

# Homework 8

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1. (a) Since  $(x^7 + 25x^6 - 25x + 5)$  is a nonconstant polynomial in  $\mathbb{Q}$ ,  $\mathbb{Q}/(x^7 + 25x^6 - 25x + 5)$  is a field if and only if  $(x^7 + 25x^6 - 25x + 5)$  is irreducible in  $\mathbb{Q}$ . We know that  $(x^7 + 25x^6 - 25x + 5)$  is irreducible in  $\mathbb{Q}$  by Eisenstein's Criterion for prime 5. Thus,  $\mathbb{Q}/(x^7 + 25x^6 - 25x + 5)$  is a field.
- (b) Consider  $\mathbb{Z}/2\mathbb{Z}$ . If  $f(x) = x^3 + 2x^2 - x + 1$ , then  $\bar{f}(x) = x^3 - x + 1$ . We know that  $\bar{f}(x)$  is irreducible in  $\mathbb{Z}/2\mathbb{Z}$  because  $\bar{f}(0) = \bar{f}(1) = 1$ . Since  $\bar{f}(x)$  is irreducible in  $\mathbb{Z}/2\mathbb{Z}$ ,  $f(x)$  is irreducible in  $\mathbb{Q}$ , which means that  $\mathbb{Q}[x]/(x^3 + 2x^2 - x + 1)$  is a field.
- (c) Only first-degree and second-degree polynomials can be irreducible in  $\mathbb{R}[x]$ . Since  $(x^5 + 42x^4 + \pi x^3 - 1729x^2 + \ln(2)x - 2019)$  is a fifth-degree polynomial, it is reducible in  $\mathbb{R}[x]$ . Therefore  $\mathbb{R}[x]/(x^5 + 42x^4 + \pi x^3 - 1729x^2 + \ln(2)x - 2019)$  is NOT a field.
2. (a) Let  $f(x) = x^3 + 2x + 1$ . We know that  $f(x)$  is irreducible in  $\mathbb{Z}/3\mathbb{Z}$  because  $f(0) = f(1) = f(2) = 1$ . Therefore,  $K$  is a field.
- (b) TODO
3. To prove that  $(a, b) = (d)$ , first let us prove that  $(a, b) \subseteq (d)$ . Let  $ar_1 + br_2$  be any element of  $(a, b)$ . Since  $d$  is the greatest common divisor of  $a$  and  $b$ ,  $a = dx$  and  $b = dy$  for some integers  $x$  and  $y$ . Therefore,  $ar_1 + br_2 = dxr_1 + dyr_2 = d(xr_1 + dyr_2)$ , which is an element of  $(d)$ . Thus,  $(a, b) \subseteq (d)$ .

Next, let us prove  $(d) \subseteq (a, b)$ . Since  $d$  is the greatest common divisor of  $a$  and  $b$ ,  $d = au + bv$  for some integers  $u$  and  $v$ . Let  $rd$  be any element of  $(d)$ . Therefore,  $rd = r(au + bv) = rau + rbv = a(ru) + b(rv) \subseteq (a, b)$ . Thus  $(d) \subseteq (a, b)$ .

Since  $(d) \subseteq (a, b)$  and  $(a, b) \subseteq (d)$ ,  $(a, b) = (d)$ .

4. Let  $a, b \in I$  such that

$$a = 2a_0 + 2a_1x + \dots + 2a_nx^n$$

and

$$b = 2b_0 + 2b_1x + \dots + 2b_nx^n$$

and  $a_i$  and  $b_i$  are integers for all  $i$ . We know that  $I$  holds under subtraction because

$$\begin{aligned} a - b &= (2a_0 + 2a_1x + \dots + 2a_nx^n) - (2b_0 + 2b_1x + \dots + 2b_nx^n) \\ &= (2a_0 - 2b_0) + (2a_1 - 2b_1)x + \dots + (2a_n - 2b_n)x^n \\ &= 2(a_0 - b_0) + 2(a_1 - b_1)x + \dots + 2(a_n - b_n)x^n \end{aligned}$$

Next let

$$c = c_0 + c_1x + \dots + c_nx^n$$

be an element of  $\mathbb{Z}[x]$ . We know that  $I$  absorbs multiplication because

$$\begin{aligned} ca &= (c_0 + c_1x + \dots + c_nx^n)(2a_0 + 2a_1x + \dots + 2a_nx^n) \\ &= 2(c_0 + c_1x + \dots + c_nx^n)(a_0 + a_1x + \dots + a_nx^n) \\ &= 2(c_0a_0 + (c_0a_1 + c_1a_0)x + (c_0a_2 + c_1a_1 + c_2a_0)x^2 + \dots + c_na_nx^n) \\ &= 2c_0a_0 + 2(c_0a_1 + c_1a_0)x + 2(c_0a_2 + c_1a_1 + c_2a_0)x^2 + \dots + 2c_na_nx^n \\ &\subseteq I \end{aligned}$$

and

$$\begin{aligned} ac &= (2a_0 + 2a_1x + \dots + 2a_nx^n)(c_0 + c_1x + \dots + c_nx^n) \\ &= 2(a_0 + a_1x + \dots + a_nx^n)(c_0 + c_1x + \dots + c_nx^n) \\ &= 2(a_0c_0 + (a_0c_1 + a_1c_0)x + (a_0c_2 + a_1c_1 + a_2c_0)x^2 + \dots + a_nc_nx^n) \\ &= 2a_0c_0 + 2(a_0c_1 + a_1c_0)x + 2(a_0c_2 + a_1c_1 + a_2c_0)x^2 + \dots + 2a_nc_nx^n \\ &\subseteq I \end{aligned}$$

Therefore,  $I$  is an ideal of  $\mathbb{Z}[x]$ .