Math 345 – Homework 1 **Due Monday, September 05, 2022**

1. Consider the differential equation

$$1 + y^2 + 2(x+1)yy' = 0.$$

   (a) Show that the ODE represents an exact ODE.

   (b) Find the general solution to the ODE.

   (c) Does a specific solution curve of the ODE pass through the point $(5, 0)$? If so, find it.

2. *The same but different...* Consider the differential equation

$$1 + y^2 + 2(x+1)yy' = 0.$$

   (a) Show that the ODE is a separable equation and find the general solution. Justify that this is the same solution found before.

   (b) Use technology and graph the associated slope field. On the picture, sketch the solution curve that passes through the point $(5, 0)$.

3. For what values of the constants m, n, and α (if any) is the following differential equation exact?

$$x^m y^2 y' + \alpha x^3 y^n = 0$$

4. Consider the ODE $M(x, y)dx + N(x, y)dy = 0$.

   (a) Let $\mu(x, y)$ be a non-vanishing function. What is the relationship between the slope field of the original ODE and the ODE $\mu M dx + \mu N dy = 0$? Justify your answer.

   (b) Why are the solution curves to the original ODE and the ODE $\mu M dx + \mu N dy = 0$? identical? Briefly explain.

5. Consider the equation $-2xydx + (3x^2 - y^2)dy = 0$.

   (a) Show that the ODE is **not** exact.

(b) Find an integrating factor that converts the ODE into an exact one.

(c) Using the integrating factor, show that the μ-multiplied ODE is exact.

(d) Find the general solution to the original ODE.

"Homogeneous" non-linear first-order equations

1. *homogeneous functions*

   def: A function $f(x,y)$ is a **homogeneous function of degree n** if given any scalar α, $f(\alpha x, \alpha y) = \alpha^n f(x,y)$.

   Determine the degree of homogeneity for the following functions.

   (a) $g(x,y) := x^3 + y^3$

   (b) $h(x,y) := \dfrac{-x}{x^2+y^2}$

   (c) $k(x,y) := \dfrac{y^2+2xy}{x^2}$

2. Prove the following proposition.

   Prop: If $f(x,y)$ is a homogeneous function of degree 0, it can always be expressed as $G(y/x)$ where $G(t)$ is a scalar function of one-variable.

   (Hint: When $x \neq 0$, $f(x,y) = (1/x)^0 f(x,y)$.)

3. Prove the following proposition.

   Prop: Let $\dfrac{dy}{dx} = f(x,y)$ be such that $f(x,y)$ is a homogeneous function of degree 0. The, through the substitution $u = y/x$, the ODE converts to a separable ODE of the form

$$\frac{du}{dx} = \frac{1}{x}\left[f(1,u) - u\right].$$

   (Hint: Differentiate the substitution $u = y/x$ or $y = ux$.)

4. Solve the ODE $\dfrac{dy}{dx} = \dfrac{y^2+2xy}{x^2}$.