Math 345 - Homework 4 **Due Friday, September 29, 2022**

1. Find the general solution for each of the differential equations.

(a) $y''' - y = 0$

Solution: This is equivalent to $D^3 - D = 0$. Using the difference of cubes formula this can be refactored as $(D - I)(D^2 + D + I)$. Applying the quadratic formula to the right-most factor yields the following:

$$\underbrace{(D-I)}_{e^x}\underbrace{\left(D - \frac{1+\sqrt{3}}{2}i\right)\left(D - \frac{1-\sqrt{3}}{2}i\right)}_{e^{-x/2}\cos(\sqrt{3}/2),\quad e^{-x/2}\sin(\sqrt{3}/2)}.$$

Therefore the general solution is $\boxed{y(x) = c_1e^x + \frac{c_2}{\sqrt{e^x}}\cos\left(\frac{\sqrt{3}}{2}x\right) + \frac{c_3}{\sqrt{e^x}}\sin\left(\frac{\sqrt{3}}{2}x\right).}$

(b) $y''' + 4y'' + 4' = 0$

Solution: This is $D^3 + 4D^2 + 4D = 0$. Factoring out a D yields $D(D^2 + 4D + 4)$ and factoring the quadratic term yields the following:

$$\underbrace{D}_{1}\ \underbrace{(D+2)^2}_{e^{-2x},\ xe^{-2x}}.$$

It follows that the general solution is $\boxed{y(x) = c_1 + \frac{c_2}{e^{2x}} + \frac{c_3x}{e^{2x}}.}$

(c) $y^{(4)} + 2y'' + y = 0$

Solution: This can be written as $D^4 + 2D^2 + I = 0$. Factoring this as a quadratic in terms of D^2 yields $(D^2 + 1)^2$. The $D^2 + 1$ factor results in $\cos x$ and $\sin x$, but because it is multiplicity 2 then we also have $x\cos x$ and $x\sin x$.

Therefore the general solution is $\boxed{y(x) = c_1\cos x + c_2x\cos x + c_3\sin x + c_4x\sin x.}$

(d) $y^{(4)} + 4y''' + 6y'' + 4y' + y = 0$

Solution: This is equivalent to $D^4 + 4D^3 + 6D^2 + 4D + I = 0$. Using the binomial theorem this can be factored into $(D + I)^4 = 0$. The $D + I$ factor yields e^{-x}, and a multiplicity of 4 results in xe^{-x}, x^2e^{-x} and x^3e^{-x}.

Therefore the general solution is $\boxed{y(x) = c_1e^{-x} + c_2xe^{-x} + c_3x^2e^{-x} + c_4x^3e^{-x}.}$

2. If m is a positive constant, find the solution of the initial value problem

$$y''' - my'' + m^2y' - m^3y = 0$$

where $f(0) = f'(0) = 0,\ f''(0) = 1.$

Solution: Begin by factoring the cubic.

$$\begin{aligned}
&D^3 - mD^2 + m^2D - m^3 = 0 \\
&D^2(D-m) + m^2(D-m) = 0 \\
&(D-m)(D^2+m^2) = 0 \\
&\underbrace{(D-m)}_{e^{mx}}\underbrace{(D+im)(D-im)}_{\cos(mx),\quad \sin(mx)} = 0
\end{aligned}$$

We can now write the general solution to the ODE and differentiate twice.

$$\begin{aligned}
y(x) &= c_1e^{mx} + c_2\cos(mx) + c_3\sin(mx) \\
y'(x) &= m\left[c_1e^{mx} - c_2\sin(mx) + c_3\cos(mx)\right] \\
y''(x) &= m^2\left[c_1e^{mx} - c_2\cos(mx) - c_3\sin(mx)\right]
\end{aligned}$$

Evaluating these functions at the initial conditions results in the following system of equations

$$\begin{aligned}
y(0) &= c_1 + c_2 = 0 \\
y'(0) &= mc_1 + mc_3 = 0 \\
y''(0) &= m^2c_1 - m^2c_2 = 1
\end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ m & 0 & m & 0 \\ m^2 & -m^2 & 0 & 1 \end{array}\right] \xRightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.5m^{-2} \\ 0 & 1 & 0 & -0.5m^{-2} \\ 0 & 0 & 1 & -0.5m^{-2} \end{array}\right]$$

Thus $c_1 = \dfrac{1}{2m^2}$, $c_2 = \dfrac{-1}{2m^2}$, and $c_3 = \dfrac{-1}{2m^2}$. It follows that the solution to the initial value problem is

$$\boxed{y(x) = \frac{1}{2m^2}\left[e^{mx} - \cos(mx) - \sin(mx)\right].}$$

3. Find a linear differential equation $Ly = 0$ with constant coefficients where $y_1(x) = e^{-2x}\cos(3x)$, $y_2(x) = x^2$, and $y_3(x) = x\sin x$ are solutions.

Solution: y_1 implies that $e^{-2x}\sin(3x)$ is also in $\ker L$. y_2 implies that $\operatorname{span}\{1, x\} \in \ker L$. Lastly, y_3 implies that $\operatorname{span}\{\sin x, \cos x, x\cos x\} \in \ker L$. Then we can combine the operators from the kernel and still remain in the kernel.

For y_1 we have $(D + (2+3i)I)(D + (2-3i)I) = D^2 + 4D + 13I$, y_2 has D^3 and y_3 has $(D^2 + I)^2$.

Multiplying out the operators results in the following

$$\begin{aligned}
&D^3(D^2 + 4D + 13I)(D^2 + I)^2 = 0\\
&D^3(D^2 + 4D + 13I)(D^4 + 2D^2 + I) = 0\\
&D^3(D^6 + 4D^5 + 15D^4 + 8D^3 + 27D^2 + 4D + 13I) = 0\\
&D^9 + 4D^8 + 15D^7 + 8D^6 + 27D^5 + 4D^4 + 13D^3 = 0
\end{aligned}$$

Hence, a linear differential equation that satisfies the constaints is

$$\boxed{y^{(9)} + 4y^{(8)} + 15y^{(7)} + 8y^{(6)} + 27y^{(5)} + 4y^{(4)} + 13y^{(3)} = 0.}$$

4. Find the general solution (by hand) of the differential equation

$$y'' - y = \frac{2}{1 + e^x}.$$

Solution: First we solve the complementary equation $y'' - y = 0$. This is equivalent to $D^2 - I = 0$, which factors into $(D + I)(D - I)$. Thus the fundamental set is $\{e^{-x}, e^x\}$ and

$$y_c = c_1 e^{-x} + c_2 e^x.$$

We want $Ly_p = \dfrac{2}{1 + e^x}$ and $y_p = v_1 e^{-x} + v_2 e^x$. Therefore

$$y_p{}' = -v_1 e^{-x} + v_1' e^{-x} + v_2 e^x + v_2' e^x.$$

Here we assume $v_1' e^{-x} + v_2' e^x = 0$. Hence $y_p{}' = -v_1 e^{-x} + v_2 e^x$. Therefore

$$y_p{}'' = \underbrace{v_1 e^{-x} + v_2 e^x}_{y_p} - v_1' e^{-x} + v_2' e^x.$$

It follows that

$$y_p{}'' - y_p = -v_1' e^{-x} + v_2' e^x = \frac{2}{1 + e^x}.$$

We now have a system of equations,

$$\underbrace{\begin{bmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{bmatrix}}_{:=A} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2/(1+e^x) \end{bmatrix}$$

Note that $\det A = W(e^{-x}, e^{x}) = 2$ and $A^{-1} = \dfrac{1}{\det A}\left[(\operatorname{tr} A)I - A\right] = \dfrac{1}{2}\begin{bmatrix} e^{x} & -e^{x} \\ e^{-x} & e^{-x} \end{bmatrix}$

Multiplying by A^{-1} on each side yields

$$\begin{aligned} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} e^{x} & -e^{x} \\ e^{-x} & e^{-x} \end{bmatrix}\begin{bmatrix} 0 \\ 2/(1+e^x) \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{-e^x}{1+e^x} & \dfrac{e^x}{1+e^x} \end{bmatrix}^T \end{aligned}$$

$$\begin{aligned} v_1' &= \frac{-e^x}{1+e^x} \\ \int v_1' \mathrm{d}x &= -\int \frac{e^x}{1+e^x}\mathrm{d}x \\ v_1 &= -\int \frac{\mathrm{d}u}{u} && (\text{Let } u = 1+e^x) \\ v_1 &= -\ln\left|1+e^x\right| \end{aligned}$$

$$\begin{aligned} v_2' &= \frac{e^x}{1+e^x} \\ v_2 &= \ln\left|1+e^x\right| \end{aligned}$$

Substituting these functions into $y_p = v_1 y_1 + v_2 y_2$ yields

$$y_p = -e^{-x}\ln\left|1+e^x\right| + e^{x}\ln\left|1+e^x\right|.$$

The general solution $y = y_c + y_p$ is

$$\boxed{y(x) = c_1 e^{-x} + c_2 e^{x} - e^{-x}\ln|1+e^x| + e^{x}\ln|1+e^x|}$$

5. Consider the non-homogeneous ODE

$$y''' + \frac{1}{x}y'' - \frac{2}{x^2}y' + \frac{2}{x^3}y = 2x.$$

(a) Verify that $\{x, x^2, 1/x\}$ form a fundamental set to the corresponding complementary ODE.

Solution: Let $y_1 := x$, $y_2 := x^2$, and $y_3 := 1/x$. Computing the Wronskian,

$$\begin{aligned} W(y_1, y_2, y_3) &= \begin{vmatrix} x & x^2 & x^{-1} \\ 1 & 2x & -x^{-2} \\ 0 & 2 & 2x^{-3} \end{vmatrix} \\ &= x \begin{vmatrix} 2x & -x^{-2} \\ 2 & 2x^{-3} \end{vmatrix} - x^2 \begin{vmatrix} 1 & -x^{-2} \\ 0 & 2x^{-3} \end{vmatrix} + x^{-1} \begin{vmatrix} 1 & 2x \\ 0 & 2 \end{vmatrix} \\ &= 6/x - 2/x + 2/x \\ &= 6/x \end{aligned}$$

Since the Wronskian is not zero, the set is linearly independent and 3 vectors in a third-order ODE span $\ker L$, thus forming a basis.

(b) Determine a particular solution of the non-homogeneous ODE.

Solution: For a particular solution we need

$$\begin{bmatrix} x & x^2 & x^{-1} \\ 1 & 2x & -x^{-2} \\ 0 & 2 & 2x^{-3} \end{bmatrix} \begin{bmatrix} v_1{}' \\ v_2{}' \\ v_3{}' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2x \end{bmatrix}.$$

Using Cramer's rule,

$$\begin{aligned} v_1{}' &= W^{-1}(y_1, y_2, y_3) \begin{vmatrix} 0 & x^2 & x^{-1} \\ 0 & 2x & -x^{-2} \\ 2x & 2 & 2x^{-3} \end{vmatrix} \\ &= \frac{x}{6}\left[0 - x^2(2x^{-1}) + x^{-1}(-4x^2)\right] \\ &= -x^2 \\ \int v_1{}'\mathrm{d}x &= \int -x^2\mathrm{d}x \\ v_1(x) &= -\frac{1}{3}x^3 \end{aligned}$$

$$\begin{aligned} v_2{}' &= W^{-1}(y_1, y_2, y_3) \begin{vmatrix} x & 0 & x^{-1} \\ 1 & 0 & -x^{-2} \\ 0 & 2x & 2x^{-3} \end{vmatrix} \\ &= \frac{x}{6}\left[x(2x^{-1}) - 0 + x^{-1}(2x)\right] \\ &= \frac{2}{3}x \\ \int v_2{}'\mathrm{d}x &= \int \frac{2}{3}x\mathrm{d}x \\ v_2 &= \frac{1}{3}x^2 \end{aligned}$$

$$\begin{aligned} v_3{}' &= W^{-1}(y_1, y_2, y_3) \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 2x \end{vmatrix} \\ &= \frac{x}{6}\left[x(4x^2) - x^2(2x) + 0\right] \\ &= \frac{1}{3}x^4 \\ \int v_3{}'\mathrm{d}x &= \int \frac{1}{3}x^4\mathrm{d}x \\ v_3 &= \frac{1}{15}x^5 \end{aligned}$$

$y_p = v_1 y_1 + v_2 y_2 + v_3 y_3$ and thus

$$\boxed{y_p(x) = \frac{1}{15}x^4.}$$