Math 325 – Homework 02
Due (via upload to Canvas) Monday, September 13, 2021 at 6 PM

1. Let $P(A) = 0.8$, $P(B) = 0.5$, and $P(A \cup B) = 0.9$

   (a) Are A and B are independent events? Explain your answer

Solution: Independent events means that $P(A \cap B) = P(A)P(B)$. To find the former, we us $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. As we have $0.9 = 0.8 + 0.5 - P(A \cap B)$, we see $P(A \cap B) = 0.4$. On the other hand, $P(A)P(B) = 0.8(0.5) = 0.4$. Hence, the events are independent.

   (b) Are A and B mutually exclusive events? Explain your answer.

Solution: Mutually exclusive implies that $A \cap B = \emptyset$. That $P(A \cap B)$ would be zero, which it is not. They are not mutually exclusive.

2. Three newspapers, A, B, and C, are published in a city and a recent survey of readers indicates the following: 20 percent read A, 16 percent read B, 14 percent read C, 8 percent read A and B, 5 percent read A and C, 4 percent read B and C, and 2 percent read A, B, and C. For one adult chosen at random, compute the probability that

   (a) he reads none of the papers

Solution: Need $1 - P(A \cup B \cup C)$. Using the formula from Homework 1,

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \\ &= 0.2 + 0.16 + 0.14 - 0.08 - 0.05 - 0.04 + 0.02 \\ &= 0.35 \end{aligned}$$

   (b) he reads exactly one of the papers

Solution: Recall that the event "A only" is written $A \cap \overline{B} \cap \overline{C} = A - (A \cap B) - (A \cap C) + (A \cap B \cap C)$. Then

$$\begin{aligned} P(A \text{ only}) &= P(A) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C) \\ &= 0.20 - 0.08 - 0.05 + 0.02 \\ &= 0.09. \end{aligned}$$

Similarly, $P(B \text{ only}) = 0.06$ and $P(C \text{ only}) = 0.07$ Then,

$$P(\text{one paper only}) = P(A \text{ only}) + P(B \text{ only}) + P(C \text{ only}) = 0.22.$$

   (c) he reads at least A and B if it is know that he reads at least one of the papers published.

Solution: This is a conditional probability. Here,

$$P(A \cap B \mid A \cup B \cup C) = \frac{P(A \cap B)}{A \cup B \cup C} = \frac{.08}{0.35} = \frac{8}{35} \approx 0.229.$$

3. Consider a set of 10 urns, nine of which each contains 3 white chips and 3 red chips, while the tenth contains 5 white chips and 1 red chip. An urn is picked at random. Then a sample of size 3 is drawn without replacement from that urn. If all three chips drawn are white, what is the probability that the urn being sampled is the one with 5 white chips.

Solution: Note that there are 9 "50%-50%" urns and one "other" urn. This question is asking the conditional probability, if we chose 3 white (W) chips, what is the likelihood we started with by selecting the "other" urn; notation $P(\text{other urn}|3W)$. Note that the reverse condition is a much easier computation,

$$P(3W|\text{ other urn}) = \frac{\binom{5}{3}}{\binom{6}{3}} = \frac{1}{2}.$$

Flipping the conditionals to compute a probability is the exact superpower that Bayes' Rule possesses. Here, we compute

$$P(\text{other urn}|3W) = \frac{P(3W|\text{ other urn})P(\text{other urn})}{P(3W|\text{ 50/50 urn})P(\text{50/50 urn}) + P(3W|\text{ other urn})P(\text{other urn})}.$$

Since $P(\text{50/50 urn}) = 9/10$, $P(\text{other urn}) = 1/10$, and $P(3W|\text{50/50 urn}) = \frac{\binom{3}{3}}{\binom{6}{3}} = \frac{1}{20}$, we get

$$P(\text{other urn}|3W) = \frac{(0.1)(0.5)}{(0.05)(0.9) + (0.1)(0.5)} = \frac{10}{19} \approx 0.526.$$

4. Let X be the outcome from rolling a biased four-sided die: $P(1) = 0.3$, $P(2) = 0.3$, $P(3) = 0.1$, and $P(4) = 0.3$. Compute the mean, variance, and standard distribution of X.

Solution:

$$\begin{aligned}
\mu &= \sum_X xp(x) \\
&= 1(0.3) + 2(0.3) + 3(0.1) + 4(0.3) \\
&= 2.4 \\
\sigma^2 &= E[X^2] - \mu^2 \text{ via theorem} \\
&= \sum_X x^2 p(x) - \mu^2 \\
&= 1^2(0.3) + 2^2(0.3) + 3^2(0.1) + 4^2(0.3) - (2.4)^2 \\
&= 1.44 \\
\sigma &= 1.2
\end{aligned}$$

5. Given $E(X+4) = 10$ and $E[(X+4)^2] = 116$, determine μ, σ^2, and $\mathrm{Var}(X-4)$. (Hint: use the linearity of the expectation operator.)

Solution: By linearity, $E(X+4) = E(X) + 4$. Hence $\mu = E(X) = 6$.

As above, $\sigma^2 = E(X^2) - \mu^2$. Note $E[(X+4)^2] = E(X^2+8X+16) = E(X^2)+8E(X)+16$. Using μ, we get $E(X^2) = 52$. Thus $\sigma^2 = 52 - 6^2 = 16$.

Lastly, $\mathrm{Var}(X-4) = E[(X-4)^2] = E[X^2] - 8\mu + 16 = 20$.