

Homework 3

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1. The set $\frac{1}{2}\mathbb{Z}$ is not a ring. A ring must be closed under multiplication. Let $a = \frac{m}{2}$ and $b = \frac{n}{2}$ for some integers m, n . We know that $ab = \frac{mn}{4}$. Assume m and n are both odd and that $m = 2x + 1$ and $n = 2y + 1$ for some integers x, y . Therefore,

$$ab = \frac{mn}{4} = \frac{(2x+1)(2y+1)}{4} = \frac{2(2xy+y)+1}{4}$$

Since ab is not an element of the set, this means that the set is not closed under multiplication.

2. This set is a ring.
Let $a = \frac{m}{2x+1}$, $b = \frac{n}{2y+1}$, $c = \frac{l}{2z+1}$ be elements of the set for some integers m, n, l, x, y , and z .
The set is closed under addition:

$$a + b = \frac{m}{2x+1} + \frac{n}{2y+1} = \frac{m(2y+1) + n(2x+1)}{2(2xy+x+y)+1}$$

Associative addition holds:

$$\begin{aligned} a + (b + c) &= \frac{m}{2x+1} + \left(\frac{n}{2y+1} + \frac{l}{2z+1} \right) \\ &= \frac{m(2y+1)(2z+1) + n(2z+1)(2x+1) + l(2y+1)(2x+1)}{(2x+1)(2y+1)(2z+1)} \\ &= \left(\frac{m}{2x+1} + \frac{n}{2y+1} \right) + \frac{l}{2z+1} = (a + b) + c \end{aligned}$$

Commutative addition holds:

$$\begin{aligned} a + b &= \frac{m}{2x+1} + \frac{n}{2y+1} \\ &= \frac{m(2y+1) + n(2x+1)}{2(2xy+x+y)+1} \\ &= \frac{n}{2y+1} + \frac{m}{2x+1} = b + a \end{aligned}$$

There exists an 0 element in the set such that:

$$a + 0 = \frac{m}{2x+1} + \frac{0}{1} = \frac{m}{2x+1} = a = \frac{0}{1} + \frac{m}{2x+1} = 0 + a$$

There is a solution to $a + x = 0$. Let $x = \frac{-m}{2x+1}$:

$$a + x = \frac{m}{2x+1} + \frac{-m}{2x+1} = \frac{m + -m}{2x+1} = \frac{0}{2x+1} = 0$$

The set is closed under multiplication:

$$ab = \frac{m}{2x+1} \cdot \frac{n}{2y+1} = \frac{mn}{(2x+1)(2y+1)} = \frac{mn}{2(2xy+x+y)+1}$$

Associative multiplication holds:

$$a(bc) = \frac{m}{2x+1} \cdot \left(\frac{n}{2y+1} \cdot \frac{l}{2z+1} \right) = \left(\frac{m}{2x+1} \cdot \frac{n}{2y+1} \right) \cdot \frac{l}{2z+1} = (ab)c$$

The Distributive Property holds:

$$\begin{aligned} a(b+c) &= \frac{m}{2x+1} \cdot \left(\frac{n}{2y+1} + \frac{l}{2z+1} \right) = \frac{m}{2x+1} \cdot \left(\frac{n(2z+1) + l(2y+1)}{(2y+1)(2z+1)} \right) \\ &= \frac{m}{2x+1} \cdot \frac{n}{2y+1} + \frac{m}{2x+1} \cdot \frac{l}{2z+1} = ab + ac \end{aligned}$$

3. TODO

4. TODO

5. Assume M is a unit. Therefore there exists some $M^{-1} \in M_2(\mathbb{Z})$ such that $MM^{-1} = I_2$. We find M^{-1} using Gaussian Elimination:

$$\begin{aligned} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array} \right] &\Leftrightarrow \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & \frac{ad-bc}{a} & \frac{-c}{a} & 1 \end{array} \right] \\ \Leftrightarrow \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array} \right] &\Leftrightarrow \left[\begin{array}{cc|cc} a & 0 & \frac{ad}{ad-bc} & \frac{-ab}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array} \right] \\ &\Leftrightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array} \right] \end{aligned}$$

Therefore $M^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$. Since $M^{-1} \in M_2(\mathbb{Z})$, each of its elements must be integers. Its elements can only be integers if $ad - bc$ evenly divides a , b , c , and d .

Let m be the greatest common denominator of a , b , c , and d . Let $a = mx$, $b = my$, $c = mz$, and $d = mw$ for some integers x , y , z , and w that are not divisible by m . As such $ad - bc = mxmw - mymz = m^2(xw - yz)$. Since m is the greatest common denominator and m^2 is also a common denominator, $m^2 \leq m$ and this is only possible if $m = 1$. Therefore, the greatest common denominator of a , b , c , and d is 1. If this is the case,

$ad - bc$ must be ± 1 since only ± 1 can evenly a , b , c , and d .

Next, assume $ad - bc = \pm 1$. By the last proof, M can only be a unit if $MM^{-1} = I_2$ and $M^{-1} \in M_2(\mathbb{Z})$. If $ad - bc = 1$, then $M^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and $MM^{-1} = I_2$. If $ad - bc = -1$, then $M^{-1} = \begin{bmatrix} -d & b \\ c & -a \end{bmatrix}$ and $MM^{-1} = I_2$.

Thus, $M \in M_2(\mathbb{Z})$ is a unit if and only if $ad - bc = \pm 1$.