MATH 307
Assignment #11
Due Friday, April 15^{th}, 2022

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. a.) Show that $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is positive.

Proof. Let v be an arbitrary vector defined as $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Computing $\langle Av, v\rangle$, we obtain

$$\begin{aligned}
\langle Av, v\rangle &= \left\langle \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} x+y+z \\ x+y+z \\ x+y+z \end{pmatrix}, (x, y, z) \right\rangle \\
&= x^2 + y^2 + z^2 + 2xy + 2xz + 2yz \\
&= (x+y+z)^2 \\
&\geq 0
\end{aligned}$$

Therefore A is positive. □

b.) Find all α such that $A = \begin{pmatrix} \alpha & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is positive.

Solution: Let v be an arbitrary vector defined as $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Computing $\langle Av, v\rangle$, we obtain

$$\begin{aligned}
\langle Av, v\rangle &= \left\langle \begin{bmatrix} \alpha & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} \alpha x + y + z \\ x \\ x \end{pmatrix}, (x, y, z) \right\rangle \\
&= \alpha^2 + 2xy + 2xz
\end{aligned}$$

Fix $x = 1$ and $z = 0$, then $\langle Av, v\rangle = \alpha + 2y$. We will show that it is always possibly to make this negative. Hence,

$$\langle Av, v\rangle = \alpha + 2y < 0 \iff 2y < -\alpha \iff y < -\frac{\alpha}{2}.$$

Therefore we choose $y = -\frac{\alpha}{2} + \varepsilon$ for some $\varepsilon > 0$. Choosing $\varepsilon = 1$, then $v = \left(1,\ -\frac{\alpha}{2} - 1, 0\right)$. Then

$$\begin{aligned} \langle Av, v\rangle &= \alpha(1)^2 + 2(1)\left(-\frac{\alpha}{2} - 1\right) + 2(1)(0) \\ &= \alpha - \frac{2\alpha}{2} - 2 \\ &= -2 \\ &< 0. \end{aligned}$$

Therefore, for any given α, we can choose a v such that $\langle Av, v\rangle < 0$ for that v. Hence, for every α, A is not a positive matrix. In other words, there exists no α such that A is positive.

c.) Show that even though all its entries are positive, the matrix $A = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$ is not positive.

Solution: Let $v := \begin{pmatrix} 0.1 \\ -0.1 \end{pmatrix}$. Then

$$\begin{aligned} \langle Av, v\rangle &= \left\langle \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} 0.1 \\ -0.1 \end{pmatrix}, \begin{pmatrix} 0.1 \\ -0.1 \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}, \begin{pmatrix} 0.1 \\ -0.1 \end{pmatrix} \right\rangle \\ &= 0 \cdot 0.1 + 0.1 \cdot (-0.1) \\ &= -0.01. \end{aligned}$$

Therefore A is not positive by counterexample.

d.) Find an example of a positive matrix some of whose entries are negative.

Solution: Let $A := \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $v = \begin{pmatrix} x \\ y \end{pmatrix}$ for some x and y. Then

$$\begin{aligned} \langle Av, v\rangle &= \left\langle \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} x - y \\ -x + y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle \\ &= x^2 - 2xy + y^2 \\ &= (x - y)^2 \\ &\geq 0 \end{aligned}$$

Since $\langle Av, v\rangle \geq 0$ for all $v = \begin{pmatrix} x \\ y \end{pmatrix}$ then A is positive, and it has a negative entry.

2. If T is a positive and invertible operator, is T^{-1} positive?

Proof. By the hypothesis that T is positive, T is also self-adjoint. Hence, by the Spectral Theorem there exists a diagonal matrix of eigenvalues. By the properties of positive operators, all eigenvalues are nonnegative. Since T is invertible by the hypothesis, the eigenvalues cannot be zero, and hence are positive. Therefore

$$\mathcal{M}(T) = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \qquad \text{and} \qquad \mathcal{M}(T^{-1}) = \begin{bmatrix} \frac{1}{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\lambda_n} \end{bmatrix}.$$

The eigenvalues of T^{-1} are again non-negative (strictly positive) and T^{-1} is clearly self adjoint. Therefore by property $b \implies a$ of positive operators, T^{-1} is positive. This also holds for complex values since the eigenvalues are nonnegative by definition of T positive. □

3. Consider the three statements:

   (a) T is self-adjoint

   (b) T is an isometry

   (c) $T^2 = I$ (such a T is called an *involution*)

   Prove that if an operator has any two of the properties, then it has the third one as well.

Proof. (a ∧ b $\implies$ c)
Suppose T is a self-adjoint isometry. Then $T^*T = I$ by properties of an isometry. Then using self-adjoint, $I = T^*T = TT = T^2$. □

Proof. (a ∧ c $\implies$ b)
Suppose T is self-adjoint and $T^2 = I$. Then $I = T^2 = TT = T^*T$. Therefore it has been shown since $T^*T = I$ is an equivalent condition of an isometry. □

Proof. (b ∧ c $\implies$ a)
Suppose T is an isometry and $T^2 = I$. Then $T^*T = I$ by properties of an isometry. And since $I = T^2$ by hypothesis, then $T^*T = T^2 = TT$. Multiplying by T on the right, we have

$$T^*T = TT \iff T^*TT = TTT \iff T^*T^2 = TT^2 \iff T^*I = TI \iff T^* = T.$$

Therefore T is self-adjoint. □

4. Prove or give a counterexample: If $T \in \mathcal{L}(V)$ and there exists an orthonormal basis $e_1, \ldots, e_n$ of V such that $\|Te_i\| = 1$ for each e_i, then T is an isometry.

Solution: Let $T = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Then $\|Te_1\| = \left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = 1$ and $\|Te_2\| = \left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = 1$. As such, the hypothesis are fulfilled but $T^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \neq I_2$. Hence T is not an isometry and the assumption is false by counterexample.

5. Suppose $T \in \mathcal{L}(V)$. Prove that there exists an isometry $S \in \mathcal{L}(V)$ such that

$$T = \sqrt{TT^*}\, S.$$

Proof.
By Polar Decomposition, there exists an isometry $S_1 \in \mathcal{L}(V)$ such that $T^* = S_1\sqrt{(T^*)^*T^*} = S_1\sqrt{TT^*}$. Taking the adjoint of both sides,

$$\begin{aligned} & T^* = S_1\sqrt{TT^*} \\ \iff & (T^*)^* = \left(S_1\sqrt{TT^*}\right)^* && \text{(adjoint both sides)} \\ \iff & T = \left(\sqrt{TT^*}\right)^* {S_1}^* && \text{(distribution of adjoint)} \end{aligned}$$

Because T^*T is a positive operator for any $T \in \mathcal{L}(V)$, then by property (b) of positive operators, T^*T is self-adjoint. Similarly, the square root of a positive operator is self-adjoint so $\sqrt{T^*T}$ is self-adjoint. Therefore, $T = \left(\sqrt{TT^*}\right)^* {S_1}^* = \sqrt{TT^*}{S_1}^*$. Since S_1 is an isometry, then ${S_1}^*$ is also an isometry by property (g). Therefore, there exists some isometry S such that $T = \sqrt{TT^*}S$. □

6. Find the singular values of the differentiation operator $D \in \mathcal{L}(\mathcal{P}_2(\mathbf{R}))$ defined by $Dp = p'$, where the inner product is $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\, dx$.
   Remark: It might be helpful to compute the matrix for D with respect to the basis $1, x, x^2$ to find eigenvalues (easy) and then compute the matrix for D again using an *orthonormal basis* for $\mathcal{P}_2(\mathbf{R})$ to compute the singular values. Use some technology for the integrations.

Solution: Using the orthonormal basis of $\mathcal{P}(2)$ from Axler's Example 6.33, then

$$\mathcal{B} = \left(\sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)\right).$$

Applying the operator to this basis, $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{\frac{1}{2}}\right) = 0$ and a change of basis on 0 is 0, hence $D(e_1) = 0$. Next, $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{\frac{3}{2}}x\right) = \sqrt{\frac{3}{2}}$. For a change in basis we have $a\sqrt{\frac{1}{2}} = \sqrt{\frac{3}{2}}$ which implies $a = \sqrt{3}$. Therefore $D(e_2) = \left(\sqrt{3}, 0, 0\right)$. Finally $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)\right) = \sqrt{\frac{45}{2}}x$. Changing this basis, $a\sqrt{\frac{3}{2}}x = \sqrt{\frac{45}{2}}x$ implies that $a = \sqrt{15}$. Thus $D(e_3) = \left(0, \sqrt{15}, 0\right)$. Therefore, the transformation matrix with respect to an orthonormal basis is

$$\mathcal{M}(D) = \begin{bmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{M}(D^*) = \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & \sqrt{15} & 0 \end{bmatrix}.$$

And hence,

$$M(D^*D) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}.$$

By properties of upper triangular matrices, the eigenvalues of D^*D are 0, 3, and 15. Therefore the singular values are $\sqrt{15}, \sqrt{3}, 0$ by proposition 7.52 (nonnegative square roots).

7. Define $T \in \mathcal{L}(\mathbf{F}^3)$ by $T(z_1, z_2, z_3) = (4z_2, 5z_3, z_1)$. Find (explicitly) an isometry $S \in \mathcal{L}(\mathbf{F}^3)$ such that $T = S\sqrt{T^*T}$.

Solution: It is clear that the matrix of T with respect to the standard basis is

$$\mathcal{M}(T) = \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 5 \\ 1 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathcal{M}(T^*) = \begin{bmatrix} 0 & 0 & 1 \\ 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}.$$

Further,

$$T^*T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 25 \end{bmatrix} \qquad \text{and} \qquad \sqrt{T^*T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

By polar decomposition we know that $T = S\sqrt{T^*T}$, so multiplying on the right by $\left(\sqrt{T^*T}\right)^{-1}$ yields $T\left(\sqrt{T^*T}\right)^{-1} = S$. Since T^*T is diagonal, the inverse is the inverse of the diagonal entries, hence

$$\left(\sqrt{T^*T}\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}.$$

Now, computing S explicity, we have

$$\begin{aligned} S &= T\left(\sqrt{T^*T}\right)^{-1} \\ &= \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 5 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \end{aligned}$$

To show that this is an isometry, we need $\|Sv\| = \|v\|$. Which for $v = (v_1, v_2, v_3)$, we have $S(v_1, v_2, v_3) = (v_2, v_3, v_1)$. Clearly $\|(v_1, v_2, v_3)\| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2} = \|(v_2, v_3, v_1)\|$. Hence S is an isometry.

8. Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that the singular values of T equal the absolute values of the eigenvalues of T, repeated appropriately.

Proof.
Since T is a self-adjoint operator, under the Spectral Theorem there exists a diagonal matrix consisting of the eigenvalues for T. Hence

$$\mathcal{M}(T) = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \quad \text{and} \quad \mathcal{M}(T^*) = \begin{bmatrix} \bar{\lambda_1} & & \\ & \ddots & \\ & & \bar{\lambda_n} \end{bmatrix}$$

therefore

$$\mathcal{M}(T^*T) = \begin{bmatrix} \bar{\lambda_1}\lambda_1 & & \\ & \ddots & \\ & & \bar{\lambda_n}\lambda_n \end{bmatrix} = \begin{bmatrix} |\lambda_1|^2 & & \\ & \ddots & \\ & & |\lambda_n|^2 \end{bmatrix}.$$

By proposition 7.52 the singular values of T are the square roots of the eigenvalues of T^*T, which are clearly $|\lambda_i|^2$. So $\sqrt{|\lambda_i|^2} = |\lambda_i|$ are the singular values. Therefore for each eigenvalue λ_i of T, there is a corresponding singular value $|\lambda_i|$ of T. □