MTH 307 - Spring 2022
Assignment #7
Due: Friday, March 4, 2022 (4pm)

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. Suppose $T \in \mathcal{L}(V)$ and $T^2 = I$ and -1 is not an eigenvalue of T. Prove that $T = I$.

Proof. Since we are given $T^2 = I$, we can subtract I from both sides to obtain $T^2 - I = 0$. Further, we can split this into

$$(T + I)((T - I)v) = 0 \quad \text{and} \quad (T - I)((T + I)v) = 0.$$

If we choose any $v \in V$ where $v \neq \vec{0}$ and $w \in V$ such that $w = (T - I)v \neq 0$, then $(T + I)w = 0$ by the first equation. Simplifying, this would imply that $Tw = -Iw$ and -1 to be an eigenvalue. This contradicts our assumption that -1 is *not* an eigenvalue. Hence, w *must* equal 0 to remove this eigenvalue. Therefore $w = 0 = (T - I)v$ then $(T - I)v = 0$. Simplifying this we obtain $Tv = Iv$ for all $v \neq 0$. If $v = 0$ then $T\vec{0} = I\vec{0}$ is certainly true. Hence $Tv = Iv$ for all $v \in V$. By function equivalency we know $T = I$ when -1 is *not* an eigenvalue. □

2. Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \text{null } P \oplus \text{range } P$.

Proof. Using the given notion that $P^2 = P$ we can simplify as follows:

$$P^2 = P \iff P^2 - P = 0 \iff P(P - I) = 0.$$

Hence for all $v \in V$, $P(P - I)v = 0$. This implies that
$\text{null}\, P \supset \{(P - I)v : v \in V\}$. Thus a typical element $v \in V$ can be rewritten as

$$v = \underbrace{Pv}_{\text{range} P} - \underbrace{(P - I)v}_{\text{null} P}.$$

Hence $V = \text{range}\, P + \text{null}\, P$. We need the intersection $\text{null}\, P \cap \text{range}\, P$, and to do this we need $\text{null}\, P$'s equality. Let $v \in \text{null}\, P$. Then $Pv = 0$ implies that (by the above equation) $v = Pv - (P - I)v = (P - I)(-v)$. Hence

$$\text{null}\, P = \{(P - I)v : v \in V\}.$$

As such, the intersection of $\text{null}\, P$ and $\text{range}\, P$ is when $Pv = (P - I)v = Pv - Iv$. Therefore $0 = -Iv$, implying that $\text{null}\, P \cap \text{range}\, P = \{\vec{0}\}$. We can apply the definition of a Direct Sum to get that $V = \text{null}\, P \oplus \text{range}\, P$. □

3. Suppose $T \in \mathcal{L}(V)$ and v is an eigenvector of T with eigenvalue λ. Suppose $p \in \mathcal{P}(\mathbf{R})$. Prove that $p(T)v = p(\lambda)v$.

Proof. Let p be of the form $p(z) = a_0 + a_1 z + \cdots + a_m z^m$. Then we have

$$\begin{aligned}
p(T)v &= \left(a_0 T^0 + a_1 T^1 + \cdots + a_m T^m\right) v && p(T) \text{ definition} \\
&= a_0 T^0 v + a_1 T^1 v + \cdots + a_m T^m v && \text{linearity of } T \\
&= a_0 \lambda^0 v + a_1 \lambda^1 v + \cdots + a_m \lambda^m v && Tv = \lambda v \\
&= (a_0 + a_1 \lambda + \cdots + a_m \lambda^m) v && \text{linearity} \\
&= p(\lambda) v && \text{definition of } p
\end{aligned}$$

□

4. Suppose W is a complex vector space and $T \in \mathcal{L}(W)$ has no eigenvalues. Prove that every subspace of W invariant under T is either $\{0\}$ or infinite-dimensional.

Proof. Suppose W is non-zero, finite dimentional, then by Theorem 5.21 it has an eigenvalue. Hence $T|_U$ has some eigenvalue λ and consequently T has that eigenvalue λ. This is a contradiction to the assumption that T has no eigenvalues. Therefore W must be $\{0\}$ or infinite dimentional to satisfy T's assumption. □

5. Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Define a function $f : \mathbf{C} \to \mathbf{R}$ by

$$f(\lambda) = \dim \operatorname{range}(T - \lambda I).$$

Prove that f is not a continuous function.

Proof. Let $\dim V = n$.
Because we are in a finite dimensional *complex* vector space we are guaranteed the existance of some eigenvalue λ_0 for T (when T is not the zero transformation). Since $T - \lambda_0 I$ is not surjective, then $f(\lambda_0) = \dim \operatorname{range} T \leq n - 1$ by the rank-nullity theorem (FTLM). If λ_1 is not an eigenvalue, then $T - \lambda_1 I$ implies $T = \lambda_1 I$. Since λ_1 is not an eigenvalue $(T - \lambda_1 I)$ is surjective. Hence $f(\lambda_1) = n$ by rank-nullity. We will assume that the output of dim is a nonnegative integer ($\dim = 0, 1, 2, \ldots$). Since we have $f = n$ and $f \leq n - 1$ for 2 different values, f is discontinuous. (The only way to be continuous is if f is constant.) □

6. Suppose $T \in \mathcal{L}(V)$ has a diagonal matrix A with respect to some basis of V and that $\lambda \in \mathbf{F}$. Prove that λ appears on the diagonal of A precisely $\dim E(\lambda, T)$ times.

Proof. Let $m = \dim V$ and $\{v_1, \ldots, v_m\}$ be a basis for V. Then denote the diagonal entries of A by $\lambda_1, \ldots, \lambda_n$.

A basis for the eigenspace is every $v_i \in V$ such that $(T - \lambda_i I)v_i = \vec{0}$ with $i \in [1, m]$. This is true only when $\lambda_i = \lambda$. We can build a basis for $E(\lambda, T)$ by appending v_i each time $\lambda_i = \lambda$. Since the dimention is the number of elements in a basis, $\dim E(\lambda, T)$ is exactly equal to the number of times $\lambda = \lambda_i$; the number of times λ appeared on the diagonal of A. □

7. Show that the function that takes $((x_1, x_2), (y_1, y_2)) \in \mathbf{R}^2 \times \mathbf{R}^2$ to $|x_1 y_1| + |x_2 y_2|$ is not an inner product on $\mathbf{R}$.

Proof. Let ϕ be defined as follows,

$$\phi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$
$$\phi\big((x_1, x_2), (y_1, y_2)\big) = |x_1 y_1| + |x_2 y_2|$$

Suppose ϕ is an inner product. Then definition of an inner product has homogeneity in the first slot, $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$. Leting $u = v = e_1$ and $\lambda = -1$, then

$$\phi(-e_1, e_1) = |-1 \cdot 1| + |0 \cdot 0| = 1.$$

and

$$-\phi(e_1, e_1) = -\left(|1 \cdot 1| + |0 \cdot 0|\right) = -1.$$

Since $1 \neq -1$ we have a contradiction to the assumption that ϕ is an inner product. Therefore ϕ is not an inner product by counterexample. □

8. Suppose $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leq \|v\|$ for every $v \in V$. Prove that $T - \sqrt{2}I$ is invertible.

Proof. Suppose that $\lambda = \sqrt{2}$ is an eigenvalue. Let $u \in V^*$. Then $Tu = \lambda u = \sqrt{2}u$ and $\|Tu\| = \left\|\sqrt{2}u\right\| = \sqrt{2}\,\|u\|$. This contadicts the assumption that $\|Tv\| \leq \|v\|$, hence $\sqrt{2}$ is not an eigenvalue. Consequestly, $(T - \sqrt{2}I)$ is invertible. □

9. Suppose $\|u\| = \|v\| = 1$ and $\langle u, v\rangle = 1$. Prove that $u = v$.

Proof. By the Cauchy-Schwarz Inequality, because $|\langle u, v\rangle| = \|u\|\,\|v\| = 1$, we get that $u = cv$ for some scalar $c \in \mathbb{F}$. Substituting $u = cv$ into $\langle u, v\rangle$,

$$\begin{aligned} 1 &= \langle u, v\rangle \\ &= \langle cv, v\rangle \\ &= c\,\langle v, v\rangle \\ &= c \cdot 1 \\ &= c. \end{aligned}$$

Therefore $c = 1$ and $u = 1 \cdot v$ implies $u = v$. $\square$