MATH 307
Assignment #10
Due Friday, April 1, 2022
CORRECTION: Removed a duplicate problem #1: 29Mar2022.

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. Suppose that T is a normal operator on V and that 3 and 4 are eigenvalues of T. Prove that there exists a vector $v \in V$ such that $\|v\| = \sqrt{2}$ and $\|Tv\| = 5$.

2. (a) Suppose that T is a self-adjoint operator on a finite-dimensional inner product space and that 2 and 3 are the only eigenvalues of T. Prove that $T^2 - 5T + 6I = 0$.

   (b) Give an example of an operator $T \in \mathcal{L}(\mathbf{C}^3)$ such that 2 and 3 are the only eigenvalues of T and $T^2 - 5T + 6I \neq 0$.

3. Suppose that T is a normal operator on V. Suppose also that $v, w \in V$ satisfy the equations
$$\|v\| = \|w\| = 2, \qquad Tv = 3v, \qquad Tw = 4w.$$
Show that $\|T(v + w)\| = 10$.

4. Suppose $T \in \mathcal{L}(V)$ is normal. Prove that range T = range T^*.

5. Consider the statement: If $T \in \mathcal{L}(V)$ and there exists an orthonormal basis $e_1, \ldots, e_n$ of V such that $\|Te_j\| = \|T^*e_j\|$ for each j, then T is normal. Show that a counterexample to the statement is given by the matrix $T = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$ with respect to the standard basis in $\mathbf{R}^2$.

6. (CST) Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Prove that T is normal if and only if all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal and
$$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$$
where $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T.

7. (CST) Prove that a normal operator on a complex inner product space is self-adjoint if and only if all its eigenvalues are real.

8. (CST) Suppose V is a complex inner product space. Prove that every normal operator on V has a square root. (An operator $S \in \mathcal{L}(V)$ is called a *square root* of S$\in \mathcal{L}(V)$ if $S^2 = T$.)