Math 345 – Homework 2 **Due Monday, September 13, 2022**

1. Express the solution of the initial value problem

$$2x\frac{dy}{dx} = y + 2x\cos x,\ \ y(1) = 0$$

as an integral.

Solution: This is first order linear. Written in standard form,

$$\frac{dy}{dx} - \frac{1}{2x}y = \cos x,\ \ y(1) = 0.$$

With $p(x) = -1/2x$,

$$\mu(x) = \exp\left(\int -\frac{dx}{2x}\right) = \frac{1}{\sqrt{x}} \text{ when } x > 0.$$

Multiplying the standard form equation by μ yield

$$\begin{aligned} \frac{1}{\sqrt{x}}y' - \frac{1}{2x^{3/2}}y &= \frac{1}{\sqrt{x}}\cos x \\ \left(\frac{1}{\sqrt{x}}y\right)' &= \frac{1}{\sqrt{x}}\cos x. \end{aligned}$$

Using the FTC,

$$\begin{aligned} \frac{1}{\sqrt{t}}y(t)\bigg|_1^x &= \int_1^x \frac{1}{\sqrt{t}}\cos t\,dt \\ \frac{1}{\sqrt{x}}y(x) - \frac{1}{\sqrt{1}}y(1) &= \int_1^x \frac{1}{\sqrt{t}}\cos t\,dt \\ \frac{1}{\sqrt{x}}y(x) &= \int_1^x \frac{1}{\sqrt{t}}\cos t\,dt \end{aligned}$$

Hence

$$y(x) = \sqrt{x}\left[\int_1^x \sqrt{t}\cos t\,dt\right] \text{ where } x > 0.$$

2. Find the general solutions of the differential equations.

(a) $x^3 + 3y - xy' = 0$

Solution: first-order linear:

$$y' - \frac{3}{x}y = x^2$$

$$\mu(x) = \exp\left(\int \frac{dx}{x}\right) = \frac{1}{x^3}$$

$$(\mu y)' = \frac{1}{x}$$

$$y(x) = x^3\left[\ln x + C\right]$$

(b) $xy^2 + 3y^2 - x^2y' = 0$

Solution: separable:

$$y' = \left(\frac{x+3}{x^2}\right)y^2$$

$$\frac{dy}{y^2} = \left(\frac{1}{x} + \frac{3}{x^2}\right)dx$$

$$-\frac{1}{y} = \ln|x| - \frac{3}{x} + C$$

$$y = \frac{x}{3 + Cx - x\ln|x|}$$

(c) $6xy^3 + 2y^4 + (9x^2y^2 + 8xy^3)y' = 0$

Solution: Hope for exactness...

$$(6xy^3 + 2y^4)_y = 18xy^2 + 8y^3 = (9x^2y^2 + 8xy^3)_x. \qquad \textit{(whew!)}$$

$$f(x,y) = \int (6xy^3 + 2y^4)dx = 3x^2y^3 + 2xy^4 + C(y).$$

Clearly $C(y)$ is independent of y here.

$$3x^2y^3 + 2xy^4 = C.$$

3. Solve the differential equation

$$(x + ye^y)\frac{dy}{dx} = 1$$

by regarding y as the independent variable rather than x.

Solution:

$$\frac{dx}{dy} = x + ye^y$$

First-order linear:

$$\frac{dx}{dy} - x = ye^y$$

Then $p(y) = -1$ and $\mu(y) = e^{-y}$.

$$(e^{-y}x)' = y$$

$$x = \frac{1}{2}y^2e^y + Ce^y$$

4. (a) Consider the ODE $y(1+x^3)y' = x^2$. Determine where in the xy-plane existence and uniqueness issues to an associated initial value problem may occur.

Solution: Here $y' = f(x,y)$ where $f(x,y) = \dfrac{x^2}{y(1+x^3)}$. Note $f(x,y)$ is undefined when $x = -1$ or $y = 0$. The partial $f_y(x,y) = -\dfrac{x^2}{y^2(1+x^3)}$ yields no new points of concern.

(b) Solve the IVP $y(1+x^3)y' = x^2$, $y(0) = y_0$ and determine how the interval in which the solution exists depends on the initial value y_0.

Solution: Equation is separable. Yields solution

$$y^2 = \frac{2}{3}\ln|1+x^3| + C.$$

For the initial condition $y(0) = y_0$, the $x = 0$ will require the domain of y restricted to $x > -1$. But the entirety of the curve

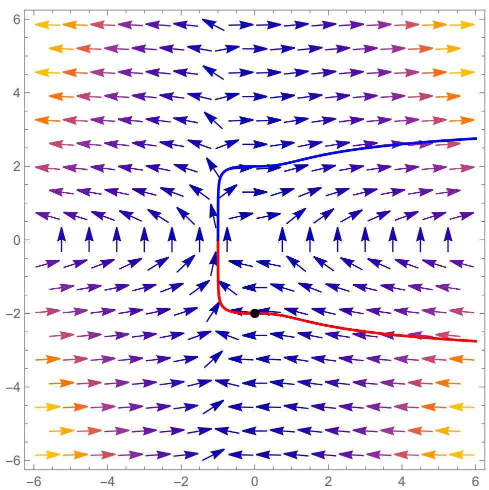
$$y^2 = \frac{2}{3}\ln|1+x^3| + y_0^2$$

is unlikely to represent y as a function of x. We need the portion of he curve that implicitly defines $y(x)$ that solves the IVP.

Solution: For example, use the initial condition $(0, -2)$. Plotting the curve in the vector field, we see would only want the red portion of the graph equation

$$y^2 = \frac{2}{3}\ln|1 + x^3| + 4.$$

What we really need to know is when does the curve "double-back" an stop representing y as a function of x. The ODE tells us that the curve has vertical slope when $y = 0$. Here, we can actually solve for this x-intercept. For this example, $y = 0$ when $x = \sqrt[3]{-1 + e^{-6}}$ (a number strictly bigger than $-a$.) For the initial condition $(0, -2)$, we would have $y(x) = -\sqrt{\frac{2}{3}\ln|1 + x^3| + 4}$ on domain $x \geq \sqrt[3]{-1 + e^{-6}}$.



In general, for $y(0) = y_0 > 0$,

$$y(x) = \sqrt{\frac{2}{3}\ln|1 + x^3| + y_0^2}, \text{ where } x \geq \sqrt[3]{-1 + \exp(-3y_0^2/2)}.$$

When $y_0 < 0$,

$$y(x) = -\sqrt{\frac{2}{3}\ln|1 + x^3| + y_0^2}, \text{ where } x \geq \sqrt[3]{-1 + \exp(-3y_0^2/2)}.$$

5. Consider the IVP $y' = ty^2$, $y(0) = 1$.

(a) Explain why this IVP has a unique solution.

Solution: $f(t, y) = ty^2$ and $f_y(t, y) = 2ty$ are continuous throughout $\mathbb{R}^2$,

(b) Covert the IVP into an equivalent integral equation.

Solution:

$$y(t) = 1 + \int_0^t sy^2(s)\, ds.$$

(c) Set up the approximate integral equation used in Picard's method and carry out the iteration for three steps.

Solution:

$$y_n(t) = 1 + \int_0^t s(y_{n-1}(s))^2\, ds.$$

Recall $y_0 = 1$, the initial condition value of y.

$$y_1(t) = 1 + \int_0^t s(1)^2\, ds = 1 + \frac{t^2}{2}$$

$$y_2(t) = 1 + \int_0^t s\left(1 + \frac{s^2}{2}\right)^2 ds = 1 + \frac{t^2}{2} + \frac{t^4}{4} + \frac{t^6}{24}$$

$$y_3(t) = 1 + \int_0^t s\left(1 + \frac{s^2}{2} + \frac{s^4}{4} + \frac{s^6}{24}\right)^2 ds = 1 + \frac{t^2}{2} + \frac{t^4}{4} + \frac{t^6}{8} + \frac{t^8}{24} + \frac{t^{10}}{96} + \frac{t^{12}}{576} + \frac{t^{14}}{8064}$$

(d) Solve the IVP by separation of variables.

Solution:

$$\frac{dy}{y^2} = t\, dt \longmapsto -\frac{1}{y} = \frac{t^2}{2} + C \longmapsto y = \frac{2}{C - t^2}$$

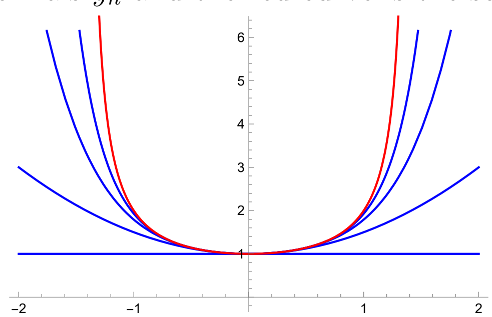
Using the IC, $y(t) = \dfrac{2}{2 - t^2}$ and the domain of t is $|t| < \sqrt{2}$.

(e) Determine the series representation of the solution to the IVP and compare it to the successive approximations computed above.

Solution: To be able to compare the solutions, we need the series representation of y. Recall $\displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ and this series converges when $|r| < 1$. Then

$$\begin{aligned} y(t) &= \frac{2}{2-t^2} \\ &= \frac{1}{1-\frac{t^2}{2}} \\ &= \sum_{n=0}^{\infty} \left(\frac{t^2}{2}\right)^2 \\ &= 1 + \frac{t^2}{2} + \frac{t^4}{4} + \frac{t^6}{8} + \frac{t^8}{16} + \frac{t^{10}}{32} + \dots \end{aligned}$$

Note this series has the same domain as $|t^2/2| < 1$ is the same as $|r| < \sqrt{2}$. We can see that the iterates appear to be converging to this series. I wouldn't do it by hand, but I would bet the fifth term of the next iterate y_4 is $\frac{t^8}{16}$. In fact, much like we did with Taylor series in Calc II, we can see the iterates converging. In the below, the bending blue curves are the polynomials y_n and the red curve is the solution to the IVP.



Reduction of Order

The general form of a second-order differential equation has the form

$$F(x, y, y', y'') = 0.$$

Reduction of order is the idea of using a change of variable to produce an equivalent, lower-order differential equation.

1. *dependent variable missing*

   When y is not explicitly present in the ODE, the ODE has the general form $F(x, y', y'') = 0$.

   (a) Explain how the introduction of the new dependent variable $p(x) = y'$ converts the second-order ODE into a first-order system in p.

Solution: Let $p = y'$, then $p' = y''$ and $F(x, y', y'') = 0$ becomes $F(x, p, p') = 0$.

   (b) Solve the equation $xy'' - y' = 3x^2$. (Note that in the end your solution for y will have two arbitrary constants; as it should.)

Solution: By above the above, the ODE can be rewritten $xp' - p = 3x^2$, a first-order linear ODE in p. Solving for p gives $p(x) = 3x^2 + Cx$. Then $y' = 3x^2 + Cx$ yields

$$y = x^3 + Cx^2 + K.$$

2. *independent variable missing*

   When x is not explicitly present in the ODE, the ODE has the general form F(y,y',y")=0. (Equations of this form are know as **autonomous**.)

   (a) Explain how the introduction of the new dependent variable $p(y) = y'$ converts the second-order ODE into a first-order system where y can be momentarily interpreted as the independent variable. (Hint: Take care with the derivative computed to replace the y'' term.)

Solution: Let $p(y) = y'$. Then $\dfrac{dp}{dx} = \dfrac{dp}{dy}\dfrac{dy}{dx}$. That is $y'' = p'y'$ or $y'' = p'p$. The ODE can be rewritten $F(y, p, pp') = 0$ and we can treat p as a function of y in solving this equation.

(b) Solve the equation $y'' + k^2 y = 0$ (where k a positive constant). (This is actually ill-posed. At some point you are going to have to choose a sign convention. This will yield a solution. The other sign convention would yield a separate solution. We will learn a much easier way to do this problem in the near future.)

Solution: Using the substitutions, we get $pp' + k^2 y = 0$, a separable equation. Solving for p is straight-forward, $p^2 = -k^2 y^2 + C$. The next step is harder. Substituting back we get the non-linear ODE

$$(y')^2 = -k^2 y^2 + C.$$

Solving for y' results in two different ODE. We consider $y' = -\sqrt{C - k^2 y^2}$ first. The ODE becomes

$$-\frac{dy}{\sqrt{C - k^2 y^2}} = dx.$$

The left-hand side is an arccosine:

$$\int -\frac{dy}{\sqrt{C - k^2 y^2}} = \int \frac{-1}{\sqrt{C}\sqrt{1 - (ky/\sqrt{c})^2}}\, dy = \frac{1}{\sqrt{C}} \int \frac{-(\sqrt{C}/k)du}{\sqrt{1 - u^2}} \text{ where } u = ky/\sqrt{C}.$$

So

$$\frac{1}{\sqrt{k}} \int \frac{-1}{\sqrt{1 - u^2}}\, du = dx$$

becomes

$$\frac{1}{k} \arccos u = x + K \text{ or } \arccos\left(\frac{ky}{\sqrt{C}}\right) = kx + K.$$

Solving for y yields

$$y(x) = \frac{\sqrt{C}}{k} \cos(kx + K).$$

When we address the positive signed equation $y' = \sqrt{C - k^2 y^2}$, the same arithmetic yields a second solution of the form

$$y(x) = \frac{\sqrt{C}}{k} \sin(kx + K).$$

(Note that two different solutions do not violate existence and uniqueness as this was not a first-order system at the start.)