Math 345 – Homework 1 **Due Monday, September 05, 2022**

1. Consider the differential equation

$$1 + y^2 + 2(x+1)yy' = 0.$$

(a) Show that the ODE represents an exact ODE.

Solution:

$$(1 + y^2)dx + 2(x+1)ydy = 0$$

Note $\frac{\partial}{\partial y}(1 + y^2) = 2y$ and $\frac{\partial}{\partial x}[2(x+1)y] = 2y$. Exact.

(b) Find the general solution to the ODE.

Solution: $g(x, y) = \int(1 + y^2)dx = x + xy^2 + C(y)$. Then $g_y(x, y) = 2xy + C'(y) = 2xy + 2y$. Hence $C(y) = y^2$ and the general solution is

$$x + xy^2 + y^2 = C.$$

(c) Does a specific solution curve of the ODE pass through the point $(5, 0)$? If so, find it.

Solution: Yes. $x + xy^2 + y^2 = 5$.

2. *The same but different...* Consider the differential equation

$$1 + y^2 + 2(x+1)yy' = 0.$$

(a) Show that the ODE is a separable equation and find the general solution. Justify that this is the same solution found before.

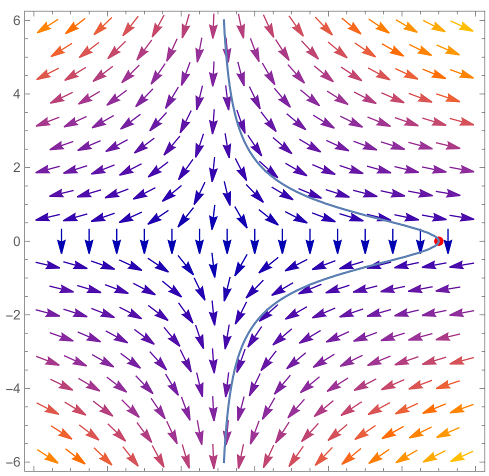
Solution:

$$\frac{dy}{dx} = -\frac{1+y^2}{2(x+1)y} = -\frac{1+y^2}{2y} \cdot \frac{1}{x+1} \qquad \text{(separable)}$$

$$\begin{aligned} \frac{2y}{1+y^2}dy &= -\frac{1}{x+1}dx \\ \ln(1+y^2) &= -\ln(x+1) + C \\ 1+y^2 &= \frac{C}{x+1} \\ (x+1)(1+y^2) &= C \\ x + xy^2 + y^2 &= C \end{aligned}$$

(b) Use technology and graph the associated slope field. On the picture, sketch the solution curve that passes through the point $(5, 0)$.

Solution: To plot the vector field, we need to use the field as defined by dy/dx in (a). Here $V(x, y) := \langle -(1+y^2), 2y(x+1) \rangle$. Via VectorPlot in Mathematica and plotting the curve in Problem 1(c),



3. For what values of the constants m, n, and α (if any) is the following differential equation exact?

$$x^m y^2 y' + \alpha x^3 y^n = 0$$

Solution: Here $\alpha x^3 y^n dx + x^m y^2 dy = 0$. We need

$$(\alpha x^3 y^n)_y = (x^m y^2)_x$$
$$\alpha n x^3 y^{n-1} = m x^{m-1} y^2$$

Need $\alpha n = m$, $3 = m - 1$, and $n - 1 = 2$. Hence $m = 4$, $n = 3$, and $\alpha = 4/3$.

4. Consider the ODE $M(x, y)dx + N(x, y)dy = 0$.

   (a) Let $\mu(x, y)$ be a non-vanishing function. What is the relationship between the slope field of the original ODE and the ODE $\mu M dx + \mu N dy = 0$? Justify your answer.

Solution: They are the same. As we did in Problem 2, the vector field is $V(x, y) = \langle -N(x, y), M(x, y)\rangle$. But multiplying the equation by the non-vanishing μ, we would get the same DE

$$\frac{dy}{dx} = \frac{-\mu M}{\mu N} = \frac{-M}{N}.$$

   (b) Why are the solution curves to the original ODE and the ODE $\mu M dx + \mu N dy = 0$? identical? Briefly explain.

Solution: If the pointwise slopes defined by both equations are identical, then the flow-lines defined by the field must also be identical.

5. Consider the equation $-2xydx + (3x^2 - y^2)dy = 0$.

(a) Show that the ODE is **not** exact.

Solution:

$$M_y = (-2xy)_y = -2x \neq N_x = (3x^2 - y^2)_x = 6x.$$

(b) Find an integrating factor that converts the ODE into an exact one.

Solution: Recall the μ equation requires $\mu = \mu_x \left(\frac{N}{M_y - N_x}\right) - \mu_y \left(\frac{M}{M_y - N_x}\right)$. Here $M_y - N_x = -8x$ and $\frac{M}{M_y - N_x} = \frac{-2xy}{-8x} = \frac{y}{4}$. So if we assume μ is independent of x, the μ equation becomes the ODE $\mu = -\frac{y}{4}\frac{d\mu}{dy}$. This is separable $(1/\mu)d\mu = (-4/y)dy$ and $\ln \mu = -4 \ln y$. Hence $\mu(y) = 1/y^4$.

(c) Using the integrating factor, show that the μ-multiplied ODE is exact.

Solution: Multiplying by μ, the ODE becomes $-2x/y^3dx + (3x^2/y^4 - 1/y^2)dy = 0$. Note that

$$(-2x/y^3)_y = 6x/y^4 \text{ and } (3x^2/y^4 - 1/y^2)_x = 6x/y^4.$$

(d) Find the general solution to the original ODE.

Solution: $g(x, y) = \int -2x/y^3dx = -x^2/y^3 + C(y)$. Then $g_y(x, y) = 6x/y^4 + C'(y)$. We need $C'(y) = -1/y^2$ and $C(y) = 1/y$. So solutions to the original DE are curves of the form

$$-\frac{x^2}{y^3} + \frac{1}{y} = C.$$

"Homogeneous" non-linear first-order equations

1. *homogeneous functions*

   def: A function $f(x, y)$ is a **homogeneous function of degree n** if given any scalar α, $f(\alpha x, \alpha y) = \alpha^n f(x, y)$.

   Determine the degree of homogeneity for the following functions.

   (a) $g(x, y) := x^3 + y^3$

Solution: $g(rx, ry) := (rx)^3 + (ry)^3 = r^3(x^3 + y^3) = r^3 g(x, y)$, degree 3

   (b) $h(x, y) := \dfrac{-x}{x^2 + y^2}$

Solution: $h(rx, ry) := \dfrac{-rx}{(rx)^2 + (ry)^2} = \dfrac{1}{r} h(x, y)$, degree -1

   (c) $k(x, y) := \dfrac{y^2 + 2xy}{x^2}$

Solution: $k(rx, ry) :== \dfrac{(ry)^2 + 2(rx)(ry)}{(rx)^2} = k(x, y)$, degree 0

2. Prove the following proposition.

   Prop: If $f(x, y)$ is a homogeneous function of degree 0, it can always be expressed as $G(y/x)$ where $G(t)$ is a scalar function of one-variable.

   (Hint: When $x \neq 0$, $f(x, y) = (1/x)^0 f(x, y)$.)

Solution: Since f homogeneous of degree 0, we have $f(rx, ry) = r^0 f(x, y) = f(x, y)$. Let $r = 1/x$. Then f can be written $f(1, y/x)$. Hence, we can write f as $G(t) = f(1, t)$ where $t = y/x$.

3. Prove the following proposition.

   Prop: Let $\dfrac{dy}{dx} = f(x, y)$ be such that $f(x, y)$ is a homogeneous function of degree 0.

The, through the substitution $u = y/x$, the ODE converts to a separable ODE of the form

$$\frac{du}{dx} = \frac{1}{x}\left[f(1,u) - u\right].$$

(Hint: Differentiate the substitution $u = y/x$ or $y = ux$.)

Solution: Let $u = y/x$. Differentiating with respect to x, we have

$$\frac{du}{dx} = \frac{y'x - y}{x^2} = \frac{1}{x}\left[\frac{dy}{dx} - \frac{y}{x}\right] = \frac{1}{x}\left[f(x,y) - \frac{y}{x}\right] = \frac{1}{x}\left[f(1,u) - u\right].$$

4. Solve the ODE $\dfrac{dy}{dx} = \dfrac{y^2 + 2xy}{x^2}$.

Solution: This is k from earlier and we showed it is homogeneous of degree 0. Moreover, $k(x,y) = \dfrac{y^2}{x^2} + \dfrac{y}{x} \to u^2 + 2u$. Using the previous proposition, we have

$$\frac{du}{dx} = \frac{1}{x}\left[u^2 + 2u - u\right] \text{ or } \frac{du}{u^2 + u} = \frac{dx}{x}.$$

Via partial fractions,

$$\ln u - \ln(u+1) = \ln x + C.$$

Or $\dfrac{u}{u+1} = Cx$. Substituting back out the u,

$$\frac{y}{y+x} = \frac{C}{x} \text{ or } y = \frac{x^2}{C - x}.$$