Math 414
Professor Carty
46 points possible

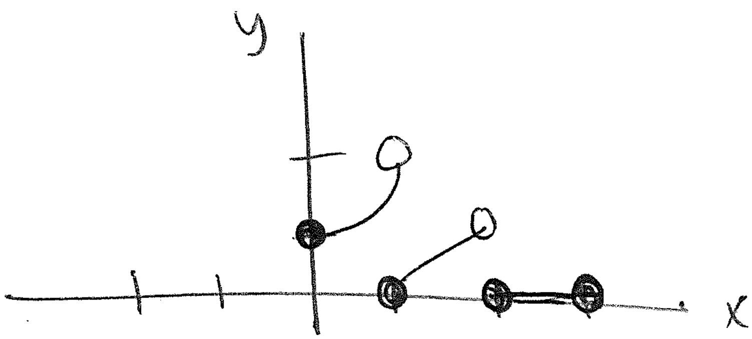
Exam 2
April 21, 2023

Name: ______________________________

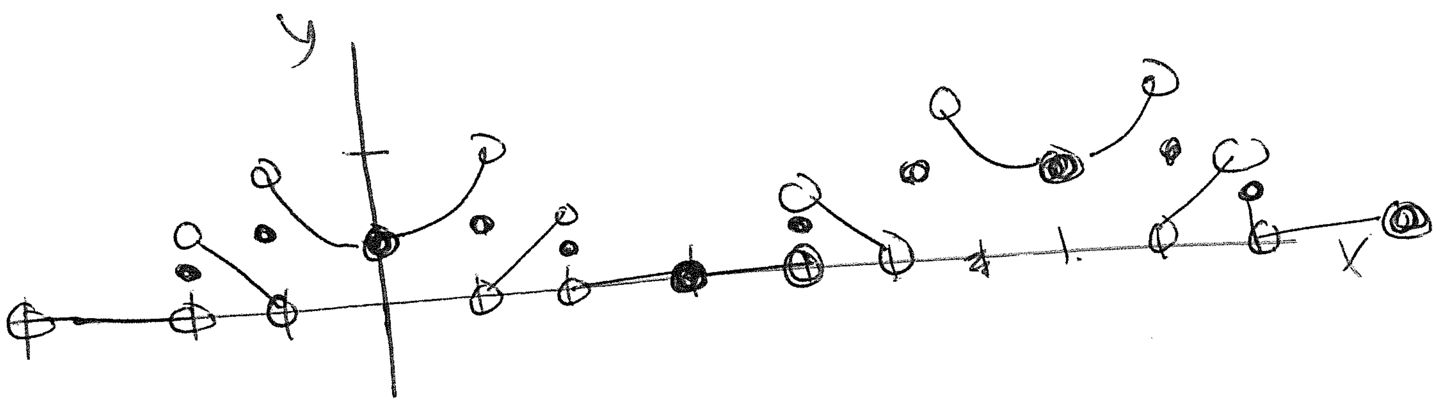
1. No hats or dark sunglasses. All hats are to be removed.
2. All book bags are to be closed and placed in a way that makes them inaccessible. Do not reach into your bag for anything during the exam. If you need extra pencils, pull them out now.
3. Be sure to print your proper name clearly.
4. *No calculators are allowed.* Watches with recording, internet, communication or calculator capabilities (e.g. a smart watch) are prohibited.
5. All electronic devices, including cell phones and other wearable devices, must be powered off and stored out of sight for the entirety of the exam.
6. If you have a question, raise your hand and I will come to you. Once you stand up, you are done with the exam. If you have to use the facilities, do so now. You will not be permitted to leave the room and return during the exam.
7. Every exam is worth a total of **45 points**. Including the cover sheet, each exam has 8 pages.
8. At 2:50, you will be instructed to put down your writing utensil. You must stop writing the exam at this time.
9. If you finish early, quietly and respectfully and in your exam. You may leave early.
10. You will hand in the paper copy of the exam on your way out of the classroom.
11. You have fifty minutes to complete the exam. I hope you do well.

1. (6 points) Let $f(x) := \begin{cases} x^2+1, & 0 \le x < 1 \\ x-1, & 1 \le x < 2 \\ 0, & 2 \le x \le 3. \end{cases}$

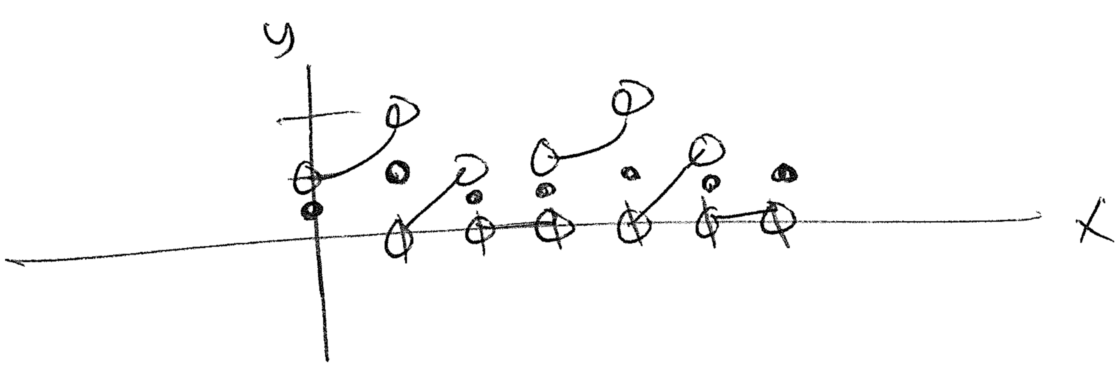
(a) Graph the function $f(x)$ on $[0,3]$.



(b) Graph two periods of the function $\hat{C}(x)$, the Fourier Cosine Series expansion of $f(x)$.



(c) In addition, assume that $f(x+3) = f(x)$ for all x. Graph two periods of the function $\hat{f}$, the general Fourier series of $f(x)$.



2. (6 points) Let $f(x)$ be defined by

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) + \sum_{n=1}^{\infty} B_n \sin(nx).$$

(a) *Explain* (you do not need to demonstrate) what we mean when we say the functions $C_n(x) := \cos(nx)$, $n = 0, 1, 2, \ldots$ and $S_k(x) := \sin(kx)$, $k = 1, 2, \ldots$ are *all* pairwise orthogonal functions over the interval $[-\pi, \pi]$.

$\langle C_n, S_k \rangle = 0$ for all n, k

$\langle C_n, C_k \rangle = 0$ for all $n \neq k$

$\langle S_n, S_k \rangle = 0$ for all $n \neq k$.

(b) Demonstrate how we used orthogonality to solve for the coefficients B_k.

Fix k, mult. by S_k and use L^2 innerproduct $\langle \, , S_k \rangle$.

$$\langle f, S_k \rangle = \frac{A_0}{2} \langle 1, S_k \rangle + \sum_{n=1}^{\infty} A_n \langle C_n, S_k \rangle + \sum_{n=1}^{\infty} B_n \langle S_n, S_k \rangle$$

$$\Rightarrow \langle f, S_k \rangle = B_k \langle S_k, S_k \rangle \text{ by } \perp.$$

$$\langle S_k, S_k \rangle = \int_{-\pi}^{\pi} \sin^2(kx)\,dx = \int_{-\pi}^{\pi} \frac{1 - \cos(2kx)}{2}\,dx$$

$$= \frac{1}{2} x \Big|_{-\pi}^{\pi} - \frac{1}{2} \sin(2kx) \left(\frac{1}{2k}\right) \Big|_{0}^{\pi}$$

$$= \pi - 0 \Rightarrow B_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\,dx$$

Potentially helpful identities: $\cos^2(\theta) = \dfrac{1 + \cos(2\theta)}{2}$ and $\sin^2(\theta) = \dfrac{1 - \cos(2\theta)}{2}$.

3. (6 points) (a) Let $u : \mathbb{R}^2 \to \mathbb{R}$. State the polar form of Laplace's equation $\Delta u = 0$.

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

(b) Evaluate the integral

$$u(r, \theta) = \frac{1 - r^2}{2\pi} \int_0^{2\pi} \frac{\cos^2\phi - \sin^2\phi}{1 - 2r\cos(\theta - \phi) + r^2}\, d\phi,$$

not by directly integrating but by considering a related problem on the disk $\{(x, y) : x^2 + y^2 < 1\}$.

This is the poisson integral formula corresponding to

$\Delta u = 0$ on $x^2 + y^2 \leq 1$

$u(x,y) = x^2 - y^2$ on $x^2 + y^2 = 1$.

Thus $u(r,\theta)$ represents the sol'n thru the disk $x^2+y^2 \leq 1$.

Here $h(1,\theta) = 1^2\cos^2\theta - 1^2\sin^2\theta \Rightarrow x^2 - y^2$ on boundary.

Since $u(x,y) = x^2 - y^2$ solves $\Delta u = 0$ and satisfies boundary equation, $u(x,y) = x^2 - y^2$ thru out disk.

Hence $u(r,\theta) = r^2\cos^2\theta - r^2\sin^2\theta$.

4. (7 points) (a) State Green's First Identity (IBP for big kids!).

$$\int_{\partial D} u \frac{\partial v}{\partial n} \, dS = \int_D \nabla u \cdot \nabla v \, dV + \int_D u \Delta v \, dV$$

(b) Prove the uniqueness (up to a constant) of solutions to Laplace's equation with the Neumann boundary condition:

$$\Delta u = 0 \text{ in } D \text{ with } \frac{\partial u}{\partial n} = 0 \text{ on } \partial D.$$

Let u, v solve BVP.

Then $w = u - v$ also solves $\Delta w = 0$ in D.

$\frac{\partial w}{\partial n} = 0$ on ∂D

Consider $w \Delta w = 0$.

$$\int_D w \Delta w = 0$$

$$\Rightarrow \underbrace{\int_{\partial D} w \frac{\partial w}{\partial n} \, dS}_{=0} - \int_D \nabla w \cdot \nabla w \, dV = 0 \quad \text{by Green's first.}$$

$$\Rightarrow \int_D \nabla w \cdot \nabla w \, dV = 0 \quad \Rightarrow \nabla w = \vec{0}$$

$\Rightarrow w = \vec{C}$, $\vec{C}$ a constant vector.

So $u - v = \vec{C}$.

5. (9 points) Show (providing all details) that the eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0, 0 < x < \pi; \\ X'(0) = 0, \\ X(\pi) = 0 \end{cases}$$

has eigenvalues and associated eigenfuntions

$$\lambda_n = \left(n + \frac{1}{2}\right)^2, \qquad X_n(x) = \cos\left(\left(n + \frac{1}{2}\right)x\right), \text{ for } n = 0, 1, 2, \ldots.$$

Hint: You may assume all the eigenvalues are real. You should then determine all the eigenvalues by considering three cases...

$\lambda = 0$, $X'' = 0 \Rightarrow X(x) = A + Bx$, $X(\pi) = 0 \Rightarrow A = -B\pi$

$X'(x) = B$, $X'(0) = 0 \Rightarrow B = 0 \Rightarrow A = 0$.

not an eigenvalue

$\lambda < 0$, $X(x) = A\cosh(\sqrt{|\lambda|}\,x) + B\sinh(\sqrt{|\lambda|}\,x)$

$X'(\cancel{x}) = A\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}\,x) + B\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}\,x)$

$X'(0) = B\sqrt{|\lambda|} = 0 \Rightarrow B = 0$

$X(\pi) = A\cosh(\sqrt{|\lambda|}\,\pi) = 0 \Rightarrow A = 0$.

not an eigenvalue.

$\lambda > 0$, $X(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$

$X'(x) = A\sqrt{\lambda}(-\sin(\sqrt{\lambda}\,x)) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\,x)$

$X'(0) = 0 \Rightarrow B = 0$.

$X(\pi) = A\cos(\sqrt{\lambda}\,\pi) = 0$

Let $A \neq 0$, $\sqrt{\lambda}\,\pi = \frac{\pi}{2} + n\pi$, $n = 0, 1, 2, \ldots$

$\Rightarrow \lambda_n = \left(\frac{1}{2} + n\right)^2$ and $X_n(x) = \cos\left(\left(n + \frac{1}{2}\right)x\right)$

6. (12 points) Use the method of separation of variables to find the solution of the IVBVP

$$\begin{cases} u_{tt} = u_{xx} - u, \; 0 < x < \pi, \; t > 0, \\ u_x(0,t) = u(\pi,t) = 0, \; t > 0 \\ u(x,0) = 2\cos\left(\frac{1}{2}x\right) - 3\cos\left(\frac{5}{2}x\right), \; 0 < x < \pi \\ u_t(x,0) = 0, \; 0 < x < \pi \end{cases}$$

Hint: Feel free to use the statement of Problem 5 (whether you solved 5, or not). Next page is blank for more room.

Let $u = X(x)T(t) \Rightarrow XT'' = X''T - XT$

$\Rightarrow \frac{T''}{T} = \frac{X''}{X} - 1$ or $\frac{T''}{T} + 1 = \frac{X''}{X} = -\lambda.$

$u_x(0,t) = 0 \Rightarrow X'(0)T(t) = 0 \Rightarrow X'(0) = 0.$

$u(\pi,t) = 0 \Rightarrow X(\pi)T(t) = 0 \Rightarrow X(\pi) = 0.$

We have the BVP See Problem 5 $X'' + \lambda X = 0$
$X(\pi) = X'(0) = 0.$

Have $\lambda_n = (n+\frac{1}{2})^2$ and $X_n(x) = \cos\left((n+\frac{1}{2})x\right)$, $n=0,1,\ldots$

For each n, $\frac{T_n''}{T_n} + 1 = -\lambda_n$, $T_n'' + \underbrace{(1+\lambda_n)}_{>0 \text{ on } n=0,1,2,\ldots} T_n = 0$

So $T_n(t) = A_n\cos(\sqrt{1+\lambda_n}\,t) + B_n\sin(\sqrt{1+\lambda_n}\,t)$

But $u_t(x,0) = 0 \Rightarrow T_n'(0) = 0 \Rightarrow B_n = 0 \; \forall n.$

So $T_n(t) = A_n\cos(\sqrt{1+\lambda_n}\,t)$

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$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\left(n+\tfrac{1}{2}\right)x\right) \cos\left(\sqrt{1+\left(n+\tfrac{1}{2}\right)^2}\; t\right)$$

$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos\left(\left(n+\tfrac{1}{2}\right)x\right)$$

$$\overset{\text{I.C.}}{=} 2\cos\left(\tfrac{1}{2}x\right) - 3\cos\left(\tfrac{5}{2}x\right)$$

$$\Rightarrow A_0 = 2, \quad A_2 = -3, \quad A_n = 0 \text{ all others}$$

$$u(x,t) = +2\cos\left(\left(0+\tfrac{1}{2}\right)x\right)\cos\left(\sqrt{1+\left(0+\tfrac{1}{2}\right)^2}\; t\right)$$
$$-3\cos\left(\tfrac{5}{2}x\right)\cos\left(\sqrt{\tfrac{29}{4}}\, t\right)$$