HW3

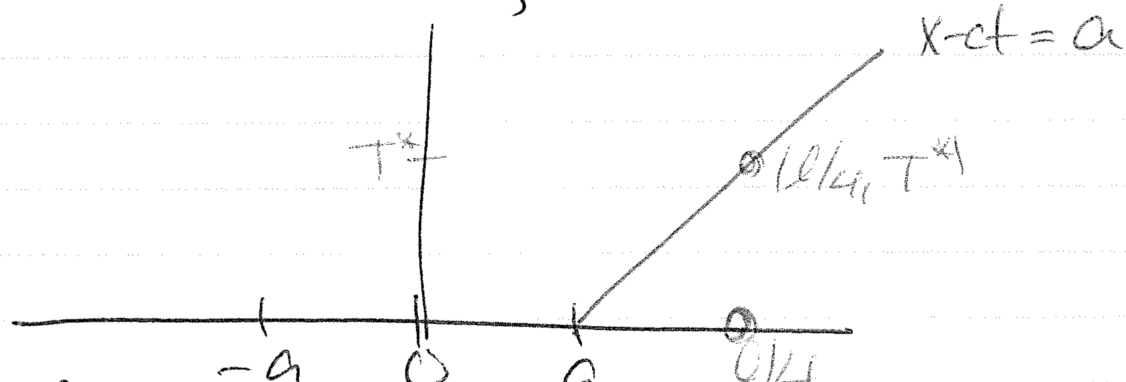
① D'Alembert's

$$u(x,t) = \frac{1}{2}\left[\log(1+(x+ct)^2) + \log(1+(x-ct)^2)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}(4+s)\,ds$$

$$= \frac{1}{2}\left[\log(1+(x+ct)^2) + \log(1+(x-ct)^2)\right] + 4t + xt$$

② (I probably should not have asked this as I derived the wave eqn thru calculus of variations and not the text's approach via tension (§1.3) but given $c = \sqrt{\frac{T}{\rho}}$ ...)

Recall info travels along characteristic lines



The info reaches $x=\ell/4$ in T^* time.

$x - ct = a \rightarrow t = \dfrac{x-a}{c}$

$$T^* = \frac{(\ell/4 - a)}{\sqrt{T/\rho}} = \sqrt{\rho/T}\,(\ell/4 - a).$$

3pd.

3. Need to show $u(-x,t) = -u(x,t)$

$$u(-x,t) = \frac{1}{2}\left[\phi(-x+ct) + \phi(-x-ct)\right] + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(s)\,ds.$$

Change variable $s \mapsto -s$, $ds \to -ds$

$$= \frac{1}{2}\left[\phi(-(x-ct)) + \phi(-(x+ct))\right] + \frac{1}{2c}\int_{x+ct}^{x-ct} \psi(-s)\cdot(-ds)$$

$$= \frac{1}{2}\left[-\phi(x-ct) - \phi(x+ct)\right] \quad \text{(by oddness of } \phi\text{)}$$
$$+ \frac{1}{2c}\int_{x+ct}^{x-ct} -\psi(s)\,(-ds) \quad \text{(by oddness of } \psi\text{)}$$

$$= -\frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x+ct}^{x-ct} \psi(s)\,ds$$

$$= -\frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] - \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

$$= -u(x,t).$$

4 $-4u_{tt} - 3u_{xt} + u_{xx} = 0$

$Lu = 0$ where $L = -4\frac{\partial^2}{\partial t^2} - 3\frac{\partial}{\partial x}\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2}$

$= \left(-4\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)$

By linearity (commutivity) sol'n $Lu=0$ solves either
① $-4u_t + u_x = 0$ or ② $u_t + u_x = 0$

For ①, characteristics are $x = -\frac{1}{4}t + c$

or $4x + t = C$

For ②, char. are $x = t + C$ or $x - t = C$.

General Sol'n is $u(x,t) = f(4x+t) + g(x-t)$

For IVP:
$u(x,0) = x^2 \Rightarrow f(4x) + g(x) = x^2$ (*)

$u_t(x,t) = f'(4x+t) - g'(x-t)$

$u'(x,0) = e^x \Rightarrow f'(4x) - g'(x) = e^x$ (**)
Diff * wrt x, $4f'(4x) + g'(x) = 2x$

⊕ $5f'(4x) = 2x + e^x$
antidiff $\frac{5}{4}\underbrace{f(4x)}_{\text{need } f(\alpha),\text{ a single variable}} = x^2 + e^x$

let $\alpha = 4x \rightarrow x = \alpha/4$

$$f(\alpha) = \frac{4}{5}\left(\left(\frac{\alpha}{4}\right)^2 + e^{\alpha/4} \right)$$

or $f(x) = \frac{x^2}{20} + \frac{4}{5}e^{x/4}$

For $g(x)$, mult (xx) by -4 and (4):

$$5g'(x) = 2x - 4e^x$$

$$g(x) = \frac{1}{5}(x^2 - 4e^x)$$

All together,

$$u(x,t) = f(4x+t) + g(x-t)$$

$$= \frac{(4x+t)^2}{20} + \frac{4}{5}e^{(4x+t)/4} + \frac{1}{5}\left((x-t)^2 - 4e^{x-t} \right)$$

(Realize now, this would have been a little easier w/ characteristic $x + \frac{1}{4}t = C$ vs. $4x+t=C$.)

3 pts

5. $u_{tt} = c^2 u_{xx}$
$u(x,0) = 0$
$u_t(x,0) = 0$

Have $E(t) = \frac{1}{2}\int_{\mathbb{R}} (u_t^2 + c^2 u_x^2)\,dx$ and we know $E(t)$ is constant.

Thus $E(t) = E(0) = \frac{1}{2}\int_{\mathbb{R}} (u_t^2(x,0) + c^2 u_x^2(x,0))\,dx$

$= \frac{1}{2}\int_{\mathbb{R}} c^2 u_x^2(x,0)\,dx$ since $u_t(x,0)=0$.

But since $u(x,0)=0$, $u_x(x,0)=0$ and
$E(t) = E(0) = 0$.

$E(t) = 0 \Rightarrow \frac{1}{2}\int_{\mathbb{R}} (u_t^2(x,t) + c^2 u_x^2(x,t))\,dx = 0 \quad \forall t$

$\Rightarrow$ integrand is zero for all x, t
$\Rightarrow$ $u_t(x,t) = u_x(x,t) = 0$
and we see $u(x,t)$ is a constant fn.
i.e. $u(x,t) = K$.

But since $u(x,0) = 0$, we get $K=0$
and $u(x,t) = 0 \quad \forall\ x \in \mathbb{R},\ t \geq 0$.

3p6

6 $u_{tt} - c^2 u_{xx} = 0$, $x \in \mathbb{R}$, $t \geq 0$.

Assume $u(x,t)$ solves the wave eqn.

a) $u(x-y, t)$, y fixed

~~then~~ $\frac{\partial^2}{\partial x^2}(u(x-y,t)) = u_{xx}(x-y,t)$

$\frac{\partial^2}{\partial t^2}(u(x-y,t)) = u_{tt}(x-y,t)$

So $u_{tt}(x-y,t) - c^2 u_{xx}(x-y,t)$

$= u_{tt}(\alpha, t) - c^2 u_{xx}(\alpha, t)$, $\alpha = x-y \in \mathbb{R}$

$= 0$. by assumption that u solves wave eqn.

b) $v = u_x(x,t)$

$v_{xx} = u_{xxx}(x,t)$

$v_{tt} = u_{xtt}(x,t)$

$v_{tt} - c^2 v_{xx} = u_{\cancel{xtt}}(x,t) - c u_{xxx}(x,t)$

$= u_{ttx}(x,t) - c u_{xxx}(x,t)$

by Clairaut's

$= \frac{\partial}{\partial x}\left(u_{tt}(x,t) - c u_{xx}(x,t)\right)$

$= \frac{\partial}{\partial x}(0)$ by assumption.

$= 0$.

c) $w = u(ax, at)$, a fixed

$w_{xx} = a^2 u_{xx}(ax, at)$, $w_{tt} = a^2 u_{tt}(ax, at)$

$$w_{tt} - c^2 w_{xx} = a^2 u_{tt}(ax, at) - c^2 a^2 u_{xx}(ax, at)$$

$$= a^2 \left[u_{tt}(ax, at) - c^2 u_{xx}(ax, at) \right]$$

$$= a^2 \left[u_{tt}(\xi, \tau) - c^2 u_{xx}(\xi, \tau) \right]$$

where $\xi = ax$, $\tau = at$.

$= a^2 [\, 0 \,]$ by assumption on u.