Matthew Wilder
MATH 307 - Spring 2022
Assignment #1
Due Friday, January 21, 2022, 4:00 PM CST

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. Label the following statements as being true or false. Provide some justification from the text for your label.

   (a) Every vector space contains a zero vector.

   True by definition

   Solution. Every vector space contains an additive identity, denoted by 0, by the 3rd part of the definition of a *vector space* (pg. 12, def 1.19).

   (b) A vector space may have more than one zero vector.

   False by contradiction.

   Proof. Suppose there exists two zero vectors, 0 and $0'$, in a vector space V. Then

$$\begin{aligned} 0 &= 0 + 0' && \text{definition of identity} \\ &= 0' + 0 && \text{commutative} \\ &= 0' && \text{definition of identity} \end{aligned}$$

   Implies $0 = 0'$, which contradicts the assumption. $\square$

   (c) In any vector space $au = bu$ implies that $a = b$.

   False by contradiction.

   Proof. Let $a, b \in \mathbb{R}$ and $u \in V$ such that V operates under normal addition and scalar multiplication in $\mathbb{R}^2$. Then, Let $a = 1$, $b = 2$, and $u = \langle 0, 0 \rangle$

$$au = bu$$

$$1 \langle 0, 0 \rangle = 2 \langle 0, 0 \rangle$$

$$\langle 1 \cdot 0, 1 \cdot 0 \rangle = \langle 2 \cdot 0, 2 \cdot 0 \rangle$$

$$\langle 0, 0 \rangle = \langle 0, 0 \rangle$$

   But $1 \neq 2$. Therefore, false by contradiction. $\square$

(d) In any vector space $au = av$ implies that $u = v$.

False by contradiction.

Proof. Let $a = 0$ and $u, v \in V$ such that V operates under normal addition and scalar multiplication in $\mathbb{R}^2$. Let $u = \langle 1, 1\rangle$ and $v = \langle 2, 2\rangle$, then

$$au = vu$$

$$\begin{aligned} 0\langle 1,1\rangle &= 0\langle 2,2\rangle \\ \langle 0\cdot 1, 0\cdot 1\rangle &= \langle 0\cdot 2, 0\cdot 2\rangle \\ \langle 0,0\rangle &= \langle 0,0\rangle \end{aligned}$$

But $u \neq v$, therefore false by contradiction $\square$

(e) In $\mathcal{P}(\mathbf{F})$ only polynomials of the same degree may be added.

False. by counterexample

Proof. Let $f(x) = 1$ and $g(x) = x$, then $f, g \in \mathcal{P}(\mathbf{F})$

$$(f+g)(x) = f(x) + g(x) = x + 1 \in \mathcal{P}(\mathbf{F})$$

But $\deg f = 0 \neq \deg g = 1$, therefore it is false by contradiction. $\square$

(f) If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree n.

False. by counterexample.

Proof. Let

$$f(x) = x^2 + 1 \qquad \text{and} \qquad g(x) = -x^2$$

Then $(f + g)(x) = 1$, and

$$\deg f = 2 \qquad \deg g = 2 \qquad \deg(f+g) = 0$$

But $0 \neq 2$, therefore it is false by counterexample. $\square$

(g) If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n.

True.

Proof. Define

$$f : \mathbb{R} \to \mathbb{R} \text{ by } f(x) = a_0x^0 + a_1x^1 + \cdots + a_nx^n$$

for some sequence $\{a_n\}$, then

$$cf = ca_0x^0 + ca_1x^1 + \cdots + ca_nx^n, \qquad c \neq 0$$

Because $c \neq 0$, no new zero terms are introduced in the sequence, thus the highest power remains unchanged. Therefore $\deg(f) = \deg(cf)$ □

(h) A nonzero element of $\mathbf{F}$ may be considered to be an element of $\mathcal{P}(\mathbf{F})$ having degree zero.

True

Proof. Let $a \in \mathbf{F}$ then define

$$f : \mathbb{R} \to \mathbb{R} \text{ by } f(x) = a = ax^0, \qquad a \neq 0$$

Then $f \in \mathcal{P}(\mathbf{F})$. And because $a \neq 0$, then $\deg(f) = 0$ □

(i) Two functions in $\mathbf{F}^S$ are equal if and only if they have the same values at each element of S.

True

Solution

If $f, g \in \mathbf{F}^S$ and $f(x) = g(x) \quad \forall \quad x \in S$, then by the definition of a function, $f = g$

2. Let $v_1, \ldots, v_4$ be four vectors in a vector space V. Verify $(v_1 + v_2) + (v_3 + v_4) = [v_2 + (v_3 + v_1)] + v_4$. Use the definition, properties, and theorems on pp.12-15 to justify each step in the transitions from the LHS to the RHS.

Proof. Recall:
Commutativity: $u + v = v + u$
Associativity: $(u + v) + w = u + (v + w)$

$$\begin{aligned}
(v_1 + v_2) + (v_3 + v_4) &= (v_3 + v_4) + (v_1 + v_2) && \text{by commutativity} \\
&= (v_4 + v_3) + (v_1 + v_2) && \text{by commutativity} \\
&= v_4 + [v_3 + (v_1 + v_2)] && \text{by associativity} \\
&= v_4 + [(v_3 + v_1) + v_2] && \text{by associativity} \\
&= v_4 + [v_2 + (v_3 + v_1)] && \text{by commutativity} \\
&= [v_2 + (v_3 + v_1)] + v_4 && \text{by commutativity}
\end{aligned}$$

□

3. Which vectors in $\mathbf{R}^3$ are linear combinations of $(1,0,-1),(0,1,1),(1,1,1)$?

   All vectors in $\mathbf{R}^3$ are linear combinations

$$a\begin{bmatrix}1\\0\\-1\end{bmatrix}+b\begin{bmatrix}0\\1\\1\end{bmatrix}+c\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}x\\y\\z\end{bmatrix} \tag{1}$$

$$a+c=x \qquad b+c=y \qquad -a+b+c=z$$

   Proof. Write the vectors in the column space of a matrix augmented with the identity.

$$\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&1&0&1&0\\-1&1&1&0&0&1\end{array}\right]$$

   Then, row reduce into the form

$$\left[\begin{array}{ccc|ccc}1&0&0&0&1&-1\\0&1&0&-1&2&-1\\0&0&1&1&-1&1\end{array}\right]$$

   Which is I_3, thus the vectors are linearly independent. A list of n linearly independent vectors in $\mathbf{R}^n$ span $\mathbf{R}^n$, thus these vectors span all of $\mathbf{R}^3$. Further, *every* vector in $\mathbf{R}^3$ is a linear combination of $(1,0,-1),(0,1,1),$ and $(1,1,1)$ □

4. Let $V=\mathbf{R}^2$ with *new* operations

$$(x,y)+(x_1,y_1)=(x+x_1,y+y_1)$$
$$c(x,y)=(cx,y)$$

   Is V a vector space? Justify.

   No, it is ***not*** a vector space. It fails the 2nd distributive law test.

   Proof. Let $\langle x,y\rangle=\langle 1,1\rangle$, $a=1$, $b=1$. Then,

$$(a+b)v=av+bv$$

$$(1+1)\langle 1,1\rangle=1\langle 1,1\rangle+1\langle 1,1\rangle$$
$$2\langle 1,1\rangle=\langle 1,1\rangle=\langle 1,1\rangle$$
$$\langle 2,1\rangle\neq\langle 2,2\rangle$$

   Therefore, false by counterexample. □

5. Let $V = \mathbf{R}^2$ with *new* operations

$$(x, y) + (x_1, y_1) = (x + x_1, 0)$$
$$a(x, y) = (ax, 0)$$

Is V a vector space? Justify.

No it is ***not*** a vector space. The default multiplicative identity is $\langle 1, 1\rangle$.

Proof. By definition of a vector space, $1v = v \;\forall\; v \in V$
Let $v = \langle 1, 1\rangle$ then

$$1\,\langle 1, 1\rangle = \langle 1, 1\rangle$$
$$\langle 1, 0\rangle \neq \langle 1, 1\rangle$$

False by counterexample. □

6. Consider $\mathbf{R}^n$ with new operations

$$v \boxplus w = v - w$$
$$a \cdot v = -av$$

Which of the parts of the definition of vector space are satisfied with these new operations?

Commutativity: **Fails**

$$v \boxplus w = w \boxplus v$$
$$v - w \neq w - v$$

Associativity: **Holds**

$$(u \boxplus v) \boxplus w = u \boxplus (v \boxplus w)$$
$$u - v \boxplus w = u \boxplus v - w$$
$$u - v - w = u - v - w$$

Additive identity: **Holds**

$$\vec{0} = \langle 0_1, \ldots, 0_n\rangle \text{ in } \mathbb{R}^n$$
$$\langle v_1, \ldots, v_n\rangle \boxplus \vec{0} = \langle v_1 - 0, \ldots, v_n - 0\rangle = \langle v_1, \ldots, v_n\rangle$$

Additive inverse: **Holds**

$$v = \langle v_1, \ldots, v_n\rangle = -v$$
$$v \boxplus (-v) = 0$$
$$v \boxplus v = 0$$

$$v - v = 0$$
$$0 = 0$$

Multiplicative identity: **Holds**

$$1_V = -1$$
$$1_V \cdot \langle v_1, \ldots, v_n \rangle = -1 \cdot \langle v_1, \ldots, v_n \rangle$$
$$1_V \cdot \langle v_1, \ldots, v_n \rangle = \langle -(-1)v_1, \ldots, -(-1)v_n \rangle$$
$$1_V \cdot \langle v_1, \ldots, v_n \rangle = \langle v_1, \ldots, v_n \rangle$$

First Distributive Property: **Holds**

$$a \cdot (\langle v_1, \ldots, v_n \rangle \boxplus \langle w_1, \ldots, w_n \rangle) = a \langle v_1, \ldots, v_n \rangle \boxplus a \langle w_1, \ldots, w_n \rangle$$
$$a \cdot \langle v_1 - w_1, \ldots, v_n - w_n \rangle = \langle -av_1, \ldots, -av_n \rangle \boxplus \langle -aw_1, \ldots, -aw_n \rangle$$
$$\langle -a(v_1 - w_1), \ldots, -a(v_n - w_n) \rangle = \langle -av_1 + aw_1, \ldots, -av_n + aw_n \rangle$$
$$\langle -av_1 + aw_1, \ldots, -av_n + aw_n \rangle = \langle -av_1 + aw_1, \ldots, -av_n + aw_n \rangle$$

Second Distributive Property: **Fails**

$$(a + b) \cdot \langle v_1, \ldots, v_n \rangle = a \cdot \langle v_1, \ldots, v_n \rangle \boxplus b \cdot \langle v_1, \ldots, v_n \rangle$$
$$\langle -(a + b)v_1, \ldots, -(a + b)v_n \rangle = \langle -av_1, \ldots, -av_n \rangle \boxplus \langle -bv_1, \ldots, -bv_n \rangle$$
$$\langle -av_1 - bv_1, \ldots, -av_n - bv_n \rangle = \langle -av_1 - (-bv_1), \ldots, -av_n - (-bv_n) \rangle$$
$$\langle -av_1 - bv_1, \ldots, -av_n - bv_n \rangle \neq \langle -av_1 + bv_1, \ldots, -av_n + bv_n \rangle$$

7. Which subsets of $\mathcal{P}(\mathbf{R})$ form a vector space? Justify.

   (a) All $p(x)$ such that $p(0) = 1$.

   Not a vector space. Let $p(x) = 1$, then under scalar multiplication with 2, $2p(x) = 2$ so it is not closed under its operations.

   (b) All $p(x)$ such that $p(0) = 0$.

   It is a vector space.
   Let S denote the set of all $p(x)$ such that $p(0) = 0$.

   Closed under addition
   Let $f, g \in S$, then we show $f + g \in S$

$$(f + g)(0) = f(0) + g(0) = 0 + 0 = 0$$

Therefore, $(f+g) \in S$ and is closed under addition

Closed under scalar multiplication
For $f \in S$ we show that $cf \in S$

$$f(0) = 0$$

$$cf(0) = c \cdot 0 = 0$$

Therefore, $cf \in p(x)$ is closed under scalar multiplication

(c) All $p(x)$ such that $2p(0) - 3p(1) = 0$.

It is a vector space.
Let S denote all $p(x)$ such that $2p(0) - 3p(1) = 0$

Closed under addition
Let f, g be functions in $\in S$

$$\begin{aligned} 2(f+g)(0) - 3(f+g)(1) &= 2[f(0)+g(0)] - 3[f(1)+g(1)] \\ &= 2f(0) + 2g(0) - 3f(1) - 3g(1) \\ &= [2f(0) - 3f(1)] + [2g(0) - 3g(1)] \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Therefore, $(f+g) \in S$ and is closed under addition

Closed under scalar multiplication.
Let $p \in S$, then we snow that $c \cdot p \in S$

$$\begin{aligned} [2cp(0) - 3cp(1)] &= c[2p(0) - 3p(1)] \\ &= c[0] \\ &= 0 \end{aligned}$$

Therefore, it is closed under scalar multiplication