Math 345
Professor Carty
70 points possible

Exam 1
October 15, 2022

Name: Key.

1. No hats or dark sunglasses. All hats are to be removed.
2. All book bags are to be closed and placed in a way that makes them inaccessible. Do not reach into your bag for anything during the exam. If you need extra pencils, pull them out now.
3. Be sure to print your proper name clearly.
4. *No calculators are allowed.* Watches with recording, internet, communication or calculator capabilities (e.g. a smart watch) are prohibited.
5. All electronic devices, including cell phones and other wearable devices, must be powered off and stored out of sight for the entirety of the exam.
6. If you have a question, raise your hand and I will come to you. Once you stand up, you are done with the exam. If you have to use the facilities, do so now. You will not be permitted to leave the room and return during the exam.
7. Every exam is worth a total of ~~**45**~~ **points**. Including the cover sheet, each exam has ~~6~~ pages.
8. At ~~9~~11:50, you will be instructed to put down your writing utensil. You must stop writing the exam at this time.
9. If you finish early, quietly and respectfully and in your exam. You may leave early.
10. You will hand in the paper copy of the exam on your way out of the classroom.
11. You have fifty minutes to complete the exam. I hope you do well.

1. (10 points) Solve the IVP

$$xy' = 2y - x^3, y(-1) = 9/5.$$

$xy' - 2y = -x^3$,

$y' - \frac{2}{x} y = -x^2 \qquad p(x) = -2/x$

$\mu(x) = \exp\left[\int p(x)\,dx\right] = x^{-2}.$

$\frac{1}{x^2} y' - \frac{2}{x^3} y = -1$

$\left(\frac{1}{x^2} y\right)' = -1 \qquad \frac{1}{x^2} y = -x + C$

$y = -x^3 + Cx^2.$

$y(-1) = 1 + C = 9/5 \Rightarrow C = 4/5.$

$y(x) = -x^3 + \frac{4}{5} x^2$

2. (10 points) (a) State the existence and uniqueness theorem for first-order linear ODE.

$y' = f(x,y)$ If $f(x,y)$ and $f_y(x,y)$ are
$y(x_0) = y_0$ continuous in a rectangle R
$(|x-x_0|<h, |y-y_0|<T)$
then there exists a unique sol'n to the IVP
on the interval $|x-x_0|<h$.

For 1st-order linear, $y' + p(x)y = q(x)$ or $f(x,y) = q(x) - p(x)y$.
As $f_y = -p(x)$, our E+U Thm say unique sol'n exist
on any intervals I s.t. $q(x), p(x)$ continuous.

(b) What does the existence and uniqueness theorem say about the initial value problem

$$xy' = 2y - x^3, y(x_0) = y_0?$$

$f(x,y) = \frac{2y}{x} - x^2$, $f_y(x,y) = \frac{2}{x}$.

Both require $x \neq 0$ for continuity.
Need to restrict domain of x s.t. that
sol'n curves stay away from $x=0$.

(From Problem 1, $x>0$ if $x_0>0$
$x<0$ if $x_0<0$)

3. (10 points) Consider an ODE of the form $y' = p(x)q(y)$.

(a) Explain why this ODE is separable.

$$\frac{dy}{dx} = \underbrace{p(x)\,q(y)}_{\text{a separable fn}} \iff \frac{dy}{q(y)} = p(x)\,dx.$$

(b) Show that all separable ODE are exact.

$$p(x)\,dx - \frac{1}{q(y)}\,dy = 0$$

Let $M(x,y) = p(x)$, $N(x,y) = -1/q(y)$.

Then $M_y = 0$ and $N_x = 0$.

$M_y = N_x$ exact.

4. (10 points) Consider the IVP

$$\frac{dy}{dt} = t^2 + y^2, \; y(1) = 2.$$

(a) Convert the IVP into an equivalent integral equation.

$$y(t) = 2 + \int_1^t \left(s^2 + (y(s))^2\right) ds$$

(b) Using Picard's Method, determine the first three Picard's iterates; $y_0(t)$, $y_1(t)$ and $y_2(t)$.

$$y_0 = 2 \qquad y_1 = 2 + \int_1^t (s^2 + 2^2)\, ds = 2 + \left(\frac{s^3}{3} + 4s\right)\Big|_1^t$$

$$= 2 + \frac{t^3}{3} + 4t - \left(\frac{1}{3} + 4\right) = \frac{t^3}{3} + 4t - \frac{7}{3}$$

$$y_2 = 2 + \int_1^t \left(s^2 + (y_1(s))^2\right) ds$$

Note $$y_1(s)^2 = \left(\frac{s^3}{3} + 4s - \frac{7}{3}\right)^2 = \frac{s^6}{9} + \frac{4}{3}s^4 - \frac{7}{9}s^3 + \frac{4}{3}s^4 + 16s^2 - \frac{28}{3}s$$

$$- \frac{7}{9}s^3 - \frac{28}{3}s + \frac{49}{3}$$

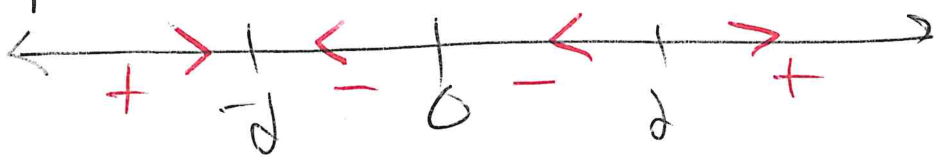
$$= s^6/9 + 8/3\, s^4 - 14/9\, s^3 + 16s^2 - 56/3\, s + 49/3$$

So $$y_2(s) = 2 + \left(\frac{s^3}{3} + \frac{s^7}{63} + \frac{8}{3}\cdot\frac{s^5}{5} - \frac{14}{9}\frac{s^4}{4} + \frac{16s^3}{3} - \frac{56}{3}\frac{s^2}{2} + \frac{49}{3}s\right)\Big|$$

5. (10 points) Consider the differential equation $dy/dt = y^2(y^2 - 4)$.

(a) Draw the phase portrait for this system and classify each of the critical (equilibrium) points.

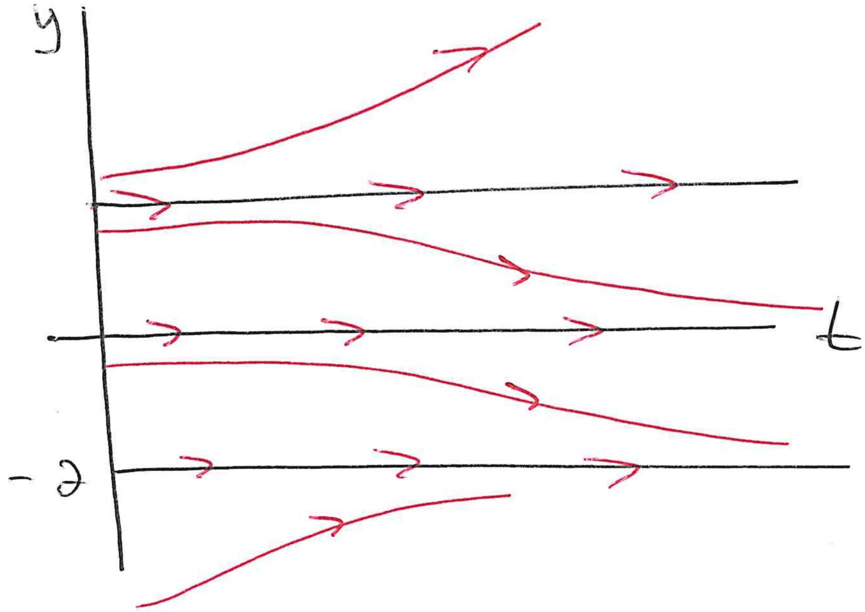
$\dot{y} = 0$ when $t=0$, $t=-2$, $t=2$.



Sign flip

$t=-2$ stable
$t=0$ semistable
$t=2$ unstable

(b) Using the phase portrait, sketch several graphs of solution curves in the ty-plane, $t \geq 0$.



6. (10 points) (a) Show that the set of functions $\{1, x, \sin x, \cos x\}$ is linearly independent.

$$W = \begin{vmatrix} 1 & x & \sin x & \cos x \\ 0 & 1 & \cos x & -\sin x \\ 0 & 0 & -\sin x & -\cos x \\ 0 & 0 & -\cos x & \sin x \end{vmatrix} = \begin{vmatrix} 1 & \cos & -\sin x \\ 0 & -\sin x & -\cos x \\ 0 & -\cos x & \sin x \end{vmatrix}$$

$$= \begin{vmatrix} -\sin x & -\cos x \\ -\cos x & \sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1$$

(b) Find an ODE for which the set $\{1, x, \sin x, \cos x\}$ forms a fundamental set.

$$D^2(D^2+1)y = 0$$

or $y'''' + y'' = 0.$

7. (10 points) Consider the ODE

$$y''' - 3y'' + 2y' = t + e^{-t}$$

(a) Find the complementary solution to the ODE.

$$(D^3 - 3D^2 + 2D)y = 0$$

$$D(D^2 - 3D + 2)y = 0$$

$$D(D-1)(D-2)y = 0$$

$$y_c(t) = C_1 + C_2 e^{t} + C_3 e^{2t}$$

(b) Find a particular solution to the ODE.

For t, $A_1 = D^2$. For e^{-t}, $A_2 = (D+1)$.

$$y_{P_1} = k_1 t + k_2 t^2 \qquad y_{P_2} = k_3 e^{-t}$$

$$y_p = k_1 t + k_2 t^2 + k_3 e^{-t}$$

$$y_p' = k_1 + 2k_2 t - k_3 e^{-t}$$

$$y_p'' = 2k_2 + k_3 e^{-t}$$

$$y_p''' = -k_3 e^{-t}$$

$$\Rightarrow -k_3 e^{-t} - 3(2k_2 + k_3 e^{-t}) + 2(k_1 + 2k_2 t - k_3 e^{-t})$$

$$= (2k_1 - 6k_2) + 4k_2 t - 6k_3 e^{-t}$$

$$k_2 = \frac{1}{4} \qquad k_3 = -1/6$$

$$2k_1 - 3/2 = 0, \quad k_1 = 3/4$$

$$y_p(t) = \frac{3}{4} t + \frac{1}{4} t^2 - \frac{1}{6} e^{-t}$$