Math 326 – Homework 04 (9.2 – 9.5)
Due (via upload to Canvas) Friday, February 18, 2022 at 11:59 PM

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from a uniform distribution on the interval $(\theta, \theta + 1)$. Let
$$\hat{\theta}_1 = \bar{X} - \frac{1}{2}, \text{ and } \hat{\theta}_2 = X_{(n)} - \frac{n}{n+1}.$$
   (a) Show that both $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased estimators of θ.
   (b) Show that both estimators are consistent estimators.
   (c) Find the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$
   (d) Which is the better estimator and why?

2. Suppose the population has a gamma distribution and we know β but α is unknown. Let $X_1, X_2, \ldots, X_n$ denote a random sample from the distribution. Determine the likelihood function, compute the factorization, and using the Factorization Theorem, show that $T = \sum_{i=1}^{n} \ln(X_i)$ is a sufficient statistic for α.

3. Let $X_1, X_2, \ldots, X_n$ be iid from a Bernoulli distribution with probability p. We are going to construct and MVUE for variance pq. Recall that in class we showed that $S = \sum X_i$ is a sufficient statistic for p.
   (a) Define the statistic
$$T = \tau(x_1, \ldots, x_n) = \begin{cases} 1, & \text{if } X_1 = 1 \text{ and } X_2 = 0 \\ 0, & \text{otherwise} \end{cases}.$$
   Show that τ is an unbiased estimator for pq.
   (b) Show that
$$P(T = 1 | S = s) = \frac{s(n-s)}{n(n-1)}.$$
   (c) Using the Rao-Blackwell Theorem states that to find an MVUE of pq, we define a new statistic $\phi(s) = E[T|S = s]$. Show that
$$\phi(s) = \frac{n}{n-1}\left[\bar{X}\left(1 - \bar{X}\right)\right]$$
   is the minimum variance unbiased estimator of pq.