MATH 307
Assignment #12
Due Friday, April 22nd, 2022

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. Suppose T is a positive operator on V. Prove that T is invertible if and only if $\langle Tv, v\rangle > 0$ for every $v \in V$ with $v \neq 0$.

   Proof. ($\Longrightarrow$)
   Suppose T is invertible. By the positivity hypothesis, there exists $R \in \mathcal{L}(V)$ such that $R^2 = T$. Hence,

$$\begin{aligned} \langle Tv, v\rangle &= \left\langle R^2 v, v\right\rangle && \text{(Substitution)} \\ &= \langle Rv, R^* v\rangle && \text{(Taking the adjoint)} \\ &= \langle Rv, Rv\rangle && \text{(Self-adjoint square roots)} \end{aligned}$$

   Which, by the definiteness property of inner products, $\langle Rv, Rv\rangle = 0$ if and only if $Rv = 0$.

   By positivity on T, then $R^2 = T$. Multiplying on the right by T^{-1}, then $R^2T^{-1} = TT^{-1} = I$. Therefore $R(RT^{-1}) = I$. Hence R is invertible and $R^{-1} = RT^{-1}$.

   Thus, by invertibility null $R = \{0\}$. Since $v \neq 0$ by hypothesis, $Rv \neq 0$. Therefore $\langle Rv, Rv\rangle > 0$ for $v \neq 0$ and hence $\langle Tv, v\rangle > 0$. □

   Proof. ($\Longleftarrow$)
   Suppose $\langle Tv, v\rangle > 0$ for $v \neq 0$. This implies that $Tv \neq 0$ and $v \neq 0$ since $\langle 0, u\rangle = 0 = \langle u, 0\rangle$ for any $u \in V$. Thus $Tv \neq 0$ for all $v \neq 0$. Hence T is injective, and thus invertible. □

2. Suppose $T \in \mathcal{L}(V)$, for an inner product space V. For $u, v \in V$, define the function of two variables $\langle u, v\rangle_T$ by
$$\langle u, v\rangle_T = \langle Tu, v\rangle.$$
Prove that $\langle\cdot,\cdot\rangle_T$ is an inner product on V if and only if T is an invertible positive operator (with respect to the original inner product $\langle\cdot,\cdot\rangle$).

Proof. ($\Longrightarrow$)
Suppose $\langle\cdot,\cdot\rangle_T$ is an inner product, we will show that T is an invertible positive operator (with respect to the original $\langle\cdot,\cdot\rangle$). By positivity of inner products, $\langle Tv, v\rangle = \langle v, v\rangle_T \geq 0$, so $\langle Tv, v\rangle \geq 0$. For self-adjoint,

$$\begin{aligned}\langle Tu, v\rangle &= \langle u, v\rangle_T \\ &= \overline{\langle v, u\rangle_T} \\ &= \overline{\langle Tv, u\rangle} \\ &= \langle u, Tv\rangle .\end{aligned}$$

Therefore T is self-adjoint and hence T is a positive operator.

For invertibility, $\langle v, v\rangle_T = 0$ if and only if $v = 0$, so $\langle v, v\rangle_T = \langle Tv, v\rangle = 0$ if and only if $v = 0^\dagger$. Suppose $Tv = 0$ for some $v \neq 0$. Then $\langle Tv, v\rangle = \langle 0, v\rangle = 0$. Hence, a contradiction to †. Thus $Tv = 0$ if and only if $v = 0$ and $\operatorname{null} T = \{0\}$. Therefore T is injective and hence invertible. □

Proof. ($\Longleftarrow$)
Suppose T is an invertible positive operator (with respect to the original $\langle\cdot,\cdot\rangle$). We will show that $\langle\cdot,\cdot\rangle_T$ is an inner product. That is, positivity, definiteness, additivity, homogeneity, and symmetry.

positivity

Since T is positive we know that $\langle Tv, v\rangle \geq 0$. Since $\langle v, v\rangle_T = \langle Tv, v\rangle$ then $\langle v, v\rangle_T \geq 0$.

definiteness

By T's positivity, $R^2 = T$ for a positive square root R of T. Then $\langle Tv, v\rangle = \left\langle R^2 v, v\right\rangle = \langle Rv, Rv\rangle$. As shown in #1, since T is positive and invertible then R is also invertible and hence injective. So $Rv = 0$ if and only if $v = 0$ and hence $\langle Rv, Rv\rangle = \langle Tv, v\rangle = \langle v, v\rangle_T = 0$ if and only if $v = 0$.

additivity in the first slot

Directly, $\langle u + v, w\rangle_T = \langle T(u + v), w\rangle = \langle Tu, w\rangle + \langle Tv, w\rangle = \langle u, w\rangle_T + \langle v, w\rangle_T$.

homogeneity in first slot

Directly, $\langle \lambda u, v\rangle_T = \langle \lambda Tu, v\rangle = \lambda \langle Tu, v\rangle = \lambda \langle u, v\rangle_T$.

conjugate symmetry

Directly with T self-adjoint, $\langle u, v\rangle_T = \langle Tu, v\rangle = \overline{\langle v, Tu\rangle} = \overline{\langle Tv, u\rangle} = \overline{\langle v, u\rangle_T}$. □

3. Suppose $S \in \mathcal{L}(V)$. Prove that the following are equivalent:

   (a) S is an isometry;

   (b) $\langle S^*u, S^*v\rangle = \langle u, v\rangle$ for all $u, v \in V$;

   (c) $S^*e_1, \ldots S^*e_m$ is an orthonormal list for every orthonormal list of vectors $e_1, \ldots, e_m$ in V;

   (d) $S^*e_1, \ldots S^*e_n$ is an orthonormal basis for some orthonormal basis $e_1, \ldots, e_n$ of V.

   We will prove $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$.

   Proof. (a $\implies$ b)
   Suppose S is an isometry. Then taking the adjoint of the right hand side, $\langle S^*u, S^*v\rangle = \langle SS^*u, v\rangle$. Since S is an isometry, then $SS^* = I$. Hence, $\langle S^*u, S^*v\rangle = \langle u, v\rangle$. □

   Proof. (b $\implies$ c)
   Suppose $\langle S^*u, S^*v\rangle = \langle u, v\rangle$ for all $u, v \in V$. Since $e_1, \ldots, e_m$ is an orthonormal list then $\left\langle e_j, e_k\right\rangle = 0$ for $j \neq k$ and 1 for $j = k$. By (b), $\left\langle S^*e_j, S^*e_k\right\rangle = \left\langle e_j, e_k\right\rangle$, hence $S^*e_1, \ldots, S^*e_m$ is also an orthonormal list in V by equality. □

   Proof. (c $\implies$ d)
   Suppose $S^*e_1, \ldots S^*e_m$ is an orthonormal list for every orthonormal list of vectors $e_1, \ldots, e_m$ in V. Then we can extend $e_1, \ldots, e_m$ to a basis of V by the Gram Schmidt procedure with and apply (c) with $m = n = \dim V$. □

   Proof. (d $\implies$ a)
   Suppose $S^*e_1, \ldots S^*e_n$ is an orthonormal basis for some orthonormal basis $e_1, \ldots, e_n$ of V. Then $\left\langle SS^*e_j, e_k\right\rangle = \left\langle S^*e_j, S^*e_k\right\rangle = \left\langle e_j, e_k\right\rangle$, which is orthonormal. Every $u, v \in V$ can be written as a unique linear combination of $e_1, \ldots, e_n$. Hence, $\langle SS^*u, v\rangle = \langle u, v\rangle$ and $SS^* = I$, a condition for an isometry. □

4. Suppose T_1, T_2 are normal operators on $\mathbf{F}^3$ and both operators have 2 , 5 , 7 as eigenvalues. Prove that there exists an isometry $S \in \mathcal{L}(\mathbf{F}^3)$ such that $T_1 = S^* T_2 S$.

Proof.

By Theorem 7.22, since T_1 are normal, then the eigenvectors of T_1 corresponding to distinct eigenvalues are orthogonal. Since there are 3 distinct eigenvalues (namely 2, 5, and 7) and $\dim V = 3$, then we have an orthonormal basis for $\mathbb{F}^3$ consisting of the eigenvectors corresponding to distinct eigenvalues from T_1. Similarly T_2 has an orthonormal basis of eigenvectors corresponding to 2, 5, and 7.

Let $B_1 := \{e_1, e_2, e_3\}$ be an orthonormal basis for $\mathbb{F}^3$ corresponding to T_1's eigenvectors. Similarly, let $B_2 := \{f_1, f_2, f_3\}$ be an orthonormal basis for $\mathbb{F}^3$ corresponding to T_2's eigenvectors. For $S \in \mathcal{L}\left(\mathbb{F}^3\right)$, define $Se_i = f_i$ for $i = 1, 2, 3$. Then for the orthonormal basis e_1, e_2, e_3 of $\mathbb{F}^3$, $Se_1, \ldots, Se_n = f_1, \ldots, f_n$ is an orthonormal basis. Hence (d) $\Longrightarrow$ (a) of 7.42 shows S is an isometry.

Therefore S^* is an isometry and $S^* = S^{-1}$. Thus $S^* f_i = S^{-1} f_i = e_i$. Then, recall that $T_1 e_i = \lambda_i e_i$ and $T_2 f_i = \lambda_i f_i$. Then

$$\begin{aligned}
T_1 e_i &= \lambda_i e_i && \text{(eigenvalues for eigenvectors)} \\
&= \lambda_i (S^* f_i) && \text{(substitution)} \\
&= S^* (\lambda_i f_i) && \text{(linearity)} \\
&= S^* (T_2 f_i) && \text{(substitution)} \\
&= S^* (T_2 S e_i) && \text{(substitution)}
\end{aligned}$$

Hence $T_1 e_i = S^* T_2 e_i$. Since e_i forms a [orthonormal] basis B_1 for $\mathbb{F}^3$, then for any vector $v \in \mathbb{F}^3$ we have $v = a_1 e_1 + a_2 e_2 + a_3 e_3$. So by linearity,

$$\begin{aligned}
T_1 v &= T_1 (a_1 e_1 + a_2 e_2 + a_3 e_3) && \text{(substitution of } v\text{)} \\
&= a_1 (T_1 e_1) + a_2 (T_1 e_2) + a_3 (T_1 e_3) && \text{(linearity)} \\
&= a_1 (S^* T_2 S e_1) + a_2 (S^* T_2 S e_2) + a_3 (S^* T_2 S e_3) && \text{(substitution of } T_1 e_i = S^* T_2 S e_i\text{)} \\
&= S^* T_2 S (a_1 e_1 + a_2 e_2 + a_3 e_3) && \text{(linearity)} \\
&= S^* T_2 S v && \text{(substitution of } v\text{)}
\end{aligned}$$

Hence $T_1 = S^* T_2 S$ and S is an isometry. $\square$

5. Fix $u, x \in V$ with $u \neq 0$. Define $T \in \mathcal{L}(V)$ by $Tv = \langle v, u\rangle x$ for every $v \in V$. Prove that

$$\sqrt{T^*T}v = \frac{||x||}{||u||}\langle v, u\rangle u$$

for every $v \in V$.

Proof.

First computing T^* we have

$$\begin{aligned}
\langle Tv, w\rangle &= \langle \langle v, u\rangle\, x,\ \ w\rangle && \text{(substitution with given } Tv = \langle v, u\rangle\, x\text{)}\\
&= \langle v, u\rangle \langle x, w\rangle && \text{(homogeneity in the first slot)}\\
&= \left\langle v,\ \overline{\langle x, w\rangle} u\right\rangle && \text{(second slot conjugate homogeneity)}\\
&= \langle v,\ \langle w, x\rangle\, u\rangle && \text{(conjugate symmetry)}\\
&= \langle v, T^* w\rangle\, . && \text{(take adjoint)}
\end{aligned}$$

Therefore $T^* w = \langle w, x\rangle\, u$. Thus

$$\begin{aligned}
T^*Tv &= T^* \langle v, u\rangle\, x && \text{(substitution)}\\
&= \langle \langle v, u\rangle\, x, x\rangle\, u && \text{(definition of } T^*\text{)}\\
&= \langle v, u\rangle \langle x, x\rangle\, u && \text{(homogeneity in first slot)}\\
&= \langle v, u\rangle\, ||x||^2 u\\
&= \langle v, u\rangle\, ||x||^2 u \frac{\langle u, u\rangle}{||u||^2} && \text{(fancy 1)}\\
&= \left(\frac{||x||}{||u||}\right)^2 \langle v, u\rangle \langle u, u\rangle\, u
\end{aligned}$$

not finished

□

6. Give an example of $T \in \mathcal{L}(\mathbf{C}^2)$ such that 0 is the only eigenvalue of T and the singular values of T are 5 , 0.

Solution: Define $T(x, y) = (5y, 0)$. Then using the standard basis

$$\mathcal{M}(T) = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathcal{M}(T) = \begin{bmatrix} 0 & 0 \\ 5 & 0 \end{bmatrix}.$$

Therefore

$$T^*T = \begin{bmatrix} 0 & 0 \\ 0 & 25 \end{bmatrix} \qquad \text{and} \qquad \sqrt{T^*T} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}.$$

Because T is an upper triangular matrix, the diagonal gives us its eigenvalues. Hence 0 is T's eigenvalue with multiplicity 2. Then by diagonal matrix properties, the eigenvalues of $\sqrt{T^*T}$ are 5 and 0. Therefore the singular values of T are 5, 0.

7. Suppose $T \in \mathcal{L}(V)$ and s is a singular value of T. Prove that there exists a vector $v \in V$ such that $\|v\| = 1$ and $\|Tv\| = s$.

Proof.

Let $s_1, \ldots, s_n$ be the singular values of T. Let $s = s_1$. Then by SVD, there exists orthonormal bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ such that

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_n \langle v, e_n \rangle f_n$$

for every $v \in V$. If we choose $v = e_1$ then its clear that $\|v\| = 1$ by normalized vector properties. Further,

$$\begin{aligned} Tv &= s_1 \langle v, e_1 \rangle f_1 + \cdots + s_n \langle v, e_n \rangle f_n \\ &= s_1 \underbrace{\langle e_1, e_1 \rangle}_{=1} f_1 + s_2 \underbrace{\langle e_1, e_2 \rangle}_{=0} f_2 + \underbrace{\cdots}_{=0} + s_n \underbrace{\langle e_1, e_n \rangle}_{=0} f_n \\ &= s_1 f_1 \\ &= s f_1. \end{aligned}$$

Then $\|Tv\|^2 = \|sf_1\|^2 = \langle sf_1, sf_1 \rangle = |s|^2 \langle f_1, f_1 \rangle = \left|s^2\right|$ since f_1 is normalized. Thus $\|Tv\| = |s|$. But singular values are non-negative by definition, hence $|s| = s$ and therefore $\|Tv\| = s$ for $v = e_1$. $\square$

8. Suppose $T \in \mathcal{L}(\mathbf{C}^2)$ is defined by $T(x, y) = (-4y, x)$. Find the singular values of T.

Solution: The transformation matrix for T with respect to the standard basis is

$$\mathcal{M}(T) = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathcal{M}(T^*) = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}.$$

Then

$$T^*T = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix} \qquad \text{and} \qquad \sqrt{T^*T} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

As such, the eigenvalues of $\sqrt{T^*T}$ are the singular values by definition. Since the eigenvalues of a diagonal matrix are the diagonals, then the singular values are 4, 1.