MATH 307
Assignment #5
Due Friday, February 18, 2022

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. Let V be a finite-dimensional vector space and let $A, B, C, D \in \mathcal{L}(V)$. Assume that $A+B$ and $A-B$ are invertible. Show that there exist X, Y so that
$$AX + BY = C$$
$$BX + AY = D.$$

2. Show that if $A \in \mathcal{L}(V)$ satisfying $A^2 - A + I = 0$, then A is invertible.

3. Assume V is a finite-dimensional vector space with $S, T, U \in \mathcal{L}(V)$. Show that if $STU = I$ then T is invertible and $T^{-1} = US$

4. Let V be a 2-dimensional vector space and let $A \in \mathcal{L}(V)$ be invertible. Show that there is a polynomial p so that $A^{-1} = p(A)$.

5. Let$V$ and W be finite-dimensional vector spaces. Fix $v \in V$. Define
$$E = \{T \in \mathcal{L}(V, W) : Tv = 0\}.$$
   (a) Show that E is a subspace of $\mathcal{L}(V, W)$.
   (b) Suppose $v \neq 0$. What is $\dim E$?

6. Let $V = \mathbf{R}^{2,2}$ be the vector space of 2×2 matrices with the usual addition and scalar multiplication of matrices. Let $W = \mathcal{P}_3(\mathbf{R})$ be the vector space of polynomials of degree less than or equal to three. Prove that V and W are isomorphic vector spaces.

7. Let V be a real vector space. $V^4 = V \times V \times V \times V$. Prove that V^4 and $\mathcal{L}(\mathbf{R}^4, V)$ are isomorphic vector spaces.