Homework 9

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1. (a) Let I be a nonempty ideal in F such that I does not equal F. Let $x \in F$ and $x \notin I$, let $a \in I$, and let a^{-1} be the multiplicative inverse of a. If $a \in I$, then $ax \in I$ and $a^{-1}ax \in I$. However, $a^{-1}ax = 1_F x = x$ and $x \notin I$ so this is a contradiction.

   (b) Since f is a homomorphism of rings, the kernel of f is an ideal of F. The only ideals of F are F and (0). If the kernel of f is (0), then f is injective. If the kernel of f is F, then f is the zero function.

2. First, assume $(p(x))$ is maximal. Let $p(x) = q_1(x)q_2(x)...q_n(x)$ be the prime factorization of $p(x)$. Because $q_i(x)$ divides $p(x)$ for some nonconstant $q_i(x)$ $(1 \leq i \leq n)$, $(p(x)) \subset (q_i(x))$. Since $(p(x))$ is maximal, either $(p(x)) = (q_i(x))$ or $F[x] = (q_i(x))$.

   If $F[x] = (q_i(x))$, then $q_i(x)$ must be a unit, which is a contradiction. Thus, $(p(x)) = (q_i(x))$. Therefore $p(x)$ must be some divisor of $q_i(x)$ such that
$$q_i(x) = p(x)q_1^{-1}(x)q_2^{-1}(x)...q_{i-1}^{-1}(x)q_{i+1}^{-1}(x)...q_n^{-1}(x).$$

   This implies that $q_1(x)$, $q_2(x)$, ... $q_{i-1}(x)$, $q_{i+1}(x)$ ... $q_n^(x)$ are units, which are nonzero constants. If $p(x)$ is the product of an irreducible polynomial $q_i(x)$ and several nonzero constant polynomials, then $p(x)$ is irreducible.

   Next, assume $p(x)$ is irreducible. Since $p(x)$ is irreducible, the quotient ring $F[x]/(p(x))$ is a field and therefore $(p(x))$ is a maximal ideal.

3. (a) By the Division Algorithm,
$$a(x) = b(x)q(x) + r(x)$$
$$0 \leq \deg r(x) < \deg b(x).$$

   By Theorem 6.1, $b(x)q(x) \in I$ and $a(x) - b(x)q(x) = r(x) \in I$.

   (b) Let $a(x)$ by any nonzero polynomial in I. By the Division Algorithm, $a(x) = p(x)q(x) + r(x)$ such that $0 \leq \deg r(x) < \deg p(x)$. However, $p(x)$ is of minimal degree, so $\deg r(x) = 0$. Therefore, $p(x)$ divides all $a(x) \in I$.

4. (a) Let $I = \{ f(x) \in F[x] \mid f(\alpha) = 0 \}$. We know that I is an ideal of $F[x]$ because for all $f_1(x)$, $f_2(x) \in I$,

$$f_1(\alpha) - f_2(\alpha) = 0 - 0 = 0$$

So $f_1(x) - f_2(x) \in I$. And, for all $g(x) \in F[x]$ and $f(x) \in I$,

$$f(\alpha)g(\alpha) = 0 * g(\alpha) = 0$$

$$g(\alpha)f(\alpha) = g(\alpha) * 0 = 0$$

So $f(x)g(x) \in I$ and $g(x)f(x) \in I$, and I is a proper ideal of $F[x]$.

Since α is a root of $p(x)$, $p(\alpha) = 0$ and $p(\alpha) \in I$. Because all ideals of $F[x]$ are principal ideals and $p(x)$ is irreducible, $I = (p(x))$ and all elements of I are divisible by $p(x)$. If $g(\alpha) = 0$, then $g(x) \in I$ and $p(x)|g(x)$.

Next, assume $p(x)|g(x)$. Then $g(x) = p(x)q(x)$ for some $q(x)$, and so

$$g(\alpha) = p(\alpha)q(\alpha) = 0.$$

Thus, $g(\alpha) = 0$ if and only if $p(x)|g(x)$.

(b) Let $I = \left\{ f(x) \in \mathbb{Q} \,\middle|\, f(\sqrt[7]{2}) = 0 \right\}$. Let $p(x) = x^2 - 2$. By Eisenstein's Criterion, $x^7 - 2$ is irreducible, and $p(\sqrt[7]{2}) = 0$ so $p(x) \in I$. By problem 4a, any nonzero $g(x) \in I$ must be divisible by $p(x)$, meaning that $\deg g \geq 7$.

(c) Let $g : F[x] \to F[\alpha]$ be a homomorphism of rings given by $g(f(x)) = f(\alpha)$ such that $f(x) \in F[x]$. Then the kernel of g is $(p(x))$. We know that g is surjective, because all $f(\alpha)$ map to some $f(x)$ so that there is an $f(x)$ for all $f(\alpha)$ such that $g(f(x)) = f(\alpha)$. Thus, by Theorem 6.13, $F[x]/(p(x))$ is isomorphic to $F[\alpha]$.