Homework 8

Madilyn Simons

1. (a) Let I be a nonempty ideal in F such that I does not equal F. Let $x \in F$ and $x \notin I$, let $a \in I$, and let a^{-1} be the multiplicative inverse of a. If $a \in I$, then $ax \in I$ and $a^{-1}ax \in I$. However, $a^{-1}ax = 1_F x = x$ and $x \notin I$ so this is a contradiction.

   (b) Since f is a homomorphism of rings, the kernel of f is an ideal of F. The only ideals of F are F and (0). If the kernel of f is (0), then f is injective. If the kernel of f is F, then f is the zero function.

2. TODO Rewrite this and replace $(q_1(x))$ with $(q_i(x))$

   First, assume $(p(x))$ is maximal. Let $p(x) = q_1(x)q_2(x)...q_n(x)$ be the prime factorization of $p(x)$. Therefore $(p(x)) \subset (q_1(x))$ since $q_1(x)$ divides $p(x)$. Since $(p(x))$ is maximal, either $(p(x)) = (q_1(x))$ or $F[x] = (q_1(x))$.

   Assume $(p(x)) = (q_1(x))$. Therefore $p(x)$ must be some divisor of $q_1(x)$ such that $q_1(x) = p(x)q_2^{-1}(x)...q_n^{-1}(x)$. This implies that $q_2(x), q_3(x), ..., q_n(x)$ are units, which are nonzero constants. If $p(x)$ is the product of an irreducible polynomial $q_1(x)$ and several nonzero constant polynomials, then $p(x)$ is irreducible.

   Now assume $F[x] = (q_1(x))$. If this is the case, then $q_1(x)$ must be a unit, which is a contradiction.

   Therefore if $(p(x))$ is maximal, then $p(x)$ is irreducible. Then the quotient ring $F[x]/(p(x))$ is all

   Next, assume $p(x)$ is irreducible. TODO

3. (a) By the Division Algorithm, $a(x) = b(x)q(x) + r(x)$ such that $0 \leq degr(x) < degb(x)$. By Theorem 6.1, $b(x)q(x) \in I$ and $a(x) - b(x)q(x) = r(x) \in I$.

   (b) Let $a(x)$ by any nonzero polynomial in I. By the Division Algorithm, $a(x) = p(x)q(x) + r(x)$ such that $0 \leq degr(x) < degp(x)$. However, $p(x)$ is of minimal degree, so $degr(x) = 0$. Therefore, $p(x)$ divides all $a(x) \in I$.