Math 325 – Homework 07
Due (via upload to Canvas) Monday, November 1, 2021 at 6 PM

1. Suppose the continuous random variable X has pdf

$$f(x) = \frac{1}{2}e^{-|x|},\ x \in \mathbb{R}.$$

   (a) Derive the moment-generating function for X.

Solution: Recall $m(t) := E(e^{tx})$. Thus,

$$\begin{aligned}
m(t) &= \int_{-\infty}^{\infty} e^{tx}\left(\frac{1}{2}e^{-|x|}\right)\,dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{tx}e^{-|x|}\,dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{(tx-|x|)}\,dx \\
&= \frac{1}{2}\int_{-\infty}^{0} e^{x(t+1)}\,dx + \frac{1}{2}\int_{0}^{\infty} e^{x(t-1)}\,dx
\end{aligned}$$

Now, let $u = x(t+1)$ in the first integral. Then, $du = (t+1)\,dx$. Similarly, let $u = x(t-1)$ in the second integral. Then, $du = (t-1)\,dx$. Then we have

$$m(t) = \frac{1}{2(t+1)}\lim_{b\to-\infty}\left(e^{x(t+1)}\right)\Big|_b^0 + \frac{1}{2(t-1)}\lim_{c\to\infty}\left(e^{x(t-1)}\right)\Big|_0^c$$

Note the first integral will only converge if $t+1>0$ and the second if $t+1<0$. For $|t|<1$,

$$\begin{aligned}
m(t) &= \frac{1}{2(t+1)}[1-0] + \frac{1}{2(t-1)}[0-1] \\
&= \frac{2t-2-(2t+2)}{(2t+2)(2t-2)} \\
&= \frac{1}{1-t^2}
\end{aligned}$$

Thus, the moment-generating function for X is $m(t) = \dfrac{1}{1-t^2}$ for $|t| \le 1$.

(b) Using the moment-generating function, find $E(X)$ and $V(X)$.

Solution: First, we compute $m'(t)$ and $m''(t)$.

$$m'(t) = \frac{2t}{(1-t^2)^2} \qquad m''(t) = \frac{8t^2}{(1-t^2)^3} + \frac{2}{(1-t^2)^2}.$$

Then, $E(X) = m'(0) = 0$. Further, $V(Y) = E(X^2) - (E(X))^2 = m''(0) - (m'(0))^2 = 2 - 0 = 2$.

2. If $E(X) = 17$ and $E(X^2) = 298$, use Chebyshev's inequality to determine a lower bound for $P(10 < X < 24)$.

Solution: As $\mu = 17$, we can see we are asked to estimate $P(|X-\mu| < 7)$. Using Chebyshev's, $P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$. To use Chebyshev's, we need the standard deviation and variance. Note $\sigma^2 = 298 - (17)^2 = 9$ and $\sigma = 3$. Moreover $7 = k \cdot 3$, or $k = 7/3$. Then

$$P(|X-\mu| < 7) \geq 1 - \frac{1}{(7/3)^2} = 1 - \frac{9}{49} = \frac{40}{49} \approx 0.816.$$

3. An urn contains 12 chips – 4 red, 3 black, and 5 white. A sample of size 3 is to be drawn without replacement. Let X denote the number of white chips in the sample; Y, the number of red.

(a) Determine the joint pdf f of X and Y.

Solution:
For each (x, y), $f(x,y) = \frac{\binom{4}{y}\cdot\binom{5}{x}\cdot\binom{3}{3-x-y}}{\binom{12}{3}}$. Hence, the pdf, $f(x,y)$ is

$X \backslash Y$	0	1	2	3
0	$\frac{1}{220}$	$\frac{12}{220}$	$\frac{18}{220}$	$\frac{4}{220}$
1	$\frac{15}{220}$	$\frac{60}{220}$	$\frac{30}{220}$	0
2	$\frac{30}{220}$	$\frac{40}{220}$	0	0
3	$\frac{10}{220}$	$\frac{12}{220}$	0	0

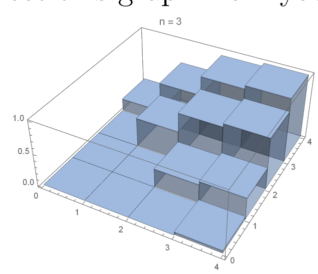
(b) Find $F(1,2)$.

Solution:

$$
\begin{aligned}
F(1,2) &= \sum_{x=0}^{1}\sum_{y=0}^{2} f(x,y) \\
&= \frac{1}{220} + \frac{12}{220} + \frac{18}{220} + \frac{15}{220} + \frac{60}{220} + \frac{30}{220} \\
&= \frac{136}{220}
\end{aligned}
$$

(c) What would the graph of $F(x,y)$ look like?

Solution: For a single variable, the cumulative distribution function for a discrete random variable is a step function. We are now considering the case in which we have two random variables. Hence, we are now in $\mathbb{R}^3$ and the graph of $z = F(x,y)$ will be a collection of rectangular prisms. It will look similar to a volume approximation picture from Calc III. While I don't expect this graph from you, it looks like:



4. Let the joint p.d.f. of X and Y be defined by

$$
f(x,y) = \frac{x+y}{32}, \qquad x = 1,2, \qquad y = 1,2,3,4.
$$

(a) $f_1(x)$, the marginal p.d.f. of X

Solution:

$$
\begin{aligned}
f_1(x) &= \sum_{y=1}^{4} \frac{x+y}{32} = \frac{x+1}{32} + \frac{x+2}{32} + \frac{x+3}{32} + \frac{x+4}{32} \\
&= \frac{4x+10}{32}, \quad x \in \{1,2\}.
\end{aligned}
$$

(b) $f_2(y)$.

Solution:

$$f_2(y) = \sum_{x=1}^{2} \frac{x+y}{32} = \frac{1+y}{32} + \frac{2+y}{32}$$
$$= \frac{3+2y}{32}, \quad y \in \{1,2,3,4\}.$$

(c) $P(X > Y)$

Solution:

$$P(X > Y) = P(X = 2, Y = 1) = f(2,1) = \frac{2+1}{32} = \frac{3}{32}$$

(d) $P(Y = 2X)$

Solution:

$$P(Y = 2X) = f(1,2) + f(2,4) = \frac{3}{32} + \frac{6}{32} = \frac{9}{32}$$

(e) $P(X + Y = 3)$

Solution:

$$P(X + Y = 3) = f(1,2) + f(2,1) = \frac{3}{32} + \frac{3}{32} = \frac{6}{32}$$

(f) $P(X \leq 3 - Y)$

Solution:

$$P(X \leq 3 - Y) = f(1,2) + f(1,1) + f(2,1)$$
$$= \frac{3}{32} + \frac{2}{32} + \frac{3}{32}$$
$$= \frac{8}{32}$$

(g) Are X and Y independent or dependent?

Solution: If X and Y are independent, then $f(x,y) = f_1(x)f_2(y)$, $x \in X$, $y \in Y$. Here,

$$f_1(x)f_2(y) = \frac{4x+10}{32} \cdot \frac{2y+3}{32} = \frac{8xy+12x+20y+30}{1024} \neq f(x,y).$$

Hence, X and Y are dependent.

5. Find the joint pdf associated with two random variables X and Y whose joint cdf is

$$F(x,y) = (1-e^{-\lambda x})(1-e^{-\lambda y}), \quad x > 0, \quad y > 0.$$

Solution: Recall, by definition (and using the support of F), we know

$$F(x,y) = \int_0^x \int_0^y f(s,t)\, dtds.$$

So, determining the PDF is double application of the FTC. That is, $f(x,y) = F_{xy}(x,y)$. Here, $F_x(x,y) = (1-e^{-\lambda y})(\lambda e^{-\lambda x})$. Then,

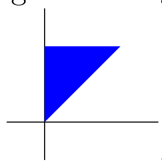
$$F_{xy}(x,y) = (\lambda e^{-\lambda x})(\lambda e^{-\lambda y}) = \lambda^2 e^{-\lambda(x+y)}.$$

Hence, the joint pdf is $f(x,y) = \lambda^2 e^{-\lambda(x+y)}$, $x > 0$, $y > 0$.

6. Consider the joint density function

$$f(x,y) = \begin{cases} 6(1-y) & 0 \leq x \leq y \leq 1 \\ 0 & \text{else} \end{cases}$$

(a) Find the marginal density functions for X and Y.

Solution: Note the region of integration is the upper-left hand portion of the unit square in the first quadrant: . Marginal density function for X:

$$f_1(x) = \int\limits_{y=-\infty}^{\infty} f(x,y)\; dy = \int\limits_{y=x}^{1} (6-6y)\; dy = (6y-3y^2)\Big|_{x}^{1}$$
$$= 3x^2 - 6x + 3, \;\; 0 \le x \le 1$$

Marginal density function for Y:

$$f_2(y) = \int\limits_{x=-\infty}^{\infty} f(x,y)\; dx = \int\limits_{x=0}^{y} (6-6y)\; dx = (6x-6xy)\Big|_{x=0}^{y}$$
$$= 6y - 6y^2, \;\; 0 \le y \le 1$$

(b) Compute $P(Y \le 1/2 | X \le 3/4)$.

Solution:

$$P(Y \le 1/2 | X \le 3/4) = \frac{P(Y \le \frac{1}{2} \cap X \le \frac{3}{4})}{P(X \le \frac{3}{4})} = \frac{\displaystyle\int_{y=0}^{\frac{1}{2}} \int_{x=0}^{y} (6-6y)\; dx\; dy}{\displaystyle\int_{x=0}^{\frac{3}{4}} (3x^2 - 6x + 3)\; dx}$$
$$= \frac{\displaystyle\int_{y=0}^{\frac{1}{2}} \left((6x-6xy)\Big|_{x=0}^{y} \right)\; dy}{(x^3 - 3x^2 + 3x)\Big|_{0}^{\frac{3}{4}}} = \frac{(3y^2 - 2y^3)\Big|_{y=0}^{\frac{1}{2}}}{(x^3 - 3x^2 + 3x)\Big|_{0}^{\frac{3}{4}}}$$
$$= \frac{\frac{1}{2}}{\frac{63}{64}} = \frac{32}{63}$$

(c) Find the conditional density function X given $Y = y$.

Solution:

$$f(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{6(1-y)}{6y(1-y)} = \frac{1}{y}, \quad 0 < y \leq 1$$

(d) Compute $P(Y \leq 1/2 | X = 3/4)$.

Solution: $P(Y \leq 1/2 | X = 3/4) = 0$. The support is $0 \leq x \leq y < 1$, and here, $x = \frac{3}{4} > \frac{1}{2}$. Hence, it is not possible for $Y \leq 1/2$ given that $x = 3/4$.

7. Suppose that X and Y are two random variables jointly distributed over the first quadrant of the xy-plane according to the pdf,

$$f(x,y) = \begin{cases} y^2 e^{-y(x+1)} & 0 \leq x, 0 \leq y \\ 0 & \text{else} \end{cases}$$

(a) Compute $P(Y^2 - X < 1)$.

Solution:

$$\begin{aligned}
P(Y^2 - X < 1) &= P(Y^2 = 1 + x) \\
&= P(-\sqrt{1+x} < Y < \sqrt{1+x}) \\
&= P(0 < Y < \sqrt{1+x})
\end{aligned}$$

$$\begin{aligned}
P(0 < Y < \sqrt{1+x}) &= \int\limits_{y=0}^{\infty} \int\limits_{x=y^2-1}^{\infty} y^2 e^{-y(x+1)} \, dx \, dy \\
&= \int\limits_{y=0}^{\infty} \left(-ye^{-y(x+1)}\right) \Big|_{x=y^2-1}^{\infty} dx \\
&= \int\limits_{y=0}^{\infty} ye^{-y^3} \, dy
\end{aligned}$$

Let $u = y^3$. Then, $du = 3y^2 \, dy$. Thus, we have

$$\begin{aligned}
P(0 < Y < \sqrt{1+x}) &= \frac{1}{3} \int\limits_{u=0}^{\infty} u^{-\frac{1}{3}} e^{-u} \, du \\
&= \frac{1}{3} \Gamma\left(\frac{2}{3}\right) \\
&\approx 0.451373
\end{aligned}$$

(b) Find the two marginal pdfs.

Solution: Marginal pdf for X:

$$f_1(x) = \int_0^\infty y^2 e^{-y(x+1)}\,dy.$$

Integrating by parts, let $u = y^2$ and $dv = e^{-y(x+1)}$. Then, $du = 2y\ dy$ and $v = -\frac{1}{1+x}e^{-y(x+1)}$. Thus,

$$\begin{aligned} f_1(x) &= \left(-\frac{y^2}{x+1}e^{-y(x+1)}\right)\Bigg|_0^\infty + \int_0^\infty \frac{2y}{x+1}e^{-y(x+1)}\,dy \\ &= \frac{2}{x+1}\int_0^\infty ye^{-y(x+1)}\,dy \end{aligned}$$

Integrating by parts again, let $u = y$ and $dv = e^{-y(x+1)}$. Then, $du = dy$ and $v = -\frac{1}{x+1}e^{-y(x+1)}$.

$$\begin{aligned} f_1(x) &= \left(-\frac{2y}{(x+1)^2}e^{-y(x+1)}\right)\Bigg|_0^\infty + \frac{2}{(x+1)^2}\int_0^\infty e^{-y(x+1)}\,dy \\ &= \left(-\frac{2}{(x+1)^3}e^{-y(x+1)}\right)\Bigg|_0^\infty \\ &= \frac{2}{(x+1)^3},\quad x \geq 0. \end{aligned}$$

Marginal pdf for Y:

$$\begin{aligned} f_2(y) &= \int_0^\infty y^2 e^{-y(x+1)}\,dx \\ &= \left(-ye^{-y(x+1)}\right)\Big|_0^\infty \\ &= ye^{-y},\quad y \geq 0 \end{aligned}$$

(c) For what values of y is the conditional density function $f(x|y)$ defined?

Solution: In order for $f(x|y)$ to be defined, we need $f_2(y)$ to be non-zero. Hence, $y > 0$.

(d) What is the conditional density function of X given that $Y = y$.

Solution:

$$\begin{aligned} f(x|y) &= \frac{f(x,y)}{f_2(y)} \\ &= \frac{y^2 e^{-y(x+1)}}{ye^{-y}} \\ &= ye^{-yx-y+y} \\ &= ye^{-yx}, \quad y > 0, x \geq 0 \end{aligned}$$