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1.) $f(x) = \frac{e^{-|x|}}{2} \quad x \in \mathbb{R}$

Split into components

$$f(x) = \begin{cases} \frac{e^{x}}{2} & x \in (-\infty, 0] \\ \frac{e^{-x}}{2} & x \in (0, \infty) \end{cases}$$

$$m(t) = \int_{\mathbb{R}} e^{tx} f(x)\, dx \quad t < 1$$

$$= \int_{-\infty}^{0} \frac{e^{tx} \cdot e^{x}}{2}\, dx + \int_{0}^{\infty} \frac{e^{tx} \cdot e^{-x}}{2}\, dx$$

$$= \frac{1}{2}\int_{-\infty}^{0} e^{x(t+1)}\, dx + \frac{1}{2}\int_{0}^{\infty} e^{x(t-1)}\, dx$$

$$\frac{1}{2}\left[\frac{e^{x(t+1)}}{(t+1)}\Bigg|_{x=-\infty}^{0} + \frac{e^{x(t-1)}}{(t-1)}\Bigg|_{x=0}^{\infty}\right]$$

$$= \frac{1}{2}\left(\left(\lim_{x\to 0} \frac{e^{x(t+1)}}{(t+1)} - \lim_{x\to -\infty} \frac{e^{x(t+1)}}{t+1}\right) + \left(\lim_{x\to\infty} \frac{e^{x(t-1)}}{t-1} - \lim_{x\to 0} \frac{e^{x(t-1)}}{t-1}\right)\right)$$

$e^{0} = 1$ $\quad e^{-\infty} = 0$

$t < 1 \Rightarrow t-1 < 0$
$\Rightarrow e^{-x}$
but
$\lim_{x\to\infty} e^{-x} = 0$

$$= \frac{1}{2}\left[\frac{1}{t+1} - 0 + 0 - \frac{1}{t-1}\right]$$

$$= \frac{1}{2}\left(\frac{1}{t+1} - \frac{1}{t-1}\right)$$

$$= \frac{1}{2}\left(\frac{t-1}{(t+1)(t-1)} - \frac{t+1}{(t-1)(t+1)}\right) = \frac{1}{2}\left(\frac{(t-1)-(t+1)}{t^2-1}\right) = \frac{1}{2}\left(\frac{-2}{t^2-1}\right)$$

$\therefore$

$$\boxed{m(t) = \frac{-1}{t^2-1}, \quad t < 1}$$

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b.

$$m(t) = \frac{-1}{t^2-1} \qquad m'(t) \overset{QR}{=} \frac{(t^2-1)(0)-(-1)(2t)}{(t^2-1)^2} = \frac{2t}{(t^2-1)^2}$$

$$E[X] = m'(0) = 0$$

$$V[X] = m''(0) - m'(0)^2 = m''(0) - 0^2 = m''(0)$$

$$m''(t) \overset{QR}{=} \frac{(t^2-1)^2(2) - (2t)(4t(t^2-1))}{(t^2-1)^4} = \frac{2(t^2-1) - 8t^2}{(t^2-1)^3}$$

$$= \frac{2t^2-2-8t^2}{(t^2-1)^3} = \frac{-6t^2-2}{(t^2-1)^3}$$

$$m''(0) = \frac{-6(0)^2-2}{(0^2-1)^3} = \frac{-2}{-1} = 2 - 0^2 = 2$$

$$\boxed{\begin{aligned} E[X] &= 0 \\ V[X] &= 2 \end{aligned}}$$

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2. $E[X] = 17, \quad E[X^2] = 298$

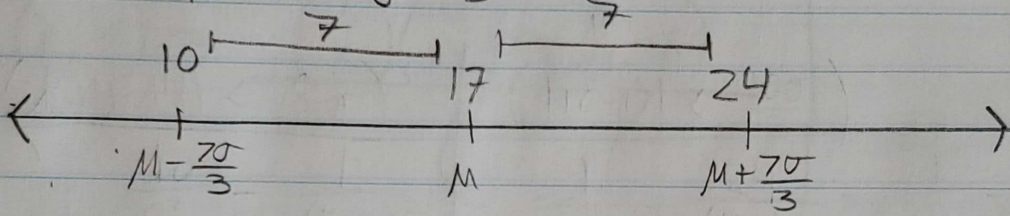
$P(10 < X < 24) = P(X < 24) - P(X < 10)$

$P(|Y - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$

$|24 - 17| = |10 - 17| = 7$

$V[X] = E[X^2] - E[X]^2 = 298 - 17^2 = 9$

$\sigma^2 = 9 \Leftrightarrow \sigma = 3$



$P\left(X < \frac{7\sigma}{3}\right) \geq 1 - \frac{1}{k^2} = 1 - \frac{1}{(7/3)^2} = 1 - \frac{1}{49/9}$

$= 1 - \frac{9}{49} = \frac{49}{49} - \frac{9}{49} = \frac{40}{49} \approx 81.63\%$

$\boxed{P(10 < X < 24) \geq 81.63\%}$

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3. Let R denote "red", B denote "black", W denote "white"

$U = \{4R, 3B, 5W\}$ $|U| = 12$

Choose 3 without replacement

X: White chips Y: Red chips

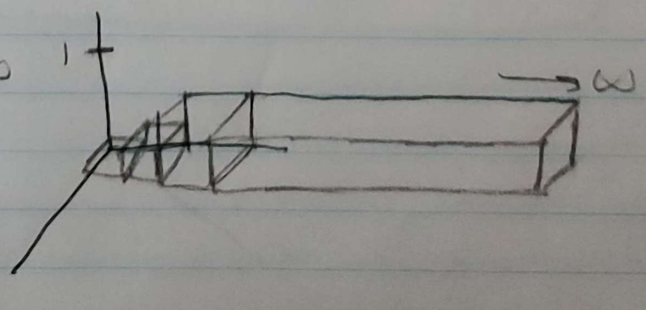
$$\boxed{f(X,Y) = \frac{\binom{5}{X}\binom{4}{Y}\binom{3}{3-X-Y}}{\binom{12}{3}}}$$

3b $F(1,2) = \sum_{X=0}^{1}\sum_{Y=0}^{2} f(X,Y) = \frac{1+12+18+15+60+30}{220} = \frac{136}{220}$

$$\boxed{= \frac{34}{55} = 61.8\overline{18}\%}$$

3c. It would look like a small city of monotone increasing height skyscrapers such that they are 1x1 bases with F(X,y) height. $\forall\ F(x,y) \ni X \geq 3,\ y \geq 3,\ F(x,y) = 1$, a flat plane, and for all other when $X+y > 3$ then its equal to a plane of that partial sum

A 1x∞ slice



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4. a.) $f(x,y) = \frac{x+y}{32}$ $x, y \in \mathbb{N}$, $x \le 2$, $y \le 4$

$$f_1(x) = \sum_y f(x,y) = \frac{x+1+x+2+x+3+x+4}{32} = \boxed{\frac{2x+5}{16}}$$

b.) $$f_2(y) = \sum_x f(x,y) = \frac{1+y+2+y}{32} = \boxed{\frac{3+2y}{32}}$$

c.) $$P(X>Y) = \{(2,1)\} = \frac{2+1}{32} = \boxed{\frac{3}{32}}$$

d. $$P(Y=2X) = \{(1,2),(2,4)\} \quad \frac{1+2+2+4}{32} = \boxed{\frac{9}{32}}$$

e. $$P(X+Y=3) = \{(1,2),(2,1)\} = \frac{1+2+1+2}{32} = \boxed{\frac{3}{16}}$$

f. $$P(X \le 3-Y) = P(X+Y \le 3) = \{(1,1),(1,2),(2,1)\}$$
$$= \frac{1+1+1+2+2+1}{32} = \boxed{\frac{1}{4}}$$

g.) $$f(x,y) \stackrel{?}{=} f_1(x) \cdot f_2(x)$$
$$\frac{2x+5}{16} \cdot \frac{3+2y}{32} = \frac{6x+4xy+15+10y}{512} \neq f(x,y)$$

$$\boxed{X \text{ and } Y \text{ are dependent}}$$

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5. $F(x,y) = (1-e^{-\lambda x})(1-e^{-\lambda y})$ $x > 0, y > 0$

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y} = F_{xy} = \frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$

$$F_x = \frac{\partial F}{\partial x} = \lambda e^{-\lambda x}(1-e^{-\lambda y})$$

$$F_{xy} = \frac{\partial F_x}{\partial y} = \lambda e^{-\lambda x} \cdot (\lambda y \cdot e^{-\lambda y})$$

$$\boxed{= \lambda^2 e^{-\lambda x - \lambda y}}$$

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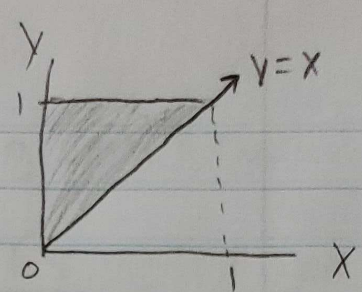

6. $f(x,y) = \begin{cases} 6-6y & 0 \le x \le y \le 1 \\ 0 & \text{else} \end{cases}$

a. $f_1(x) = \int_S f(x,y)\,dy \qquad f_2(y) = \int_S f(x,y)\,dx$

$f_1(x) = \int_{y=x}^{1} 6-6y\,dy = 6y-3y^2\Big|_x^1 = \boxed{3x^2-6x+3,\ x \in [0,1]}$

$f_2(y) = \int_{x=0}^{y} 6-6y\,dx = 6x-6xy\Big|_{x=0}^{x=y} = \boxed{6y-6y^2,\ y \in [0,1]}$

b. $P(Y \le \frac{1}{2} \mid X \le \frac{3}{4}) = \dfrac{P(Y \le \frac{1}{2}, X \le \frac{3}{4})}{P(X \le 3/4)} =$

$P(X \le \frac{3}{4}) = \int_0^{3/4} f_1(x)\,dx = \int_0^{3/4} 3x^2-6x+3\,dx = x^3-3x^2+3x\Big|_{x=0}^{3/4}$

$= (3/4)^3 - 3(3/4)^2 + 3(3/4)$

$= \underline{63/64}$

$P(Y \le \frac{1}{2}, X \le \frac{3}{4}) = \int_{y=0}^{1/2} \int_{x=0}^{y} 6-6y\,dxdy$

$= \int_{y=0}^{1/2} 6x-6xy\Big|_0^y = \int_0^{1/2} 6y-6y^2 = 3y^2-2y^3\Big|_0^{1/2} = 3(\frac{1}{2})^2 - 2(\frac{1}{2})^3$

$= \frac{3}{4} - \frac{1}{4} = \underline{1/2}$

$\dfrac{1/2}{63/64} = \frac{1}{2} \cdot \frac{64}{63} = \frac{32}{63}$ $\boxed{\approx 50.79\%}$

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6\. c.) conditional density function X given $Y = y$:

$$f(x \mid y) = \frac{f(x,y)}{f_2(y)} = \frac{6-6y}{6y-6y^2} = \frac{1-y}{y-y^2}$$

$$= \frac{1-y}{y(1-y)} = \boxed{\frac{1}{y}}$$

d.) $P(Y \leq \frac{1}{2} \mid X = \frac{3}{4})$

Recall: Domain condition: $0 \leq x \leq y \leq 1$

$\because x \leq y, \ y \geq x$

but $x = 3/4 \therefore \quad y \geq 3/4$

and $\forall y \in [0, 1/2], \ y \ngeq 3/4$, (not in the domain)

the probability of such a y occuring is exactly 0, more specifically, it is <u>impossible</u>

$$\boxed{\therefore P(Y \leq \tfrac{1}{2} \mid X = 3/4) = 0}$$