Math 325 Section 01 **Exam 2**
Professor Carty **November 3, 2021**
40 points possible

Name: ______________________________

1. No hats or dark sunglasses. All hats are to be removed.
2. All book bags are to be closed and placed in a way that makes them inaccessible. Do not reach into your bag for anything during the exam. If you need extra pencils, pull them out now.
3. Be sure to print your proper name clearly.
4. Calculators are optional, but if you choose to use one you may ONLY use the battery or solarpowered Texas Instruments BA35 model calculator, the BA II Plus, the BA II Plus Professional, the TI30Xa or TI30X II (IIS solar or IIB battery), or TI-30X MultiView (XS Solar or XB Battery).
5. All electronic devices, including cell phones and other wearable devices, must be silenced or powered off and stored out of sight for the entirety of the exam.
6. If you have a question, raise your hand and I will come to you. Once you stand up, you are done with the exam. If you have to use the facilities, do so now. You will not be permitted to leave the room and return during the exam.
7. Every exam is worth a total of **40 points**. Including the cover sheet, each exam has 7 pages.
8. At 11:45, you will be instructed to put down your writing utensil. Anyone seen continuing to write after this announcement will have their exam marked and lose all points on the page they are writing on. At this time, you will use your phone to take PDFs of the pages and upload them into Canvas as a single PDF document (the same way you do with homework.) Solutions must be uploaded by 11:55.
9. If you finish early, quietly and respectfully perform the preceding tasks. You may leave early.
10. You will hand in the paper copy of the exam on your way out of the classroom.
11. You have forty-five minutes to complete the exam. I hope you do well.

1. (10 points) Weekly CPU time used by an accounting firm has a density function (measured in hours) given by

$$f(y) = \begin{cases} \frac{3}{4}y(2-y), & 0 \le y \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Find the cumulative distribution function $F(y)$.

$$F(y) = \int_0^y \frac{3}{4}t(2-t) = \frac{3}{4}\int_0^y (2t - t^2) \overset{FTC}{=} \frac{3}{4}\left(t^2 - \frac{t^3}{3}\right)\Big|_0^y$$

$$= \frac{3}{4}\left(y^2 - \frac{y^3}{3}\right)$$

$$F(y) = \begin{cases} \frac{3}{4}\left(y^2 - \frac{y^3}{3}\right) & , 0 \le y \le 2 \\ 0 & , y < 0 \\ 1 & , y > 2 \end{cases}$$

(b) Find $P(Y \le 1.5)$.

$$P(Y \le 1.5) = F(1.5) = \frac{3}{4}\left(\left(\frac{3}{2}\right)^2 - \frac{(3/2)^3}{3}\right)$$

(c) Find the mean of the distribution.

$$E[Y] = \int_0^2 y f(y) = \frac{3}{4}\int_0^2 (2y^2 - y^3) \overset{FTC}{=} \frac{3}{4}\cdot\left(\frac{2y^3}{3} - \frac{y^4}{4}\right)\Big|_0^2$$

$$= \frac{3}{4}\cdot\left(\frac{16}{3} - \frac{16}{4}\right) = 4 - 3 = 1$$

2. (6 points) The length of time to failure (in hundreds of hours) for a transistor is a random variable T with distribution function given by

$$F(t) := \begin{cases} 1 - e^{-t^2}, & 0 \le t \le \infty \\ 0, & \text{elsewhere} \end{cases}$$

(a) Find the probability that the transistor lasts at least 50 hours.

50 hours $\rightarrow 0.5 = t$.

$$P(t \geq 0.5) = 1 - P(t \leq 0.5)$$
$$= 1 - [1 - e^{-(1/2)^2}]$$
$$= e^{-1/4}$$
$$\approx 0.7788$$

(b) Determine the probability density function for the random variable T.

$$\frac{d}{dt}(F(t)) = f(t).$$

$$f(t) = \begin{cases} -e^{-t^2} \cdot (-2t) = 2te^{-t^2}, & 0 \leq t \leq \infty \\ 0, & \text{else.} \end{cases}$$

$\mu = 520 \quad \sigma = 130$

3. (7 points) The scores on the SAT Math subject exam for 2021 were approximately Normally distributed with a mean of 520 and standard deviation of 130.

   (a) What percent of scores are between 400 and 600?

$$P(400 < S < 600) = P\left(\frac{400-520}{130} < Z < \frac{600-520}{130}\right)$$

$$= P\left(-\frac{12}{13} < Z < \frac{8}{13}\right)$$

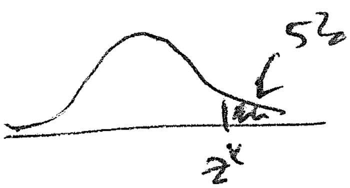
$$= 1 - P(Z > 8/13) - P(Z > 12/13)$$

$$= 1 - P(Z > 0.615) - P(Z > 0.923)$$

$$= 1 - 0.2676 - 0.1788$$

$$= 0.5536.$$

   (b) How high must a student score in order to place in the top 5% of all students taking the SAT Math?



$\Longrightarrow$ By Table 4, $z^* = 1.65$

$$\frac{S-520}{130} = 1.65$$

$$S = 1.65(130) + 520 = 734.5.$$

4. (5 points) A random variable Y has a chi-square distribution with $\nu = 8$ degrees of freedom.

(a) What is the mean, variance and standard deviation of Y?

$\chi^2 \Rightarrow$ a Gamma w/ $\beta = 2$, $\alpha = \nu/2 = 8/2 = 4$.

Then $\mu = \alpha\beta = 8$

$\sigma^2 = \alpha\beta \cdot \beta = 16$

$\sigma = 4$.

(b) Use Chebyshev's inequality to find an interval about the mean for which the probability Y will lie within it is at least 0.75.

Cheb: $P(|Y-8| < k \cdot 4) > 1 - \frac{1}{k^2}$ ~~$\frac{3}{4}$~~

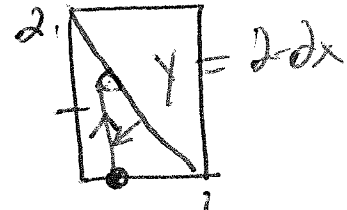
Want $1 - \frac{1}{k^2} = \frac{3}{4} \Rightarrow k = 2$.

So $|Y-8| < 2 \cdot 4$ or $|Y-8| < 8$

i.e. $0 \le Y \le 16$.

5. (12 points) Suppose that random variables X and Y have the density function

$$f(x,y) = \begin{cases} x(2-y), & 0 \le x \le 1, 0 \le y \le 2 \\ 0, & \text{elsewhere} \end{cases}$$



(a) Find the probability of the event $2X + Y \le 2$.

$$P(Y \le 2-2x) = \int_{x=0}^{1} \int_{y=0}^{2-2x} x(2-y)\,dy\,dx$$

$$= \int_0^1 -x\frac{(2-y)^2}{2}\Big|_{0=y}^{2-2x} dx = -\frac{1}{2}\int_0^1 x\left[(2x)^2 - 4\right] dx$$

$$= -\frac{1}{2}\int_0^1 (4x^3 - 4x)\,dx = -\frac{1}{2}\left(x^4 - 2x^2\right)\Big|_0^1$$

$$= -\frac{1}{2}(1-2) = \frac{1}{2}.$$

(b) Find the marginal density function of X.

$$f_1(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_0^2 x(2-y)\,dy$$

$$= -x\frac{(2-y)^2}{2}\Big|_0^2 = -x \cdot 0 + x \cdot \frac{4}{2} = 2x$$

(c) Find $P(X > 0.5)$.

$$P(X > 1/2) = \int_{1/2}^{1} f_1(x)\,dx = \int_{1/2}^{1} 2\,dx \overset{FTC}{=} 2x\Big|_{1/2}^{1} = \frac{3}{4}$$

(d) Find $P(Y > 15|X = 0.5)$.

$$f(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{x(2-y)}{2x} = \frac{1}{2}(2-y)$$

$$P(Y > 1.5 | X = 0.5) = \int_{y=1.5}^{2} \frac{1}{2}(2-y)\,dy$$

$$\overset{FTC}{=} -\frac{(2-y)^2}{4}\Big|_{1.5}^{2} = 0 - -\frac{(2-3/2)^2}{4} = \frac{1}{16}.$$