Math 345 - Homework 5 **Due Friday, October 7, 2022**

1. Find the general solution of each of the differential equations.

(a) $y''' - 3y'' + 3y' - y = 4e^t$

Solution: Solving the complementary problem y_c first is $y''' - 3y'' + 3y' - y = 0$. Using the binomial theorem this is $(D-1)^3 = 0$ and therefore the fundemental set of y_c is $\{e^t, te^t, t^2e^t\}$. Since $4e^t$ is a solution to $D-1$ we will use it as an annihilator A.

$$\begin{aligned} A(D-1)^3 &= A4e^t \\ (D-1)^4 &= 0 \end{aligned}$$

Therefore $\{e^t, te^t, t^2e^t, \underbrace{t^3e^t}_{y_p}\}$ is the fundemental set of y_a. Then $y_c = c_1e^t + c_2te^t + c_3t^2e^t$ and $y_p = c_4t^3e^t$. Differentiating y_p,

$$\begin{aligned} y_p &= c_4t^3e^t \\ y_p' &= 3c_4t^2e^t + c_4t^3e^t \\ y_p'' &= 6c_4te^t + 3c_4t^2e^t + 3c_4t^2e^t + c_4t^3e^t \\ &= 6c_4te^t + 6c_4t^2e^t + c_4t^3e^t \\ y_p''' &= 6c_4te^t + 6c_4e^t + 6c_4t^2e^t + 12c_4te^t + c_4t^3e^t + 3c_4t^2e^t \\ &= 6c_4e^t + 18c_4te^t + 9c_4t^2e^t + c_4t^3e^t \end{aligned}$$

Substituting into the original ODE,

$$\begin{aligned} y''' - 3y'' + 3y' - y &= 4e^t \\ c_4e^t\left[\underbrace{\left[6 + 18t + 9t^2 + t^3\right]}_{y'''} - 3\underbrace{\left[6t + 6t^2 + t^3\right]}_{y''} + 3\underbrace{\left[3t^2 + t^3\right]}_{y'} - \underbrace{\left[t^3\right]}_{y}\right] &= 4e^t \\ 6c_4e^t &= 4e^t \\ c_4 &= \frac{2}{3} \end{aligned}$$

Therefore $y_p = \frac{2}{3}t^3e^t$ and $y = c_1e^t + c_2te^t + c_3t^2e^t + \frac{2}{3}t^3e^t$

(b) $y^{(4)} + 2y'' + y = 3\sin t - 5\cos t$

Solution: This can be factored into $(D^2+1)^2$ and hence y_c has the fundemental solution set of $\{\sin t, t\sin t, \cos t, t\cos t\}$. Since $3\sin t - 5\cos t$ is a solution to $A = D^2 + 1$, we will let A be the annihilator. Thus $(D^2+1)^3 = 0$ and the fundemental solutions for y_a are $\{\sin t, t\sin t, t^2\sin t, \cos t, t\cos t, t^2\cos t\}$

Differentiating y_p,

$$\begin{aligned}
y_p &= c_5 t^2 \sin t + c_6 t^2 \cos t \\
y_p' &= 2c_5 t \sin t + c_5 t^2 \cos t + 2c_6 t \cos t - c_6 t^2 \sin t \\
y_p'' &= c_5 \left[-t^2 \sin t + 4t \cos t + 2\sin(t)\right] + c_6 \left[2\cos t - 4t \sin t - t^2 \cos t\right] \\
y_p''' &= c_5 \left[-t^2 \cos t - 6t \sin t + 6\cos t\right] + c_6 \left[-6\sin t - 6t \cos t + t^2 \sin t\right] \\
y_p^{(4)} &= c_5 \left[t^2 \sin t - 8t \cos t - 12\sin t\right] + c_6 \left[-12\cos t + 8t \sin t + t^2 \cos t\right]
\end{aligned}$$

And clearly $y_p^{(4)} + 2y_p'' + y_p = -8c_5 \sin t - 8c_6 \cos t = 3\sin t - 5\cos t$. Letting $\sin t$ and $\cos t$ be vector component variables, $\langle -8c_5, -8c_6 \rangle = \langle 3, -5 \rangle$ so $c_5 = -\frac{3}{8}$ and $c_6 = \frac{5}{8}$. It follows that the general solution is

$$y(t) = c_1 \sin t + c_2 \cos t + c_3 t \sin t + c_4 t \cos t - \frac{3}{8} t^2 \sin t + \frac{5}{8} t^2 \cos t$$

(c) $y''' + 4y' = t + \cos t + e^{-2t}$

Solution: Factoring yields $D(D^2+4)$ so the fundemental set for y_c is $\{1, \sin(2t), \cos(2t)\}$. For constructing an annihilator, t implies D^2, $\cos t$ implies D^2+1, and e^{-2t} implies $D+2$. Hence $A = D^2(D^2+1)(D+2)$ and $D^3(D^2+4)(D^2+1)(D+2) = 0$. So the fundemental set for y_a is

$$\{1, t, t^2, \sin 2t, \cos 2t, \sin t, \cos t, e^{-2t}\}.$$

Differentiating y_p,

$$\begin{aligned}
y_p &= d_1 t + d_2 t^2 + d_3 \sin t + d_4 \cos t + d_5 e^{-2t} \\
y_p' &= 2d_2 t + d_3 \cos t - d_4 \sin t - 2d_5 e^{-2t} \\
y_p'' &= 2d_2 - d_3 \sin t - d_4 \cos t + 4d_5 e^{-2t} \\
y_p''' &= -d_3 \cos t + d_4 \sin t - 8d_5 e^{-2t}
\end{aligned}$$

Then

$$\begin{aligned}
y''' + 4y' &= t + \cos t + e^{-2t} \\
\left[-d_3 \cos t + d_4 \sin t - 8d_5 e^{-2t}\right] + 4\left[2d_2 t + d_3 \cos t - d_4 \sin t - 2d_5 e^{-2t}\right] &= t + \cos t + e^{-2t} \\
8d_2 t + 3d_3 \cos t - 3d_4 \sin t - 16d_5 e^{-2t} &= t + \cos t + e^{-2t}
\end{aligned}$$

Therefore $d_1 = 0,\ d_2 = \frac{1}{8},\ d_3 = \frac{1}{3},\ d_4 = 0,\ d_5 = -\frac{1}{16}$ and hence the general solution is

$$y(t) = c_1 + c_2 \sin(2t) + c_3 \cos(2t) + \frac{1}{8} t^2 + \frac{1}{3} \sin t - \frac{1}{16} e^{-2t}$$

2. Find the solution of the given IVP

$$y^{(4)} + 2y'' + y = 3\sin t - 5\cos t, \quad y(0) = y'(0) = 0, \quad y''(0) = y'''(0) = 1.$$

Solution: We have the general solution from (1b)

$$y(t) = c_1 \sin t + c_2 \cos t + c_3 t \sin t + c_4 t \cos t - \frac{3}{8}t^2 \sin t + \frac{5}{8}t^2 \cos t.$$

Differentiating that gives us

$$\begin{aligned}
y' &= c_1 \cos t - c_2 \sin t + c_3 \sin t + c_3 t \cos t + c_4 \cos t - c_4 t \sin t - \frac{3}{4}t \sin t - \frac{3}{8}t^2 \cos t + \frac{5}{4}t \cos t - \frac{5}{8}t^2 \sin t \\
y'' &= -c_1 \sin t - c_2 \cos t + c_3 \left[2\cos t - t\sin t\right] - c_4 \left[2\sin t + t\cos t\right] \\
&\quad - \frac{3}{4}\sin t - \frac{t\,(3\cos t + 5\sin 5)}{2} + \frac{5}{4}\cos t + \frac{t^2(3\sin t - 5\cos t)}{8} \\
y''' &= -\frac{9}{4}\cos t - \frac{15}{4}\sin t - c_1 \cos t - c_2 \sin t - 3c_3 \sin t - 3c_4 \cos t \\
&\quad - \frac{9}{4}t \sin t - \frac{15}{4}t\cos t - c_3 t \cos t + c_4 t \sin t + \frac{3}{8}t^2 \cos t + \frac{5}{8}t^2 \sin t
\end{aligned}$$

But by the initial conditions, $y(0) = c_2 = 0$, and $y'(0) = c_1 + c_4 = 0$. Then $y''(0) = 2c_3 + \frac{5}{4} = 1$ implies $c_3 = -\frac{1}{8}$ and $y'''(0) = -2c_4 - \frac{9}{4} = 1$ implies $c_4 = -\frac{13}{8}$. This can be back substituted to find $c_1 = \frac{13}{8}$. Hence the solution to the IVP is

$$y(t) = \frac{1}{8}\left[13\sin t - t\sin t - 13t\cos t - 3t^2 \sin t + 5t^2 \cos t\right]$$

3. Convert the given IVP to its equivalent first-order linear initial value system.

$$y^{(4)} + 2y'' + y = 3\sin t - 5\cos t, \quad y(0) = y'(0) = 0, \quad y''(0) = y'''(0) = 1.$$

Solution: We make the following substitutions:

$$y_1 = y \qquad y_1' = y_2 \qquad y_2' = y_3 \qquad y_3' = y_4 \qquad y_4' = -2y_3 - y_1$$

And then we create the following linear system:

$$\begin{bmatrix} \dot{y}_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3\sin t - 5\cos t \end{bmatrix}$$

4. Show that the function $f(t,x) = 5 \sin t \cos x$ is Lipschitz on its domain. Be sure to clearly justify your answer.

Solution: there is none

$$\{a^n b^{n+1} \mid n \geq 0\}$$