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1. 60 customers/Hour

$X = \#$ of customers in a 5 min interval, $\lambda = 5$

a.) $P(X=0) = \dfrac{\lambda^{y} e^{-\lambda}}{y!} = \dfrac{5^0 e^{-5}}{0!} = \dfrac{1}{e^5}$

$= 0.006738$

b.) we want to find λ s.t. $\dfrac{\lambda^0 e^{-\lambda}}{0!} = \dfrac{1}{2} = \dfrac{1}{e^{\lambda}} \Rightarrow e^{\lambda} = 2$

$\therefore \lambda = \ln(2)$

The time interval would need to be exactly $\ln(2)$ minutes

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2. $n = 20$ $p = 0.05$

a.) $\binom{n}{y} p^y q^{n-y}$

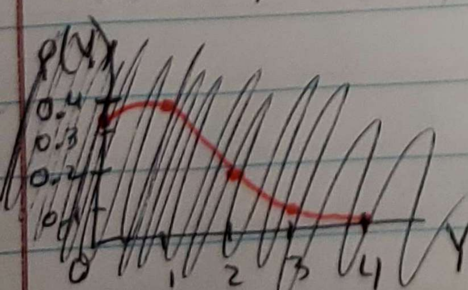
$P(Y=0) = \binom{20}{0}(0.05)^0(0.95)^{20} = 0.3585$

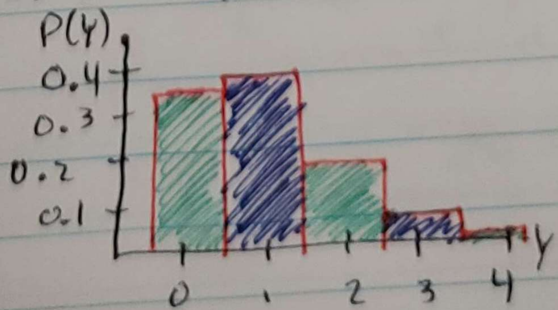
$P(Y=1) = \binom{20}{1}(0.05)^1(0.95)^{19} = 0.3774$

$P(Y=2) = \binom{20}{2}(0.05)^2(0.95)^{18} = 0.1887$

$P(Y=3) = \binom{20}{3}(0.05)^3(0.95)^{17} = 0.0596$

$P(Y=4) = \binom{20}{4}(0.05)^4(0.95)^{16} = 0.0133$





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2b $\lambda = np$. $n = 20$ $p = 0.05$

$20 \cdot 0.05 = 1 = \lambda$

$$\frac{\lambda^y e^{-\lambda}}{y!} = \frac{1^y e^{-1}}{y!} = \frac{1}{e \cdot y!} = p(y)$$

Poisson	Binomial	difference
$P(Y=0) = 0.36788$	0.3585	0.00938
$P(Y=1) = 0.36788$	0.3774	-0.009521
$p(Y=2) = 0.1839397$	0.1887	-0.004760
$p(Y=3) = 0.061313$	0.0596	0.001713
$p(Y=4) = 0.01533$	0.0133276	0.002002

$\boxed{\text{The poisson approximations are within 0.01 or 1\% or roughly 0.5\% average}}$

3 (a)

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3. a.) $p(y) = \binom{n}{y} p^y q^{n-y}$

$$m(t) = \sum_{y=0}^{n} e^{ty} p(y) = \sum_{y=0}^{n} e^{ty} \cdot \frac{n!}{y!(n-y)!} \cdot p^y \cdot q^{n-y}$$

$$= \sum_{y=0}^{n} e^{ty} \cdot p^y \cdot \frac{n! q^{n-y}}{y!(n-y)!} = \sum_{y=0}^{n} (pe^t)^y \cdot \frac{n!}{y!(n-y)!} q^{n-y}$$

$$= \sum_{y=0}^{n} \binom{n}{y} q^{n-y} (pe^t)^y$$

using binomial series formula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

we can simplify to

$(q + pe^t)^n$, but $q = 1-p$, $\therefore$ $((1-p)+pe^t)^n$ holds true directly.

b.) $E[Y] = m'(0)$ $\quad m'(t) = ((1-p)+pe^t)^n \frac{d}{dt}$

$$n[(1-p)+pe^t]^{n-1} \cdot [0+pe^t] = npe^t [(1-p)+pe^t]^{n-1} = m'(t)$$

$$m'(0) = npe^0 [(1-p)+pe^t]^{n-1} = np[(1-p)+pe^0]^{n-1} = np[1]^{n-1}$$

$$m'(0) = \boxed{np = E[Y]}$$

$$m''(t) = \frac{d}{dt}\left(npe^t[(1-p)+pe^t]^{n-1}\right)$$

$$= npe^t \left[(n-1)[(1-p)+pe^t]^{n-2} pe^t\right] + [(1-p)+pe^t]^{n-1} \cdot [npe^t]$$

$$= \underline{npe^t} \left\{[(1-p)+pe^t]^{n-2}(n-1)pe^t + [(1-p)+pe^t]^{n-1}\right\}$$

$$m''(0) = np \cdot \{(n-1)p+1\} = np \cdot \{np-p+1\} = np \cdot \{np-(1-q)+1\}$$

$$= np \cdot \{np-1+q+1\}$$

$$\boxed{= np \cdot (np+q) = E[Y^2]}$$

$$V[Y] = E[Y^2] - (E[Y])^2 = np(np+q) - (np)^2$$

$$= (n^2p^2 + npq) - n^2p^2 = \boxed{npq = V[Y]}$$

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4. geo $p(y) = pq^{y-1}$

$\sum ar^{n-1} = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$

a.

$$m(t) = \sum_{1=y}^{\infty} e^{ty} pq^{y-1}$$

$$= \sum_{y=1}^{\infty} e^{ty} pq^y q^{-1} = \sum_{y=1}^{\infty} (qe^t)^y \frac{p}{q} = \frac{p}{q} \sum_{y=1}^{\infty} (qe^t)^y$$

reindex $\frac{p}{q} \cdot (qe^t) \sum_{y=0}^{\infty} (qe^t)^{y-1} = \frac{pqe^t}{q} \cdot \left(\frac{1}{1-qe^t} \right) = \frac{pe^t}{1-qe^t}$

b.) $E[Y] = m'(0)$, $m'(t) = \frac{(1-qe^t)(pe^t) - (pe^t)(-qe^t)}{(1-qe^t)^2}$

$$m'(0) = \frac{(1-q)p + pq}{(1-q)^2} = \frac{p - pq + pq}{p^2} = \frac{p}{p^2} = \boxed{\frac{1}{p} = E[Y]}$$

$$E[Y^2] = \sum_{y=1}^{\infty} q^{y-1} p y^2 = p \sum_{y=1} q^{y-1} y^2 = p \sum_{y=1} \underbrace{y^2 (1-p)^{y-1}}_{\frac{-d}{dp}\left(y(1-p)^y\right)}$$

$$= p \sum_{y=1}^{\infty} \frac{-d}{dp} y(1-p)^y = p \cdot \frac{-d}{dp} \sum_{y=1}^{\infty} y(1-p)^y$$

converges

$$= -p \frac{d}{dp} \sum_{y=1}^{\infty} y(1-p)^{y-1} \cdot p \left(\frac{1-p}{p} \right) = -p \frac{d}{dp} \frac{1-p}{p} \underbrace{\sum_{y=1}^{\infty} y(1-p)^{y-1}}_{= 1/p}$$

$$= -p \frac{d}{dp} \frac{1-p}{p^2} = -p \frac{d}{dp} \left[\frac{1}{p^2} - \frac{1}{p} \right] = -p \left[-2p^{-3} + p^{-2} \right]$$

$$= \frac{2}{p^2} - \frac{1}{p} = \boxed{\frac{2-p}{p^2} = E[Y^2]}$$

$$V[Y] = E[Y^2] - (E[Y]^2) = \frac{2-p}{p^2} - \frac{1}{p^2} = \boxed{\frac{1-p}{p^2} = V[Y]}$$

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5. $m(t) = \frac{2}{5}e^{t} + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t}$

$m'(t) = \frac{2}{5}e^{t} + \frac{2}{5}e^{2t} + \frac{6}{5}e^{3t}$ $\quad m'(0) = \frac{2}{5} + \frac{2}{5} + \frac{6}{5} = 2 = \mu$

$m''(t) = \frac{2}{5}e^{t} + \frac{4}{5}e^{2t} + \frac{18}{5}e^{3t}$ $\quad m''(0) = \frac{2}{5} + \frac{4}{5} + \frac{18}{5} = \frac{24}{5}$

$\text{Variance} = \frac{24}{5} - 2^2 = \frac{4}{5}$ variance

$$P(Y=y) = \begin{cases} 2/5, & y = 1 \text{ or } y = 3 \\ 1/5, & y = 2 \end{cases}$$

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6. a.) poisson is $e^{\lambda(e^t-1)}$; $e^{5.6(e^t-1)}$ $\quad \lambda = 5.6$

$p(Y=3) = \frac{5.6^3 e^{-5.6}}{3!} = 0.108234$

b) $$p(Y=y) = \begin{cases} 0.25 & y=1 \\ 0.35 & y=3 \\ 0.40 & y=5 \end{cases}$$ $\quad P(Y=3) = 0.35$

(.) $\frac{pe^t}{1-qe^t}$ geo $\quad p = 0.35 \quad p(y) = 0.35 \cdot 0.65^{y-1}$

$P(Y=3) = 0.35 \cdot (0.65)^2$

0.147875

d. $(pe^t + q)^n = (0.35e^t + 0.65)^{14}$

$n = 14 \quad p = 0.35 \quad P(Y=3) = \binom{14}{3} 0.35^3 0.65^{11} = 0.13656$