# 326 Homework 5

(1) a) $\mu = \int_0^\infty y \cdot \frac{2\theta^2}{y^3}\,dy = 2\theta^2 \int_0^\infty \frac{1}{y^2}\,dy$

$= 2\theta^2\left(-\frac{1}{y}\Big|_\theta^\infty\right) = 2\theta^2\left(0 + \frac{1}{\theta}\right) = 2\theta$

$\mu_1' = m_1' \Rightarrow 2\theta = \bar{X} \Rightarrow \hat{\theta} = \frac{\bar{X}}{2}$.

b) Note $E[\hat{\theta}] = \frac{1}{2n}\sum E(X_i) = \frac{n2\theta}{2n} = \theta$.

unbiased.

For $V(\hat{\theta})$, need $E[X^2]$.

$E[X^2] = \int_0^\infty y^2 \cdot \frac{2\theta^2}{y^3}\,dy = 2\theta^2 \int_0^\infty \frac{1}{y}\,dy$

which diverges by the p-test.

$E[X^2]$ is infinite, hence so is $V(\hat{\theta})$,

2. $L(x_1, x_2, \ldots, x_n | \theta) = \frac{1}{\theta} e^{-x_1/\theta} \cdot \frac{1}{\theta} e^{-x_2/\theta} \cdot \ldots \cdot \frac{1}{\theta} e^{-x_n/\theta}$

$$= \frac{1}{\theta^n} e^{-\Sigma x_i/\theta}.$$

Let $S = \Sigma x_i$.

Then $L(x_1, \ldots, x_n | \theta) = g(S, \theta) \cdot h(x_1, \ldots, x_n)$

where $g(S|\theta) = \frac{1}{\theta^n} e^{-S/\theta}$ and $h(\vec{x}) = 1$.

For MLE, $\frac{dg}{d\theta} = 0$

$$\frac{-n}{\theta^{n+1}} e^{-S/\theta} + \frac{1}{\theta^n} e^{-S/\theta} \cdot \frac{S}{\theta^2} = 0$$

$$\frac{e^{-S/\theta}}{\theta^{n+1}} \left( -n + \frac{S}{\theta} \right) = 0$$

$$-\theta n + S = 0 \Rightarrow \theta = S/n$$

$$\hat{\theta} = \frac{S}{n} = \overline{X}.$$

Then, since $f(x) = x^2$ is 1-1 fn on $[0, \infty)$, by the invariance properties of MLEs, the MLE of $\theta^2$ is $(\overline{X})^2$.

$$\hat{\varphi} = (\overline{X})^2 \text{ is the MLE of } \theta^2.$$

3a. $L(x_1, \ldots, x_n | \theta) = \frac{1}{\theta} r x_1^{r-1} e^{-x_1^r/\theta} \cdot \frac{1}{\theta} r x_2^{r-1} e^{-x_2^r/\theta} \cdots \frac{1}{\theta} r x_n^{r-1} e^{-x_n^r/\theta}$

$= \left(\frac{r}{\theta}\right)^n e^{-\sum x_i^r/\theta} \cdot x_1^{r-1} x_2^{r-1} \cdots x_n^{r-1}$

$= \left(\frac{r}{\theta}\right)^n e^{-un/\theta} \cdot (x_1 \cdots x_n)^{r-1}$ where $U = \frac{\sum x_i^r}{n}$.

$= g(\theta, u) \cdot h(\vec{x})$

where $g(\theta, u) = \left(\frac{r}{\theta}\right)^n e^{-nu/\theta}$, $h(\vec{x}) = (x_1 \cdots x_n)^{r-1}$.

b) By the Factorization Thm, $U$ is a sufficient statistic.

c) $\frac{dg}{d\theta} = n\left(\frac{r}{\theta}\right)^{n-1}\left(\frac{-r}{\theta^2}\right) e^{-nu/\theta} + \left(\frac{r}{\theta}\right)^n e^{-nu/\theta} \cdot \frac{nu}{\theta^2}$

$= -n \frac{r^n}{\theta^{n+1}} e^{-nu/\theta} + n \frac{r^n}{\theta^{n+2}} e^{-nu/\theta} u$

$= \frac{n e^{-nu/\theta}}{\theta^{n+2}} (-\theta + u)$

$\frac{dg}{d\theta} = 0$ when $\theta = u$.

The MLE of $\theta$ is $\hat{\theta} = U = \frac{\sum x_i^r}{n}$.

d) $E(u) = \frac{1}{n} \sum E(x_i^r)$.

$$E(x^r) = \int_0^\infty x^r \cdot \frac{1}{\theta} r x^{r-1} e^{-x^r/\theta} \, dx$$

Let $u = x^r$

$du = r x^{r-1} dr$

$$E(x^r) = \int_0^\infty u \cdot \frac{1}{\theta} e^{-u/\theta} \, du$$

Let $\alpha = u$ $\quad d\beta = \frac{1}{\theta} e^{-u/\theta} du$

$d\alpha = du$ $\quad \beta = -e^{-u/\theta}$

$$= -ue^{-u/\theta} \Big|_0^\infty + \int_0^\infty e^{-u/\theta} du$$

$$= 0 \quad + \quad -\theta e^{-u/\theta} \Big|_0^\infty$$

$$= - - \theta$$

$$= \theta.$$

e) By Rao-Blackwell Thm, since $U$ is both unbiased and sufficient, $U$ is the MVUE.

f) $V(U) = \frac{1}{n^2} V(\Sigma x_i^r) = \frac{n V(x_i^r)}{n^2} = \frac{V(x_i^r)}{n}$.

Note $V(x_i^r) = E(x_i^{2r}) - \underbrace{(E(x_i^r))^2}_{= \theta^2 \text{ by (d)}}$

$E(x_i^{2r}) = \int_0^\infty u^2 \cdot \frac{1}{\theta} e^{-u/\theta} du$ by same u-sub.

$< \infty$, finite via application of IBP twice.

Hence $V(\cancel{X_i^r})$ is finite.

g) And $\lim_{n \to \infty} V(U) = \lim_{n \to \infty} \frac{V(x_i^r)}{n} = 0$.

Thus $U$ is a consistent estimator.

i.e $U \to \theta$ in probability.