Homework 3

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1. The set $\frac{1}{2}\mathbb{Z}$ is not a ring. A ring must be closed under multiplication. Let $a = \frac{m}{2}$ and $b = \frac{n}{2}$ for some integers m, n. We know that $ab = \frac{mn}{4}$. Assume m and n are both odd and that $m = 2x + 1$ and $n = 2y + 1$ for some integers x, y. Therefore,

$$ab = \frac{mn}{4} = \frac{(2x+1)(2y+1)}{4} = \frac{2(2xy+y)+1}{4}$$

Since ab is not an element of the set, this means that the set is not closed under multiplication.

2. This set is a ring.
Let $a = \frac{m}{2x+1}$, $b = \frac{n}{2y+1}$, $c = \frac{l}{2z+1}$ be elements of the set for some integers m, n, l, x, y, and z.
The set is closed under addition:

$$a + b = \frac{m}{2x+1} + \frac{n}{2y+1} = \frac{2my + 2nx + m + n}{2(2xy + x + y) + 1}$$

Associative addition holds:

$$a + (b + c) = \frac{m}{2x+1} + (\frac{n}{2y+1} + \frac{l}{2z+1})$$

$$= \frac{m(2y+1)(2z+1) + n(2z+1)(2x+1) + l(2y+1)(2x+1)}{(2x+1)(2y+1)(2z+1)}$$

$$= (\frac{m}{2x+1} + \frac{n}{2y+1}) + \frac{l}{2z+1} = (a + b) + c$$

Commutative addition holds:

$$a + b = \frac{m}{2x+1} + \frac{n}{2y+1}$$

$$= \frac{m(2y+1) + 2(nx+1)}{2(2xy+x+y)+1}$$

$$= \frac{n}{2y+1} + \frac{m}{2x+1} = b + a$$

Next, we can prove the existence of 0_S by letting $0_S = 0$:

$$a + 0_S = a + 0 = a = 0 + a = 0_S + a$$

There is a solution to $a + x = 0$. Let $x = \frac{-m}{2x+1}$ for any integers m, x:

$$a + x = \frac{m}{2x+1} + \frac{-m}{2x+1} = \frac{m + -m}{2x+1} = \frac{0}{2x+1} = 0$$

The set is closed under multiplication:

$$ab = \frac{m}{2x+1} \cdot \frac{n}{2y+1} = \frac{mn}{(2x+1)(2y+1)} = \frac{mn}{2(2xy+x+y)+1}$$

Associative multiplication holds:

$$a(bc) = \frac{m}{2x+1} \cdot (\frac{n}{2y+1} \cdot \frac{l}{2z+1}) = (\frac{m}{2x+1} \cdot \frac{n}{2y+1}) \cdot \frac{l}{2z+1} = (ab)c$$

The Distributive Property holds:

$$a(b+c) = \frac{m}{2x+1} \cdot (\frac{n}{2y+1} + \frac{l}{2z+1}) = \frac{m}{2x+1} \cdot (\frac{n(2z+1) + l(2y+1)}{(2y+1)(2z+1)})$$

$$= \frac{m}{2x+1} \cdot \frac{n}{2y+1} + \frac{m}{2x+1} \cdot \frac{l}{2z+1} = ab + ac$$

3. The set is not a ring as it is not closed under multiplication.
Let $\frac{m}{6x+3}$, $\frac{n}{6y+4}$ be elements of the set for some integers m, n, x, and y. As such,

$$\frac{m}{6x+3} \cdot \frac{n}{6y+4} = \frac{mn}{6(6xy+4x+3y+2)}$$

4. Assume a, b are elements in $R_1 \cap R_2$. This implies a, b are in R_1. Since R_1 is a subring, $a+b$ is in R_1. Similarly, a, b are in R_2 and $a+b$ is in R_2 as R_2 is also a subring. Since $a+b$ is in R_1 and R_2, $a+b$ is in $R_1 \cap R_2$ and $R_1 \cap R_2$ is closed under addition.

Since R_1 and R_2 are subrings, ab is in R_1 and ab is in R_2. Therefore ab is in $R_1 \cap R_2$ and $R_1 \cap R_2$ is closed under multiplication.

Thus, $R_1 \cap R_2$ is a subring of R (and therefore a ring).

$R_1 \cup R_2$ is not necessarily a ring. Let R_1 and R_2 be subrings of R such that R_2 is not a subring of R_1. Let a be an element of R_1, but not of R_2. Let b be any element of R_2.

If $R_1 \cup R_2$ is a ring, $a+b$ is in $R_1 \cup R_2$. By definition of union, $a+b$ is in R_1 or $a+b$ is in R_2. If $a+b$ is in R_1, then $(a+b)-a$ is also in R_1 (since R_1 is closed under addition). However, this implies b is in R_1. If any element in R_2 is an element of R_1, then R_2 is a subring of R_1, which is a contradiction. If $a+b$ is in R_2, then $(a+b)-b$ is also in R_2. This implies that a is in R_2, which is another contradiction. Therefore, if two subrings exist such that neither is a subset of the other, their union is not a ring.

5. Assume M is a unit. Therefore there exists some $M^{-1} \in M_2(\mathbb{Z})$ such that $MM^{-1} = I_2$. We find M^{-1} using Gaussian Elimination:

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \Leftrightarrow \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & \frac{ad-bc}{a} & \frac{-c}{a} & 1 \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \Leftrightarrow \left[\begin{array}{cc|cc} a & 0 & \frac{ad}{ad-bc} & \frac{-ab}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

Therefore $M^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$. Since $M^{-1} \in M_2(\mathbb{Z})$, each of its elements must be integers. Its elements can only be integers if $ad-bc$ evenly divides a, b, c, and d.

Let m be the greatest common divisor of a, b, c, and d. Let $a = mx$, $b = my$, $c = mz$, and $d = mw$ for some integers x, y, z, and w that are not divisible by m. As such $ad - bc = mxmw - mymz = m^2(xw - yz)$. Since m is the greatest common denominator and m^2 is also a common denominator, $m^2 \leq m$ and this is only possible if $m = 1$. Therefore, the greatest common divisor of a, b, c, and d is 1. If this is the case, $ad - bc$ must be ± 1 since only ± 1 can evenly a, b, c, and d.

Next, assume $ad - bc = \pm 1$. By the last proof, M can only be a unit if $MM^{-1} = I_2$ and $M^{-1} \in M_2(\mathbb{Z})$. If $ad-bc = 1$, then $M^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and $MM^{-1} = I_2$. If $ad-bc = -1$, then $M^{-1} = \begin{bmatrix} -d & b \\ c & -a \end{bmatrix}$ and $MM^{-1} = I_2$.

Thus, $M \in M_2(\mathbb{Z})$ is a unit if and only if $ad - bc = \pm 1$.