Math 345 – Homework 4 Due Friday, September 30, 2022

1. Find the general solution of each of the differential equations.

   (a) $y''' - y = 0$

Solution: Here $L = D^3 - 1 = (D-1)(D^2 + D + 1)$ has roots $r = 1$ and $r = \dfrac{-1 \pm i\sqrt{3}}{2}$:

$$y(x) = C_1 e^{-x} + C_2 e^{-x/2} \cos(x\sqrt{3}/2) + C_3 e^{-x/2} \sin(x\sqrt{3}/2).$$

   (b) $y''' + 4y'' + 4y' = 0$

Solution: Here $L = D^3 + D^2 + 4D = D(D+2)^2$ has roots $r = 0$ and $r = -2$, repeated:

$$y(x) = C_1 e^{0x} + C_2 e^{-2x} + C_3 x e^{-2x}.$$

   (c) $y^{(4)} + 2y'' + y = 0$

Solution: Here $L = D^4 + 2D^2 + 1 = (D^2 + 1)^2$ has roots $r = \pm i$:

$$y(x) = C_1 \cos x + C_2 \sin(x) + C_3 x \cos x + C_3 x \sin(x).$$

   (d) $y^{(4)} + 4y''' + 6y'' + 4y' + y = 0$

Solution: Here $L = D^4 + 4D^3 + 6D^2 + 4D + 1 = (D+1)^4$ has roots $r = -1$, repeated four times:

$$y(x) = C_1 e^{-x} + C_2 x e^{-x} + C_3 x^2 e^{-x} + C_4 x^3 e^{-x}.$$

2. If m is a positive constant, find the solution of the initial value problem

$$y''' - my'' + m^2y' - m^3y = 0$$

where $f(0) = f'(0) = 0$, $f''(0) = 1$.

Solution: $L = D^3 - mD^2 + m^2D - m^3 = (D-m)(D^2+m^2)$ with roots $r = m$ and $r = \pm mi$:

$$y_c(x) = C_1e^{mx} + C_2\cos(mx) + C_3\sin(mx).$$

To solve the IVP we need

$$\begin{aligned} y'(x) &= mC_1e^{mx} - C_2\sin(mx) + C_3\cos(mx) \\ y''(x) &= m^2C_1e^{mx} - C_2\cos(mx) - C_3\sin(mx). \end{aligned}$$

Evaluating these functions at the initial conditions results in the following system of equations

$$\begin{aligned} y(0) &= C_1 + C_2 = 0 \\ y'(0) &= mC_1 + mC_3 = 0 \\ y''(0) &= m^2C_1 - m^2C_2 = 1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ m & 0 & m & 0 \\ m^2 & -m^2 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.5m^{-2} \\ 0 & 1 & 0 & -0.5m^{-2} \\ 0 & 0 & 1 & -0.5m^{-2} \end{array}\right]$$

The solution to the initial value problem is

$$y_c(x) = \frac{1}{2m^2}e^{mx} - \frac{1}{2m^2}\cos(mx) - \frac{1}{2m^2}\sin(mx).$$

3. Find a linear differential equation $Ly = 0$ with constant coefficients where $y_1(x) = e^{-2x}\cos 3x$, $y_2(x) = x^2$, and $y_3(x) = x\sin x$ are solutions.

Solution: For y_1, this comes from the root $r = -2 \pm 3i$. The factor in the differential operator is $(D-(-2+3i))(D-(-2-3i) = D^2 + 4D + 13$.

For y_2, this comes from the root $r = 0$ repeated twice and the factor is D^3.

For y_3, this comes from the root $r = \pm i$ repeated and the factor it $(D^2+1)^2$.

Altogether, an ODE with these solutions in the fundamental set is

$$(D^2 + 4D + 13)(D^2 + 1)^2 D^3 y = 0.$$

4. Find the general solution (by hand) of the differential equation

$$y'' - y = \frac{2}{1+e^x}.$$

Solution: First we solve the complementary equation $y'' - y = 0$. This is equivalent to $D^2 - I = 0$, which factors into $(D+I)(D-I)$. Thus the fundamental set is $\{e^{-x}, e^x\}$ and

$$y_c = C_1 e^{-x} + C_2 e^x.$$

We seek a solution via variation of parameters of the form $y_p(x) = v_1 e^{-x} + v_2 e^x$. This yields the system

$$\begin{bmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2/(1+e^x) \end{bmatrix}.$$

Solving the system yields

$$v_1' = \frac{-e^x}{1+e^x} \text{ and } v_2' = \frac{e^{-x}}{1+e^x}.$$

A little u-sub and a little partial fractions later,

$$v_1(x) = -\ln(1+e^x) \text{ and } v_2(x) = -x - e^{-x} + \ln(1+e^x).$$

This yield the general solution to the non-homogeneous problem

$$\begin{aligned} y(x) &= C_1 e^{-x} + C_2 e^x - \ln(1+e^x)e^{-x} + (-x - e^{-x} + \ln(1+e^x))e^x \\ &= C_1 e^{-x} + C_2 e^x - \ln(1+e^x)e^{-x} - xe^x - 1 + \ln(1+e^x)e^x. \end{aligned}$$

5. Consider the non-homogenous ODE

$$y''' + \frac{1}{x}y'' - \frac{2}{x^2}y' + \frac{2}{x^3}y = 2x.$$

(a) Verify that $\{x, x^2, 1/x\}$ form a fundamental set to the corresponding complementary ODE.

Solution: Let $y_1 := x$, $y_2 := x^2$, and $y_3 := 1/x$. First we need to show that they actually solve the ODE. (To solve this problem, one would use the Method of Frobenius. We won't cover that technique in this class.)

$y_1' = 1$, $y_1'' = 0$ and the ODE is

$$0 + \frac{1}{x}0 - \frac{2}{x^2}(1) + \frac{2}{x^3}x = -\frac{2}{x^2} + \frac{2}{x^2} = 0.$$

$y_2' = 2x$, $y_2'' = 2$, and $y_2''' = 0$ and the ODE is

$$0 + \frac{1}{x}(2) - \frac{2}{x^2}(2x) + \frac{2}{x^3}x^2 = \frac{2}{x} - \frac{4}{x} + \frac{2}{x} = 0.$$

$y_3' = -1/x^2$, $y_3'' = 2/x^3$, and $y_3''' = -6/x^4$ the ODE is

$$-\frac{6}{x^4} + \frac{1}{x}\left(\frac{2}{x^3}\right) - \frac{2}{x^2}\left(\frac{-1}{x^2}\right) + \frac{2}{x^3}\left(\frac{1}{x}\right) = 0.$$

For linear independence, we compute the Wronskian:

$$\begin{aligned} W(y_1, y_2, y_3) &= \begin{vmatrix} x & x^2 & 1/x \\ 1 & 2x & -1/x^2 \\ 0 & 2 & 2/x^3 \end{vmatrix} \\ &= x\begin{vmatrix} 2x & -1/x^2 \\ 2 & 2/x^3 \end{vmatrix} - \begin{vmatrix} x^2 & 1/x \\ 2 & 2/x^3 \end{vmatrix} \\ &= \frac{6}{x} \\ &\neq 0. \end{aligned}$$

(b) Determine a particular solution of the non-homogenous ODE.

Solution:

We seek a solution via variation of parameters of the form $y_p(x) = v_1 x + v_2 x^2 + v_3 \frac{1}{x}$. This yields the system

$$\begin{bmatrix} x & x^2 & 1/x \\ 1 & 2x & -1/x^2 \\ 0 & 2 & 2/x^3 \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2x \end{bmatrix}.$$

Using Cramer's Rule to solve the system yields

$$v_1' = \frac{1}{W(y_1, y_2, y_3)} \begin{vmatrix} 0 & x^2 & 1/x \\ 0 & 2x & -1/x^2 \\ 2x & 2 & 2/x^3 \end{vmatrix} = -x^2.$$

$$v_2' = \frac{1}{W(y_1, y_2, y_3)} \begin{vmatrix} x & 0 & 1/x \\ 1 & 9 & -1/x^2 \\ 0 & 2x & 2/x^3 \end{vmatrix} = \frac{2}{3}x$$

$$v_3' = \frac{1}{W(y_1, y_2, y_3)} \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 2x \end{vmatrix} = \frac{1}{3}x^4.$$

$$v_1(x) = -\frac{1}{3}x^3, v_2(x) = \frac{1}{3}x^2 \text{ and } v_3(x) = \frac{1}{15}x^5.$$

This yield the general solution to the non-homogeneous problem

$$\begin{aligned} y_p(x) &= v_1 x + v_2 x^2 + v_3 \frac{1}{x} \\ &= -\frac{1}{3}x^3 \cdot x + \frac{1}{3}x^2 \cdot x^2 + \frac{1}{15}x^5 \cdot \frac{1}{x} \\ &= \frac{1}{15}x^4 \end{aligned}$$