Math 325 – Homework 06
Due (via upload to Canvas) Friday, October 22, 2021 at 6 PM

1. (exponential distributions are memoryless).
   (a) Let Y have an exponential distribution with mean $\beta > 0$. Show that

$$P(Y > y_1 + y_2 | Y > y_1) = P(Y > y_2).$$

Solution: Recall that $P(Y > y_1 + y_2 | Y > y_1) = \dfrac{P(Y > y_1 + y_2 \cap Y > y_1)}{Y > y_1)}$. But, as $y_1 + y_2 > y_1$, $P(Y > y_1 + y_2 | Y > y_1) = \dfrac{P(Y > y_1 + y_2)}{P(Y > y_1)}$. Note that for any $r > 0$,

$$P(Y > r) = \int_r^\infty \frac{1}{\beta} \exp[-t/\beta]\, dt = -\exp[-t/\beta]\Big|_r^\infty = e^{-r/\beta}.$$

Hence,

$$P(Y > y_1 + y_2 | Y > y_1) = \frac{e^{-(y_1+y_2)/\beta}}{e^{-y_1/\beta}} = e^{-y_2/\beta} = P(Y > y_2).$$

   (b) Exponential distributions possess the same "memoryless" property that geometric probability distributions for a discrete random variable did. Explain what we mean by "memoryless".

Solution: This is akin to what we saw with Bernoulli events and the Binomial distribution. The number of heads in the future has absolutely nothing to do with the number of heads that occurred in the past. Here, no matter when you start the clock (y_1) the probability of seeing something y_2 time later has no bearing on the probability. The past does not impact any future probability: memoryless.

2. A candy maker produces mints that have a label weight of 20.4 grams. Assume that the distribution of the weights of these mints is $N(21.37, 0.16)$. (*For the below, I suggest using Table 4.* ***I reserve the right to handout Table 4 for use on hour exam or the final.*** *Be sure you can use it.*)

   (a) Let Y denote the weight of a single mint selected at random from the production line. Find the probability that the mint weights more than the label weight.

Solution: Want $P(Y > 20.4)$. Here $\sigma^2 = 0.16$ and $\sigma = 0.4$. Using z-scores (as opposed to R), $z = \dfrac{20.4 - 21.37}{0.4} = -2.425$. Using 2.43, $P(Y > 20.4) = 1 - 0.0075 = 0.9925$.

   (b) Find an interval about the mean within which you expect 90% of all mints to weight.

Solution: If 90% is in the middle, then 10% is in the tails and 5% is in the upper tail. We see that the corresponding z-score is $z = 1.645$. Then, we need the Y such that $\pm 1.645 = \dfrac{Y - 21.37}{0.4}$, or $Y = 21.37 \pm 0.658$.

   (c) Suppose that 15 mints are selected at random and weighted. Let X equal the number of mints that weight less than 20.91 grams. Find $P(X \leq 2)$.

Solution: Each mint can be interpreted as a Bernoulli event with the probability of success being $P(Y < 20.91)$. Since $P(Y < 20.91) = P(Y < \frac{20.91-21.37}{0.4}) = P(Y < -1.15) = P(Y > 1.15) = 0.1271$.
Then

$$\begin{aligned} P(X \leq 2) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= (0.1271)^0(1 - 0.1271)^{15} + 15(0.1271)^1(0.8729)^{14} + \binom{15}{2}(0.1271)^2(0.8729)^{13} \\ &= 0.7041796. \end{aligned}$$

3. Telephone calls enter a college switchboard at a mean rate of 2/3 call per minute according to a Poisson process. Let Y denote the waiting time until the tenth call arrives.

   (a) What is the p.d.f. of Y?

Solution: This is a Γ-distribution with $\alpha = 10$ and $\beta = 1/\lambda = 3/2$. Note $\Gamma 10 = 9!$. So $f(y) = \frac{y^9 e^{-2y/3}}{(3/2)^1 09!}$ on the support $[0, \infty)$.

   (b) What are the mean and variance of Y?

Solution: By theorem, $\mu = \alpha\beta = 15$ and $\sigma^2 = \alpha\beta^2 = 90/4 = 45/2$.

   (c) What is the probability that it take less than 5 minutes for the 10th call to come in?

Solution: $P(Y \leq 5) = \int_0^5 f(y)\, dy$. Having no desire to do integration by parts 9 times,

$$P(Y \leq 5) = \text{pgamma}(5, 10, 2/3) \approx 0.002356375.$$

4. Consider $f(x) = cx^3(1-x)^6$, $0 < x < 1$.

(a) Determine the constant c such that $f(x) = cx^3(1-x)^6$, $0 < x < 1$, is a p.d.f.

Solution: We could integrate the PDF over $[0,1]$ and determine c, but we also could recognize this as a beta distribution. Here $\alpha - 1 =$ and $\beta - 1 = 6$. Hence $\alpha = 4$ and $\beta = 7$. Then

$$c = \frac{1}{B(4,7)} = \frac{\Gamma(4+7)}{\Gamma(4)\Gamma(7)} = \frac{10!}{3! \cdot 6!} = 840.$$

(b) Determine the mean and variance for the associated random variable X.

Solution: By theorem, $\mu = \dfrac{\alpha}{\alpha+\beta} = \dfrac{4}{11}$ and $\sigma^2 = \dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \dfrac{4 \cdot 7}{11^2 \cdot 12} = \frac{7}{363}$.

(c) Compute the distribution function $F(x)$.

Solution: Note $F(x) = 0$ when $x < 0$. For $x \in [0,1]$, need to compute $F(x) = \int_0^x 840t^3(1-t)^6\,dt$. We could compute this using the binomial formula, but integrating 7 terms seems a bit much. Using the substitution $u = 1-t$, $t = 1-u$ yields only 4 terms in the antiderivative.

$$\begin{aligned}
F(x) &= \int_1^{1-x} 840(1-u)^3u^6(-du) \\
&= 840\int_1^{1-x} (1-3u+3u^2-u^3)u^6\,du \\
&= 840\left(\frac{u^7}{7} - 3\frac{u^8}{8} + 3\frac{u^9}{9} - \frac{u^{10}}{10}\right)\Bigg|_{1-x}^{1} \\
&= 1 - 120(1-x)^7 + 315(1-x)^8 - 280(1-x)^9 + 84(1-x)^{10}
\end{aligned}$$

In the end,

$$F(x) = \begin{cases} 0 & , x < 0 \\ 1 - 120(1-x)^7 + 315(1-x)^8 - 280(1-x)^9 + 84(1-x)^{10} & , 0 \le x \le 1 \\ 1 & , x > 1 \end{cases}$$

(d) Find $P(X > 0.4)$.

Solution: Using our work above,

$$P(X > 0.4) = 1 - P(X < 0.4) = 1 - F(2/5) \approx 1 - 0.6177 = 0.3823.$$

5. Let Z be a standard normal random variable. We will show that the random variable Z^2 has a gamma distribution.

   (a) State $f(z)$, the PDF associated with Z.

Solution: Let Z be a standard normal variable; from $N(0,1)$. Then, the pdf of Z is

$$f(z) = \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-z^2}{2}\right], \; -\infty < z < \infty.$$

   (b) Let $Y = Z^2$, use the PDF for z and construct the CDF for the random variable Y. (Note that you are not being asked to evaluate the resultant integral. Simply construct the CDF.)

Solution: Consider the random variable $Y = Z^2$. We need to compute Y's CDF; $G(y) = P(Y \le y)$.
The CDF of y is

$$\begin{aligned} G(y) &= P(Z^2 \le y) \\ &= P(-\sqrt{y} \le Z \le \sqrt{y}) \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} f(z)\, dz \\ &= \int_{-\infty}^{\sqrt{y}} f(z)\, dz + \int_{-\infty}^{-\sqrt{y}} f(z)\, dz \\ &= F(\sqrt{y}) - F(-\sqrt{y}), . && \text{(where } F \text{ is the cdf of } Z\text{)} \end{aligned}$$

   (c) Determine the PDF for $Y = Z^2$ and show that it a gamma distribution and determine the parameters α and β.

Solution: The PDF of Y is determined by differentiating the CDF for Y.

$$\begin{aligned}
g(y) &= G'(y) \\
&= \frac{1}{2\sqrt{y}}F'(\sqrt{y}) + \frac{1}{2\sqrt{y}}F'(-\sqrt{y}) \\
&= \frac{1}{2\sqrt{y}}f(\sqrt{y}) + \frac{1}{2\sqrt{y}}f(-\sqrt{y}) \\
&= \frac{1}{2\sqrt{y}}\left(\frac{1}{\sqrt{2\pi}}e^{-y/2}\right) + \frac{1}{2\sqrt{y}}\left(\frac{1}{\sqrt{2\pi}}e^{-y/2}\right) \\
&= \frac{1}{\sqrt{y}\sqrt{2\pi}}e^{-y/2} \\
&= \frac{y^{-\frac{1}{2}}e^{-y/2}}{\sqrt{2}\sqrt{\pi}} \\
&= \frac{y^{-\frac{1}{2}}e^{-y/2}}{2^{(1/2)}\Gamma(1/2)}
\end{aligned}$$

We see Z^2 has the gamma PDF with parameters $\alpha = 1/2$, $\beta = 2$.