z - scores $= \frac{y-\mu}{\sigma}$

Matt

1. (10 points) Weekly CPU time used by an accounting firm has a density function (measured in hours) given by

$$f(y) = \begin{cases} \frac{3}{4}y(2-y), & 0 \le y \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Find the cumulative distribution function $F(y)$.

$\int_S f(y)\,dy = F(y)$

$\frac{3}{4}\int_0^y 2t - t^2\,dt = \frac{3}{4}\left[t^2 - \frac{t^3}{3}\right]_0^y = \frac{3}{4}\left[y^2 - y^3/3\right]$

$F(y) = \frac{3}{4}\left(y^2 - \frac{y^3}{3}\right)$

(b) Find $P(Y \le 1.5)$.

$P(Y \le 1.5) = F(1.5)$

$\frac{3}{4}\left(1.5^2 - \frac{(1.5)^3}{3}\right) = \boxed{0.84375}$

(c) Find the mean of the distribution.

$F(y) = 0.5$ $\qquad 0.5 = \frac{3}{4}\left(y^2 - \frac{y^3}{3}\right)$

$\frac{2}{3} = y^2 - y^3/3$

$\frac{y^3}{3} - y^2 + \frac{2}{3} = 0$

let $y = 1$

$\frac{1}{3} - 1 + \frac{2}{3} = 0$

1 is a root of y $\quad \boxed{\mu = 1}$

2. (6 points) The length of time to failure (in hundreds of hours) for a transistor is a random variable T with distribution function given by

$$F(t) := \begin{cases} 1 - e^{-t^2}, & 0 \le t \le \infty \\ 0, & \text{elsewhere} \end{cases}$$

(a) Find the probability that the transistor lasts at least 50 hours.

$$1-e^{-50^2} = 1-e^{-2500} = 1 - \frac{1}{1-e^{2500}} \approx 1 + \frac{1}{\infty}$$

$$\approx 1 + 0$$

$$1 - 1 = 0$$

$$P(t \geq 50) \approx 0$$

(b) Determine the probability density function for the random variable T.

find $f(t)$

$$F(t) = 1 - e^{-t^2}$$

$$f(t) = -\frac{d}{dt} e^{-t^2}$$ ~~has no elementary antiderivative~~

but it has a derivative

$$-\left(e^{-t^2}\right)\left(\frac{d}{dt} - t^2\right)$$

$$-e^{-t^2} \cdot -2t$$

$$2te^{t^2} = f(t)$$

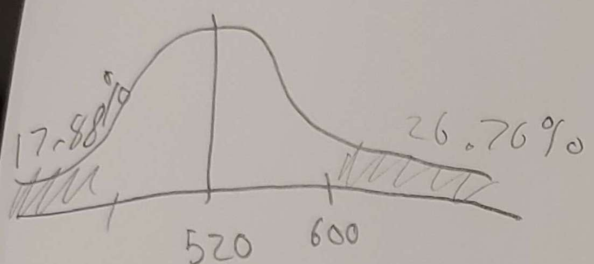
3. (7 points) The scores on the SAT Math subject exam for 2021 were approximately Normally distributed with a mean of 520 and standard deviation of 130.

(a) What percent of scores are between 400 and 600?

$\mu = 520 \quad \sigma = 130$

$z\text{-score} = \frac{y-\mu}{\sigma} = \frac{y-520}{130}$

P 500 ± 100



$\frac{600-520}{130}$ $\frac{8}{13}$ $400-520/130$ $\frac{-12}{13}$

0.62

$\frac{12}{13}$

0.2676

$1-26.76=17.88\%$

0.923 → 0.1788

~~0.5536 or 55.4%~~

(b) How high must a student score in order to place in the top 5% of all students taking the SAT Math?

z = 1.645

$1.645 = \frac{y-520}{130}$

$213.85 = y - 520$

~~$y = 233.9$~~

$y = 734$

4. (5 points) A random variable Y has a chi-square distribution with $\nu = 8$ degrees of freedom.

   (a) What is the mean, variance and standard deviation of Y?

   ???

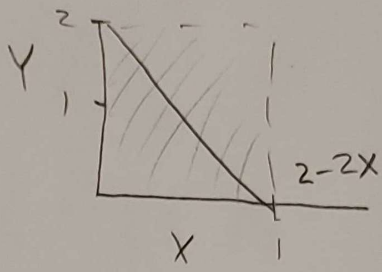
   (b) Use Chebyshev's inequality to find an interval about the mean for which the probability Y will lie within it is at least 0.75.

   $1 - \frac{1}{k^2}$

5. (12 points) Suppose that random variables X and Y have the density function

$$f(x,y) = \begin{cases} x(2-y), & 0 \le x \le 1, 0 \le y \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Find the probability of the event $2X + Y \le 2$.



$P(Y \le 2-2x$

$$\int_{x=0}^{1}\int_{y=0}^{2-2x} 2x - xy \, dy\, dx = \int_{x=0}^{1} 2xy - \frac{xy^2}{2}\Big|_0^{2-2x} dx$$

$$2x(2-2x) - \frac{x(2-2x)^2}{2}$$

$$\int_0^1 4x - 4x^2 - \frac{x}{2}\left(4 - 4x + 4x^2\right) dx$$

$$\int_0^1 4x - 4x^2 - 2x + 2x^2 - 2x^3 = x^2 - \frac{2}{3}x^3 - \frac{1}{2}x^4\Big|_0^1$$

$$2x - 2x^2 - 2x^3 \qquad 1 - \frac{2}{3} - \frac{1}{2} = \boxed{\frac{-1}{6}}$$ rip

(b) Find the marginal density function of X.

$$f(x) = \int_y y f(x,y)$$

$$= \int_{y=0}^{x} y(x(2-y))\, dy = \int_0^x y(2x - xy)\, dy = \int_0^x 2xy - xy^2\, dy$$

$$= xy^2 - \frac{xy^3}{3}\Big|_{y=0}^{x} \qquad \boxed{f_1(x) = x^3 - \frac{x^4}{3}}$$

(c) Find $P(X > 0.5)$.

$P(X>0.5) = 1 - P(X<0.5)$

$$\int_{y=0}^{2}\int_{x=0}^{1/2} 2x - xy \, dx\, dy = x^2 - \frac{x^2 y}{2} \Big|_0^{1/2}$$

$$= \int_0^2 \frac{1}{4} - \frac{1}{8}y \, dy$$

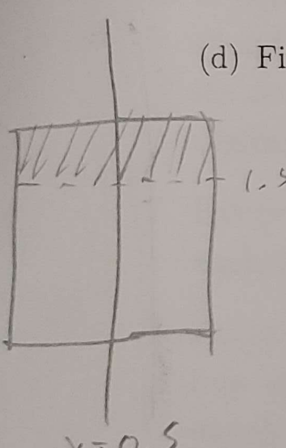
$$\frac{1}{4}y - \frac{1}{16}y^2 \Big|_0^2$$

$$\frac{2}{4} - \frac{4}{16} = 0.5 - 0.25 = 0.25$$

$1 - 0.25$

$= \boxed{75\%}$

(d) Find $P(Y > 15|X = 0.5)$.



$P(Y>1.5 \mid x=0.5)$

$f(x,y) = 2x - xy$

$$\int_{y=1.5}^{2} 2(0.5) - 0.5y = \int_{1.5}^{2} 1 - 0.5y \, dy$$

$$y - \frac{1}{4}y^2 \Big|_{1.5}^{2}$$

$1 - 0.9375$

0.0625

$\boxed{1/16}$