Math 345 - Homework 8

1. For the following matrices A:
   (a) Do the eigenvalue decomposition and determine the associated eigenvectors
   (b) Determine the transition matrix P whose column space is the (generalized) eigenspaces of A.
   (c) Compute e^{At}
   (d) Use P to compute (up to sign) Jordon Canonical form of each.

A. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$

(a) $(A - \lambda I) = 0$ yields $\lambda^3(\lambda - 10) = 0$, so $\lambda_1 = 10$ and $\lambda_2 = \lambda_3 = \lambda_4 = 0$.

For λ_1, $(A - \lambda_1 I)v_1 = 0$ can be row reduced to

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -0.25 & 0 \\ 0 & 1 & 0 & -0.50 & 0 \\ 0 & 0 & 1 & -0.75 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

x_4 is a free variable, and letting it equal 4 gives the eigenvector $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

Then $(A - \lambda_2 I)^3 = 0 = A^3$. Note $A^3 = 100A$. Row reduction leads to

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

So x_2, x_3, and x_4 are free variables. The generalized eigenvectors must have the form $x_1 + x_2 + x_3 + x_4 = 0$. Clearly $v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$, and $v_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ are linearly independent and in $\ker A^3$.

(b) Now constructing the transition matrix P from generalized eigenvectors, we have

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{bmatrix}.$$

(c) To find e^At we will use $Pe^{\Lambda t}P^{-1}$ where Λ is a diagonal matrix of eigenvalues from A.

$$\Lambda = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad e^{\Lambda t} = \begin{bmatrix} e^{10t} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \textit{clearly } P^{-1} = \frac{1}{10}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -8 & 2 & 2 \\ 3 & 3 & -7 & 3 \\ 4 & 4 & 4 & -6 \end{bmatrix}.$$

It follows that

$$e^{At} = Pe^{\Lambda t}P^{-1} = \frac{1}{10}\begin{bmatrix} e^{10t}+9 & e^{10t}-1 & e^{10t}-1 & e^{10t}-1 \\ 2e^{10t}-2 & 2e^{10t}+8 & 2e^{10t}-2 & 2e^{10t}-2 \\ 3e^{10t}-3 & 3e^{10t}-3 & 3e^{10t}+7 & 3e^{10t}-3 \\ 4e^{10t}-4 & 4e^{10t}-4 & 4e^{10t}-4 & 4e^{10t}+6 \end{bmatrix}$$

(d) Since $A = PJP^{-1}$ then certainly $P^{-1}AP = J$. Hence

$$\begin{aligned} J &= P^{-1}AP \\ &= \frac{1}{10}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -8 & 2 & 2 \\ 3 & 3 & -7 & 3 \\ 4 & 4 & 4 & -6 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

B. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$

(a) The characteristic polynomial is just $\lambda^4 = 0$. Then $(A - \lambda_i I) = A - 0I = A$. Row reducing to find the kernel of A gives

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Giving the equations $x_1 + x_4 = 0$ and $x_2 - x_3 = 0$. This can give us two eigenvectors $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$. Next, finding generalized eigenvectors from $\ker A^2$ has the row reduction

$$\left[\begin{array}{cccc|c} 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Part of this kernel ($x_2 - x_3 = 0$) is already in $\ker A$, but now x_1 and x_4 are free variables, so e_1 and e_4 could be generalized eigenvectors. But, only one of them can be used since $v_1 \in \operatorname{span}\{e_1, e_4\}$. Letting $v_3 \equiv e_1$. Then the next power $A^3 = 0$, so $\ker A^3 = V$, meaning we just need a $v_4 \notin \operatorname{span}\{v_1, v_2, v_3\}$. It follows that $v_4 = e_2$ satisfies this condition.

(b) The transition matrix is

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

(c) To find $e^A t$ we will use $e^{At} = e^{(S+N)t}$ with $S = PDP^{-1}$ and where D is a diagonal matrix of eigenvalues from A. Since $D = 0_{4,4}$, then $S = 0$. Therefore $N = A - S = A$. It follows that $e^{At} = e^{Nt}$ where $N \equiv A$ is nilpotent of degree 3. Thus

$$\begin{aligned} e^{At} &= I + tA + \frac{t^2}{2}A^2 + 0 + \ldots \\ &= I + t\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ t & \left(1 - \frac{t^2}{2}\right) & \frac{t^2}{2} & t \\ t & -\frac{t^2}{2} & \left(\frac{t^2}{2} + 1\right) & t \\ 0 & -t & t & 1 \end{bmatrix} \end{aligned}$$

(d)

$$
\begin{aligned}
J &= P^{-1}AP \\
&= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

C. $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$

(a) The characteristic polynomial is $\lambda^4 - 8\lambda^3 + 26\lambda^2 - 40\lambda + 24 = 0$, which has roots $2 \pm i\sqrt{2}$ and 2 with multiplicity 2. Let $\lambda_1 = 2 + i\sqrt{2},\ \lambda_2 = 2 - i\sqrt{2}$ and $\lambda_3 = \lambda_4 = 2$.

$$(A - \lambda_1 I)v_1 = 0 \implies \begin{bmatrix} -i\sqrt{2} & 1 & 0 & 0 \\ 0 & -i\sqrt{2} & 0 & 0 \\ 0 & 0 & -i\sqrt{2} & -1 \\ 0 & 0 & 2 & -i\sqrt{2} \end{bmatrix} v_1 = 0 \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{i\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

thus $x_3 = \frac{i\sqrt{2}}{2}x_4$. Letting $x_4 = 1$ then we get $v_1 = \begin{pmatrix} 0 \\ 0 \\ \frac{i\sqrt{2}}{2} \\ 1 \end{pmatrix}$ and conjugate $v_2 = \begin{pmatrix} 0 \\ 0 \\ \frac{-i\sqrt{2}}{2} \\ 1 \end{pmatrix}$.

For λ_3 and λ_4, $(A - 2I)$ row reduces to $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Thus the only free variable is x_1, so we can let $v_3 = e_1$. We then need to use $(A - 2I)^2$ for the last generalized eigenvector. This is a diagonal matrix and it row reduces to $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Thus x_1 and x_2 are free variables, but since $e_1 \equiv v_3$ (already in the span), we can instead use $v_4 = e_2$.

(b) Then putting the real lambdas first so P is invertible, we get the transition matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{i\sqrt{2}}{2} & -\frac{i\sqrt{2}}{2} \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

(c) Since $A = P$ and

$$\begin{aligned} S &= PDP^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{i\sqrt{2}}{2} & -\frac{i\sqrt{2}}{2} \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 + i\sqrt{2} & 0 \\ 0 & 0 & 0 & 2 - i\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{i\sqrt{2}}{2} & \frac{1}{2} \\ 0 & 0 & \frac{i\sqrt{2}}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & & \\ 0 & 2 & & \\ & & 2 & -2 \\ & & 2 & 2 \end{bmatrix} \end{aligned}$$

Then $N = A - S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix}$ which is clearly nilpotent of degree 2. Then $e^{At} = e^{(S+N)t} = e^{St}.e^{Nt}$. By the exponential definition we know $e^{Nt} = I + Nt + 0 + \cdots = \begin{bmatrix} 1 & t & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$. Then reverse-engineering the notes, $e^{St} = \begin{bmatrix} e^{2t} & 0 & & \\ 0 & e^{2t} & & \\ & & e^{2t}\cos(2t) & -e^{2t}\sin(2t) \\ & & e^{2t}\sin(2t) & e^{2t}\cos(2t) \end{bmatrix}$. Lastly $e^{At} = e^{St}.e^{Nt} = e^{Nt}.e^{St}$ which is

$$e^{At} = \begin{bmatrix} e^{2t} & te^{2t} & & \\ 0 & e^{2t} & & \\ & & e^{2t}\cos(2t) & -e^{2t}\sin(2t) \\ & & e^{2t}\sin(2t) & e^{2t}\cos(2t) \end{bmatrix}$$

(d)

$$\begin{aligned} J &= P^{-1}AP \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{i\sqrt{2}}{2} & \frac{1}{2} \\ 0 & 0 & \frac{i\sqrt{2}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{i\sqrt{2}}{2} & -\frac{i\sqrt{2}}{2} \\ 0 & 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & & \\ 0 & 2 & & \\ & & 2 + i\sqrt{2} & 0 \\ & & 0 & 2 - i\sqrt{2} \end{bmatrix} \end{aligned}$$

D. $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

(a) It has the characteristic polynomial $\lambda^4 - 4\lambda^3 + 8\lambda^2 - 8\lambda + 4$ and eigenvalues $\lambda_j = 1 \pm i$. Then row reducing $(A - (1-i)I)$ gives

$$\begin{bmatrix} 1 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Meaning x_3 and x_4 are free variables, with $x_1 = ix_2$. For $x_2 = 1$ we have the eigenvector $\vec{v_1} = \begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix} \implies u_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},\ v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Then we look at $(A - (1-i)I)^2$ for the generalized eigenvectors. Row reducing that gives

$$\begin{bmatrix} 1 & -i & 0 & \frac{1}{2}(i-1) \\ 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $x_3 = ix_4$ and $\frac{1}{2}(i-1)x_4 = ix_2 - x_1$.

Letting $x_4 = 1$ yields $\vec{v_2} = \begin{pmatrix} 0.5 \\ 0.5 \\ i \\ 1 \end{pmatrix} \implies u_2 = \begin{pmatrix} 0.5 \\ 0.5 \\ 0 \\ 1 \end{pmatrix},\ v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

(b)

$$P = \begin{bmatrix} v_1 & u_1 & v_2 & u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c)

(d)

2.

3.

4.