MTH 325 :: Homework 8 :: MATTHEW_WILDER

1.) /**

* roll a fair 6-sided die twice

* Let X denote the outcome of the first roll and

* Let Y denote the outcome of the sum of the two rolls

*/

a.) $\mu_X = E[X] = \frac{1}{6}\sum_{i=1}^{6} i = \frac{1}{\cancel{6}} \cdot \frac{\cancel{6}(6+1)}{2} = \boxed{\frac{7}{2}}$

c.) $\mu_Y = E[Y] = \frac{1}{36}\sum_{i=1}^{6}\sum_{j=1}^{6}(i+j) = \frac{1}{36}\sum_{i=1}^{6} 6i + \frac{6(6+1)}{2} =$

$= \frac{1}{36}\sum_{i=1}^{6} 6i+21 = \frac{1}{36}\left(6(21)+\sum_{i=1}^{6} 6i\right) = \frac{1}{6}\left(21+\sum_{i=1}^{6} i\right) = \frac{1}{6}(21+21)$

$= \frac{1}{6}(42) = \boxed{7}$

out of order

b.) σ_X^2 need to find $E[X^2] = \frac{1}{6}\sum_{i=1}^{6} i^2 = \frac{1}{6}\left(\frac{\cancel{6}(6+1)(2\cdot 6+1)}{\cancel{6}}\right)$

$= \frac{1}{6}(7\cdot 13) = \frac{91}{6}$ $\quad V[X] = E[X^2]-(E[X])^2$

$= \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \boxed{\frac{35}{12}} = \sigma_X^2$

d. σ_Y^2 need $E[Y^2] = \frac{1}{36}\cdot\sum_{i=1}^{6}\sum_{j=1}^{6}(i+j)^2 = \frac{1}{36}\sum_{i=1}^{6}\sum_{j=1}^{6} i^2+2ij+j^2$

$= \frac{1}{36}\left(91\cdot 6 + 6\cdot 91 + 2\sum_{i=1}^{6}\sum_{j=1}^{6} ij\right) = \frac{1}{36}\left(1092 + 2\sum_{i=1}^{6} i \sum_{j=1}^{6} j\right)$

$= \frac{1}{36}\left(1092 + 2\sum_{i=1}^{6} 21i\right) = \frac{1}{36}\left(1092+42(21)\right) = \frac{1974}{36} = \frac{329}{6} = E[Y^2]$

$V[Y] = E[Y^2] - (E[Y])^2 = \frac{329}{6} - 49 = \boxed{\frac{35}{6}} = \sigma_Y^2$

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1 e. recall: $\mu_x = \frac{7}{2}$ $\mu_y = 7$ $X: x \in [1,6]$

$Y: y \in [2,12]$

$\sigma_x^2 = \frac{35}{12}$ $\sigma_y^2 = \frac{35}{6}$

$\text{cov}(X,Y) = E[XY] - \mu_x \mu_y$ $\mu_x \mu_y = 49/2$

$$36 E[XY] = \sum_{i=1}^{6} \sum_{j=1}^{6} i(i+j) = \sum_{i=1}^{6} i \cdot \sum_{j=1}^{6} i+j = \sum_{i=1}^{6} i \cdot \left(6i + \sum_{j=1}^{6} j\right)$$

$$= \sum_{i=1}^{6} i \cdot (6i+21) = \sum_{i=1}^{6} 6i^2 + 21i = 6\sum_{i=1}^{6} i^2 + 21\sum_{i=1}^{6} i$$

$$= 6 \cdot \frac{6(6+1)(2 \cdot 6+1)}{6} + 21\left(\frac{6(6+1)}{2}\right) = 546 + 441 = 987$$

$$\Rightarrow E[XY] = \frac{987}{36} = \frac{329}{36} \Rightarrow \text{cov}(X,Y) = 329/36 - 49/2 = \boxed{35/12}$$

f. $$\rho = \frac{\text{cov}(X,Y)}{\sigma_x \sigma_y} = \frac{35/12}{\sqrt{35/12} \cdot \sqrt{35/6}} = \frac{35/12}{} = \frac{35}{12} \cdot \frac{\sqrt{72}}{35} = \frac{\sqrt{72}}{12} = \frac{6\sqrt{2}}{12}$$

$$\boxed{= \frac{\sqrt{2}}{2}}$$

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2. Assume that X, Y, and Z are random variables with

$E[X] = 2$ $\quad V[X] = 4$ $\quad cov(X, Y) = -1$

$E[Y] = -1$ $\quad V[Y] = 6$ $\quad cov(X, Z) = 1$

$E[Z] = 4$ $\quad V[Z] = 8$ $\quad cov(Y, Z) = 0$

Find $E[3X + 4Y - 6Z]$

and $V[3X + 4Y - 6Z]$

$E[3X+4Y-6Z] = 3E[X] + 4E[Y] - 6E[Z]$

$= 3(2) + 4(-1) - 6(4) = 6 - 4 - 24 = \boxed{-22}$

$V[3X+4Y-6Z]$

$= 3^2 V[X] + 4^2 V[Y] + (-6)^2 V[Z]$

$+2[(3\cdot 4)cov(X,Y) + (3\cdot -6)cov(X,Z) + (4\cdot -6)cov(Y,Z)]$

$= 9\cdot 4 + 16\cdot 6 + 36\cdot 8$

$+2[12\cdot(-1) - 18(1) - 24(0)]$

$= 36 + 96 + 36\cdot 8 + 2(-12 - 18 + 0) = \boxed{360}$

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3. Y = # of tosses until all sides are observed (6 sided die)

$\therefore Y = Y_1 + Y_2 + \cdots + Y_6$

where Y_i is the number of tosses for the next unique face to be observed w/ $Y_1 = 1$

a. $E[Y] = E[Y_1] + E[Y_2] + \cdots + E[Y_6]$

$E[Y_1] = 1$ Y_1 = geo w/ $p = 6/6$ $\mu = \frac{1}{6/6} = 1$

$E[Y_2]$: Y_2 = geo w/ $p = \frac{5}{6}$ $\mu = \frac{1}{5/6} = \frac{6}{5}$

$E[Y_3]$: Y_3 geo w/ $p = \frac{4}{6}$ $\mu = \frac{1}{4/6} = \frac{3}{2}$

$E[Y_4]$: Y_4 geo w/ $p = 3/6$ $\mu = \frac{1}{3/6} = 2$

$E[Y_5]$: Y_5 geo w/ $p = 2/6$ $\mu = \frac{1}{2/6} = 3$

$E[Y_6]$: Y_6 geo w/ $p = 1/6$ $\mu = 1/_{1/6} = 6$

$E[Y] = 1 + \frac{6}{5} + \frac{3}{2} + 2 + 3 + 6 = \boxed{\frac{147}{10}}$ ≈ 15 rolls

b.) $COV(Y_i, Y_j) \ni i \neq j$

$= E[Y_i Y_j] - \mu_{Y_i}\mu_{Y_j}$ and by independence

$= E[Y_i]E[Y_j] - \mu_{Y_i}\mu_{Y_j} = \mu_{Y_i}\mu_{Y_j} - \mu_{Y_i}\mu_{Y_j} = \boxed{0}$

c. $V[Y] = \sum_{i=1}^{6} \cancel{Y_i}^{1} V[Y_i] + 2\sum\sum_{i<j} \cancel{y_i y_j} \overset{0}{COV(Y_i Y_j)}$

$= \sum_{i=1}^{6} V[Y_i] =$ $V = \frac{q}{p^2}$

$0 + \frac{1/6}{(5/6)^2} + \frac{2/6}{(4/6)^2} + \cancel{\frac{3/6}{(3/6)^2}}\,2 + \frac{4/6}{(2/6)^2} + \frac{5/6}{(1/6)^2}$

3c continued

$$= 2 + \frac{1}{6}\cdot\frac{36}{25} + \frac{2}{6}\cdot\frac{36}{16} + \frac{4}{6}\cdot\frac{36}{4} + \frac{5}{6}\cdot\frac{36}{1}$$

$$= 2 + \frac{6}{25} + \frac{12}{16} + 6 + 30 = 38 + \frac{3}{4} + \frac{24}{100}$$

$$= 38 + 0.75 + 0.24 = \boxed{38.99}$$

3d Give $I = (y-r, y+r) \ni P(Y=y) \geq 0.75$

$E[Y] = 14.7 \quad V[Y] = 38.99$

By chebyshev's, $1 - \frac{1}{k^2} \geq 0.75 \Rightarrow k = 2$

$\therefore \quad I = (\mu - 2\sigma, \mu + 2\sigma)$

$\sigma = \sqrt{\sigma^2} = \sqrt{var(Y)} = \sqrt{38.99}$, $\mu = 14.7$

$$\boxed{\therefore \quad I = (14.7 - 2\sqrt{38.99}, 14.7 + 2\sqrt{38.99})}$$

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4a

$-1 \le p \le 1 \Rightarrow p^2 \in [0,1]$

proof:

$x^2 \ge 0 \ \forall \ x \in \mathbb{R}$ as a known lemma and because the bounds of p are -1 and $+1$, p^2 can be at most 1, $(1)^2 = 1$ and $c^2 < c \ni |c| < 1$ so we are compactly bounded between 0 and 1 inclusive. $\square$

4b. Show that $E[U^2] \ge 0 \ni U = aX + bY \ \forall \ a,b \in \mathbb{R}$

proof:

$V(U) = E(U^2) - (E[U])^2 \ge 0$ by definition of variance

$\therefore \ E(U^2) = V(U) + (E[U])^2$

and $V(U) \ge 0$, $(E[U])^2 \ge 0$, $\therefore \ V(U) + (E[U])^2 \ge 0$

and by the equality, $E(U^2) \ge 0$

by direct proof

$\square$

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4c. show that $(E[XY])^2 \leq E[X^2] \cdot E[Y^2]$

Suppose $E[(aX-Y)^2] \ni a \in \mathbb{R}$

Then $E[(aX-Y)^2] \geq 0$ by part (b) $E[u^2] \geq 0$

$= E[(aX)^2 - 2aXY + Y^2] \geq 0$

$= E[(aX)^2] + E[Y^2] - 2E[aXY]$

$= a^2 E[X^2] - 2aE[XY] + E[Y^2]$

using $ax^2 + bx^1 + cx^0 = 0$

denote $a = E[X^2]$
$b = E[XY] \cdot -2$ solved in terms of a
$c = E[Y^2]$

$\because E[u^2] \geq$ it has 0 or 1 real roots
$\therefore b^2 - 4ac \leq 0$ by the quadratic formula

$(-2E[XY])^2 - 4(E[X^2])(E[Y^2]) \leq 0$

$\Rightarrow 4E[XY]^2 \leq 4E[X^2]E[Y^2]$

$\Rightarrow E[XY]^2 \cancel{\geq} E[X^2]E[Y^2]$
$\leq$

$\square$

4d. $\operatorname{cov}[X,Y] = E[(X-\mu_X)(Y-\mu_Y)]$

$V[X] = E[(X-\mu_X)^2]$

$V[Y] = E[(Y-\mu_Y)^2]$

4e. Let $u = (X-\mu_X)$

$v = (Y-\mu_Y)$

from part (c): ~~E[XY~~

$$E[uv]^2 \le E[u^2]E[v^2]$$

$$\Rightarrow E[uv]^2 \le V[X]V[Y]$$

$$\Rightarrow |E[uv]| \le \sqrt{\sigma_x^2 \sigma_y^2} = \sqrt{\sigma_x^2}\cdot\sqrt{\sigma_y^2} = \sigma_x \sigma_y$$

$$\Rightarrow |E[(X-\mu_X)(Y-\mu_Y)]| \le \sigma_x \sigma_y$$

$$\Rightarrow |\operatorname{cov}[X,Y]| \le \sigma_x \sigma_y$$

$$\Rightarrow \frac{|\operatorname{cov}[X,Y]|}{\sigma_x \sigma_y} \le 1$$

$$\Rightarrow |\rho| \le 1$$

$$\therefore -1 \le \rho \le 1 \quad \square$$

I'm running out of colors D: