Math 345 – Homework 5 **Due Friday, October 7, 2022**

1. Find the general solution of each of the differential equations.

   (a) $y''' - 3y'' + 3y' - y = 4e^t$

Solution: Solve the associated homogeneous problem first; $Ly - 0$ where $L = (D-1)^3$. Yields complementary solution

$$y_c(t) = C_1 e^t + C_2 t e^t + C_3 t^2 e^t.$$

To annihilate $f(t)$ we use another $D-1$. Hence the form of the particular solution is $y_p(t) = K t^3 e^t$. Plugging the candidate into the non-homogeneous ODE we see K is required to be 2/3. Hence, the general solution is

$$y(t) = C_1 e^t + C_2 t e^t + C_3 t^2 e^t + \frac{2}{3} t^3 e^t.$$

   (b) $y^{(4)} + 2y'' + y = 3\sin t - 5\cos t$

Solution: Here $L = (D^2+1)^2$ and the complementary solution is the

$$\operatorname{span}\{\cos t, \sin t, t\cos t, t\sin t\}.$$

To annihilate either term on the right, need another (D^2+1). Hence $y_p(t) = K_1 t^2 \cos t + K_2 t^2 \sin t$. Into the ODE and solving for the coefficients yields $K_1 = 5/8$ and $K_2 = -3/8$. The general solution is

$$y(t) = C_1 \cos t + C_2 \sin t + C_3 t \cos t + C_4 t \sin t + \frac{5}{8} t^2 \cos t - \frac{3}{8} t^2 \sin t.$$

(c) $y''' + 4y' = t + \cos t + e^{-2t}$

Solution: Here $Ly = f_1(t) + f_2(t) + f_3(t)$ where $f_1(t) = t$, $f_2(t) = \cos t$, and $f_3(t) = e^{-2t}$. Usually, the easiest thing to do is to find individual y_{p_i} for each $f_i(t)$ and sum them in forming the general solution. Here $L = D(D^2 + 4)$ and the complementary solution is $\operatorname{span}\{1, \cos(2t), \sin(2t)\}$. The annihilators, in order, are $A_1 = D^2$, $A_2 = D^2 + 1$ and $A_3 = D + 2$. Due to L, the particular solutions will have form $y_{p_1}(t) = At + Bt^2$, $y_{p_2}(t) = Ct\cos t + Et\sin t$, and $y_{p_3}(t) = Fe^{-2t}$. Plugging in and solving yields

$$y(t) = C_1 + C_2\sin(2t) + C_3\cos(2t) + \frac{1}{8}t^2 + \frac{1}{3}\sin t - \frac{1}{16}e^{-2t}.$$

2. Find the solution of the given IVP

$$y^{(4)} + 2y'' + y = 3\sin t - 5\cos t,\ y(0) = y'(0) = 0,\ y''(0) = y'''(0) = 1.$$

Solution: Start with the general solution given above in (b). (Sadly we have to differentiate it a lot.)

$$y(t) = C_1\cos t + C_2\sin t + C_3 t\cos t + C_4 t\sin t + \frac{5}{8}t^2\cos t - \frac{3}{8}t^2\sin t.$$

Note $y(0) = C_1$. Thus the first IC requires $C_1 = 0$. So,

$$y(t) = C_2\sin t + C_3 t\cos t + C_4 t\sin t + \frac{5}{8}t^2\cos t - \frac{3}{8}t^2\sin t.$$

Then

$$y'(t) = C_2\cos t + C_3(\cos t - t\sin t) + C_4(\sin t + t\cos t) + \frac{5}{8}(2t\cos t - t^2\sin t) - \frac{3}{8}(2t\sin t + t^2\cos t)$$

and $y'(0) = 0$ implies $C_2 + C_3 = 0$, or $C_3 = -C_2$. Hence,

$$\begin{aligned} y'(t) &= -C_3\cos t + C_3(\cos t - t\sin t) + C_4(\sin t + t\cos t) + \frac{5}{8}(2t\cos t - t^2\sin t) - \frac{3}{8}(2t\sin t + t^2\cos t) \\ &= -C_3(t\sin t) + C_4(\sin t + t\cos t) + \frac{5}{8}(2t\cos t - t^2\sin t) - \frac{3}{8}(2t\sin t + t^2\cos t) \end{aligned}$$

Continuing,

$$\begin{aligned} y''(t) = &-C_3(\sin t + t\cos t) + C_4(2\cos t - t\sin t) \\ &+ \frac{5}{8}(2\cos t - 4t\sin t - t^2\cos t) - \frac{3}{8}(2\sin t + 4t\cos t - t^2\sin t) \end{aligned}$$

and $y''(0) = 2C_4 + \frac{5}{4} = 1$. Or $C_4 = -1/8$.

Solution: Then,

$$
\begin{aligned}
y''(t) &= -C_3(\sin t + t\cos t) - \frac{1}{8}(2\cos t - t\sin t) \\
&\quad + \frac{5}{8}(2\cos t - 4t\sin t - t^2\cos t) - \frac{3}{8}(2\sin t + 4t\cos t - t^2\sin t) \\
&= -C_3(\sin t + t\cos t) - \frac{1}{4}\cos t + \frac{1}{8}t\sin t \\
&\quad + \frac{5}{4}\cos t - \frac{5}{2}t\sin t - \frac{5}{8}t^2\cos t - \frac{3}{4}\sin t - \frac{3}{2}t\cos t + \frac{3}{8}t^2\sin t \\
&= -C_3(\sin t + t\cos t) + \cos t - \frac{3}{4}\sin t - \frac{11}{8}t\sin t - \frac{3}{2}t\cos t - \frac{5}{8}t^2\cos t + \frac{3}{8}t^2\sin t
\end{aligned}
$$

Last one,

$$
\begin{aligned}
y'''(t) &= -C_3(2\cos t - t\sin t) - \sin t - \frac{3}{4}\cos t - \frac{11}{8}(\sin t + t\cos t) - \frac{3}{2}(\cos t - t\sin t) \\
&\quad - \frac{5}{8}(2t\cos t - t^2\sin t) + \frac{3}{8}(2t\sin t + t^2\cos t)
\end{aligned}
$$

and $y'''(0) = -2C_3 - \frac{3}{4} - \frac{3}{2} = 1$. Or $C_3 = -\frac{13}{8}$ and $C_2 = \frac{13}{8}$. Altogether, the solution to the IVP is

$$y(t) = \frac{13}{8}\sin t - \frac{13}{8}t\cos t - \frac{1}{8}t\sin t + \frac{5}{8}t^2\cos t - \frac{3}{8}t^2\sin t.$$

3. Convert the given IVP to its equivalent first-order linear initial value system.

$$y^{(4)} + 2y'' + y = 3\sin t - 5\cos t,\ \ y(0) = y'(0) = 0,\ \ y''(0) = y'''(0) = 1.$$

Solution: Let $y_1 = y$, $y_2 = y'$, $y_3 = y''$, $y_4 = y'''$. This yields the system $y_1' = y_2$, $y_2' = y_3$, $y_3' = y_4$ and $y_4' = y^{(4)}$ is governed by the ODE. The ODE $y^{(4)} = -2y'' - y + 3\sin t - 5\cos t$ yields $y_4' = -2y_3 - y_1 + 3\sin t - 5\cos t$. Converting the initial condition yields $y_1(0) = y_2(0) = 0$, $y_3(0) = y_4(0) = 1$. As a system

$$\begin{aligned}
y_1' &= y_2 \\
y_2' &= y_3 \\
y_3' &= y_4 \\
y_4' &= -y_1 - 2y_3 + 3\sin t - 5\cos t
\end{aligned}$$

where $y_1(0) = y_2(0) = 0$, $y_3(0) = y_4(0) = 1$.

Definition: A function $f(t,x)$ on a domain $D \subset \mathbb{R} \times \mathbb{R}^n$ is said to satisfy a **Lipschitz condition** with respect to x on D if there exists a constant $K > 0$ such that

$$|f(t,x_a) - f(t,x_b)| \leq K|x_a - x_b|, \text{ for any } (t,x_a),(t,x_b) \in D.$$

(Colloquially, we just say that "f is Lipschitz". K is called the *Lipschitz condition.* (And, yes, the 8 year old in all of us loves this name.))

4. Show that the function $f(t,x) = 5\sin t \cos x$ is Lipschitz on its domain. Be sure to clearly justify your answer.

Solution: By properties of $\sin\theta$,

$$|f(t,x_a) - f(t,x_b)| = |5\sin t||\cos x_a - \cos x_b| \leq 5|\cos x_a - \cos x_b|.$$

To complete the problem, we use the Mean Value Theorem. Since $\cos x$ is differentiable on all of $\mathbb{R}$, on the closed interval $[x_b, x_a]$, there exists a c in (x_b, x_a) such that

$$(\cos x)'\big|_{x=c} = \frac{\cos x_a - \cos x_b}{x_a - x_b}.$$

So

$$|\cos x_a - \cos x_b| = |\sin c||x_a - x_b| \leq |x_a - x_b|.$$

Putting the inequalities together yields

$$|f(t,x_a) - f(t,x_b)| \leq 5|\cos x_a - \cos x_b|.$$