

# Homework 7

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1. If  $f(x) = x^7 - 7x + 20$ , then

$$f(x+1) = x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 14$$

By Eisenstein's Criterion,  $f(x+1)$  is irreducible in  $\mathbb{Q}[x]$  for  $p = 7$ , and so  $f(x)$  is irreducible in  $\mathbb{Q}[x]$ .

2. Let  $f(x) = x^5 - 4x^4 + 4x^3 + 3x^2 - 26x + 21$ . If  $p = 2$ , then  $\bar{f}(x) = x^5 + x^2 + 1$  in  $\mathbb{Z}_p[x]$ . Since  $\bar{f}(0) = \bar{f}(1) = 1$ ,  $\bar{f}(x)$  is irreducible in  $\mathbb{Z}_p[x]$ . Therefore,  $f(x)$  is irreducible in  $\mathbb{Q}[x]$ .

3. (a) Since  $-1+3i$  is a root of  $f(x)$ ,  $-1-3i$  is also a root of  $f(x)$ . Therefore

$$f(x) = (x+1+3i)(x+1-3i)h(x) = (x^2+2x+10)h(x)$$

for some polynomial  $h(x)$ .

By division,

$$h(x) = f(x)/(x^2+2x+10) = x^4 + x^2 - 6$$

Thus,

$$f(x) = (x^2+2x+10)(x^4+x^2-6) = (x^2+2x+10)(x^2+3)(x^2-2)$$

We know that  $(x^2+2x+10)$  and  $(x^2-2)$  are irreducible in  $\mathbb{Q}[x]$  by Eisenstein's Criterion for prime  $p = 2$ , and  $(x^2+3)$  is irreducible in  $\mathbb{Q}[x]$  by Eisenstein's Criterion for prime  $p = 3$ . Therefore, the irreducible factorization of  $f(x)$  in  $\mathbb{Q}[x]$  is

$$f(x) = (x^2+2x+10)(x^2+3)(x^2-2)$$

- (b) In  $\mathbb{R}[x]$ ,

$$f(x) = (x^2+2x+10)(x^2+3)(x^2-2) = (x^2+2x+10)(x^2+3)(x+\sqrt{2})(x-\sqrt{2})$$

We know that  $(x^2+2x+10)$  and  $(x^2+3)$  are irreducible in  $\mathbb{R}[x]$  because a polynomial of the form  $f(x) = ax^2 + bx + c$  is irreducible in  $\mathbb{R}[x]$  if  $b^2 - 4ac < 0$ . Therefore, the irreducible factorization of  $f(x)$  in  $\mathbb{R}[x]$  is

$$(x^2+2x+10)(x^2+3)(x+\sqrt{2})(x-\sqrt{2})$$

(c) In  $\mathbb{C}[x]$ , the irreducible factorization of  $f(x)$  is

$$f(x) = (x + 1 + 3i)(x + 1 - i)(x + \sqrt{3}i)(x - \sqrt{3}i)(x + \sqrt{2})(x - \sqrt{2})$$

We know that each of these polynomials are irreducible in  $\mathbb{C}[x]$  because they all have degree 1.