HW 10 MTH 325 Matthew Wilder

1.) $Y_1, \dots Y_n$ ind. unif dist variable on $[0, \theta]$
Find the pdf of

$$Y_{(n)} = \max\{Y_1, \dots, Y_n\}$$

mean $= \frac{\theta}{2}$

PDF of a single uni dist $= \frac{1}{b-a}$ by uniform def

$\therefore \frac{1}{\theta - 0} = \frac{1}{\theta}$

$$f(y_i) = \begin{cases} 1/\theta & y_i \in [0, \theta] \\ 0 & \text{e.w.} \end{cases}$$ as pdf

To get CDF we integrate wrt y_i $\therefore$

$$\int_0^{y_i} \frac{1}{\theta} dy_i = \frac{1}{\theta}\int_0^{y_i} dy_i = \frac{1}{\theta}\left[y_i \Big|_{y_i=0}^{y_i=y_i} \right]$$

$$= \frac{1}{\theta}\left[y_i - 0\right] = \frac{y_i}{\theta}$$

$$\therefore F(y_i) = \begin{cases} 0 & y_i < 0 \\ y_i/\theta & y_i \in [0, \theta] \\ 1 & y_i > \theta \end{cases}$$

and then $Y_{(n)} = n\left[F(y_n)\right]^{n-1} f(y_n)$ by definition

and $= n\left[y_n/\theta\right]^{n-1} \frac{1}{\theta}$ by substitution

$$= n \cdot \frac{(y_n)^{n-1}}{\theta^{n-1}} \cdot \frac{1}{\theta} = n \cdot \frac{(y_n)^{n-1}}{\theta^n}$$

$\square$

MTH 325

HW10 Matt Wilder

2. 10 sample points from uni dist $\in [0,25] = I$

a.) $P(Y_{10} < 20)$ $\left(\frac{20-0}{25-0}\right)$

$P(Y < 20) = 0.8$

$P(Y_{10} < 20)$ means that ALL 10 trials resulted in a value <20, which is an 80% chance each

$\therefore$ p(all 10 are less than 20) $= (0.8)^{10}$

binom w/ $p = 0.8$

$q = 0.2$

$n = 10$

$$P(Y_{10} < 20) = 0.8^{10}$$
$$\approx 10.74\%$$

b.) $E[Y_{10}]$ $n = 10$

$g_{10}(y_{10}) = 10 \cdot \frac{y^9}{25^{10}}$ by problem 1

$$\int_0^{25} 10 \frac{y^9}{25^{10}} \cdot y\, dy = \frac{10}{25^{10}} \cdot \int_0^{25} y^{10} dy$$

$$= \frac{10}{25^{10}} \cdot \frac{y^{11}}{11} \Big|_0^{25} = \frac{10}{11} \cdot \frac{25^{11}}{25^{10}}$$

$$= \frac{10}{11} \cdot 25 = \frac{250}{11}$$

$$= 22.\overline{72}\%$$

MTH 325 HW10

Matt Wilder

3. (a) 2% defects, inspect $n \geq N \ni |f(n) - f(n+1)| < \varepsilon$

Let f_D denote number of defects

98% confidence that f_D differs by 0.02 by < 0.05

$$P(|f_D - 0.02| < 0.05) \geq 0.98 = 1 - \frac{pq}{n\varepsilon^2}$$

$$= 1 - \frac{0.02 \cdot 0.98}{n \cdot 0.05^2}$$

$\therefore$

$$0.02 = \frac{0.02 \cdot 0.98}{n \cdot 0.05^2}$$ get rid of the 1

$$\Rightarrow \quad 1 = \frac{0.98}{n \cdot 0.05^2}$$ div by 0.02

$$n = \frac{0.98}{0.05^2}$$ multiply by n

$$\boxed{n = 392}$$

b.) $$P(|f_D - p| < \overset{\varepsilon}{0.05}) \geq 0.98 = 1 - \frac{pq}{n\varepsilon^2}$$

$$0.02 = \frac{pq}{n \cdot 0.05^2} \longleftarrow = 1 - \frac{pq}{n(0.05)^2}$$

$$n = \frac{pq}{0.02 \cdot 0.05^2} = \frac{p(1-p)}{1/20{,}000} = 20{,}000(p - p^2) = 0.98 \equiv n(p)$$

$$\max(20{,}000(p - p^2)) \Rightarrow 20{,}000(1 - 2p) = 0 \equiv n'(p)$$

$\therefore p = 1/2$ is max

and $$n\left(\tfrac{1}{2}\right) = 20{,}000\left(\tfrac{1}{2} - \left(\tfrac{1}{2}\right)^2\right) = 20{,}000 \cdot \tfrac{1}{4}$$

$$= \boxed{5{,}000 = n}$$

$\square$

MTH 325 HW10 Matt Wilder

4 prob of mean of random sample w/ $n=15$ from a dist with pdf $f(x) = 3x^2 \ni x \in [0,1]$, 0 e.w., is between 3/5 and 4/5

$$f(x) = \begin{cases} 3x^2 & x \in [0,1] \\ 0 & \text{e.w.} \end{cases}$$

$$P\left(\frac{3}{5} \le \bar{\mu} \le \frac{4}{5}\right)$$

By CLT $X \equiv N(\mu, \sigma^2)$

$$\mu = E[X] = \int_0^1 3x^2 x = \int_0^1 3x^3 = \frac{3}{4}x^4 \Big|_0^1 = \frac{3}{4}$$

$$\sigma^2 = E[x^2] - [E(x)]^2 = \int_0^1 3x^4\,dx - \left(3/4\right)^2$$

$$= \frac{3}{5} - \left(\frac{3}{4}\right)^2$$

$$= 3/80 = 0.0375 = V[X]$$

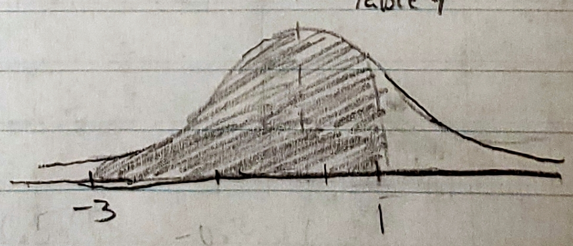
So $X \equiv N(3/4, 3/80)$

$n=15$, $\frac{3}{80} \cdot n^{-1} = \frac{1}{400}$, $N\left(0, \frac{1}{400}\right) \Rightarrow \sigma = \frac{1}{20}$

$$P(3/5 \le \bar{X} \le 4/5) \equiv P\left(\frac{0.6-0.75}{1/20} \le Z \le \frac{0.8-0.75}{1/20}\right)$$

Table 4

$P(-3 \le Z \le 1)$



$P(Z \ge -3) = 1 - P(Z \ge 3)$

$\overset{\text{Table 4}}{=} 1 - 0.00135 = 0.99865$

$P(Z \le 1) = 1 - P(Z \ge 1) \overset{T4}{=} 1 - 0.1587 = 0.8413$

$0.8413 - 0.00135 = \boxed{0.83995}$

MTH 325 HW10 Matt Wilder

#5 avg less than 86 mg/mi of NOX and NMOG over the first 150k miles

Let $E =$ NOX + NOMG, with $N(80, 4^2 \text{ mg/mi})$

(a) $P(\mu_E \geq 86 \text{ mg/mi}) = \text{Table4}\left(\frac{86-80}{4}\right)$
$= \text{Table 4}(1.5)$
$= 0.0668 = \boxed{6.68\%}$

(b) Let $n=25$

$P(\mu_E \geq 86) = \text{Table 4}\left(\frac{86-80}{4/\sqrt{25}}\right) = \text{Table4}(7.5)$
$= \boxed{0\%}$

CLT

(c) For $n=25$, what level $L \ni P(\mu_E > L) = 0.01$

Then the z-score $= \frac{L-80}{4/\sqrt{25}} = \frac{5}{4} \cdot (L-80) = z$

by $\text{Table 4}^{-1}(0.01) \approx 2.33$

and $2.33 = \frac{5}{4}(L-80)$

$\Rightarrow 1.864 = L-80$

$\Rightarrow \boxed{L = 81.864 \text{ mg/mi}}$