Math 345
Professor Carty
70 points possible

Exam 2
December 2, 2022

Name: ______________________________

1. No hats or dark sunglasses. All hats are to be removed.

2. All book bags are to be closed and placed in a way that makes them inaccessible. Do not reach into your bag for anything during the exam. If you need extra pencils, pull them out now.

3. Be sure to print your proper name clearly.

4. *No calculators are allowed.* Watches with recording, internet, communication or calculator capabilities (e.g. a smart watch) are prohibited.

5. All electronic devices, including cell phones and other wearable devices, must be powered off and stored out of sight for the entirety of the exam.

6. If you have a question, raise your hand and I will come to you. Once you stand up, you are done with the exam. If you have to use the facilities, do so now. You will not be permitted to leave the room and return during the exam.

7. Every exam is worth a total of **70 points**. Including the cover sheet, each exam has 7 pages.

8. At 2:50, you will be instructed to put down your writing utensil. You must stop writing the exam at this time.

9. If you finish early, quietly and respectfully and in your exam. You may leave early.

10. You will hand in the paper copy of the exam on your way out of the classroom.

11. You have fifty minutes to complete the exam. I hope you do well.

1. (10 points) Consider the ODE

$$y''' - 3y'' + 2y' = t + e^{-t}.$$

Transform the given initial value problem into its equivalent linear system in matrix form. (Do NOT solve the system.)

$X_1 = Y \quad , \quad X_1' = Y' \quad \longrightarrow \quad X_1' = X_2$

$X_2 = Y' \quad , \quad X_2' = Y'' \quad \longrightarrow \quad X_2' = X_3$

$X_3 = Y'' \quad , \quad X_3' = Y''' \quad \rightarrow \quad X_3' = 3y'' - 2y' + t + e^{-t}$

$X_3' = 3X_3 - 2X_2 + t + e^{-t}$

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ t + e^{-t} \end{pmatrix}$$

2. (20 points) Let $\mathbf{\Phi}(t) = \frac{1}{3}\begin{bmatrix} e^{4t}+2e^t & e^{4t}-e^t \\ 2e^{4t}-2e^t & 2e^{4t}+e^t \end{bmatrix}$.

(a) Show that $\mathbf{\Phi}(t)$ is a fundamental matrix for the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$.

$$\Phi'(t) = \frac{1}{3}\begin{bmatrix} 4e^{4t}+2e^t & 4e^{4t}-e^t \\ 8e^{4t}-2e^t & 8e^{4t}+e^t \end{bmatrix}$$

$$A\Phi = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} e^{4t}+2e^t & e^{4t}-e^t \\ 2e^{4t}-2e^t & 2e^{4t}+e^t \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 4e^{4t}+2e^t & 4e^{4t}-e^t \\ 8e^{4t}-2e^t & 8e^{4t}+e^t \end{bmatrix}$$

$$\det\Phi = \left(\frac{1}{3}\right)^2\left[(e^{4t}+2e^t)(2e^{4t}+e^t) - (2e^{4t}-2e^t)(e^{4t}-e^t)\right]$$

$$= \frac{1}{9}\left[2e^{8t}+5e^{5t}+2e^{2t} - (2e^{8t}-4e^{5t}+2e^{2t})\right]$$

$$= \frac{1}{9}\left[9e^{5t}\right] = e^{5t} \neq 0 \text{ for all } t.$$

(b) Prove that $\mathbf{\Phi}(t)$ is $e^{\mathbf{A}t}$.

We know e^{At} solves $\dot{X} = AX$ and $e^{A0} = I_2$.

Since $\Phi(0) = \frac{1}{3}\begin{bmatrix} 1+2 & 1-1 \\ 2-2 & 2+1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = I_2$,

by existence and uniqueness thm, $\Phi(t) = e^{At}$.

(c) Compute a particular solution to the non-homogeneous system

$$\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 3e^t \\ 0 \end{bmatrix}.$$

$$X_p(t) = e^{At} \int e^{-At} f(t)\, dt$$

$$e^{-At} f(t) = \frac{1}{3} \begin{bmatrix} e^{-4t} + 2e^{-t} & e^{-4t} - e^{-t} \\ 2e^{-4t} - 2e^{-t} & 2e^{-4t} + e^{-t} \end{bmatrix} \begin{bmatrix} 3e^t \\ 0 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 3e^{-3t} + 6 \\ 6e^{-3t} - 6 \end{bmatrix} = \begin{bmatrix} e^{-3t} + 2 \\ 2e^{-3t} - 2 \end{bmatrix}$$

$$\int e^{-At} f(t)\, dt = \begin{bmatrix} -\frac{1}{3} e^{-3t} + 2t \\ -\frac{2}{3} e^{-3t} - 2t \end{bmatrix}$$

$$X_p(t) = \frac{1}{3} \begin{bmatrix} e^{4t} + 2e^t & e^{4t} - e^t \\ 2e^{4t} - 2e^t & 2e^{4t} + e^t \end{bmatrix} \begin{bmatrix} -\frac{1}{3} e^{-3t} + 2t \\ -\frac{2}{3} e^{-3t} - 2t \end{bmatrix}$$

3. (15 points) Let $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

(a) Find the stable, unstable, and center subspaces for $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

$$A - \lambda I = \underbrace{(-2-\lambda)}_{\lambda_1 = -2}\left[(2-\lambda)^2 + 1\right]$$

$(2-\lambda)^2 = -1$, $2-\lambda = \pm i$, $\lambda = 2 \pm i$

$\lambda_1 = -2$: $A + 2I = \begin{bmatrix} 4 & 1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad V_1 = e_3$

$\lambda = 2+i$: ~~$A + 2iI$~~ $A - (2+i)I = \begin{bmatrix} -i & 1 & 0 \\ -1 & -i & 0 \\ 0 & 0 & i \end{bmatrix}$ $\quad \vec{V} = \langle 1, i, 0 \rangle$

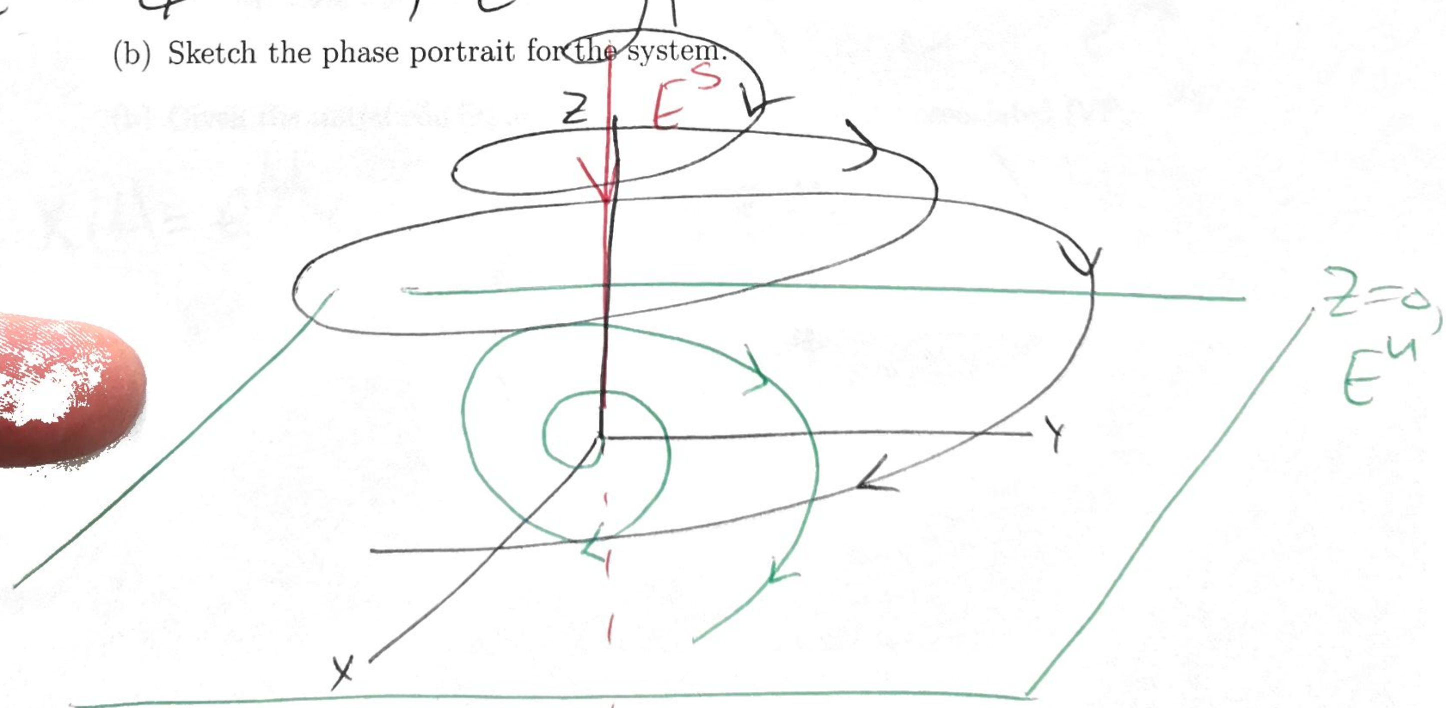
$= \langle 1,0,0 \rangle + i\langle 0,1,0 \rangle$

$\Rightarrow V_2 = \langle 0,1,0 \rangle$, $u_2 = \langle 1,0,0 \rangle$

$E^s = \text{span}\{e_3\}$.

$E^c = \emptyset$, $E^u = \text{span}\{e_1, e_2\}$

(b) Sketch the phase portrait for the system.



4. (15 points) Consider the system $\mathbf{x}' = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{x}$.

(a) Determine a fundamental matrix for the given system.

$$A = D + N = 2I + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Clearly D and N commute, $DN = 2N = ND$.

So $e^{At} = e^{Dt} e^{Nt}$. $N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $N^3 = [0]$

$$e^{At} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \left(I + tN + \frac{t^2}{2} N^2 \right)$$

$$= e^{2t} \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{2t} & te^{2t} & t^2 e^{2t}/2 \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix}$$

(b) Given the initial condition $\mathbf{x}(0) = \langle -1, 0, 3 \rangle$, solve the associated IVP.

$$x(t) = e^{At} x_0 = \begin{pmatrix} -e^{2t} + 3t^2 e^{2t}/2 \\ 3te^{2t} \\ 3e^{2t} \end{pmatrix}$$

5. (10 points) Consider the first-order system

$$\dot{x} = 4x(y-1)$$
$$\dot{y} = y(x+x^2).$$

(a) Determine all the critical points of the system.

$\dot{x}=0 \Rightarrow x=0$ or $y=1$

$\dot{y}=0$ and $x=0 \Rightarrow y$ free. entire y-axis ($x=0$) are fixed point!

$\dot{y}=0$ and $y=1 \Rightarrow x+x^2=0 \Rightarrow x=0$ (done already)

or $x=-1$.

isolated fixed point @ $(-1,1)$

(b) For the critical points found in (a), use the Linearization Theorem and classify the critical points, if possible. If it is not possible, clearly explain why.

We can not characterize non-isolated critical points ($x=0$).

Classify $(-1,1)$

$$Df(x,y) = \begin{bmatrix} 4(y-1) & 4x \\ y(1+2x) & x+x^2 \end{bmatrix}, \quad Df(-1,1) = \begin{bmatrix} 0 & -4 \\ -1 & 0 \end{bmatrix}$$

$$|Df(-1,1) - \lambda I| = \lambda^2 - 4 = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = -2$$

$(-1,1)$ is a saddle point