Homework 5

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1. Assume that if $f(x)$, $g(x)$, $s(x)$, $d(x)$, $p(x)$ in $S[x]$ satisfy

$$f(x) + g(x) = s(x)$$

$$f(x) - g(x) = d(x)$$

$$f(x)g(x) = p(x)$$

then, for any α in R,

$$f(\alpha) + g(\alpha) = s(\alpha)$$

$$f(\alpha) - g(\alpha) = d(\alpha)$$

$$f(\alpha)g(\alpha) = p(\alpha)$$

We can prove that φ_α preserves addition as such:

$$\varphi_\alpha(f) + \varphi_\alpha(g) = f(\alpha) + g(\alpha) = s(\alpha) = \varphi_\alpha(s) = \varphi_\alpha(f + g)$$

We can prove that φ_α preserves multiplication as such:

$$\varphi_\alpha(f)\varphi_\alpha(g) = f(\alpha)g(\alpha) = p(\alpha) = \varphi_\alpha(p) = \varphi_\alpha(fg)$$

Thus, φ_α is a ring homomorphism.

2. Let $f(\alpha)$, $g(\alpha)$ be elements in $S[\alpha]$. This implies that $f(x)$ and $g(x)$ are elements of $S[x]$. Since S is a ring, its elements are closed under addition, implying that $S[x]$ preserves addition as well. Let $f(x) + g(x) = s(x)$ and $f(x) - g(x) = d(x)$. Since $s(x)$ and $d(x)$ are in $S[x]$, $s(\alpha) = f(\alpha) + g(\alpha)$ and $d(\alpha) = f(\alpha) - g(\alpha)$ are elements of $S[\alpha]$ and $S[\alpha]$ preserves addition.

Similarly, since S preserves multiplication, we can assume $p(x) = f(x)g(x)$ is an element of $S[x]$. Therefore $p(\alpha) = f(\alpha)g(\alpha)$ is an element of $S[\alpha]$ and $S[alpha]$ preserves multiplication.

Thus, $S[\alpha]$ is a subring of R.

3. Let $a(x) = (a_0+a_1x+...+a_nx^n)$, $b(x) = (b_0+b_1x+...+b_nx^m)$ be elements of $R_1[x]$.

Since F is a ring homomorphism, we can prove G preserves addition as such:

$$\begin{aligned}
G(a(x)+b(x)) &= G((a_0+a_1x+...+a_nx^n)+(b_0+b_1x+...+b_mx^m)) \\
&= G(a_0+a_1x+...+a_nx^n+b_0+b_1x+...+b_mx^m) \\
&= F(a_0)+F(a_1)x+...+F(a_n)x^n+F(b_0)+F(b_1)x+...+F(b_m)x^m \\
&= (F(a_0)+F(a_1)x+...+F(a_n)x^n)+(F(b_0)+F(b_1)x+...+F(b_m)x^m) \\
&= G(a_0+a_1x+...+a_nx^n)+G(b_0+b_1x+...+b_mx^m) \\
&= G(a(x))+G(b(x))
\end{aligned}$$

We can prove G preserves multiplication as such:

$$\begin{aligned}
G((ab)(x)) &= G((a_0+a_1x+...+a_nx^n)(b_0+b_1x+...+b_mx^m)) \\
&= G(a_0b_0+(a_0b_1+a_1b_0)x+(a_0b_2+a_1b_1+a_2b_0)x^2+...+a_nb_mx^{x+m}) \\
&= F(a_0b_0)+F(a_0b_1+a_1b_0)x+F(a_0b_2+a_1b_1+a_2b_0)x^2+...+F(a_nb_m)x^{x+m} \\
&= (F(a_0)+F(a_1)x+...+F(a_n)x^n)(F(b_0)+F(b_1)x+...+F(b_m)x^m) \\
&= G(a_0+a_1x+...+a_nx^n)G(b_0+b_1x+...+b_mx^m) \\
&= G(a(x))G(b(x))
\end{aligned}$$

Therefore G is also a a ring homomorphism.