# Math 345 – Homework 7 Due Monday, October 31, 2022

1. Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

   (a) Show that $\mathbf{A}$ is a nilpotent matrix.

**Solution:** $\mathbf{A}^2 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{A}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

   (b) Compute $e^{\mathbf{A}}$ exactly.

**Solution:**

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{1}{2}\mathbf{A}^2 = \begin{bmatrix} 1 & 1 & 7/2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Compute $e^{\mathbf{A}t}$ when $\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

**Solution:** $e^{\mathbf{A}t} = \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}$

3. Consider the upper triangular matrix $\mathbf{U} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Decompose $\mathbf{U}$ into a diagonal matrix $D$ and a nilpotent matrix $N$.

**Solution:**

$$\mathbf{U} = \mathbf{D} + \mathbf{N} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $\mathbf{N}^2 = \mathbf{0}$.

(b) Use the previous problem to compute $e^{\mathbf{U}}$.

**Solution:** As discussed in class, we really should have shown that $\mathbf{D}$ and $\mathbf{N}$ commute to use this approach. They do

$$\mathbf{DN} = \mathbf{DN} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence,

$$\begin{aligned} e^{\mathbf{A}} &= e^{\mathbf{D}} e^{\mathbf{N}} = e^{\mathbf{D}}(\mathbf{I} + \mathbf{N}) \\ &= \begin{bmatrix} e^2 & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e \end{bmatrix} \left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} e^2 & e^2 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e \end{bmatrix} \end{aligned}$$

(c) Will this technique work for all upper triangular matrices? Justify your answer.

**Solution:** As much discussed after Josh's counterexample, it will not. Must also have commutativity.

4. Prove that if matrices $\mathbf{A}$ and $\mathbf{B}$ are similar matrices, then $\mathbf{A}$ and $\mathbf{B}$ have the same set of eigenvalues.

**Solution:** We can show that they have the same characteristic polynomial. Hence, the same eigenvalues. We have $\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}$. Hence

$$\begin{aligned}
\det(\mathbf{A} - \lambda\mathbf{I}) &= \det(\mathbf{P}^{-1}\mathbf{B}\mathbf{P} - \lambda\mathbf{I}) \\
&= \det(\mathbf{P}^{-1}\mathbf{B}\mathbf{P} - \lambda\mathbf{P}^{-1}\mathbf{P}) \\
&= \det(\mathbf{P}^{-1}(\mathbf{B} - \lambda\mathbf{I})\mathbf{P}) \\
&= \det(\mathbf{P}^{-1})\det(\mathbf{B} - \lambda\mathbf{I})\det(\mathbf{P}) \\
&= \det(\mathbf{B} - \lambda\mathbf{I}).
\end{aligned}$$

5. Compute $e^{\mathbf{A}t}$ where

A. $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 6 & 2 & 0 \end{bmatrix}$ B. $\mathbf{A} = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 2 & 0 \\ -2 & 2 & 2 \end{bmatrix}$ C. $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

**Solution:** This point in the course, we only have Putzer's Algorithm. As these are all $3 \times 3$ matrices, once we order the eigenvalues, the solutions will have the form

$$e^{\mathbf{A}t} = r_1(t)\mathbf{P}_0 + r_2(t)\mathbf{P}_1 + r_3(t)\mathbf{P}_2.$$

**Solution:**

**A**:

The eigenvalue is $\lambda_1 = -2$, $\lambda_2 = 1$, and $\lambda_3 = 3$. Then $\mathbf{P}_0 = \mathbf{I}$,

$$\mathbf{P}_1 = \mathbf{A} - (-2)\mathbf{I} = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 3 & -1 \\ 6 & 2 & 2 \end{bmatrix},$$

and

$$\mathbf{P}_2 = (\mathbf{A} - (-2)\mathbf{I})(\mathbf{A} - (1)\mathbf{I}) = \begin{bmatrix} 8 & 5 & 1 \\ 0 & 0 & 0 \\ 16 & 10 & 8 \end{bmatrix},$$

For the coefficient functions, we solve the recursive IVP. For $r_1'(t) = -2r_1(t)$, $r_1(0) = 1$, we get $r_1(t) = e^{-2t}$. For $r_2'(t) = r_2(t) + r_1(t) = r_2(t) + e^{-2t}$, $r_2(0) = 0$, we get $r_2(t) = \frac{1}{3}e^{-2t}\left(e^{3t} - 1\right)$. Lastly, $r_2'(t) = 3r_3(t) + \frac{1}{3}e^{-2t}\left(e^{3t} - 1\right)$, $r_3(0) = 0$, we get $r_3(t) = \frac{e^{-2t}}{15} - \frac{e^t}{6} + \frac{e^{3t}}{10}$.

$$\begin{aligned} e^{\mathbf{A}t} &= r_1(t)\mathbf{P}_0 + r_2(t)\mathbf{P}_0 + r_3(t)\mathbf{P}_2 \\ &= e^{-2t}\mathbf{I} + \left(\frac{1}{3}e^{-2t}\left(e^{3t} - 1\right)\right)\begin{bmatrix} 3 & 1 & 1 \\ 2 & 3 & -1 \\ 6 & 2 & 2 \end{bmatrix} + \left(\frac{e^{-2t}}{15} - \frac{e^t}{6} + \frac{e^{3t}}{10}\right)\begin{bmatrix} 8 & 5 & 1 \\ 0 & 0 & 0 \\ 16 & 10 & 8 \end{bmatrix} \\ &= \begin{pmatrix} \frac{8e^{-2t}}{15} - \frac{e^t}{3} + \frac{4e^{3t}}{5} & \frac{e^{3t}}{2} - \frac{e^t}{2} & -\frac{4}{15}e^{-2t} + \frac{e^t}{6} + \frac{e^{3t}}{10} \\ \frac{2e^t}{3} - \frac{2}{3}e^{-2t} & e^t & \frac{e^{-2t}}{3} - \frac{e^t}{3} \\ -\frac{14}{15}e^{-2t} - \frac{2e^t}{3} + \frac{8e^{3t}}{5} & e^{3t} - e^t & \frac{7e^{-2t}}{15} + \frac{e^t}{3} + \frac{e^{3t}}{5} \end{pmatrix} \end{aligned}$$

**Solution: B**: Point of this one is that Putzer's works with repeated eigenvalues. Here $\lambda = 2$, repeated twice. (This computation is easier than the last. Procedure exactly the same.)

$$e^{\mathbf{A}t} = \begin{pmatrix} e^{2t} & -2e^{2t}t & 0 \\ 0 & e^{2t} & 0 \\ -2e^{2t}t & e^{2t}\left(2t^2 + 2t\right) & e^{2t} \end{pmatrix}$$

**Solution:** **C**: Point of this one is that Putzer's works with complex eigenvalues. Here $\lambda_1 = 3$, and $\lambda_{2,3} = 1 \pm 2i$. (This computation is a bit much. Procedure exactly the same.)

$$e^{\mathbf{A}t} = \begin{pmatrix} e^t\cos(2t) & e^t\sin(2t) & \frac{e^{3t}}{4} + \frac{1}{4}e^t\sin(2t) - \frac{1}{4}e^t\cos(2t) \\ -e^t\sin(2t) & e^t\cos(2t) & -\frac{e^{3t}}{4} + \frac{1}{4}e^t\sin(2t) + \frac{1}{4}e^t\cos(2t) \\ 0 & 0 & e^{3t} \end{pmatrix}$$

6. Find the solution of $\vec{x}' = \mathbf{A}\vec{x}$, $\vec{x}(0) = \langle 1, -1, 1\rangle$ for each $\mathbf{A}$ in Exercise 5.

**Solution:** For each of these, we just need $\vec{x}(t) = e^{\mathbf{A}t}\vec{x}(0)$.

$$\mathbf{A}\text{: } \vec{x}(t) = \begin{pmatrix} \frac{4e^{-2t}}{15} + \frac{e^t}{3} + \frac{2e^{3t}}{5} \\ -\frac{1}{3}e^{-2t} - \frac{2e^t}{3} \\ -\frac{7}{15}e^{-2t} + \frac{2e^t}{3} + \frac{4e^{3t}}{5} \end{pmatrix}$$

$$\mathbf{B}\text{: } \vec{x}(t) = \begin{pmatrix} e^{2t}(2t+1) \\ -e^{2t} \\ e^{2t}\left(-2t^2 - 4t + 1\right) \end{pmatrix}$$

$$\mathbf{C}\text{: } \vec{x}(t) = \begin{pmatrix} \frac{1}{4}e^t\left(e^{2t} - 3\sin(2t) + 3\cos(2t)\right) \\ -\frac{1}{4}e^t\left(e^{2t} + 3\sin(2t) + 3\cos(2t)\right) \\ e^{3t} \end{pmatrix}$$

7. Verify the Cayley-Hamilton theorem for Exercise 5B.

**Solution:** Here $\mathbf{A} = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 2 & 0 \\ -2 & 2 & 2 \end{bmatrix}$. The characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = -\lambda^3 + 5\lambda^2 - 11\lambda + 15$. Want to show $-\mathbf{A}^3 + 5\mathbf{A}^2 - 11\mathbf{A} + 15\mathbf{I} = 0$. Note

$$\mathbf{A}^2 = \begin{pmatrix} -3 & 4 & 4 \\ -4 & -3 & -2 \\ 0 & 0 & 9 \end{pmatrix}, \mathbf{A}^3 = \begin{pmatrix} -11 & -2 & 9 \\ 2 & -11 & -10 \\ 0 & 0 & 27 \end{pmatrix}$$

$$\begin{aligned} & -\mathbf{A}^3 + 5\mathbf{A}^2 - 11\mathbf{A} + 15\mathbf{I} \\ = - & \begin{pmatrix} -11 & -2 & 9 \\ 2 & -11 & -10 \\ 0 & 0 & 27 \end{pmatrix} + 5 \begin{pmatrix} -3 & 4 & 4 \\ -4 & -3 & -2 \\ 0 & 0 & 9 \end{pmatrix} - 11 \begin{pmatrix} 2 & -2 & 0 \\ 0 & 2 & 0 \\ -2 & 2 & 2 \end{pmatrix} + 15\mathbf{I} \\ & = \mathbf{0} \end{aligned}$$

8. We are going to revisit 5A and do it the traditional engineering ODE way.

   (a) Order and label your eigenvalues as you did in Problem 5. For each eigenvalue $\lambda_i$, find an associated eigenvector $\mathbf{v}_i$.

**Solution:** The eigensystem is $\lambda_1 = -2$, $\lambda_2 = 1$, and $\lambda_3 = 3$, with $\mathbf{v}_1 = \langle -4, 5, 7\rangle$, $\mathbf{v}_2 = \langle 1, -2, 2\rangle$, and $\mathbf{v}_3 = \langle 1, 0, 2\rangle$, respectively.

   (b) Construct a fundamental matrix $\mathbf{\Phi}(t) = [e^{\lambda_1}\mathbf{v}_1, e^{\lambda_2}\mathbf{v}_2, e^{\lambda_3}\mathbf{v}_3]$ and prove that $\mathbf{\Phi}(t)$ is not $e^{t\mathbf{A}}$. Note: $\mathbf{\Phi}(t)\vec{c}$ is the form of the solution in lower-level courses.

**Solution:**

$$\mathbf{\Phi}(t) = \begin{pmatrix} -4e^{-2t} & e^{t} & e^{3t} \\ 5e^{-2t} & -2e^{t} & 0 \\ 7e^{-2t} & 2e^{t} & 2e^{3t} \end{pmatrix}.$$

Note

$$\mathbf{\Phi}(0) = \begin{pmatrix} -4 & 1 & 1 \\ 5 & -2 & 0 \\ 7 & 2 & 2 \end{pmatrix} \neq \mathbf{I}.$$

Hence $\mathbf{\Phi}(t)$ is not $e^{t\mathbf{A}}$.

   (c) Diagonalize $A$. That is, use the eigenvalues to construct a diagonal matrix $\mathbf{D}$ and the eigenvectors to construct a change of basis matrix $\mathbf{S}$ such that $\mathbf{D} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$. (Show this these matrices do the job intended.)

**Solution:** Here $\mathbf{S} = \mathbf{\Phi}(0) = \begin{pmatrix} -4 & 1 & 1 \\ 5 & -2 & 0 \\ 7 & 2 & 2 \end{pmatrix}$ and $\mathbf{S}^{-1} = \begin{pmatrix} -\frac{2}{15} & 0 & \frac{1}{15} \\ -\frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \\ \frac{4}{5} & \frac{1}{2} & \frac{1}{10} \end{pmatrix}$. Note

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(d) Using your diagonalization, compute $e^{t\mathbf{A}}$ using $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$. That is, construct $\mathbf{S}e^{t\mathbf{D}}\mathbf{S}^{-1}$ and show that this matrix is $\mathbf{I}$ when $t = 0$.

**Solution:** Using $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$ and the fact that $e^{t\mathbf{A}} = e^{t\mathbf{S}\mathbf{D}\mathbf{S}^{-1}} = \mathbf{S}e^{t\mathbf{D}}\mathbf{S}^{-1}$.

$$\begin{aligned}
e^{t\mathbf{A}} &= \mathbf{S}e^{t\mathbf{D}}\mathbf{S}^{-1} \\
&= \mathbf{S} \begin{pmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix} \mathbf{S}^{-1} \\
&= \begin{pmatrix} -4e^{-2t} & e^{t} & e^{3t} \\ 5e^{-2t} & -2e^{t} & 0 \\ 7e^{-2t} & 2e^{t} & 2e^{3t} \end{pmatrix} \mathbf{S}^{-1} \\
&= \begin{pmatrix} \frac{1}{15}e^{-2t}\left(-5e^{3t} + 12e^{5t} + 8\right) & \frac{1}{2}e^{t}\left(e^{2t} - 1\right) & \frac{1}{30}e^{-2t}\left(5e^{3t} + 3e^{5t} - 8\right) \\ \frac{2}{3}e^{-2t}\left(e^{3t} - 1\right) & e^{t} & -\frac{1}{3}e^{-2t}\left(e^{3t} - 1\right) \\ \frac{2}{15}e^{-2t}\left(-5e^{3t} + 12e^{5t} - 7\right) & e^{t}\left(e^{2t} - 1\right) & \frac{1}{15}e^{-2t}\left(5e^{3t} + 3e^{5t} + 7\right) \end{pmatrix}
\end{aligned}$$

Checking the final result, evaluating the righthand side at $t = 0$,

$$\begin{pmatrix} \frac{1}{15}\left(-5 + 12 + 8\right) & \frac{1}{2}\left(1 - 1\right) & \frac{1}{30}\left(5 + 3 - 8\right) \\ \frac{2}{3}\left(1 - 1\right) & 1 & -\frac{1}{3}\left(1 - 1\right) \\ \frac{2}{15}\left(-5 + 12 - 7\right) & e^{t}\left(1 - 1\right) & \frac{1}{15}\left(5 + 3 + 7\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

9. Show that if the real part of each eigenvalue of $\mathbf{A}$ is negative, then every solution of $\vec{x}' = \mathbf{A}\vec{x}$ satisfies $\lim_{t\to\infty} \vec{x}(t) = 0$.

**Solution:** Let $\mathbf{A}$ be $n \times n$ and order the eigenvalues in ascending order $\lambda_1 < \lambda_2 < \ldots < \lambda_n < 0$. The solution to the system is of the form

$$e^{t\mathbf{A}} = r_1(t)\mathbf{I} + r_2(t)\mathbf{P}_1 + \ldots + r_n(t)\mathbf{P}_{n-1}.$$

Consider the sequence of coefficient functions $r_i(t)$. Here $r_1(t) = e^{\lambda_1 t}$. Note $r_1(t) \to 0$ as $t \to \infty$. Here $r_2'(t) = \lambda_2 r_2(t) + r_1(t)$ and $r_2(t) = \dfrac{e^{\lambda_1 t}}{\lambda_1 - \lambda_2} + Ce^{\lambda_2 t}$. As both $\lambda < 0$, this coefficient limits to zero as $t \to \infty$. Note that this pattern continues, the $k$th coefficient will be a linear combination of the first $e^{\lambda_i t}$ (with potential polynomial multiplication in $t$ if repeated eigenvalues). Hence, $r_i(t) \to 0$ for all $t \to \infty$ and $e^{t\mathbf{A}} \to \mathbf{0}$).

10. Find the solution of the initial value problem.

$$\vec{x}' = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 0 & 0 \\ 2 & -2 & 3 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}, \ \vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

**Solution:** Need to solve the associated homogeneous part first,

$$\vec{x}' = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 0 & 0 \\ 2 & -2 & 3 \end{bmatrix} \vec{x}.$$

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 3$, $\lambda_2 = 2$, and $\lambda_3 = 1$. Computing

$$x_c(t) = e^{\mathbf{A}t} = \begin{pmatrix} 2e^{2t} - e^t & e^t - e^{2t} & 0 \\ 2e^{2t} - 2e^t & 2e^t - e^{2t} & 0 \\ e^{3t} - e^t & e^t - e^{3t} & e^{3t} \end{pmatrix}.$$

Using variation of parameters, $x_p(t) = e^{\mathbf{A}t} \int_0^t e^{-\mathbf{A}s} \mathbf{f}(s)\, ds$ where $\mathbf{f}(s) = \langle 1, s, s^2 \rangle$. Here $e^{-\mathbf{A}s}\mathbf{f}(s) = \begin{pmatrix} \left(e^{-s} - e^{-2s}\right) s + 2e^{-2s} - e^{-s} \\ \left(2e^{-s} - e^{-2s}\right) s + 2e^{-2s} - 2e^{-s} \\ e^{-3s}s^2 + \left(e^{-s} - e^{-3s}\right) s + e^{-3s} - e^{-s} \end{pmatrix}$ and

$$\int_0^t e^{-\mathbf{A}s}\mathbf{f}(s)\, ds = \begin{pmatrix} \frac{1}{4} e^{-2t} \left(-4e^t t + 2t + 3e^{2t} - 3\right) \\ \frac{1}{4} e^{-2t} \left(-8e^t t + 2t + 3e^{2t} - 3\right) \\ \frac{1}{27} \left(e^{-3t} \left(-3t \left(3t + 9e^{2t} - 1\right) - 8\right) + 8\right) \end{pmatrix}.$$

**Solution:**

Then the particular solution is

$$x_p(t) = e^{\mathbf{A}t} \int_0^t e^{-\mathbf{A}s}\mathbf{f}(s)\,ds = \begin{pmatrix} \frac{1}{4}\left(-2t + 3e^{2t} - 3\right) \\ \frac{3}{4}\left(-2t + e^{2t} - 1\right) \\ \frac{1}{27}\left(-9t^2 - 24t + 8e^{3t} - 8\right) \end{pmatrix}.$$

Using the these parts and the initial condition, the solution to the IVP is

$$\begin{aligned} x(t) &= x_c(t)x_0 + x_p(t) \\ &= \begin{pmatrix} 2e^{2t} - e^t & e^t - e^{2t} & 0 \\ 2e^{2t} - 2e^t & 2e^t - e^{2t} & 0 \\ e^{3t} - e^t & e^t - e^{3t} & e^{3t} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{4}\left(-2t + 3e^{2t} - 3\right) \\ \frac{3}{4}\left(-2t + e^{2t} - 1\right) \\ \frac{1}{27}\left(-9t^2 - 24t + 8e^{3t} - 8\right) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4}\left(e^t\left(11e^t - 4\right) - 2t - 3\right) \\ \frac{1}{4}\left(e^t\left(11e^t - 8\right) - 6t - 3\right) \\ \frac{1}{27}\left(-3t(3t+8) - 27e^t + 89e^{3t} - 8\right) \end{pmatrix}. \end{aligned}$$