Homework 8

Madilyn Simons

1. (a) Since $(x^7+25x^6-25x+5)$ is a nonconstant polynomial in $\mathbb{Q}$, $\mathbb{Q}/(x^7+25x^6-25x+5)$ is a field if and only if $(x^7+25x^6-25x+5)$ is irreducible in $\mathbb{Q}$. We know that $(x^7+25x^6-25x+5)$ is irreducible in $\mathbb{Q}$ by Eisenstein's Criterion for prime 5. Thus, $\mathbb{Q}/(x^7+25x^6-25x+5)$ is a field.

   (b) Consider $\mathbb{Z}/2\mathbb{Z}$. If $f(x) = x^3+2x^2-x+1$, then $\overline{f}(x) = x^3-x+1$. We know that $\overline{f}(x)$ is irreducible in $\mathbb{Z}/2\mathbb{Z}$ because $\overline{f}(0) = \overline{f}(1) = 1$. Since $\overline{f}(x)$ is irreducible in $\mathbb{Z}/2\mathbb{Z}$, $f(x)$ is irreducible in $\mathbb{Q}$, which means that $\mathbb{Q}[x]/(x^3+2x^2-x+1)$ is a field.

   (c) Only first-degree and second-degree polynomials can be irreducible in $\mathbb{R}[x]$. Since $(x^5+42x^4+\pi x^3-1729x^2+ln(2)x-2019)$ is a fifth-degree polynomial, it is reducible in $\mathbb{R}[x]$. Therefore $\mathbb{R}[x]/(x^5+42x^4+\pi x^3-1729x^2+ln(2)x-2019)$ is NOT a field.

2. (a) Let $f(x) = x^3+2x+1$. We know that $f(x)$ is irreducible in $\mathbb{Z}/3\mathbb{Z}$ because $f(0) = f(1) = f(2) = 1$. Therefore, K is a field.

3. To prove that $(a,b) = (d)$, first let us prove that $(a,b) \subseteq (d)$. Let $ar_1 + br_2$ be any element of (a,b). Since d is the greatest common divisor of a and b, $a = dx$ and $b = dy$ for some integers x and y. Therefore, $ar_1+br_2 = dxr_1+dyr_2 = d(xr_1+dyr_2)$, which is an element of (d). Thus, $(a,b) \subseteq (d)$.

   Next, let us prove $(d) \subseteq (a,b)$. Since d is the greatest common divisor of a and b, $d = au+bv$ for some integers u and v. Let rd be any element of (d). Therefore, $rd = r(au+bv) = rau+rbv = a(ru)+b(rv) \subseteq (a,b)$. Thus $(d) \subseteq (a,b)$.

   Since $(d) \subseteq (a,b)$ and $(a,b) \subseteq (d)$, $(a,b) = (d)$.

4. Let a, $b \in I$ such that

$$a = 2a_0 + 2a_1x + ... + 2a_nx^n$$

and

$$b = 2b_0 + 2_b1x + ...2b_nx^n$$

and a_i and b_i are integers for all i. We know that I holds under subtraction because

$$\begin{aligned} a - b &= (2a_0 + 2a_1 x + ... + 2a_n x^n) - (2b_0 + 2_b 1x + ...2b_n x^n) \\ &= (2a_0 - 2b_0) + (2a_1 - 2b_1)x + ... + (2a_n - 2b_n)x^n \\ &= 2(a_0 - b_0) + 2(a_1 - b_1)x + ... + 2(a_n - b_n)x^n \end{aligned}$$

Next let

$$c = c_0 + c_1 x + ... + c_n x^n$$

be an element of $\mathbb{Z}[x]$. We know that I absorbs multiplication because

$$\begin{aligned} ca &= (c_0 + c_1 x + ... + c_n x^n)(2a_0 + 2a_1 x + ... + 2a_n x^n) \\ &= 2(c_0 + c_1 x + ... + c_n x^n)(a_0 + a_1 x + ... + a_n x^n) \\ &= 2(c_0 a_0 + (c_0 a_1 + c_1 a_0)x + (c_0 a_2 + c_1 a_1 + c_2 a_0)x^2 + ... + c_n a_n x^n) \\ &= 2c_0 a_0 + 2(c_0 a_1 + c_1 a_0)x + 2(c_0 a_2 + c_1 a_1 + c_2 a_0)x^2 + ... + 2c_n a_n x^n \\ &\subseteq I \end{aligned}$$

and

$$\begin{aligned} ac &= (2a_0 + 2a_1 x + ... + 2a_n x^n)(c_0 + c_1 x + ... + c_n x^n) \\ &= 2(a_0 + a_1 x + ... + a_n x^n)(c_0 + c_1 x + ... + c_n x^n) \\ &= 2(a_0 c_0 + (a_0 c_1 + a_1 c_0)x + (a_0 c_2 + a_1 c_1 + a_2 c_0)x^2 + ... + a_n c_n x^n) \\ &= 2a_0 c_0 + 2(a_0 c_1 + a_1 c_0)x + 2(a_0 c_2 + a_1 c_1 + a_2 c_0)x^2 + ... + 2a_n c_n x^n \\ &\subseteq I \end{aligned}$$

Therefore, I is an ideal of $\mathbb{Z}[x]$.

5. I claim that $I = (0, x^2 - 7)$. We know that I is finitely generated because for any $g(x) \in I$ can be generated from $f(x) = x^2 - 7$ as such

$$\begin{aligned} g(7) &= f(7)q(7) + r(7) \\ 0 &= 0 * q(7) + r(7) \\ 0 &= 0 + r(7) \\ 0 &= r(7) \end{aligned}$$

Therefore I is finitely generated.