Math 325 – Homework 05
Due (via upload to Canvas) Friday, October 8, 2021 at 6 PM

1. Let the random variable Y be the length of life (in hours) of an electron tube. Suppose that a reasonable probability model for Y is given by

$$f(x) = \frac{1}{1000}e^{-x/1000}.$$

(a) Show that $f(x)$ is a proper probability model.

Solution: Clearly $f(y) > 0$ for all $y \geq 0$. Moreover

$$\int_0^\infty f(x) = \lim_{R\to\infty} \int_0^R \frac{1}{1000}e^{-x/1000} = \lim_{R\to\infty} -e^{-x/1000}\Big|_0^R = \lim_{R\to\infty} \left(-e^{-R/1000} + 1\right) = 1.$$

(b) What is the probability that a tube lasts more than 1500 hours?

Solution:

$$P(X > 150) = \int_{1500}^\infty f(x) = \lim_{R\to\infty} \left(-e^{-R/1000} + e^{-1500/1000}\right) \approx 0.2231.$$

(c) How long do we expect the average tube to last?

Solution:

$$\begin{aligned} E[X] &= \int_0^\infty xf(x) = \int_0^\infty x\frac{1}{1000}e^{-x/1000} \\ &= -xe^{-x/1000}\Big|_0^\infty - \int_0^\infty -e^{-x/1000} && \text{(by parts)} \\ &= 0 + \int_0^\infty e^{-x/1000} && \text{(by L'Hopital's Rule)} \\ &= -1000e^{-x/1000}\Big|_0^\infty \\ &= 1000 \end{aligned}$$

2. Consider the cumulative distribution function for the random variable Y,

$$F(y) = \begin{cases} 0, & y < 1 \\ 1/6, & 1 \leq y < 3 \\ 2/3, & 3 \leq y < 4 \\ 1, & y \geq 4 \end{cases}$$

(a) Is Y a continuous random variable? Justify your answer.

Solution:
No. The CDF is not continuous, hence the PDF is not continous.

(b) Determine the probability distribution function of Y.

Solution:

$$f(1) = \frac{1}{6}, \quad f(3) = \frac{2}{3} - \frac{1}{6} = \frac{1}{2}, \quad f(4) = \frac{1}{3}.$$

(c) Compute the probability $P(Y = 2)$.

Solution: It has to be zero. This is a discrete probability distribution.

(d) What is the moment generating function associated with the random variable Y?

Solution:

$$m(t) = \frac{1}{6}e^{t} + \frac{1}{2}e^{3t} + \frac{1}{3}e^{4t}.$$

3. Let Y be a continuous random variable with pdf f given by

$$f(y) = \begin{cases} ay, & 0 \leq y < 1 \\ a, & 1 \leq y < 2 \\ -ay + 3a, & 2 \leq y \leq 3 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Determine the constant a.

Solution:

$$\begin{aligned} \int_{\mathbb{R}} f(y) &= \int_0^1 ay + \int_1^2 a + \int_2^3 (-ay + 3a) \\ &= \frac{a}{2} + a - \frac{9a}{2} + 9a + \frac{4a}{2} - 6a \\ &= 2a \end{aligned}$$

As the total probability needs to be 1, we see $a = 1/2$.

(b) Determine the distribution function F and sketch its graph.

Solution:

$F(y) = \int\limits_{-\infty}^{y} f(t)\,dt$ Thus, for $0 \leq y < 1$,

$$F(y) = \int\limits_{-\infty}^{y} f(t)\,dt = \int\limits_{0}^{y} \frac{1}{2}t\,dt = \frac{1}{4}t^2\Big|_0^y = \frac{1}{4}y^2.$$

Similarly, for $1 \leq y \leq 2$,

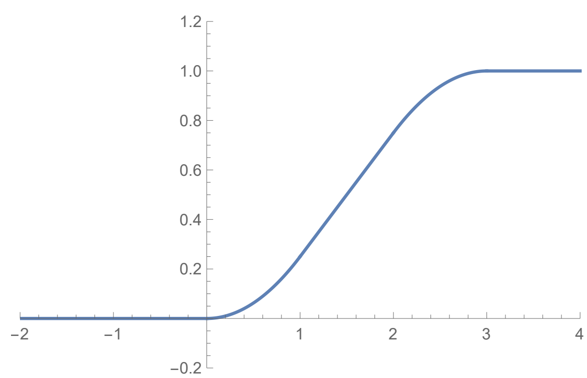
$$F(y) = \int\limits_{0}^{1} \frac{1}{2}t\,dt + \int\limits_{1}^{y} \frac{1}{2}\,dt = \frac{1}{4} + \left(\frac{1}{2}t\right)\Big|_1^y = \frac{1}{2}y - \frac{1}{4}.$$

Finally, for $2 \leq y \leq 3$,

$$\begin{aligned} F(y) &= \int\limits_{0}^{1} \frac{1}{2}t\,dt + \int\limits_{1}^{2} \frac{1}{2}\,dt + \int\limits_{2}^{y} \left(-\frac{1}{2}t + \frac{3}{2}\right)\,dt \\ &= \frac{1}{4} + \frac{1}{2} + \left(-\frac{1}{4}t^2 + \frac{3}{2}t\right)\Big|_2^y \\ &= -\frac{1}{4}y^2 + \frac{3}{2}y - \frac{5}{4}. \end{aligned}$$

Therefore, $F(y) = \begin{cases} 0, & y < 0 \\ \frac{1}{4}y^2, & 0 \leq y < 1 \\ \frac{1}{2}y - \frac{1}{4}, & 1 \leq y < 2 \\ -\frac{1}{4}y^2 + \frac{3}{2}y - \frac{5}{4}, & 2 \leq y < 3 \\ 1, & y > 3 \end{cases}$

Sketch of $z = F(y)$:



(c) If X_1, X_2 and X_3 are three independent observations for Y, what is the probability that exactly one of these three numbers is larger than 1.5?

Solution:

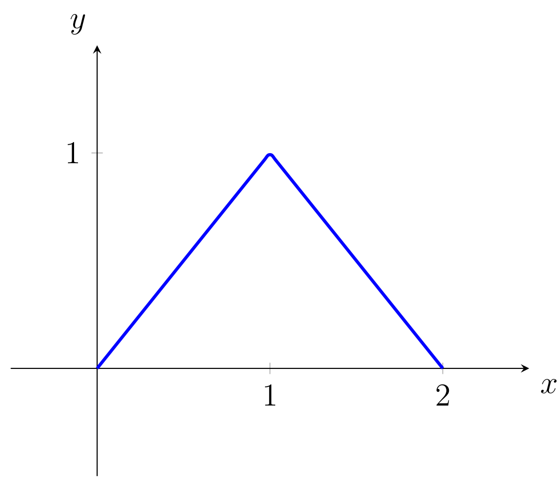
$$\begin{aligned} P(Y > 1.5) &= 1 - P(Y \leq 1.5) \\ &= 1 - F(1.5) \\ &= 1 - \left[\frac{1}{2}(1.5) - \frac{1}{4}\right] \\ &= \frac{1}{2} \end{aligned}$$

Now, we let success=the observation is greater than 1.5, and failure= the observation is less than or equal to 1.5. Then, P(success)=$\frac{1}{2}$ and P(failure)=$\frac{1}{2}$. We can now think of this as a binomial experiment where $n = 3$ and Y =number of observations greater than 1.5. Hence, $P($ exactly one of X_1, X_2, X_3 is larger than 1.5$)$ = $\binom{3}{1}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 = \frac{3}{8}$.

4. Let the p.d.f. of X be $f(x) = 1 - |x - 1|$, $0 \leq x \leq 2$.

(a) Plot $f(x)$.

Solution:



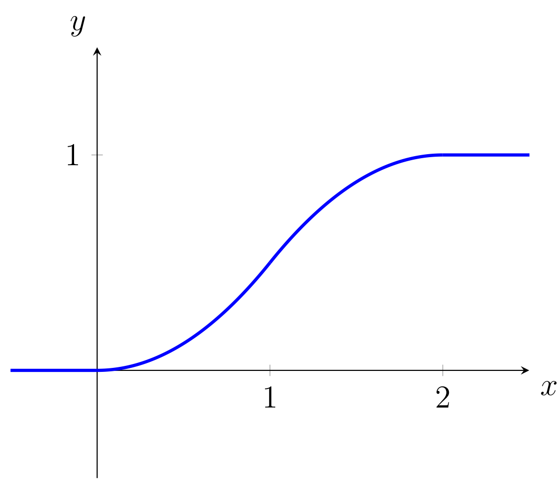
(b) Find the cumulative distribution function for the random variable X and plot F.

Solution: When $0 \leq x \leq 1$,

$$F(x) = P(X \leq x) = \int_0^x t\,dt = x^2/2.$$

When $1 \leq x \leq 2$,

$$F(x) = \int_0^1 t\,dt + \int_1^x (2-t)\,dt = -1 + 2x - \frac{x^2}{2}.$$



(c) Find the value of x that corresponds to the third quartile.

Solution: $F(x) = 0.75$ requires solving $-1 + 2x - x^2/2 = 3/4$ yields $x = 2 \pm 1/\sqrt{2}$. But only the answer in $[0, 2]$ makes sense. Hence $x = 2 - 1/\sqrt{2}$.

5. The p.d.f. of Y is $f(y) = c/y^2$, $1 < y < \infty$, zero elsewhere.

   (a) Find the value of c so that f is a p.d.f.

Solution:

$$\begin{aligned}\int_{\mathbb{R}} f(y) &= \lim_{R\to\infty} \int_1^R c/y^2 \\ &= \lim_{R\to\infty} (-c/R + c) \\ &= c\end{aligned}$$

For a proper probability distribution, we need $c = 1$

   (b) Show that $E[Y]$ does not exist for this distribution.

Solution:

$$E[y] = \int_1^\infty y f(y) = \int_1^\infty \frac{1}{y},$$

which diverges by the p-test. (Or you could compute the integral like we did in (a).)

6. A circle of radius r has area $A = \pi r^2$. If a random circle has a radius that is uniformly distributed on the interval $(0, 2)$, what are the mean an variance of the area of the circle.

Solution: As r is uniformly distributed on $(0,2)$, we have the probability distribution for r of $f(r) = 1/2$, $r \in (0,2)$. Then expected value for the area is

$$E[A(r)] = \int_0^2 \pi r^2 f(r)\,dr = \frac{\pi}{2}\int_0^2 r^2\,dr = \frac{\pi}{2}\cdot\frac{8}{3} = \frac{4\pi}{3}.$$

For variance, we use $\sigma^2 = E[A^2(r)] - (E[A(r)])^2$.

$$E[A^2(r)] = \frac{1}{2}\int_0^2 \pi^2 r^4\,dr = \frac{\pi^2}{2}\cdot\frac{32}{5} = \frac{16\pi^2}{5}.$$

Then

$$\sigma^2 = \frac{16\pi^2}{5} - \frac{16\pi^2}{9} = \frac{64\pi^2}{45}.$$