Math 345 - Homework 3 **Due Friday, September 23, 2022**

1. Rewrite the following differential equations as equivalent integral equations.

   (a) $x'(t) = \sin t \cos 3t + x^6(t), \qquad x(0) = 4$

   For the IC, $t_0 = 0$ and $x_0 = 4$. Using the formula in the notes,

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))\mathrm{d}s$$
$$x(t) = 4 + \int_{0}^{t} \left(\sin(s)\cos(3s) + x^6(s)\right) \mathrm{d}s$$

   (b) $x''(t) = t^4 \cos 3x(t) + x^6(t), \qquad x(1) = 4, \qquad x'(1) = 3$

   For the first IC of $x'(1) = 3$, the formula gives

$$x'(t) = x'(t_0) + \int_{t_0}^{t} f(s, x(s))\mathrm{d}s$$
$$x'(t) = 3 + \int_{1}^{t} s^4 \cos(3x(s)) + x^6(s)\mathrm{d}s$$

   Then for the second IC of $x(1) = 4$, the formula (using a different variable) gives

$$x(t) = x(t_0) + \int_{t_0}^{t} f(r, x(r))\mathrm{d}r$$
$$x(t) = 4 + \int_{1}^{t} \left[3 + \int_{1}^{r} s^4 \cos(3x(s)) + x^6(s)\mathrm{d}s\right]\mathrm{d}r$$
$$x(t) = 3t + 1 + \int_{1}^{t} \int_{1}^{r} s^4 \cos(3x(s)) + x^6(s)\mathrm{d}s\mathrm{d}r$$

2. Show that $x''(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad x'(t_0) = x_1$ is equivalent to $x(t) = x_0 + x_1(t - t_0) + \int_{t_0}^{t}(t-s)f(s,x(s))\mathrm{d}s$. (Hint: Consider changing the order of integration of your resultant double integral)

$$x'(t) = x'(t_0) + \int_{t_0}^{t} f(s,x(s))\mathrm{d}s$$
$$x'(t) = x_1 + \int_{t_0}^{t} f(s,x(s))\mathrm{d}s$$

Then

$$x(t) = x(t_0) + \int_{t_0}^{t}\left[x_1 + \int_{t_0}^{t} f(s,x(s))\mathrm{d}s\right]\mathrm{d}s$$
$$x(t) = x_0 + \int_{t_0}^{t} x_1\mathrm{d}s + \int_{t_0}^{t}\left[\int_{t_0}^{t} f(s,x(s))\mathrm{d}s\right]\mathrm{d}s$$
$$x(t) = x_0 + x_1 s\Big|_{t_0}^{t} + \int_{t_0}^{t}\left[\int_{t_0}^{t} f(s,x(s))\mathrm{d}s\right]\mathrm{d}s$$
$$x(t) = x_0 + x_1\cdot t - x_1\cdot t_0 + \int_{t_0}^{t}\left[\int_{t_0}^{t} f(s,x(s))\mathrm{d}s\right]\mathrm{d}s$$
$$x(t) = x_0 + x_1(t - t_0) + \int_{t_0}^{t}\left[\int_{t_0}^{t} f(s,x(s))\mathrm{d}s\right]\mathrm{d}s$$

3. Consider the autonomous system

$$\frac{dy}{dt} = y(y-1)(y-2), \quad y_0 \geq 0$$

(a) Determine the critical (equilibrium) points, and classify each one as asymptotically stable, unstable, or semistable.

Solution: Since this is conveniently factored, the critical points are 0, 1, and 2.

Since $y(0) = 0$ and the leading coefficient of the cubic is positive, $\frac{dy}{dt} < 0$ for all $y < 0$.

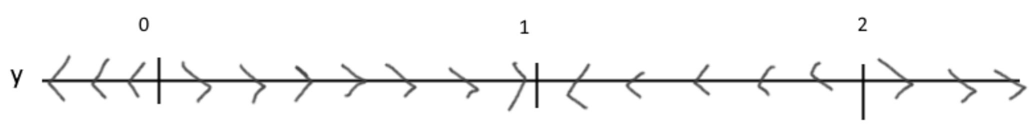
Since $y(1) = 0$, we also know that $\frac{dy}{dt} > 0$ for all $y \in (0, 1)$ by some theorem in calc 1 or mth420 that deals with roots.

Similarly with the theorem used previously, we know that $\frac{dy}{dt} < 0$ for all $y \in (1, 2)$.

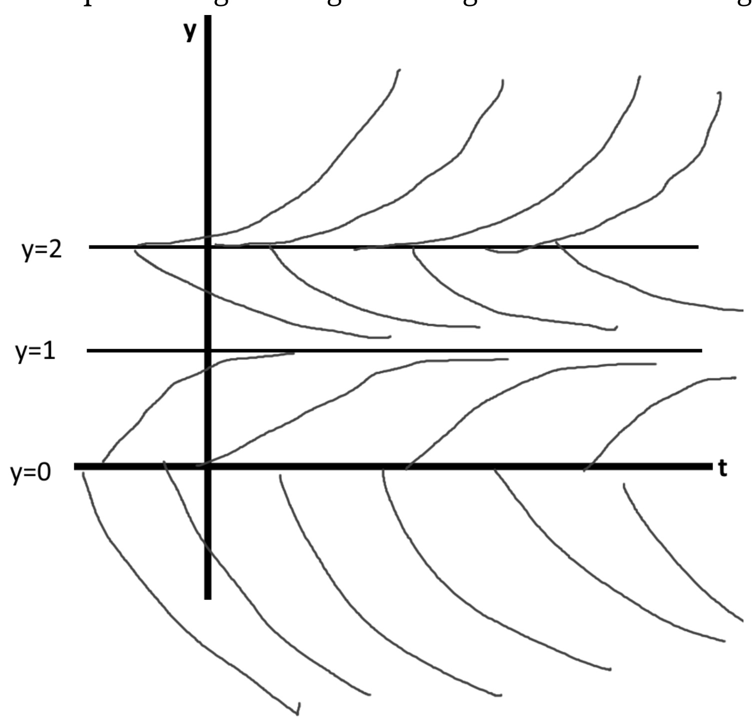
Lastly it is also clear that for $\frac{dy}{dt} > 0$ for all $y > 2$ by properties of cubics with positive leading coefficients.

For $y = 0$, it is unstable because any small change diverges from 0. For $y = 1$ is stable because any small change converges back to 1. And for $y = 2$ it is unstable because any small change diverges from 2.

(b) Sketch the phase diagram



(c) Use the phase diagram to give a rough sketch of the integral curves



4. Let $y_1(t) := t^2$ and $y_2(t) := t^{-1}$

   (a) Show that each y_i solves the differential equation $t^2y'' - 2y = 0, \quad t > 0$

   Solution: For y_1, $y_1{}' = 2t$ and $y_1{}'' = 2$. Then

$$t^2 \cdot \underbrace{2}_{y_1{}''} - 2 \cdot \underbrace{t^2}_{y_1} = 0 \quad \Longrightarrow \quad 0 = 0.$$

   For y_2, $y_2{}' = -t^{-2}$ and $y_2{}'' = 2t^{-3}$. Then

$$t^2 \cdot \underbrace{2t^{-3}}_{y_2{}''} - 2 \cdot \underbrace{t^{-1}}_{y_1} = 0 \quad \Longrightarrow \quad 0 = 0.$$

   (b) Show that $\{y_1, y_2\}$ forms a fundamental set of the ODE.

   Solution:

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} t^2 & t{-}1 \\ 2t & -t^{-2} \end{vmatrix} \\ &= (t^2)(-t^{-2}) - (2t)(t^{-1}) \\ &= -1 - 2 \\ &\neq 0 \end{aligned}$$

   Which means that all y_i are linearly independent and 2 LI solutions on a 2nd order ODE span the kernel.

   (c) Verify the Principle of Superposition. That is, $y(t) := C_1y_1 + C_2y_2$ is the general solution to the ODE.

$$\begin{aligned} t^2(y(t)'') - 2(y(t)) &= t^2(2C_1 + 2C_2t^{-3}) - 2(C_1t^2 + C_2t^{-1}) \\ &= 2t^2C_1 + 2C_2t^{-1} - 2t^2C_1 + 2C_2t^{-1} \\ &= 0 \end{aligned}$$

5. Let $y_1(t) := 1$ and $y_2(t) := \sqrt{t}$

   (a) Show that each y_i solves the differential equation $yy'' + (y')^2 = 0, \quad t > 0.$

   Solution: For y_1, $y' = 1' = 0$ and $y'' = 0$. So $1 \cdot 0 + (0)^2 = 0$ is true.

   For y_2, $y' = \frac{1}{2\sqrt{t}}$ and $y'' = -\frac{1}{4t^{3/2}}$. Then

$$\sqrt{t} \cdot -\frac{1}{4t^{3/2}} + \left[\frac{1}{2\sqrt{t}}\right]^2 = -\frac{1}{4t} + \frac{1}{4\sqrt{t^2}} = 0$$

(b) Show that y_1 and y_2 are linearly independent.

$$W(y_1, y_2) = \begin{vmatrix} 1 & \sqrt{t} \\ 0 & \frac{1}{2\sqrt{t}} \end{vmatrix} = \frac{1}{2\sqrt{t}} \neq 0$$

(c) Let $y_3 = \sqrt{2t+3}$. Show that y_3 is another solution to the ODE.

Solution: $(y_3)' = 1/\sqrt{2t+3}$ and $(y_3)'' = -1/(2t+3)^{3/2}$

$$\sqrt{2t+3} \cdot -1/(2t+3)^{3/2} + (1/\sqrt{2t+3})^2 = -\frac{1}{2t+3} + \frac{1}{2t+3} = 0$$

(d) Show that y_3 can not be written as a linear combination of y_1 and y_2. Why does this result not violate the Principle of Superposition?

$$\begin{aligned} W(y_1, y_2, y_3) &= \begin{vmatrix} 1 & \sqrt{t} & \sqrt{2t+3} \\ 0 & \frac{1}{2\sqrt{t}} & 1/\sqrt{2t+3} \\ 0 & -\frac{1}{4t^{3/2}} & -1/(2t+3)^{3/2} \end{vmatrix} \\ &= \frac{1}{2\sqrt{t}} \cdot -1/(2t+3)^{3/2} - 1/\sqrt{2t+3} \cdot -\frac{1}{4t^{3/2}} - \sqrt{t}[0] + \sqrt{2t+3}[0] \\ &= \frac{1}{2\sqrt{t}} \cdot -1/(2t+3)^{3/2} - \frac{1}{\sqrt{2t+3}} \cdot -\frac{1}{4t^{3/2}} - \sqrt{t} \\ &\neq 0 \end{aligned}$$

Hence they are linearly independent. This doesn't violate the principle because $y_3 \notin \operatorname{span}\{y_1, y_2\}$

(e) Solve the ODE. Hint: Consider the substitution $u = yy'$.

6. Verify that $\{e^t, e^{-1}, e^{-2t}\}$ for a fundamental set of solutions to the ODE $y''' + 2y'' - y' - 2y = 0$. That is, (i) show each solve the ODE and (ii) the functions are linearly independent.

(i) .

(ii) Let $y_1 := e^t$, $y_2 := e^{-t}$, and $y_3 := e^{-2t}$. Then

$$y_1' = y_1'' = e^t \qquad \text{and} \qquad y_2' = -e{-}t,\ y_2'' = e^{-t} \qquad \text{and} \qquad y_3' = -2e^{-2t}, y_3'' = 4e^{-2t}$$

Then

$$\begin{aligned} W(y_1, y_2, y_3) &= \begin{vmatrix} e^t & e^{-t} & e^{-2t} \\ e^t & -e{-}t & -2e^{-2t} \\ e^t & e^{-t} & 4e^{-2t} \end{vmatrix} \\ &= -6e^{-2t} \neq 0 \end{aligned}$$