Math 326 – Homework 02 (8.1 – 8.4)
Due (via upload to Canvas) Wednesday, February 02, 2022 at 11:59 PM

1. Suppose Y_1, Y_2, Y_3, and Y_4 are an iid random sample with distribution $\mathrm{N}(\mu, \sigma^2)$. For each of the following, determine which of the following estimators of μ are unbiased. Then for each unbiased estimator, calculate the mean square error. Which estimator has the lowest MSE?

   A. Y_1 B. $Y_1 + Y_2$ C. $\dfrac{Y_1 + 2Y_2 + 2Y_3 + Y_4}{6}$ D. $\overline{Y}$

2. In class we explored the estimators $\hat{p}_1 = \overline{Y}$ and $\hat{p}_2 = \dfrac{\sum Y_i + 1}{n+2}$ for population proportion p. For $n = 15$, what values of p is $\hat{p}_2$ the better estimator with respect to MSE?

3. Suppose $Y_1, Y_2, Y_3, \ldots Y_n$ is an iid random sample from a distribution with the following density function
$$f(y) = 3y^2/\theta^3 \text{ on support } 0 < y < \theta.$$
   Consider two estimators of θ: $\hat{\theta}_1 = \overline{Y}$ and $\hat{\theta}_2 = \max(Y_1, Y_2, Y_3, \ldots Y_n)$.

   (a) Show that $\hat{\theta}_1$ is a biased estimator of θ.

   (b) Define a multiple of $\hat{\theta}_1$ that is an unbiased estimator of θ. Call this new estimator $\tilde{\theta}_1$.

   (c) Compute the MSE for $\tilde{\theta}_1$.

   (d) Show that $\hat{\theta}_2$ is a biased estimator of θ.

   (e) Define a multiple of $\hat{\theta}_2$ that is an unbiased estimator of θ. Call this new estimator $\tilde{\theta}_2$.

   (f) Compute the MSE for $\tilde{\theta}_2$.

4. In a study of the relationship between birth order and college success, an investigator found that 126 in a sample of 180 college graduates were firstborn or only children; in a sample of 100 nongraduates of comparable age and socioeconmomic background, the number of firstborn or only children was 54. Estimate the difference in the proportions of firstborn or only children for two populations from which these samples were drawn. Give a bound for the error of estimation.

5. Suppose $\hat{\theta}_1$, $\hat{\theta}_2$, and $\hat{\theta}_3$ are all unbiased estimators of θ. Suppose that $\mathrm{Var}(\hat{\theta}_i) = 1 + i$ for $i = 1, 2, 3$. Let $X = a\hat{\theta}_1 + b\hat{\theta}_2 + c\hat{\theta}_3$, where a, b, and c are non-negative constants with $a + b + c = 1$.

   (a) Show that X is unbiased for θ.

   (b) Assuming that the $\hat{\theta}_i$'s are independent, find a, b, and c that minimize $\mathrm{Var}(X)$.