Math 325 – Homework 04
Due (via upload to Canvas) Tuesday, September 28, 2021 at 6 PM

1. Assume that customers enter a store at the rate of 60 persons per hour.
   (a) What is the probability that during a 5-minute interval no one will enter the store?

Solution: 60 persons per hour=1 person per minute. Thus, for a 5-minute interval,

$$\lambda = \frac{5 \text{ people}}{5 \text{ minutes}}.$$

This is a Poisson distribution. Thus, $p(0) = \frac{5^0 e^{-5}}{0!} = e^{-5} \approx 0.006738$.

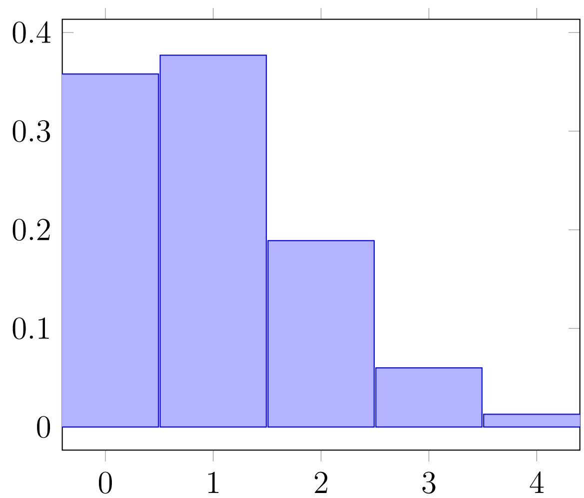
   (b) What time interval is such that the probability is 1/2 that no one will enter the store during that interval?

Solution: We first need to find λ such that $p(0) = \frac{1}{2}$. Thus, we need to solve $\frac{1}{2} = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-\lambda}$ for λ. Taking the natural log of both sides, we have $\ln(\frac{1}{2}) = \ln(e^{-\lambda}) \implies \lambda \approx \frac{0.693147181 \text{ people}}{0.693147181 \text{ minutes}}$.
Hence, the time interval such that the probability is $\frac{1}{2}$ that no one will enter the store during that interval is approximately 0.693147181 minutes $\approx$ 41.59 seconds.

2. *Poisson approximations of binomial distributions*
For binomial distributions with very small probabilities of *success* (p near zero), the histogram of probabilities becomes very skewed towards the *failure* side. An early application of the Poisson distribution was to use it to approximate binomial distributions when p was small. Consider a binomial experiment for $n = 20$, $p = 0.05$

(a) Sketch the histogram associated with this distribution. Clearly state the binomial probabilities for $Y = 0, 1, 2, 3,$ and 4.

Solution: This is a binomial distribution. Hence, $p(y) = \binom{20}{y}(0.05)^y(0.95)^{n-y}$.
Thus, $p(0) \approx 0.358486,\ p(1) \approx 0.377354,\ p(2) \approx 0.188677,\ p(3) \approx 0.0595821,$
$p(4) \approx 0.0133276.$



(b) Now calculate the probabilities for $Y = 0, 1, 2, 3,$ and 4 by using the Poisson approximation with $\lambda = np$ and compare these to the actual probabilities in (a).

Solution: For a Poisson distribution, $p(y) = \frac{\lambda^y e^{-\lambda}}{y!}$. Here, $\lambda = np = (20)(0.05) = 1$.
Thus, $p(0) \approx 0.36788,\ p(1) \approx 0.36788, p(2) \approx 0.183939,\ p(3) \approx 0.0613,\ p(4) \approx 0.015328.$
These probabilities are very similar to the binomial probabilities that we found in part (a). Hence, the Poisson distribution is a good approximation of the binomial distribution for small p.

3. Let Y denote a binomially distributed random variable with probability of success p and n trials.

   (a) Show (via derivation) that the moment-generating function for Y is $m(t) = [(1 - p) + pe^t]^n$

Solution: Need to simplify $m(t) = \sum_y e^{ty} p(y)$, when $p(y) = \binom{n}{y} p^y q^{n-y}$, $y = 0, 1, \ldots, n$.

$$\begin{aligned} m(t) &= \sum_y e^{ty} \binom{n}{y} p^y q^{n-y} \\ &= \sum_y \binom{n}{y} (pe^t)^y q^{n-y} \\ &= (pe^t + q)^n. \end{aligned}$$

   (b) Use the moment-generating function to find $E[Y]$, $E[Y^2]$, and $V[Y]$.

Solution: Here $m'(t) = n(pe^t + q)^{n-1} \cdot pe^t$ and $\mu = m'(0) = n(p+q)^{n-1} p = np$. Then $m''(t) = n(n-1)(pe^t + q)^{n-2} \cdot pe^t \cdot pe^t + n(pe^t + q)^{n-1} \cdot pe^t$ and $m''(t) = n(n-1)pp + np = n^2p^2 - np^2 + np$. Hence $E[Y^2] = n^2p^2 - np^2 + np$ and

$$V[Y] = m''(0) - [m'(0)]^2 = n^2p^2 - np^2 + np - (np)^2 = np(-p+1) = npq.$$

4. (a) If Y has a geometric distribution with probability of success P, show (via derivation) that the moment-generating function for Y is

$$m(t) = \frac{pe^t}{1 - qe^t}.$$

Solution:

$$\begin{aligned} m(t) &= \sum_y e^{ty} p(y) = \sum_{y=1}^{\infty} e^{ty} q^{y-1} p = pe^t \sum_{y=1}^{\infty} e^{t(y-1)} q^{y-1} \\ &= pe^t \sum_{y=1}^{\infty} (e^t q)^{y-1} = pe^t \frac{1}{1 - qe^t}, \end{aligned}$$

provided $|qe^t| < 1$ or $h = \ln(1/q)$.

(b) Use (a) to find $E[Y]$, $E[Y^2]$, and $V[Y]$.

Solution: Have $m(t) = \dfrac{pe^t}{1 - qe^t}$.

$$m'(t) = \frac{pe^t(1 - qe^t) - pe^t(-qe^t)}{(1 - qe^t)^2} = \frac{pe^t}{(1 - qe^t)^2}.$$

Then $\mu = m'(0) = \dfrac{p}{(1-q)^2} = \dfrac{p}{p^2} = \dfrac{1}{p}$.

Note $m'(t) = \dfrac{m(t)}{1 - qe^t}$. Then

$$m''(t) = \frac{m'(t)(1 - qe^t) - m(t)(-qe^t)}{(1 - qe^t)^2}$$

and

$$m''(0) = \frac{m'(0)(1 - q) + m(0)q}{(1 - q)^2} = \frac{(1/p)p + q}{p^2} = \frac{1 + q}{p^2}.$$

Variance is

$$\sigma^2 = m''(0) - [m'(t)]^2 = \frac{1 + q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}.$$

5. If the moment-generating function of Y is

$$m(t) = \frac{2}{5}e^t + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t}$$

find the mean, variance, and p.d.f. of Y.

Solution: Using m,

$$m'(t) = \frac{2}{5}e^t + \frac{2}{5}e^{2t} + \frac{6}{5}e^{3t} \text{ and } m''(t) = \frac{2}{5}e^t + \frac{4}{5}e^{2t} + \frac{18}{5}e^{3t}.$$

So

$$\mu = m'(0) = \frac{2}{5} + \frac{2}{5} + \frac{6}{5} = 2 \text{ and } m''(t) = \frac{24}{5}.$$

Then

$$\sigma^2 = m''(0) - [m'(0)]^2 = \frac{24}{5} - 4 = \frac{4}{5}.$$

To determine the distribution, appeal to the definition of $m(t) = \sum_y e^{ty}p(y)$. We see the admissible values for Y are $\{1, 2, 3\}$ and that $p(1) = p(3) = 2/5$, $p(2) = 1/5$.

6. Find $P(X = 3)$ if the moment-generating function of X is

(a) $m(t) = e^{5.6(e^t - 1)}$

Solution: This is Poisson with $\lambda = 5.6$. Then $p(3) = \dfrac{(5.6)^3 e^{-5.6}}{3!} \approx 0.1082$.

(b) $m(t) = 0.25e^t + 0.35e^{3t} + 0.40e^{5t}$

Solution: Like Problem 4, this is the distribution for the outcomes $Y \in \{1, 3, 5\}$ and $p(3) = 0.35$.

(c) $m(t) = \dfrac{0.35e^t}{1 - 0.65e^t}$

Solution: By Problem 4, this is a geometric distribution with $p = 0.35$ and $q = 0.65$. Then

$$p(3) = (0.65)^2(0.35) \approx 0.147875.$$

(d) $m(t) = (0.65 + 0.35e^t)^{14}$

Solution: This is a Binomial distribution with $n = 14$ and $p = 0.35$. Then

$$p(3) = \binom{14}{3}(0.35)^3(0.65)^{11} \approx 0.13659.$$