Math 325 - Homework 09

Matthew Wilder

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1. Let Y_1 and Y_2 be discrete random variables with joint probability function $p(y_1, y_2)$. Prove that $E(Y_1) = E(E(Y_1|Y_2))$

   Proof. By definition of expected values, we know that $E(E(Y_1|Y_2))$ can be rewritten as

   $$\sum_{y_1}\sum_{y_2} p(y_1) \cdot p(y_1, y_2)$$

   And since $p(y_1|y_2) = \frac{p(y_1,y_2)}{p(y_2)}$, we can solve for $p(y_1, y_2)$ to get

   $$p(y_1, y_2) = p(y_2) \cdot p(y_1|y_2)$$

   Substituting into the previous sum we obtain,

   $$\sum_{y_1}\sum_{y_2} p(y_1) \cdot p(y_2) \cdot p(y_1|y_2)$$

   Because $p(y_1)$ is held constant with respect to y_2, we can factor it out of the inner sum to obtain

   $$\sum_{y_1} p(y_1) \sum_{y_2} p(y_2) \cdot p(y_1|y_2)$$

   Using definition of $E(g(Y_1)|Y_2 = y_2) = \sum_{y_1} g(y_1) \cdot p(y_1|y_2)$, we can substitute the inner sum with $E(Y_1|Y_2)$, thus obtaining

   $$\sum_{y_1} p(y_1)E(Y_1|Y_2)$$

   Which, again, by definition of $E(Y_1|Y_2)$, we know that this is equivalent to $(E(Y_1)$

   $$\boxed{\therefore E(Y_1) = E(E(Y_1|Y_2)}$$

   □

2. Suppose that Y has a binomial distribution with parameters n and p but that p varies from day to day according to the beta distribution with parameters α and β. Show that

   (a) $E(Y) = \frac{n\alpha}{\alpha+\beta}$

Proof. We will rewrite $E(Y)$ as $E(E(Y|p))$ and for any given p, Y is a binomial distribution.

Thus, the expected value, $E(Y|p) = np$ by definition of a binomial distribution.

Therefore, we obtain $E(np)$. But, because n is held constant, we can factor it out by linearity to obtain $n \cdot E(p)$.

Finally, because p varies via a beta distribution, we know that $E(p) = \frac{\alpha}{\alpha+\beta}$ by definition. Thus, $nE(p) = n\frac{\alpha}{\alpha+\beta}$

$$\boxed{\therefore E(Y) = \frac{n\alpha}{\alpha+\beta}}$$

□

(b) $Var(Y) = \frac{n\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Proof. By definition, we know that $Var(Y) = E(Var(Y|p)) + Var(E(Y|p))$

Using the $E(Y|p)$ obtained from part (a), we can simplify the expression into

$$Var(Y) = E(Var(Y|p)) + Var(np)$$

And by the definition of variance, the n can be pulled out to obtain

$$Var(Y) = E(Var(Y|p)) + n^2 Var(p)$$

Then, by definition from the beta distribution, we can substitute in for $Var(P)$

$$Var(Y) = E(Var(Y|p)) + \frac{n^2 \cdot \alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Since Y is a binomial distribution, therefore $Var(Y|p) = npq$ by definition. Then

$$\begin{aligned}
E(Var(Y|p)) &= E(npq) && \text{by definition of variance}\\
&= nE(pq) && \text{by linearity of expectation}\\
&= nE(p(1-p)) && \text{by } q = 1-p\\
&= nE(p-p^2) && \text{by distribution}\\
&= nE(p) - nE(p^2) && \text{by linearity of expectation}
\end{aligned}$$

Now we need to compute the value of $E(p^2)$

$$\begin{aligned}
Var(p) &= E(p^2) - [E(p)]^2 && \text{by definition of variance} \\
E(p^2) &= Var(p) + [E(p)]^2 && \text{solve for } E(p^2) \\
&= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} + \left[\frac{\alpha}{\alpha+\beta}\right]^2 && \text{by definitions of Beta distribution} \\
&= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} + \frac{\alpha^2}{(\alpha+\beta)^2} && \text{by distribution of squared term} \\
&= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} + \frac{\alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} && \text{multiply by a fancy 1 } \left(\frac{\alpha+\beta+1}{\alpha+\beta+1}\right) \\
&= \frac{\alpha\beta + \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} && \text{add fractions with same denominator}
\end{aligned}$$

$$\therefore E(p^2) = \frac{\alpha\beta + \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Returning to $E(Var(Y|p)) = nE(p) - nE(p^2)$, we know $E(p)$ is $\frac{\alpha}{\alpha+\beta}$ by definition of a beta distribution. Using the $E(p^2)$ that we just computed and $E(Var(Y|p) = nE(p) - nE(p^2)$ we get

$$E(Var(Y|p) = n\frac{\alpha}{\alpha+\beta} - n\frac{\alpha\beta + \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Recall that $Var(Y) = E(Var(Y|p)) + \frac{n^2\cdot\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$. Substituting in the above value of $E(Var(Y|p))$ we obtain the following:

$$Var(Y) = n\frac{\alpha}{\alpha+\beta} - n\frac{\alpha\beta+\alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} + \frac{n^2\cdot\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= n\left[\frac{\alpha}{\alpha+\beta} - \frac{\alpha\beta+\alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} + \frac{n\cdot\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}\right] \text{ Factor out } n$$

$$= n\left[\frac{\alpha(\alpha+\beta)(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} - \frac{\alpha\beta+\alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} + \frac{n\cdot\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}\right] \text{ Multiply fancy 1}$$

$$= n\left[\frac{\alpha(\alpha+\beta)(\alpha+\beta+1) - \Big(\alpha\beta+\alpha^2(\alpha+\beta+1)\Big) + n\cdot\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}\right] \text{ Combine fractions}$$

$$= n\left[\frac{\Big(\alpha^3+2\alpha^2\beta+\alpha^2+\alpha\beta^2+\alpha\beta\Big) - \Big(\alpha\beta+\big(\alpha^3+\alpha^2\beta+\alpha^2\big)\Big) + n\cdot\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}\right] \text{ Distribute terms}$$

$$= n\left[\frac{\alpha^2\beta+\alpha\beta^2+n\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}\right] \text{ Combine like terms}$$

$$= n\left[\frac{\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}\right] \text{ Factor out } \alpha\beta$$

$$= \frac{n\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)} \text{ Multiply in the } n$$

$$\boxed{\therefore Var(Y) = \frac{n\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}}$$

□

3. Suppose that X and Y have a joint uniform density over the unit square. Find the pdf for their product, that is, find $f_U(u)$, where $U = XY$

Answer. Using the given fact that X and Y are over a unit square, we can define

$$f(x,y) = \begin{cases} 1 & x,y \in [0,1] \\ 0 & elsewhere \end{cases}$$

When $X, Y = (0,0)$, $U = 0$
When $X, Y = (1,0)$, $U = 0$
When $X, Y = (0,1)$, $U = 0$
When $X, Y = (1,1)$, $U = 1$
$\therefore U \in$ [0,1]

By our given, $U = XY$, we can substitute in for U and solve for Y to get a CDF in the following form

$$
\begin{aligned}
F_U(u) &= P(U \le u) && \text{by definition of CDF}\\
&= P(XY \le u) && \text{by substitution } U = XY\\
&= P\Big(Y \le \frac{u}{X}\Big) && \text{solving for } Y\\
&= \Bigg(1 - P\Big(Y \ge \frac{u}{X}\Big)\Bigg) && \text{By definition that probability sums to 1}\\
&= 1 - \int_u^1 \int_{\frac{u}{x}}^1 f(x,y)\,dy\,dx && \text{by definition of the CDF}\\
&= 1 - \int_u^1 \int_{\frac{u}{x}}^1 1\,dy\,dx && \text{since } f(x,y) = 1 \text{ on the support}\\
&= 1 - \int_u^1 \Big(y\Big|_{y=\frac{u}{x}}^1\Big)\,dx && \text{By FTC}\\
&= 1 - \int_u^1 1 - \frac{u}{x}\,dx && \text{By evaluation}\\
&= 1 - \Bigg(x - u \cdot ln|x|\Bigg|_{x=u}^1\Bigg) && \text{By FTC}\\
&= 1 - \Big[\big(1 - u \cdot ln(1)\big) - \big(u - u \cdot ln(u)\big)\Big] && \text{By evaluation}\\
&= 1 - \Big[\big(1 - 0\big) - \big(u - u \cdot ln(u)\big)\Big] && \text{By } ln(1) = 0\\
&= 1 - \Big(1 - u + u \cdot ln(u)\Big) && \text{By simplification}\\
&= u - u \cdot ln(u),\ u > 0 && \text{By simplification}
\end{aligned}
$$

We can now write the piecewise CDF as

$$
F_U(u) = \begin{cases} 0 & u \in (-\infty, 0] \\ u - u \cdot ln(u) & u \in (0, 1] \\ 1 & u \in (1, \infty) \end{cases}
$$

Differentiating the CDF, we can obtain the PDF. The derivative of $u - u \cdot ln(u)$ is a chain rule, $1 - \big(u \cdot \frac{du}{u} ln(u) + \frac{du}{d} u \cdot ln(u)\big) = 1 - \big(u \cdot \frac{1}{u} + 1 \cdot ln(u)\big) = 1 - (1 + ln(u)) = -ln(u)$, and there derivative of constants 1 and 0 are both 0.

$$
\boxed{\therefore f_U(u) = \begin{cases} -ln(u) & u \in (0, 1] \\ 0 & elsewhere \end{cases}}
$$

4. Let $X_1, X_2, ..., X_n$ be independent random variables with Poisson distributions and with parameter λ_i, respectively. Let $U = X_1 + \cdots + X_n$. Show that U has a Poisson distribution and determine U's parameter λ_U

Proof. Let $m(t)$ denote the moment generating function for a Poisson distribution. Then

$$
m(t) = e^{\lambda(e^t - 1)}
$$

We know that $m_U(t)$ will be the product of the the n-many $m_x(t)$ moment generating functions that U consists of, therefore:

$$m_U(t) = \prod_{i=1}^{n} m_{x_i}(t) = m_{x_1}(t) \cdot m_{x_2}(t) \cdots m_{x_n}(t)$$

Each m_{x_i} term is a Poisson distribution by definition, and thus each m_{x_i} has the moment generating function $m(t) = e^{\lambda_i(e^t-1)}$. We can substitute this into the product to obtain the following:

$$m_U(t) = \prod_{i=1}^{n} e^{\lambda_i(e^t-1)}$$

By the laws of exponents, this can be rewritten into the following form,

$$m_U(t) = e^{\sum_{i=1}^{n} \lambda_i(e^t-1)}$$

And the $(e^t - 1)$ term is independent of i, therefore we can factor it out of the sum to obtain

$$m_U(t) = e^{(e^t-1)\sum_{i=1}^{n} \lambda_i}$$

We define λ_U as $\sum_{i=1}^{n} \lambda_i$ and thus the moment generating function of U can be rewritten as

$$m_U(t) = e^{\lambda_U(e^t-1)}$$

Which, letting λ_U be the coefficient in the Poisson moment generating function $m(t) = e^{\lambda(e^t-1)}$, then $\lambda_U = \lambda$ is exactly the form of the Poisson moment generating function.

$$\boxed{\therefore m_U(t) = e^{\lambda_U(e^t-1)}}$$

$\square$

5. Let $X_1, X_2, ..., X_n$ denote a random sample of size n from a distribution which is normal $N(\mu, \sigma)$. Define $U = \sum_{i=1}^{\infty} \left(\frac{X_i-\mu}{\sigma}\right)^2$. Show that U has a χ^2 with n degrees of freedom. (Hint: See Homework 6, Problem 5.)

Proof. Let $Z_i = \frac{X_i-\mu}{\sigma}$, then $U = \sum_{i=1}^{n} Z_i^2$. By Homework 6, Problem 5, we know that Z_i^2 has a gamma distribution with $\alpha = \frac{1}{2}$ and $\beta = 2$.

A gamma distribution with $\alpha = \frac{1}{2}$ and $\beta = 2$ is a chi-squared distribution (χ^2) with 1 degree of freedom by the definitions of Γ and χ^2 distributions.

The moment generating function for Z_i^2 is $(1-2t)^{-\frac{1}{2}}$, (1 degree of freedom). Because U is a sum of n of these distributions, the moment generating function $m_U(t)$ is a product of each component's mgf, therefore

$$m_U(t) = \prod_{i=1}^{n} m_{x_i}(t) = m_{x_1}(t) \cdot m_{x_2}(t) \cdots m_{x_n}(t)$$

And then $m_{x_i}(t)$ is $(1-2t)^{-\frac{1}{2}}$ $\forall\ i \in [1, n]$, so

$$m_U(t) = \prod_{i=1}^{n} (1-2t)^{-\frac{1}{2}}$$

Since every term has the same base of $(1-2t)$, multiplication becomes the sum of their exponents, therefore

$$m_U(t) = (1-2t)^{\sum_{i=1}^{n} -\frac{1}{2}}$$

Using the definition that $\sum_{i=1}^{n} c = nc$ we can rewrite this to become

$$m_U(t) = (1-2t)^{-\frac{n}{2}}$$

A χ^2 distribution with n degrees of freedom has the moment generating function $m(t) = (1-2t)^{-\frac{n}{2}}$. But our computed $m_U(t)$ also has this exact moment generating function. By the uniqueness of the moment generating function, U is a χ^2 distribution with n degrees of freedom.

$$\boxed{\therefore m_U(t) = (1-2t)^{-\frac{n}{2}} = \chi_n^2}$$

$\square$