MTH 414 - Spring 2023
Assignment #9
Due: Monday May 1st 2023 (2:00 PM)

1. Prove the following properties. Given $f(x) \in S(\mathbb{R})$ and $\mathcal{U} : f \to \hat{f}$.

   (a) $\mathcal{U}(f(x+h)) = \mathcal{U}(e^{2\pi i h x} f(x))$

   Proof. Let $g(x) := f(x+h)$. Then

$$\begin{aligned}
\mathcal{U}(f(x+h)) &= \hat{f}(x+h) \\
&= \hat{g}(x) \\
&= \int g(t) e^{-2\pi i x t} \mathrm{d}t \\
&= \int f(t+h) e^{-2\pi i x t} \mathrm{d}t \\
&= \int f(u) e^{-2\pi i x (u-h)} \mathrm{d}u && (u = t+h) \\
&= e^{2\pi i x h} \int f(u) e^{-2\pi i x u} \mathrm{d}u \\
&= e^{2\pi i x h} \hat{f}(x) \\
&= \mathcal{U}(e^{2\pi i h x} f(x))
\end{aligned}$$

□

   (b) $(f * g)(x) = (g * f)(x)$

   Proof.

$$\begin{aligned}
(f * g)(x) &= \int_{\mathbb{R}} f(x-t) g(t) \mathrm{d}t \\
u = x - t, &\quad t = x - u, \quad \mathrm{d}t = -\mathrm{d}u \\
&= -\int_{u(t=-\infty)}^{u(t=\infty)} f(u) g(x-u) \mathrm{d}u \\
&= -\int_{\infty}^{-\infty} g(x-u) f(u) \mathrm{d}u \\
&= \int_{\mathbb{R}} g(x-u) f(u) \mathrm{d}u \\
&= (g * f)(x)
\end{aligned}$$

□

   (c) $\dfrac{\mathrm{d}}{\mathrm{d}x}((f * g)(x)) = \left(\dfrac{\mathrm{d}f}{\mathrm{d}x} * g\right)(x)$

Proof.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left((f*g)(x)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{\mathbb{R}} f(x-t)g(t)\mathrm{d}t\right)\\ &= \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}x} f(x-t)g(t)\mathrm{d}t && \text{(since integrating wrt } t\text{, not } x)\\ &= \left(\frac{\mathrm{d}f}{\mathrm{d}x} * g\right)(x)\end{aligned}$$

□

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\left((f*g)(x)\right) = \left(f * \frac{\mathrm{d}g}{\mathrm{d}x}\right)(x)$

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((f*g)(x)\right) \overset{(b)}{=} \frac{\mathrm{d}}{\mathrm{d}x}\left((g*f)(x)\right) \overset{(c)}{=} \left(\frac{\mathrm{d}g}{\mathrm{d}x} * f\right)(x) \overset{(b)}{=} \left(f * \frac{\mathrm{d}g}{\mathrm{d}x}\right)(x)$$

□

(e) $\mathcal{U}(e^{-2\pi i a x} f(x)) = \hat{f}(\gamma - a)$

Proof. I changed the exponent sign, I believe it was wrong.

$$\begin{aligned}\hat{f}(\gamma - a) &= \hat{g}(\gamma)\\ &= \int_{\mathbb{R}} g(t)e^{-2\pi i x t}\mathrm{d}t\\ &= \int_{\mathbb{R}} f(t-a)e^{-2\pi i x t}\mathrm{d}t\\ &= \int_{\mathbb{R}} f(u)e^{-2\pi i x(u+a)}\mathrm{d}u && (u = t-a,\ t = u+a)\\ &= e^{-2\pi i x a}\int_{\mathbb{R}} f(u)e^{-2\pi i x u}\mathrm{d}u\\ &= e^{-2\pi i x a}\hat{f}(x)\end{aligned}$$

□

(f) $-\frac{\mathrm{d}}{\mathrm{d}\gamma}\hat{f}(\gamma) = \mathcal{U}(2\pi i x f(x))$

Proof.

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}\gamma}\hat{f}(\gamma) &= -\frac{\mathrm{d}}{\mathrm{d}\gamma}\left(\int_{\mathbb{R}} f(\gamma)e^{-2\pi i \gamma x}\mathrm{d}\gamma\right)\\ &= \left(\int_{\mathbb{R}} f(\gamma) - \frac{\mathrm{d}}{\mathrm{d}\gamma}e^{-2\pi i \gamma x}\mathrm{d}\gamma\right)\\ &= -\left(\int_{\mathbb{R}} f(\gamma) - 2\pi i x e^{-2\pi i \gamma x}\mathrm{d}\gamma\right)\\ &= 2\pi i x\left(\int_{\mathbb{R}} f(\gamma)e^{-2\pi i \gamma x}\mathrm{d}\gamma\right)\\ &= \mathcal{U}(2\pi i x f(\gamma))\end{aligned}$$

□

(g) $\int_{\mathbb{R}} \hat{f}(x)g(x) = \int_{\mathbb{R}} f(x)\hat{g}(x)$

Proof.

$$\begin{aligned}
\int_{\mathbb{R}} \hat{f}(x)g(x) &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(t)e^{-2\pi ixt}\mathrm{d}t \right) g(x)\mathrm{d}x \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(t)e^{-2\pi ixt}g(x)\mathrm{d}t \right) \mathrm{d}x \\
&\overset{\text{Fubini}}{=} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(t)e^{-2\pi ixt}g(x)\mathrm{d}x \right) \mathrm{d}t \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-2\pi ixt}g(x)\mathrm{d}x \right) f(t)\mathrm{d}t \\
&= \int_{\mathbb{R}} \hat{g}(t)f(t)\mathrm{d}t \\
&= \int_{\mathbb{R}} f(x)\hat{g}(x)\mathrm{d}x
\end{aligned}$$

□

2. Find the Fourier Transform of the function $f(x) = xe^{-kx^2}$.

Solution:

In order to use the derivitive formulas, first want to compute $\mathcal{F}\left(e^{-kx^2}\right)$.

$$\mathcal{F}\left(e^{-kx^2}\right) = \int_{\mathbb{R}} e^{-kx^2}e^{-2\pi itx}\mathrm{d}x = \int_{\mathbb{R}} e^{-(kx^2+2\pi itx)\mathrm{d}x}$$

This has a quadratic in x, $kx^2 + 2\pi itx$. Completing the square gives

$$kx^2 + 2\pi itx + \left(\frac{\pi it}{\sqrt{k}}\right)^2 - \left(\frac{\pi it}{\sqrt{k}}\right)^2 = \left(\sqrt{k}x + \left(\frac{\pi it}{\sqrt{k}}\right)\right)^2 - \left(\frac{\pi it}{\sqrt{k}}\right)^2$$

Thus we have

$$\int_{\mathbb{R}} \exp\left\{-\left[\left(\sqrt{k}x + \left(\frac{\pi it}{\sqrt{k}}\right)\right)^2 - \left(\frac{\pi it}{\sqrt{k}}\right)^2\right]\right\}\mathrm{d}x = \exp\left\{\left(\frac{\pi it}{\sqrt{k}}\right)^2\right\}\int_{\mathbb{R}} \exp\left\{-\left(\sqrt{k}x + \left(\frac{\pi it}{\sqrt{k}}\right)\right)^2\right\}\mathrm{d}x$$

Letting $u = \sqrt{k}x + \left(\frac{\pi it}{\sqrt{k}}\right)$ gives $\frac{1}{\sqrt{k}}\mathrm{d}u = \mathrm{d}x$ so

$$\mathcal{F}\left(e^{-kx^2}\right) = \frac{1}{\sqrt{k}}\exp\left\{\left(\frac{\pi it}{\sqrt{k}}\right)^2\right\}\int_{\mathbb{R}} e^{-u^2}\mathrm{d}u = \frac{1}{\sqrt{k}}\exp\left\{-\frac{\pi^2t^2}{k}\right\}\sqrt{\pi} = \frac{\sqrt{\pi}}{\sqrt{k}}e^{-\pi^2t^2/k}$$

Next, $\int xe^{-kx^2}\mathrm{d}x = \left(\frac{e^{-kx^2}}{-2k}\right)$. It follows that

$$\mathcal{F}\left(xe^{-kx^2}\right) = \mathcal{F}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{e^{-kx^2}}{-2k}\right)\right) = 2\pi it\mathcal{F}\left(\frac{1}{-2k}\left(e^{-kx^2}\right)\right) = -\frac{\pi it}{k}\mathcal{F}\left(e^{-kx^2}\right) = \frac{-\pi\sqrt{\pi}it}{k\sqrt{k}}e^{-\pi^2t^2/k}$$

3. Use the Fourier transform and the convolution theorem to solve the biharmonic heat equation:

$$\begin{cases} u_t = -\Delta^2 u & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) \end{cases}$$

Solution:

$\hat{u}_t(\gamma,t) = k(2\pi i\gamma)^4 \hat{u}(\gamma,t)$

$$\frac{\hat{u}_t(\gamma,t)}{\hat{u}(\gamma,t)} = 16\pi^4\gamma^4 \implies \hat{u}(\gamma,t) = C(\gamma)e^{-16\pi^4\gamma^4 kt}$$

By the initial condition, $\hat{u}(\gamma,0) = C(\gamma)e^0 = C(\gamma) = \hat{f}(\gamma)$. Hence

$$\hat{u}(\gamma,t) = e^{-16\pi^4\gamma^4 kt}\hat{f}(\gamma)$$

This is a product of Schwartz class functions, so we know inverses exist. Let $k(x,t) = \mathcal{U}^{-1}(e^{-16\pi^4\gamma^4 kt})$ then

$$\mathcal{U}^{-1}\left(\hat{\mathcal{U}}(x,t)\right) = \mathcal{U}^{-1}(\hat{k}(\gamma,t)\hat{f}(\gamma)) = k(x,t) * f(x)$$

$$u(x,t) = \int_{\mathbb{R}} k(x-y,t)f(y)\,\mathrm{d}y$$

We can compute k further. In order to use a Gauss Kernel, we want $(-16\pi^4\gamma^4 kt) = \pi(\zeta)^2\gamma^2$. Let $\zeta = (4\pi\gamma\sqrt{\pi kt})$ then

$$e^{-\pi(4\pi\gamma\sqrt{\pi kt})^2\gamma^2} = \hat{k}(\gamma,t).$$

Therefore

$$\mathcal{U}^{-1}\left(\hat{k}(\gamma,t)\right) = G_{4\pi\gamma\sqrt{\pi kt}}(x) = \frac{1}{4\pi\gamma\sqrt{\pi kt}}\exp\left\{\frac{-x^2}{16\pi^2\gamma^2 kt}\right\}.$$

Finally

$$u(x,t) = \int_{\mathbb{R}} \frac{1}{4\pi\gamma\sqrt{\pi kt}}\exp\left\{\frac{-(x-y)^2}{16\pi^2\gamma^2 kt}\right\} f(y)\,\mathrm{d}y$$

4. Solve the following Laplace's equation on a infinite strip:

$$\begin{cases} \Delta u = 0 & 0 < x < L,\ y \in \mathbb{R} \\ u(0,y) = g_1(y) \\ u(L,y) = g_2(y) \end{cases}$$

Solution: