Math 345 – Homework 8 **Due Monday, November 16, 2022**

1. For the following matrices A:
   (a) do the eigenvalue decomposition and determine the associated eigenvectors,
   (b) determine the transition matrix P who is columns space the (generalized) eigenspaces of A,
   (c) compute e^{At}, and
   (d) show use P to compute the (up to sign) Jordan Canonical form of each.

A. $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{pmatrix}$ B. $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}$ C. $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 2 & 2 \end{pmatrix}$ D. $\begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

Solution: **A**: This matrix is diagonalizable. The eigensystem is $\lambda \in \{10, 0, 0, 0\}$ with corresponding eigenvectors $\mathbf{v}_1 = \langle 1, 2, 3, 4 \rangle$, $\mathbf{v}_2 = \langle -1, 0, 0, 1 \rangle$, $\mathbf{v}_3 = \langle -1, 0, 1, 0 \rangle$, and $\mathbf{v}_4 = \langle -1, 1, 0, 0 \rangle$. We form the change of basis matrix

$$\mathbf{P} = [\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3 \mathbf{v}_4] = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 2 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 4 & 1 & 0 & 0 \end{pmatrix} \text{ and } \mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $e^{\mathbf{A}t} = e^{\mathbf{P}\mathbf{D}\mathbf{P}^{-1}t} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}$;

$$e^{\mathbf{A}t} = \begin{pmatrix} \frac{e^{10t}}{10} + \frac{9}{10} & \frac{e^{10t}}{10} - \frac{1}{10} & \frac{e^{10t}}{10} - \frac{1}{10} & \frac{e^{10t}}{10} - \frac{1}{10} \\ \frac{e^{10t}}{5} - \frac{1}{5} & \frac{e^{10t}}{5} + \frac{4}{5} & \frac{e^{10t}}{5} - \frac{1}{5} & \frac{e^{10t}}{5} - \frac{1}{5} \\ \frac{3e^{10t}}{10} - \frac{3}{10} & \frac{3e^{10t}}{10} - \frac{3}{10} & \frac{3e^{10t}}{10} + \frac{7}{10} & \frac{3e^{10t}}{10} - \frac{3}{10} \\ \frac{2e^{10t}}{5} - \frac{2}{5} & \frac{2e^{10t}}{5} - \frac{2}{5} & \frac{2e^{10t}}{5} - \frac{2}{5} & \frac{2e^{10t}}{5} + \frac{3}{5} \end{pmatrix}.$$

Solution: **B**: This matrix has a defective eigenspace. The eigensystem is $\lambda = 0$, repeated four times with corresponding eigenvectors $\mathbf{v}_1 = \langle -1, 0, 0, 1\rangle$, and $\mathbf{v}_2 = \langle 0, 1, 1, 0\rangle$. To find the generalized eigenvectors, we look at the null space of powers of $\mathbf{A}\lambda\mathbf{I} = \mathbf{A}$. Here $\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ has the null space spanned by $\{\langle 0,0,0,1\rangle, \langle 0,1,1,0\rangle, \langle 1,0,0,0\rangle\}$. Note that $\mathbf{v}_1 = -\mathbf{e}_1 + \mathbf{e}_4$. So we can choose either of these standard basis vectors for our first generalized eigenvector. Let $\mathbf{v}_3 = \mathbf{e}_1$. Here $\mathbf{A}^3 = \mathbf{0}$. For the second generalized eigenvector, we need any vector that is not in the span of the null space of $\mathbf{A}^2$. One choice is $\mathbf{v}_4 = \langle 0, 1, 0, 1\rangle$.

$$\mathbf{P} = [\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3\mathbf{v}_4] = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Here we have that $\mathbf{A}$ is nilpotent and that the block form is $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{0}$ Then

$$e^{\mathbf{A}t} = \mathbf{I} + t\mathbf{A} + \frac{t}{2}\mathbf{A}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 - \frac{t^2}{2} & \frac{t^2}{2} & t \\ t & -\frac{t^2}{2} & \frac{t^2}{2} + 1 & t \\ 0 & -t & t & 1 \end{pmatrix}.$$

Solution: **C**: This matrix has also has a defective eigenspace. The eigensystem is $\lambda \in \{2 \pm i, 2, 2\}$, with corresponding complex eigenvectors $\langle 0, 0, i, 1\rangle$, $\langle 0, 0, -i, 1\rangle$, and $\mathbf{e}_1$. For real eigenvectors for $\mathbf{P}$, we choose $\mathbf{u}_1 = \mathbf{e}_4$, $\mathbf{v}_1 = \mathbf{e}_3$, and $\mathbf{u}_3 = \mathbf{e}_1$. To find the generalized eigenvectors associated with $\lambda = 2$, we look at the null space of $(\mathbf{A} - 2\mathbf{I})^2$. Here $(\mathbf{A} - 2\mathbf{I})^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ has the null space spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$. We choose $\mathbf{u}_3 = \mathbf{e}_2$ and form

$$\mathbf{P} = [\mathbf{v}_1 \mathbf{u}_1 \mathbf{u}_2 \mathbf{u}_3] = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Note that the block form is

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

and that it is not block diagonal. To compute $e^{\mathbf{A}t}$, we use the decomposition $\mathbf{A} = \mathbf{S} + \mathbf{N}$ where

$$\mathbf{S} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P} \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \mathbf{P}^{-1} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Then $\mathbf{A} - \mathbf{S} = \mathbf{N} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Solution:

We then get $e^{\mathbf{D}t} = \begin{pmatrix} e^{2t}\cos(t) & -e^{2t}\sin(t) & 0 & 0 \\ e^{2t}\sin(t) & e^{2t}\cos(t) & 0 & 0 \\ 0 & 0 & e^{2t} & 0 \\ 0 & 0 & 0 & e^{2t} \end{pmatrix}$ and $e^{\mathbf{N}t} = \mathbf{I}+t\mathbf{N} = \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

All together,

$$\begin{aligned} e^{\mathbf{A}t} &= e^{\mathbf{S}t}e^{\mathbf{N}t} \\ &= \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}e^{\mathbf{N}t} \\ &= \begin{pmatrix} e^{2t} & e^{2t}t & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & e^{2t}\cos(t) & -e^{2t}\sin(t) \\ 0 & 0 & e^{2t}\sin(t) & e^{2t}\cos(t) \end{pmatrix}. \end{aligned}$$

Solution: D: This matrix also has a defective eigenspace with repeated $\lambda = 1 \pm i$, with corresponding complex eigenvectors $\langle i, 1, 0, 0\rangle$, and $\langle -i, 1, 0, 0\rangle$. For real eigenvectors for $\mathbf{P}$, we choose $\mathbf{u}_1 = \mathbf{e}_2$, and $\mathbf{v}_1 = \mathbf{e}_1$. To find the generalized eigenvectors associated with $\lambda = 1 + i$, we look at the null space of $(\mathbf{A} - (1+i)\mathbf{I})^2$. Here $(\mathbf{A} - (1+i)\mathbf{I})^2 = \begin{pmatrix} -2 & 2i & -1-2i & -1 \\ -2i & -2 & 1-2i & -1 \\ 0 & 0 & -2 & 2i \\ 0 & 0 & -2i & -2 \end{pmatrix}$ has the null space spanned by the complex eigenvectors $\langle i, 1, 0, 0\rangle$, and $\langle 1-i, 0, 2i, 2\rangle$. We then choose $\mathbf{v}_2 = \langle -1, 0, 2, 0\rangle$ and $\mathbf{u}_2 = \langle 1, 0, 0, 2\rangle$.

$$\mathbf{P} = [\mathbf{v}_1\mathbf{u}_1\mathbf{v}_2\mathbf{u}_2] = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Note that the block form is

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & -1 & 1 & \frac{1}{2} \\ 1 & 1 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and that it is not block diagonal. To compute $e^{\mathbf{A}t}$, we use the decomposition $\mathbf{A} = \mathbf{S} + \mathbf{N}$ where

$$\mathbf{S} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}\mathbf{P}^{-1} = \begin{pmatrix} 1 & -1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then $\mathbf{A} - \mathbf{S} = \mathbf{N} = \begin{pmatrix} 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Solution:

Note $N^2 = \mathbf{0}$. We then get $e^{\mathbf{D}t} = \begin{pmatrix} e^t\cos(t) & -e^t\sin(t) & 0 & 0 \\ e^t\sin(t) & e^t\cos(t) & 0 & 0 \\ 0 & 0 & e^t\cos(t) & -e^t\sin(t) \\ 0 & 0 & e^t\sin(t) & e^t\cos(t) \end{pmatrix}$ and

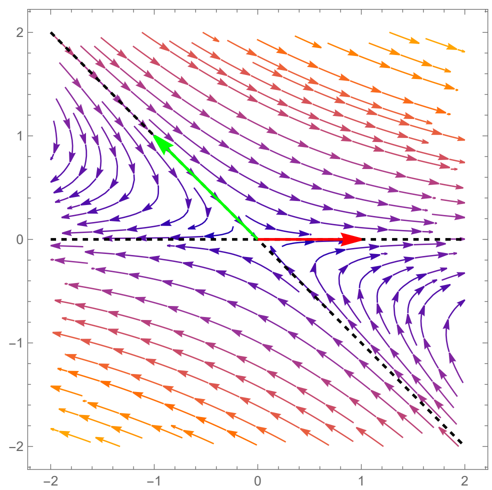
$e^{\mathbf{N}t} = \mathbf{I} + t\mathbf{N} = \begin{pmatrix} 1 & 0 & \frac{t}{2} & -\frac{t}{2} \\ 0 & 1 & \frac{t}{2} & \frac{t}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. All together,

$$\begin{aligned} e^{\mathbf{A}t} &= e^{\mathbf{S}t}e^{\mathbf{N}t} \\ &= \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}e^{\mathbf{N}t} \\ &= \begin{pmatrix} e^t\cos(t) & -e^t\sin(t) & \frac{1}{2}e^t(-t\sin(t)+\sin(t)+t\cos(t)) & -\frac{1}{2}e^t((t-1)\sin(t)+t\cos(t)) \\ e^t\sin(t) & e^t\cos(t) & \frac{1}{2}e^t((t+1)\sin(t)+t\cos(t)) & \frac{1}{2}e^t(t\cos(t)-(t+1)\sin(t)) \\ 0 & 0 & e^t\cos(t) & -e^t\sin(t) \\ 0 & 0 & e^t\sin(t) & e^t\cos(t) \end{pmatrix}. \end{aligned}$$

2. For the following, find the stable unstable and center subspaces for $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. Then sketch the phase portrait in each of these cases.

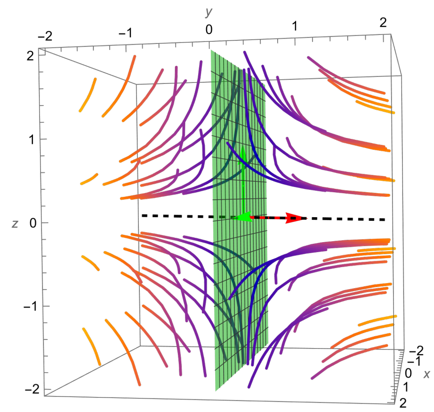
A. $\begin{pmatrix} 2 & 4 \\ 0 & -2 \end{pmatrix}$ B. $\begin{pmatrix} -1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ C. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 2 & 2 \end{pmatrix}$

Solution: **A**: The eigensystem is $\lambda_1 = -2$, $\mathbf{v}_1 = \langle -1, 1 \rangle$ and $\lambda_2 = 2$, $\mathbf{v}_2 = \langle 1, 0 \rangle$. The equilibrium point at the origin is a saddle point. The stable manifold $E^S = \text{span}\{\mathbf{v}_1\}$ and unstable manifold $E^U = \text{span}\{\mathbf{v}_2\}$ (represented below by the dashed lines and colored vectors).

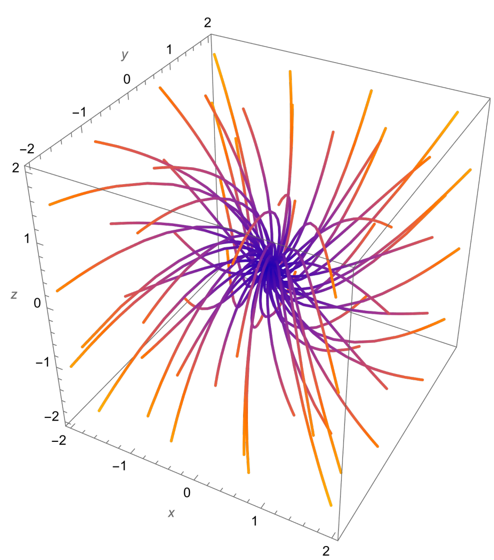


Solution:

B: The eigensystem is $\lambda_1 = 2$, $\mathbf{v}_1 = \langle 2, -1, -1\rangle$ and $\lambda_2 = -1$ (repeated), $\mathbf{v}_2 = \mathbf{e}_1$ and $\mathbf{v}_3 = \mathbf{e}_3$. The equilibrium point at the origin is again saddle point. The stable manifold $E^S = \text{span}\{\mathbf{e}_1, \mathbf{e}_3\}$, is the green plane with its spanning set and unstable manifold $E^U = \text{span}\{\langle 1, 1, 0\rangle\}$ is the dashed line with the spanning vector in red.



C: The eigensystem is $\lambda_1 = 2$, $\mathbf{u}_1 = \mathbf{e}_1$ and $\lambda_{2,3} = 2 \pm \sqrt{2}i$, with $\mathbf{v}_2 = \langle 0, 1/\sqrt{2}, 0\rangle$ and $\mathbf{u}_2 = \mathbf{e}_3$. The equilibrium point at the origin is again hyperbolic point. As the real part of all the eigenvalues are positive, all of $\mathbb{R}^3$ is the unstable manifold.



3. Find the stable, unstable and center subspaces for the systems $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ as they are defined in Problem 1 C and D.

Solution: **C**: The real part of every λ is 2. Hence, all of $\mathbb{R}^4$ is the unstable manifold.

D: same answer. (This was a silly question).

4. Consider the autonomous non-linear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ where

$$\mathbf{f}(x_1, x_2, x_3) = \langle x_1 + x_1x_2^2 + x_1x_3^2, -x_1 + x_2 - x_2x_3 + x_1x_2x_3, x_2 + x_3 - x_1^2 \rangle.$$

(a) Find the rest points of the non-linear system.

Solution: $\mathbf{f}(x_1, x_2, x_3) = (0, 0, 0)$ at $(0, 0, 0)$ or $(0, -1, 1)$.

(b) Find the derivative $D\mathbf{f}(\mathbf{x})$.

Solution:

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 1 + x_2^2 + x_3^2 & 2x_1x_2 & 2x_1x_3 \\ -1 + x_2x_3 & 1 - x_3 + x_1x_3 & -x_2 + x_1x_2 \\ -2x_1 & 1 & 1 \end{pmatrix}$$

(c) Find the linearization of $\mathbf{f}(\mathbf{x})$ at each of the rest points.

Solution: At the origin, $D\mathbf{f}(0,0,0) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ and the linear system is

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

At the $(0,-1,1)$, $D\mathbf{f}(0,0,0) = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and the linear system is

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 + 1 \\ x_3 - 1 \end{pmatrix}$$