MATH 307
Assignment #7
Due Friday, March 4, 2022

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. Suppose $T \in \mathcal{L}(V)$ and $T^2 = I$ and -1 is not an eigenvalue of T. Prove that $T = I$.

2. Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \operatorname{null} P \oplus \operatorname{range} P$.

3. Suppose $T \in \mathcal{L}(V)$ and v is an eigenvector of T with eigenvalue λ. Suppose $p \in \mathcal{P}(\mathbf{R})$. Prove that $p(T)v = p(\lambda)v$.

4. Suppose W is a complex vector space and $T \in \mathcal{L}(W)$ has no eigenvalues. Prove that every subspace of W invariant under T is either $\{0\}$ or infinite-dimensional.

5. Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Define a function $f : \mathbf{C} \to \mathbf{R}$ by
$$f(\lambda) = \dim \operatorname{range}(T - \lambda I).$$
Prove that f is not a continuous function.

6. Suppose $T \in \mathcal{L}(V)$ has a diagonal matrix A with respect to some basis of V and that $\lambda \in \mathbf{F}$. Prove that λ appears on the diagonal of A precisely $\dim E(\lambda, T)$ times.

7. Show that the function that takes $((x_1, x_2), (y_1, y_2)) \in \mathbf{R}^2 \times \mathbf{R}^2$ to $|x_1 y_1| + |x_2 y_2|$ is not an inner product on $\mathbf{R}$.

8. Suppose $T \in \mathcal{L}(V)$ is such that $\|Tv\| \le \|v\|$ for every $v \in V$. Prove that $T - \sqrt{2}I$ is invertible.

9. Suppose $\|u| = \|v\| = 1$ and $\langle u, v \rangle = 1$. Prove that $u = v$.