Math 326 Section 01
Professor Carty
60 points possible

Exam 1
March 2, 2022

Name: Matthew Wilder Matthew Wilder

1. No hats or dark sunglasses. All hats are to be removed.
2. All book bags are to be closed and placed in a way that makes them inaccessible. Do not reach into your bag for anything during the exam. If you need extra pencils, pull them out now.
3. Be sure to print your proper name clearly.
4. Calculators are optional, but if you choose to use one you may ONLY use the battery or solarpowered Texas Instruments BA35 model calculator, the BA II Plus, the BA II Plus Professional, the TI30Xa or TI30X II (IIS solar or IIB battery), or TI-30X MultiView (XS Solar or XB Battery).
5. All electronic devices, including cell phones and other wearable devices, must be silenced or powered off and stored out of sight for the entirety of the exam.
6. If you have a question, raise your hand and I will come to you. Once you stand up, you are done with the exam. If you have to use the facilities, do so now. You will not be permitted to leave the room and return during the exam.
7. Every exam is worth a total of **60 points**. Including the cover sheet, each exam has 6 pages.
8. At 11:45, you will be instructed to put down your writing utensil. Anyone seen continuing to write after this announcement will have their exam marked and lose all points on the page they are writing on. At this time, you will use your phone to take PDFs of the pages and upload them into Canvas as a single PDF document (the same way you do with homework.) Solutions must be uploaded by 10:55.
9. If you finish early, quietly and respectfully perform the preceding tasks. You may leave early.
10. You will hand in the paper copy of the exam on your way out of the classroom.
11. You have forty-five minutes to complete the exam. I hope you do well.

1. (5 points) In a study to compare the perceived effects of two pain relievers, 200 patients were given medicine A, of whom 90% found relief, and 300 patients were given medicine B with 80% experiencing relief. Find a 95% confidence interval for the difference in population proportions experiencing relief between A and B.

$\mu_A - \mu_B$ large sample, $z = 1.960$

$$\sqrt{\frac{p_1(1-p_1)}{200} + \frac{p_2(1-p_2)}{300}}$$

$$.9 - .8 \pm \sqrt{\frac{0.9 \cdot 0.1}{200} + \frac{0.8 \cdot 0.2}{300}}$$

(0.1 ± 0.0313)

2. (10 points) Air trapped in amber from the Cretaceous era (75 million years ago) may suggest that the composition of our atmosphere has changed. Nine different samples have been obtained and the gas tested for the percentage of nitrogen in the atmosphere. We will treat these as a random sample.

1.645
1.96
2.6

(a) Given that $\bar{X} = 59.6\%$ and $S^2 = 39.13$ %, compute a 99% confidence interval on the nitrogen level in the ancient atmosphere. (FYI, the nitrogen level of our air is 78.1% today.)

$n = 9$

$$\bar{X} \pm t_{0.005}(8) \cdot \sigma$$

$$59.6\% \pm 3.355 \cdot \sqrt{39.13}$$

$$59.6\% \pm 20.98790$$

$(38.61\%,\ 80.59\%)$

(b) Construct a 90% confidence interval for the population variance σ^2.

3. (20 points) Suppose Y_1, Y_2, Y_3, Y_4 is an iid random sample from a exponential distribution with unknown rate parameter $\beta > 0$:

$$f(y) = \frac{1}{\beta} e^{-y/\beta}, 0 < y < \infty.$$

$\mu = a\beta = 1\beta = \beta$

exp is Gamma$(1, \beta)$

Consider the two estimators of β: $\hat{\theta}_1 = \overline{Y}$ and $\hat{\theta}_2 = \dfrac{2Y_1 + 3Y_2}{5}$.

$\sigma^2 = a\beta^2 = \beta^2$

(a) Show that $\hat{\theta}_2$ is an unbiased estimator of β.

\widehat{\theta}_2 is the better hat

Need to show $E[\hat{\theta}_2] = \theta$, $E[\hat{\theta}_2] = \frac{2}{5}E[Y_1] + \frac{3}{5}E[Y_2]$

$\mu = \beta$

$$= \frac{2}{5}\beta + \frac{3}{5}\beta$$

$$= \frac{5}{5}\beta$$

$$= \beta$$

$\therefore$ unbiased $\square$

(b) Determine the efficiency of $\hat{\theta}_2$ relative to $\hat{\theta}_1$.

$$\text{eff}(\hat{\theta}_2, \hat{\theta}_1) = \frac{Var(\hat{\theta}_1)}{Var(\hat{\theta}_2)} = \frac{\beta^2/n^2}{13\beta^2/n^2} = \boxed{\frac{1}{13}}$$

$$Var(\hat{\theta}_1) = Var(\overline{Y}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2}Var(Y_i) = \frac{\beta^2}{n^2}$$

$$Var(\hat{\theta}_2) = Var\left(\frac{2}{5}Y_1 + \frac{3}{5}Y_2\right) = \frac{4}{25}Var(Y_1) + \frac{9}{25}Var(Y_2) = \frac{4\beta^2 + 9\beta^2}{25} = 13\beta^2/25$$

(c) Now consider a third estimator of β, $\hat{\theta}_3 = \min(Y_1, Y_2, Y_3, Y_4)$. Show that the distribution of $\hat{\theta}_3$ is also exponentially distributed. (Recall the general density function for min order statistic is $f_{(1)}(y) = n[1 - F(y)]^{n-1} f(y)$.)

$n = 4$ $\qquad f(y) = \frac{1}{\beta} e^{-y/\beta}$ $\qquad 0 < y < \infty$

$f_{(1)}(y) = 4[1 - F(y)]^3 \left(\frac{1}{\beta} e^{-y/\beta}\right)$

$F(y) = \int_0^y \frac{1}{\beta} e^{-y/\beta} dy$

$= \frac{1}{\beta} \int_0^y e^{-y/\beta} dy$

$u = -y/\beta$

$du = -1/\beta \, dy$

$-\int_0^y \frac{1}{\beta} e^{-y/\beta} dy$

$= -\int_0^y e^u du$

$-u\Big]_0^y \to \frac{y}{\beta}\Big]_0^y \quad \frac{y}{\beta}$

$\int_0^y \frac{1}{\beta} e^{-y/\beta} dy$ $\qquad u = \frac{-y}{\beta} \Rightarrow y = -\beta u$

$u = -y/\beta$

$du = -1/\beta \, dy$

$\int_0^{-\beta u} -e^u du$

$= -\left[e^u\right]_0^{-\beta u} = -\left[e^{-\beta u} - 1\right]$

$-e^{-\beta\left(\frac{-y}{\beta}\right)} + 1$

$-e^y + 1 = F(y)$

$f_1(y) = 4e^{3y} \frac{1}{\beta} e^{-y/\beta}$

exponential

(d) Show that $\hat{\theta}_3$ is a biased estimator and compute the mean square error of $\hat{\theta}_3$.

$MSE() = Var + Bias^2$ $\qquad Bias = \hat{\theta} - \theta$

$\mu = \alpha\beta, \ \sigma^2 = \alpha\beta^2$

$E[f_1(y)] \stackrel{?}{=} 4e^{3y} \mu \neq \mu$ biased

4. (25 points) Suppose that $X_1, \ldots, X_n$ is a iid sample from a Rayleigh distribution with parameter $\theta > 0$ unknown:

$$f(y) = \frac{2x}{\theta} e^{-x^2/\theta}, 0 < x < \infty.$$

Note that $E(X) = \sqrt{\pi\theta}/2$, $E(X^2) = \theta$, $E(X^3) = 3\sqrt{\pi\theta^3}/4$, and $E(X^4) = \theta^4/2$. (You do not need to prove these facts.)

(a) Find the method of moments estimator θ_{MOM} for θ.

(b) Find and simplify the likelihood function $L(x_1, x_2, \ldots, x_n|\theta)$, complete the factorization, and determine a sufficient statistic for θ.

(c) Find the maximum likelihood estimator θ_{MLE} for θ.

$L = g \cdot h$

(d) Show that the maximum likelihood estimator of θ is consistent.

$\lim_{n \to \infty} Var = 0$

(e) Is the maximum likelihood estimator a minimum variance unbiased estimator? Briefly explain your answer.