MATH 307
Assignment #6
Due Friday, February 25, 2022
Revision: 2/21/2022: Fixed typo in problem #3: $U + V$ should be $U + W$.

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. Label the following statements as being true or false. Provide some justification from the text for your label.
   (a) Every linear operator on an n-dimensional vector space has n distinct eigenvalues.
   (b) If a linear operator on a vector space over $\mathbf{R}$ has one eigenvector, then it has an infinite number of eigenvectors.
   (c) There exists a square matrix with no eigenvectors.
   (d) Eigenvalues must be nonzero scalars.
   (e) Any two eigenvectors are linearly independent.
   (f) The sum of two eigenvalues of a linear operator T is also an eigenvalue of T.
   (g) Linear operators on infinite-dimensional vectors spaces never have eigenvalues.
   (h) The sum of two eigenvectors of an operator T is always an eigenvector of T.

2. Consider the operator $T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ acting on $\mathbf{R}^2$. How many subspaces are there that are invariant under T?

3. If U and W are invariant subspaces for $T \in \mathcal{L}(V)$ then $U + W$ is invariant for T.

4. In $\mathbf{R}^2$, let T be the reflection across the line $y = x$.
   (a) Write the matrix A that represents T relative to the standard basis.
   (b) Determine two invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ for T such that $\mathbf{R}^2 = \mathcal{M} \oplus \mathcal{N}$ where neither $\mathcal{M}$ nor $\mathcal{N}$ is the zero subspace.
   (c) Write a basis $\{u_1, u_2\}$ for $\mathbf{R}^2$ so that $\mathcal{M} = \text{span } (u_1)$ and $\mathcal{N} = \text{span } (u_2)$.
   (d) Write the matrix B that represents T relative to the basis $\{u_1, u_2\}$.

5. (a) Let $V = \mathbf{R}^2$. Find eigenvalues and eigenvectors for the linear operator T defined by $T(x, y) = (2y, x)$.
   (b) Let $V = \mathbf{R}^2$. Find eigenvalues and eigenvectors for the linear operator T defined by $T(x, y) = (-2y, x)$.

6. Let $T \in \mathcal{L}(V)$ and $S \in \mathcal{L}(V)$ be invertible.
   (a) Show that T and $S^{-1}TS$ have the same eigenvalues.
   (b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$.

7. Show that the operator $T \in \mathcal{L}(\mathbf{C}^\infty)$ defined by

$$T(z_1, z_2, \ldots) = (0, z_1, z_2, \ldots)$$

has no eigenvalues.

8. Suppose $T \in \mathcal{L}(V)$ and there exists nonzero vectors v and w so that

$$Tv = 3w \qquad \text{and} \qquad Tw = 3v.$$

Prove that 3 or -3 is an eigenvalue of T.