Math 345 – Homework 3 **Due Friday, September 23, 2022**

1. Rewrite the following differential equations as equivalent integral equations.

(a) $x'(t) = \sin t \cos 3t + x^6(t)$, $x(0) = 4$

Solution:

$$x(t) = \int_0^t (\sin s \cos 3s + x^6(s))\, ds + 4$$

(b) $x''(t) = t^4 \cos 3x(t) + x^6(t)$, $x(1) = 4$, $x'(1) = 3$

Solution:

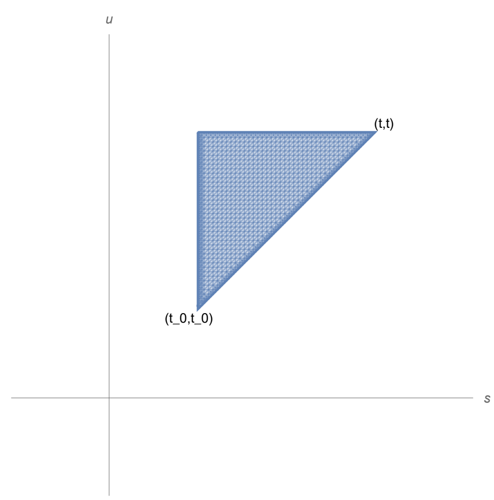
$$\begin{aligned} x'(t) &= \int_1^t (s^4 \cos 3x(s) + x^6(s))\, ds + 3 \\ x(t) &= \int_1^t \int_1^u (s^4 \cos 3x(s) + x^6(s))\, ds du + \int_1^t 3ds + 4 \\ &= \int_1^t \int_1^u (s^4 \cos 3x(s) + x^6(s))\, ds du + 3t + 1 \end{aligned}$$

2. Show that $x''(t) = f(t, x(t))$, $x(t_0) = x_0$, $x'(t_0) = x_1$ is equivalent to $x(t) = x_0 + x_1(t - t_0) + \int_{t_0}^{t} (t-s) f(s, x(s))\, ds$. (Hint: Consider changing the order of integration of you resultant double integral. (Also, for the engineers, you may have seen this formula if you have worked with Green's functions.))

Solution: By the same technique in Problem 1(b),

$$\begin{aligned} x(t) &= \int_{t_0}^{t} \int_{t_0}^{u} f(s, x(s))\, ds du + \int_{t_0}^{t} x_1 ds + x_0 \\ &= \int_{t_0}^{t} \int_{t_0}^{u} f(s, x(s))\, ds du + x_1(t - t_0) + x_0 \end{aligned}$$

The region of integration describes a triangle in the su-plane. Here $t_0 \le u \le t$ and for fixed u, s ranges from t_0 to u.



Switching the order of integration,

$$\int_{t_0}^{t} \int_{t_0}^{u} f(s, x(s))\, ds du = \int_{t_0}^{t} \int_{s}^{t} f(s, x(s))\, du ds = \int_{t_0}^{t} (t - s) f(s, x(s))\, ds$$

and

$$x(t) = \int_{t_0}^{t} (t - s) f(s, x(s))\, ds + x_1(t - t_0) + x_0.$$

3. Consider the autonomous system

$$dy/dt = y(y-1)(y-2), \; y_0 \geq 0.$$

(a) determine the critical (equilibrium) points, and classify each one as asymptotically stable, unstable or semistable

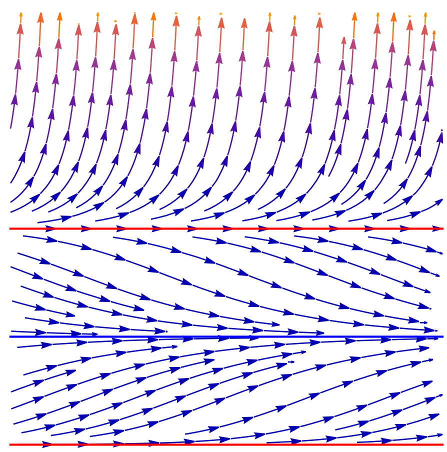
Solution: Here the critical points are $y = 0$, $y = 1$, and $y = 2$. Note that $y' > 0$ on $(0, 1)$ and $(2, \infty)$. So $y = 0$ and $y = 2$ are unstable equilibria and $y = 1$ is stable.

(b) sketch the phase diagram

Solution:

(c) use the phase diagram to give a rough sketch of the integral curves

Solution:



4. Let $y_1(t) := t^2$ and $y_2(t) := t^{-1}$.

(a) Show that each y_i solves the differential equation $t^2y'' - 2y = 0$, $t > 0$.

Solution: $y_1(t) := t^2$, $y_1'(t) := 2t$, and $y_1''(t) := 2$,

$$t^2(2) - 2(t^2) = 0.$$

$y_2(t) := 1/t$, $y_2(t) := -1/t^2$, and $y_2(t) := 2/t^3$,

$$t^2(2/t^3) - 2(1/t) = 0.$$

(b) Show that $\{y_1, y_2\}$ forms a fundamental set for the ODE.

Solution: We already have shown they solve the ODE. To show they are linearly independent,

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^2 & 1/t \\ 2t & -1/t^2 \end{vmatrix} = -1 - 2 = -3.$$

(c) Verify the Principle of Superposition. That is, $y(t) := C_1y_1 + C_2y_2$ is the general solution to the ODE.

Solution:

$$\begin{aligned} t^2(C_1y_1 + C_2y_2)'' - 2(C_1y_1 + C_2y_2) &= t^2(C_1y_1'' + C_2y_2'') - 2C_1y_1 - 2C_2y_2 \\ &= C_1(t^2y_1'' - 2y_1)) + C_2(t^2y_2'' - 2y_2)) \\ &= C_1(0) + C_2(0) && \text{(by (a))} \end{aligned}$$

5. Let $y_1(t) := 1$ and $y_2(t) := \sqrt{t}$.

(a) Show that each y_i solves the differential equation $yy'' + (y')^2 = 0$, $t > 0$.

Solution: $y_1(t) := 1$, $y_1'(t) := 0$, and $y_1''(t) := 0$,

$$y_1 y_1'' + (y_1')^2 = 1(0) + (0)^2 = 0.$$

$y_2(t) := \sqrt{t}$, $y_2(t) := 1/2t^{1/2}$, and $y_2(t) := -1/4t^{3/2}$,

$$y_2 y_2'' + (y_2')^2 = \sqrt{t}(-1/4t^{3/2}) + (1/2t^{1/2})^2 = -1/4t + 1/4t = 0.$$

(b) Show that y_1 and y_2 are linearly independent.

Solution:

$$W(y_1, y_2) = \begin{vmatrix} 1 & \sqrt{t} \\ 0 & -1/2\sqrt{t} \end{vmatrix} = -1/2\sqrt{t}.$$

Note W is non-zero for all $t > 0$.

(c) Let $y_3 = \sqrt{2t+3}$. Show that y_3 is another solution to the ODE.

Solution: $y_3(t) := \sqrt{2t+3}$, $y_3'(t) := 1/\sqrt{2t+3}$, and $y_1''(t) := -1/(2t+3)^{3/2}$,

$$y_3 y_3'' + (y_3')^2 = \sqrt{2t+3}(-1/(2t+3)^{3/2}) - (1/\sqrt{2t+3})^2 = 0.$$

(d) Show that y_3 can not be written an a linear combination of y_1 and y_2. Why does this result not violate the Principle of Superposition?

Solution: Assume $y_3 = C_1 y_1 + C_2 y_2$. Then this expression must hold for all $t > 0$. At $t - 1$, we have $\sqrt{1+3} = C_1 + C_2\sqrt{1}$. Or $2 = C_1 + C_2$. At $t = 4$, $\sqrt{7} = C_1 + 2C_2$. This yields $C_1 = \sqrt{7}$ and $C_2 = 2 - \sqrt{7}$ and the claim that $y_3(t) = \sqrt{7} + (2-\sqrt{7})\sqrt{t}$. But this is clearly not true. For example, if $t = 6$, $y_3(6) = 3$, but $\sqrt{7} + (2-\sqrt{7})\sqrt{3} \neq 3$. Hence, y_3 can not be written as a linear combination of y_1 and y_2.

The Principle of Superposition is a property guaranteed to linear differential operator. Hence linear combinations of solutions are always solutions. Here, the operator (and hence differential equation) is not linear. We can show this by proving the operator is not linear.

$$\begin{aligned}
&(C_1y_1 + C_2y_2)(C_1y_1 + C_2y_2)'' + ((C_1y_1 + C_2y_2)')^2 \\
&= (C_1y_1 + C_2y_2)(C_1y_1'' + C_2y_2'') + (C_1y_1' + C_2y_2')(C_1y_1' + C_2y_2') \\
&= C_1^2y_1y_1'' + C_1C_2y_1''y_2 + C_1C_2y_1y_2'' + C_2^2y_2y_2'' + C_1^2(y_1')^2 + 2C_1C_2y_1'y_2' + C_2^2(y_2')^2 \\
&= C_1^2(y_1y_1'' + (y_1')^2) + +C_2^2(y_2y_2'' + (y_2')^2) + C_1C_2y_1''y_2 + C_1C_2y_1y_2'' + 2C_1C_2y_1'y_2' \\
&\neq C_1^2(y_1y_1'' + (y_1')^2) + +C_2^2(y_2y_2'' + (y_2')^2)
\end{aligned}$$

(e) Solve the ODE. Hint: Consider the substitution $u = yy'$.

Solution: Note $u' = yy'' + y'y'$. Hence the ODE can be written $u' = 0$ and $u = C$ yields $yy' = C$. Solving this separable equation, we get $y^2 = Ct + D$. (Note that specific choices of C and D result in y's above.

6. Verify that $\{e^t, e^{-t}, e^{-2t}\}$ form a fundamental set of solutions to the ODE $y''' + 2y'' - y' - 2y = 0$. That is, (i) show each solve the ODE and (ii) the functions are linearly independent.

Solution:

$$y_1 = e^t : \ (e^t)''' + 2(e^t)'' - (e^t)' - 2(e^t) = e^t + 2e^t - e^t - 2e^t = 0.$$

$$y_2 = e^{-t} : \ (e^{-t})''' + 2(e^{-t})'' - (e^{-t})' - 2(e^{-t}) = -e^{-t} + 2e^{-t} - (-e^{-t}) - 2e^{-t} = 0.$$

$$y_3 = e^{-2t} : \ (e^{-2t})''' + 2(e^{-2t})'' - (e^{-2t})' - 2(e^{-2t}) = -8e^{-2t} + 2(4e^{-2t}) - (-2e^{-2t}) - 2e^{-2t} = 0.$$

To show they are linearly independent,

$$\begin{aligned}
W(y_1, y_2, y_3) &= \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \\
&= \begin{vmatrix} e^t & e^{-t} & e^{-2t} \\ e^t & -e^{-t} & -2e^{-2t} \\ e^t & e^{-t} & 4e^{-2t} \end{vmatrix} \\
&= e^t \begin{vmatrix} -e^{-t} & -2e^{-2t} \\ e^{-t} & 4e^{-2t} \end{vmatrix} - e^{-t} \begin{vmatrix} e^t & -2e^{-2t} \\ e^t & 4e^{-2t} \end{vmatrix} + e^{-2t} \begin{vmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{vmatrix} \\
&= e^t(-4e^{-3t} + 2e^{-3t}) - e^{-t}(4e^{-t} + 2e^{-t}) + e^{-2t}(2) \\
&= -6e^{-2t}.
\end{aligned}$$

7. Let y_1 and y_2 be solutions of $y'' + p(t)y' + q(t)y = 0$, p and q continuous on an open interval I.

(a) Prove that if y_1 and y_2 are zero at the same point in I, then they cannot be a fundamental set of solutions on I.

Solution: If y_1 and y_2 form a fundamental set on I, then the Wronskian $W(y_1, y_2)(t) = y_1y_2' - y_1'y_2$ is non-zero on all of I. However, by Abel's Theorem,

$$W(y_1, y_2)(t) = C\exp\left(-\int p(t)\,dt\right).$$

So if $y_1(a) = y_2(a) = 0$ at any point in I, then Abel's theorem states that $C = 0$. Hence, y_1 and y_2 are linearly dependent at every point in I.

(b) Prove that if y_1 and y_2 have and extrema at the same point in I, then they cannot be a fundamental set of solutions on I.

Solution: This is really the same question using $y_1'(a) = y_2'(a) = 0$ at any point in I.

8. Let the linear differential operator L be defined by

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y,$$

where $a_i \in \mathbb{R}$.

(a) Find $L[t^n]$.

Solution: $L[t^n] = \sum_{k=0}^{n} a_k \left(\frac{n!}{k!}\right) t^k$

(b) Find $L[e^{rt}]$.

Solution: $L[e^{rt}] = (a_0 r^n + a_1 r^{n-1} + \ldots + a_{n-1} r + a_n) e^{rt}$

(c) Determine four solutions of the equation $y^{(4)} - 5y'' + 4y = 0$. Do these four solutions form a fundamental set of solutions? Why?

Solution: Using (b), $(r^4 - 5r^2 + 4)e^{rt} = 0$. Any solution would require r be a root of $r^4 - 5r^2 + 4$. Since

$$r^4 - 5r^2 + 4 = (r^2 - 4)(r^2 - 1) = (r + 2)(r - 2)(r + 1)(r - 1)/$$

we have the four solutions

$$e^{-2t}, e^{2t}, e^{-t}, \text{ and } e^t.$$

They are linearly independent, hence a fundamental set over $\mathbb{R}$. Compute the Wronskian:

$$\begin{vmatrix} e^{-2t} & e^{2t} & e^{-t} & e^t \\ -2e^{-2t} & 2e^{2t} & -e^{-t} & e^t \\ 4e^{-2t} & 4e^{2t} & -e^{-t} & e^t \\ -8e^{-2t} & 8e^{2t} & e^{-t} & e^t \end{vmatrix} = 72$$

(According to the amazing Mathematica command

"Wronskian[Exp[-2 t], Exp[2 t], Exp[-t], Exp[t], t]".)

9. (Problem 5 original version) Let $y_1(t) := 1$ and $y_2(t) := \sqrt{t}$.

(a) Show that each y_i solves the differential equation $yy'' + (y')^2 = 0$, $t > 0$.

Solution: $y_1(t) := 1$, $y_1'(t) := 0$, and $y_1''(t) := 0$,

$$y_1 y_1'' + (y_1')^2 = 1(0) + (0)^2 = 0.$$

$y_2(t) := \sqrt{t}$, $y_2(t) := 1/2t^{1/2}$, and $y_2(t) := -1/4t^{3/2}$,

$$y_2 y_2'' + (y_2')^2 = \sqrt{t}(-1/4t^{3/2}) + (1/2t^{1/2})^2 = -1/4t + 1/4t = 0.$$

(b) Show that $y(t) := C_1 y_1 + C_2 y_2$ is not, in general, a solution to the ODE.

Solution:

$$\begin{aligned}
&(C_1y_1 + C_2y_2)(C_1y_1 + C_2y_2)'' + ((C_1y_1 + C_2y_2)')^2 \\
&= (C_1y_1 + C_2y_2)(C_1y_1'' + C_2y_2'') + (C_1y_1' + C_2y_2')(C_1y_1' + C_2y_2') \\
&= C_1^2 y_1y_1'' + C_1C_2y_1''y_2 + C_1C_2y_1y_2'' + C_2^2y_2y_2'' + C_1^2(y_1')^2 + 2C_1C_2y_1'y_2' + C_2^2(y_2')^2 \\
&= C_1^2(y_1y_1'' + (y_1')^2) + +C_2^2(y_2y_2'' + (y_2')^2) + C_1C_2y_1''y_2 + C_1C_2y_1y_2'' + 2C_1C_2y_1'y_2' \\
&= C_1^2(0) + +C_2^2(0) + C_1C_2(0)y_2 + C_1C_2(1)(-1/4t^{3/2}) + 2C_1C_2(0)y_2' \quad \text{(by (a))} \\
&= -C_1C_2/4t^{3/2} \\
&\neq 0 \\
&\text{(for arbitrary constants)}
\end{aligned}$$

(c) Why does your last result not violate the Principle of Superposition.

Solution: The computation above shows that the differential operator associated with this question is non-linear. The Principle of Superposition is only guaranteed when the differential operator is a linear operator.