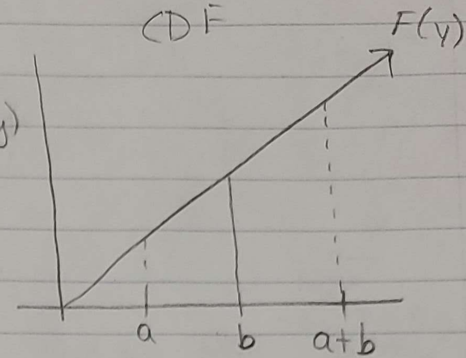
Math 325 - Homework 6 - Matthew Wilder

1.)

exp = gamma $\alpha = 1$ $\quad \lambda e^{-\lambda x}$

$$\exp = \frac{e^{-y/\beta}}{\beta} \qquad CDF = 1 - e^{-\lambda x}$$



denote $a = y_1$, $b = y_2$

$$P(Y > a+b \mid Y > a) = P(Y > b)$$

Proof.

$F(y)$ is a monotone increasing function

$$P(Y > a+b \mid Y > a) = \frac{P(Y>a+b) \cap P(Y>a)}{P(Y>a)} = \frac{P(Y>a+b)}{P(Y>a)}$$

$$= \int_{a+b}^{\infty} f(y) \Big/ \int_{a}^{\infty} f(y) = P(Y > a+b) / P(Y > a)$$

$$= 1 - P(Y < a+b) / 1 - P(Y < a)$$

$$= 1 - [1 - e^{-\lambda(a+b)}] / 1 - [1 - e^{-\lambda a}]$$

$$= e^{-\lambda(a+b)} / e^{-\lambda a}$$

$$= e^{-\lambda a} e^{-\lambda b} / e^{-\lambda a} = e^{-\lambda b} = 1 - F(b)$$

$$= 1 - P(Y < b)$$

$$= P(Y > b)$$

$\square$

b.) To be memoryless it means that the probability of an event occuring is the same regardless of the previous results/events.
Further, as an example, the probability of flipping tails is still 50%, even if you flipped 5 heads in a row prior

$= e \cdot e$

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2.) Labeled at 20.4g

$N(21.37, 0.16)$
$N(\mu, \sigma^2)$

a.) $P(Y > 20.4) = 1 - P(Y < 20.4) = 1 - \int_0^{20.4} \frac{1}{\sqrt{0.16}\sqrt{2\pi}} e^{\left[\frac{-(y-21.37)^2}{2(0.16)^2}\right]} dy$

$= 1 - \frac{1}{\sqrt{0.32\pi}} \int_0^{20.4} \exp\left[\frac{-y^2 + 42.74y - 456.68}{0.0512}\right]$

On second thought, Table 4...

$Z = \frac{Y - \mu}{\sigma} = \frac{20.4 - 21.37}{\sqrt{0.16}} = -2.425$

$P(Z > -2.425) = P(Z < +2.425) = 1 - P(Z > 2.425)$

at $Z = 2.43$, 0.0075

$1 - 0.0075 = 0.9925$

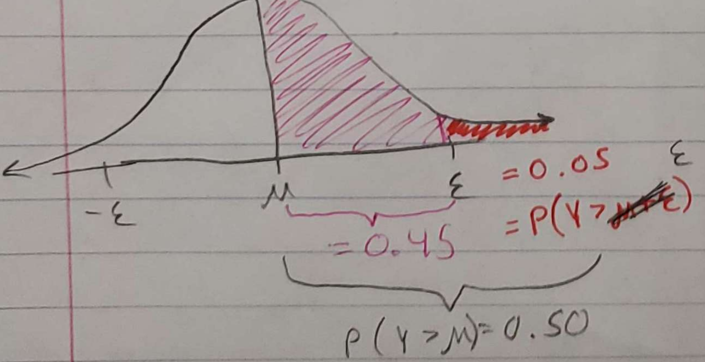
$\boxed{\approx 99.25\%}$

b.) Find $I \ni P(I) = 0.90$

$I \in (\mu - \varepsilon, \mu + \varepsilon)$

$P(Y > \mu) = 0.50$
$P(Y < \mu) = 0.50$

$p(Y > \varepsilon) = \frac{1 - 0.9}{2} = 0.05$



$Z^{-1}(0.05) \approx \pm 1.645$

$\pm 1.645 = \frac{Y - 21.37}{\sqrt{0.16}}$

$Y = \pm 1.645\sqrt{0.16} + 21.37$

$Y = 22.028$, or 20.712

$\boxed{I = (20.712, 22.028)}$

C.) 15 mints selected and weighed

$X =$ mint $\ni$ weight $< 20.91g$

$P(X \leq 2)$

For single mint: $P(W < 20.91)$

$Z = \frac{Y - \mu}{\sigma} = \frac{20.91 - 21.37}{\sqrt{0.16}} = -1.15 \quad P(Z > -1.15) = 1 - P(Z > 1.15)$

$Z(1.15) = 0.1251$ = Probability of weighing less than 20.91g

$1 - 0.1251 = 0.8749$ = Prop of $> 20.91g$

$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$ ← Success = picking a low weight

$= \binom{15}{0} 0.1251^0 \, 0.8749^{15}$

$+ \binom{15}{1} 0.1251^1 \, 0.8749^{14}$

$+ \binom{15}{2} 0.1251^2 \, 0.8749^{13}$

$= 0.71279$ or 71.28% chance that 0, 1, or 2 mints weigh below 20.91g

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3.) $\frac{2}{3}$ calls per minute   denote wait time until 10^{th} call by Y

a.) PDF of Y, aka $f(y)$ Gamma Distribution $\alpha = 10$

$$\beta = \frac{1}{\lambda} = \frac{1}{2/3} = \frac{3}{2}$$

$$f(y) = \left[\frac{1}{\Gamma(10)\left(\frac{3}{2}\right)^{10}}\right] y^{9} e^{2y/3} = \left[\frac{y^{9} e^{2y/3}}{\frac{167403915}{8}}\right] =$$

$$f(y) = \frac{8}{167{,}403{,}915} y^{9} e^{2y/3}$$

b.) $\alpha = 10$ $\beta = \frac{3}{2}$

mean: $\mu = \alpha\beta = 15$

variance $= \alpha\beta^{2} = 22.5$

c.) $$P(Y < 5) = \int_{0}^{5} f(y)\,dy = \frac{8}{167403915} \int_{0}^{5} y^{9} e^{2y/3}\,dy$$

pgamma(5, 10, $\frac{1}{2/3}$) = 0.002356375

$3/2$

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4.) $f(x) = cx^3(1-x)^6 \quad x \in (0,1)$

a.) $\int_0^1 cx^3(1-x)^6 dx = 1$

$= c\int_0^1 x^3(1-x)^6 dx = c\int_0^1 x^3[1x^0 - 6x^1 + 15x^2 - 20x^3 + 15x^4 - 6x^5 + 1x^6]$

$= c\int_0^1 x^3 - 6x^4 + 15x^5 - 20x^6 + 15x^7 - 6x^8 + x^9$

$= c\left[\frac{x^4}{4} - \frac{6x^5}{5} + \frac{15x^6}{6} - \frac{20x^7}{7} + \frac{15x^8}{8} - \frac{6x^9}{9} + \frac{x^{10}}{10}\right]_0^1$

$c\left[\frac{1}{4} - \frac{6}{5} + \frac{15}{6} - \frac{20}{7} + \frac{15}{8} - \frac{6}{9} + \frac{1}{10}\right]$

$c\left[\frac{1}{840}\right] = 1 \Rightarrow C = 840$

b.) $\alpha = 4 \quad \beta = 7$ oh... that would make 4a easier :(

$\mu = \frac{4}{11}$ $\quad \sigma^2 = \frac{4 \cdot 7}{(4+7)^2(4+7+1)} = \frac{28}{121(12)} = \boxed{\frac{7}{363}}$ Variance

c.) $F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 840\left[\frac{x^4}{4} - \frac{6x^5}{5} + \frac{15x^6}{6} - \frac{20x^7}{7} + \frac{15x^8}{8} - \frac{6x^9}{9} + \frac{x^{10}}{10}\right] & x \in (0,1) \\ 1 & x \ge 1 \end{cases}$

So #1 long way either way :/

d.) $P(x > 0.4) = 1 - P(x < 0.4)$

$= 1 - F(0.4) = 1 - 0.6177 = 0.3822806$

38.23%

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5.) Z : normal r.v (standard)

Z^2 : gamma dist

$$f(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(z-\mu)^2}{2\sigma^2}\right] \quad z \in \mathbb{R}$$

but $\mu = 0,\ \sigma = 1$ (standard

a.) $$f(z) = \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-z^2}{2}\right]$$

b.) Let $Y = Z^2$

$$f(z)^2 = \left(\frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(z-\mu)^2}{2\sigma^2}\right]\right)^2 =$$

$$= \frac{1}{\sigma^2 2\pi} \exp\left[\frac{-2(z-\mu)^2}{2\sigma^2}\right] = \frac{1}{\sigma^2 2\pi} e^{\frac{-(z-\mu)^2}{\sigma^2}}$$

$$= \frac{1}{2} \cdot \frac{1}{\sigma^2 \pi} e^{\left(\frac{-(z-\mu)^2}{\sigma^2}\right.}$$

$$\text{CDF} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\left(\frac{-(z-\mu)^2}{\sigma^2}\right.}}{\sigma^2}\, dz$$

c.) $$\text{PDF} = \frac{\partial}{\partial z}(\text{CDF}) \overset{FTC}{=} \frac{1}{2\pi} e^{\left(\frac{-(z-\mu)^2}{\sigma^2}\right)}$$

$\frac{1}{2\pi}$

$2\pi = \Gamma(a)\beta^a$

$= F(1)\beta^1$

$= \beta^1$

$\alpha = 1$

$\beta = 2\pi$