MTH 326 - Spring 2022
Assignment #5
Due: Friday, February 25, 2022 (23:59)

1. Suppose $X_1, X_2, \dots, X_n$ are iid with the common density function

$$f(x) = \frac{2\theta^2}{x^3}, \quad x > \theta$$

where θ is unknown.

(a) Find the method of moments estimator of θ.

Solution: As per the notes,

$$\underbrace{\mu'_k = E[\,X^k\,]}_{\text{population}} \qquad \text{and} \qquad \underbrace{m'_k = \frac{1}{n}\sum X_i^k}_{\text{data set}}$$

Computing μ'_1, we have

$$\begin{aligned}
\mu'_1 &= E[\,X\,] \\
&= \int_\theta^\infty x \cdot \frac{2\theta^2}{x^3}\,dx \\
&= 2\theta^2 \int_\theta^\infty x^{-2}\,dx \\
&= 2\theta^2 \left[-\frac{1}{x}\right]_{x=\theta}^{x=\infty} \\
&= 2\theta^2 \left[-\lim_{x\to\infty}\frac{1}{x} + \frac{+1}{\theta}\right] \\
&= \frac{0 + 2\theta^2}{\theta} \\
&= 2\theta
\end{aligned}$$

Then for m'_k,

$$\begin{aligned}
m'_k = \frac{1}{n}\sum X_i^k &= 2\theta \\
\iff m'_k &= \bar{X} = 2\theta \\
\iff & \boxed{\hat{\theta} = \frac{\bar{X}}{2}}
\end{aligned}$$

(b) Find the bias and variance of your estimator.

Solution: By the definition for Bias, $B = E[\,\hat{\theta}\,] - \theta$. Substituting in our estimator from part (a),

$$
\begin{aligned}
B[\,\hat{\theta}\,] &= E[\,\hat{\theta}\,] - \theta && \text{Definition of Bias} \\
&= E\left(\frac{\bar{X}}{2}\right) - \theta && \text{Substituting from (a)} \\
&= \frac{1}{2}E[\,\bar{X}\,] - \theta && \text{Linearity of E} \\
&= \frac{1}{2}\left[E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)\right] - \theta && \text{Definition of } \bar{X} \\
&= \frac{1}{2n}\sum_{i=1}^{n} E[\,X_i\,] - \theta && \text{Linearity of } E \\
&= \frac{1}{2n}\sum_{i=1}^{n} 2\theta - \theta && E[\,X\,] \text{ from (a)} \\
&= \frac{n \cdot 2\theta}{2n} - \theta && \text{Sum } n \text{ times} \\
&= \theta - \theta && \text{Cancel } 2n \\
&= 0
\end{aligned}
$$

$$\boxed{\text{Therefore } \hat{\theta} \text{ is an unbiased estimator.}}$$

As for variance,

$$V[\,X_i\,] = E[\,X_i^2\,] - E[\,X_i\,]^2$$

$$
\begin{aligned}
E[\,X^2\,] &= \int_{\theta}^{\infty} x^2 \cdot \frac{2\theta}{x^3}\,dx \\
&= 2\theta^2 \int_{\theta}^{\infty} \frac{1}{x}\,dx \\
&= 2\theta^2 \left[\ln x\right]_{x=\theta}^{x=\infty} \\
&= 2\theta^2 \left[\lim_{x\to\infty} \ln x - \ln\theta\right] \\
&= \infty
\end{aligned}
$$

And $E[\,X_i\,]^2 = (2\theta)^2 = 4\theta^2$. Therefore

$$V[\,X_i\,] = \infty - 4\theta^2 = \infty.$$

$$\boxed{\text{Therefore the variance is infinite.}}$$

2. Suppose that $X_1, X_2, \ldots, X_n$ are a random sample from an exponentially distributed population with unknown mean θ. Find the maximum likelihood estimator of the population variance θ^2.

Solution: An exponential distribution is a Gamma distribution with $\alpha = 1$, therefore

$$f(x) = \lambda e^{-\lambda x} \qquad x \in [0, \infty) \qquad \mu_x = \frac{1}{\lambda}$$

If θ is the mean then $\theta = \frac{1}{\lambda}$. Solving for λ, $\lambda = \frac{1}{\theta}$. Substituting this into the exponential formula,

$$f(x) = \frac{e^{-x/\theta}}{\theta}.$$

Now we can start to find the likelihood estimator,

$$\begin{aligned}
L(X_1, \ldots, X_n \mid \theta) &= \prod_{i=1}^{n} f(X_i \mid \theta) \\
&= \frac{e^{-x_1/\theta}}{\theta} \times \cdots \times \frac{e^{-x_n/\theta}}{\theta} \\
&= \frac{e^{-x_1/\theta} \times \cdots \times e^{-x_n/\theta}}{\theta^n} \\
&= \frac{e^{\frac{-x_1 - x_2 - \cdots - x_n}{\theta}}}{\theta^n} \\
&= \frac{e^{-\bar{X}\,/\,\theta}}{\theta^n}
\end{aligned}$$

Then, taking the log likelihood function,

$$\begin{aligned}
\ln\left(L(X_1, \ldots, X_n \mid \theta)\right) &= \ln\left(\frac{e^{-\bar{X}\,/\,\theta}}{\theta}\right) \\
&= \ln\left(\frac{1}{\theta^n}\right) + \ln\left(e^{-\bar{X}\,/\,\theta}\right) \\
&= \ln\left(\theta^{-n}\right) + \left(\frac{-\bar{X}}{\theta}\right) \ln\left(e^1\right) \\
&= -n \ln \theta + \left(\frac{-\bar{X}}{\theta}\right) \cdot (1) \\
&= -n \ln \theta - \frac{\bar{X}}{\theta}.
\end{aligned}$$

Differentiating said log likelihood function,

$$\begin{aligned}
\frac{\partial \ln L}{\partial \theta} &= \frac{\partial}{\partial \theta}(-n \ln \theta) - \frac{\partial}{\partial \theta}\left(\frac{\bar{X}}{\theta}\right) \\
&= -\frac{n}{\theta} + \frac{\bar{X}}{\theta^2}
\end{aligned}$$

To maximize (or minimize) θ, we let $-\frac{n}{\theta} + \frac{\bar{X}}{\theta^2} = 0$ and solve.

$$\begin{aligned} -\frac{n}{\theta} + \frac{\bar{X}}{\theta^2} = 0 &\iff \frac{\bar{X}}{\theta^2} = \frac{n}{\theta} \\ &\iff \bar{X}\theta = n\theta^2 \\ &\iff \bar{X} = n\theta \\ &\iff \theta = \frac{\bar{X}}{n} \end{aligned}$$

By the invariance property, the MLE of θ^2 is the square of the MLE of θ. So,

$$\begin{aligned} &\text{MLE}(\theta^2) = (\text{MLE}(\theta))^2 \\ \iff &\boxed{\text{MLE}(\theta^2) = \left(\frac{\bar{X}}{n}\right)^2} \end{aligned}$$

3. Let $X_1, X_2, \ldots, X_n$ denote a remote sample from the density function given by

$$f(x) = \left(\frac{1}{\theta}\right) r x^{r-1} e^{-x^r/\theta}, \qquad x > 0, \qquad \theta > 0$$

where r is a known positive constant. Consider the statistic defined by $U = \frac{1}{n}\sum X_i^r$.

(a) Find and simplify the likelihood function $L(x_1, x_2, ..., x_n \mid \theta)$ and complete the factorization.

$$\begin{aligned}
L(X_1, \ldots, X_n \mid \theta) &= f(X_1 \mid \theta) \times \cdots \times f(X_n \mid \theta) \\
&= \prod_{i=1}^{n} f(X_i \mid \theta) \\
&= \prod_{i=1}^{n} \frac{1}{\theta} f(X_i) \\
&= \frac{1}{\theta^n} \prod_{i=1}^{n} f(X_i) \\
&= \frac{1}{\theta^n} \prod_{i=1}^{n} r x_i^{r-1} e^{-x_i^r/\theta} \\
&= \frac{e^{-\bar{X}^r/\theta} r^n}{\theta^n} \prod_{i=1}^{n} x_i^{r-1}
\end{aligned}$$

To factorize, we will let

$$g(U, \theta) = \frac{1}{\theta^n} e^{-\bar{X}^r/\theta} \qquad \text{and} \qquad h(X_1, \ldots, X_n) = r^n \prod_{i=1}^{n} x_i^{r-1}$$

(b) Show that U is a sufficient statistic for θ.

Since $U = \frac{1}{n}\sum X_i^r$, then

$$g(U, \theta) = \frac{1}{\theta^n} e^{-\bar{X}^r/\theta} \qquad \text{By factorization theorem.}$$

Since $\bar{X}$ is in g, then $\bar{X} = \sum_{i=1}^{n} X_i^r$ is sufficient by the factorization theorem.

(c) Show that U is the MLE for θ.

$$L(X_1, \ldots, X_n \mid \theta) = \frac{r^n \cdot e^{-\bar{X}^r/\theta}}{\theta^n} \prod_{i=1}^{n} x_i^{r-1}$$

$$\begin{aligned}
\ln\left(L(X_1, \ldots, X_n \mid \theta)\right) &= \ln\left(\frac{1}{\theta}\right)^n + \ln r^n + \ln\left(\prod_{i=1}^{n} x_i^{r-1}\right) + \ln\left(e^{-\bar{X}^r/\theta}\right) \\
&= -n \ln \theta + n \ln r + (r-1) \ln \bar{X} - \frac{\bar{X}^r}{\theta}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial \theta}\left[\ln\Big(L(X_1, \ldots, X_n \mid \theta)\Big)\right] &= -\frac{n}{\theta} + 0 + 0 + \frac{\bar{X}^r}{\theta^2} \\
&= 0 \\
&\iff \frac{\bar{X}^r}{\theta^2} = \frac{n}{\theta} \\
&\iff \theta = \frac{\bar{X}^r}{n}
\end{aligned}$$

(d) Show that U is an unbiased estimator of θ.

$$\begin{aligned}
E[\,U\,] &= E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} E[\,X_i\,] \\
&= \frac{1}{n} \cdot n\theta \\
&= \theta
\end{aligned}$$

Therefore U is unbiased.

(e) What can we now conclude about the estimator U.

Because U is unbiased, sufficient, and has the MLE; we can conlude that U is the MVUE (minimum variance unbiased estimator).

(f) Explain why $V(U)$ is finite. (You do not have to compute it, but clearly explain why it is finite.)

The variance of U is finite because we showed U to be an MVUE and all one-variable functions with an MVUE guarantee the existance of finite variance.

Analytically, suppose that U had infinite variance, then U couldn't have an MVUE since the minimum *variance* of infinity would be infinite; and thus not the minimum.

(g) Show that U is a consistent estimator of θ.

We first need the variance:

$$\begin{aligned} V[\,U\,] &= V\left(\frac{1}{n}\sum_{i=1}^{n} X_i^r\right) \\ &= \frac{1}{n^2}\sum_{i=1}^{n} V\left(X_i^r\right) \end{aligned}$$

Taking the limit,

$$\lim_{n\to\infty} V[\,U\,] = \lim_{n\to\infty}\left(\frac{1}{n^2}\sum_{i=1}^{n} V\left(X_i^r\right)\right)$$

We know that $\lim_{n\to\infty}\dfrac{1}{n^2} = 0$. Since we showed in (f) that $V[\,U\,]$ is finite, $V[\,X\,]$ is also finite since U is a sum of X's. 0 times a finite quantity is going to be 0. Therefore $V[\,U\,] = 0$. Since the variance is zero, U is a consistent estimator of θ.