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MATH 307 - Spring 2022
Assignment #5
Due Friday, 02-18-22, 16:00 CST

For each Problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. Let V be a finite-dimensional vector space and let $A, B, C, D \in \mathcal{L}(V)$. Assume that $A + B$ and $A - B$ are invertible. Show that there exist X, Y so that

$$\begin{aligned} AX + BY &= C \\ BX + AY &= D. \end{aligned}$$

Proof. Adding the two equations together, and using the property that $(A + B)^{-1}$ exists, we get

$$\begin{aligned} & AX + BY + BX + AY = C + D \\ \iff & A(X + Y) + B(X + Y) = C + D \\ \iff & (A + B)(X + Y) = C + D \\ \iff & (A + B)^{-1}(A + B)(X + Y) = (A + B)^{-1}(C + D) \\ \iff & X + Y = (A + B)^{-1}(C + D). \end{aligned}$$

Subtracting the two equations from each other and using the property that $(A - B)^{-1}$ exists, we get

$$\begin{aligned} & AX + BY - BX - AY = C - D \\ \iff & AX - AY - BX + BY = C - D \\ \iff & AX - AY - (BX - BY) = C - D \\ \iff & A(X - Y) - B(X - Y) = C - D \\ \iff & (A - B)(X - Y) = C - D \\ \iff & (A - B)^{-1}(A - B)(X - Y) = (A - B)^{-1}(C - D) \\ \iff & X - Y = (A - B)^{-1}(C - D). \end{aligned}$$

Adding the results from each system together,

$$\begin{aligned} & X + Y + X - Y = (A + B)^{-1}(C + D) + (A - B)^{-1}(C - D) \\ \iff & 2X = (A + B)^{-1}(C + D) + (A - B)^{-1}(C - D) \\ \iff & X = \frac{(A + B)^{-1}(C + D) + (A - B)^{-1}(C - D)}{2}. \end{aligned}$$

From our first system of equations,

$$\begin{aligned}
&X + Y = (A+B)^{-1}(C+D) \\
\iff &Y = (A+B)^{-1}(C+D) - X \\
\iff &Y = (A+B)^{-1}(C+D) - \frac{(A+B)^{-1}(C+D) + (A-B)^{-1}(C-D)}{2} \\
\iff &Y = \frac{2(A+B)^{-1}(C+D)}{2} + \frac{-(A+B)^{-1}(C+D) - (A-B)^{-1}(C-D)}{2} \\
\iff &Y = \frac{2(A+B)^{-1}(C+D) - (A+B)^{-1}(C+D) - (A-B)^{-1}(C-D)}{2} \\
\iff &Y = \frac{(A+B)^{-1}(C+D) - (A-B)^{-1}(C-D)}{2}.
\end{aligned}$$

Meanwhile, back at the ranch, we can write C as

$$\begin{aligned}
C &= \left[\frac{(C+D)+(C-D)}{2}\right] \\
&= \left[\frac{(A+B)(A+B)^{-1}(C+D) + (A-B)(A-B)^{-1}(C-D)}{2}\right] \\
&= \left[\frac{A(A+B)^{-1}(C+D) + B(A+B)^{-1}(C+D) + A(A-B)^{-1}(C-D) - B(A-B)^{-1}(C-D)}{2}\right] \\
&= \left[\frac{A(A+B)^{-1}(C+D) + A(A-B)^{-1}(C-D) + B(A+B)^{-1}(C+D) - B(A-B)^{-1}(C-D)}{2}\right] \\
&= A\left[\frac{(A+B)^{-1}(C+D) + (A-B)^{-1}(C-D)}{2}\right] + B\left[\frac{(A+B)^{-1}(C+D) - (A-B)^{-1}(C-D)}{2}\right] \\
&= AX + BY.
\end{aligned}$$

Smilarly, we can write D as

$$\begin{aligned}
D &= \left[\frac{(C+D)-(C-D)}{2}\right] \\
&= \left[\frac{(A+B)(A+B)^{-1}(C+D) - (A-B)(A-B)^{-1}(C-D)}{2}\right] \\
&= \left[\frac{(A+B)(A+B)^{-1}(C+D) + (B-A)(A-B)^{-1}(C-D)}{2}\right] \\
&= \left[\frac{B(A+B)^{-1}(C+D) + A(A+B)^{-1}(C+D) + B(A-B)^{-1}(C-D) - A(A-B)^{-1}(C-D)}{2}\right] \\
&= \left[\frac{B(A+B)^{-1}(C+D) + B(A-B)^{-1}(C-D) + A(A+B)^{-1}(C+D) - A(A-B)^{-1}(C-D)}{2}\right] \\
&= B\left[\frac{(A+B)^{-1}(C+D) + (A-B)^{-1}(C-D)}{2}\right] + A\left[\frac{(A+B)^{-1}(C+D) - (A-B)^{-1}(C-D)}{2}\right] \\
&= BX + AY.
\end{aligned}$$

Therefore $AX + BY = C$ and $BX + AY = D$ have been shown. $\square$

2. Show that if $A \in \mathcal{L}(V)$ satisfying $A^2 - A + I = 0$, then A is invertible.

Proof.

$$\begin{aligned} A^2 - A + I = 0 &\iff I = A - A^2 \\ &\iff I = A(I - A) \\ &\iff I = (I - A)A \end{aligned}$$

Therefore A is invertible and $A^{-1} = I - A$. □

3. Assume V is a finite-dimensional vector space with $S, T, U \in \mathcal{L}(V)$. Show that if $STU = I$ then T is invertible and $T^{-1} = US$

Proof. Utilizing that $AA^{-1} = I \iff A^{-1}A = I$, it can be shown directly that

$$STU = I \iff S(TU) = I \iff (TU)S = I \iff T(US) = I \iff T^{-1} = US.$$

□

4. Let V be a 2-dimensional vector space and let $A \in \mathcal{L}(V)$ be invertible. Show that there is a polynomial p so that $A^{-1} = p(A)$.

Let matrix A be of the form

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then the characteristic polynomial of A is

$$A^2 - \mathrm{Tr}(A)A + \det(A)A^0 = 0$$

Simplifying and solving for A^{-1},

$$\begin{aligned}
&A^2 - \mathrm{tr}(A)A + \det(A)A^0 = 0 \\
\iff\ &A^2 - (a+d)A + (ad-bc)A^0 = 0 \\
\iff\ &A^2A^{-1} - (a+d)AA^{-1} + (ad-bc)A^0A^{-1} = 0A^{-1} \\
\iff\ &A - (a+d)I_2 + (ad-bc)A^{-1} = 0 \\
\iff\ &(ad-bc)A^{-1} = (a+d)I_2 - A \\
\iff\ &A^{-1} = \frac{(a+d)I_2}{ad-bc} - \frac{A}{ad-bc} \\
\iff\ &A^{-1} = A^0\frac{a+d}{ad-bc} - A\frac{1}{ad-bc}
\end{aligned}$$

Therefore

$$A^{-1} = p(A) = A^0\frac{a+d}{ad-bc} - A\frac{1}{ad-bc}$$

5. Let V and W be finite-dimensional vector spaces. Fix $v \in V$. Define

$$E = \{T \in \mathcal{L}(V, W) : Tv = 0\}.$$

   (a) Show that E is a subspace of $\mathcal{L}(V, W)$.

   (b) Suppose $v \neq 0$. What is $\dim E$?

6. Let $V = \mathbf{R}^{2,2}$ be the vector space of 2×2 matrices with the usual addition and scalar multiplication of matrices. Let $W = \mathcal{P}_3(\mathbf{R})$ be the vector space of polynomials of degree less than or equal to three. Prove that V and W are isomorphic vector spaces.

Proof. Let B be a basis for V such that

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Then let β be a basis for W such that

$$\beta = \left\{1, x, x^2, x^3\right\}.$$

Then $\dim V = 4$ and $\dim W = 4$. Since $\dim V = \dim W$, then V is isomorphic to W (by 3.59, page 82). $\square$

7. Let V be a real vector space. $V^4 = V \times V \times V \times V$. Prove that V^4 and $\mathcal{L}(\mathbf{R}^4, V)$ are isomorphic vector spaces.

Proof. Define a map Φ as

$$\Phi : V^4 \to \mathcal{L}(\mathbb{R}^4, V)$$

$$\Phi(v) = T_v : \mathbb{R}^4 \to V$$

$$\text{Where } T_v(x_1, x_2, x_3, x_4) = x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4$$

with $v = (v_1, v_2, v_3, v_4) \in V^4$

Then there is an inverse map ϕ so that $T \in \mathcal{L}(\mathbb{R}^4, V)$,

$$\phi : \mathcal{L}(\mathbb{R}^4, V) \to V^4$$

$$\phi(T) = (T(e_1), T(e_2), T(e_3), T(e_4))$$

With e_i denoting the standard basis elements for $\mathbb{R}^4$.

Next we will show the composition of functions $\Phi \circ \phi$ and $\phi \circ \Phi$ are the identity.

$$\begin{aligned}(\Phi \circ \phi)(T)\overbrace{(x_1, x_2, x_3, x_4)}^{\in \mathbb{R}^4} &= \Phi(\overbrace{T(e_1), T(e_2), T(e_3), T(e_4)}^{\in V^4})(x_1, x_2, x_3, x_4)\\ &= x_1T(e_1) + x_2T(e_2) + x_3T(e_3) + x_4T(e_4)\\ &= T(x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4)\\ &= T(x_1, x_2, x_3, x_4)\end{aligned}$$

For the other direction, we will let $v = (v_1, v_2, v_3, v_4) \in V^4$. Then,

$$\begin{aligned}(\phi \circ \Phi)\overbrace{(v_1, v_2, v_3, v_4)}^{\in V^4} &= (\Phi(v)(e_1), \Phi(v)(e_2), \Phi(v)(e_3), \Phi(v)(e_4))\\ &= (v_1, v_2, v_3, v_4)\end{aligned}$$

What remains to show is that Φ *or* ϕ is a homomorphism. Showing either case will result in an isomorphism based on the compositions and inverses shown above. So, for some scalars a, b and vectors $v, w \in V^4$,

$$\begin{aligned}&\Phi(av + bw)(x_1, x_2, x_3, x_4)\\ &= \Phi(a(v_1, v_2, v_3, v_4) + b(w_1, w_2, w_3, w_4))(x_1, x_2, x_3, x_4)\\ &= \Phi(av_1 + bw_1,\ av_2 + bw_2,\ av_3 + bw_3,\ av_4 + bw_4)(x_1, x_2, x_3, x_4)\\ &= x_1av_1 + x_1bw_1 + x_2av_2 + x_2bw_2 + x_3av_3 + x_3bw_3 + x_4av_4 + x_4bw_4\\ &= a(x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4) + b(x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4)\\ &= a\Phi(v)(x_1, x_2, x_3, x_4) + b\Phi(w)(x_1, x_2, x_3, x_4)\end{aligned}$$

Therefore V^4 is isomorphic to $\mathcal{L}(\mathbb{R}^4, V)$. □