MATH 307
Assignment #10
Due Friday, April 1st, 2022

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. Suppose that T is a normal operator on V and that 3 and 4 are eigenvalues of T. Prove that there exists a vector $v \in V$ such that $\|v\| = \sqrt{2}$ and $\|Tv\| = 5$.

   Proof.
   Let $v_1, v_2 \in V$ be the eigenvectors corresponding to the eigenvalues 3 and 4 respectively. Then, by proposition 7.22 (since T is normal and 3 and 4 are distinct eigenvalues), v_1 and v_2 are orthogonal. Let $e_1, e_2 \in V$ be eigenvectors of T such that $e_1 := \frac{v_1}{\|v_1\|} \in E(3,T)$ and $e_2 := \frac{v_2}{\|v_2\|} \in E(4,T)$. Then, $\{e_1, e_2\}$ is a set of orthonormal eigenvectors of T in V.

   Let $v \in V$ denote the linear combination $v = e_1 + e_2$. Applying the Pythagorean Theorem (since e_1 and e_2 are orthogonal), we get that

$$
\begin{aligned}
\|v\| &= \sqrt{\|v\|^2} && \text{(Algebra)} \\
&= \sqrt{\|e_1 + e_2\|^2} && \text{(Substitution)} \\
&= \sqrt{\|e_1\|^2 + \|e_2\|^2} && \text{(Pythagorean Theorem)} \\
&= \sqrt{1+1} && \text{(normalized vectors)} \\
&= \sqrt{2}.
\end{aligned}
$$

   Therefore, there exists a vector $v \in V$ such that $\|v\| = \sqrt{2}$.

   We will now show that $\|Tv\| = 5$. Using the previous $v, e_1, e_2 \in V$, then

$$
\begin{aligned}
\|Tv\| &= \sqrt{\|Tv\|^2} && \text{(Algebra)} \\
&= \sqrt{\|Te_1 + Te_2\|^2} && \text{(Substitution and linearity)} \\
&= \sqrt{\|Te_1\|^2 + \|Te_2\|^2} && \text{(Pythagorean Theorem)} \\
&= \sqrt{\|3e_1\|^2 + \|4e_2\|^2} && (Tv = \lambda v) \\
&= \sqrt{(|3|\,\|e_1\|)^2 + (|4|\,\|e_2\|)^2} && \text{(Property of norms)} \\
&= \sqrt{9\,\|e_1\|^2 + 16\,\|e_2\|^2} && \text{(Simplify)} \\
&= \sqrt{9\sqrt{1^2} + 16\sqrt{1^2}} && \text{(Normalized vectors)} \\
&= 5 && \text{(Simplify)}
\end{aligned}
$$

   Hence, for $v \in V$ such that $v = e_1 + e_2$, then $\|Tv\| = 5$. Therefore, there exists a vector $v \in V$ such that $\|v\| = \sqrt{2}$ and $\|Tv\| = 5$. □

2. (a) Suppose that T is a self-adjoint operator on a finite-dimensional inner product space and that 2 and 3 are the only eigenvalues of T. Prove that $T^2 - 5T + 6I = 0$.

Proof.
Because self-adjoint implies normal, then regardless of if $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, V has an orthonormal basis consisting of eigenvectors of T of which has a diagonal matrix representation by either Spectral Theorem. Hence by direct proof,

$$
\begin{aligned}
T^2 - 5T + 6I &= (T - 2I)(T - 3I) \\
&= \left[\begin{pmatrix} 2 & & & & & \\ & \ddots & & & & \\ & & 2 & & & \\ & & & 3 & & \\ & & & & \ddots & \\ & & & & & 3 \end{pmatrix} - 2I\right] \cdot \left[\begin{pmatrix} 2 & & & & & \\ & \ddots & & & & \\ & & 2 & & & \\ & & & 3 & & \\ & & & & \ddots & \\ & & & & & 3 \end{pmatrix} - 3I\right] \\
&= \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & & & & & \\ & \ddots & & & & \\ & & -1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \cdot -1 & & & & & \\ & \ddots & & & & \\ & & 0 \cdot -1 & & & \\ & & & 1 \cdot 0 & & \\ & & & & \ddots & \\ & & & & & 1 \cdot 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix} \\
&= 0
\end{aligned}
$$

Therefore $T^2 - 5T + 6I = 0$ (the zero transformation) for all $v \in V$ if 2 and 3 are the only eigenvalues. □

(b) Give an example of an operator $T \in \mathcal{L}(\mathbf{C}^3)$ such that 2 and 3 are the only eigenvalues of T and $T^2 - 5T + 6I \neq 0$.

Solution:

Let the transformation matrix of operator T be be denoted by

$$\mathcal{M}(T) := \begin{bmatrix} 2 & \pi & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Then, it is clear by properties of upper-triangular matrices that 2 (with multiplicity 2), and 3 are the eigenvalues of T. Additionally, $T^2 = \begin{bmatrix} 4 & 4\pi & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$, and $5T$ and $6I$ are obvious.

Hence,

$$\begin{aligned} T^2 - 5T + 6I &= \begin{bmatrix} 4 & 4\pi & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} - \begin{bmatrix} 10 & 5\pi & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 15 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\pi & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Which certainly is not the zero transformation.

3. Suppose that T is a normal operator on V. Suppose also that $v, w \in V$ satisfy the equations

$$\|v\| = \|w\| = 2, \qquad Tv = 3v, \qquad Tw = 4w.$$

Show that $\|T(v+w)\| = 10$.

Solution:

By our hypotheses, $v \in E(3,T)$ and $w \in E(2,T)$. By proposition 7.22 (since T is normal and 3 and 4 are distict eigenvalues), then v and w are orthogonal. Hence,

$$\begin{aligned}
\|T(v+w)\| &= \sqrt{\|T(v+w)\|^2} && \text{(Algebra)}\\
&= \sqrt{\|Tv+Tw\|^2} && \text{(Linearity)}\\
&= \sqrt{\|3v+4w\|^2} && (Tv = \lambda v \text{ substitution (hypotheses)})\\
&= \sqrt{\|3v\|^2 + \|4w\|^2} && \text{(Pythagorean Theorem)}\\
&= \sqrt{3^2\,\|v\|^2 + 4^2\,\|w\|^2} && \text{(Properties of norms)}\\
&= \sqrt{9\,\|v\|^2 + 16\,\|w\|^2} && \text{(Simplify)}\\
&= \sqrt{9\cdot 2^2 + 16\cdot 2^2} && \text{(Substitution (hypotheses))}\\
&= \sqrt{36+64} && \text{(Simplify)}\\
&= 10 && \text{(Simplify)}
\end{aligned}$$

Therefore, under the hypotheses, $\|T(v+w)\| = 10$.

4. Suppose $T \in \mathcal{L}(V)$ is normal. Prove that range T = range T^*.

Proof.

Because T is normal, by proposition 7.20, $\|Tv\| = \|T^*v\|$ for all v. Since $Tw = 0$ for any vector $w \in \operatorname{null} T$ and $Tv = T^*v$, then $w \in \operatorname{null} T^*$. Similarly, it can be shown that for $T^*u = 0$ for $u \in \operatorname{null} T^*$ and $T^*v = Tv$, then $u \in \operatorname{null} T$. Hence $\operatorname{null} T = \operatorname{null} T^*$.

Then, using the table of properties regarding the null space and range of T and T^* (7.7),

$$\begin{aligned}
\operatorname{range} T &= (\operatorname{null} T^*)^\perp && \text{(7.7 d)}\\
&= (\operatorname{null} T)^\perp && (\operatorname{null} T = \operatorname{null} T^*)\\
&= \operatorname{range} T^* && \text{(7.7 b)}
\end{aligned}$$

Therefore $\operatorname{range} T = \operatorname{range} T^*$ for a normal operator $T \in \mathcal{L}(V)$.

□

5. Consider the statement: If $T \in \mathcal{L}(V)$ and there exists an orthonormal basis $e_1, \ldots, e_n$ of V such that $\|Te_j\| = \|T^*e_j\|$ for each j, then T is normal. Show that a counterexample to the statement is given by the matrix $T = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$ with respect to the standard basis in $\mathbf{R}^2$.

Solution:

We have $\mathcal{M}(T^*) = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$. Then,

For e_1,

$$\|Te_1\| = \left\| \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\| = 1,$$

and

$$\|T^*e_1\| = \left\| \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\| = 1.$$

So $\|Te_1\| = \|T^*e_1\|$ holds. As for e_2,

$$\|Te_2\| = \left\| \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\| = 5,$$

and

$$\|T^*e_1\| = \left\| \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\| = 5.$$

So $\|Te_2\| = \|T^*e_2\|$ holds, and hence the hypothesis $\left\|Te_j\right\| = \left\|T^*e_j\right\|$ holds for each j. The conclusion then states that T is normal, that is $TT^* = T^*T$. Checking this,

$$TT^* = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix},$$

and

$$T^*T = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

Since $TT^* \neq T^*T$, then T is **not** normal. Hence, a counterexample.

6. (CST) Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Prove that T is normal if and only if all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal and

$$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$$

where $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T.

Proof. ($\Longrightarrow$)

Since T is normal, by the Complex Spectral Theorem there exists an orthonormal basis of V consisting of eigenvectors. The eigenvectors corresponding to distinct eigenvalues is a subset of that orthonormal basis, and hence are all orthonormal, and thus orthogonal.

By proposition 5.38 (page 156) the sum of eigenspaces for distinct eigenvalues is a direct sum. Hence, $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$.

□

Proof. ($\Longleftarrow$)

For every index $i \in [1, m]$ we can form an orthonormal basis of $E(\lambda_i, T)$. Since this basis is formed from vectors inside an eigenspace, then are themselves eigenvectors. Since V is a direct sum of these eigenspaces, of which are orthonormal, we have an orthnormal basis of eigenvectors. Thus, by the Complex Spectral Theorem $[(b) \implies (a)]$, T is normal.

□

7. (CST) Prove that a normal operator on a complex inner product space is self-adjoint if and only if all its eigenvalues are real.

Proof. ($\Longrightarrow$)

By the Complex Spectral Theorem, a normal operator can be expressed as a diagonal matrix consisting of the eigenvalues. Let

$$\mathcal{M}(T) = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

for eigenvalues λ_i. Then by definition of T^* (conjugate transpose),

$$\mathcal{M}(T^*) = \begin{bmatrix} \bar{\lambda}_1 & & \\ & \ddots & \\ & & \bar{\lambda}_n \end{bmatrix}.$$

By the ($\Longrightarrow$) hypothesis, $T = T^*$, which implies $\lambda_1 = \bar{\lambda}_1, \ldots, \lambda_n = \bar{\lambda}_n$, which is only true if $\lambda_i \in \mathbb{R}$. Hence if T is self-adjoint, then all of its eigenvalues are real. □

Proof. ($\Longleftarrow$)

By the reasoning of the forward direction, we have that

$$\mathcal{M}(T) = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \quad \text{and} \quad \mathcal{M}(T^*) = \begin{bmatrix} \bar{\lambda}_1 & & \\ & \ddots & \\ & & \bar{\lambda}_n \end{bmatrix}.$$

Under the ($\Longleftarrow$) hypothesis, we assume $\lambda_i \in \mathbb{R}$, therefore $\bar{\lambda}_i = \lambda_i$ for every index $i \in [1, n]$. Hence,

$$M(T^*) = \begin{bmatrix} \bar{\lambda}_1 & & \\ & \ddots & \\ & & \bar{\lambda}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \mathcal{M}(T),$$

which is the definition of self-adjoint.

□

8. (CST) Suppose V is a complex inner product space. Prove that every normal operator on V has a square root. (An operator $R \in \mathcal{L}(V)$ is called a *square root* of T if $R^2 = T$.)

Proof. Let T be an arbitrary normal operator on V. Then By the Complex Spectral Theorem, since T is normal, there exists an orthonormal basis for V consisting of T's eigenvectors, which can further be used to create a diagonal matrix. Hence, the transformation matrix of T can be expressed as:

$$\mathcal{M}(T) = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

with λ_i denoting an eigenvalue of T and $\dim V = n$.

Now, let the transformation matrix of R be defined as

$$\mathcal{M}(R) = \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix}.$$

Then,

$$\mathcal{M}(R^2) = \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \mathcal{M}(T).$$

Hence, R is the square root of T. Note that the square root of any complex number is indeed a complex number (and its opposite) (this is left as an exercise for a MTH 403 student). $\square$