Homework 8

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1. (a) Since $(x^7+25x^6-25x+5)$ is a nonconstant polynomial in $\mathbb{Q}$, $\mathbb{Q}/(x^7+25x^6-25x+5)$ is a field if and only if $(x^7+25x^6-25x+5)$ is irreducible in $\mathbb{Q}$. We know that $(x^7+25x^6-25x+5)$ is irreducible in $\mathbb{Q}$ by Eisenstein's Criterion for prime 5. Thus, $\mathbb{Q}/(x^7+25x^6-25x+5)$ is a field.

   (b) Consider $\mathbb{Z}/2\mathbb{Z}$. If $f(x) = x^3+2x^2-x+1$, then $\overline{f}(x) = x^3-x+1$. We know that $\overline{f}(x)$ is irreducible in $\mathbb{Z}/2\mathbb{Z}$ because $\overline{f}(0) = \overline{f}(1) = 1$. Since $\overline{f}(x)$ is irreducible in $\mathbb{Z}/2\mathbb{Z}$, $f(x)$ is irreducible in $\mathbb{Q}$, which means that $\mathbb{Q}[x]/(x^3+2x^2-x+1)$ is a field.

   (c) Only first-degree and second-degree polynomials can be irreducible in $\mathbb{R}[x]$. Since $(x^5+42x^4+\pi x^3-1729x^2+ln(2)x-2019)$ is a fifth-degree polynomial, it is reducible in $\mathbb{R}[x]$. Therefore $\mathbb{R}[x]/(x^5+42x^4+\pi x^3-1729x^2+ln(2)x-2019)$ is NOT a field.

2. (a) Let $f(x) = x^3+2x+1$. We know that $f(x)$ is irreducible in $\mathbb{Z}/3\mathbb{Z}$ because $f(0) = f(1) = f(2) = 1$. Therefore, K is a field.

   (b) TODO

3. To prove that $(a,b) = (d)$, first let us prove that $(a,b) \subseteq (d)$. Let $ar_1 + br_2$ be any element of (a,b). Since d is the greatest common divisor of a and b, $a = dx$ and $b = dy$ for some integers x and y. Therefore, $ar_1+br_2 = dxr_1+dyr_2 = d(xr_1+dyr_2)$, which is an element of (d). Thus, $(a,b) \subseteq (d)$.

   Next, let us prove $(d) \subseteq (a,b)$. Since d is the greatest common divisor of a and b, $d = au+bv$ for some integers u and v. Let rd be any element of (d). Therefore, $rd = r(au+bv) = rau+rbv = a(ru)+b(rv) \subseteq (a,b)$. Thus $(d) \subseteq (a,b)$.

   Since $(d) \subseteq (a,b)$ and $(a,b) \subseteq (d)$, $(a,b) = (d)$.