Math 345 – Homework 3 **Due Friday, September 23, 2022**

1. Rewrite the following differential equations as equivalent integral equations.

   (a) $x'(t) = \sin t \cos 3t + x^6(t)$, $x(0) = 4$

   (b) $x''(t) = t^4 \cos 3x(t) + x^6(t)$, $x(1) = 4$, $x'(1) = 3$

2. Show that $x''(t) = f(t, x(t))$, $x(t_0) = x_0$, $x'(t_0) = x_1$ is equivalent to $x(t) = x_0 + x_1(t - t_0) + \int_{t_0}^{t} (t-s) f(s, x(s))\, ds$. (Hint: Consider changing the order of integration of you resultant double integral. (Also, for the engineers, you may have seen this formula if you have worked with Green's functions.))

3. Consider the autonomous system

$$dy/dt = y(y-1)(y-2), \;\; y_0 \geq 0.$$

   (a) determine the critical (equilibrium) points, and classify each one as asymptotically stable, unstable or semistable

   (b) sketch the phase diagram

   (c) use the phase diagram to give a rough sketch of the integral curves

4. Let $y_1(t) := t^2$ and $y_2(t) := t^{-1}$.

   (a) Show that each y_i solves the differential equation $t^2 y'' - 2y = 0$, $t > 0$.

   (b) Show that $\{y_1, y_2\}$ forms a fundamental set for the ODE.

   (c) Verify the Principle of Superposition. That is, $y(t) := C_1 y_1 + C_2 y_2$ is the general solution to the ODE.

5. Let $y_1(t) := 1$ and $y_2(t) := \sqrt{t}$.

   (a) Show that each y_i solves the differential equation $yy'' + (y')^2 = 0$, $t > 0$.

   (b) Show that y_1 and y_2 are linearly independent.

   (c) Let $y_3 = \sqrt{2t+3}$. Show that y_3 is another solution to the ODE.

(d) Show that y_3 can not be written an a linear combination of y_1 and y_2. Why does this result not violate the Principle of Superposition?

(e) Solve the ODE. Hint: Consider the substitution $u = yy'$.

6. Verify that $\{e^t, e^{-t}, e^{-2t}\}$ form a fundamental set of solutions to the ODE $y''' + 2y'' - y' - 2y = 0$. That is, (i) show each solve the ODE and (ii) the functions are linearly independent.

7. Let y_1 and y_2 be solutions of $y'' + p(t)y' + q(t)y = 0$, p and q continuous on an open interval I.

   (a) Prove that if y_1 and y_2 are zero at the same point in I, then they cannot be a fundamental set of solutions on I.

   (b) Prove that if y_1 and y_2 have and extrema at the same point in I, then they cannot be a fundamental set of solutions on I.

8. Let the linear differential operator L be defined by

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y,$$

   where $a_i \in \mathbb{R}$.

   (a) Find $L[t^n]$.

   (b) Find $L[e^{rt}]$.

   (c) Determine four solutions of the equation $y^{(4)} - 5y'' + 4y = 0$. Do these four solutions form a fundamental set of solutions? Why?