MATH 307
Assignment #13
Due Monday, May 2, 2022
CORRECTION: Due date is Monday 5/2; Problem #5 inserted "null" in the equation for V. April 26.

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. Define $T \in \mathcal{L}(\mathbf{C}^2)$ by $T(w, z) = (0, w)$. Find all generalized eigenvectors of T.

2. Define $T \in \mathcal{L}(\mathbf{C}^2)$ by $T(w, z) = (z, -w)$. Find the generalized eigenspaces corresponding to the distinct eigenvalues of T. (Note Example 5.8 is an analogous transformation.)

3. Suppose $T \in \mathcal{L}(V)$ and $\alpha, \beta \in \mathbf{F}$ with $\alpha \neq \beta$. Prove that $G(\alpha, T) \cap G(\beta, T) = \{0\}$.

4. Suppose that $T \in \mathcal{L}(\mathbf{C}^3)$ is defined by $T(z_1, z_2, z_3) = (z_2, z_3, 0)$. Prove that T has no square root. More precisely, prove that there does not exist $S \in \mathcal{L}(\mathbf{C}^3)$ such that $S^2 = T$.

5. Suppose that $T \in \mathcal{L}(V)$ is not nilpotent. Let $n = \dim V$. Show that
   $V = \text{null } T^{n-1} \oplus \text{range } T^{n-1}$.

6. Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible. Prove that T and $S^{-1}TS$ have the same eigenvalues with the same multiplicities.

7. Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Prove that V has a basis consisting of eigenvectors of T if and only if every generalized eigenvector of T is an eigenvector of T.

8. Define $N \in \mathcal{L}(\mathbf{F}^5)$ by

   $$N(x_1, x_2, x_3, x_4, x_5) = (2x_2, 3x_3, -x_4, 4x_5, 0).$$

   Find a square root of $I + N$.

9. Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Prove that there exists $D, N \in \mathcal{L}(V)$ such that $T = D + N$, the operator D is diagonalizable, N is nilpotent, and $DN = ND$.