326 Homework 4

1. a) $E(\hat{\theta}_1) = E(\bar{X}) - 1/2 = \frac{1}{n}\sum E(X_i) - \frac{1}{2}$

$= \frac{1}{n} \cdot \frac{(\theta+1)+\theta}{2} \cdot n - \frac{1}{2} = \frac{2\theta+1}{2} - \frac{1}{2} = \theta.$

$E(\hat{\theta}_2) = E(X_{(n)}) - \frac{n}{n+1}.$

Need $E(X_{(n)})$: $F_{(n)}(x) = [P(X < x)]^n = \left[\int_\theta^x 1\,dt\right]^n$

$= (x-\theta)^n$

Then $f_{(n)}(x) = n(x-\theta)^{n-1}$

$E[X_{(n)}] = \int_\theta^{\theta+1} x \cdot n(x-\theta)^{n-1}\,dx$, let $u = x-\theta$

$= \int_0^1 (u+\theta) \cdot n u^{n-1}\,du.$

$= n\int_0^1 (u^n + \theta u^{n-1})\,du$

$= \frac{n}{n+1} + \theta$

Hence $E(\hat{\theta}_2) = \frac{n}{n+1} + \theta - \frac{n}{n+1} = \theta.$

b) In both cases, need to show $V(\hat{\theta}_i) \to 0$ as $n \to \infty$.

$$V(\hat{\theta}_1) = V(\bar{X}) = V\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n^2} \cdot n \cdot \frac{1}{12} = \frac{1}{12n}.$$

$V(\hat{\theta}_1) \to 0$ as $n \to \infty \Rightarrow \hat{\theta}_1 \to \theta$ in probability.

$$V(\hat{\theta}_2) = V(X_{(n)}) = E(X_{(n)}^2) - [E(X_{(n)})]^2.$$

As before, $E(X_{(n)}^2) = \int_{\theta}^{\theta+1} x^2 \cdot n(x-\theta)^{n-1}\,dx$

$$= \frac{n}{n+2} + \frac{2\theta n}{n+1} + \theta^2 \quad \text{by } u = x-\theta \text{ again.}$$

So $V(\hat{\theta}_2) = \left(\frac{n}{n+2} + \frac{2\theta n}{n+1} + \theta^2\right) - \left(\frac{n}{n+1} + \theta\right)^2$

$$= \frac{n}{n+2} - \frac{n^2}{(n+1)^2}$$

$$= \frac{n}{(n+2)(n+1)^2}$$

Then $\lim_{n\to\infty} V(\hat{\theta}_2) = \lim_{n\to\infty} \frac{n}{n^3 + \text{l.o.t.}} = 0.$

$\hat{\theta}_2$ also consistent.

c) $\text{Eff}(\hat{\theta}_1, \hat{\theta}_2) = \dfrac{V(\hat{\theta}_2)}{V(\hat{\theta}_1)} = \dfrac{\frac{n}{(n+2)(n+1)^2}}{\frac{1}{12n}} =$

$= \dfrac{12n^2}{(n+2)(n+1)^2}$

d) If Eff > 1 , $\hat{\theta}_1$ better.
But for small n, Eff > 1 while for large n $\hat{\theta}_2$ is better. $\left(\text{Clear } \lim_{n \to \infty} \text{Eff}(\hat{\theta}_1, \hat{\theta}_2) = 0 \right)$.

Note $n=7$, $\text{Eff}(\hat{\theta}_1, \hat{\theta}_2) = \dfrac{49}{48} > 0$

but $n=8$, $\text{Eff}(\hat{\theta}_1, \hat{\theta}_2) = \dfrac{128}{135} < 0$.

So $\hat{\theta}_1$ better if $n = 2, \ldots, 7$, after that $\hat{\theta}_2$ is better.

2. gamma$(\alpha,\beta) \Rightarrow \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta}$

$$L(x_1, x_2, \ldots x_n|\alpha) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x_1^{\alpha-1} e^{-x_1/\beta} \cdot \frac{1}{\Gamma(\alpha)\beta^\alpha} x_2^{\alpha-1} e^{-x_2/\beta} \cdots \frac{1}{\Gamma(\alpha)\beta^\alpha} x_n^{\alpha-1} e^{-x_n/\beta}$$

$$= \underbrace{\left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n x_1^{\alpha-1} x_2^{\alpha-1} \cdots x_n^{\alpha-1}}_{\text{this will be } g \text{ as all the } \alpha\text{'s are here}} \cdot \underbrace{e^{-\sum x_i/\beta}}_{h \text{ is data and known constant}}$$

$$= \left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n \exp(\ln(x_1^{\alpha-1} x_2^{\alpha-1} \cdots x_n^{\alpha-1})) \cdot \exp(-\sum x_i/\beta)$$

$$= \left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n \exp\left[(\alpha-1)\sum \ln x_i\right] \cdot \exp(-\sum x_i/\beta).$$

Let $T = \sum_1^n \ln x_i$

Then $g(T,\alpha) = \frac{1}{(\Gamma(\alpha)\beta^\alpha)^n} e^{(\alpha-1)T}$, $h(\vec{x}) = \exp(-\sum x_i/\beta)$.

By the Factorization Thm, $T = \sum \ln x_i$ is a sufficient statistic.

3a. $p(x_i \mid p) = p^{x_i}(1-p)^{1-x_i}$, $x_i = 0, 1$

$$E[T] = 1 \cdot P(X_1 = 1 \text{ and } X_2 = 0) + 0 \cdot P(\text{not}(X_1 = 1 \text{ and } X_2 = 0))$$
$$= P(X_1 = 1) P(X_2 = 0)$$
$$= p(1-p) \quad \text{or} \quad pq.$$

b. $$P(T=1 \mid S=s) = P(X_1 = 1, X_2 = 0 \mid X_1 + X_2 + \cdots + X_n = s)$$
$$= \frac{P(X_1 = 1, X_2 = 0 \text{ and } 1 + 0 + X_3 + X_4 + \cdots + X_n = s)}{P(X_1 + X_2 + \cdots + X_n = s)}$$
$$= \frac{P(X_1 = 1, X_2 = 0 \text{ and } X_3 + X_4 + \cdots + X_n = s-1)}{P(X_1 + X_2 + \cdots + X_n = s)}$$
$$= \frac{P(X_1 = 1) P(X_2 = 0) P(X_3 + \cdots + X_n = s-1)}{P(X_1 + X_2 + \cdots + X_n = s)}$$

$\sim \text{binom}(n-2, p) @ s-1$

$\sim \text{binom}(n, p) @ s.$

Since arbitrary binomial is $p(y) = \binom{n}{y} p^y q^{n-y}$, we get

$$P(T=1 \mid S=s) = \frac{p \cdot (1-p) \cdot \binom{n-2}{s-1} p^{s-1} (1-p)^{(n-2)-(s-1)}}{\binom{n}{s} p^s (1-p)^{n-s}}$$

$$= \frac{\binom{n-2}{s-1}}{\binom{n}{s}} \cdot \underbrace{p\, p^{s-1} p^{-s}}_{=1} \underbrace{(1-p)(1-p)^{n-1-s} (1-p)^{s-n}}_{=1}$$

$$= \frac{(n-2)!}{(n-s-1)!\,(s-1)!} \cdot \frac{(n-s)!\,s!}{n!}$$

$$= \frac{1}{n(n-1)} \cdot s \cdot n-s$$

$$= \frac{s(n-s)}{n(n-1)}.$$

c) $E[T|S] = 0 \cdot P(T=0|S=s) + 1 \cdot P(T=1|S=s)$

$$= \frac{s(n-s)}{n(n-1)}$$

$$= \frac{s}{n} \cdot \frac{n(1-s/n)}{n-1}$$

$$= \frac{n}{n-1} \cdot \bar{x} \cdot (1-\bar{x}).$$