Math 345 – Homework 7 **Due Monday, October 31, 2022**

1. Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

   (a) Show that $\mathbf{A}$ is a nilpotent matrix.

   (b) Compute $e^{\mathbf{A}}$ exactly.

2. Compute $e^{\mathbf{A}t}$ when $\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

3. Consider the upper triangular matrix $\mathbf{U} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   (a) Decompose $\mathbf{U}$ into a diagonal matrix D and a nilpotent matrix N.

   (b) Use the previous problem to compute $e^{\mathbf{U}}$.

   (c) Will this technique work for all upper triangular matrices? Justify your answer.

4. Prove that if matrices $\mathbf{A}$ and $\mathbf{B}$ are similar matrices, then $\mathbf{A}$ and $\mathbf{B}$ have the same set of eigenvalues.

5. Compute $e^{\mathbf{A}t}$ where

   A. $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 6 & 2 & 0 \end{bmatrix}$ B. $\mathbf{A} = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 2 & 0 \\ -2 & 2 & 2 \end{bmatrix}$ C. $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

6. Find the solution of $\vec{x}' = \mathbf{A}\vec{x}$, $\vec{x}(0) = \langle 1, -1, 1 \rangle$ for each $\mathbf{A}$ in Exercise 5.

7. Verify the Cayley-Hamilton theorem for Exercise 5B.

8. We are going to revisit 5A and do it the traditional engineering ODE way.

(a) Order and label your eigenvalues as you did in Problem 5. For each eigenvalue λ_i, find an associated eigenvector $\mathbf{v}_i$.

(b) Construct a fundamental matrix $\mathbf{\Phi}(t) = [e^{\lambda_1}\mathbf{v}_1, e^{\lambda_2}\mathbf{v}_2, e^{\lambda_3}\mathbf{v}_3]$ and prove that $\mathbf{\Phi}(t)$ is not $e^{t\mathbf{A}}$. Note: $\mathbf{\Phi}(t)\vec{c}$ is the form of the solution in lower-level courses.

(c) Diagonalize A. That is, use the eigenvalues to construct a diagonal matrix $\mathbf{D}$ and the eigenvectors to construct a change of basis matrix $\mathbf{S}$ such that $\mathbf{D} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$. (Show this these matrices do the job intended.)

(d) Using your diagonalization, compute $e^{t\mathbf{A}}$ using $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$. That is, construct $\mathbf{S}e^{t\mathbf{D}}\mathbf{S}^{-1}$ and show that this matrix is $\mathbf{I}$ when $t = 0$.

9. Show that if the real part of each eigenvalue of $\mathbf{A}$ is negative, then every solution of $\vec{x}' = \mathbf{A}\vec{x}$ satisfies $\lim_{t\to\infty} \vec{x}(t) = 0$.

10. Find the solution of the initial value problem.

$$\vec{x}' = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 0 & 0 \\ 2 & -2 & 3 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$