

Homework 6

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1. To prove that "is an associate of" is an equivalence relation on R , we must show that it is reflexive, symmetric, and transitive.

First, since R is a commutative ring, there exists some identity element in R , 1_R , such that $a1_R = a$. We can say that 1_R is a unit since $(1_R)(1_R) = 1_R$. Since $a1_R = a$, a is an associate of a and "is an associate of" is reflexive.

Next, let $a, b \in R$ such that a is an associate of b . Therefore, there exist some unit, u , in R such that $au = b$. Since u is a unit, there also exists some unit $v \in R$ such that $uv = 1_R$.

If $a = bu$, then

$$av = (bu)v$$

$$av = buv$$

$$av = b(uv)$$

$$av = b1_R$$

$$av = b$$

Therefore, b is an associate of a and "is an associate of" is reflexive.

Finally, let a be an associate of b and let b be an associate of c for some elements $a, b, c \in R$. Therefore, $a = bu$ and $b = cv$ for some units $u, v \in R$.

We notice that

$$a = bu$$

$$a = (cv)u$$

$$a = c(vu)$$

We know that vu is a unit because there exists some $v^{-1}, u^{-1} \in R$ such that $vv^{-1} = 1_R$ and $uu^{-1} = 1_R$ since v and u are units. Thus, $(vu)(v^{-1}u^{-1}) = (vv^{-1})(uu^{-1}) = 1_R$ and so vu is a unit. Since vu is a unit, a is an associate of c and "is an associate of" is transitive.

2. (a) There are 4 different quadratic polynomial in $(\mathbb{Z}/2\mathbb{Z})[x]$:

$$x^2$$

$$x^2 + 1$$

$$x^2 + x$$

$$x^2 + x + 1$$

Let $f(x)$ be any quadratic polynomial in $(\mathbb{Z}/2\mathbb{Z})[x]$. We know that If $f(x) = x^2$, then $f(x)$ is reducible because 0 is a root. If $f(x) = x^2 + 1$, then $f(x)$ is reducible because 1 is a root. If $f(x) = x^2 + x$, then $f(x)$ is reducible because 1 is a root.

If $f(x) = x^2 + x + 1$, then $f(x)$ is irreducible because $f(x)$ has no roots in $(\mathbb{Z}/2\mathbb{Z})[x]$:

$$f(0) = 0^2 + 0 + 1 = 1$$

$$f(1) = 1^2 + 1 + 1 = 3 = 1$$

Therefore the only irreducible quadratic polynomial in $(\mathbb{Z}/2\mathbb{Z})[x]$ is $x^2 + x + 1$.

- (b) No, because $f(1) = 0$.
- (c) Yes, because $g(0) = g(1) = 1$.
- (d) Yes, because $h(0) = h(1) = 1$.