MATH 307
Assignment #9
Due Friday, March 25, 2022

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. Suppose V is finite-dimensional and $P \in \mathcal{L}(V)$ is such that (1) $P^2 = P$ and (2) every vector in null P is orthogonal to every vector in range P. Prove that there exists a subspace U of V such that $P = P_U$.
   Hint: For $v \in V$, write $v = Pv + (v - Pv)$.

   Proof. We will first show that $V = \operatorname{range} P \oplus \operatorname{null} P$. Let $u \in \operatorname{range} P$ and $w \in \operatorname{null} P$. Then we will show that every $v \in V$ can be expressed as $v = u + w$. Taking $P(v - Pv)$ we have $Pv - P^2v = Pv - Pv$ by hypothesis (1), which is clearly zero. Therefore $(v - Pv) \in \operatorname{null} P$. As for Pv, we know that $Pv \in \operatorname{range} P$ by definition of range. Therefore we can rewrite v as the sum of subsets

$$v = \underbrace{Pv}_{\in \operatorname{range} P} + \underbrace{(v - Pv)}_{\in \operatorname{null} P}.$$

   To show it is a unique linear combination (direct sum), we need $\operatorname{null} P \cap \operatorname{range} P = \{0\}$. For every $x \in \operatorname{null} P \cap \operatorname{range} P$, we have $Px = 0$ and $x = Py$ for some $y \in V$. By hypothesis (1), $P^2y = Py = P(Py) = Px = 0$. Thus, $\operatorname{null} P \cap \operatorname{range} P = \{0\}$ and

$$v = \operatorname{range} P \oplus \operatorname{null} P.$$

   Hence every $v \in V$ can be written as a unique linear combination $v = Pv + (v - Pv)$. By hypothesis (2), we have $\operatorname{null} P \subseteq (\operatorname{range} P)^{\perp}$. For $U := \operatorname{range} P$ (a subspace), then

$$P_U v = P_U(Pv + v - Pv) = P_U \underbrace{(Pv)}_{\in U} + P_U \underbrace{(v - Pv)}_{\in U^{\perp}} = Pv + 0 = Pv.$$

   $\square$

2. Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U is invariant under T if and only if $P_U T P_U = T P_U$.

Proof. ($\Longrightarrow$) (If U is invariant under T then $P_U T P_U = T P_U$.)
For any $v \in V$ we have $P_U v \in U$ by properties of P_U (6.55 (d)). By our hypothesis $T(T_U v) \in U$, therefore

$$P_U \underbrace{(TP_U v)}_{\in U} = TP_U v$$

by property 6.55 (b). □

Proof. ($\Longleftarrow$) (If $P_U T P_U = T P_U$ then U is invariant under T.)
For any vector $u \in U$ we have $Tu = v + w$ for some $v \in U$ and $w \in U^\perp$. Then we would first have

$$\begin{aligned} P_U T P_U &= P_U T u \\ &= P_U(v + w) \\ &= \underbrace{P_U v}_{\in U} + \underbrace{P_U w}_{\in U^\perp} \\ &= v + 0 \\ &= v. \end{aligned}$$

Secondly, we would have $TP_U u = Tu = v+w$. Therefore under our hypothesis, $v = v+w$, which implies $w = 0$. Then $Tu = v + 0 = v \in U$, hence U is invariant under T. □

Hence U is invariant under T if and only if $P_U T P_U = T P_U$.

3. In $\mathbf{R}^4$, let
$$U = \operatorname{span}\left((0,0,1,1),(1,2,1,1)\right).$$
Find $u \in U$ such that $\|u - (1,3,5,4)\|$ is as small as possible.

Solution: Let B be an orthonormal basis of U. Let $v_1, v_2 \in \mathbb{R}^4$ equal $v_1 = (0,0,1,1)$ and $v_2 = (1,2,1,1)$. Using the Gram-Schmidt procedure on v_1 and v_2,

$$e_1 = \frac{1}{\|v_1\|}v_1 = \frac{1}{\sqrt{2}}(0,0,1,1)$$

and

$$\begin{aligned} e_2 &= \frac{v_2 - \langle v_2, e_1\rangle e_1}{\|v_2 - \langle v_2, e_1\rangle e_1\|} \\ &= \frac{v_2 - \frac{1}{2}\langle v_2, v_1\rangle v_1}{\left\|v_2 - \frac{1}{2}\langle v_2, v_1\rangle v_1\right\|} \\ &= \frac{v_2 - \frac{1}{2}(0+0+1+1)v_1}{\left\|v_2 - \frac{1}{2}(0+0+1+1)v_1\right\|} \\ &= \frac{v_2 - v_1}{\|v_2 - v_1\|} \\ &= \frac{1}{\sqrt{5}}(1,2,0,0). \end{aligned}$$

Then $B := \{e_1, e_2\}$. Then the closest point to $u \in U$ to $w := (1,3,5,4)$ is

$$\begin{aligned} u &= \langle w, e_1\rangle e_1 + \langle w, e_2\rangle e_2 \\ &= \frac{5+4}{2}(0,0,1,1) + \frac{1+6}{5}(1,2,0,0) \\ &= \left(\frac{7}{5}, \frac{14}{5}, \frac{9}{2}, \frac{9}{2}\right). \end{aligned}$$

4. Assume $T \in \mathcal{L}(V)$ for a complex vector space V. Prove that T is self-adjoint if and only if all eigenvalues for T are real.

Proof. ($\Longrightarrow$) (If T is self-adjoint, then all the eigenvalues for T are real.)

Let λ be an eigenvalue of T and $v \in V \backslash 0$ such that $Tv = \lambda v$. Then

$$\begin{aligned} \lambda \|v\|^2 &= \langle \lambda v, v \rangle \\ &= \langle Tv, v \rangle \\ &= \langle v, T^* v \rangle \\ &= \langle v, Tv \rangle \qquad \text{by hypothesis} \\ &= \langle v, \lambda v \rangle \\ &= \bar{\lambda} \langle v, v \rangle \\ &= \bar{\lambda} \|v\|^2 . \end{aligned}$$

Therefore $\lambda = \bar{\lambda}$, hence $\lambda \in \mathbb{R}$. □

Proof. ($\Longleftarrow$) (If all the eigenvalues for T are real, then T is self-adjoint.) By the hypothesis, $\lambda = \bar{\lambda}$ for an eigenvalue λ. Let $v \in V$ be an eigenvector so that $Tv = \lambda v$. Then,

$$\begin{aligned} \langle Tv, v \rangle &= \langle \lambda v, v \rangle \\ &= \lambda \langle v, v \rangle \\ &= \left\langle v, \bar{\lambda} v \right\rangle \\ &= \langle v, \lambda v \rangle \qquad \text{by hypothesis} \\ &= \langle v, Tv \rangle . \end{aligned}$$

Therefore $\langle Tv, v \rangle = \langle v, Tv \rangle$ is true for eigenvector $v \in V$. We are not guaranteed anything else under these assumptions.

False by counterexample: Let

$$\mathcal{M}(T) := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{M}(T^*) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

The eigenvalues of T are $\lambda = 1$ with multiplicity 2. However, $\mathcal{M}(T) \neq \mathcal{M}(T^*)$. Hence a counterexample. □

5. If $T \in \mathcal{L}(V)$ is self-adjoint and if $T^2 v = 0$, then $Tv = 0$

Proof. If we have $T^2 v = 0$, then this is equivilent to $T(Tv) = 0$. Taking the inner product of Tv with itself, $\langle Tv, Tv\rangle$, we will show this is zero. By our hypothesis,

$$\begin{aligned}
\langle Tv, Tv\rangle &= \langle v, T^*Tv\rangle \\
&= \left\langle v, T^2 v\right\rangle \\
&= \langle v, 0\rangle \qquad \text{by hypothesis} \\
&= 0.
\end{aligned}$$

Hence $\langle Tv, Tv\rangle = 0$, which implies that $Tv = 0$. □

6. Suppose $T \in \mathcal{L}(V, W)$. Prove that

   (a) T is injective if and only if T^* is surjective.

   Proof. ($\Longleftrightarrow$)
   By definition of injective, $\operatorname{null} T = \{0\}$. Using 7.7 properties,

$$\begin{aligned}
&\operatorname{null} T = \{0\} && \text{(hypothesis)} \\
\Longleftrightarrow\ &(\operatorname{range} T^*)^\perp = \{0\} && \text{(c)} \\
\Longleftrightarrow\ &\operatorname{range} T^* = \{0\}^\perp = W && \text{(perp of both)} \\
\Longleftrightarrow\ &T^* \text{ is surjective.} && \text{(definition of surjective)}
\end{aligned}$$

□

   (b) T is surjective if and only if T^* is injective.

   Proof. ($\Longleftrightarrow$)
   By definition of injective, $\operatorname{null} T^* = \{0\}$. Using 7.7 properties,

$$\begin{aligned}
&\operatorname{null} T^* = \{0\} && \text{(hypothesis)} \\
\Longleftrightarrow\ &(\operatorname{range} T)^\perp = \{0\} && \text{(a)} \\
\Longleftrightarrow\ &(\operatorname{range} T) = \{0\}^\perp = W && \text{(perp of both)} \\
\Longleftrightarrow\ &T \text{ is surjective.} && \text{(definition of surjective)}
\end{aligned}$$

□

7. Suppose $S, T \in \mathcal{L}(V)$ are self-adjoint. Prove that ST is self-adjoint if and only if $ST = TS$.

Proof. ($\Longrightarrow$) (If ST is self-adjoint then $ST = TS$.)

$$\begin{aligned} ST &= (ST)^* && \text{(}ST\text{ self adjoint hypothesis)} \\ &= T^*S^* && \text{(property e)} \\ &= TS. && (T = T^* \text{ and } S = S^* \text{ hypothesis)} \end{aligned}$$

Hence if ST is self adjoint then $ST = TS$. $\square$

Proof. ($\Longleftarrow$) (If $ST = TS$ then ST is self-adjoint.)

$$\begin{aligned} (ST)^* &= T^*S^* && \text{(property e)} \\ &= TS && (T = T^* \text{ and } S = S^* \text{ hypothesis)} \\ &= ST. && (ST = TS \text{ hypothesis)} \end{aligned}$$

Hence if $ST = TS$ then $(ST)^* = ST$. In other words, ST is self-adjoint. $\square$

8. Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that there is a subspace U of V such that $P = P_U$ if and only if P is self-adjoint.

Proof. ($\Longrightarrow$) (If there is a subspace U of V such that $P = P_U$ then P is self-adjoint)

□

Proof. ($\Longleftarrow$) (If P is self-adjoint then there exists a subspace U of V such that $P = P_U$) Since P is self-adjoint under the hypothesis, $V = \text{range}\, P + \text{null}\, P$. By the logic of problem #1, $V = \text{range}\, P \oplus \text{null}\, P$. Let $U := \text{range}\, P$ and $v \in V$. Then for some $u \in U$ and some $w \in U^{\perp}$, $v = u + w$. We have $Pw = 0$ by null space definition. So

$$\begin{aligned} P_U v &= P_U(u + w) \\ &= P_U u + P_U w \\ &= u + 0 \\ &= u \\ &= Pu + 0 \\ &= Pu + Pw \\ &= Pv. \end{aligned}$$

□