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MATH 307 - Spring 2022
Assignment #4
Due Friday, 02-11-22, 16:00 CST

For each Problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. (a) Find linear map $T : \mathbf{R}^4 \to \mathbf{R}^4$ so that range $T =$ null T.

Solution: Define $T : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$T(a,\ b,\ c,\ d) = (0,\ 0,\ a,\ b).$$

Then range $T = \{(0,\ 0,\ c,\ d) \mid c,\ d \in \mathbb{R}\}$
and null $T = \{(0,\ 0,\ c,\ d) \mid c,\ d \in \mathbb{R}\}$.

Therefore range $T =$ null T.

(b) Show that there is no linear map $T : \mathbf{R}^5 \to \mathbf{R}^5$ so that range $T =$ null T.

Solution: Suppose there exists a transformation T from $\mathbb{R}^5 \to \mathbb{R}^5$ such that range $T =$ null T. Then this implies that $\dim \operatorname{range} T = \dim \operatorname{null} T$. By the Fundemental Theorem of Linear Maps,

$$\dim\ V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Because $V = \mathbb{R}^5$ and dim $\mathbb{R}^5 = 5$, we can substitute in for dim V to get

$$5 = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Now, if we let $x = \dim \operatorname{range} T = \dim \operatorname{null} T$, then we can substitute in to obtain

$$5 = x + x.$$

Solving for x, we see that $x = 2.5$. Hence, if such a map T existed, then $\dim \operatorname{range} T = \dim \operatorname{null} T = 2.5$. But a fractal dimension is not possible in our course, therefore no such map T can exist.

2. Find a 4×4 matrix M so that the range of M is spanned by $(1, 0, 1, 0)$ and $(0, 1, 0, 1)$.

Solution: Let

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Then the range of M is

$$\begin{aligned} \operatorname{range} M &= \{M(a,\ b,\ c,\ d)^T : a,\ b,\ c,\ d \in R\} \\ &= \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} : a,\ b,\ c,\ d \in \mathbb{R} \right\} \\ &= \{(a,\ b,\ a,\ b)^T : a,\ b \in \mathbb{R}\} \\ &= \{a(1,\ 0,\ 1,\ 0)^T + b(0,\ 1,\ 0,\ 1)^T : a,\ b \in \mathbb{R}\} \end{aligned}$$

Which is by definition the span of $(1,\ 0,\ 1,\ 0)$ and $(0,\ 1,\ 0,\ 1)$. Therefore,

$$\operatorname{range} M = \operatorname{span}\{(1,\ 0,\ 1,\ 0),\ (0,\ 1,\ 0,\ 1)\}.$$

3. (a) Give an example of a linear map on a three-dimensional space with a two-dimensional range.

Solution: Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(x,\ y,\ z) = (x,\ y,\ 0).$$

Then $\operatorname{range} T = \{(x,\ y,\ 0) \mid x, y \in \mathbb{R}\}$ and $\dim \operatorname{range} T = 2$.

(b) Give an example of a linear map on a three-dimensional space with a two-dimensional null-space.

Solution: Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(x,\, y,\, z) = (x,\, 0,\, 0).$$

Then $\operatorname{null} T = \{(0,\, y,\, z) \mid y, z \in \mathbb{R}\}$ and $\dim \operatorname{null} T = 2$.

4. Let $T : V \to V$ be a linear map with a one-dimensional range. Prove that $T^2 = cT$ for some scalar c. (This means that $T(Tv) = cTv$ for all $v \in V$.)

Proof. Fix $v \in V$.

If $\dim \operatorname{range} T = 1$ then there is only one element in the basis of the range. Therefore, there exists some element $w \in V$ such that $\operatorname{range} T = \operatorname{span} w$. Then,

$$Tv \in \operatorname{range} T = \operatorname{span} w = \{\alpha w : \alpha \in \mathbb{F}\}. \tag{4.1}$$

By (4.1), there exists scalars β and γ in $\mathbb{F}$ such that $Tv = \beta w$ and $Tw = \gamma w$. Therefore,

$$\begin{aligned}
T(Tv) &= T(\beta w) && \text{By } Tv = \beta w. \\
&= \beta T w && \text{By homogeneity of } T. \\
&= \beta\gamma w && \text{By } Tw = \gamma w. \\
&= \gamma\beta w && \text{By commutativity over } \mathbb{F}. \\
&= \gamma T v. && \text{By } Tv = \beta w.
\end{aligned}$$

Letting $c = \gamma$ it has been shown that $T(Tv) = cTv$ (for some scalar c) when $T : V \to V$ has a one dimensional range. $\square$

5. Let $D \in \mathcal{L}(\mathcal{P}_3(\mathbf{R}), \mathcal{P}_2(\mathbf{R}))$ denote the differentiation map $Dp = p'$. Example 3.34 gives the matrix of D with respect to the usual bases for $\mathcal{P}_3(\mathbf{R})$ and $\mathcal{P}_2(\mathbf{R})$.
   Find two new bases for $\mathcal{P}_3(\mathbf{R})$ and $\mathcal{P}_2(\mathbf{R})$ so that the matrix for D with respect to these bases is
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Solution: Fix the basis of $\mathcal{P}_2(\mathbf{R})$ to be $\{1, x, x^2\}$. Pulling from the columns of the given matrix as coefficients for with this basis, we can find the preimage from the definitions of derivitives. Thus,

$$1(1) + 0(x) + 0(x^2) = 1 = \frac{d}{dx}(x) = D(x).$$
$$0(1) + 1(x) + 0(x^2) = x = \frac{d}{dx}\left(\frac{x^2}{2}\right) = D\left(\frac{x^2}{2}\right).$$
$$0(1) + 0(x) + 1(x^2) = x^2 = \frac{d}{dx}\left(\frac{x^3}{3}\right) = D\left(\frac{x^3}{3}\right).$$
$$0(1) + 0(x) + 0(x^2) = 0 = \frac{d}{dx}(1) = D(1).$$

Therefore our ordered basis for $\mathcal{P}_3(\mathbf{R}) = (x,\ \frac{x^2}{2},\ \frac{x^3}{3},\ 1)$.

6. The general operation of finding an antiderivative is not a linear map because of the "$+C$" which means that any function has infinitely many antiderivatives. Let's define a linear map from $\mathcal{P}_3(\mathbf{R})$ to $\mathcal{P}_4(\mathbf{R})$ that avoids ambiguity. Let $A(a_0+a_1x+a_2x^2+a_3x^3) = a_0x + (a_1/2)x^2 + (a_2/3)x^3 + (a_3/4)x^4$.

   (a) Find the matrix of A with respect to the standard bases for $\mathcal{P}_3(\mathbf{R})$ and $\mathcal{P}_4(\mathbf{R})$.

Solution: Using the definition of antiderivatives, we need to write the antiderivative of each element in the standard basis of $\mathcal{P}_3(\mathbf{R})$ as a linear combination of the standard basis of $\mathcal{P}_4(\mathbf{R})$. Thus,

$$\begin{aligned}
A(x^0) &= \frac{x^1}{1} = 0(1) + 1(x) + 0(x^2) + 0(x^3) + 0(x^4). \\
A(x^1) &= \frac{x^2}{2} = 0(1) + 0(x) + \frac{1}{2}(x^2) + 0(x^3) + 0(x^4). \\
A(x^2) &= \frac{x^3}{3} = 0(1) + 0(x) + 0(x^2) + \frac{1}{3}(x^3) + 0(x^4). \\
A(x^3) &= \frac{x^4}{4} = 0(1) + 0(x) + 0(x^2) + 0(x^3) + \frac{1}{4}(x^4).
\end{aligned}$$

Turning the coefficients on the right hand side of the four bases into column vectors, we have

$$\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1/2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1/3\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1/4\end{pmatrix}.$$

Turning these into the columns of a 5x4 matrix, A, we get

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}.$$

(b) Find new bases for $\mathcal{P}_3(\mathbf{R})$ and $\mathcal{P}_4(\mathbf{R})$ so that the matrix for A with respect to the new bases is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Solution: We essentially need to reverse the methodology from part (a). We will fix our basis of $\mathcal{P}_3(\mathbf{R})$ to be $\{1, x, x^2, x^3\}$ and the basis of $\mathcal{P}_4(\mathbf{R})$ to be some $\{a, b, c, d, e\}$. Using its columns of A as coefficients, we need

$$\begin{aligned} A(1) &= \frac{x^1}{1} = 1(a) + 0(b) + 0(c) + 0(d) + 0(e). \\ A(x^1) &= \frac{x^2}{2} = 0(a) + 1(b) + 0(c) + 0(d) + 0(e). \\ A(x^2) &= \frac{x^3}{3} = 0(a) + 0(b) + 1(c) + 0(d) + 0(e). \\ A(x^3) &= \frac{x^4}{4} = 0(a) + 0(b) + 0(c) + 1(d) + 0(e). \end{aligned}$$

From this, we can see that

$$\begin{aligned} a &= x^1, \\ b &= x^2/2, \\ c &= x^3/3, \text{ and} \\ d &= x^4/4. \end{aligned}$$

And our remaining term, e, in order to complete the basis of $\mathcal{P}_4(\mathbf{R})$, must be a constant term. Therefore we'll let $e = \pi$.

Thus, we have a fixed and derived basis,

$$\begin{aligned} \mathcal{P}_3(\mathbf{R}) &= \{1,\ x,\ x^2,\ x^3\} \quad \text{and} \\ \mathcal{P}_4(\mathbf{R}) &= \left\{x,\ \frac{x^2}{2},\ \frac{x^3}{3},\ \frac{x^4}{4},\ \pi\right\}. \end{aligned}$$