MTH 307 - Spring 2022
Assignment #6
Due: Friday, February 25, 2022 (4pm)

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. Label the following statements as being true or false. Provide some justification from the text for your label.

   (a) Every linear operator on an n-dimensional vector space has n distinct eigenvalues.

   > **Solution - False:** Some eigenvalues can have a multiplicity greater than 1, such as the identity matrix I_2 has the eigenvalues $\lambda = 1$ and $\lambda = 1$. Thus, it has a single *distinct* eigenvalue $\lambda = 1$ with multiplicity 2.

   (b) If a linear operator on a vector space over $\mathbf{R}$ has one eigenvector, then it has an infinite number of eigenvectors.

   > **Solution - True:** Let v be an eigenvector, then any non-zero vector in span$\{v\}$ is also an eigenvector. (All non-zero scalar multiples of v.) There are uncountably infinite non-zero scalars, c, in $\mathbb{R}$ or $\mathbb{C}$ such that $cv \in \text{span}\{v\}$, therefore the statement is true.

(c) There exists a square matrix with no eigenvectors. Eigenvalues must be nonzero scalars.

Solution - True: Example. Let $T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ in the field $\mathbb{R}$. Then for some $x, y \in \mathbb{R}$ such that $x \neq 0$ or $y \neq 0$, and for a vector $v = \langle x, y \rangle$, $(T - \lambda I_2)v = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$. Then we have the following system of equations:

$$-\lambda x - y = 0 \qquad\qquad x - \lambda y = 0.$$

The second equation can be rewritten as $x = \lambda y$ and substituting into the first we get

$$\begin{aligned} -\lambda(\lambda y) - y = 0 &\iff \quad -\lambda^2 y - y = 0 \\ &\iff \quad -\lambda^2 = 1 \\ &\iff \quad \lambda = \sqrt{-1} \notin \mathbb{R}. \end{aligned}$$

Thus the first equation yields zero eigenvalues and thus zero eigenvectors. Next, the first equation can be rewritten as $y = -\lambda x$. Substituting that value into the second equation, we get

$$\begin{aligned} x = \lambda y &\iff \quad x = -\lambda^2 x \\ &\iff \quad 1 = -\lambda^2 \\ &\iff \quad \lambda = \sqrt{-1} \notin \mathbb{R}. \end{aligned}$$

Therefore neither equation yields an eigenvalue or eigenvector and hence we have an example of a square matrix with no eigenvectors.

(d) Eigenvalues must be nonzero scalars.

Solution - False: Counterexample. Let $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in the field $\mathbb{R}$. Then for some $x, y \in \mathbb{R}$ such that $x \neq 0$ or $y \neq 0$, and for a vector $v = \langle x, y \rangle$, $(T - \lambda I_2)v = \begin{bmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$. Then we have the following system of equations:

$$(1-\lambda)x = 0 \qquad\qquad -\lambda y = 0.$$

Looking at the second equation, the only way it is true is if $\lambda = 0$, therefore we have a contradiction that there exists no non-zero eigenvalue.

(e) Any two eigenvectors are linearly independent.

Solution - False: Taking the matrix setup from (d), we have that $\lambda = 0$ is an eigenvalue. Computing an eigenvector, we first have

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda v = 0v.$$

From here we can see that $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a valid eigenvector for $\lambda = 0$ since $Tv = 0v$ holds true. Additionally, $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ would also be an eigenvector (or any non-zero scalar). But these are linearly dependent, hence a contradiction.

(f) The sum of two eigenvalues of a linear operator T is also an eigenvalue of T.

Solution - False: Consider the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. It has eigenvalues $\lambda = 1$ and $\lambda = 2$ by the triangular matrix diagonal property. But $2 + 1 = 3$ is not an eigenvalue since we can only have 2 eigenvalues and we already showed them to be 1 and 2.

(g) Linear operators on infinite-dimensional vectors spaces never have eigenvalues.

Define $T \in \mathcal{L}(\mathbb{R}^\infty)$ by

$$T(x_1, x_2, \dots) = (x_1, 0, \dots).$$

Then T has an eigenvalue $\lambda = 1$ with eigenspace of $\operatorname{span}(1, 0, \dots)$. Hence, a contradiction.

(h) The sum of two eigenvectors of an operator T is always an eigenvector of T.

Solution - False: Using the same setup for T as in part (d), we have $\lambda = 1$ and $\lambda = 2$ as the 2 distinct eigenvalues with $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as respective eigenvectors. But the sum, $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not in either eigenspace. Hence a contradiction to the assumption.

2. Consider the operator $T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ acting on $\mathbf{R}^2$. How many subspaces are there that are invariant under T?

Solution: First, the trival subspaces of span$\{0\}$ and $\mathbb{R}^2$ hold true. Then using the triangular matrix property to get the eigenvalues $\lambda = 0$ and $\lambda = 1$, we can compute eigenvectors for them.

For $\lambda = 0$,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1x \\ 0y \end{pmatrix} = 0v = 0 \quad \text{implies} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \operatorname{span}(\langle 0, 1 \rangle).$$

For $\lambda = 1$,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1x \\ 0y \end{pmatrix} = 1v = v \quad \text{implies} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \operatorname{span}(\langle 1, 0 \rangle).$$

So there are 4 distinct invariant subspaces,

$$\operatorname{span}(\vec{0}) \qquad \operatorname{span}(\langle 0, 1 \rangle) \qquad \operatorname{span}(\langle 1, 0 \rangle) \qquad \mathbb{R}^2$$

3. If U and W are invariant subspaces for $T \in \mathcal{L}(V)$ then $U + W$ is invariant for T.

Proof. Let $u \in U$ and $w \in W$. Then $T(u + w) = T(u) + T(w)$ by linearity.

By our assumption of invariance, $T(u) \in U$ and $T(w) \in W$. Therefore $T(u) + T(w) \in U + W$ by the definition of Sum of Subspaces. This implies $T(u + w) \in U + W$ by linearity. Hence, $U + W$ is invariant under T. $\square$

4. In $\mathbf{R}^2$, let T be the reflection across the line $y = x$.

(a) Write the matrix A that represents T relative to the standard basis.

Solution:

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

$$T(x, y) = (y, x)$$

Computing the images of the standard basis,

$$T(e_1) = (0, 1) \qquad \text{and} \qquad T(e_2) = (1, 0).$$

Therefore,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(b) Determine two invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ for T such that $\mathbf{R}^2 = \mathcal{M} \oplus \mathcal{N}$ where neither $\mathcal{M}$ nor $\mathcal{N}$ is the zero subspace.

Solution: Let $v \in \mathbb{R}^2$ denoted by $v = \langle x, y \rangle$ such that $v \neq \vec{0}$, then

$$\begin{aligned}(A - \lambda I_2)v = \vec{0} &\iff \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\iff -\lambda x + y = 0 \quad \text{and} \quad x - \lambda y = 0 \\ &\iff x = \lambda y \quad \text{and} \quad y = \lambda x \\ &\iff x = \lambda^2 x \quad \text{and} \quad y = \lambda^2 y \\ &\iff \lambda = \pm 1.\end{aligned}$$

For $\lambda = 1$, $Av = 1v$ implies that $y = x$ and $x = y$, so we get an eigenvectors of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda = -1$, $Av = -v$ implies that $y = -x$ and $x = -y$, so we get an eigenvectors of $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Therefore we can let $\mathcal{M} = \text{span} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and let $\mathcal{N} = \text{span} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

These are linearly independent and invariant under T because $\mathcal{M} \cap \mathcal{N} = \vec{0}$. Therefore, $\mathbb{R}^2 = \mathcal{M} \oplus \mathcal{N}$.

(c) Write a basis $\{u_1, u_2\}$ for $\mathbf{R}^2$ so that $\mathcal{M} = \text{span } (u_1)$ and $\mathcal{N} = \text{span } (u_2)$.

Solution: Using the conclusion of part (b), we can let the basis be

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

(d) Write the matrix B that represents T relative to the basis $\{u_1, u_2\}$.

Solution:

$$T \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad T \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Using these as the columns in matrix B,

$$B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

5. (a) Let $V = \mathbf{R}^2$. Find eigenvalues and eigenvectors for the linear operator T defined by $T(x, y) = (2y, x)$.

Let A be a transformation matrix for T, then

$$A = [T(e_1) \quad T(e_2)] = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}.$$

Let $v \in \mathbb{R}^2$ denoted by $v = \langle x, y \rangle$ such that $v \neq \vec{0}$, then

$$\begin{aligned}
(A - \lambda I_2)v = \vec{0} &\iff \begin{bmatrix} -\lambda & 2 \\ 1 & -\lambda \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
&\iff -\lambda x + 2y = 0 \quad \text{and} \quad x - \lambda y = 0 \\
&\iff 2y = \lambda x \quad \text{and} \quad x = \lambda y \\
&\iff 2y = \lambda^2 y \quad \text{and} \quad x = \frac{\lambda^2 x}{2} \\
&\iff 2 = \lambda^2 \quad \text{and} \quad 1 = \frac{\lambda^2}{2} \\
&\iff \lambda = \pm\sqrt{2}.
\end{aligned}$$

Therefore the eigenvalues are $\lambda = \sqrt{2}$ and $\lambda = -\sqrt{2}$.

Computing eigenvectors for $\lambda = \sqrt{2}$,

$$\begin{aligned}
Av = \lambda v &\iff \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\sqrt{2} \\ y\sqrt{2} \end{pmatrix} \\
&\iff 2y = x\sqrt{2} \quad \text{and} \quad x = y\sqrt{2}
\end{aligned}$$

Which we can therefore conclude an eigenvector $\begin{pmatrix} 2 \\ \sqrt{2} \end{pmatrix}$ for $\lambda = \sqrt{2}$.

Computing eigenvectors for $\lambda = -\sqrt{2}$,

$$\begin{aligned}
Av = \lambda v &\iff \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x\sqrt{2} \\ -y\sqrt{2} \end{pmatrix} \\
&\iff 2y = -x\sqrt{2} \quad \text{and} \quad x = -y\sqrt{2}
\end{aligned}$$

Which we can therefore conclude an eigenvector $\begin{pmatrix} 2 \\ -\sqrt{2} \end{pmatrix}$ for $\lambda = -\sqrt{2}$.

(b) Let $V = \mathbf{R}^2$. Find eigenvalues and eigenvectors for the linear operator T defined by $T(x, y) = (-2y, x)$.

Let A be a transformation matrix for T, then

$$A = [T(e_1) \quad T(e_2)] = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}.$$

Let $v \in \mathbb{R}^2$ denoted by $v = \langle x, y \rangle$ such that $v \neq \vec{0}$, then

$$\begin{aligned}
(A - \lambda I_2)v = \vec{0} &\iff \begin{bmatrix} -\lambda & -2 \\ 1 & -\lambda \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
&\iff -\lambda x - 2y = 0 \quad \text{and} \quad x - \lambda y = 0 \\
&\iff 2y = -x\lambda \quad \text{and} \quad x = \lambda y \\
&\iff 2y = -\lambda^2 y \quad \text{and} \quad x = -\frac{\lambda^2 x}{2} \\
&\iff 2 = -\lambda^2 \quad \text{and} \quad 1 = -\frac{\lambda^2}{2} \\
&\iff \lambda = - \pm \sqrt{2} \\
&\iff \lambda = \pm\sqrt{2}.
\end{aligned}$$

Therefore the eigenvalues are $\lambda = \sqrt{2}$ and $\lambda = -\sqrt{2}$.

Computing eigenvectors for $\lambda = \sqrt{2}$,

$$\begin{aligned}
Av = \lambda v &\iff \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\sqrt{2} \\ y\sqrt{2} \end{pmatrix} \\
&\iff 2y = x\sqrt{2} \quad \text{and} \quad x = y\sqrt{2}
\end{aligned}$$

Which we can therefore conclude an eigenvector $\begin{pmatrix} 2 \\ \sqrt{2} \end{pmatrix}$ for $\lambda = \sqrt{2}$.

Computing eigenvectors for $\lambda = -\sqrt{2}$,

$$\begin{aligned}
Av = \lambda v &\iff \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x\sqrt{2} \\ -y\sqrt{2} \end{pmatrix} \\
&\iff 2y = -x\sqrt{2} \quad \text{and} \quad x = -y\sqrt{2}
\end{aligned}$$

Which we can therefore conclude an eigenvector $\begin{pmatrix} 2 \\ -\sqrt{2} \end{pmatrix}$ for $\lambda = -\sqrt{2}$.

6. Let $T \in \mathcal{L}(V)$ and $S \in \mathcal{L}(V)$ be invertible.

   (a) Show that T and $S^{-1}TS$ have the same eigenvalues.

Proof. Assume that λ is the eigenvalue of T. By the definition of eigenvalue, for some vector $v \in V$, we have $Tv = \lambda v$. And, again, the the definition of eigenvalue, we will show that $S^{-1}TSv = \lambda v$.

$$\begin{aligned}
S^{-1}TSv = \lambda v &\iff S^{-1}T(SS^{-1})v = S^{-1}\lambda v && \text{Multiply } S^{-1} \\
&\iff S^{-1}Tv = S^{-1}\lambda v && \text{Identity, } SS^{-1} = I \\
&\iff S^{-1}\lambda v = S^{-1}\lambda v && \text{Eigenvalue of } T \\
&\iff \lambda S^{-1}v = \lambda S^{-1}v && \text{Linearity of } S \\
&\iff \lambda S^{-1}Sv = \lambda S^{-1}Sv && \text{Multiply } S \\
&\iff \lambda v = \lambda v && \text{Identity} \\
&\iff \lambda v = Tv && \text{Substitute } Tv
\end{aligned}$$

□

   (b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$.

We can manipulate the equation as follows,

$$\begin{aligned}
S^{-1}TSv = \lambda v &\iff SS^{-1}TSv = S\lambda v && \text{Multiply } S \\
&\iff TSv = \lambda Sv && \text{Identity, linearity.}
\end{aligned}$$

Which, by the definition of eigenvector, $Tv = \lambda v$, if we substitute in v with Sv to the above manipulation, we get that Sv is an *eigenvector* of T.

7. Show that the operator $T \in \mathcal{L}(\mathbf{C}^\infty)$ defined by

$$T(z_1, z_2, \ldots) = (0, z_1, z_2, \ldots)$$

has no eigenvalues.

Proof. Let $v \in \mathbb{C}^\infty$ such that $v \neq \vec{0}$. Denote v by $v = (z_1, z_2, \ldots)$ for some $z_i \in \mathbb{C}$. By the definition of an eigenvalue, we would have

$$\begin{aligned} Tv &= \lambda v && \text{Definition of eigenvalue.} \\ T(z_1, z_2, \ldots) &= \lambda(z_1, z_2, \ldots) && \text{Substitute in for } v. \\ (0, z_1, \ldots) &= (\lambda z_1, \lambda z_2, \ldots) && \text{Definition of } T \text{ and distribution of } \lambda. \end{aligned}$$

Which implies that $\lambda z_1 = 0$ and $z_i = \lambda z_{i+1}$ (for index $i \geq 1$). Therefore $\lambda = 0$ or $\lambda \neq 0$.

Case 1: $\lambda = 0$. Means that $z_1 = 0$ and by $z_i = \lambda z_{i+1}$ we would have $0 = z_1 = z_2 = \ldots$. This is a contradiction to the eigenvalue assumption that $v \neq \vec{0}$.

Case 2: $\lambda \neq 0$. Means that $z_1 \neq 0$ and by $z_i = \lambda z_{i+1}$ we would have $0 = z_1 = z_2 = \ldots$. This is a contradiction to the assumption that $z_1 \neq 0$. $\square$

8. Suppose $T \in \mathcal{L}(V)$ and there exists nonzero vectors v and w so that

$$Tv = 3w \qquad \text{and} \qquad Tw = 3v.$$

Prove that 3 or -3 is an eigenvalue of T.

Proof. Because T is linear, for some $v, w \in V$, $T(v+w) = T(v) + T(w)$.

$$\begin{aligned} T(v+w) &= T(v) + T(w) && \text{Linearity.} \\ &= 3w + 3v && \text{Given.} \\ &= 3(w+v) && \text{Factor out 3.} \\ &= 3(v+w) && \text{Commutativity on } V \end{aligned}$$

Therefore 3 is an eigenvalue since $T(v+w) = 3(v+w)$.

For the next eigenvalue, we will take a look at $T(v-w)$.

$$\begin{aligned} T(v-w) &= T(v) - T(w) && \text{Linearity.} \\ &= 3w - 3v && \text{Given.} \\ &= -3(-w+v) && \text{Factor out 3.} \\ &= 3(v-w) && \text{Commutativity on } V \end{aligned}$$

Therefore -3 is an eigenvalue since $T(v-w) = -3(v-w)$. $\square$