Math 345 – Homework 6 **Due Wednesday, October 19, 2022**

1. The equations of motions for two coupled pendulums are

$$\ddot{\theta}_1 + k(\theta_1 - \theta_2) + \omega_1^2\theta_1 = 0$$
$$\ddot{\theta}_2 + k(\theta_2 - \theta_1) + \omega_2^2\theta_2 = 0$$

   where θ_i is the angle of the pendulums shaft from equilibrium, k the spring constant of the coupler, and ω_i is the natural frequency of the pendulum when uncoupled. Convert the system of coupled ODE to a first-order system.

2. Consider the linear system

$$\begin{aligned} y_1' &= y_2 + 4y_3 \\ y_2' &= -y_1 - 2y_3 \\ y_3' &= y_3 \end{aligned}$$

   (a) Convert the linear system to the equivalent matrix equation.

   (b) Show that a fundamental matrix exists for the given system via Liouville's formula.

   (c) Show that

$$\Phi(t) := \begin{bmatrix} \sin t & \cos t & e^t \\ \cos t & -\sin t & -3e^t \\ 0 & 0 & e^t \end{bmatrix}$$

   is a fundamental matrix for the system.

   (d) Compute $\det \Phi(t)$ and show that the conclusion of Liouville's theorem holds.

3. Consider $y'' - 3y' + 2y = 0$.

   (a) Find a fundamental set for the given scalar equation.

   (b) Convert the scalar equation to an equivalent first order system.

   (c) Show that the matrix associated with the Wronskian of your fundamental set forms a fundamental matrix for the linear system.

4. (a) If $\Phi(t)$ is a fundamental matrix for $x' = Ax$ and C is a nonsingular matrix of the same dimension, show that $\Phi(t)C$ is a fundamental matrix.

   (b) Show that if $\Phi(t)$ and $\Psi(t)$ are fundamental matrices for $x' = Ax$, then there is a constant, nonsingular matrix C such that $\Phi(t)C = \Psi(t)$.