Homework 3

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1. TODO
2. TODO
3. TODO
4. TODO
5. Assume M is a unit. Therefore there exists some $M^{-1} \in M_2(\mathbb{Z})$ such that $MM^{-1} = I_2$. We find M^{-1} using Gaussian Elimination:

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \Leftrightarrow \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & \frac{ad-bc}{a} & \frac{-c}{a} & 1 \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \Leftrightarrow \left[\begin{array}{cc|cc} a & 0 & \frac{ad}{ad-bc} & \frac{-ab}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

Therefore $M^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$. Since $M^{-1} \in M_2(\mathbb{Z})$, each of its elements must be integers. Its elements can only be integers if $ad - bc$ evenly divides a, b, c, and d.

First, assume a, b, c, and d are relatively prime. If so, then the only integers that evenly divide all four are ± 1. Therefore $ad - bc$ must be ± 1.

Otherwise, a, b, c, and d are not relatively prime. Let m be the greatest common factor of the a, b, c, and d. Now, let $a = mx$, $b = my$, $c = mz$ and $d = mw$ for some integers x, y, z, and w. As such, $ad-bc = m^2xw - m^yz = m^2(xw-yz)$. This means that m^2 and $(xw-yz)$ must also be common factors of a, b, c, and d. If m is the greatest common factor, then $m^2 \leq m$. This is only possible if $m = 1$. Likewise, $|(xw - yz)| \leq |m|$. Since $m = 1$, $(xw - yz)$ must be ± 1 or 0, but it can't be 0 since $m^2 \cdot 0 = 0$, which can't divide any integer. Since $m^2 = 1$ and $xw - yz = \pm 1$, $ad - bc$ must be 1.