Homework 8

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1. (a) Since $(x^7 + 25x^6 - 25x + 5)$ is a nonconstant polynomial in $\mathbb{Q}$[x], $\mathbb{Q}[x]/(x^7 + 25x^6 - 25x + 5)$ is a field if and only if $(x^7 + 25x^6 - 25x + 5)$ is irreducible in $\mathbb{Q}$[x]. We know that $(x^7 + 25x^6 - 25x + 5)$ is irreducible in $\mathbb{Q}$[x] by Eisenstein's criterion for prime 5. Thus, $\mathbb{Q}[x]/(x^7 + 25x^6 - 25x + 5)$ is a field.

   (b) Consider $\mathbb{Z}/2\mathbb{Z}$. If $f(x) = x^3 + 2x^2 - x + 1$, then $\overline{f}(x) = x^3 - x + 1$. We know that $\overline{f}(x)$ is irreducible in $\mathbb{Z}/2\mathbb{Z}$ because $\overline{f}(0) = \overline{f}(1) = 1$ and $\deg \overline{f}(x) = 3$. Since $\overline{f}(x)$ is irreducible in $\mathbb{Z}/2\mathbb{Z}$, $f(x)$ is irreducible in $\mathbb{Q}$, which means that $\mathbb{Q}[x]/(x^3 + 2x^2 - x + 1)$ is a field.

   (c) Only first-degree and second-degree polynomials can be irreducible in $\mathbb{R}[x]$. Since $(x^5 + 42x^4 + \pi x^3 - 1729x^2 + \ln(2)x - 2019)$ is a fifth-degree polynomial, it is reducible in $\mathbb{R}[x]$. Therefore $\mathbb{R}[x]/(x^5 + 42x^4 + \pi x^3 - 1729x^2 + ln(2)x - 2019)$ is NOT a field.

2. (a) Let $f(x) = x^3 + 2x + 1$. We know that $f(x)$ is irreducible over $\mathbb{Z}/3\mathbb{Z}$ because $f(0) = f(1) = f(2) = 1$ and $\deg f(x) = 3$. Therefore, K is a field.

   (b) We know that $f(t)$ does not have any roots in K because $a \in \mathbb{Z}/3\mathbb{Z}$ is a root of $f(t)$ if and only if $t - a$ is a factor of $f(t)$. By definition of $\mathbb{Z}/3\mathbb{Z}$, a can only be 0, 1, or 2. However, by the Division Algorithm,

$$t^3 + 2t + 1 = t(t^2 + 2) + 1,$$

$$t^3 + 2t + 1 = (t + 1)(t^2 - t) + 1,$$

   and

$$t^3 + 2t + 1 = (t + 2)(t^2 - 2t) + 1.$$

   Therefore, $f(t)$ has no roots.

3. To prove that $(a, b) = (d)$, first let us prove that $(a, b) \subseteq (d)$. Let $ar_1 + br_2$ be any element of (a, b). Since d is the greatest common divisor of a and b, $a = dx$ and $b = dy$ for some integers x and y. Therefore, $ar_1 + br_2 = dxr_1 + dyr_2 = d(xr_1 + dyr_2)$, which is an element of (d). Thus, $(a, b) \subseteq (d)$.

   Next, let us prove $(d) \subseteq (a, b)$. Since d is the greatest common divisor of a and b, $d = au + bv$ for some integers u and v. Let rd be any element of (d). Therefore, $rd = r(au + bv) = rau + rbv = a(ru) + b(rv) \in (a, b)$. Thus $(d) \subseteq (a, b)$.

Since $(d) \subseteq (a, b)$ and $(a, b) \subseteq (d)$, $(a, b) = (d)$.

4. Let $f(x) \in I$ and $g(x) \in \mathbb{Z}[x]$ such that

$$f(x) = a_0 + a_1x + a_2x^2 + ... + a_{n-1}x^{n-1} + 2a_nx^n$$

and

$$g(x) = b_0 + b_1x + b_2x^2 + ... + b_{m-1}x^{m-1} + 2b_mx^m.$$

Then,

$$f(x)*g(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b1 + a_2b_0) + ... + 2a_nb_mx^{n+m}$$

and

$$g(x)*f(x) = b_0a_0 + (b_0a_1 + b_1a_0)x + (b_0a_2 + b_1a1 + b_2a_0) + ... + 2b_ma_nx^{m+n}.$$

Since the leading coefficients of $f(x) * g(x)$ and $g(x) * f(x)$ are both even, $f(x) * g(x) \in I$ and $g(x) * f(x) \in I$. Therefore, I is an ideal of $\mathbb{Z}[x]$.

5. Let $f(x)$, $g(x) \in I$ such that $g(x) \neq 0_F$. By the Division Algorithm, $g(x) = f(x)q(x) + r(x)$ such that $r(x) = 0_F$ or $\deg r(x) < \deg g(x)$. By the definition of I,

$$g(7) = f(7)q(7) + r(7)$$
$$0 = f0 * q(7) + r(7)$$
$$0 = r(7).$$

Therefore, $r(x) \in I$ and $f(x)$ divides $g(x)$. Therefore, I is finitely-generated.