

Homework 5

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1. Assume that if $f(x), g(x), s(x), d(x), p(x)$ in $S[x]$ satisfy

$$f(x) + g(x) = s(x)$$

$$f(x) - g(x) = d(x)$$

$$f(x)g(x) = p(x)$$

then, for any α in R ,

$$f(\alpha) + g(\alpha) = s(\alpha)$$

$$f(\alpha) - g(\alpha) = d(\alpha)$$

$$f(\alpha)g(\alpha) = p(\alpha)$$

We can prove that ϕ_α is closed under addition as such:

$$\phi_\alpha(f) + \phi_\alpha(g) = f(\alpha) + g(\alpha) = s(\alpha) = \phi_\alpha(s)$$

$$\phi_\alpha(f) - \phi_\alpha(g) = f(\alpha) - g(\alpha) = d(\alpha) = \phi_\alpha(d)$$

We can prove that ϕ_α is closed under multiplication as such:

$$\phi_\alpha(f)\phi_\alpha(g) = f(\alpha)g(\alpha) = p(\alpha) = \phi_\alpha(p)$$

Thus, ϕ_α is a ring homomorphism.

2. Let $f(\alpha), g(\alpha)$ be elements in $S[\alpha]$. This implies that $f(x)$ and $g(x)$ are elements of $S[x]$. Since S is a ring, its elements are closed under addition, implying that $S[x]$ is closed under addition as well. Let $f(x) + g(x) = s(x)$ and $f(x) - g(x) = d(x)$. Since $s(x)$ and $d(x)$ are in $S[x]$, $s(\alpha) = f(\alpha) + g(\alpha)$ and $d(\alpha) = f(\alpha) - g(\alpha)$ are elements of $S[\alpha]$ and $S[\alpha]$ is closed under addition.

Similarly, since S is closed under multiplication, we can assume $p(x) = f(x)g(x)$ is an element of $S[x]$. Therefore $p(\alpha) = f(\alpha)g(\alpha)$ is an element of $S[\alpha]$ and $S[\alpha]$ is closed under multiplication.

Thus, $S[\alpha]$ is a subring of R .

3. Let $(a_0 + a_1x + \dots + a_nx^n)$, $(b_0 + b_1x + \dots + b_nx^n)$ be elements of $R_1[x]$.

We can prove G is closed under addition as such:

$$\begin{aligned}
& G((a_0 + a_1x + \dots + a_nx^n) + (b_0 + b_1x + \dots + b_nx^n)) \\
&= G(a_0 + a_1x + \dots + a_nx^n + b_0 + b_1x + \dots + b_nx^n) \\
&= F(a_0) + F(a_1)x + \dots + F(a_n)x^n + F(b_0) + F(b_1)x + \dots + F(b_n)x^n \\
&= (F(a_0) + F(a_1)x + \dots + F(a_n)x^n) + (F(b_0) + F(b_1)x + \dots + F(b_n)x^n) \\
&= G(a_0 + a_1x + \dots + a_nx^n) + G(b_0 + b_1x + \dots + b_nx^n)
\end{aligned}$$

We can prove G is closed under multiplication as such:

$$\begin{aligned}
& G((a_0 + a_1x + \dots + a_nx^n)(b_0 + b_1x + \dots + b_nx^n)) \\
&= G(a_0b_0 + \dots + a_nb_nx^{n+m}) \\
&= F(a_0b_0) + \dots + F(a_nb_n)x^{n+m} \\
&= F(a_0)F(b_0) + \dots + F(a_n)F(b_n)x^{n+m} \\
&= (F(a_0) + F(a_1)x + \dots + F(a_n)x^n)(F(b_0) + F(b_1)x + \dots + F(b_n)x^n) \\
&= G(a_0 + a_1x + \dots + a_nx^n)G(b_0 + b_1x + \dots + b_nx^n)
\end{aligned}$$

Therefore G is also a ring homomorphism.