10)(x^2 + 3)(x + \sqrt{2})(x - \sqrt{2}).$$

(c) In $\mathbb{C}[x]$, the irreducible factorization of $f(x)$ is

$$f(x) = (x + 1 + 3i)(x + 1 - 3i)(x + \sqrt{3}i)(x - \sqrt{3}i)(x + \sqrt{2})(x - \sqrt{2}).$$

We know that each of these polynomials are irreducible in $\mathbb{C}[x]$ because they all have degree 1.