Homework 5

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1. To prove φ_α is closed under addition, first let f, g be any elements of $S[x]$. We know that $f+g$ is also an element of $S[x]$ because the coefficients of f and g are elements of S, which is a ring. This implies that the coefficients of f and g are closed under addition, so the coefficients of $f+g$ are also in S.

   With this having be said, we can prove φ_α is closed under addition as such:

   $$\varphi_\alpha(f+g) = (f+g)(\alpha) = f(\alpha) + g(\alpha) = \varphi_\alpha(f) + \varphi_\alpha(g)$$

   We know $f \cdot g$ is an element of $S[x]$ since S is closed under multiplication, so the coefficients of $f \cdot g$ are elements of S.

   We can prove φ_α is closed under multiplication as such:

   $$\varphi_\alpha(f \cdot g) = (f \cdot g)(\alpha) = f(\alpha) \cdot g(\alpha) = \varphi_\alpha(f) \cdot \varphi_\alpha(g)$$

   Thus, φ_α is a ring homomorphism.

2. Let $f(\alpha)$, $g(\alpha)$ be any elements in $S[\alpha]$. By the definition of $S[\alpha]$, $f(x)$, $g(x)$ as elements of $S[x]$. Since $S[x]$ is a ring homomorphism, $(f+g)(x)$ is an element of $S[x]$. If $(f+g)(x)$ is an element of $S[x]$, then $(f+g)(\alpha)$ is an element of $S[\alpha]$ and $S[\alpha]$ is closed under addition.

   Similarly, $f(x)g(x)$ is an element of $S[x]$. This implies that $f(\alpha)g(\alpha)$ is an element of $S[\alpha]$ and so $S[\alpha]$ is closed under multiplication.

   For $S[\alpha]$ to be a subring of R, 0_R must be an element of $S[\alpha]$. Define the function $f : S[x] \to R$ by $f(x) = 0_R$. As such, $f(\alpha) = 0_R$ is an element of $S[\alpha]$.

   Let $f(x) = a_0 + a_1x + ... + a_nx^n$ be any element of $S[x]$. By definition, a_i is an element of R for all non-negative integers i, and for all a_i, there exists a b_i in R such that $a_i + b_i = 0_R$. Therefore, there exists a function $g(x) = b_0 + b_1x + ... + b_nx^n$ in $S[x]$ such that $f(x) + g(x) = 0_R$. As a direct result, for any $f(\alpha)$ in $S[\alpha]$ there exists a $g(\alpha)$ such that $f(\alpha) + g(\alpha) = 0_R$.

   Thus, $S[\alpha]$ is a subring of R.

3. Let $(a_0 + a_1x + ... + a_nx^n)$, $(b_0 + b_1x + ... + b_nx^n)$ be elements of $R_1[x]$.

   We can prove G is closed under addition as such:

$$G((a_0 + a_1x + ... + a_nx^n) + (b_0 + b_1x + ... + b_nx^n))$$
$$= G(a_0 + a_1x + ... + a_nx^n + b_0 + b_1x + ... + b_nx^n)$$
$$= F(a_0) + F(a_1)x + ... + F(a_n)x^n + F(b_0) + F(b_1)x + ... + F(b_n)x^n$$
$$= (F(a_0) + F(a_1)x + ... + F(a_n)x^n) + (F(b_0) + F(b_1)x + ... + F(b_n)x^n)$$
$$= G(a_0 + a_1x + ... + a_nx^n) + G(b_0 + b_1x + ... + b_nx^n)$$

   We can prove G is closed under multiplication as such:

$$G((a_0 + a_1x + ... + a_nx^n)(b_0 + b_1x + ... + b_nx^n))$$
$$= G(a_0b_0 + ... + a_nb_mx^{n+m})$$
$$= F(a_0b_0) + ... + F(a_nb_m)x^{n+m}$$
$$= F(a_0)F(b_0) + ... + F(a_n)F(b_m)x^{n+m}$$
$$= (F(a_0) + F(a_1)x + ... + F(a_n)x^n)(F(b_0) + F(b_1)x + ... + F(b_m)x^m)$$
$$= G(a_0 + a_1x + ... + a_nx^n)G(b_0 + b_1x + ... + b_nx^n)$$

   Therefore G is also a a ring homomorphism.