326 Homework 2.

1A. $E[Y_1] = \mu$, unbiased

$MSE(Y_1) = V(Y_1) + B(Y_1)^2 = \sigma^2$.

B. $E[Y_1 + Y_2] = 2\mu$. Biased

$B(Y_1 + Y_2) = 2\mu - \mu = \mu$.

C. $E\left(\frac{Y_1 + 2Y_2 + 2Y_3 + Y_4}{6}\right) = \frac{\mu + 2\mu + 2\mu + \mu}{6} = \mu$. unbiased

$V\left(\frac{Y_1 + 2Y_2 + 2Y_3 + Y_4}{6}\right) = \frac{\sigma^2 + 4\sigma^2 + 4\sigma^2 + \sigma^2}{36} = \frac{5\sigma^2}{18}$

$MSE\left(\frac{Y_1 + 2Y_2 + 2Y_3 + Y_4}{6}\right) = V(\ ) + B(\ ) = \frac{5\sigma^2}{18}$

D. $E[\bar{Y}] = \mu$. $V[\bar{Y}] = \frac{\sigma^2}{4}$ as $n=4$.

$MSE(\bar{Y}) = V[\bar{Y}] = \sigma^2/4$ lowest.

2. Have $MSE(\hat{p}_1) = \dfrac{p-p^2}{n} \Rightarrow \dfrac{p-p^2}{15}$

$MSE(\hat{p}_2) = \dfrac{1+(n-4)p+(4-n)p^2}{(n+2)^2} \Rightarrow \dfrac{1+11p-11p^2}{289}$.

If $\hat{p}_2$ is better, then $MSE(\hat{p}_1) - MSE(\hat{p}_2) > 0$.

$D(p) = MSE(\hat{p}_1) - MSE(\hat{p}_2) = \dfrac{-15+124p-124p^2}{4335}$

$D(p) = 0$ when $p = \dfrac{31 \pm 4\sqrt{31}}{62}$.

Since $D(1/2) = 44 > 0$, $\hat{p}_2$ is the better estimator when

$$\frac{31-4\sqrt{31}}{62} < p < \frac{31+4\sqrt{31}}{62}.$$

3. a) $E(\bar{Y}) = E(Y) = \int_0^\theta y \cdot \frac{3y^2}{\theta^3} dy = \frac{3}{\theta^3}\int_0^\theta y^3 dy = \frac{3}{4}\theta.$

$\neq \theta \Rightarrow$ biased

b) Let $\tilde{\theta}_1 = \frac{4}{3}\hat{\theta}_1 = \frac{4}{3}\bar{Y}.$

c) Since $B(\tilde{\theta}_1) = 0$, $MSE(\tilde{\theta}_1) = V(\tilde{\theta}_1)$

$V(\tilde{\theta}_1) = V\left(\frac{4}{3}\bar{Y}\right) = \frac{16}{9}V(\bar{Y}) = \frac{16}{9}\cdot\frac{V(Y)}{n}.$

For $V(Y)$, use $E(Y^2) - E(Y)^2$.

$E(Y^2) = \int_0^\theta y^2 \cdot \frac{3y^2}{\theta^3} dy = \frac{3}{\theta^3}\int_0^\theta y^4 dy = \frac{3\theta^2}{5}.$

$V(Y) = \frac{3\theta^2}{5} - \left(\frac{3}{4}\theta\right)^2 = \frac{3}{80}\theta^2.$

So $MSE(\tilde{\theta}_1) = \frac{16}{9}\cdot\frac{3\theta^2/80}{n} = \frac{\theta^2}{15n}$

d) For max order stat, we have

$P(Y_{(n)} \leq y) = [F(y)]^n$ w/ pdf

$m(u) = n[F(u)]^{n-1}\cdot f(u).$

For $F(u) = \int_0^u \frac{3y^2}{\theta^3} dy = \left(\frac{y}{\theta}\right)^3$

and $m(u) = n\left(\frac{y}{\theta}\right)^{3(n-1)}\cdot\frac{3y^2}{\theta^3} = \frac{3ny^{3n-1}}{\theta^{3n}},$

$0 < u < \theta$

Then $E(\hat{\theta}_2) = \int_0^\theta u \cdot \frac{3n u^{3n-1}}{\theta^{3n}} du$

$= \frac{3n}{\theta^{3n}} \int_0^\theta u^{3n} du = \frac{3n}{3n+1}\theta \neq \theta.$

e) $\tilde{\theta}_2 = \frac{3n+1}{3n} \hat{\theta}_2$.

f) As before, $MSE(\tilde{\theta}_2) = V(\tilde{\theta}_2) = \left(\frac{3n+1}{3n}\right)^2 V(\hat{\theta}_2)$

For $V(\hat{\theta}_2)$, $E(U^2) = \int_0^\theta u^2 \cdot \frac{3n u^{3n-1}}{\theta^{3n}} du = \frac{3n}{3n+2}\theta^2$.

So $MSE(\tilde{\theta}_2) = \left(\frac{3n+1}{3n}\right)^2 \left[\frac{3n}{3n+2}\theta^2 - \left(\frac{3n}{3n+1}\theta\right)^2\right]$

$= \theta^2 \left[\frac{(3n+1)^2}{3n(3n+2)} - 1\right]$

$= \frac{\theta^2}{3n(3n+2)}$

Note $\tilde{\theta}_2$ is a better estimator than $\tilde{\theta}_1$ when $n > 1$.

4. p_1 : proportion of graduates who are firstborn

p_2 : proportion of non-graduates who are firstborn.

p_1 estimated by $\hat{p}_1 = \frac{126}{180} = \frac{70}{100}$

Note $n_1 = 180$ and $\sigma_{\hat{p}_1}^2 = \frac{p_1 q_1}{n_1} \approx \frac{(0.7)(0.3)}{180}$

p_2 estimated by $\hat{p}_2 = \frac{54}{100}$

Note $n_2 = 100$ and $\sigma_{\hat{p}_2}^2 \approx \frac{(0.54)(0.46)}{100}$

So $p_1 - p_2$ estimated by

$$\hat{p}_1 - \hat{p}_2 \pm 2\sqrt{\sigma_{\hat{p}_1}^2 + \sigma_{\hat{p}_2}^2}$$

$$(.7 - .54) \pm 2(.0604)$$

$$0.16 \pm 0.1208 \text{ or } (0.04, 0.28)$$

5. a) $E(x) = aE(\hat{\theta}_1) + bE(\hat{\theta}_2) + cE(\hat{\theta}_3)$

$= a\theta + b\theta + c\theta$

$= (a+b+c)\theta$

$= \theta.$

b) $V(x) = a^2 V(\hat{\theta}_1) + b^2 V(\hat{\theta}_2) + c^2 V(\hat{\theta}_3)$ since indy.

$= a^2(1+1) + b^2(1+2) + c^2(1+3)$

$= 2a^2 + 3b^2 + 4c^2.$

Minimize $V(x) = 2a^2 + 3b^2 + 4c^2$

Subject to: $a+b+c=1.$

$\nabla V = \langle 4a, 6b + 8c \rangle$

Lagrange: $\nabla V \parallel \langle 1,1,1 \rangle$

i.e. $4a = \lambda$ $4a = 6b \Rightarrow b = \frac{2}{3}a$

$6b = \lambda$,

$8c = \lambda$ $4a = 8c \Rightarrow c = \frac{1}{2}a.$

Then $a + \frac{2}{3}a + \frac{1}{2}a = 1$, $6a + 4a + 3a = 6$, $a = 6/13$.

$b = 4/13$, $c = 3/13$.