MATH 307
Assignment #11
Due Friday, April 15, 2022

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. (a) Show that $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is positive.

   (b) Find all α such that $A = \begin{pmatrix} \alpha & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is positive.

   (c) Show that even though all its entries are positive, the matrix $A = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$ is not positive.

   (d) Find an example of a positive matrix some of whose entries are negative.

2. If T is a positive and invertible operator, is T^{-1} positive?

3. Consider the three statements:

   (a) T is self-adjoint

   (b) T is an isometry

   (c) $T^2 = I$ (such a T is called an *involution*)

   Prove that if an operator has any two of the properties, then it has the third one as well.

4. Prove or give a counterexample: If $T \in \mathcal{L}(V)$ and there exists an orthonormal basis $e_1, \ldots, e_n$ of V such that $\|Te_i\| = 1$ for each e_i, then T is an isometry.

5. Suppose $T \in \mathcal{L}(V)$. Prove that there exists an isometry $S \in \mathcal{L}(V)$ such that

$$T = \sqrt{TT^*}\, S.$$

6. Find the singular values of the differentiation operator $D \in \mathcal{L}(\mathcal{P}_2(\mathbf{R}))$ defined by $Dp = p'$, where the inner product is $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\, dx$.
   Remark: It might be helpful to compute the matrix for D with respect to the basis $1, x, x^2$ to find eigenvalues (easy) and then compute the matrix for D again using an *orthonormal basis* for $\mathcal{P}_2(\mathbf{R})$ to compute the singular values. Use some technology for the integrations.

7. Define $T \in \mathcal{L}(\mathbf{F}^3)$ by $T(z_1, z_2, z_3) = (4z_2, 5z_3, z_1)$. Find (explicitly) an isometry $S \in \mathcal{L}(\mathbf{F}^3)$ such that $T = S\sqrt{T^*T}$.

8. Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that the singular values of T equal the absolute values of the eigenvalues of T, repeated appropriately.