Homework 8

1a. Begin w/ $\Delta u = f$.

Recall $\Delta u = \nabla \cdot \nabla u$.

So $$\int_D \Delta u \, dV = \int_D f \, dV$$

By Div. Thm $$\int_D \Delta u \, dV = \int_D \nabla \cdot (\nabla u) \, dV$$

$$= \int_{\partial D} (\nabla u \cdot n) \, dS \quad \text{or} \quad \int_{\partial D} \frac{\partial u}{\partial n} \, dS.$$

So $$\int_{\partial D} \frac{\partial u}{\partial n} \, dS = \int_D f \, dV$$

By Neuman Conditions,

$$\int_{\partial D} g \, dS = \int_D f \, dV$$ must be true for any soln u to the non-homogeneous PDE.

b. For homogeneous case, we set $\int_{\partial D} g \, dS = 0$.

Hence the average value of $g(x,y,z)$ on ∂D must be zero.

c.

8pd

2. a. u harmonic $\Rightarrow$ $\Delta u = 0$ on disk $r < 2$.

By max (min) principle, max (min) occurs on the boundary

$$u(x,y)\Big|_{x^2+y^2=4} = u(2,\theta) = \frac{3}{2}(2\cos\theta)(2\sin\theta) + 1$$

$$u(2,\theta) = 6\cos\theta\sin\theta + 1$$
$$= 3\sin 2\theta + 1$$

Calc I: $u' = 6\cos 2\theta = 0$ when $2\theta = \frac{\pi}{2} + n\pi$

$$\theta = \frac{\pi}{4} + \frac{n\pi}{2}.$$

4 critical # in parametrization $0 \le \theta < 2\pi$

$$\theta_1 = \frac{\pi}{4},\ \theta_2 = \frac{3\pi}{4},\ \theta_3 = \frac{5\pi}{4},\ \theta_4 = \frac{7\pi}{4}.$$

$u(2, \pi/4) = u(2, 5\pi/4) = 3\sin(\pi/2) + 1 = 4$ MAX

$u(2, 3\pi/4) = u(2, 7\pi/4) = 3\sin(3\pi/2) + 1 = -2$. min.

b. By the Mean Value Property,

$$u(0,0) = \frac{1}{2\pi}\int_{x^2+y^2=4} h(x,y)\,ds$$

$$= \frac{1}{2\pi}\int_0^{2\pi} h(2,\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (3\sin 2\theta + 1)\,d\theta$$

$$= \frac{1}{2\pi}\left(-\frac{3}{2}\cos 2\theta + \theta\right)\Big|_0^{2\pi}$$

$$= 1$$ (Not a surprise. mean of the max + min.)

3. Let u and w solve $\begin{cases} \Delta u = 0 & \text{in } D \\ \frac{\partial u}{\partial n} + a(x)u(x) = 0 & \text{on } \partial D \end{cases}$ w/ $a(x) > 0$ on ∂D

Define $v = u - w$.

Note v solve the exact same BVP.

① $\Delta v = \Delta u - \Delta w \Rightarrow$ $\Delta v = 0$

② $\frac{\partial v}{\partial n} = \nabla v \circ n$

$$= (\nabla u - \nabla w) \circ n$$
$$= \nabla u \circ n - \nabla w \circ n$$
$$= \frac{\partial u}{\partial n} - \frac{\partial w}{\partial n}$$
$$= -a(x)u - (-a(x)w)$$
$$= -a(x)(u - w)$$
$$= -a(x)v$$ $\Rightarrow \frac{\partial v}{\partial n} + a(x)v = 0.$

For uniqueness, need to show $v = 0$.

Consider $v\Delta v = 0$

$$\Rightarrow \int_D v\Delta v \, dV = 0$$

$$\int_{\partial D} v\frac{\partial v}{\partial n} dS - \int_D \nabla v \circ \nabla v \, dV = 0$$ by Green's 1st!

$$\int_{\partial D} v\,(-a(x)\,v)\,dS = \int_D \nabla v \cdot \nabla v \, dV$$

$$-\int_{\partial D} a(x)\,v^2(x)\,dS = \int_D |\nabla v|^2 \, dV$$

(LHS) (RHS)

Note integrand of LHS is non-negative, hence LHS is non-positive. Since $a(x) > 0$.

But RHS is non-negative.

Equality means $\int_{\partial D} a(x)\,v^2(x)\,dS = \int_D |\nabla v|^2 \, dV = 0.$

Well $\int_D |\nabla v|^2 dV = 0 \Rightarrow \nabla v = \vec{0}$ thru out D
i.e. v is constant.

$v = C$ a constant, then $\int_{\partial D} a(x)\, C^2 \, dS = 0$

$\Rightarrow C = 0$ (since $a(x) > 0$ on ∂D).

We get $v = 0$ or $u = w$.

$\langle p5 \rangle$

4. At a minimum, the directional derivative in the direction of v needs to be zero for any v.

That is, $\lim_{\epsilon \to 0} \frac{E[u+\epsilon v] - E[u]}{\epsilon} = 0$.

Let $v \in C^1(\bar{D})$.

$$E[u+\epsilon v] = \frac{1}{2}\int_D |\nabla(u+\epsilon v)|^2 - \int_{\partial D} h(u+\epsilon v)$$

$$= \underline{\frac{1}{2}\int_D |\nabla u|^2} + \epsilon\int_D \nabla u \cdot \nabla v + \frac{\epsilon^2}{2}\int_D |\nabla v|^2 - \underline{\int_{\partial D} hu} - \epsilon\int_{\partial D} hv$$

(from class notes)

$$= E[u] + \epsilon\int_D \nabla u \cdot \nabla v - \epsilon\int_{\partial D} hv + \frac{\epsilon^2}{2}\int_D |\nabla v|^2$$

So $\lim_{\epsilon \to 0} \frac{E[u+\epsilon v] - E[u]}{\epsilon} = \int_D \nabla u \cdot \nabla v - \int_{\partial D} hv$

$= \int_D \nabla v \cdot \nabla u - \int_{\partial D} hv$ (move derivs off v)

$= \int_{\partial D} v\frac{\partial u}{\partial n} - \int_D v\Delta u - \int_{\partial D} hv$ Green's 1st.

$= \int_{\partial D} vh - \int_D v\Delta u - \int_{\partial D} hv$ provided $\frac{\partial u}{\partial n} = h$!

$= -\int_D v\Delta u \, dV$.

Sp6

At a min, $\int_D v \Delta u = 0$ for any and all $v \in C^1$.

$\therefore$ $\Delta u = 0$ thruout D, where $\frac{\partial u}{\partial n} = h$.

i.e. u solves Laplace's Eqn.