MATH 307
Assignment #4
Due Friday, February 11, 2022
**Revised 07Feb2022 to correct typo in problem 5. Fixed a LaTeX typo at the beginning of the first sentence to state that D is a linear transformation from $\mathcal{P}_3(\mathbf{R})$ to $\mathcal{P}_2(\mathbf{R})$. And corrected parts (a) and (b) of #6 changing D to A.

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. (a) Find linear map $T : \mathbf{R}^4 \to \mathbf{R}^4$ so that range $T =$ null T.
   (b) Show that there is no linear map $T : \mathbf{R}^5 \to \mathbf{R}^5$ so that range $T =$ null T.
2. Find a 4×4 matrix M so that the range of M is spanned by $(1, 0, 1, 0)$ and $(0, 1, 0, 1)$.
3. (a) Give an example of a linear map on a three-dimensional space with a two-dimensional range.
   (b) Give an example of a linear map on a three-dimensional space with a two-dimensional null-space.
4. Let $T : V \to V$ be a linear map with a one-dimensional range. Prove that $T^2 = cT$ for some scalar c. (This means that $T(Tv) = cTv$ for all $v \in V$.)
5. Let $D \in \mathcal{L}(\mathcal{P}_3(\mathbf{R}), \mathcal{P}_2(\mathbf{R}))$ denote the differentiation map $Dp = p'$. Example 3.34 gives the matrix of D with respect to the usual bases for $\mathcal{P}_3(\mathbf{R})$ and $\mathcal{P}_2(\mathbf{R}$.
   Find two new bases for $\mathcal{P}_3(\mathbf{R})$ and $\mathcal{P}_2(\mathbf{R}$ so that the matrix for D with respect to these bases is
   $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$
6. The general operation of finding an antiderivative is not a linear map because of the "$+C$" which means that any function has infinitely many antiderivatives. Let's define a linear map from $\mathcal{P}_3(\mathbf{R})$ to $\mathcal{P}_4(\mathbf{R})$ that avoids ambiguity. Let $A(a_0 + a_1x + a_2x^2 + a_3x^3) = a_0x + (a_1/2)x^2 + (a_2/3)x^3 + (a_3/4)x^4$.
   (a) Find the matrix of A with respect to the standard bases for $\mathcal{P}_3(\mathbf{R})$ and $\mathcal{P}_4(\mathbf{R})$.
   (b) Find new bases for $\mathcal{P}_3(\mathbf{R})$ and $\mathcal{P}_4(\mathbf{R})$ so that the matrix for A with respect to the new bases is
   $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$