MATH 307
Assignment #9
Due Friday, March 25, 2022

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. Suppose V is finite-dimensional and $P \in \mathcal{L}(V)$ is such that (1) $P^2 = P$ and (2) every vector in null P is orthogonal to every vector in range P. Prove that there exists a subspace U of V such that $P = P_U$.
   Hint: For $v \in V$, write $v = Pv + (v - Pv)$.

2. Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U is invariant under T if and only if $P_U T P_U = T P_U$.

3. In $\mathbf{R}^4$, let
   $$U = \operatorname{span}\left((0,0,1,1),(1,2,1,1)\right).$$
   Find $u \in U$ such that $\|u - (1,3,5,4)\|$ is as small as possible.

4. Assume $T \in \mathcal{L}(V)$ for a complex vector space V. Prove that T is self-adjoint if and only if all eigenvalues for T are real.

5. If $T \in \mathcal{L}(V)$ is self-adjoint and if $T^2 v = 0$, then $Tv = 0$

6. Suppose $T \in \mathcal{L}(V, W)$. Prove that

   (a) T is injective if and only if T^* is surjective.

   (b) T is surjective if and only if T^* is injective.

7. Suppose $S, T \in \mathcal{L}(V)$ are self-adjoint. Prove that ST is self-adjoint if and only if $ST = TS$.

8. Suppose $T \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that there is a subspace U of V such that $P = P_U$ if and only if P is self-adjoint.