Homework 5

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1. Assume that if $f(x)$, $g(x)$, $s(x)$, $d(x)$, $p(x)$ in $S[x]$ satisfy

$$f(x) + g(x) = s(x)$$

$$f(x) - g(x) = d(x)$$

$$f(x)g(x) = p(x)$$

then, for any α in R,

$$f(\alpha) + g(\alpha) = s(\alpha)$$

$$f(\alpha) - g(\alpha) = d(\alpha)$$

$$f(\alpha)g(\alpha) = p(\alpha)$$

We can prove that ϕ_α preserves addition as such:

$$\phi_\alpha(f) + \phi_\alpha(g) = f(\alpha) + g(\alpha) = s(\alpha) = \phi_\alpha(s)$$

$$\phi_\alpha(f) - \phi_\alpha(g) = f(\alpha) - g(\alpha) = d(\alpha) = \phi_\alpha(d)$$

We can prove that ϕ_α preserves multiplication as such:

$$\phi_\alpha(f)\phi_\alpha(g) = f(\alpha)g(\alpha) = p(\alpha) = \phi_\alpha(p)$$

Thus, ϕ_α is a ring homomorphism.

2. Let $f(\alpha)$, $g(\alpha)$ be elements in $S[\alpha]$. This implies that $f(x)$ and $g(x)$ are elements of $S[x]$. Since S is a ring, its elements are closed under addition, implying that $S[x]$ preserves addition as well. Let $f(x) + g(x) = s(x)$ and $f(x) - g(x) = d(x)$. Since $s(x)$ and $d(x)$ are in $S[x]$, $s(\alpha) = f(\alpha) + g(\alpha)$ and $d(\alpha) = f(\alpha) - g(\alpha)$ are elements of $S[\alpha]$ and $S[\alpha]$ preserves addition.

Similarly, since S preserves multiplication, we can assume $p(x) = f(x)g(x)$ is an element of $S[x]$. Therefore $p(\alpha) = f(\alpha)g(\alpha)$ is an element of $S[\alpha]$ and $S[alpha]$ preserves multiplication.

Thus, $S[\alpha]$ is a subring of R.

3. Let $a(x) = (a_0 + a_1 x + ... + a_n x^n)$, $b(x) = (b_0 + b_1 x + ... + b_n x^m)$ be elements of $R_1[x]$.

   Since F is a ring homomorphism, we can prove G preserves addition as such:

$$\begin{aligned}
G(a(x) + b(x)) &= G((a_0 + a_1 x + ... + a_n x^n) + (b_0 + b_1 x + ... + b_m x^m)) \\
&= G(a_0 + a_1 x + ... + a_n x^n + b_0 + b_1 x + ... + b_m x^m) \\
&= F(a_0) + F(a_1)x + ... + F(a_n)x^n + F(b_0) + F(b_1)x + ... + F(b_m)x^m \\
&= (F(a_0) + F(a_1)x + ... + F(a_n)x^n) + (F(b_0) + F(b_1)x + ... + F(b_m)x^m) \\
&= G(a_0 + a_1 x + ... + a_n x^n) + G(b_0 + b_1 x + ... + b_m x^m) \\
&= G(a(x)) + G(b(x))
\end{aligned}$$

   We can prove G preserves multiplication as such:

$$\begin{aligned}
G((ab)(x)) &= G((a_0 + a_1 x + ... + a_n x^n)(b_0 + b_1 x + ... + b_m x^m)) \\
&= G(a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + ... + a_n b_m x^{x+m}) \\
&= F(a_0 b_0) + F(a_0 b_1 + a_1 b_0)x + F(a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + ... + F(a_n b_m)x^{x+m} \\
&= (F(a_0) + F(a_1)x + ... + F(a_n)x^n)(F(b_0) + F(b_1)x + ... + F(b_m)x^m) \\
&= G(a_0 + a_1 x + ... + a_n x^n)G(b_0 + b_1 x + ... + b_m x^m) \\
&= G(a(x))G(b(x))
\end{aligned}$$

   Therefore G is also a a ring homomorphism.