MATH 307
Assignment #13
Due Monday, May 2^{nd}, 2022

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. Define $T \in \mathcal{L}(\mathbf{C}^2)$ by $T(w,z) = (0,w)$. Find all generalized eigenvectors of T.

   Solution: Using the standard basis, $\mathcal{M}(T) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. For eigenvalue $\lambda_1 = 0$ we have $(0,1)$ as an eigenvector. Then for $\mathcal{M}(T^2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ with eigenvalue $\lambda_2 = 0$, we have $(1,0)$ as an eigenvector. Then all the generalized eigenvectors of T are $\operatorname{span}(e_1, e_2)$.

2. Define $T \in \mathcal{L}(\mathbf{C}^2)$ by $T(w,z) = (z,-w)$. Find the generalized eigenspaces corresponding to the distinct eigenvalues of T. (Note Example 5.8 is an analogous transformation.)

   Solution: Using the standard basis, $\mathcal{M}(T) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. So since $\lambda^2 + 1 = 0$ we have $\pm i$ as the eigenvalues. For $\lambda = i$, the corresponding eigenvector is $(w,-wi)$. For $\lambda = -i$, the corresponding eigenvector is (w,wi).

   Hence, $G(i,T) \equiv \operatorname{span}\{(1,-i)\}$ and $G(-i,T) \equiv \operatorname{span}\{(1,i)\}$.

3. Suppose $T \in \mathcal{L}(V)$ and $\alpha, \beta \in \mathbf{F}$ with $\alpha \neq \beta$. Prove that $G(\alpha,T) \cap G(\beta,T) = \{0\}$.

   Proof. By theorem 8.13 (Linearly independent generalized eigenvectors), since we have distinct α and β by hypothesis, then $G(\alpha,T)$ and $G(\beta,T)$ are also form linearly independent subspaces. Then the only thing left in common is the zero vector. Hence $G(\alpha,T) \cap G(\beta,T) = \{0\}$. $\square$

4. Suppose that $T \in \mathcal{L}(\mathbf{C}^3)$ is defined by $T(z_1, z_2, z_3) = (z_2, z_3, 0)$. Prove that T has no square root. More precisely, prove that there does not exist $S \in \mathcal{L}(\mathbf{C}^3)$ such that $S^2 = T$.

Proof. (By contradiction)
Suppose there exists a square root S such that $S^2 = T$. Since $S^2 = T$, with the matrix of T with respect to the standard basis being

$$\mathcal{M}(T) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

Then by commutivity $TS = (S^2)S = S(S^2) = ST$. Denote S by some matrix

$$\mathcal{M}(S) := \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}.$$

So

$$TS = \begin{bmatrix} d & e & f \\ g & h & j \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad ST = \begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix}.$$

As such, $d = g = h = 0$ by commutivity. Similarly,

$$a = e = j := x_1 \quad \text{and} \quad b = f := x_2.$$

Substituting back into S,

$$S := \begin{bmatrix} x_1 & x_2 & c \\ 0 & x_1 & x_2 \\ 0 & 0 & x_1 \end{bmatrix} \quad \text{and} \quad S^2 = \begin{bmatrix} {x_1}^2 & 2x_1x_2 & 2cx_1 + {x_2}^2 \\ 0 & {x_1}^2 & 2x_1x_2 \\ 0 & 0 & {x_1}^2 \end{bmatrix} \underbrace{=}_{\text{assumption}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = T.$$

By our assumption, $S^2 = T$, so all entries must match. In particular, ${x_1}^2$ must equal 0. As such, $S^2_{1,2} = 2x_1x_2 = 0 \neq T_{1,2}$. Hence a contradiction. $\square$

5. Suppose that $T \in \mathcal{L}(V)$ is not nilpotent. Let $n = \dim V$. Show that
$V = \operatorname{null} T^{n-1} \oplus \operatorname{range} T^{n-1}$.

Proof. Since T is not nilpotent we know $\operatorname{null} T^n \oplus \operatorname{range} T^n = V$ by theorem. We know that $\operatorname{null} T \neq V$ by the subset chaining. (Since if it equalled V then it would have to be nilpotent (which its not by hypothesis).) We claim that $\operatorname{null} T^{n-1} = \operatorname{null} T^n$ implies that $\dim \operatorname{range} T^{n-1} = \dim \operatorname{range} T^n$.
Suppose that $\operatorname{null} T^{n-1} \neq \operatorname{null} T^n$. Then $\{0\} = \operatorname{null} I \subset \operatorname{null} T \subset \cdots \subset \operatorname{null} T^{n-1} \subset \operatorname{null} T^n$, which would imply that T is nilpotent (since the sequence continues till $\dim V$). Hence a contradiction to the not-nilpotent hypothesis. That is, we know *at least* $\operatorname{null} T^{n-1} = T^n$, and by rank-nullity theorem, $\operatorname{range} T^{n-1} = \operatorname{range} T^n$. Hence we can substitute into the known equation with

$$\underbrace{\operatorname{null} T^n}_{=\operatorname{null} T^{n-1}} \oplus \underbrace{\operatorname{range} T^n}_{=\operatorname{range} T^{n-1}} = V$$

to obtain

$$V = \operatorname{null} T^{n-1} \oplus \operatorname{range} T^{n-1}.$$

□

6. Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible. Prove that T and $S^{-1}TS$ have the same eigenvalues with the same multiplicities.

Proof. Since T and $S^{-1}TS$ are similar matrices by change of bases. Then the characteristic polynomials of similar matrices are the same. Hence the eigenvalues are the same (and the multiplicities also). □

7. Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Prove that V has a basis consisting of eigenvectors of T if and only if every generalized eigenvector of T is an eigenvector of T.

Proof. ($\Longrightarrow$) Since we have a basis of eigenvectors ($\dim V$ of them) and generalized eigenvectors are a superset of an already spanning set, then the set of generalized eigenvectors is the same as the set of eigenvectors. Note that the number of linearly independent generalized eigenvectors cannot exceed $\dim V$. □

Proof. ($\Longleftarrow$) Since by our assumption all of the generalized eigenvectors are equivalent to the eigenvectors. Theorem 8.23 says that we have a basis of generalized eigenvectors (which are assumed to equal the eigenvectors), hence we have a basis of eigenvectors. □

8. Define $N \in \mathcal{L}(\mathbf{F}^5)$ by
$$N(x_1, x_2, x_3, x_4, x_5) = (2x_2, 3x_3, -x_4, 4x_5, 0).$$
Find a square root of $I + N$.

Solution: The matrix with respect to the standard basis for N is

$$\mathcal{M}(N) = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{M}(I+N) = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We are looking for $R^2 = I + N$ for some square root R. Since N is nilpotent, applying Theorem 8.31 we have

$$\begin{array}{l} \quad (I + a_1N + a_2N^2 + \cdots + a_{m-1}N^{m-1}) \\ \times\, (I + a_1N + a_2N^2 + \cdots + a_{m-1}N^{m-1}) \\ \hline \end{array}$$

$$= I + (2a_1N) + \left[2a_2N^2 + a_1^2N^2\right] + \left[2a_3N^3 + 2a_1a_2N^3\right] + \left[2a_4N^4 + 2a_1a_3N^4 + {a_2}^2N^4\right] + \underbrace{0 + \cdots}_{\text{nilpotent}}$$

To make the right hand side of this equal zero, we want $I + (2a_1N) = I + N$, hence $a_1 = \frac{1}{2}$.
Then for the second term we want $(2a_2 + {a_1}^2) = 0 = 2a_2 + \frac{1}{4}$ which implies $a_2 = -\frac{1}{8}$.
For the third term we want $2a_3 + 2a_1a_2 = 0 = 2a_3 - \frac{1}{8}$. Hence $a_3 = \frac{1}{16}$.
Finally for the fourth term, we want $2a_4 + 2a_1a_3 + {a_2}^2 = 0 = 2a_4 + \frac{1}{16} + \frac{1}{64}$ so $a_4 = -\frac{5}{128}$.

Then $\sqrt{I+N} = I + \frac{1}{2}N - \frac{1}{8}N^2 + \frac{1}{16}N^3 - \frac{5}{128}N^4$.

9. Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Prove that there exists $D, N \in \mathcal{L}(V)$ such that $T = D + N$, the operator D is diagonalizable, N is nilpotent, and $DN = ND$.

Proof. By Theorem 8.29 there exists a basis of V where T is a block diagonal matrix of the form

$$T = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{bmatrix} \quad \text{and} \quad A_j = \begin{bmatrix} \lambda_j & & * \\ & \ddots & \\ 0 & & \lambda_j \end{bmatrix}$$

Then we can decompose A_j into

$$D_j = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_j \end{bmatrix} \quad \text{and} \quad N_j = \begin{bmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{bmatrix}$$

Then commutivity clearly holds for D_j and N_j since

$$D_j N_j = \lambda_j I N_j = \lambda_j N_j = N_j(\lambda_j) = N_j(\lambda I) = N_j D_j;$$

Then to form D and N we have

$$D = \begin{bmatrix} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_1 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} \lambda_m & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix} \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \end{bmatrix}.$$

Clearly $T = D + N$. And for every $j \in \{1, \ldots, m\}$ we have $D_j N_j = N_j D_j$, then $DN = ND$. □