Homework 3

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1. TODO
2. TODO
3. TODO
4. TODO
5. Assume M is a unit. Therefore there exists some $M^{-1} \in M_2(\mathbb{Z})$ such that $MM^{-1} = I_2$. We find M^{-1} using Gaussian Elimination:

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \Leftrightarrow \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & \frac{ad-bc}{a} & \frac{-c}{a} & 1 \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \Leftrightarrow \left[\begin{array}{cc|cc} a & 0 & \frac{ad}{ad-bc} & \frac{-ab}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

Therefore $M^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$. Since $M^{-1} \in M_2(\mathbb{Z})$, each of its elements must be integers. Its elements can only be integers if $ad-bc$ evenly divides a, b, c, and d.

Let m be the greatest common denominator of a, b, c, and d. Let $a = mx$, $b = my$, $c = mz$, and $d = mw$ for some integers x, y, z, and w that are not divisible by m. As such $ad - bc = mxmw - mymz = m^2(xw - yz)$. Since m is the greatest common denominator and m^2 is also a common denominator, $m^2 \leq m$ and this is only possible if $m = 1$. Therefore, the greatest common denominator of a, b, c, and d is 1. If this is the case, $ad - bc$ must be ± 1 since only ± 1 can evenly a, b, c, and d.

Next, assume $ad - bc = \pm 1$. By the last proof, M can only be a unit if $MM^{-1} = I_2$ and $M^{-1} \in M_2(\mathbb{Z})$. If $ad - bc = 1$, then $M^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and $MM^{-1} = I_2$. If $ad - bc = -1$, then $M^{-1} = \begin{bmatrix} -d & b \\ c & -a \end{bmatrix}$ and $MM^{-1} = I_2$.

Thus, $M \in M_2(\mathbb{Z})$ is a unit if and only if $ad - bc = \pm 1$.