Homework 8

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1. (a) Since $(x^7+25x^6-25x+5)$ is a nonconstant polynomial in $\mathbb{Q}$, $\mathbb{Q}/(x^7+25x^6-25x+5)$ is a field if and only if $(x^7+25x^6-25x+5)$ is irreducible in $\mathbb{Q}$. We know that $(x^7+25x^6-25x+5)$ is irreducible in $\mathbb{Q}$ by Eisenstein's Criterion for prime 5. Thus, $\mathbb{Q}/(x^7+25x^6-25x+5)$ is a field.

   (b) Consider $\mathbb{Z}/2\mathbb{Z}$. If $f(x) = x^3+2x^2-x+1$, then $\overline{f}(x) = x^3-x+1$. We know that $\overline{f}(x)$ is irreducible in $\mathbb{Z}/2\mathbb{Z}$ because $\overline{f}(0) = \overline{f}(1) = 1$. Since $\overline{f}(x)$ is irreducible in $\mathbb{Z}/2\mathbb{Z}$, $f(x)$ is irreducible in $\mathbb{Q}$, which means that $\mathbb{Q}[x]/(x^3+2x^2-x+1)$ is a field.

   (c) Only first-degree and second-degree polynomials can be irreducible in $\mathbb{R}[x]$. Since $(x^5+42x^4+\pi x^3-1729x^2+ln(2)x-2019)$ is a fifth-degree polynomial, it is reducible in $\mathbb{R}[x]$. Therefore $\mathbb{R}[x]/(x^5+42x^4+\pi x^3-1729x^2+ln(2)x-2019)$ is NOT a field.

2. (a) Let $f(x) = x^3+2x+1$. We know that $f(x)$ is irreducible in $\mathbb{Z}/3\mathbb{Z}$ because $f(0) = f(1) = f(2) = 1$. Therefore, K is a field.

   (b) TODO

3. To prove that $(a,b) = (d)$, first let us prove that $(a,b) \subseteq (d)$. Let $ar_1 + br_2$ be any element of (a,b). Since d is the greatest common divisor of a and b, $a = dx$ and $b = dy$ for some integers x and y. Therefore, $ar_1+br_2 = dxr_1+dyr_2 = d(xr_1+dyr_2)$, which is an element of (d). Thus, $(a,b) \subseteq (d)$.

   Next, let us prove $(d) \subseteq (a,b)$. Since d is the greatest common divisor of a and b, $d = au+bv$ for some integers u and v. Let rd be any element of (d). Therefore, $rd = r(au+bv) = rau+rbv = a(ru)+b(rv) \subseteq (a,b)$. Thus $(d) \subseteq (a,b)$.

   Since $(d) \subseteq (a,b)$ and $(a,b) \subseteq (d)$, $(a,b) = (d)$.

4. Let a, $b \in I$ such that

$$a = 2a_0 + 2a_1x + ... + 2a_nx^n$$

and

$$b = 2b_0 + 2_b1x + ...2b_nx^n$$

and a_i and b_i are integers for all i. We know that I holds under subtraction because

$$\begin{aligned}
a - b &= (2a_0 + 2a_1x + ... + 2a_nx^n) - (2b_0 + 2_b1x + ...2b_nx^n) \\
&= (2a_0 - 2b_0) + (2a_1 - 2b_1)x + ... + (2a_n - 2b_n)x^n \\
&= 2(a_0 - b_0) + 2(a_1 - b_1)x + ... + 2(a_n - b_n)x^n
\end{aligned}$$

Next let

$$c = c_0 + c_1x + ... + c_nx^n$$

be an element of $\mathbb{Z}[x]$. We know that I absorbs multiplication because

$$\begin{aligned}
ca &= (c_0 + c_1x + ... + c_nx^n)(2a_0 + 2a_1x + ... + 2a_nx^n) \\
&= 2(c_0 + c_1x + ... + c_nx^n)(a_0 + a_1x + ... + a_nx^n) \\
&= 2(c_0a_0 + (c_0a_1 + c_1a_0)x + (c_0a_2 + c_1a_1 + c_2a_0)x^2 + ... + c_na_nx^n) \\
&= 2c_0a_0 + 2(c_0a_1 + c_1a_0)x + 2(c_0a_2 + c_1a_1 + c_2a_0)x^2 + ... + 2c_na_nx^n \\
&\subseteq I
\end{aligned}$$

and

$$\begin{aligned}
ac &= (2a_0 + 2a_1x + ... + 2a_nx^n)(c_0 + c_1x + ... + c_nx^n) \\
&= 2(a_0 + a_1x + ... + a_nx^n)(c_0 + c_1x + ... + c_nx^n) \\
&= 2(a_0c_0 + (a_0c_1 + a_1c_0)x + (a_0c_2 + a_1c_1 + a_2c_0)x^2 + ... + a_nc_nx^n) \\
&= 2a_0c_0 + 2(a_0c_1 + a_1c_0)x + 2(a_0c_2 + a_1c_1 + a_2c_0)x^2 + ... + 2a_nc_nx^n \\
&\subseteq I
\end{aligned}$$

Therefore, I is an ideal of $\mathbb{Z}[x]$.

5. I claim that $I = (x^2 - 7)$. Let $g(x)$, $f(x)$ be elements of I such that $g(x) = f(x)q(x) + r(x)$ and the degree of $r(x)$ is less than the degree of $f(x)$. We know that $g(x)$ is generated by $f(x)$ because

$$\begin{aligned}
g(7) &= f(7)q(7) + r(7) \\
0 &= 0 * q(7) + r(7) \\
0 &= 0 + r(7) \\
0 &= r(7)
\end{aligned}$$

Therefore I is finitely generated.