Math 414 – Homework 9 **Due Monday, May 1, 2023**

Chapter 12ish: Fourier Transform

1. Prove the following properties. Given $f(x) \in S(\mathbb{R})$ and $\mathcal{U} : f \to \hat{f}$.

   (a) (translation invariance of $\mathcal{U}$) $\mathcal{U}(f(x+h)) = \mathcal{U}(f(x))$ whenever $h \in \mathbb{R}$.

   (b) (convolution operator is commutative) $(f * g)(x) = (g * f)(x)$.

   (c) $\frac{d}{dx}\left((f * g)(x)\right) = \left(\frac{df}{dx} * g\right)(x)$.

   (d) $\frac{d}{dx}\left((f * g)(x)\right) = \left(f * \frac{dg}{dx}\right)(x)$.

   (e) $\mathcal{U}(e^{2\pi i a x} f(x)) = \hat{f}(\gamma - a)$

   (f) $-\frac{d}{d\gamma}\hat{f}(\gamma) = \mathcal{U}(2\pi i x f(x))$

   (g) ("pass the hat") $\int_{\mathbb{R}} \hat{f}(x) g(x)\, dx = \int_{\mathbb{R}} f(x) \hat{g}(x)\, dx$

2. Find the Fourier Transform of the function $f(x) = xe^{-kx^2}$. (Hint: Using properties of the Fourier transform and results from lecture is easier than computing via the definition.)

3. Use the Fourier transform and the convolution theorem to solve the biharmonic heat equation:

$$u_t = -\Delta^2 u, x \in \mathbb{R}, t > 0$$
$$u(x, 0) = f(x).$$

   (Hint: After solving an appropriate ODE in t, you should have a product of Schwartz class functions in the transform variable. How do we inverse transform products? Also, you will not know the closed form of one of the inverses. Explain why it exists, define it, and carry on the calculation formally.)

4. Solve the following Laplace's equation on a infinite strip:

$$\Delta u = 0, 0 < x < L, y \in \mathbb{R},$$
$$u(0, y) = g_1(y),$$
$$u(L, y) = g_2(y).$$

The idea here is to use a Fourier transform in the y variable to reduce the problem to an ODE in the x variable (depending parametrically on the Fourier transform variable γ). Note: Like the PDE above, don't worry if you can't invert the Fourier transforms that arise. Just call them something and carry them along.