Math 325 – Homework 08
Due (via upload to Canvas) Friday, November 12, 2021 at 6 PM

1. Roll a fair six-sided die twice. Let X denote the outcome on the first role, and let Y equal the sum of the two rolls.

Solution: First note

$p(x,y)$	$y=2$	3	4	5	6	7	8	9	10	11	12
$x=1$	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0	0
2	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0
3	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0
4	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0
5	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0
6	0	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36

.

and the marginal distributions are

$$p_1(x) = 1/6, \text{ for all } x,$$

and

$p_2(y)$	$y=2$	3	4	5	6	7	8	9	10	11	12
	1/36	2/36	3/36	4/36	5/36	6/36	5/6	4/36	3/36	2/36	1/36

.

Find

(a) μ_x

Solution:

$$\mu_x = \sum_{x=1}^{6} x p_1(x) = \frac{1}{6} \cdot \frac{6(7)}{2} = \frac{7}{2}.$$

(b) σ_x^2

Solution: We use our computation theorem $\sigma_x^2 = E[X^2] - (\mu_x)^2$.

$$E[X^2] = \sum_{x=1}^{6} x^2 p_1(x) = \frac{1}{6} \cdot \frac{6(7)(13)}{6} = \frac{91}{6}$$

Then

$$\sigma_x^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}.$$

(c) μ_y

Solution:

$$\begin{aligned}
\mu_Y &= \sum_{y=2}^{12} y p_2(y) \\
&= 1(1/36) + 2(2/36) + 3(3/36) + \ldots + 10(3/36) + 11(2/36) + 12(1/36) \\
&= 7
\end{aligned}$$

(d) σ_y^2

Solution:

$$\begin{aligned}
E[Y^2] &= \sum_{y=2}^{12} y^2 p_2(y) \\
&= 1(1/36) + 4(2/36) + 9(3/36) + \ldots + 100(3/36) + 121(2/36) + 144(1/36) \\
&= 329/6
\end{aligned}$$

Then

$$\sigma_y^2 = \frac{329}{6} - (7)^2 = \frac{35}{6}.$$

(e) $\mathrm{Cov}(X, Y)$

Solution: Using the computation theorem, $\mathrm{Cov}(X,Y) = E[XY] - \mu_x\mu_y$:

$$\begin{aligned}
E[XY] &= \sum_{x=1}^{6}\sum_{y=x+1}^{x+6} xy\, p(x,y) \\
&= \frac{1}{36}\sum_{x=1}^{6}\left(x\sum_{y=x+1}^{x+6} y\right) \\
&= \frac{1}{36}\sum_{x=1}^{6} x\,(6x+21) \\
&= \frac{6}{36}\sum_{x=1}^{6} x^2 + \frac{21}{36}\sum_{x=1}^{6} x \\
&= \frac{6}{36}\cdot\frac{6(7)(13)}{6} + \frac{21}{36}\cdot\frac{6(7)}{2} \\
&= 329/12
\end{aligned}$$

Thus, $\mathrm{Cov}(X,Y) = \frac{329}{12} - \left(\frac{7}{2}\cdot 7\right) = \frac{35}{12}$.

(f) ρ

Solution:

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_x\sigma_y} = \frac{35/12}{\left(\sqrt{35/12}\right)\left(\sqrt{35/6}\right)} = \frac{1}{\sqrt{2}}.$$

2. Assume that Y_1, Y_2, and Y_3 are random variables, with

$$E(Y_1) = 2, E(Y_2) = -1, E(Y_3) = 4, \mathrm{Var}(Y_1) = 4, \mathrm{Var}(Y_2) = 6, \mathrm{Var}(Y_3) = 8,$$

$$\mathrm{Cov}(Y_1, Y_2) = -1, \mathrm{Cov}(Y_1, Y_3) = 1, \mathrm{Cov}(Y_2, Y_3) = 0.$$

Find $E(3Y_1 + 4Y_2 - 6Y_3)$ and $\mathrm{Var}(3Y_1 + 4Y_2 - 6Y_3)$.

Solution:

$$E(3Y_1 + 4Y_2 - 6Y_3) = 3E(Y_1) + 4E[Y_2] - 6E[Y_3] = 3(2) + 4(-1) - 6(4) = -22$$

$$\begin{aligned}
\mathrm{Var}(3Y_1 + 4Y_2 - 6Y_3) &= 9V(Y_1) + 16V(Y_2) + 36V(Y_3) \\
&\quad + 2[3(4)\mathrm{Cov}(Y_1, Y_2) + 3(-6)\mathrm{Cov}(Y_1, Y_3) + 4(-6)\mathrm{Cov}(Y_2, Y_3)] \\
&= 9(4) + 16(6) + 36(8) \\
&\quad + 2[3(4)(-1) + 3(-6)(1)) + 4(-6)(0)] \\
&= 360
\end{aligned}$$

3. If Y denotes the number of tosses of the die until you observe each of the six faces, $Y = Y_1 + Y_2 + \ldots + Y_6$ where Y_1 is the trial on which the first face is tossed, Y_2 is the number of additional tosses required to get a face different than the first,..., Y_6 is the number of tosses required to get the last remaining face after all other faces have been observed.

   (a) What is the expected number of tosses required to have rolled all the faces?

Solution: Note that each roll is an independent Bernoulli event. Roll 1 is always a "success". If Y_1 is the number of rolls until the first face,

$$P(Y_1 = 1) = 1.$$

For Face 2, each roll has

$$P(\text{"success"}) = 5/6 \text{ and } P(\text{Face 1 again}) = 1/6.$$

Then Y_2, the number of rolls until the second face, has the geometric distribution

$$p(y_2) = \frac{5}{6}\left(\frac{1}{6}\right)^{y_2-1}, y_2 = 1, 2, \ldots.$$

In this fashion, Y_3, the number of rolls until the second face, has the geometric distribution

$$p(y_3) = \frac{4}{6}\left(\frac{2}{6}\right)^{y_3-1}, y_3 = 1, 2, \ldots,$$

etc. Then, for $Y = Y_1 + \ldots + Y_6$, linearity yields

$$E[Y] = E[Y_1] + E[Y_2] + \ldots + E[Y_6] = 1 + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + 1 = 14.7$$

   (b) Determine the $\text{Cov}(Y_i, Y_j)$ when $i \neq j$.

Solution: As mentioned above, each roll is independent and each Y_i is independent (geometric distributions are memoryless processes). Thus, $\text{Cov}(Y_i, Y_j) = 0$ when $i \neq j$.

   (c) Find $\text{Var}(Y)$.

Solution: By part b, $\text{Cov}(Y_i, Y_j) = 0$, $i \neq j$. Also, for each Y_i, $i > 1$ we have a geometric distribution and the $\sigma_i^2 = q_i/p_i^2$. Thus,

$$
\begin{aligned}
V(Y) &= V(Y_1) + V(Y_2) + V(Y_3) + V(Y_4) + V(Y_5) + V(Y_6) \\
&= 0 + \frac{1 - \frac{5}{6}}{\left(\frac{5}{6}\right)^2} + \frac{1 - \frac{4}{6}}{\left(\frac{4}{6}\right)^2} + \frac{1 - \frac{1}{2}}{\left(\frac{1}{4}\right)^2} + \frac{1 - \frac{1}{3}}{\left(\frac{1}{9}\right)^2} \\
&= 0 + 0.24 + 0.75 + 2 + 6 + 30 \\
&= 38.99.
\end{aligned}
$$

(d) Give an interval that will contain Y with probability at least 75%.

Solution: We need to use Chebyshev's Theorem, $P(|Y - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$.
From part a, we have $\mu = 14.7$ and from part b we have $\sigma = \sqrt{38.99} \approx 6.244$. Thus, $P(|Y - 14.7| < k(6.244)) \geq 0.75$.
This implies $1 - \frac{1}{k^2} = 0.75 \implies k = 2$.
Hence, we have $|Y - 14.7| < 2(6.244) = 12.488 \implies 2.212 < Y < 27.188.$. But the lower bound given by Chebyshev's is nonsense as it take a minimum of 6 rolls to get six faces. So

$$6 \leq Y \leq 27.19.$$

4. An important fact about correlation is the following theorem: If X and Y are jointly distributed random variables with finite variances, then

$$-1 \leq \rho \leq 1.$$

(a) If $-1 \leq \rho \leq 1$, what can you say about ρ^2?

Solution: $\rho^2 \leq 1$.

(b) Let $U = aX + bY$, any linear combination of X and Y. Explain why $E[U^2] \geq 0$ always.

Solution: By definition, $E[U^2] = \iint_S U^2 f(x,y)dA$. As the integrand is a non-negative function, so must be the value of the integral.

(c) Use the expectation $E[(aX - Y)^2]$, where a is any real number, to show

$$(E[XY])^2 \leq E[X^2]E[Y^2].$$

(This is the hardest part.)

Solution: Note

$$E[(aX - Y)^2] = a^2 E[X^2] - 2aE[XY] + E[Y^2].$$

Note that this results in a quadratic function in the parameter a; $Q9a) = E[X^2]a^2 - 2E[XY]a + E[Y^2]$. Moreover, by (b), the polynomial $Q(a) \geq 0$ for all values of a. In other words, the quadratic has either one (repeated) root or no real roots. Equivalently, the discriminant, must be less than or equal to zero.

$$(-2E[XY])^2 - 4E[X^2]E[Y^2] \leq 0.$$

A little rearrangement yields $(E[XY])^2 \leq E[X^2]E[Y^2]$.

(d) State the *definitions* of $\mathrm{Cov}(X,Y)$, $V(X)$, and $V(Y)$.

Solution:

$$\mathrm{Cov}(X,Y) = \iint_{\mathbb{R}^2} (x - \mu_X)(y - \mu_y) f(x,y)dA = E[(X - \mu_X)(Y - \mu_Y)]$$

and

$$V(X) = \iint_{\mathbb{R}^2} (x - \mu_X)^2 f(x,y)dA = E[(X - \mu_X)^2] \text{ and } V(Y) = E[(Y - \mu_Y)^2].$$

(e) Use the above and prove the theorem.

Solution: Using our inequality in (c), we can consider the linear change in variables $X \to X - \mu_X$ and $Y \to Y - \mu_y$ (computing moments about the means). Then

$$(E[(X - \mu_X)(Y - \mu_Y)])^2 \leq E[(X - \mu_X)^2]E[(Y - \mu_Y)^2].$$

Using (d), we recognize this moments as the definitions of of $\text{Cov}(X, Y)$, $V(X)$, and $V(Y)$;

$$[\text{Cov}(X, Y)]^2 \leq V(X)V(Y), \text{ or } \frac{[\text{Cov}(X, Y)]^2}{V(X)V(Y)} \leq 1.$$

By the definition of ρ, this last inequality is $\rho^2 \leq 1$.