MATH 307
Assignment #3
Due Friday, February 4, 2022
**Revised 01Feb2022 to correct typo in problem 2. Changed u_1, u_2 to v_1, v_2.

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. Suppose that the vectors v_1, v_2, v_3, v_4 are a basis for V. Show that the vectors $v_1 - v_2, v_1 + v_2, v_3 + v_4, v_4$ also form a basis for V.

2. Suppose that the vectors v_1, v_2, v_3, v_4 is a basis for V. Let U be a subspace of V. Assume $v_1, v_2 \in U$ but neither v_3 nor v_4 are in U. Is v_1, v_2 a basis for U? Justify.

3. Let $v_1 = (-1, 1, 2) \in \mathbf{R}^3$. Construct two bases for $\mathbf{R}^3$: $\{v_1, v_2, v_3\}$ and $\{v_1, v_2', v_3'\}$ so that $\{v_2, v_3, v_3'\}$ is also a basis.

4. (a) Under what conditions on the scalar ξ do the vectors $(1, 1, 1)$ and $(1, \xi, \xi^2)$ form a basis for $\mathbf{R}^3$?

   (b) Under what conditions on the scalar ξ do the vectors $(0, 1, \xi)$, $(\xi, 0, 1)$, and $(\xi, 1, 1 + \xi)$ form a basis for $\mathbf{R}^3$?

5. Let $V = \mathcal{P}(\mathbf{R})$ be the vector space of all polynomials with real coefficients. If p is any polynomial, let Tp be the polynomial defined by $(Tp)(x) = p(x + 1) - p(x)$. Show that T is a linear transformation.

6. Let $V = \mathcal{P}_4(\mathbf{R})$, the vector space of polynomials of degree at most four.
   Let $U = \{p \in V : p(1) = p(3)\}$

   (a) Find a basis of U.

   (b) Extend the basis in (a) to a basis of V.

   (c) Find a subspace W of V so that $V = U \oplus W$.

7. Suppose $a, b, c \in \mathbf{R}$. Define $T : \mathcal{P}(\mathbf{R}) \to \mathbf{R}^3$ by

$$Tp = \left(2p(5) - 5p'(1) + ap(1)p(3)\ ,\ \int_1^4 x^2 p(x)\, dx + be^{p(0)}\ ,\ p(2) + c\right).$$

   Show that T is linear if and only if $a = b = c = 0$.

8. (a) Find an example of a function $\varphi : \mathbf{R}^2 \to \mathbf{R}$ that is homogeneous but not additive (and hence not linear).

   (b) Find an example of a function $\varphi : \mathbf{C}^2 \to \mathbf{C}$ that is additive but not homogeneous (and hence not linear).