## Math 325 – Homework 01
## Due (via upload to Canvas) Friday, September 3, 2021 at 6 PM

1. Which of the following relationships are true? Justify your answer.

   (a) $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$

**Solution:** True. This is simply the distributive law.

   (b) $(A \cup B) = ((A \cap \overline{B}) \cup B)$

**Solution:** True. $((A \cap \overline{B}) \cup B) = (A \cup B) \cap (\overline{B} \cup B) = (A \cup B) \cap U = A \cup B$.

   (c) $\overline{A} \cup B = A \cup B$

**Solution:** False. If true, then $(\overline{A} \cup B) \cap (A \cup B) = A \cup B$. But by the distributive law, $(\overline{A} \cup B) \cap (A \cup B) = (\overline{A} \cap A) \cup B = \emptyset$.

   (d) $(\overline{A \cup B}) \cap C = \overline{A} \cap \overline{B} \cap \overline{C}$

**Solution:** False. By DeMorgan's, $\overline{A \cup B} = \overline{A} \cap \overline{B}$. Then $(\overline{A \cup B}) \cap C = \overline{A} \cap \overline{B} \cap C$.

   (e) $(A \cap B) \cap (\overline{B} \cap C) = \emptyset$

**Solution:** True. $(A \cap B) \cap (\overline{B} \cap C) = A \cap (B \cap \overline{B}) \cap C = A \cap \emptyset \cap C$.

2. Let $A$, $B$, and $C$ be three events associated with an experiment. Express the following verbal statements in set notation.

   (a) At least one of the three events occurs.

**Solution:** $A$ or $B$ or $C$ yields $A \cup B \cup C$

   (b) Exactly one of the events occurs

**Solution:** For example, $A$ alone yields $A$ and not $B$ and not $C$. i.e. $A \cap \overline{B} \cap \overline{C}$. All together, $A$ alone or $B$ alone or $C$ alone is $(A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C)$.

   (c) Events $A$ and $C$ occur, but not $B$

**Solution:** $(A \cap C) \cap \overline{B}$

   (d) Not more than two of the events occur simultaneously.

**Solution:** There is an easy way and a hard way to do this. The hard would be to union the answers in (a) with the sets when exactly two occur. The easy way is to recognize that this is the complement of all three events occuring simultaneously. That is $\overline{A \cap B \cap C}$

3. (a) Use set theory (i.e. decompose the event into mutually exclusive sets and use Axiom 3) to prove that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

**Solution:** Recall $A = A \cap U = A \cap (B \cup \overline{B}) = (A \cap B) \cup (A \cap \overline{B})$. Since the rightmost sets are clearly disjoint, $P(A) = P(A \cap B) + P(A \cap \overline{B})$, (EQN 1). Similarly $P(B) = P(A \cap B) + P(\overline{A} \cap B)$, (EQN 2). Adding these equations, we have

$$P(A) + P(B) = P(A \cap B) + P(A \cap \overline{B}) + P(A \cap B) + P(\overline{A} \cap B).$$

Note the last three sets in this expression are disjoint. Moreover,

$$(A \cap \overline{B}) \cup (A \cap B) \cup (\overline{A} \cap B) = A \cup B$$

. Hence, $P(A) + P(B) = P(A \cap B) + P(A \cup B)$.

(b) Use set theory to prove that if $A \subset B$, then $P(A) \leq P(B)$.

**Solution:** Note $B = (B \cap A) \cup (B \cap \overline{A})$. Hence $P(B) = P(B \cap A) + P(B \cap \overline{A})$. But, since $A \subset B$, we have $A \cap B = A$. Thus $P(B) = P(A) + P(B \cap \overline{A})$ and $P(B) \geq P(A)$.

(c) Use (a) and show that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

**Solution:** Applying (a) twice,

$$\begin{aligned} P(A \cup B \cup C) &= P(A \cup B) + P(C) - P((A \cup B) \cap C) \\ &= P(A) + P(B) - P(A \cap B) + P(C) - P((A \cup B) \cap C). \end{aligned}$$

But $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. So applying (a) again,

$$\begin{aligned} P((A \cup B) \cap C) &= P((A \cap C) \cup (B \cap C)) \\ &= P(A \cap C) + P(B \cap C) - P((A \cap C) \cap (B \cap C)) \\ &= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C) \end{aligned}$$

Combing these two identities completes the proof.

4. Let $A$ and $B$ be events in a sample space with $P(A) = x$, $P(B) = y$, and $P(A \cap B) = z$. Express each of the following probabilities in terms of $x$, $y$ and $z$.

   (a) $P(\bar{A} \cap \bar{B})$

**Solution:** Note $U = (A \cup B) \cup \overline{A \cup B}$. By DeMorgan's laws, $U = (A \cup B) \cup (\overline{A} \cap \overline{B})$. Thus $1 = P(A \cup B) + P(\overline{A} \cap \overline{B})$ and $P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$. Then, by Problem 3(a), $P(\bar{A} \cap \bar{B}) = 1 - [P(A) + P(B) - P(A \cap B)] = 1 - x - y + z$.

   (b) $P(A \cup \bar{B})$

**Solution:** Note, $A \cup \bar{B} = \bar{B} \cup (A \cap B)$. So $P(A \cup \bar{B}) = P(\bar{B}) + P(A \cap B)$. Using $P(\bar{B}) = 1 - P(B)$, $P(A \cup \bar{B}) = 1 - y + z$.

   (c) Is it possible for $x = 0.17$, $y = 0.13$ and $z = 0.30$. Briefly explain your answer.

**Solution:** Absolutely not. As $A \cap B \subset A$, we know $P(A \cap B) \leq P(A)$. Clearly, $0.30 \not\leq 0.17$.

5. A combination lock has 8 dials with five positions, say $a$, $b$, $c$, $d$, and $e$.

   (a) How many different combinations are there to this lock?

**Solution:** $5^8$

   (b) How many combinations are available if no two adjacent dials are set in the same position?

**Solution:** After the first, only 4 choices for each dial. Thus $5 \cdot 4^7$.

   (c) How many combinations are available if only positions $a$ and $b$ are used, and these are used equally often?

**Solution:** Must use four $a$'s and four $b$'s. We need to count the number of ways to order this set. There are 8! ways to order the 8 elements of the set. But for the four $a$'s, we don't care what order they are in individually and there are 4! ways to order them. So 8!/4!. Similarly, there are 4! ways to order the $b$'s. Hence, in total, there are $\frac{8!}{4!4!} = \binom{8}{4}$.

   (d) How many combinations are available if you use four $a$, three $b$, and one $c$ in the combination?

**Solution:** Using same logic in (b), this yields the multi-nomial coefficient, $\binom{8}{4 \cdot 3 \cdot 1}$.

   (e) How many combinations are available if you use four of one type, three of a second, and one other?

**Solution:** In (d), we counted how many ways to do this for a fixed set of three letters. But there are $5 \cdot 4 \cdot 3$ different ways to choose a set of three letters. Hence, $5 \cdot 4 \cdot 3\binom{8}{4 \cdot 3 \cdot 1}$ different combinations.

6. A shipment of 1500 washers contains 400 defective and 1100 nondefective items. Two-hundred washers are chosen at random (without replacement) and classified.

   (a) What is the probability that exactly 75 defective items are found?

**Solution:** In choosing the 75 defective in the set of 200, there is $\binom{400}{75}$ ways to do so. In choosing the rest of the 200, there is $\binom{1100}{125}$. Hence there are $\binom{400}{75}\binom{1100}{125}$ ways to choose a set with exactly 75 defective. In total, there are $\binom{1500}{200}$ different ways to choose any set of 200. So

$$P(\text{exactly 75 defective}) = \frac{\binom{400}{75}\binom{1100}{125}}{\binom{1500}{200}} \approx 0.0000936516$$

   (b) What is the probability that at least 3 defective items are found?

**Solution:** Note that $P(\text{at least 3 defective}) = 1 - P(\text{no more than 2 defective})$. So

$$P(\text{at least 3}) = 1 - [P(\text{exactly zero}) + P(\text{exactly one}) + P(\text{exactly two})].$$

Using the counting method from (a),

$$P(\text{at least 3}) = 1 - \left[\frac{\binom{1100}{200}}{\binom{1500}{200}} + \frac{\binom{400}{1}\binom{1100}{199}}{\binom{1500}{200}} + \frac{\binom{400}{2}\binom{1100}{198}}{\binom{1500}{200}}\right].$$

(A probability so close to 100% that Mathematica is telling me the numerical approximation of above is 1.)