MTH 326 - Spring 2022
Assignment #6
Due: Wednesday March 9, 2022 (11:59pm)

1. Suppose that we want to study whether students enrolled in a campus wellness program get more sleep than the average college student. (This problem assumes that the wellness group comes from the same general population as the other college students.) Suppose that the average student gets 6.1 hours of sleep per day and denote $\mu$ denote the average sleep gotten by a student in the wellness population.

   (a) State the hypothesis test for the test you are designing.

   **Solution:**

   We are testing whether the wellness group gets *more* sleep than the general population. Hence, we are testing $\mu_a > \mu_0$. The null hypothesis is that there is no difference in means, thus $\mu_a = \mu_0$.

$$H_0 : \mu_0 = \mu_a$$
$$H_a : \mu_0 < \mu_a$$

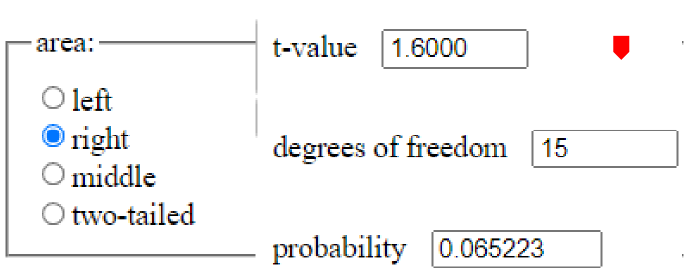
   Assume the standard *deviation* is known as $\sigma = 0.5$ hour and that the underlying sleep distribution is normal and that there are 16 students in the wellness program. We define the rejection region to be $\bar{Y} > 6.3$.

   (b) Find $\alpha$, the probability of Type I error.

   **Solution:**

   We will show the probability that $H_0$ is rejected when it is true. (df = 15)

$$\begin{aligned}\alpha &= \Pr(\text{Type I error}) \\ &= \Pr(\bar{Y} > 6.3 \mid \mu_0 = 6.1) \\ &= \Pr\left(t > \frac{6.3 - 6.1}{0.5/\sqrt{16}}\right) \\ &= \Pr(t > 1.6) \\ &\approx 0.065223 \qquad \text{(By WA “technology”)}\end{aligned}$$



(c) Find $\beta$, the probability of Type II error at $\mu = 6.3, 6.5$, and $6.7$.

**Solution:**

For the probability of a Type 2 error, we want the left tail probability.

When $\mu = 6.3$,

$$\mathcal{T} = \frac{\bar{Y} - \mu}{S/\sqrt{n}} = \frac{6.3 - 6.3}{0.5/\sqrt{15}} = 0$$

And $\Pr(\mathcal{T} < 0) = 0.5$.
$\beta = 0.5$

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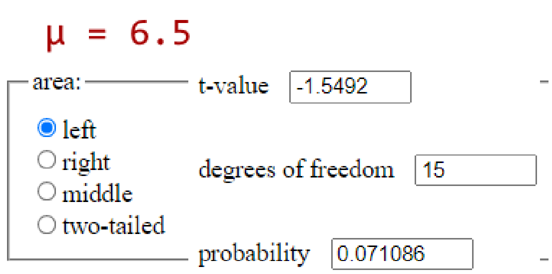
When $\mu = 6.5$,

$$\mathcal{T} = \frac{\bar{Y} - \mu}{S/\sqrt{n}} = \frac{6.3 - 6.5}{0.5/\sqrt{15}} = -1.54919$$

And $\Pr(Z < -1.54919) \approx 0.071087$.




$\beta = 0.071087$

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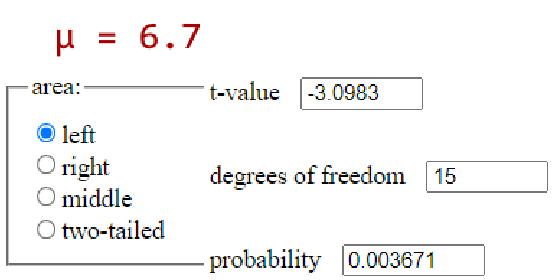
When $\mu = 6.7$,

$$\mathcal{T} = \frac{\bar{Y} - \mu}{S/\sqrt{n}} = \frac{6.3 - 6.7}{0.5/\sqrt{15}} = -3.09838$$

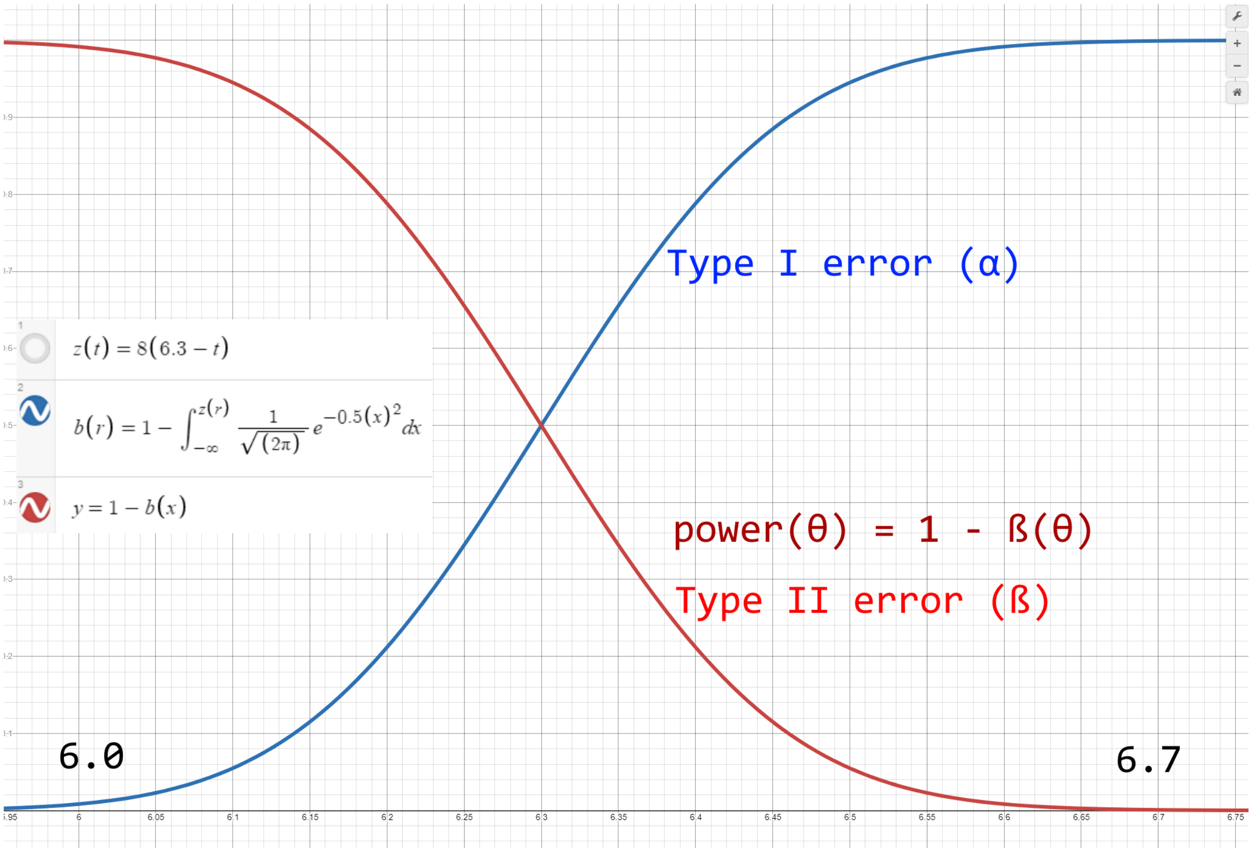
And $\Pr(Z < -3.09838) \approx 0.003671$.




$\beta = 0.003671$

(d) Sketch the power function for this test.

**Solution:** The following Desmos graph is has the red line as the power function $\text{power}(\theta) = 1 - \beta(\theta)$. This is approximately the same shape as the respective T distribution with 15 degrees of freedom, but was easier to set up the integral with normal.



2. Lifetimes of certain brand of switch follows (approximately) a normal distribution with mean 100 hours. Five switches of a new brand are obtained and tested and their lifetimes measured to be 120, 101, 114, 95, and 130 hours. Does this provide strong evidence that the new switches have a longer average lifespan?

**Solution:**

$$H_0 : \mu_0 = 100 \text{ hours}$$
$$H_a : \mu_a > 100 \text{ hours}$$

The sample mean is $\mu = 112$. The sample standard deviation is

$$\begin{aligned} S^2 &= \frac{\sum (x_i - \bar{x})^2}{n-1} \\ &= \frac{(95-112)^2 + (101-112)^2 + (114-112)^2 + (120-112)^2 + (130-112)^2}{5-1} \\ &= 200.5 \end{aligned}$$

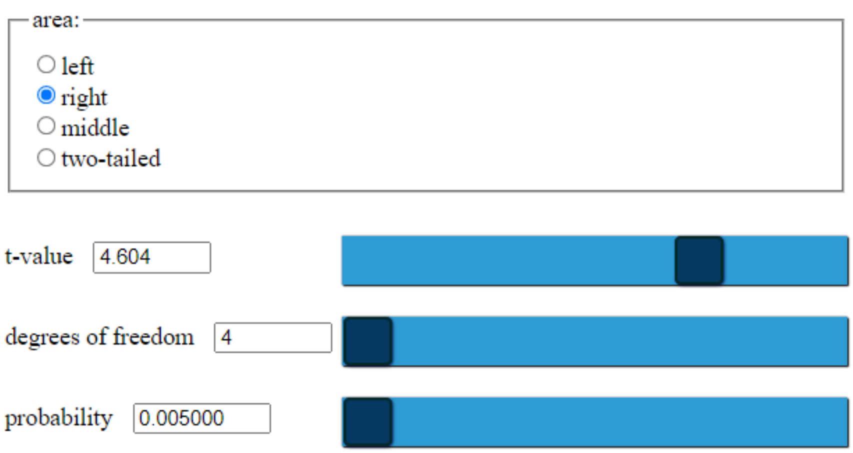
and $S \approx 14.1598$. Then our test statistic is

$$\mathcal{T} = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{112 - 100}{14.1598/\sqrt{5}} \approx 1.895.$$

Using an $\alpha = 0.01$ significance level with $n - 1 = 4$ degrees of freedom. Via Table 5, $t_{0.005}(4) = 4.604$. The respective $p$ value is

$$\begin{aligned} p &= \Pr(\mathcal{T} \geq 1.895 \mid \mu_0 = 100) \\ &< \Pr(\mathcal{T} \geq 4.604) \\ &= 0.005 \qquad \text{(WebAssign “technology”)} \\ &< \alpha \end{aligned}$$

Hence, we reject $H_0$ and conclude that there is sufficient evidence that the lifespan has increased.



3. A federal regulatory agency hypothesises that the average length of a stay in the hospital is in excess of 5 days. A pilot data set had standard *deviation* of 3.1 days. Using this as the population standard error, how large a sample would you need in designing a test with $\alpha = 0.01$ and $\beta = 0.05$ if the true average is 5.5 days.

**Solution:** We assume this follows a normal distribution and let

$$H_0 : \mu_0 = 5$$
$$H_a : \mu_a = 5.5$$

Then $Z_\alpha = Z_{0.01} \approx 2.33$ and $Z_\beta = Z_{0.05} \approx 1.645$. Using the following formula,

$$n = \left\lceil \left( \frac{Z_\alpha + Z_\beta}{\mu_a - \mu_0} \right)^2 \cdot S^2 \right\rceil$$

we can substitute in our data to obtain

$$n = \left\lceil \left( \frac{2.33 + 1.645}{5.5 - 5} \right)^2 \cdot 3.1^2 \right\rceil = \lceil 607.376 \rceil = 608.$$

Therefore the sample will need to be at least size $n = 608$.

4. A study was performed to compare two cholesterol-reducing drugs. Observations of the number of units of cholesterol reduction were recorded for 12 subjects receiving Drug A and 14 subjects receiving Drug B:

| | **Drug A** | **Drug B** |
|---|---|---|
| $n$ | 12 | 14 |
| Mean | 5.64 | 5.03 |
| stnd *deviation* | 1.25 | 1.82 |

Researchers are interested in testing if the drugs appear to be *different* in their average cholesterol reduction.

(a) State the assumptions needed for the independent samples $t$ test to be valid.

**Solution:**

- Population must be at least 20 times the sample size
- The variance of the two populations are the same.
- The two random samples are independent.
- The samples were selected from normal populations.

(b) Perform the $t$-test, find the $p$-value, and state the conclusion using a 5% significance level.

**Solution:** Using the pooled estimator, and since we assume $\mu_0 = \mu_a$ then $\mu_0 - \mu_a = 0$.

$$S_p^2 = \frac{\sum (X_i - \bar{X})^2 + \sum (Y_i - \bar{Y})^2}{n_1 + n_2 - 2}.$$

$$T = \frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}.$$

$$\frac{(5.64 - 5.03) - 0}{\sqrt{\frac{(12-1)\cdot 1.25^2 + (14-1)\cdot 1.82^2}{12+14-2}}\sqrt{\frac{1}{12} + \frac{1}{14}}} \approx 0.97865.$$

Using technology, $p = 0.337516$ Since $p > \alpha = 0.05$ we accept $H_0$ and conclude that there is **not** enough evidence to indicate a difference between the two drugs.