Math 325 – Homework 09
Due (via upload to Canvas) Monday, November 22, 2021 at 6 PM

1. Let Y_1 and Y_2 be discrete random variables with joint probability function $p(y_1, y_2)$. Prove that $E(Y_1) = E(E(Y_1|Y_2))$.

Solution:

$$\begin{aligned}
E(Y_1) &= \sum_{y_1}\sum_{y_2} y_1 f(y_1, y_2) \\
&= \sum_{y_1}\sum_{y_2} y_1 f(y_1|y_2) f_2(y_2) \\
&= \sum_{y_2}\left[\sum_{y_1} y_1 f(y_1|y_2)\right] f_2(y_2) \\
&= \sum_{y_2} E(Y_1|Y_2) f_2(y_2) \\
&= E(E(Y_1|Y_2)).
\end{aligned}$$

2. Suppose that Y has a binomial distribution with parameters n and p but that p varies from day to day according to the beta distribution with parameters α and β. Show that
   (a) $E(Y) = n\alpha/(\alpha + \beta)$,

Solution:

$$\begin{aligned}
E(Y) &= E(E(Y|p)) \\
&= E(np) \\
&= nE(p) \\
&= \frac{n\alpha}{\alpha + \beta}.
\end{aligned}$$

   (b) $\text{Var}(Y) = \dfrac{n\alpha\beta(\alpha + \beta + n)}{(\alpha + \beta)^2(\alpha + \beta + 1)}$.

Solution:

$$
\begin{aligned}
V(Y) &= V(E(Y|p)) + E(V(Y|p)) \\
&= V(np) + E(np(1-p)) \\
&= n^2V(p) + nE(p) - nE(p^2) \\
&= n^2V(p) + nE(p) - n[V(p) + (E(p))^2] \\
&= (n^2-n)V(p) + nE(p) + n(E(p))^2 \\
&= (n^2-n)\left(\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}\right) + n\left(\frac{\alpha}{\alpha+\beta}\right) - n\frac{\alpha^2}{(\alpha+\beta)^2} \\
&= (n^2-n)\left(\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}\right) + n\frac{\alpha\beta}{(\alpha+\beta)^2} \\
&= \frac{(n^2-n)\alpha\beta + n\alpha\beta(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} \\
&= \frac{n^2\alpha\beta + n\alpha^2\beta + n\alpha\beta^2}{(\alpha+\beta)^2(\alpha+\beta+1)} \\
&= \frac{n\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}.
\end{aligned}
$$

3. Suppose that X and Y have a joint uniform density over the unit square. Find the pdf for their product - that is, find $f_U(u)$, where $U = XY$.

Solution: Given that $f(x,y) = \begin{cases} 1, & 0 \le x \le 1, \;\; 0 \le y \le 1 \\ 0, & \text{else} \end{cases}$

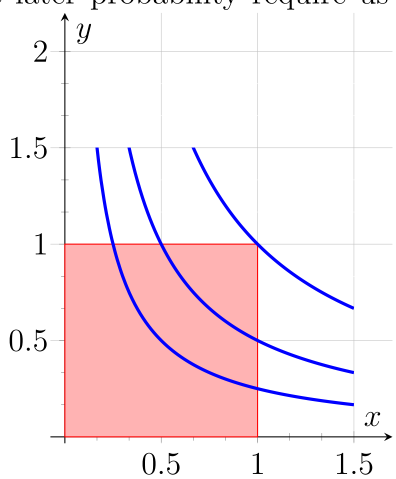
Let $U = XY$. Then CDF for U is:

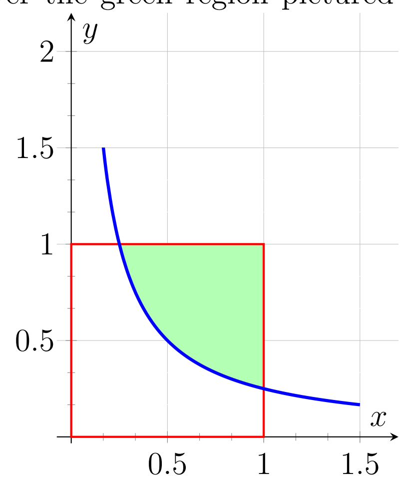
$$F_U(u) = P(U \le u) = P(XY \le u) = P\left(Y \le \frac{u}{X}\right).$$

Now, we consider the sketch of f's support below. If $u \le 0$ or $u > 1$, there will no intersection with the support. For $0 < u \le 1$, we see the curve $y = u/x$ intersects the support of $f(x,y)$. Below are the curves for $u = 1$, $1/2$, and $1/4$ respectively. The inequality $y \le u/x$ corresponds to the portion of the support that contains the origin (the white region). However, choosing to integrate over this region will always require us to compute two separate double integrals. Instead, we can consider

$$P(Y \le u/X) = 1 - P(Y \ge u/X),$$

and the later probability require us to integrate over the green region pictured below.





Hence,

$$\begin{aligned} F_U(u) &= 1 - \int_{y=u}^{1} \int_{x=\frac{u}{y}}^{1} 1 \; dx \; dy \\ &= 1 - \left[\int_{y=u}^{1} \left(1 - \frac{u}{y}\right) \; dy \right] \\ &= 1 - (y - u \ln|y|) \Big|_{y=u}^{1} \\ &= u - u\ln(u), \;\; 0 < u \le 1. \end{aligned}$$

Then, differentiating with respect to u, we get $U(u) = \begin{cases} -\ln(u), & 0 < u \le 1 \\ 0, & \text{else} \end{cases}$

4. Let $X_1, X_2, \ldots, X_n$ be independent random variables with Poisson distributions and with parameter λ_i, respectively. Let $U = X_1 + \ldots + X_n$. Show that U has a Poisson distribution and determine U's parameter λ_U.

Solution: We know that each X_i has a Poisson distribution with parameter λ_i. Hence, for each X_i, $m_{X_i}(t) = \exp[\lambda_i(e^t - 1)]$. Thus,

$$\begin{aligned} m_U(t) &= m_{X_1}(t) \times m_{X_2}(t) \times ... \times m_{X_n}(t) \\ &= \exp[\lambda_1(e^t - 1)] \times \exp[\lambda_2(e^t - 1)] \times ... \times \exp[\lambda_n(e^t - 1)] \\ &= \exp\left[\sum_{i=1}^{n} \lambda_i(e^t - 1)\right] \\ &= \exp\left[(e^t - 1)\sum_{i=1}^{n} \lambda_i\right] \end{aligned}$$

Thus, U has a Poisson distribution, as this is the moment-generating function for the Poisson distribution where $\lambda_U = \sum_{i=1}^{n} \lambda_i$.

5. Let $X_1, X_2, \ldots, X_n$ denote a random sample of size n from a distribution which is normal $N(\mu, \sigma)$. Define $U = \sum_{1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2$. Show that U has a χ^2 with n degrees of freedom. (Hint: See Homework 6, Problem 5.)

Solution: Let $Y_i = \left(\frac{X_i - \mu}{\sigma}\right)^2$. In Homework 6, Problem 5 we showed each Y_i is a Gamma distributed with $\alpha = 1/2$ and $\beta = 2$. That is, Y_i is χ^2-distributed with 1 degree of freedom and the moment generating function is

$$m_{Y_i}(t) = (1 - 2t)^{-\frac{1}{2}}.$$

Now, the moment generating function for $U = Y_1 + Y_2 + \ldots + Y_n$ is

$$\begin{aligned} m_U(t) &= m_{Z_1^2}(t) \times m_{Z_2^2}(t) \times ... \times m_{Z_n^2}(t) \\ &= (1 - 2t)^{-\frac{1}{2}} \times (1 - 2t)^{-\frac{1}{2}} \times ... \times (1 - 2t)^{-\frac{1}{2}} \\ &= (1 - 2t)^{-\frac{n}{2}}. \end{aligned}$$

This is the moment-generating function for the χ^2 distribution with n degrees of freedom. Thus, by the uniqueness of moment-generating functions, U has the χ^2 distribution with n degrees of freedom.