MATH 307
Assignment #8
Due Friday, March 11, 2022

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. Prove that
$$16 \leq (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$
for all positive numbers a, b, c, d.
Hint: find two vectors having lots of square roots; compute an inner product and also use Cauchy-Schwarz.

Solution: Direct proof.

Proof. Let scalars $a, b, c, d \in \mathbb{R}^+$ and $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^4$ such that $\vec{\mathbf{v}} := \left[\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}\right]$ and $\vec{\mathbf{w}} := \left[\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{d}}\right]$. By the Cauchy-Schwarz Inequality, $|\langle\vec{\mathbf{v}}, \vec{\mathbf{w}}\rangle| \leq \|\vec{\mathbf{v}}\| \, \|\vec{\mathbf{w}}\|$.
Simplifying the left hand side,

$$\begin{aligned} |\langle\vec{\mathbf{v}}, \vec{\mathbf{w}}\rangle| &= \left|\left\langle \left[\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}\right],\ \left[\tfrac{1}{\sqrt{a}}, \tfrac{1}{\sqrt{b}}, \tfrac{1}{\sqrt{c}}, \tfrac{1}{\sqrt{d}}\right]\right\rangle\right| \\ &= \left|\frac{\sqrt{a}}{\sqrt{a}} + \frac{\sqrt{b}}{\sqrt{b}} + \frac{\sqrt{c}}{\sqrt{c}} + \frac{\sqrt{d}}{\sqrt{d}}\right| \\ &= 4. \end{aligned}$$

Simplifying the right hand side,

$$\begin{aligned} \|\vec{\mathbf{v}}\| \, \|\vec{\mathbf{w}}\| &= \sqrt{\sqrt{a}^2 + \sqrt{b}^2 + \sqrt{c}^2 + \sqrt{d}^2} \cdot \sqrt{\left(\frac{1}{\sqrt{a}}\right)^2 + \left(\frac{1}{\sqrt{b}}\right)^2 + \left(\frac{1}{\sqrt{c}}\right)^2 + \left(\frac{1}{\sqrt{d}}\right)^2} \\ &= \sqrt{a+b+c+d} \cdot \sqrt{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}. \end{aligned}$$

Hence, by Cauchy-Schwarz, $4 \leq \sqrt{a+b+c+d} \cdot \sqrt{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}$. Squaring both sides of this inequality, we see that $16 \leq (a+b+c+d) \cdot \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$. $\square$

2. Prove or disprove: there is an inner product on $\mathbf{R}^2$ such that the associated norm is given by
$$\|(x,y)\| = \max\{|x|,|y|\}$$
for all $(x,y) \in \mathbf{R}^2$.

Solution: False by counterexample.

Proof. Let $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^2$ such that $\vec{\mathbf{v}} = (2,2)$ and $\vec{\mathbf{w}} = (2,-2)$. Computing the norms, we obtain $\|\vec{\mathbf{v}}\| = \max\{|2|,|2|\} = 2$ and $\|\vec{\mathbf{w}}\| = \max\{|2|,|-2|\} = 2$.

Taking the norm of the sum $\vec{\mathbf{v}} + \vec{\mathbf{w}}$,

$$\begin{aligned}\|\vec{\mathbf{v}} + \vec{\mathbf{w}}\| &= \|(2,2) + (2,-2)\| \\ &= \|(4,0)\| \\ &= \max\{|4|,|0|\} \\ &= 4.\end{aligned}$$

Since $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = (2,2)\cdot(2,-2) = 2\cdot 2 + 2\cdot(-2) = 0$, $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are orthogonal. Hence, by the Pythagorean Theorem, $\|\vec{\mathbf{v}} + \vec{\mathbf{w}}\|^2 = \|\vec{\mathbf{v}}\|^2 + \|\vec{\mathbf{w}}\|^2$. However, $4^2 = 16 \neq 8 = 2^2 + 2^2$. Thus a contradiction to the Pythagorean Theorem. □

3. Suppose V is a real inner product space. Prove that
$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4} \qquad \text{for all } u, v \in V$$

Solution: Direct proof.

Proof. The following utilizes the fact that we are acting on a real vector space and the conjugates are trivial. As such, $\langle \vec{\mathbf{u}}, \vec{\mathbf{v}}\rangle = \langle \vec{\mathbf{v}}, \vec{\mathbf{u}}\rangle$ and $\langle \vec{\mathbf{u}}, -\vec{\mathbf{v}}\rangle = -\langle \vec{\mathbf{u}}, \vec{\mathbf{v}}\rangle$.

$$\begin{aligned}&\frac{\|\vec{\mathbf{u}} + \vec{\mathbf{v}}\|^2 - \|\vec{\mathbf{u}} - \vec{\mathbf{v}}\|^2}{4} \\ &= \frac{1}{4}\Big[\langle \vec{\mathbf{u}} + \vec{\mathbf{v}}, \vec{\mathbf{u}} + \vec{\mathbf{v}}\rangle - \langle \vec{\mathbf{u}} - \vec{\mathbf{v}}, \vec{\mathbf{u}} - \vec{\mathbf{v}}\rangle\Big] \\ &= \frac{1}{4}\Big[\|\vec{\mathbf{u}}\| + \langle \vec{\mathbf{u}}, \vec{\mathbf{v}}\rangle + \langle \vec{\mathbf{v}}, \vec{\mathbf{u}}\rangle + \|\vec{\mathbf{v}}\| - \Big(\|\vec{\mathbf{u}}\| + \langle \vec{\mathbf{u}}, -\vec{\mathbf{v}}\rangle + \langle -\vec{\mathbf{v}}, \vec{\mathbf{u}}\rangle + \langle -\vec{\mathbf{v}}, -\vec{\mathbf{v}}\rangle\Big)\Big] \\ &= \frac{1}{4}\Big[\|\vec{\mathbf{u}}\| + \langle \vec{\mathbf{u}}, \vec{\mathbf{v}}\rangle + \langle \vec{\mathbf{v}}, \vec{\mathbf{u}}\rangle + \|\vec{\mathbf{v}}\| - \|\vec{\mathbf{u}}\| - \langle \vec{\mathbf{u}}, -\vec{\mathbf{v}}\rangle - \langle -\vec{\mathbf{v}}, \vec{\mathbf{u}}\rangle - \langle -\vec{\mathbf{v}}, -\vec{\mathbf{v}}\rangle\Big] \\ &= \frac{1}{4}\Big[\|\vec{\mathbf{u}}\| + \langle \vec{\mathbf{u}}, \vec{\mathbf{v}}\rangle + \langle \vec{\mathbf{u}}, \vec{\mathbf{v}}\rangle + \|\vec{\mathbf{v}}\| - \|\vec{\mathbf{u}}\| + \langle \vec{\mathbf{u}}, \vec{\mathbf{v}}\rangle + \langle \vec{\mathbf{v}}, \vec{\mathbf{u}}\rangle - \|\vec{\mathbf{v}}\|\Big] \\ &= \frac{1}{4}\Big[\langle \vec{\mathbf{u}}, \vec{\mathbf{v}}\rangle + \langle \vec{\mathbf{u}}, \vec{\mathbf{v}}\rangle + \langle \vec{\mathbf{u}}, \vec{\mathbf{v}}\rangle + \langle \vec{\mathbf{u}}, \vec{\mathbf{v}}\rangle\Big] \\ &= \langle \vec{\mathbf{u}}, \vec{\mathbf{v}}\rangle\end{aligned}$$

□

4. Show that if $a_1, \dots a_n \in \mathbf{R}$, then the square of the average of $a_1, \dots, a_n$ is less than or equal to the average of $a_1^2, \dots, a_n^2$.

Solution: Direct proof:

Proof. Let $\bar{A} := \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^2$ and $\bar{B} := \frac{1}{n}\sum_{i=1}^{n} a_i^2$. Will will show that $\bar{A} \leq \bar{B}$.

Let $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^n$ such that $\vec{\mathbf{v}} = (a_1, \dots, a_n)$ and $\vec{\mathbf{w}} = (1_1, \dots, 1_n)$. By the Cauchy-Schwarz Inequality we have $|\langle \vec{\mathbf{v}}, \vec{\mathbf{w}} \rangle| \leq \|\vec{\mathbf{v}}\| \, \|\vec{\mathbf{w}}\|$. Computing the left hand side,

$$\begin{aligned}
|\langle \vec{\mathbf{v}}, \vec{\mathbf{w}} \rangle| &= |\langle (a_1, \dots, a_n), (1, \dots, 1) \rangle| \\
&= |1a_1 + \cdots + 1a_n| \\
&= |a_1 + \cdots + a_n| \,.
\end{aligned}$$

Computing the right hand side,

$$\begin{aligned}
\|\vec{\mathbf{v}}\| \, \|\vec{\mathbf{w}}\| &= \sqrt{a_1^2 + \cdots + a_n^2} \cdot \sqrt{1_1 + \cdots + 1_n} \\
&= \sqrt{n \left(a_1^2 + \cdots + a_n^2\right)}.
\end{aligned}$$

Hence, $|a_1 + \cdots + a_n| \leq \sqrt{n \left(a_1^2 + \cdots + a_n^2\right)}$. Algebraically manipulating this inequality,

$$\begin{aligned}
|a_1 + \cdots + a_n| &\leq \sqrt{n \left(a_1^2 + \cdots + a_n^2\right)} \\
\left(|a_1 + \cdots + a_n|\right)^2 &\leq n \left(a_1^2 + \cdots + a_n^2\right) && \text{(square both sides)} \\
\frac{1}{n^2}\left(|a_1 + \cdots + a_n|\right)^2 &\leq \frac{1}{n}\left(a_1^2 + \cdots + a_n^2\right) && \text{(divide by } n^2\text{)} \\
\frac{1}{n^2}\left(a_1 + \cdots + a_n\right)^2 &\leq \frac{1}{n}\left(a_1^2 + \cdots + a_n^2\right) && \text{(property of absolute value)} \\
\left(\frac{a_1 + \cdots + a_n}{n}\right)^2 &\leq \left(\frac{a_1^2 + \cdots + a_n^2}{n}\right) && \text{(factor in } n\text{)} \\
\bar{A} &\leq \bar{B}.
\end{aligned}$$

$\square$

5. Convert $\mathcal{P}_2([0,1])$ into an inner product space by writing $\langle p, q\rangle = \int_0^1 p(x)\overline{q(x)}\,dx$ for $p, q \in \mathcal{P}_2([0,1])$. Find a complete orthonormal set in that space.

Solution: Direct computation via Gram Schmidt.

Let $\mathcal{P}_2([0,1])$ have the ordered basis $\{1, x, x^2\}$ denoted by $\{v_1, v_2, v_3\}$. Using the Gram Schmidt process, we will compute e_1, e_2 and e_3.

For $e_1 = \dfrac{v_1}{\|v_1\|}$, we compute as follows.

$$\begin{aligned}
e_1 &= \frac{v_1}{\|v_1\|} \\
&= \frac{1}{\|1\|} \\
&= \frac{1}{\langle 1, 1\rangle} \\
&= \frac{1}{\int_0^1 1 \cdot 1\,dx} \\
&= \frac{1}{x\big|_{x=0}^{x=1}} \\
&= \frac{1}{1-0} \\
&= 1.
\end{aligned}$$

Next, let $\alpha = v_2 - \langle v_2, e_1 \rangle e_1$. Then $e_2 = \dfrac{v_2 - \langle v_2, e_1 \rangle e_1}{\|v_2 - \langle v_2, e_1 \rangle e_1\|} = \dfrac{\alpha}{\|\alpha\|}$. Computing α,

$$\begin{aligned}
\alpha &= v_2 - \langle v_2, e_1 \rangle e_1 \\
&= x - \langle x, 1 \rangle \cdot 1 \\
&= x - 1 \int_0^1 1x \, dx \\
&= x - \left[\frac{x^2}{2} \right]_{x=0}^{x=1} \\
&= x - \frac{1}{2} \left[(1)^2 - (0)^2 \right] \\
&= x - \frac{1}{2}.
\end{aligned}$$

And computing the norm,

$$\begin{aligned}
\|\alpha\| &= \left\| x - \frac{1}{2} \right\| \\
&= \sqrt{\left\langle x - \frac{1}{2},\ x - \frac{1}{2} \right\rangle} \\
&= \sqrt{\int_0^1 \left(x - \frac{1}{2} \right)^2 dx} \\
&= \sqrt{\frac{1}{3} \left[\left(x - \frac{1}{2} \right)^3 \right]_{x=0}^{x=1}} \\
&= \sqrt{\frac{1}{3} \left[\left(1 - \frac{1}{2} \right)^3 - \left(0 - \frac{1}{2} \right)^3 \right]} \\
&= \sqrt{\frac{1}{3} \left[\frac{1}{8} - \left(-\frac{1}{8} \right) \right]} \\
&= \sqrt{\frac{1}{12}} \\
&= \frac{1}{2\sqrt{3}}.
\end{aligned}$$

And hence,

$$\begin{aligned}
e_2 &= \frac{\alpha}{\|\alpha\|} \\
&= \frac{x - \frac{1}{2}}{\frac{1}{2\sqrt{3}}} \\
&= 2\sqrt{3} \left(x - \frac{1}{2} \right).
\end{aligned}$$

Lastly, let $\beta = v_3 - \langle v_3, e_1 \rangle e_1 - \langle v_3 \rangle e_2 e_2$. Then $e_3 = \dfrac{\beta}{\|\beta\|}$. Computing β,

$$\begin{aligned}
\beta &= v_3 - \langle v_3, e_1 \rangle e_1 - \langle v_3, e_2 \rangle e_2 \\
&= x^2 - \langle x^2, 1 \rangle - \left\langle x^2, 2\sqrt{3}\left(x - \frac{1}{2}\right) \right\rangle \cdot 2\sqrt{3}\left(x - \frac{1}{2}\right) \\
&= x^2 - \int_0^1 x^2\,dx - 2\sqrt{3} \cdot 2\sqrt{3}\left(x - \frac{1}{2}\right) \int_0^1 \left(x^3 - \frac{x^2}{2}\right) dx \\
&= x^2 - \int_0^1 x^2\,dx - (12x - 6) \int_0^1 \left(x^3 - \frac{x^2}{2}\right) dx \\
&= x^2 - \left[\frac{x^3}{3}\right]_{x=0}^{x=1} - (12x - 6)\left[\frac{x^4}{4} - \frac{x^3}{6}\right]_{x=0}^{x=1} \\
&= x^2 - \frac{1}{3} - (12x - 6)\left[\frac{1}{12}\right] \\
&= x^2 - \frac{1}{3} - \left(x - \frac{1}{2}\right) \\
&= x^2 - x + \frac{1}{6}.
\end{aligned}$$

And

$$\begin{aligned}
\|\beta\| &= \sqrt{\langle \beta, \beta \rangle} \\
&= \sqrt{\int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx} \\
&= \sqrt{\int_0^1 \left(x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36}\right) dx} \\
&= \sqrt{\left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{4x^3}{9} - \frac{x^2}{6} + \frac{x}{36}\right]_{x=0}^{x=1}} \\
&= \sqrt{\frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}} \\
&= \sqrt{\frac{36 - 90 + 80 - 30 + 5}{180}} \\
&= \frac{1}{\sqrt{180}} \\
&= \frac{1}{6\sqrt{5}}.
\end{aligned}$$

Hence $e_3 = \dfrac{\beta}{\|\beta\|} = \dfrac{x^2 - x + \frac{1}{6}}{(6\sqrt{5})^{-1}} = 6\sqrt{5}\left(x^2 - x + \dfrac{1}{6}\right)$. Therefore the orthonormal basis of $\mathcal{P}_2([0,1])$ is

$$\left\{1, \quad 2\sqrt{3}\left(x - \frac{1}{2}\right), \quad 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right)\right\}.$$

6. Suppose U is the subspace of $\mathbf{R}^4$ defined by

$$U = \text{span}\,((1,2,3,-4),(-5,4,3,2)).$$

Find an orthonormal basis of U and an orthonormal basis of $U^\perp$.

Solution: Applying Gram Schmidt 4 times.

Before we do any Gram-Schmidt-ing we need to extend U to a basis for $\mathbb{R}^4$. We'll choose $\hat{\imath}$ and $\hat{\jmath}$ and verify that these are infact linear independent, thus forming a basis for $\mathbb{R}^4$. To do this, we will apply the definition of linearly independent and show that the zero vector is only obtainable from 0 coefficients.

$$\vec{0} = a\begin{bmatrix}1\\2\\3\\-4\end{bmatrix} + b\begin{bmatrix}-5\\4\\3\\2\end{bmatrix} + c\underbrace{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}_{\hat{\imath}} + d\underbrace{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}}_{\hat{\jmath}}$$

Therefore

$$\begin{aligned} a - 5b + c &= 0 \\ 2a + 4b + d &= 0 \\ 3a + 3b &= 0 \\ -4a + 2b &= 0. \end{aligned}$$

Which, the last 2 rows imply that $b = -a$ and $b = 2a$. Thus $2a = -a$ which is only possible if $a = 0$. Similarly, since $b = -a = -0$, $b = 0$. By the first row, $0 - 5(0) + c = 0$ implies $c = 0$. Similarly, the second row $2(0) + 4(0) + d = 0$ implies $d = 0$. Hence, $a = b = c = d = 0$ and thus the set is linearly independent and forms a basis of $\mathbb{R}^4$.

Denote the basis $\left\{(1,2,3,-1),\ (-5,4,3,2),\ \hat{\imath},\ \hat{\jmath}\right\}$ by $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4\}$

For our first application,

$$\begin{aligned} \vec{\mathbf{e_1}} &= \frac{\vec{\mathbf{v}}_1}{\|\vec{\mathbf{v}}_1\|} = \frac{\vec{\mathbf{v}}_1}{\sqrt{\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_1}} = \frac{(1,2,3,-4)}{\sqrt{(1,2,3,-4)\cdot(1,2,3,-4)}} = \frac{(1,2,3,-4)}{\sqrt{1^2+2^2+3^2+(-4)^2}} \\ &= \frac{1}{\sqrt{30}}(1,2,3,-4). \end{aligned}$$

For the second application, let $\vec{\mathbf{u}}_2$ denote $\vec{\mathbf{v}}_2 - \langle \vec{\mathbf{v}}_2, \vec{\mathbf{e_1}} \rangle\, \vec{\mathbf{e_1}}$.

We will first compute the value of $\vec{\mathbf{u}}_2$.

$$\begin{aligned}
\vec{\mathbf{u}}_2 &= \vec{\mathbf{v}}_2 - \langle \vec{\mathbf{v}}_2, \vec{\mathbf{e_1}} \rangle \vec{\mathbf{e_1}} \\
&= \vec{\mathbf{v}}_2 - (\vec{\mathbf{v}}_2 \cdot \vec{\mathbf{e_1}}) \vec{\mathbf{e_1}} \\
&= \vec{\mathbf{v}}_2 - \frac{1}{\sqrt{30}}(-5 \cdot 1 + 4 \cdot 2 + 3 \cdot 3 + 2 \cdot (-4)) \vec{\mathbf{e_1}} \\
&= \vec{\mathbf{v}}_2 - \frac{4}{\sqrt{30}} \vec{\mathbf{e_1}} \\
&= (-5, 4, 3, 2) - \frac{4}{\sqrt{30}} \left(\frac{1}{\sqrt{30}}(1, 2, 3, -4) \right) \\
&= (-5, 4, 3, 2) - \frac{4}{30}(1, 2, 3, -4) \\
&= \left(-\frac{77}{15},\ \frac{56}{15},\ \frac{39}{15},\ \frac{38}{15} \right) \\
&= \frac{1}{15} \left(-77,\ 56,\ 39,\ 38 \right)
\end{aligned}$$

Then

$$\begin{aligned}
\vec{\mathbf{e_2}} &= \frac{\vec{\mathbf{u}}_2}{\|\vec{\mathbf{u}}_2\|} = \frac{\vec{\mathbf{u}}_2}{\sqrt{\vec{\mathbf{u}}_2 \cdot \vec{\mathbf{u}}_2}} = \vec{\mathbf{u}}_2 \cdot \frac{1}{\sqrt{\vec{\mathbf{u}}_2 \cdot \vec{\mathbf{u}}_2}} \\
&= \vec{\mathbf{u}}_2 \cdot \frac{1}{\sqrt{\frac{1}{15}\left(-77,\ 56,\ 39,\ 38\right) \cdot \frac{1}{15}\left(-77,\ 56,\ 39,\ 38\right)}} \\
&= \vec{\mathbf{u}}_2 \cdot \frac{15}{\sqrt{\left(-77,\ 56,\ 39,\ 38\right) \cdot \left(-77,\ 56,\ 39,\ 38\right)}} \\
&= \vec{\mathbf{u}}_2 \cdot \frac{15}{\sqrt{12{,}030}} \\
&= \frac{1}{15}\left(-77,\ 56,\ 39,\ 38\right) \cdot \frac{15}{\sqrt{12{,}030}} \\
&= \frac{1}{\sqrt{12{,}030}}\left(-77,\ 56,\ 39,\ 38\right)
\end{aligned}$$

For the third application, let $\vec{\mathbf{u}}_3$ denote $\vec{\mathbf{v}}_3 - \langle \vec{\mathbf{v}}_3, \vec{\mathbf{e_1}} \rangle \vec{\mathbf{e_1}} - \langle \vec{\mathbf{v}}_3, \vec{\mathbf{e_2}} \rangle \vec{\mathbf{e_2}}$. Computing $\vec{\mathbf{u}}_3$,

$$
\begin{aligned}
\vec{\mathbf{u}}_3 &= \vec{\mathbf{v}}_3 - \langle \vec{\mathbf{v}}_3, \vec{\mathbf{e_1}} \rangle \vec{\mathbf{e_1}} - \langle \vec{\mathbf{v}}_3, \vec{\mathbf{e_2}} \rangle \vec{\mathbf{e_2}} \\
&= \vec{\mathbf{v}}_3 - (\vec{\mathbf{v}}_3 \cdot \vec{\mathbf{e_1}}) \vec{\mathbf{e_1}} - (\vec{\mathbf{v}}_3 \cdot \vec{\mathbf{e_2}}) \vec{\mathbf{e_2}} \\
&= \vec{\mathbf{v}}_3 - \left(1 \cdot \frac{1}{\sqrt{30}} + 0 + 0 + 0\right) \vec{\mathbf{e_1}} - \left(1 \cdot \frac{-77}{\sqrt{12{,}030}} + 0 + 0 + 0\right) \vec{\mathbf{e_2}} \\
&= \vec{\mathbf{v}}_3 - \left(\frac{1}{\sqrt{30}}\right)\left(\frac{1}{\sqrt{30}}\right)(1,2,3,-4) - \left(\frac{-77}{\sqrt{12{,}030}}\right)\left(\frac{1}{\sqrt{12{,}030}}\right)(-77,\ 56,\ 39,\ 38) \\
&= \vec{\mathbf{v}}_3 - \frac{1}{30}(1,2,3,-4) + \frac{77}{12{,}030}(-77,56,39,38) \\
&= (1,0,0,0) - \frac{1}{30}(1,2,3,-4) + \frac{77}{12{,}030}(-77,56,39,38) \\
&= \frac{12{,}030}{12{,}030}\hat{\imath} - \frac{401}{12{,}030}(1,2,3,-4) + \frac{77}{12{,}030}(-77,56,39,38) \\
&= \frac{1}{12{,}030}(5700,3510,1800,4530) \\
&= \frac{1}{401}(190,117,60,151)
\end{aligned}
$$

Then

$$
\begin{aligned}
\vec{\mathbf{e_3}} &= \frac{\vec{\mathbf{u}}_3}{\|\vec{\mathbf{u}}_3\|} = \frac{\vec{\mathbf{u}}_3}{\sqrt{\vec{\mathbf{u}}_3 \cdot \vec{\mathbf{u}}_3}} = \vec{\mathbf{u}}_3 \cdot \frac{1}{\sqrt{\vec{\mathbf{u}}_3 \cdot \vec{\mathbf{u}}_3}} \\
&= \vec{\mathbf{u}}_3 \frac{1}{\sqrt{\frac{1}{401}(190,117,60,151) \cdot \frac{1}{401}(190,117,60,151)}} \\
&= 401\vec{\mathbf{u}}_3 \cdot \frac{1}{\sqrt{(190,117,60,151) \cdot (190,117,60,151)}} \\
&= 401\vec{\mathbf{u}}_3 \cdot \frac{1}{\sqrt{190^2 + 117^2 + 60^2 + 151^2}} \\
&= 401\vec{\mathbf{u}}_3 \cdot \frac{1}{\sqrt{76{,}190}} \\
&= \frac{1}{401}(190,117,60,151) \cdot \frac{401}{\sqrt{76{,}190}} \\
&= \frac{1}{\sqrt{76{,}190}}(190,117,60,151)
\end{aligned}
$$

Lastly, for $\vec{\mathbf{e_4}}$, let $\vec{\mathbf{u}}_4$ denote $\vec{\mathbf{v}}_4 - \langle \vec{\mathbf{v}}_4, \vec{\mathbf{e_1}} \rangle \vec{\mathbf{e_1}} - \langle \vec{\mathbf{v}}_4, \vec{\mathbf{e_2}} \rangle \vec{\mathbf{e_2}} - \langle \vec{\mathbf{v}}_4, \vec{\mathbf{e_3}} \rangle \vec{\mathbf{e_3}}$. Computing $\vec{\mathbf{u}}_4$,

$$\begin{aligned}
\vec{\mathbf{u}}_4 &= \vec{\mathbf{v}}_4 - \langle \vec{\mathbf{v}}_4, \vec{\mathbf{e_1}} \rangle \vec{\mathbf{e_1}} - \langle \vec{\mathbf{v}}_4, \vec{\mathbf{e_2}} \rangle \vec{\mathbf{e_2}} - \langle \vec{\mathbf{v}}_4, \vec{\mathbf{e_3}} \rangle \vec{\mathbf{e_3}} \\
&= \hat{j} - \left\langle \hat{j}, \vec{\mathbf{e_1}} \right\rangle \vec{\mathbf{e_1}} - \left\langle \hat{j}, \vec{\mathbf{e_2}} \right\rangle \vec{\mathbf{e_2}} - \left\langle \hat{j}, \vec{\mathbf{e_3}} \right\rangle \vec{\mathbf{e_3}} \\
&= \hat{j} - (\hat{j} \cdot \vec{\mathbf{e_1}})\vec{\mathbf{e_1}} - (\hat{j} \cdot \vec{\mathbf{e_2}})\vec{\mathbf{e_2}} - (\hat{j} \cdot \vec{\mathbf{e_3}})\vec{\mathbf{e_3}} \\
&= \hat{j} - \left(\frac{2}{\sqrt{30}}\right) \vec{\mathbf{e_1}} - \left(\frac{56}{\sqrt{12{,}030}}\right) \vec{\mathbf{e_2}} - \left(\frac{117}{\sqrt{76{,}190}}\right) \vec{\mathbf{e_3}} \\
&= \hat{j} - \left(\frac{2}{30}\right)(1,2,3,-4) - \left(\frac{56}{12{,}030}\right)(-77,\ 56,\ 39,\ 38) - \left(\frac{117}{76{,}190}\right)(190,117,60,151) \\
&= \hat{j} - \left(\frac{1}{15}\right)(1,2,3,-4) - \left(\frac{28}{6015}\right)(-77,\ 56,\ 39,\ 38) - \left(\frac{117}{76{,}190}\right)(190,117,60,151) \\
&= \frac{228{,}570}{228{,}570}\hat{j} - \left(\frac{15238}{228{,}570}\right)(1,2,3,-4) - \left(\frac{1444}{228{,}570}\right)(-77,\ 56,\ 39,\ 38) \\
&\qquad - \left(\frac{351}{228{,}570}\right)(190,117,60,151) \\
&= \frac{1}{228{,}570}(29260,\ 76173,\ -123090,\ -46921)
\end{aligned}$$

Then computing $\vec{\mathbf{e_4}}$,

$$\begin{aligned}
\vec{\mathbf{e_4}} &= \frac{\vec{\mathbf{u}}_4}{\|\vec{\mathbf{u}}_4\|} = \frac{\vec{\mathbf{u}}_4}{\sqrt{\vec{\mathbf{u}}_4 \cdot \vec{\mathbf{u}}_4}} = \vec{\mathbf{u}}_4 \cdot \frac{1}{\sqrt{\vec{\mathbf{u}}_4 \cdot \vec{\mathbf{u}}_4}} \\
&= 228570\vec{\mathbf{u}}_4 \cdot \frac{1}{\sqrt{29260^2 + 76173^2 + (-123090)^2 + (-46921)^2}} \\
&= 228570\vec{\mathbf{u}}_4 \cdot \frac{1}{\sqrt{24{,}011{,}201{,}870}} \\
&= \frac{228570}{228570}(29260,\ 76173,\ -123090,\ -46921) \cdot \frac{1}{\sqrt{24{,}011{,}201{,}870}} \\
&= \frac{1}{\sqrt{24{,}011{,}201{,}870}}(29260,\ 76173,\ -123090,\ -46921)
\end{aligned}$$

Hence our orthonormal basis of U is

$$\left\{ \frac{1}{\sqrt{30}}(1,2,3,-4), \quad \frac{1}{\sqrt{12{,}030}}(-77,\ 56,\ 39,\ 38) \right\}$$

And our orthonormal basis of $U^{\perp}$ is

$$\left\{ \frac{1}{\sqrt{76{,}190}}(190,117,60,151), \quad \frac{1}{\sqrt{24{,}011{,}201{,}870}}(29260,\ 76173,\ -123090,\ -46921) \right\}$$