Homework 7

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1. If $f(x) = x^7 - 7(x+1) + 20$, then
$$f(x+1) = x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 14$$
By Eisenstein's Criterion, $f(x+1)$ is irreducible in $\mathbb{Q}[x]$, and so $f(x)$ is irreducible in $\mathbb{Q}[x]$.

2. Let $f(x) = x^5 - 4x^4 + 4d^3 + 3x^2 - 26x + 21$. If $p = 2$, then $\overline{f}(x) = x^5 + x^2 + 1$ in $\mathbb{Z}_p[x]$. Since $\overline{f}(0) = \overline{f}(1) = 1$, $\overline{f}(x)$ is irreducible in $\mathbb{Z}_p[x]$. Therefore, $f(x)$ is irreducible in $\mathbb{Q}[x]$.

3. (a) Since $-1+3i$ is a root of $f(x)$, $-1-3i$ is a root of $f(x)$. Therefore
$$f(x) = (x+1+3i)(x+1-3i)(h(x)) = (x^2+2x+10)(h(x))$$
for some polynomial $h(x)$. By division,
$$h(x) = f(x)/(x^2+2x+10) = x^4 + x^2 - 6$$
Thus,
$$f(x) = (x^2+2x+10)(x^4+x^2-6) = (x^2+2x+10)(x^2+3)(x^2-2)$$
We know that $(x^2+2x+10)$ and (x^2-2) are irreducible in $\mathbb{Q}[x]$ by Eisenstein's Criterion for prime $p=2$, and (x^2+3) is irreducible in $\mathbb{Q}[x]$ for prime $p=3$. Therefore, the irreducible factorization of $f(x)$ in $\mathbb{Q}[x]$ is
$$f(c) = (x^2+2x+10)(x^2+3)(x^2-2)$$
(b) In $\mathbb{R}[x]$,
$$f(x) = (x^2+2x+10)(x^2+3)(x^2-2) = (x^2+2x+10)(x^2+3)(x+\sqrt{2})(x-\sqrt{2})$$
We know that $(x^2+2x+10)$ and (x^2+3) are irreducible $\mathbb{R}[x]$ because a polynomial of the form $f(x) = ax^2+bx+c$ is irreducible in $\mathbb{R}[x]$ if $b^2-4ac<0$. Therefore, the irreducible factorization of $f(x)$ in $\mathbb{R}[x]$ is
$$(x^2+2x+10)(x^2+3)(x+\sqrt{2})(x-\sqrt{2})$$
(c) In $\mathbb{C}[x]$, the irreducible factorization of $f(x)$ is
$$f(x) = (x+1+3i)(x+1-i)(x+\sqrt{3}i)(x-\sqrt{3}i)(x+\sqrt{2})(x-\sqrt{2})$$
We know that each of these polynomials are irreducible in $\mathbb{C}[x]$ because they all have degree 1.