Math 345 – Homework 9 Due Monday, December 5, 2022

1. Classify the equilibrium points of the Lorenz equation

$$\mathbf{x}'(t) = \langle y - x, \mu x - y - xz, xy - z \rangle$$

for $\mu > 0$.

   (a) At what value of the parameter μ do two new equilibrium points "bifurcate" from the equilibrium point at the origin?

   (b) Classify the equilibrium point in the regime when there is only one.

   (c) Classify the equilibrium point at the origin in the regime when there are three.

2. Consider the system

$$\dot{x} = y, \dot{y} = -x + (1 - x^2 - y^2)y.$$

   (a) Let D be any disc $x^2 + y^2 \leq R^2$. Explain why the system satisfies the hypotheses of the Existence and Uniqueness theorem throughout D.

   (b) By substitution, show that $x(t) = \sin t$, $y(t) = \cos t$ is an exact solution of the system.

   (c) Now consider an IVP where (x_0, y_0) is in the unit disk. Without doing any calculations, explain why the solution must satisfy $x^2(t) + y^2(t) < 1$ for all t.

   (d) Use linearization to classify the equilibrium at the origin, if possible.

   (e) We want to show that the solution curve $x^2 + y^2 = 1$ is semi-attracting. Consider the any circle $x^2 + y^2 = R^2$, $R > 0$

      i. Find an outward pointing normal to the circle. (This is a Calc III question.)

      ii. Use the dot product and show that the vector field defined by the system is always inwardly pointing through the boundary of the circle.

   (f) Give a rough sketch of the flow lines for the system.

3. For each autonomous ODE, find the values of r at which bifurcations occur, classify them as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Finally, sketch the bifurcation diagram of fixed points x_e versus r.

   (a) $\dot{x} = 1 + rx + x^2$

   (b) $\dot{x} = rx - \ln(1 + x)$

   (c) $\dot{x} = x + \frac{rx}{1+x^2}$

4. TBD