Math 325 – Homework 10
Due (via upload to Canvas) Monday, December 06, 2021 at 6 PM

1. Let $Y_1, Y_2, \ldots, Y_n$ be independent, uniformly distributed variable on the interval $[0, \theta]$. Find the probability distribution function of $Y_{(n)} = \max\{Y_1, Y_2, \ldots, Y_n\}$

Solution: For Y_i uniformly distributed, $f(y_i) = \begin{cases} \frac{1}{\theta}, & 0 \leq y_i \leq \theta \\ 0, & \text{else} \end{cases}$. The associated CDF for each Y_i is

$$F(y_i) = \begin{cases} 0, & y_i < 0 \\ y_i/\theta, & 0 \leq y_i \leq \theta \\ 1, & y_i > \theta \end{cases} .$$

Using the derivations from class, the max-order statistic is defined by

$$g_{(n)}(y) = n[F(y)]^{n-1} f(y) = \frac{ny^{n-1}}{\theta^n},$$

on the support $0 \leq y \leq \theta$.

2. Suppose 10 sample points are taken from a population with uniform distribution on $[0, 25]$.
   (a) What is the probability that the largest data point is less than 20?

Solution: Relating to Problem 1, $\theta = 25$ and $n = 10$. Then $f(y_i) = \begin{cases} \frac{1}{25}, & 0 \le y_i \le 25 \\ 0, & \text{else} \end{cases}$, the associated CDF for each Y_i is

$$F(y_i) = \begin{cases} 0, & y_i < 0 \\ y_i/25, & 0 \le y_i \le 25 \\ 1, & y_i > 25 \end{cases},$$

and the max-order statistic is defined by

$$g_{(10)}(y) = 10[F(y)]^9 f(y) = \frac{10y^9}{25^n},$$

on the support $0 \le y \le 25$. Thus,

$$P(Y_n < 20) = \int_{y=0}^{20} \frac{10y^9}{25^{10}}\, dy = \left. \frac{10y^{10}}{10(25)^{10}} \right|_{y=0}^{20} = \frac{20^{10}}{25^{10}} \approx 0.10737.$$

(b) What is the expected value of $Y_{(10)}$?

Solution:

$$E[Y_{(10)}] = \int_0^{25} y \cdot f_{(10)}(y)\, dy = \int_0^{25} \frac{10y^{10}}{25^{10}}\, dy = \frac{10(25)^{11}}{25^{10}(11)} = \frac{250}{11} \approx 22.73.$$

3. (a) Items are produced in such a manner that 2% turn out defective. A large number of such items, say n, are inspected and the relative frequency of defectives, say f_D, is recorded. How large should n be in order that the probability is at least 0.98 that f_D differs from 0.02 by less than 0.05.

Solution: This is an application of the Law of Large Numbers. Here $P(D) = 0.02$, $P(\sim D) = 0.98$, $\epsilon = 0.05$ and we seek n such that

$$P(|f_D - 0.02| < 0.05) \ge 1 - pq/n\epsilon^2 = 0.98.$$

Solving $1 - (0.02)(0.98)/n(0.05)^2 = 0.98$ yields $n = 392$.

(b) Answer (a) above if 0.02, the probability of obtaining a defective item, is replaced by p which is assumed to be unknown.

Solution: General p yield the equation $1 - p(1-p)/n(0.05)^2 = 0.98$. Then $p(1-p) = (0.00005)n$ or $n = 20000p(1-p)$. To choose n for any p, we need to choose p that maximizes the expression for n. Here $n(p) = 20000(p - p^2)$ and $n'(p) = 20000(1-2p)$. Then $n'(p) = 0$ when $p = 1/2$. To show that this is a maximum, consider that $n''(p) = -40000$. As $n''(p) < 0$ always, $p = 1/2$ is the location of a global maximum. In the end, choosing $n \geq 20000(1/2)^2 = 5000$ will do the job.

4. Compute an approximate probability that the mean of a random sample of size 15 from a distribution having p.d.f. $f(x) = 3x^2$, $0 \leq x \leq 1$, zero elsewhere, is between 3/5 and 4/5.

Solution: To use the Central Limit Theorem, we need μ and σ for random variable X.

$$\mu = \int_0^1 x \cdot 3x^2\,dx = \int_0^1 3x^3\,dx = \left.\frac{3x^4}{4}\right|_0^1 = \frac{3}{4}.$$

For σ,

$$E[X^2] = \int_0^1 3x^4\,dx = \left.\frac{3x^5}{5}\right|_0^1 = \frac{3}{5}$$

and

$$\sigma^2 = E[X^2] - \mu^2 = \frac{3}{80}.$$

Then the random variable $\bar{X}$ is approximately distributed by the Normal distribution with mean 3/4 and variance $\sigma^2/n = 1/400$. Then the probability that $\bar{X}$ is between 3/5 and 4/5 is found using Normal distribution methods.

$$\begin{aligned} P\left(\frac{3}{5} < \bar{X} < \frac{4}{5}\right) &= P\left(\frac{3/5 - 3/4}{1/20} < Z < \frac{4/5 - 3/4}{1/20}\right) \\ &= P(-3 < Z < 1) \\ &= 0.83995 \text{ by Table 4, or} \\ &= 0.8385 \text{ by the empirical rule} \end{aligned}$$

5. In 2017, the entire fleet of light-duty vehicles sold in the US by each manufacturer must emit an average of no more than 86 milligrams per mile (mg/mi) of nitrogen oxides (NOX) and nonmethane organic gases (NMOG)over the useful life (150,000 mile of driving) of the vehicle. Let E=NOX+NOMG. For a specific vehicle, the manufacturer assumes the E is normally distributed with mean 80 mg/mi and standard deviation 4 mg/mi.

(a) What is the probability that a single vehicle of this model emits more than 86 mg/mi of E?

Solution: For a single vehicle, we use the Normal distribution $N(80, 16)$.

$$P(E > 86) = P\left(Z > \frac{86 - 80}{4}\right) = P(Z > 1.5) = 0.0668.$$

(b) The company selects 25 vehicles to sell to a construction company. What is the probability that the average E exceeds 86 mg/mi?

Solution: For $n = 25$, $\bar{E}$ is distributed by $N(80, 16/25)$, $\sigma = 4/5$ here. Then

$$P(\bar{E} > 86) = P\left(Z > \frac{86 - 80}{4/5}\right) = P(Z > 7.5) \approx 0.$$

(c) For the company's fleet of 25 vehicles, what is the emissions level L such that the probability that the average E level for the fleet is greater than L is only 0.01?

Solution: For $n = 25$, we want L such that $P(\bar{E} > L) = 0.01$. Using the same sampling distribution from (b), equivalent question is $P\left(Z > \frac{L-80}{4/5}\right) = 0.01$. This requires a reverse Table 4 calculation. We need the critical number z^* that corresponds to a 98% confidence interval. Here $z^* = 2.33$ and

$$\frac{L - 80}{4/5} = 2.33 \text{ or } L = 81.864 \text{ mg}.$$