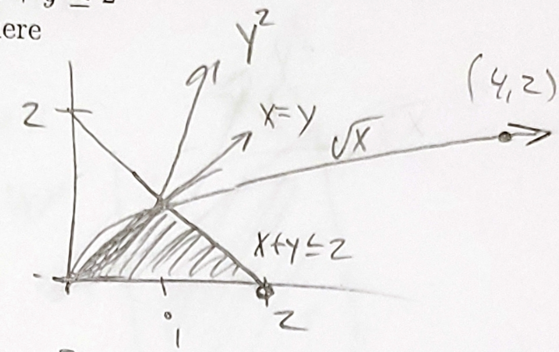
1. (12 points) Suppose that random variables X and Y have the density function

$$f(x,y) = \begin{cases} 6x^2y, & 0 \le x \le y, x+y \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Find the probability of the event $Y^2 \le X$.



$Y^2 \le X$

$|Y| \le \sqrt{X}$

(b) Find the marginal density function of Y.

$$f_2(y) = \int_{x=0}^{2} 6x^2 y \, dy \qquad \left. \frac{6x^2y^2}{2} \right|_0^2$$

$$\left. 3x^2y^2 \right|_{y=0}^{2}$$

$$f_2(y) = 12y^2$$

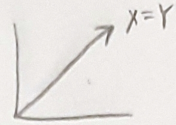
(c) Find $P(X < 1/3 | Y = 3/2)$.

$$\left. \frac{6x^2y^2}{2} \right|_0^y \quad 3x^2y^2$$

$$\frac{\int_{x=0}^{1/3} \int_{y=0}^{x} 6x^2 y \, dy dx}{f_2(3/2)} = \frac{\int_{x=0}^{1/3} 3x^4 dx = \left. \frac{3x^5}{5} \right|_0^{1/3}}{12(3/2)^2}$$

$$= \frac{\frac{3(1/3)^5}{5}}{12 \cdot \frac{9}{4}} \qquad \frac{1/81}{27} \qquad \frac{27}{81} = \boxed{\frac{1}{3}}$$

$u = X - Y \rightarrow$



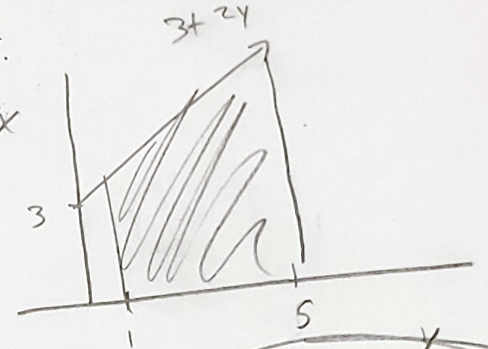
2. (8 points) Let Y be a random variable with a density function given by

$$f(y) = (3/2)y^2, \text{ on support } [-1, 1].$$

$\frac{3}{2}y^2$

Consider the random variable $U = 3 + 2Y$.

(a) Determine the cumulative distribution function for U.



$Y = -1, \quad u = 1$ CDF u/x

$Y = 1 \quad u = 5$

$u \in [1, 5]$

$F(u)$

$$\int_1^5 \int_0^{\frac{u-3}{2}} \frac{3}{2}y^2 \, dy \, du = \int_1^5 \left[\frac{y^3}{2} \Big|_{y=0}^{\frac{u-3}{2}} \right] du$$

$$= \frac{1}{2} \int_1^5 \left(\frac{u-3}{2}\right)^3 du = \frac{1}{16} \int_1^5 (u-3)^3 du$$

$$= \frac{u^4}{4} - \frac{u^3}{3} + \frac{27}{2}u^2 + 27u \Big|_1^5$$

$$G(u) = \begin{cases} 0 & \text{if } u < 1 \\ \frac{u^4}{4} - \frac{u^3}{3} + \frac{27}{2}u^2 + 27u & u \in [1, 5] \\ 1 & \text{if } u > 5 \end{cases}$$

$2y + 3 = u$

$y = \frac{u-3}{2}$

(b) Determine the probability density function of U.

differentiate

$$\left(\frac{u^4}{4} - \frac{u^3}{3} + \frac{27}{2}u^2 + 27u\right) \frac{dG}{du}$$

$(u-3)(u-3)(u-3)$

$(u^2 - 6u + 9)(u-3)$

$u^3 - 3u^2 - 6u^2 + 18u + 9u - 27$

$u^3 - 9u^2 + 27u + 27$

$$g(u) = \begin{cases} u^3 - 9u^2 + 27u + 27 & u \in [1, 5] \\ 0 & \text{e.w.} \end{cases}$$

3. (5 points) Let Y_1, Y_2, and Y_3 denote a random sample of size 3 from a gamma distribution with $\alpha = 7$ and $\beta = 5$.

(a) Find the moment-generating function of $S = Y_1 + Y_2 + Y_3$.

$$m(t) = (1-\beta t)^{-\alpha}$$

$$(1-5t)^{-7}$$

$$m_s(t) = \prod m_i(t) = \prod_{i=1}^{3} (1-5t)^{-7}$$

$$\therefore m_s(t) = (1-5t)^{-21}$$

(b) Determine the probability density function for the random variable S.

Let $S = A$

$$f(A) = \left[\frac{1}{6! \cdot 5^7}\right] A^6 e^{-A/5}$$

$n=3$ $\frac{n!}{(k-1)!(n-k)!} F(y)^{k-1}(1-F(y))^{n-k} f(y)$

4. (5 points) Let X_1, X_2, and X_3 be independent, exponentially distributed random variables with mean $\beta = 3$. Find the density function of $Y = \min\{X_1, X_2, X_3\}$.

Gamma w/ $\alpha = 1$

$\frac{6}{2!}$

$f(x_i) = \frac{1}{3} e^{-x_i/3}$

$3\left[1-F(y)\right]^2 \frac{1}{3} e^{-y/3} = \left[1-F(y)\right]^2 e^{-y/3}$

$F(y) \stackrel{?}{=} \frac{1}{3}\int_0^\infty e^{-x_i/3}\, dx_i = -3e^{-y/3}\Big|_{y=0}^{\infty}$

$0 - [-3(1)]$

$= 3$

$[(1-3)]^2 = 4$

$4e^{-y/3} = \min\{X_1\ X_2\ X_3\}$

5. Let X and Y be uncorrelated random variables and consider $A = X+Y$ and $B = X-Y$.

(a) (3 points) Determine $E(XY) - \mu_X\mu_Y$. $=$ cov ?

$\boxed{0}$

(b) (4 points) Find the Cov(A, B) in terms of the variances of X and Y.

$\text{cov}(A,B) = E[AB] - \mu_A\mu_B - 2\,\text{cov}(XY)$

something in terms of XY which is indep. $\therefore \emptyset$

$\text{cov}(A,B) = E[AB] - \mu_A\mu_B$

(c) (3 points) Determine Var(B) in terms of the variances of X and Y.

$\text{Var}(B) = V[X] - V[Y]$

$\sigma_A \quad \sigma_B$

(d) (3 points) **Extra Credit:** Find an expression for the correlation coefficients between A and B.

$\sqrt{V[A]} = \sqrt{V[X] + V[Y]}$

$\sqrt{V[B]} = \sqrt{V[X] - V[Y]}$

$\sqrt{(X+Y)} \cdot \sqrt{(X-Y)} \quad \sqrt{X^2 - XY - Y^2}$

$$\frac{E[AB] - \mu_A \mu_B}{E\left(\sqrt{X^2 - XY - Y^2}\right)}$$