Math 345 - Homework 1 **Due Monday, September 05, 2022**

1. Consider the differential equation

$$1 + y^2 + 2(x+1)yy' = 0.$$

(a) Show that the ODE represents an exact ODE.

First, since $y' \equiv \dfrac{dy}{dx}$, we can rewrite the equation and define functions $M(x, y)$ and $N(x, y)$ as

$$\underbrace{(1+y^2)}_{M}\,dx + \underbrace{2(x+1)y}_{N}\,dy = 0.$$

Then $\dfrac{\partial M}{\partial y} = 2y$ and $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}(2xy + 2y) = 2y$. Since $M_y = N_x$, it is an exact ODE.

(b) Find the general solution to the ODE.

Continuing with the same M and N, integrating these gives

$$\int M dx = x + xy^2 + C(y) \text{ and}$$
$$\int N dy = xy^2 + y^2 + C(x).$$

Since $f(x, y) = \int M dx = \int N dy$,

$$x + xy^2 + C(y) = xy^2 + y^2 + C(x)$$
$$\iff x + C(y) = y^2 + C(x)$$

which implies that $C(x) = x$ and $C(y) = y^2$. Hence, the general solution is $f(x, y) = x + xy^2 + y^2 = C$.

(c) Does a specific solution curve of the ODE pass through the point $(5, 0)$? If so, find it.

Yes, it exists. Evaluating $f(5, 0)$ yields $5 = C$. So, substituting this into the general solution gives $f(x, y) = x + xy^2 + y^2 = 5$.

2. *The same but different…* Consider the differential equation

$$1 + y^2 + 2(x+1)\,y\,y' = 0.$$

(a) Show that the ODE is a separable equation and find the general solution. Justify that this is the same solution found before.

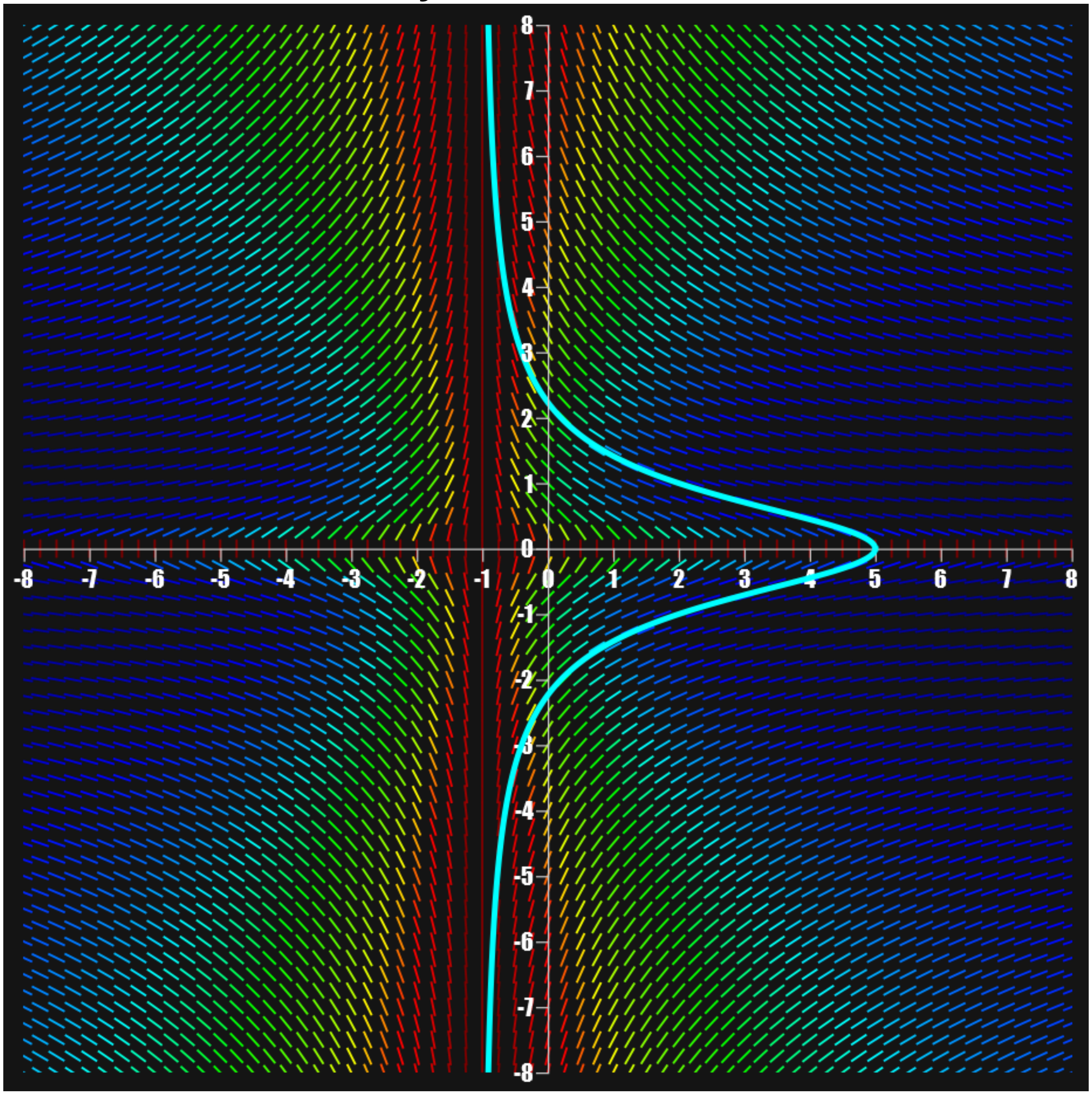
Rewriting it with leibniz notation and then simplifying,

$$\begin{aligned}
& (1+y^2)\mathrm{d}x + 2(x+1)y\ \mathrm{d}y = 0 \\
\iff\quad & (1+y^2)\mathrm{d}x = -2(x+1)y\ \mathrm{d}y \\
\iff\quad & \frac{1}{-2(x+1)}\mathrm{d}x = \frac{y}{1+y^2}\mathrm{d}y \\
\iff\quad & \int \frac{1}{-2(x+1)}\mathrm{d}x = \int \frac{y}{1+y^2}\mathrm{d}y && (\text{let } u_1 = x+1,\ u_2 = 1+y^2) \\
\iff\quad & \frac{-1}{2}\int \frac{\mathrm{d}u_1}{u_1} = \frac{1}{2}\int \frac{\mathrm{d}u_2}{u_2} \\
\iff\quad & \frac{-1}{2}\ln|u_1| = \frac{1}{2}\ln|u_2| + C_1 \\
\iff\quad & \frac{-1}{2}\ln|x+1| = \frac{1}{2}\ln\left|1+y^2\right| + C_1 \\
\iff\quad & e^{\frac{-1}{2}\ln|x+1|} = e^{\frac{1}{2}\ln\left|1+y^2\right|+C_1} \\
\iff\quad & e^{\ln\left|(x+1)^{-0.5}\right|} = e^{\ln\left|(1+y^2)^{0.5}\right|}e^{C_1} \\
\iff\quad & (x+1)^{-0.5} = C_2(1+y^2)^{0.5} \\
\iff\quad & \frac{1}{\sqrt{x+1}} = C_2\sqrt{1+y^2} \\
\iff\quad & \sqrt{1+y^2}\sqrt{x+1} = \frac{1}{C_2} \\
\iff\quad & (1+y^2)(x+1) = \frac{1}{C_2^2} \\
\iff\quad & y^2 + xy^2 + x = C
\end{aligned}$$

Which is the same as before, $f(x,y) = y^2 + xy^2 + x = C$

(b) Use technology and graph the associated slope field. On the picture, sketch the solution curve that passes through the point $(5, 0)$.

(Disclaimer: I have not implemented an ODE solver so the colors are based on the slope field vectors, NOT the vectors of y)



3. For what values of the constants m, n, and α (if any) is the following differential equation exact?

$$x^m y^2 y' + \alpha x^3 y^n = 0$$

Rewriting y' and defining functions M and N gives us

$$\underbrace{\alpha x^3 y^n}_{M} \, \mathrm{d}x + \underbrace{x^m y^2}_{N} \, \mathrm{d}y = 0.$$

In order to be exact, we need $M_y = N_x$. Currently $M_y = \alpha x^3 n y^{n-1}$ and $N_x = m x^{m-1} y^2$. To match y's exponents, we need $n - 1 = 2$, which implies $n = 3$. Similarly, to match x's exponents, we need $m - 1 = 3$, which implies that $m = 4$. This results in the equation $4x^3 y^2 = 3\alpha x^3 y^2$, further implying that $\alpha = \frac{4}{3}$.

Thus, $n = 3$, $m = 4$, and $\alpha = \frac{4}{3}$ makes this ODE exact.

4. Consider the ODE $M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$.

   (a) Let $\mu(x, y)$ be a non-vanishing function. What is the relationship between the slope field of the original ODE and the ODE $\mu M \mathrm{d}x + \mu N \mathrm{d}y = 0$? Justify your answer.

   The slope fields are the same because the equation is being algebraically manipulated with multiplication of the same function on each side. μ also needs to be non-vanishing to prevent the limits at infinity from being zero where they otherwise wouldn't be.

   (b) Why are the solution curves to the original ODE and the ODE $\mu M \mathrm{d}x + \mu N \mathrm{d}y = 0$ identical? Briefly explain.

   I believe this is true because integration and derivation are linear operators and can be scaled by scalar functions independent of the differential or integral. This most certainly is not the case for a multivariable μ, but that's why we make μ independent of one variable, rendering it a function on just one variable.

5. Consider the equation $-2xy\mathrm{d}x + (3x^2 - y^2)\mathrm{d}y = 0$.

   (a) Show that the ODE is **not** exact.

   We define M and N as

$$\underbrace{-2xy}_{M} \, \mathrm{d}x + \underbrace{(3x^2 - y^2)}_{N} \, \mathrm{d}y = 0.$$

   Then $M_y = -2x$ and $N_x = 6x$. Since $M_y \neq N_x$ the ODE is not exact.

(b) Find an integrating factor that converts the ODE into an exact one.

To find the integrating factor we will use the formula $\mu = \dfrac{\mu_x N - \mu_y M}{M_y - N_x}$. From part (a) we have $M, N, M_y,$ and N_x so we can do some algebra.

$$\begin{aligned}
\mu &= \frac{\mu_x N - \mu_y M}{M_y - N_x} \\
&= \frac{\mu_x (3x^2 - y^2) - \mu_y(-2xy)}{-2x - 6x} && \text{(substition)} \\
&= \frac{\mu_x (3x^2 - y^2)}{-8x} - \frac{\mu_y 2xy}{8x} && \text{(linearity)} \\
&= \frac{\mu_x (3x^2 - y^2)}{-8x} - \frac{\mu_y y}{4} && \text{(like terms)}
\end{aligned}$$

From here, we can apply case 2 of integrating factors and let μ be a function solely in terms of y. Consequently, $\mu_x = 0$, so $\mu = \dfrac{-\mu_y y}{4} = \dfrac{-y}{4}\dfrac{\mathrm{d}\mu}{\mathrm{d}y}$. Now we do some basic calculus to finish finding μ.

$$\begin{aligned}
& \mu = \frac{-y}{4}\frac{\mathrm{d}\mu}{\mathrm{d}y} \\
\iff \quad & \frac{1}{y} = \frac{-1}{4\mu}\frac{\mathrm{d}\mu}{\mathrm{d}y} && \text{(divide by } \mu y\text{)} \\
\iff \quad & \frac{1}{y}\mathrm{d}y = \frac{-1}{4\mu}\,\mathrm{d}\mu && \text{(multiply by d}y\text{)} \\
\iff \quad & \int \frac{1}{y}\mathrm{d}y = \int \frac{-1}{4\mu}\,\mathrm{d}\mu \\
\iff \quad & \ln|y| + C = \frac{-1}{4}\ln|\mu| \\
\iff \quad & \ln|\mu| = -4\ln|y| = \ln\left|y^{-4}\right| \\
\iff \quad & \mu = y^{-4} + C && (e \text{ of both sides})
\end{aligned}$$

Letting $C = 0$ gives us a single integrating factor of $\mu = y^{-4}$.

(c) Using the integrating factor, show that the μ-multiplied ODE is exact.

Multiplying both sides by μ yields

$$\underbrace{-2xy^{-3}}_{M}\,\mathrm{d}x + \underbrace{(3x^2y^{-4} - y^{-2})}_{N}\,\mathrm{d}y = 0.$$

Then $M_y = -2(-3)y^{-4} = -6y^{-4}$ and $N_x = 6xy^{-4}$. Since $M_y = N_x$, the ODE is exact.

(d) Find the general solution to the original ODE.

Using the M and N from part (c), integrating for $\int M\mathrm{d}x$ and $\int N\mathrm{d}y$ gives

$$\begin{array}{rr} \displaystyle\int M\mathrm{d}x & \displaystyle\int N\mathrm{d}y \\ \displaystyle\int -2xy^{-3}\mathrm{d}x & \displaystyle\int 3x^2y^{-4} - y^{-2}\mathrm{d}y \\ -x^2y^{-3} + C(y) & -x^2y^{-3} + y^{-1} + C(x). \end{array}$$

Therefore $C(y) = y^{-1} + C$ and $C(x) = 0$. Hence, $f(x, y) = -x^2y^{-3} + y^{-1} + C$. This solution can be verified as follows,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\left[-x^2y^{-3} + y^{-1} = -C\right] \\ &3x^2y^{-4}y' - 2xy^{-3} - y^{-2}y' = 0 \\ &y' = \frac{2xy^{-3}}{3x^2y^{-4} - y^{-2}} \\ &y' = \frac{2xy^2}{3x - y^2} = \frac{\mathrm{d}y}{\mathrm{d}x} \end{aligned}$$

“Homogeneous” non-linear first-order equations

6. *homogeneous functions*

def: A function $f(x, y)$ is a **homogeneous function of degree** n if given any scalar α, $f(\alpha x, \alpha y) = \alpha^n f(x, y)$.

Determine the degree of homogeneity for the following functions.

(a) $g(x, y) := x^3 + y^3$

$$\begin{aligned} g(\alpha x, \alpha y) &= (\alpha x)^3 + (\alpha y)^3 \\ &= \alpha^3 x^3 + \alpha^3 y^3 \\ &= \alpha^3(x^3 + y^3) \\ &= \alpha^3 f(x, y) \end{aligned}$$

Therefore g is a homogeneous function of degree 3.

(b) $h(x, y) := \dfrac{-x}{x^2 + y^2}$

$$\begin{aligned} h(\alpha x, \alpha y) &= \frac{-\alpha x}{(\alpha x)^2 + (\alpha y)^2} \\ &= \frac{-x}{\alpha x^2 + \alpha y^2} \\ &= \frac{-x}{\alpha(x^2 + y^2)} \\ &= \alpha^{-1} h(x, y) \end{aligned}$$

Therefore h is a homogeneous function of degree -1.

(c) $k(x, y) := \dfrac{y^2 + 2xy}{x^2}$

$$\begin{aligned} k(\alpha x, \alpha y) &= \frac{(\alpha y)^2 + 2(\alpha x)(\alpha y)}{(\alpha x)^2} \\ &= \frac{\alpha^2 y^2 + 2\alpha^2 xy}{\alpha^2 x^2} \\ &= \alpha^0 \frac{y^2 + 2xy}{x^2} \\ &= \alpha^0 k(x, y) \end{aligned}$$

Therefore k is a homogeneous function of degree 0.

7. Prove the following proposition

   Prop: If $f(x, y)$ is a homogeneous function of degree 0, it can always be expressed as $G(y/x)$ where $G(t)$ is a scalar function of one-variable.

   (Hint: When $x \neq 0$, $f(x, y) = (1/x)^0 f(x, y)$.)

   Proof. For $x \neq 0$, let $\alpha = 1/x$. Then by our assumptions

   $$f(x, y) = \alpha^0 f(x, y) = f(\alpha x, \alpha y) = f\left(\frac{x}{x}, \frac{y}{x}\right) = f\left(1, \frac{y}{x}\right).$$

   Since the first parameter of f is constant we can define a scalar function $G(t)$ using only the second parameter $t = \frac{y}{x}$. Then $G(t) = G(y/x) = f(x, y)$.

   For $x = 0$, $G(y/x)$ would be undefined, but $f(x, y) = f(0, y)$ which is a function of just y, so $G(t)$ works with $y = t$. □

8. Prove the following proposition

   Prop: Let $\frac{dy}{dx} = f(x, y)$ be such that $f(x, y)$ is a homogeneous function of degree 0. Then, through the substitution $u = y/x$, the ODE converts to a separable ODE of the form

   $$\frac{du}{dx} = \frac{1}{x}\left[f(1, u) - u\right].$$

   (Hint: Differentiate the substitution $u = y/x$ or $y = ux$.)

   Proof. Using the previous proposition with our hypothesis of degree 0 means that $f(x, y) = f(1, u)$, we will use this substitution later. Let $y = ux$. Differentiating both sides yields $dy = x du + u dx$ by the product rule. Returning to our original $f(x, y)$ and substituting dy with the former gives the desired result.

   $$\begin{aligned}
   \frac{dy}{dx} &= f(x, y) && \text{(ODE assumption)} \\
   \frac{x du + u dx}{dx} &= f(x, y) && \text{(substitution of } dy\text{)} \\
   \frac{du}{dx} \cdot x + u &= f(x, y) && \text{(simplify fraction)} \\
   \frac{du}{dx} \cdot x &= f(x, y) - u && \text{(subtract } u\text{)} \\
   \frac{du}{dx} &= \frac{1}{x}\left[f(x, y) - u\right] && \text{(divide by } x\text{)} \\
   \frac{du}{dx} &= \frac{1}{x}\left[f(1, u) - u\right] && \text{(substitution with previous proposition)}
   \end{aligned}$$

   □

9. Solve the ODE $\dfrac{dy}{dx} = \dfrac{y^2 + 2xy}{x^2}$.

Solution: By #6(c) we know that $f(x, y) := \frac{y^2+2xy}{x^2}$ is a homogeneous function of degree 0. Given these conditions, we can apply the proposition from #8 to find

$$\begin{aligned} \frac{du}{dx} &= \frac{1}{x}\left[f(1,u) - u\right] \\ \frac{du}{dx} &= \frac{1}{x}\left[u^2 + u\right] \end{aligned}$$

Now, using separation of variables yields

$$\begin{aligned} \frac{1}{u^2+u}du &= \frac{1}{x}dx \\ \int \frac{1}{u^2+u}du &= \int \frac{1}{x}dx \end{aligned}$$

Doing partial fractions on the u integrand gives

$$\frac{1}{u^2+u} = \frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1} \iff A(u+1) + Bu = 1 \implies A = 1 \wedge B = -1.$$

Hence,

$$\begin{aligned} &\int \frac{1}{u}du - \int \frac{1}{u+1}du = \int \frac{1}{x}dx \\ &\ln|u| - \ln|u+1| + C = \ln|x| \\ &\exp\left[\ln|u| - \ln|u+1| + C\right] = \exp\left[\ln|x|\right] \\ &\frac{Cu}{u+1} = x \\ &\frac{Cyx^{-1}}{yx^{-1}+1} = x && \text{(substitution of } u = y/x\text{)} \\ &Cyx^{-1} = y + x && \text{(multiply by } yx^{-1}+1\text{)} \\ &C = \frac{y}{yx^{-1}} + \frac{x}{yx^{-1}} && \text{(divide by } yx^{-1}\text{)} \\ &C = x + \frac{x^2}{y} && \text{(simplify)} \\ &\frac{x^2}{y} = C - x \\ &y = \frac{x^2}{C-x} \end{aligned}$$

So the general solution to the ODE is $y(x) = \dfrac{x^2}{C-x}$.