Math 345 – Homework 9 **Due Monday, December 5, 2022**

1. Classify the equilibrium points of the Lorenz equation

$$\mathbf{x}'(t) = \langle y - x, \mu x - y - xz, xy - z \rangle$$

for $\mu > 0$.

(a) At what value of the parameter μ do two new equilibrium points "bifurcate" from the equilibrium point at the origin?

Solution: $x'(t) = 0$, requires $y = x$. This and $y'(t) = 0$, yields $x(\mu - 1 - z) = 0$. Hence $x = 0$ or $z = \mu - 1$. Note $z'(t) = 0$ yields $x^2 = z$. When $x = 0$, this corresponds to the origin $(0, 0, 0)$ always being a critical point. If $z = \mu - 1$, we get $x = y = \pm\sqrt{\mu - 1}$. Hence the two additional equilibria occur at $(\pm\sqrt{\mu - 1}, \pm\sqrt{\mu - 1}, \mu - 1)$ when $\mu > 1$.

(b) Classify the equilibrium point in the regime when there is only one.

Solution:

$$D_{\mathbf{f}}(x, y, z) = \begin{pmatrix} -1 & 1 & 0 \\ \mu - z & -1 & x \\ y & x & -1 \end{pmatrix} \text{ and } D_{\mathbf{f}}(0, 0, 0) - \lambda \mathbf{I} = \begin{pmatrix} -\lambda - 1 & 1 & 0 \\ \mu & -\lambda - 1 & 0 \\ 0 & 0 & -\lambda - 1 \end{pmatrix}.$$

$\det D_{\mathbf{f}}(0, 0, 0) - \lambda \mathbf{I} = (-\lambda - 1)[(-\lambda - 1)^2 - \mu]$. This yields the eigenvalues $\lambda_1 = -1$, $\lambda_{2,3} = -1 \pm \sqrt{\mu}$. When $0 < \mu < 1$, the origin is a stable sink.

(c) Classify the equilibrium point at the origin in the regime when there are three.

Solution: When $\mu = 1$, semistable sink (one zero eigenvalue). When $\mu > 1$, the origin becomes a saddle point, two stable and one unstable direction.

2. Consider the system

$$\dot{x} = y, \dot{y} = -x + (1 - x^2 - y^2)y.$$

(a) Let D be any disc $x^2 + y^2 \leq R^2$. Explain why the system satisfies the hypotheses of the Existence and Uniqueness theorem throughout D.

Solution: The vector field $\mathbf{f}(x, y) = \langle y, -x + (1 - x^2 - y^2)y \rangle$ is continuous and differentiable on any disk of radius R.

(b) By substitution, show that $x(t) = \sin t$, $y(t) = \cos t$ is an exact solution of the system.

Solution: The $\dot{x} = 0$ condition is trivial. $\dot{y} = -\sin t$. Evaluating the second component of the vector field on the path,

$$-\sin t + (1 - \sin^2 t - \cos^2 t)\cos t = -\sin t.$$

Hence $x(t) = \sin t$, $y(t) = \cos t$ parameterizes a flow line for the non-linear system. Note, this is the parametrization of the unit circle.

(c) Now consider an IVP where (x_0, y_0) is in the unit disk. Without doing any calculations, explain why the solution must satisfy $x^2(t) + y^2(t) < 1$ for all t.

Solution: Flow lines / solution curves can only intersect at rest points. That is, the flow line that is the unique solution to any IVP can not cross curve $x^2 + y^2 = 1$ unless the initial condition was on the unit circle to start. Hence $x^2(t) + y^2(t) < 1$ for all t.

(d) Use linearization to classify the equilibrium at the origin, if possible.

Solution: The system is self-linearizable:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -y(x^2+y^2) \end{pmatrix},$$

a linear part plus a cubic part. The eigenvalues for $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ are $\lambda_{1,2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. This characterizes a spiral source. More interesting, together with (c), we see that the spirals spiral out, but must the limiting to the unit circle. (The proper term for this is that the solution curve parametrized by $(\sin t, \cos t)$ is a *limit cycle*.)

(e) We want to show that the solution curve $x^2 + y^2 = 1$ is semi-attracting. Consider the any circle $x^2 + y^2 = R^2$, $R > 0$

i. Find an outward pointing normal to the circle. (This is a Calc III question.)

ii. Use the dot product and show that the vector field defined by the system is always inwardly pointing through the boundary of the circle.

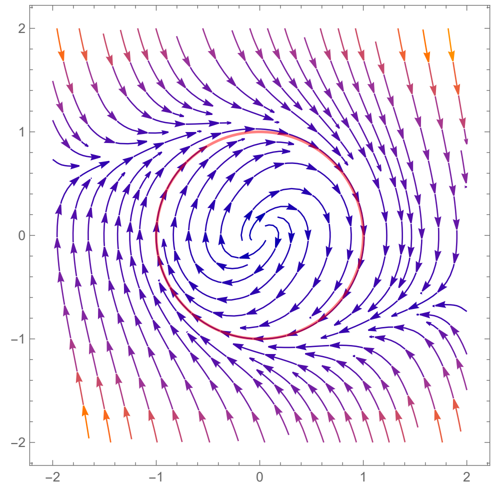
Solution: There was a typo here. We needed $R > 1$ for this to work. From calculus that for a differentiable f, the gradient of f, is normal to the level curve/surface $f(\mathbf{x}) = C$. Here, letting $g(x, y) = x^2 + y^2$, $\nabla g = \langle 2x, 2y \rangle$ and this is an outward pointing normal on the circle. To show that the flow is always inward, we can use the dot product. Recall $\dot{\vec{a}\vec{b}} = ||\vec{a}||||\vec{b}|| \cos\theta$, where θ is the angle between the vectors. Thus, if the dot product of $\mathbf{f}$ and ∇g is always negative, the vector fields flow will always be inward. Here,

$$\mathbf{f} \cdot \nabla g = 2y^2(1 - x^2 - y^2) = 2y^2(1 - R^2) < 0 \text{ when } x^2 + y^2 = R^2.$$

This shows that the flow outside of the unit disk is also flowing to the unit circle.

(f) Give a rough sketch of the flow lines for the system.

Solution:

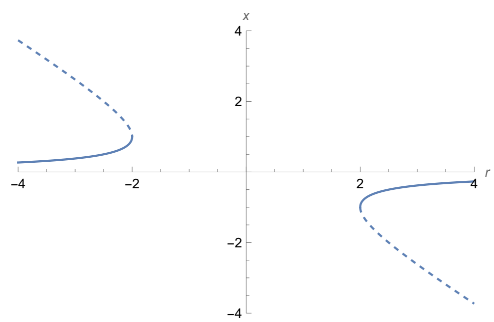


Note the unit circle is in red. Due to the clockwise flow of the solution in (b) and the continuity of the vector field, we can determine the flow for the entire field.

3. For each autonomous ODE, find the values of r at which bifurcations occur, classify them as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Finally, sketch the bifurcation diagram of fixed points x_e versus r.

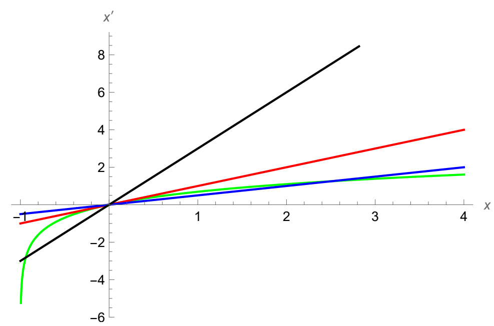
   (a) $\dot{x} = 1 + rx + x^2$

Solution: $\dot{x} = 0$ when $x = \dfrac{-r \pm \sqrt{r^2 - 4}}{2}$. Note that there are no critical numbers in the regime $|r| < 2$. At $r = \pm 2$, the system undergoes a saddle-node bifurcation. For example, when $r > 2$, the quadratic $\dot{x} = 1 + rx + x^2 < 0$ on $\dfrac{-r - \sqrt{r^2 - 4}}{2} < x < \dfrac{-r + \sqrt{r^2 - 4}}{2}$. Hence, $x = \dfrac{-r - \sqrt{r^2 - 4}}{2}$ is a stable node and $x = \dfrac{-r + \sqrt{r^2 - 4}}{2}$ is an unstable node. Analysis when $r < -2$, yields the same kind of analysis. The critical point close to the origin will be stable.

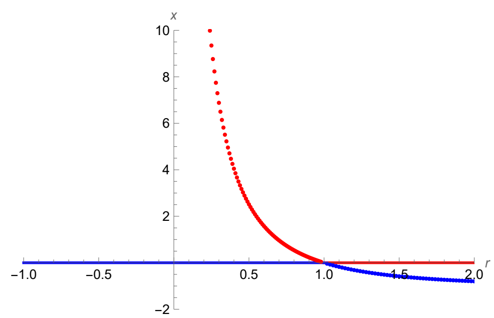


(b) $\dot{x} = rx - \ln(1+x)$

Solution: Note that for any r, $x = 0$ is an equilibrium. For $r \leq 0$, the only fixed point is at $x = 0$. We can see this by considering the intercepts of $\dot{x} = rx$ and $\dot{x} = \ln(1+x)$. The only intercept of the line and the logarithmic function will be at the origin. Moreover, as $rx - \ln(1+x) > 0$ on $x \in (-1, 0)$ and $rx - \ln(1+x) < 0$ on $x \in (0, \infty)$, the equilibrium point $x = 0$ is always stable. When $r > 0$, the change occurs when $r = 1$. As we can see (and check easily with derivatives) $\dot{x} = rx$ and $\dot{x} = \ln(1+x)$ are tangent at $(0, 0)$ when $r = 1$. When $r \in (0, 1)$, we get a graph like the blue line below. In this regime, $\dot{x} > 0$ when $x < 0$ and to the left of the intersection point. Hence, $x = 0$ is stable and the other intersection is an unstable equilibria. This reverses when $r > 1$. We get a transcritical bifurcation when at $r = 1$.



Below (using mathematica and numerical solvers) is a graph of the bifurcation diagram. Blue represents the stable nodes and red the unstable nodes.



(c) $\dot{x} = x + \dfrac{rx}{1+x^2}$

Solution: This is a pitchfork bifurcation. For any r, $x = 0$, and $x = \pm\sqrt{-1-r}$ are the roots. Of course, there are only three real roots when $r < -1$. Checking the sign of $\dot{x}$, we see the stable nodes (blue) and the unstable nodes (red).