Math 345 – Homework 6 **Due Wednesday, October 19, 2022**

1. The equations of motions for two coupled pendulums are

$$\ddot{\theta}_1 + k(\theta_1 - \theta_2) + \omega_1^2\theta_1 = 0$$
$$\ddot{\theta}_2 + k(\theta_2 - \theta_1) + \omega_2^2\theta_2 = 0$$

where θ_i is the angle of the pendulums shaft from equilibrium, k the spring constant of the coupler, and ω_i is the natural frequency of the pendulum when uncoupled. Convert the system of coupled ODE to a first-order system.

Solution: Let $y_1 = \theta_1$, $y_1' = \dot{\theta}_1 = y_2$, and

$$y_2' = \ddot{\theta}_1 = -k(\theta_1 - \theta_2) - \omega_1^2\theta_1.$$

For θ_2, let $y_3 = \theta_2$. Then $y_3' = \dot{\theta}_2 = y_4$, and

$$y_4' = \ddot{\theta}_1 = -k(\theta_2 - \theta_1) - \omega_2^2\theta_2.$$

This yields the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= (\omega_1^2 - k)y_1 + ky_3 \\ y_3' &= y_4 \\ y_4' &= ky_1 + (\omega_2^2 - k)y_3. \end{aligned}$$

2. Consider the linear system

$$
\begin{aligned}
y_1' &= y_2 + 4y_3 \\
y_2' &= -y_1 - 2y_3 \\
y_3' &= y_3
\end{aligned}
$$

(a) Convert the linear system to the equivalent matrix equation.

Solution:

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} \text{ where } \mathbf{A} = \begin{bmatrix} 0 & 1 & 4 \\ -1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

(b) Show that a fundamental matrix exists for the given system via Liouville's formula.

Solution: The constant matrix $\mathbf{A}$ is continuous for all t. We know we can construct a solution matrix $\mathbf{X}(t)$ at a point $t = t_0$ such that $\mathbf{X}(t_0) = \mathbf{I}$. By Liouville's,

$$
\begin{aligned}
\det \mathbf{X}(t) &= \det \mathbf{X}(t_0) \exp\left[\int_{t_0}^{t} \operatorname{tr}\mathbf{A}(s)\, ds\right] \\
&= \exp\left[\int_{t_0}^{t} 1\, ds\right] \\
&= e^{t-t_0}.
\end{aligned}
$$

Thus $\det \mathbf{X}(t) \neq 0$ for all t and t_0. Thus, a fundamental matrix exists.

(c) Show that

$$\Phi(t) := \begin{bmatrix} \sin t & \cos t & e^t \\ \cos t & -\sin t & -3e^t \\ 0 & 0 & e^t \end{bmatrix}$$

is a fundamental matrix for the system.

Solution: If $\Phi(t)$ is a fundamental matrix, then it satisfies the matrix DE $\Phi'(t) = \mathbf{A}\Phi(t)$. Note

$$\Phi'(t) := \begin{bmatrix} \cos t & -\sin t & e^t \\ -\sin t & -\cos t & -3e^t \\ 0 & 0 & e^t \end{bmatrix}$$

On the other side,

$$\mathbf{A}\Phi(t) = \begin{bmatrix} 0 & 1 & 4 \\ -1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin t & \cos t & e^t \\ \cos t & -\sin t & -3e^t \\ 0 & 0 & e^t \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t & e^t \\ -\sin t & -\cos t & -3e^t \\ 0 & 0 & e^t \end{bmatrix}.$$

To show the columns are linearly independent (expanding across the third row),

$$\det \Phi(t) = e^t \begin{vmatrix} \cos t & -\sin t \\ -\sin t & -\cos t \end{vmatrix} = -e^t.$$

(d) Compute $\det \Phi(t)$ and show that the conclusion of Liouville's theorem holds.

Solution: Note $\det \Phi(t_0) = -e^{t_0}$ and $\mathrm{tr}\mathbf{A} = 1$. Then

$$\begin{aligned} \det \Phi(t_0)\exp\left[\int_{t_0}^{t} \mathrm{tr}\mathbf{A}\, ds\right] &= (-e^{t_0})\exp\left[\int_{t_0}^{t} 1\, ds\right] \\ &= -e^{t_0}e^{t-t_0} \\ &= -e^t \\ &= \det \Phi(t). \end{aligned}$$

3. Consider $y'' - 3y' + 2y = 0$.

(a) Find a fundamental set for the given scalar equation.

Solution: Here $L = D^2 - 3D + 2 = (D-1)(D-2)$ and the fundamental set is $\{e^t, e^{2t}\}$.

(b) Convert the scalar equation to an equivalent first order system.

Solution: Let $y_1 = y$ and $y_2 = y'$. This yields the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -2y_1 + 3y_2, \end{aligned}$$

or $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$.

(c) Show that the matrix associated with the Wronskian of your fundamental set forms a fundamental matrix for the linear system.

Solution: Consider $\mathbf{X}(t) = \begin{bmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}$. Need to show $\mathbf{X}'(t) = \mathbf{A}\mathbf{X}(t)$. Left hand side, $\mathbf{X}'(t) = \begin{bmatrix} e^t & 2e^{2t} \\ e^t & 4e^{2t} \end{bmatrix}$. Right hand side,

$$\mathbf{A}\mathbf{X}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix} = \begin{bmatrix} e^t & 2e^{2t} \\ -2e^t + 3e^t & -2e^{2t} + 3(2e^{2t}) \end{bmatrix} = \mathbf{X}'(t).$$

Lastly,

$$\det \mathbf{X}(t) = \begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix} = e^{3t} \neq 0 \text{ for all } t.$$

Hence the matrix is non-singular.

4. (a) If $\Phi(t)$ is a fundamental matrix for $x' = Ax$ and C is a nonsingular matrix of the same dimension, show that $\Phi(t)C$ is a fundamental matrix.

Solution: Need to show solves the associated matrix DE $X' = AX$ and that $\det \Phi(t)C \neq 0$ on some interval. As C is a constant non-singular matrix, $\det C \neq 0$ and C^{-1} exists. As $\frac{d}{dt}(\Phi(t)) = A\Phi(t)$, we have

$$\frac{d}{dt}(\Phi(t)C) = \Phi'(t)C = (A\Phi(t))C = A(\Phi(t)C).$$

Additionally, $\det(\Phi(t)C) = \det \Phi(t) \det C \neq 0$ as $\Phi(t)$ is a fundamental matrix.

(b) Show that if $\Phi(t)$ and $\Psi(t)$ are fundamental matrices for $x' = Ax$, then there is a constant, nonsingular matrix C such that $\Phi(t)C = \Psi(t)$.

Solution: Consider any IVP $x(t_0) = x_0$. By existence and uniqueness, there exist a constant vector $\mathbf{c}$ such that $x(t) = \Phi(t)\mathbf{c}$ and $x(0) = \Phi(0)\mathbf{c} = x_0$. Similarly, there exists a $\mathbf{v}$ such that $x(t) = \Psi(t)\mathbf{v}$ and such that $x(0) = \Psi(0)\mathbf{v} = x_0$. Hence $\Phi(0)\mathbf{c} = \Psi(0)\mathbf{v}$, or $\mathbf{c} = \Phi(0)^{-1}\Psi(0)\mathbf{v}$. Then

$$x(t) = \Phi(t)\mathbf{c} = \Phi(t)(\Phi(0)^{-1}\Psi(0)\mathbf{v}) = (\Phi(t)\Phi(0)^{-1}\Psi(0))\mathbf{v}.$$

By part (a), $\Phi(t)\Phi(0)^{-1}\Psi(0)$ is another fundamental matrix. But by applying the existence and uniqueness again, the fact that $\mathbf{v}$ is the constant vector required when using $\Psi(t)$ to solve the IVP, we have that $\Psi(t) = \Phi(t)\Phi(0)^{-1}\Psi(0)$. That is, the matrix $C = \Phi(0)^{-1}\Psi(0)$.