Matthew Wilder
MATH 307 - Spring 2022
Assignment #2
Due Friday, 01-28-22, 16:00 CST

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. Label the following statements as being true or false. Provide some justification from the text for your label.

   (a) If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V.

   Solution:

   True, by Definition 1.32 (page 18), $U \subseteq V$ is called a **subspace** of V if U is also a *vector space*.

   (b) The empty set is a subspace of every vector space.

   Solution:

   False, by 1.34 (page 18) *Conditions for a subspace*, a subspace must contain the additive identity, 0, but $0 \notin \{\}$. Therefore, the empty set is not a subspace.

   (c) If V is a vector space other than the zero vector space $\{0\}$, then V contains a subspace W such that $W \neq V$.

   Solution:

   True. $\{0\}$ is a vector space, therefore it is always a subspace of an arbitrary vector space V, since by our assumptions $V \supset \{0\}$ and every vector space must contain 0 by definition. Letting $W \equiv \{0\}$ then $|V| > |W| = 1$ for all $V \neq \{0\}$. The set of order 0, the empty set $\{\}$, is not a subspace by part (b). Thus V cannot have order 1 or 0, and therefore $|V| \geq 2$. The trivial zero vector space is a proper subspace ($W \neq V$) since $|V| \neq |W|$.

(d) The intersection of any two subsets of V is a subspace of V.

Solution:

False. Suppose $V \equiv \mathbb{R}^2$. Then Suppose U and W are subsets of V such that $U \equiv \{\}$ and $W \equiv \mathbb{R}^2$. Then $U \cap W = \{\}$, but by 1b the empty set is not a vector space and thus cannot be a subspace by definition of subspace.

2. Prove that the intersection of two subspaces U and W of a vector space V is a subspace of V.

Proof. We need to show that all 3 conditions for a subspace (theorem 1.34) hold for the subset $U \cap W$ of V.

We will first show that $0 \in U \cap W$. Since $0 \in U$ and $0 \in W$, then by the definition of intersection, $0 \in U \cap W$.

Next we show that $U \cap W$ is closed under addition. Taking two arbitrary vectors, $u, w \in U \cap W$. Then by the definition of intersection, $u, w \in U$ and $u, w \in W$. Since U is a subspace, $u + w \in U$. Similarly, since W is a subspace, $u + w \in W$. Thus $u + w \in U \cap W$ and it is closed under addition.

Last, we show that it closed under scalar multiplication. For some scalar $a \in \mathbb{F}$ and $u \in U \cap V$. Then $u \in U$ and $u \in W$. By definition of U and W being subspaces, $au \in U$ and $au \in W$, therefore $au \in U \cap W$ and $U \cap W$ is closed under scalar multiplication.

All three subspace conditions hold from Theorem 1.34, therefore $U \cap W$ is a subspace of V. $\square$

3. Prove that the union of two subspaces U and W of a vector space V is a subspace of V if and only if one of of the subspaces is contained in the other.

Proof. ($\Longrightarrow$) If $U \cup W$ is a subspace of V, then $(U \subseteq W) \vee (W \subseteq U)$.

Suppose the contrary, that is, if $(U \nsubseteq W) \wedge (W \nsubseteq U)$, then
$(\exists\ u \in U \mid u \notin W) \wedge (\exists\ w \in W \mid w \notin U)$.

If $u + w \in U \cup W \ \Rightarrow\ (u + w \in U) \vee (u + w \in W)$ by union definition

Case: $u + w \in U \ \Rightarrow\ u + w + (-u) = w \in U$ which is a contradiction.
Case: $u + w \in W \ \Rightarrow\ u + w + (-w) = u \in W$ which is a contradiction.

$\therefore\ (U \subseteq W) \vee (W \subseteq U)$

($\Longleftarrow$) If $(U \subseteq W) \vee (W \subseteq U)$ then $U \cup W$ is a subspace of V

Case: $U \subseteq W \ \Rightarrow\ U \cup W = W$, which is by definition a subspace of V
Case: $W \subseteq U \ \Rightarrow\ U \cup W = U$, which is by definition a subspace of V

$\therefore U \cup W$ is a subspace of V $\square$

4. Let V be the vector space of 2×2 matrices with the usual operation of addition and scalar multiplication as seen in MTH 207. (We are *not* considering multiplication of matrices in this exercise.)

   Let W_1 be the set of matrices in V of the form $\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$ and let W_2 be the set of matrices in V of the form $\begin{bmatrix} a & b \\ -a & c \end{bmatrix}$.

   (a) Prove that W_1 and W_2 are subspaces of V.

Solution: We will show each separately. Need to show that W_1 contains the additive identity, closed under addition, and closed under scalar multiplication (Theorem 1.34).

Identity:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W_1$$

Closed under addition: Let $u, v \in W_1$ be defined as

$$u = \begin{bmatrix} x & -x \\ y & z \end{bmatrix} \qquad v = \begin{bmatrix} \alpha & -\alpha \\ \beta & \gamma \end{bmatrix}$$

Then $u + v \in W_1$ since

$$\begin{bmatrix} x & -x \\ y & z \end{bmatrix} + \begin{bmatrix} \alpha & -\alpha \\ \beta & \gamma \end{bmatrix} = \begin{bmatrix} x+\alpha & -x-\alpha \\ y+\beta & z+\gamma \end{bmatrix} = \begin{bmatrix} x+\alpha & -(x+\alpha) \\ y+\beta & z+\gamma \end{bmatrix} \in W_1$$

Closed under scalar multiplication: Let $a \in \mathbb{F}$, $u \in W_1$, then

$$au = a\begin{bmatrix} x & -x \\ y & z \end{bmatrix} = \begin{bmatrix} ax & -ax \\ ay & az \end{bmatrix} = \begin{bmatrix} ax & -(ax) \\ ay & az \end{bmatrix} \in W_1$$

Therefore W_1 is a subspace of V since $W_1 \subseteq V$ and all 3 conditions hold.

We will now show the same for W_2

Identity:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W_2$$

Closed under addition: Let $u, v \in W_2$ be defined as

$$u = \begin{bmatrix} a & b \\ -a & c \end{bmatrix} \qquad v = \begin{bmatrix} x & y \\ -x & z \end{bmatrix}$$

Then $u + v \in W_2$ since

$$\begin{bmatrix} a & b \\ -a & c \end{bmatrix} + \begin{bmatrix} x & y \\ -x & z \end{bmatrix} = \begin{bmatrix} a+x & b+y \\ -a-x & c+z \end{bmatrix} = \begin{bmatrix} a+x & b+y \\ -(a+x) & c+z \end{bmatrix} \in W_2$$

Closed under scalar multiplication: Let $r \in \mathbb{F}$, $u \in W_2$, then

$$ru = r\begin{bmatrix} a & b \\ -a & c \end{bmatrix} = \begin{bmatrix} ra & rb \\ -ra & rc \end{bmatrix} = \begin{bmatrix} ra & rb \\ -(ra) & rc \end{bmatrix} \in W_2$$

Therefore W_2 is a subspace of V since $W_2 \subseteq V$ and all 3 conditions hold.

(b) Describe the subspace $W_1 \cap W_2$.

Solution: The subspace of $W_1 \cap W_2$ would be all $M_{2,2}(\mathbb{F})$ of the form

$$\begin{bmatrix} x & -x \\ -x & y \end{bmatrix}$$

Since W_1 requires that m_{12} be $-m_{11}$, whereas row 2 has no restrictions, and W_2 requires m_{21} be $-m_{11}$ placing a new restriction on m_{21}, and column 2 adds no restrictions. Thus, the restrictions applied to $W_1 \cap W_2$ are that $m_{11} = -m_{12} = -m_{21}$

(c) Show that the subspace $W_1 + W_2$ is all of V.

Solution.

Proof. For $v \in V$, let $v = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$. For $w \in W_1$ and $w_2 \in W_2$. Let $v = w_1 + w_2$, then

$$w_1 = \begin{bmatrix} a & -a \\ b & c \end{bmatrix} \qquad w_2 = \begin{bmatrix} d & e \\ -d & f \end{bmatrix}$$

$$v = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix} = \begin{bmatrix} a+d & -a+e \\ b-d & c+f \end{bmatrix} = w_1 + w_2$$

Therefore,

$$v_1 = a + d \qquad v_2 = -a + e \qquad v_3 = b - d \qquad v_4 = c + f$$

Let $d = f = 0$ then,

$$v_1 = a \qquad v_2 = -a + e \qquad v_3 = b \qquad v_4 = c$$

Solving for a, b, c, d, e, f in terms of v_1, v_2, v_3, v_4, we get

$$a = v_1 \qquad b = v_3 \qquad c = v_4 \qquad d = 0 \qquad e = v_1 + v_2 \qquad f = 0$$

From the equations, $v = w_1 + w_2 \ \forall \ v \in V$, $w_1 \in W_1$, $w_2 \in W_2$. Thus $V \subseteq W_1 + W_2$. Since $w_1, w_2 \in V$ by definition of subset, then $w_1 + w_2 \in V \ \therefore \ W_1 + W_2 \subseteq V$.

Hence, $W_1 + W_2 = V$ $\square$

5. Prove or give a counterexample: if U_1, U_2, W are subspaces of the vector space V such that $V = U_1 \oplus W$ and $V = U_2 \oplus W$, then $U_1 = U_2$.

False by counterexample.

Proof. Let $V = \mathbb{R}^2$, U_1, U_2, W be subspaces such that

$$U_1 = \{(x, 0) : x \in \mathbb{R}\} \qquad U_2 = \{(0, y) : y \in \mathbb{R}\} \qquad W = \{(x, x) : x \in \mathbb{R}\}$$

Then the condition that $V = U_1 \oplus W$ and $V = U_2 \oplus W$ holds. To show that, let $v \in V$, $u_1 \in U_1$, and $w \in W$, then $U_1 \cap W = \{0\}$ and spans V since

$$v = \underbrace{(x - y, 0)}_{u_1 \in U_1} + \underbrace{(y, y)}_{w \in W} = (x, y)$$

Similarly for $U_2 \oplus W = V$, with $v \in V$, $u_2 \in U_2$, and $w \in W$, with $U_2 \cap W = \{0\}$, it spans V since

$$v = \underbrace{(0, y - x)}_{u_2 \in U_2} + \underbrace{(x, x)}_{w \in W} = (x, y)$$

All the assumed conditions have been satisfied, but $U_1 \neq U_2$ since $U_1 \cap U_2 = \{0\}$. Thus the original claim is false. $\square$

6. Let $V = \mathbf{R}^3$ – the usual 3D space from Calc III. Let U be the x-axis. Define W to be the subspace spanned by $(1, 0, 1)$. Show that the usual xz-plane is the direct sum $U \oplus W$.

Solution: Rewriting U and W in set notation we get

$$U = \{(a, 0, 0) : a \in \mathbb{R}\} \qquad W = \{(b, 0, b) : b \in \mathbb{R}\}$$

For some $v \in xz$-plane, it can be written as

$$v = \underbrace{(x - z, 0, 0)}_{u \in U} + \underbrace{(z, 0, z)}_{w \in W} = (x, 0, z) \in xz\text{-plane}$$

To show that it is a direct sum, we need $U \cap W = \{0\}$.

$$U \cap W = \{(a, 0, 0)\} \wedge \{(b, 0, b)\} = \{(a = b, 0, b = 0)\} \Rightarrow a = b = 0 \Rightarrow \{(0, 0, 0)\}$$

7. Suppose that the vectors v_1, v_2, v_3, v_4 span the vector space V. Show that the vectors $v_1 - v_2, v_1 + v_2, v_3 + v_4, v_4$ also span V.

Proof. Let S denote span$((v_1 - v_2), (v_1 + v_2), (v_3 + v_4), v_4)$ We need to show that $\{v_1, v_2, v_3, v_4\} \in S$

$$v_1 = \frac{1}{2}\left[(v_1 - v_2) + (v_1 + v_2)\right] = \frac{1}{2}\left[2v_1\right] = v_1$$

$$\therefore\ v_1 \in S$$

$$v_2 = (v_1 + v_2) + \underbrace{(-v_1)}_{\in S} = v_2$$

$$\therefore\ v_2 \in S$$

$$v_4 \in S \quad \text{without computation}$$

$$v_3 = (v_3 + v_4) + \underbrace{(-v_4)}_{\in S} = v_3$$

$$\therefore\ v_3 \in S$$

Since $\{v_1, v_2, v_3, v_4\} \in S$ and span$(v_1, v_2, v_3, v_4) = V$, then $S \supseteq V$ □