MATH 307
Assignment #12
Due Friday, April 22, 2022

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. Suppose T is a positive operator on V. Prove that T is invertible if and only if $\langle Tv, v\rangle > 0$ for every $v \in V$ with $v \neq 0$.

2. Suppose $T \in \mathcal{L}(V)$, for an inner product space V. For $u, v \in V$, define the function of two variables $\langle u, v\rangle_T$ by
$$\langle u, v\rangle_T = \langle Tu, v\rangle.$$
Prove that $\langle \cdot, \cdot\rangle_T$ is an inner product on V if and only if T is an invertible positive operator (with respect to the original inner product $\langle \cdot, \cdot\rangle$).

3. Suppose $S \in \mathcal{L}(V)$. Prove that the following are equivalent:
   (a) S is an isometry;
   (b) $\langle S^*u, S^*v\rangle = \langle u, v\rangle$ for all $u, v \in V$;
   (c) $S^*e_1, \ldots S^*e_m$ is an orthonormal list for every orthonormal list of vectors $e_1, \ldots, e_m$ in V;
   (d) $S^*e_1, \ldots S^*e_n$ is an orthonormal basis for some orthonormal basis $e_1, \ldots, e_n$ of V.

4. Suppose T_1, T_2 are normal operators on $\mathbf{F}^3$ and both operators have 2 , 5 , 7 as eigenvalues. Prove that there exists an isometry $S \in \mathcal{L}(\mathbf{F}^3)$ such that $T_1 = S^*T_2S$.

5. Fix $u, x \in V$ with $u \neq 0$. Define $T \in \mathcal{L}(V)$ by $Tv = \langle v, u\rangle x$ for every $v \in V$. Prove that
$$\sqrt{T^*T}v = \frac{\|x\|}{\|u\|}\langle v, u\rangle u$$
for every $v \in V$.

6. Give an example of $T \in \mathcal{L}(\mathbf{C}^2)$ such that 0 is the only eigenvalue of T and the singular values of T are 5 , 0.

7. Suppose $T \in \mathcal{L}(V)$ and s is a singular value of T. Prove that there exists a vector $v \in V$ such that $\|v\| = 1$ and $\|Tv\| = s$.

8. Suppose $T \in \mathcal{L}(\mathbf{C}^2)$ is defined by $T(x, y) = (-4y, x)$. Find the singular values of T.