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MATH 307 - Spring 2022
Assignment #3
Due Friday, 02-04-22, 16:00 CST

For each problem, include the statement of the problem. Leave a blank line. At the beginning of the next line, write **Solution** or **Proof** – as appropriate.

1. Suppose that the vectors v_1, v_2, v_3, v_4 are a basis for V. Show that the vectors $v_1 - v_2, v_1 + v_2, v_3 + v_4, v_4$ also form a basis for V.

Solution: Let $S \equiv \text{span}\{(v_1 - v_2), (v_1 + v_2), (v_3 + v_4), v_4\}$ We need to show that $\{v_1, v_2, v_3, v_4\} \in S$

$$\frac{1}{2}\Big[\underbrace{(v_1 - v_2)}_{\in S} + \underbrace{(v_1 + v_2)}_{\in S}\Big] = \frac{1}{2}[2v_1] = v_1$$

Therefore $v_1 \in S$

$$\underbrace{(v_1 + v_2)}_{\in S} + \underbrace{(-v_1)}_{\in S} = v_2$$

Therefore $v_2 \in S$

$$\underbrace{1v_4}_{\in S} = v_4$$

Therefore $v_4 \in S$

$$\underbrace{(v_3 + v_4)}_{\in S} + \underbrace{(-v_4)}_{\in S} = v_3$$

Therefore $v_3 \in S$

Since $v_1, v_2, v_3, v_4 \in S$ and is a basis, then $S \equiv V$. Theorem 2.31 says every spanning list contains a basis, and Theorem 2.35 says that all bases must have the same length. Since the length of the basis $\{v_1, v_2, v_3, v_4\}$ is 4, and the length of spanning list $\{(v_1 - v_2), (v_1 + v_2), (v_3 + v_4), v_4\}$ is 4, that spanning list must also form a basis for V.

2. Suppose that the vectors v_1, v_2, v_3, v_4 is a basis for V. Let U be a subspace of V. Assume $v_1, v_2 \in U$ but neither v_3 nor v_4 are in U. Is v_1, v_2 a basis for U? Justify.

Solution. No. v_1, v_2 will not always form a basis for U

Proof. Let $V \equiv \mathbb{R}^4$ and v_1, v_2, v_3, v_4 be defined as follows,

$$\begin{aligned} v_1 &= (1,0,0,0) \\ v_2 &= (0,1,0,0) \\ v_3 &= (0,0,1,0) \\ v_4 &= (0,0,0,1) \end{aligned}$$

This is the trivial basis, since for some $(w, x, y, z) \in \mathbb{R}^4$

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = a \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{v_1} + b \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{v_2} + c \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{v_3} + d \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{v_4}$$

$$a = w \qquad b = x \qquad c = y \qquad d = z$$

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = w \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{v_1} + x \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{v_2} + y \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{v_3} + z \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{v_4}$$

Then we let $U \equiv \{(u_1, u_2, u_3, u_4) \mid u_3 = u_4\}$. Thus, $v_1 \in U$ and $v_2 \in U$ holds since $0 = 0$. But $v_3 \notin U$ since $1 \neq 0$ and similarly $v_4 \notin U$ since $0 \neq 1$. v_1 and v_2 do not span U since, for example, no such $av_1 + bv_2 = (1,1,1,1) \in U$, which violates the definition of a basis being a *list of vectors in V that is linearly independent and <u>spans</u> V*. Thus v_1, v_2 is **not** a basis for U. □

3. Let $v_1 = (-1, 1, 2) \in \mathbf{R}^3$. Construct two bases for $\mathbf{R}^3$: $\{v_1, v_2, v_3\}$ and $\{v_1, v_2', v_3'\}$ so that $\{v_2, v_3, v_3'\}$ is also a basis.

Solution: Let B_1 and B_2 be bases of $\mathbb{R}^3$ such that

$$B_1 = \{v_1, v_2, v_3\} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$B_2 = \{v_1, v_2', v_3'\} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Then, from the problem statement, let B_3 be a third basis for $\mathbb{R}^3$ such that

$$B_3 = \{v_2, v_3, v_3'\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Then, $\operatorname{span} B_1 = \mathbb{R}^3$ since

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a \underbrace{\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}}_{v_1} + b \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{v_2} + c \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{v_3} \in \operatorname{span} B_1$$

Which implies that

$$\begin{aligned} x &= -a + b \\ y &= a + c \\ z &= 2a \end{aligned}$$

Solving for $a, b,$ and c in terms of $x, y,$ and z. We get

$$\begin{aligned} a &= \frac{z}{2} \\ b &= x + \frac{z}{2} \\ c &= y - \frac{z}{2} \end{aligned}$$

Therefore, substituting back, we get

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{z}{2} \underbrace{\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}}_{v_1} + \frac{2x + z}{2} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{v_2} + \frac{2y - z}{2} \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{v_3} \in \operatorname{span} B_1$$

Since $\operatorname{span} B_1 = \mathbb{R}^3$, the length of B_1 is 3, and $\dim \mathbb{R}^3 = 3$, then by Theorem 2.42, B_1 is a basis of $\mathbb{R}^3$ since $3 = 3$.

Next, $\operatorname{span} B_2 = \mathbb{R}^3$ since

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a \underbrace{\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}}_{v_1} + b \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{v_2'} + c \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{v_3'} \in \operatorname{span} B_2$$

Which implies that

$$\begin{aligned} x &= -a + b \\ y &= a \\ z &= 2a + c \end{aligned}$$

Solving for a, b, and c in terms of x, y, and z. We get

$$\begin{aligned} a &= y \\ b &= x + y \\ c &= z - 2y \end{aligned}$$

Therefore,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \underbrace{\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}}_{v_1} + (x + y) \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{v_2'} + (z - 2y) \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{v_3'} \in \operatorname{span} B_2$$

Because $\operatorname{span} B_2 = \mathbb{R}^3$, and the length of B_2 equals $\dim \mathbb{R}^3$, by Theorem 2.42, B_2 is a basis of $\mathbb{R}^3$

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Lastly, $\operatorname{span} B_3 = \mathbb{R}^3$ since

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{v_2} + b \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{v_3} + c \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{v_3'} \in \operatorname{span} B_3$$

Which implies that

$$\begin{aligned} x &= a \\ y &= b \\ z &= c \end{aligned}$$

Substituting in for a, b and c we get,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{v_2} + y \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{v_3} + z \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{v_3'} \in \operatorname{span} B_3$$

Because $\operatorname{span} B_3 = \mathbb{R}^3$, and the length of B_3 equals $\dim \mathbb{R}^3$, by Theorem 2.42, B_3 is a basis of $\mathbb{R}^3$

4. (a) Under what conditions on the scalar ξ do the vectors $(1,1,1)$ and $(1,\xi,\xi^2)$ form a basis for $\mathbf{R}^3$?

Solution: Under no condition does ξ form a basis for the two vectors. If we assumed that we choose a ξ such that $\{(1,1,1),(1,\xi,\xi^2)\}$ is linearly independent (a requirement for a basis), then by Theorem 2.39 the length of the list must equal the dimension of V. In this instance,

$$\dim \mathbb{R}^3 = 3 \neq 2 = \left|\{(1,1,1),(1,\xi,\xi^2)\}\right|$$

Hence, no ξ will form a basis of $\mathbb{R}^3$ since the length would be less than the dimension of $\mathbb{R}^3$ anyways.

(b) Under what conditions on the scalar ξ do the vectors $(0,1,\xi)$, $(\xi,0,1)$, and $(\xi,1,1+\xi)$ form a basis for $\mathbf{R}^3$?

Solution: Let $v_1 = (0,1,\xi)$, $v_2 = (\xi,0,1)$, and $v_3 = (\xi,1,1+\xi)$. Under no condition does ξ form a basis because for every $\xi \in \mathbb{R}$, v_3 is a linear combination of $v_1 + v_2$.

$$\underbrace{(0,1,\xi)}_{v_1} + \underbrace{(\xi,0,1)}_{v_2} = \underbrace{(\xi,1,1+\xi)}_{v_3}$$

Hence, the set of vectors is linearly **dependent**, and by definition of a basis, cannot form a basis.

5. Let $V = \mathcal{P}(\mathbf{R})$ be the vector space of all polynomials with real coefficients. If p is any polynomial, let Tp be the polynomial defined by $(Tp)(x) = p(x+1) - p(x)$. Show that T is a linear transformation.

Solution:
First we will show additivity,

$$\begin{aligned} T\big(f+g\big)(x) &= \big(f+g\big)(x+1) - \big(f+g\big)(x) \\ &= f(x+1) + g(x+1) - \Big(f(x) + g(x)\Big) \\ &= f(x+1) + g(x+1) - f(x) - g(x) \\ &= f(x+1) - f(x) + g(x+1) - g(x) \end{aligned}$$

$$Tf(x) + Tg(x) = f(x+1) - f(x) + g(x+1) - g(x)$$

Since $T\big(f+g\big)(x) = Tf(x) + Tg(x)$, addition is preserved.

Now we will show homogeneity,

$$\begin{aligned} T\big(\lambda f\big)(x) &= \big(\lambda f\big)(x+1) - \big(\lambda f\big)(x) \\ &= \lambda f(x+1) - \lambda f(x) \end{aligned}$$

$$\begin{aligned} \lambda\left(Tf(x)\right) &= \lambda\left(f(x+1) - f(x)\right) \\ &= \lambda f(x+1) - \lambda f(x) \end{aligned}$$

Since $T\big(\lambda f\big)(x) = \lambda Tf(x)$, homogeneity is preserved.

These two conditions together satisfy the definition of a linear map. Therefore T is a linear transformation.

6. Let $V = \mathcal{P}_4(\mathbf{R})$, the vector space of polynomials of degree at most four.
   Let $U = \{p \in V : p(1) = p(3)\}$

   (a) Find a basis of U.

Solution: Since $p(1) = p(3)$, we can start by finding $p(x) : p(1) = p(3) = 0$ and then adding the constant function $p(x) = c$ to remove the zero constraint and return to U. Let $S \subset U$ such that $S \equiv \{p(x) \in V : p(1) = p(3) = 0\}$

$$(x-1)(x-3) \in S \in U$$

$$x(x-1)(x-3) \in S \in U$$

$$x^2(x-1)(x-3) \in S \in U$$

$$1 \in U$$

We can show that the list $\{1,\ (x-1)(x-3),\ x(x-1)(x-3),\ x^2(x-1)(x-3)\}$ is linearly independent since

$$a + b(x-1)(x-3) + cx(x-1)(x-3) + dx^2(x-1)(x-3) = 0$$

Since there is no degree 4 term on the RHS, $d = 0$, similarly since there are no degree 3 terms on the RHS, $c = 0$, no degree 2 terms on the RHS implies $b = 0$, and thus we are left with $a = 0$. So the only wait to obtain the zero polynomial is trivially, thus the list is linearly independent. This means that $\dim U \geq 4$. But since $U \subseteq P$, $\dim U \leq \dim V = 5$ (by 2.38). Suppose that $\dim U = 5$, then $x \in U$, which is a contradiction. Therefore $\dim U \leq 4$. But by the linear independence of the former list of elements, $\dim U \geq 4$. Hence $4 \leq \dim U \leq 4$, and thus $\dim U = 4$. Then by Theorem 2.39 $\{1,\ (x-1)(x-3),\ x(x-1)(x-3),\ x^2(x-1)(x-3)\}$ is a basis for U since it is linearly independent and its length equals $\dim U$.

   (b) Extend the basis in (a) to a basis of V.

Solution: Using the explanation from part (a) that said $x \notin U$. Thus we can add x to the list and maintain linear independence.
Let $L \equiv \{1,\ x,\ (x-1)(x-3),\ x(x-1)(x-3),\ x^2(x-1)(x-3)\}$ Then $|L| = 5$, and $\dim V = 5$. Since L is linearly independent, and $|L| = \dim V$, then L is a basis by Theorem 2.39.

(c) Find a subspace W of V so that $V = U \oplus W$.

Let $W = \operatorname{span}(x)$, then by Theorem 2.43,

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$$

By part (a), $\dim U = 4$. We can easily see that $\dim W = 1$. $U \cap W = \{0\}$ and $\dim(0) = 0$. Therefore, substituting back in we see that

$$\dim(U + W) = 4 + 1 - 0 = 5$$

But because $U \cap W = \{0\}$, $U \oplus W = U + W$ (by 1.45).
Hence, $\dim(U \oplus W) = \dim(U + W) = 5$. And we know that $\dim V = 5$. Therefore,

$$\dim(U \oplus W) = \dim V$$

And since $U \oplus W \subseteq V$,

$$U \oplus W = V$$

7. Suppose $a, b, c \in \mathbf{R}$. Define $T : \mathcal{P}(\mathbf{R}) \to \mathbf{R}^3$ by

$$Tp = \left(2p(5) - 5p'(1) + ap(1)p(3)\ ,\ \int_1^4 x^2 p(x)\, dx + be^{p(0)}\ ,\ p(2) + c\right).$$

Show that T is linear if and only if $a = b = c = 0$.

Proof. ($\Longrightarrow$) If T is linear, then $a = b = c = 0$.

$$\begin{aligned}
T(p+q) &= \begin{pmatrix} 2(p+q)(5) - 5(p+q)'(1) + a(p+q)(1) \cdot (p+q)(3) \\ \int_1^4 x^2 (p+q)(x)\, dx + be^{(p+q)(0)} \\ (p+q)(2) + c \end{pmatrix} \\
&= \begin{pmatrix} 2p(5) + 2q(5) - 5p'(1) - 5q'(1) + a\left[\big(p(1) + q(1)\big)\big(p(3) + q(3)\big)\right] \\ \int_1^4 x^2 p(x) + x^2 q(x)\, dx + be^{p(0)+q(0)} \\ p(2) + q(2) + c \end{pmatrix} \\
&= \begin{pmatrix} 2p(5) + 2q(5) - 5p'(1) - 5q'(1) + a\left[\big(p(1) + q(1)\big)\big(p(3) + q(3)\big)\right] \\ \int_1^4 x^2 p(x)\, dx + \int_1^4 x^2 q(x)\, dx + be^{p(0)+q(0)} \\ p(2) + q(2) + c \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
Tp + Tq &= \begin{pmatrix} 2p(5) - 5p'(1) + ap(1)p(3) \\ \int_1^4 x^2 p(x)\, dx + be^{p(0)} \\ p(2) + c \end{pmatrix} + \begin{pmatrix} 2q(5) - 5q'(1) + aq(1)q(3) \\ \int_1^4 x^2 q(x)\, dx + be^{q(0)} \\ q(2) + c \end{pmatrix} \\
&= \begin{pmatrix} 2p(5) + 2q(5) - 5p'(1) - 5q'(1) + ap(1)p(3) + aq(1)q(3) \\ \int_1^4 x^2 p(x)\, dx + \int_1^4 x^2 q(x)\, dx + be^{p(0)} + be^{q(0)} \\ p(2) + c + q(2) + c \end{pmatrix} \\
&= \begin{pmatrix} 2p(5) + 2q(5) - 5p'(1) - 5q'(1) + ap(1)p(3) + aq(1)q(3) \\ \int_1^4 x^2 p(x)\, dx + \int_1^4 x^2 q(x)\, dx + be^{p(0)} + be^{q(0)} \\ p(2) + c + q(2) + c \end{pmatrix} \\
&= \begin{pmatrix} 2p(5) + 2q(5) - 5p'(1) - 5q'(1) + ap(1)p(3) + aq(1)q(3) \\ \int_1^4 x^2 p(x)\, dx + \int_1^4 x^2 q(x)\, dx + 2be^{p(0)} \\ p(2) + q(2) + 2c \end{pmatrix}
\end{aligned}$$

In order for additivity to hold, each component must be equal. For the 1st component,

$$\begin{aligned}
&2p(5) + 2q(5) - 5p'(1) - 5q'(1) + a\left[\big(p(1) + q(1)\big)\big(p(3) + q(3)\big)\right] \\
&= 2p(5) + 2q(5) - 5p'(1) - 5q'(1) + ap(1)p(3) + aq(1)q(3)
\end{aligned}$$

Simplifying, we get

$$a\left[\big(p(1) + q(1)\big)\big(p(3) + q(3)\big)\right] = ap(1)p(3) + aq(1)q(3)$$

Distributing the left hand side we get

$$ap(1)p(3) + aq(1)q(3) + ap(1)q(3) + aq(1)p(3) = ap(1)p(3) + aq(1)q(3)$$

Simlifying further we reach,

$$ap(1)q(3) + aq(1)p(3) = 0$$

Thus, for the first component to hold linearity under addition, a must be 0.

For the second component, linearity implies that

$$\int_1^4 x^2 p(x)\,dx + \int_1^4 x^2 q(x)\,dx + be^{p(0)+q(0)} = \int_1^4 x^2 p(x)\,dx + \int_1^4 x^2 q(x)\,dx + 2be^{p(0)}$$

Simplifying out the integrals leaves us with

$$be^{p(0)+q(0)} = 2be^{p(0)+q(0)}$$

Which implies $b = 0$ for linearity.

For the third component, linearity implies that

$$p(2) + q(2) + c = p(2) + q(2) + 2c$$

Simplifying for

$$c = 2c$$

Implies that $c = 0$ to preserve linearity.

Therefore, in order for addition to be linear, $a = b = c = 0$. It suffices to show just addition since it restricts all 3 variables to one value. □

Proof. ($\Longleftarrow$) If $a = b = c = 0$ then T is linear. That means that

$$Tp = \left(2p(5) - 5p'(1)\,,\; \int_1^4 x^2 p(x)\,dx\,,\; p(2)\right).$$

It has been shown in the forward direction that this holds for additivity when $a = b = c = 0$. Thus, we only need to show the remaining property of homogeneity.

$$\begin{aligned} T(\lambda p) &= \left(2\lambda p(5) - 5\lambda p'(1)\,,\; \int_1^4 x^2 \lambda p(x)\,dx\,,\; \lambda p(2)\right) \\ &= \left(2\lambda p(5) - 5\lambda p'(1)\,,\; \lambda \int_1^4 x^2 p(x)\,dx\,,\; \lambda p(2)\right) \end{aligned}$$

$$\begin{aligned} \lambda\,(Tp) &= \lambda \left(2p(5) - 5p'(1)\,,\; \int_1^4 x^2 p(x)\,dx\,,\; p(2)\right) \\ &= \left(2\lambda p(5) - 5\lambda p'(1)\,,\; \lambda \int_1^4 x^2 p(x)\,dx\,,\; \lambda p(2)\right) \end{aligned}$$

Hence, $T(\lambda p) = \lambda(Tp)$ and therefore T is linear by definition. □

It has been shown that T is linear if and only if $a = b = c = 0$

8. (a) Find an example of a function $\varphi : \mathbf{R}^2 \to \mathbf{R}$ that is homogeneous but not additive (and hence not linear).

Solution: Let $\phi : \mathbb{R}^2 \to R$ by defined as,

$$\phi(x, y) = \sqrt{x^3 + y^3}$$

Then it is homogeneous since

$$\begin{aligned}
\phi(\lambda(x, y)) &= \phi(\lambda x, \lambda y) \\
&= \sqrt[3]{(\lambda x)^3 + (\lambda y)^3} \\
&= \sqrt[3]{\lambda^3 x^3 + \lambda^3 y^3} \\
&= \sqrt[3]{\lambda^3 (x^3 + y^3)} \\
&= \lambda \sqrt[3]{x^3 + y^3} \\
&= \lambda \phi(x, y)
\end{aligned}$$

However, it is not additive because if we take $v_1 = (x_1, y_1) = (1, 0)$ and $v_2 = (x_2, y_2) = (0, 1)$ then

$$\begin{aligned}
\phi(v_1 + v_2) &= \phi((1, 0) + (0, 1)) \\
&= \phi(1, 1) \\
&= \sqrt[3]{1^3 + 1^3} \\
&= \sqrt[3]{2}
\end{aligned}$$

And

$$\begin{aligned}
\phi(v_1) + \phi(v_2) &= \phi(1, 0) + \phi(0, 1) \\
&= \sqrt[3]{1^3 + 0^3} + \sqrt[3]{0^3 + 1^3} \\
&= \sqrt[3]{1} + \sqrt[3]{1} \\
&= 2
\end{aligned}$$

Therefore $\phi(v_1 + v_2) \neq \phi(v_1) + \phi(v_2)$. Hence ϕ is not additive.

(b) Find an example of a function $\varphi : \mathbf{C}^2 \to \mathbf{C}$ that is additive but not homogeneous (and hence not linear).

Solution:

$$\phi : \mathbb{C}^2 \to C$$

$$(a + bi\,,\; c + di) \to ai + b$$

This is additive because if we
Let $x = (a_1 + b_1 i\,,\; c_1 + d_1 i)$ and
Let $y = (a_2 + b_2 i\,,\; c_2 + d_2 i)$

$$\begin{aligned}
\phi(x + y) &= \phi((a_1 + a_2) + (b_1 + b_2)i\,,\; (c_1 + c_2) + (d_1 + d_2)i) \\
&= (b_1 + b_2) + (a_1 + a_2)i \\
&= b_1 + a_1 i + b_2 + a_2 i \\
&= \phi(x) + \phi(y)
\end{aligned}$$

But it is not homogeneous because if we let $\lambda = i \in \mathbb{C}$, $a = 1$, $b = c = d = 0$, then

$$\phi(\lambda(1, 0)) = \phi(i(1,\, 0)) = \phi(i,\, 0) = 1$$

But

$$\lambda(\phi(1, 0)) = i(\phi(1, 0)) = i(i) = -1$$

Therefore $\lambda(\phi(v)) \neq \phi(\lambda v)$, hence ϕ is not homogeneous.