Math 325 – Homework 03
Due (via upload to Canvas) Tuesday, September 21, 2021 at 6 PM

1. On a 15-question multiple-choice test, there are four possible answers, of which one is correct. Suppose that a student guesses on each question. Let Y equal the number of correct answers.

   (a) How is Y distributed?

Solution: Binomial distribution with $n = 15$, $p = 1/4$, and

$$p(y) = \binom{15}{y}\left(\frac{1}{4}\right)^{y}\left(\frac{3}{4}\right)^{15-y}, \; y = 0, 1, 2, \ldots, 15.$$

   (b) Give the values of $E[Y]$ and $V[Y]$.

Solution:

$$\mu = np = 15(1/4) = 15/4 = 3.75, \quad \sigma^2 = npq = (3.75)(3/4) = 45/16 = 2.8125$$

   (c) Find $P(Y \leq 5)$.

Solution:

$$P(Y \leq 5) = \sum_{y=0}^{5} p(y) \approx 0.8516319.$$

   (d) If it takes a score of 50% to pass the exam. What is the probability that the student passes.

Solution: Need a score of 8 or better.

$$P(Y \geq 8) = 1 - P(Y \leq 7) \approx 0.01729984.$$

2. A random variable Y has a binomial distribution with mean 84 and variance 36. Find $P(Y \geq 50)$.

Solution: Have $\mu = np = 84$ and $\sigma^2 = npq = 36$. So $q = 36/84 = 3/7$, $p = 1 - q = 4/7$, $n = \mu/p = 147$, and

$$p(y) = \binom{147}{y}\left(\frac{4}{7}\right)^y\left(\frac{3}{7}\right)^{147-y}, \ y = 0, 1, 2, \ldots, 147.$$

$$P(Y \geq 50) = 1 - P(Y \leq 49) = 1 - \sum_{y=0}^{49} p(y) \approx 1.$$

3. From past experience it is known that 4% of accounts in a large accounting population are in error.

   (a) What is the probability that the first account in error is found on the 5th try?

Solution: Here a "success" is finding an account in error and $P(S) = 0.04$ and the distribution is the geometric distribution is

$$p(y) = (0.96)^{y-1}(0.04).$$

Then $P(Y = 5) = (0.96)^4(0.04) = 0.0339739$.

   (b) What is the probability that the first account in error occurs in the first five accounts audited?

Solution: $P(Y \leq 5) = \sum_{y=1}^{5} p(y) = 0.184627$.

4. According to a representative for an automobile manufacturer, the company uses 5000 lock-and-key combinations on its vehicles. Suppose that you find a key for one of these cars.

   (a) Give the expected number of vehicles that you would have to check to find one that your key will fit.

Solution: This is a geometric distribution with $p = 1/5000$ and the mean of the geometric distribution is given by $\mu = 1/p = 5000$. Expect to need to check 5000 vehicles.

(b) Give the probability that exactly 2000 vehicles would have to be checked to find one that your key fit.

Solution:
$$P(Y = 2000) = (4999/5000)^{1999}(1/5000) \approx 0.0001340855.$$

(c) Give the probability that at most 2000 vehicles would have to be checked to find one that your key fit.

Solution:
$$P(Y \leq 2000) \approx 0.3297068.$$

(d) Give the probability that you would have to check at least 3000 vehicles to find one that your key fit.

Solution:
$$P(Y > 3000) = 1 - P(Y \leq 3000) \approx 0.5486689.$$

5. In an NBA championship series, the team which wins four games out of seven will be the winner. Suppose that team A has probability 0.54 of winning over the team B, and the teams A and B face each other in the championship games.

   (a) What is the probability that team A will win the series in six games?

Solution: This is a negative binomial distribution where success is Team A winning with probability $p = 0.54$ and $r = 4$ with

$$p_4(y) = \binom{y-1}{3}(0.54)^4(0.46)^{y-4}, \; r = 4, 5, 6 \text{ or } 7.$$

Then "win in 6" is $P(Y = 6) = \binom{5}{3}(0.54)^4(0.46)^2 \approx 0.179925$.

   (b) What is the probability that team A will win the series?

Solution:

$$\begin{aligned} P(\text{A wins series}) &= P(Y \geq 4) \\ &= P(Y = 4) + P(Y = 5) + P(Y = 6) + P(Y = 7) \\ &\approx 0.586942. \end{aligned}$$

6. A pediatrician wishes to recruit 5 couples, each of whom is expecting their first child, to participate in a new childbirth regimen. She anticipates that 20% of all couples she asks will agree. What is the probability that 15 couples must be asked before 5 are found who agree to participate?

Solution: We need 5 success in 15 trials. We can interpret this as 5 wins in a 15 tries, i.e. a negative binomial probability distribution with $p - 0.20$ and $r = 5$

$$p_5(15) = \binom{14}{4}(0.20)^5(0.80)^{10} \approx 0.03439414.$$