MATH 307
Assignment #2
Due Friday, January 28, 2022

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. Label the following statements as being true or false. Provide some justification from the text for your label.
   (a) If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V.
   (b) The empty set is a subspace of every vector space.
   (c) If V is a vector space other than the zero vector space $\{0\}$, then V contains a subspace W such that $W \neq V$.
   (d) The intersection of any two subsets of V is a subspace of V.
2. Prove that the intersection of two subspaces U and W of a vector space V is a subspace of V.
3. Prove that the union of two subspaces U and W of a vector space V is a subspace of V if and only if one of of the subspaces is contained in the other.
4. Let V be the vector space of 2×2 matrices with the usual operation of addition and scalar multiplication as seen in MTH 207. (We are *not* considering multiplication of matrices in this exercise.)
   Let W_1 be the set of matrices in V of the form $\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$ and let W_2 be the set of matrices in V of the form $\begin{bmatrix} a & b \\ -a & c \end{bmatrix}$.
   (a) Prove that W_1 and W_2 are subspaces of V.
   (b) Describe the subspace $W_1 \cap W_2$.
   (c) Show that the subspace $W_1 + W_2$ is all of V.
5. Prove or give a counterexample: if U_1, U_2, W are subspaces of the vector space V such that $V = U_1 \oplus W$ and $V = U_2 \oplus W$, then $U_1 = U_2$.
6. Let $V = \mathbf{R}^3$ – the usual 3D space from Calc III. Let U be the x-axis. Define W to be the subspace spanned by $(1, 0, 1)$. Show that the usual xz-plane is the direct sum $U \oplus W$.
7. Suppose that the vectors v_1, v_2, v_3, v_4 span the vector space V. Show that the vectors $v_1 - v_2, v_1 + v_2, v_3 + v_4, v_4$ also span V.