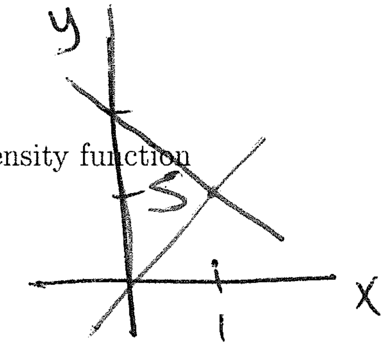
Math 325 Section 01 **Exam 3**
Professor Carty **December 1, 2021**
43 points possible

Name: ____

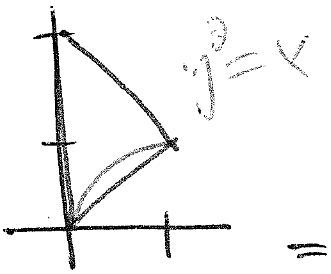
1. No hats or dark sunglasses. All hats are to be removed.
2. All book bags are to be closed and placed in a way that makes them inaccessible. Do not reach into your bag for anything during the exam. If you need extra pencils, pull them out now.
3. Be sure to print your proper name clearly.
4. Calculators are optional, but if you choose to use one you may ONLY use the battery or solarpowered Texas Instruments BA35 model calculator, the BA II Plus, the BA II Plus Professional, the TI30Xa or TI30X II (IIS solar or IIB battery), or TI-30X MultiView (XS Solar or XB Battery).
5. All electronic devices, including cell phones and other wearable devices, must be silenced or powered off and stored out of sight for the entirety of the exam.
6. If you have a question, raise your hand and I will come to you. Once you stand up, you are done with the exam. If you have to use the facilities, do so now. You will not be permitted to leave the room and return during the exam.
7. Every exam is worth a total of **40 points**. Including the cover sheet, each exam has 7 pages.
8. At 11:45, you will be instructed to put down your writing utensil. Anyone seen continuing to write after this announcement will have their exam marked and lose all points on the page they are writing on. At this time, you will use your phone to take PDFs of the pages and upload them into Canvas as a single PDF document (the same way you do with homework.) Solutions must be uploaded by 11:55.
9. If you finish early, quietly and respectfully perform the preceding tasks. You may leave early.
10. You will hand in the paper copy of the exam on your way out of the classroom.
11. You have forty-five minutes to complete the exam. I hope you do well.

1. (12 points) Suppose that random variables X and Y have the density function



$$f(x,y) = \begin{cases} 6x^2y, & 0 \le x \le y, x+y \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Find the probability of the event $Y^2 \le X$.

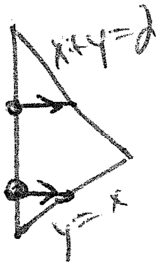


$$P(Y^2 \le X) = \int_0^1 \int_{y=x}^{\sqrt{x}} 6x^2 y\,dy\,dx$$

$$= \int_0^1 3x^2y^2 \Big|_{y=x}^{\sqrt{x}} = \int_0^1 (3x^3 - 3x^4)\,dx$$

$$= \left(\frac{3x^4}{4} - \frac{3x^5}{5}\right)\Big|_0^1 = \frac{3}{4} - \frac{3}{5} = \frac{3}{20}.$$

(b) Find the marginal density function of Y.



2 parts: $0 \le y \le 1$, $f_Y(y) = \int_{x=0}^{y} 6x^2y\,dx = 2x^3y \Big|_0^y = 2y^4$

$1 \le y \le 2$, $f_Y(y) = \int_0^{2-y} 6x^2y\,dx = 2(2-y)^3y$

$f_Y(y) = 0$ elsewhere.

(c) Find $P(X < 1/3|Y = 3/2)$.

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{6x^2y}{2(2-y)^3y}, \text{ since } y>1. \quad f(x|y) = \frac{3x^2}{(2-y)^3}$$

$$f(x|3/2) = \frac{3x^2}{(1/2)^3} = 24x^2$$

$$P(X \le 1/3 \mid Y=3/2) = \int_0^{1/3} 24x^2\,dx = 8x^3 \Big|_0^{1/3} = 8/27$$

2. (8 points) Let Y be a random variable with a density function given by

$$f(y) = (3/2)y^2, \text{ on support } [-1, 1].$$

Consider the random variable $U = 3 + 2Y$.

(a) Determine the cumulative distribution function for U.

$$F(u) = P(U \leq u) = P(3 + 2Y \leq u) = P\left(Y \leq \frac{u-3}{2}\right)$$

Note: need $-1 \leq \frac{u-3}{2} \leq 1$, $-2 \leq u-3 \leq 2$, $\boxed{1 \leq u \leq 5}$

Then $F(u) = \int_{-1}^{\frac{u-3}{2}} \frac{3}{2} y^2 \, dy = \frac{y^3}{2} \Big|_{-1}^{\frac{u-3}{2}} = \frac{1}{2}\left(\frac{u-3}{2}\right)^3 + \frac{1}{2}.$

on $1 \leq u \leq 5$

$$F(u) = \begin{cases} 0, & u < 1 \\ \frac{1}{2}\left(\frac{u-3}{2}\right)^3 + \frac{1}{2}, & 1 \leq u \leq 5 \\ 1, & u \geq 5 \end{cases}$$

(b) Determine the probability density function of U.

$$f(u) = F'(u) = \frac{3}{2}\left(\frac{u-3}{2}\right)^2 \cdot \frac{1}{2}$$

on support $[1, 5]$.

3. (5 points) Let Y_1, Y_2, and Y_3 denote a random sample of size 3 from a gamma distribution with $\alpha = 7$ and $\beta = 5$.

(a) Find the moment-generating function of $S = Y_1 + Y_2 + Y_3$.

For Y_i, $m_i(t) = (1-5t)^{-7}$

Then $m_S(t) = m_1(t) \times m_2(t) \times m_3(t)$

$= [(1-5t)^{-7}]^3$

$= (1-5t)^{-21}$

(b) Determine the probability density function for the random variable S.

S is gamma w/ $\alpha = 21$, $\beta = 5$

and

$$f(s) = \frac{1}{\Gamma(21) 5^{21}} s^{20} e^{-s/5}, \quad s \geq 0$$

4. (5 points) Let X_1, X_2, and X_3 be independent, exponentially distributed random variables with mean $\beta = 3$. Find the density function of $Y = \min\{X_1, X_2, X_3\}$.

$\alpha = 1$, $\beta = 3$

$f(x) = \frac{1}{3} e^{-x/3}$ for each X_i

Recall min order stat: $Y_1(y) = n[1 - F(y)]^{n-1} f(y)$

need $F(y)$: $F(y) = P(Y \leq y) = \int_0^y \frac{1}{3} e^{-x/3} dx$

$= -e^{-x/3} \Big|_0^y = -e^{-y/3} + 1.$

For $n = 3$,

$Y_1(y) = 3[1 - (-e^{-y/3} + 1)]^2 \cdot \left(\frac{1}{3} e^{-y/3}\right)$

$= 3(e^{-2y/3})\left(\frac{1}{3} e^{-y/3}\right)$

$= e^{-y}, \quad y > 0.$

5. Let X and Y be uncorrelated random variables and consider $A = X+Y$ and $B = X-Y$.

(a) (3 points) Determine $E(XY) - \mu_X\mu_Y$.

$$E(XY) - \mu_x \mu_y = E((X-\mu_x)(Y-\mu_y))$$
$$= Cov(X,Y)$$
$$= 0 \quad \text{as "uncorrelated"}$$

(b) (4 points) Find the Cov(A, B) in terms of the variances of X and Y.

$$Cov(A,B) = Cov(X+Y, X-Y)$$
$$= Cov(X,X) + Cov(X,-Y) + Cov(Y,X) + Cov(Y,-Y)$$
$$= V(X) - Cov(X,Y) + Cov(X,Y) - V(Y)$$
$$= V(X) - V(Y)$$

(c) (3 points) Determine Var(B) in terms of the variances of X and Y.

$$V(B) = V(X-Y) = V(X) + (-1)^2 V(Y) + 2\,Cov(X,-Y)$$
$$= V(X) + V(Y) - 2\,Cov(X,Y)$$
$$= V(X) + V(Y)$$

(d) (3 points) **Extra Credit:** Find an expression for the correlation coefficients between A and B.

Recall $\rho = \dfrac{Cov(A,B)}{\sigma_A \sigma_B} = \dfrac{Cov(A,B)}{\sqrt{V(A)}\sqrt{V(B)}}$.

Note $V(A) = V(X+Y) = V(X)+V(Y) + 2Cov(X,Y)$
$= V(X)+V(Y)$
$= V(B)$

So $\rho = \dfrac{V(X)-V(Y)}{V(X)+V(Y)}$.