MTH 414 - Spring 2023
Homework #1 - Calculus/ODE Review
Due: Monday, January 23, 2023 (2:00PM)

1. (partial differentiation)

   For the function $f(x, y) = \sin(xy) - x^3 y + xy^4 - 12$

   (a) compute f_x and f_y

   Solution:

   - $f_x = y\cos(xy) - 3x^2 y + y^4$
   - $f_y = x\cos(xy) - x^3 + 4xy^3$

   (b) compute f_{xx}, f_{xy}, f_{yy} and f_{yx}

   Solution:

   - $f_{xx} = -y^2\sin(xy) - 6xy$
   - $f_{yy} = -x^2\sin(xy) + 12xy^2$
   - $f_{xy} = f_{yx} = \cos(xy) - xy\sin(xy) - 3x^2 + 4y^3$

2. (differential operators)

   If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(z) = \cos(z)+z^2$ and $u : \mathbb{R}^2 \to \mathbb{R}$ is defined by $u(x, y) = x^2 y+2x+y^3$, compute

   (a) $\partial_x f(u(x, y))$

   Solution: $f \circ u = \cos\left(x^2 y + 2x + y^3\right) + (x^2 y + 2x + y^3)^2$. Then,

$$
\begin{aligned}
\partial_x (f \circ u) &= \partial_x \left(\cos\left(x^2 y + 2x + y^3\right) + (x^2 y + 2x + y^3)^2\right) \\
&= \frac{\partial}{\partial x}\left(\cos\left(x^2 y + 2x + y^3\right)\right) + \frac{\partial}{\partial x}\left((x^2 y + 2x + y^3)^2\right) \\
&= \left(-\sin\left(x^2 y + 2x + y^3\right) \cdot \frac{\partial}{\partial x}\left(x^2 y + 2x + y^3\right)\right) + \left(2(x^2 y + 2x + y^3) \cdot \frac{\partial}{\partial x}\left(x^2 y + 2x + y^3\right)\right) \\
&= \frac{\partial}{\partial x}\left(x^2 y + 2x + y^3\right)\left(-\sin\left(x^2 y + 2x + y^3\right) + 2(x^2 y + 2x + y^3)\right) \\
&= 2(xy + 1)\left(-\sin\left(x^2 y + 2x + y^3\right) + 2(x^2 y + 2x + y^3)\right)
\end{aligned}
$$

   (b) $\nabla f(u(x, y))$

   Solution: $f \circ u = \cos\left(x^2 y + 2x + y^3\right) + (x^2 y + 2x + y^3)^2$. Then,

$$
\begin{aligned}
\nabla(f \circ u) &= \begin{pmatrix} \partial_x (f \circ u) \\ \partial_y (f \circ u) \end{pmatrix} \\
&= \begin{pmatrix} 2(xy + 1)\left(-\sin\left(x^2 y + 2x + y^3\right) + 2(x^2 y + 2x + y^3)\right) \\ (3y^2 + x^2)\left(-\sin\left(x^2 y + 2x + y^3\right) + 2(x^2 y + 2x + y^3)\right) \end{pmatrix} \\
&= \left(-\sin\left(x^2 y + 2x + y^3\right) + 2(x^2 y + 2x + y^3)\right)\begin{pmatrix} 2(xy + 1) \\ (3y^2 + x^2) \end{pmatrix}
\end{aligned}
$$

3. (the Laplacian)

   Consider the function $f(x, y) = \log(x^2 + y^2)$. Show that $f_{xx}+f_{yy} = 0$ and determine the domain in $\mathbb{R}^2$ for which your calculation is valid

   Solution:

$$
\begin{aligned}
f_x &= 2x \Big/ \left(\ln(10)\cdot(x^2+y^2)\right) \\
f_{xx} &= \frac{\partial}{\partial x}\left(\underbrace{2x}_{f} \Big/ \underbrace{\left(\ln(10)\cdot(x^2+y^2)\right)}_{g}\right) \\
&= \frac{f'g - fg'}{g^2} \\
&= \frac{\left(2\ln(10)(x^2+y^2)\right) - \left(4x^2\ln(10)\right)}{\ln^2(10)(x^2+y^2)^2} \\
&= \frac{2(x^2+y^2) - 4x^2}{\ln(10)(x^2+y^2)^2} \\
&= \frac{-2x^2+2y^2}{\ln(10)(x^2+y^2)^2} \\
f_{yy} &= \frac{2x^2-2y^2}{\ln(10)(x^2+y^2)^2} \\
f_{xx}+f_{yy} &= \frac{(-2x^2+2y^2)+(2x^2-2y^2)}{\ln(10)(x^2+y^2)^2} = 0.
\end{aligned}
$$

   Domain $D := \{(x, y) \mid (x, y) \neq (0, 0)\}$

4. (first-order linear) Let $y(t)$ and $r \in \mathbb{R}, r \neq 0$. Find the general solution to the ODE.

$$ry' + 2y = e^{t^2}$$

   Solution: We would like to have the form $y' + p(t)y = q(t)$. We can rewrite the equation as

$$y' + \underbrace{\frac{2}{r}y}_{p(t)y} = \underbrace{\frac{1}{r}e^{t^2}}_{q(t)}$$

   Then $\mu(t) = \exp\left(\int p(t)dt\right) = \exp\left(\frac{2t}{r}\right)$. Next,

$$
\begin{aligned}
y &= \frac{1}{\mu(t)}\left(\int \mu(t)q(t)\,\mathrm{d}t + C\right) \\
&= \frac{1}{\exp(2t/r)}\left(\int \exp(2t/r)\cdot\frac{\exp\left(e^{t^2}\right)}{r}\,\mathrm{d}t + C\right) \\
&= \frac{\frac{1}{r}\int \exp\left(\frac{2t}{r}+t^2\right)\mathrm{d}t + C}{\exp(2t/r)}
\end{aligned}
$$

5. (initial value problem)

   (a) Given any $\alpha \in \mathbb{R}$, solve the initial value problem

$$y' = y^2 \cos(t), \;\; y(0) = \alpha.$$

   Solution: This is a separable equation.

$$\begin{aligned} y' = \frac{\mathrm{d}y}{\mathrm{d}t} &= y^2 \cos(t) \\ \iff \frac{1}{y^2}\,\mathrm{d}y &= \cos(t)\,\mathrm{d}t \\ \iff -\frac{1}{y} &= \sin(t) + C \\ \iff y(t) &= \frac{1}{-\sin(t) + C} \\ \implies y(0) &= \frac{1}{C} = \alpha \\ \iff C &= \frac{1}{\alpha} \end{aligned}$$

   Hence the general solution is $y = \dfrac{1}{\frac{1}{\alpha} - \sin(t)}$.

   (b) For what values of α is the solution defined for all time? (Hint: You may need to treat $\alpha = 0$ and $\alpha \neq 0$ separately.)

   Solution: The solution is undefined when the denominator, $\frac{1}{\alpha} - \sin(t) = 0$. Rearranging this into $\frac{1}{\alpha} = \sin(t)$, this has an infinite number of solutions when $\frac{1}{\alpha} \in [-1, 1]$. This occurs when $|\alpha| \geq 1$, therefore the solution exists for all time when $|\alpha| < 1$ and $\alpha \neq 0$ (since we cannot divide by zero).

   Thus, a solution exists for all t when $|\alpha| < 1 \wedge \alpha \neq 0$.

6. (second-order constant coefficient)

   Find the general solution to the differential equation

$$y'' + 2y' + 5y = 0$$

   Solution: $D^2 + 2D + 5I = 0$ has roots $D = (-1 \pm 2i)I$.

   Therefore, $y(t) = c_1 e^{-t} \sin(2t) + c_2 e^{-t} \cos(2t)$.

7. (solution spaces)

Consider the second order linear homogeneous ODE of the form

$$y'' + P(x)y' + Q(x)y = 0$$

where P and Q are defined on some interval I in $\mathbb{R}$. Show that the set of all solutions to this ODE forms a vector space. That is, verify each of the following:

(a) $y = 0$ is a solution

Proof. $y = 0 \implies y' = 0 \implies y'' = 0$, hence

$$\begin{aligned} y'' + P(x)y' + Q(x)y &= 0 + 0 \cdot P(x) + 0 \cdot Q(x) && \text{(substitution)} \\ &= 0 + 0 + 0 && \text{(properties of zero)} \\ &= 0 && \text{(additive identity on } \mathbb{R}\text{)} \end{aligned}$$

$\therefore \; y = 0$ is a solution. $\square$

(b) Given any two solutions y_1 and y_2 and scalars $\alpha, \beta \in \mathbb{R}$, the function $\alpha y_1 + \beta y_2$ is a solution of the ODE.

Proof. Let $f := \alpha y_1 + \beta y_2$, then $f' = \alpha y_1' + \beta y_2'$ and $f'' = \alpha y_1'' + \beta y_2''$. Substituting the proposed solution, f, into the differential equation gives

$$\begin{aligned} y'' + P(x)y' + Q(x)y &= (\alpha y_1'' + \beta y_2'') + P(x)(\alpha y_1' + \beta y_2') + Q(x)(\alpha y_1 + \beta y_2) \\ &= \alpha y_1'' + \beta y_2'' + P(x)\alpha y_1' + P(x)\beta y_2' + Q(x)\alpha y_1 + Q(x)\beta y_2 \\ &= \left(\alpha y_1'' + P(x)\alpha y_1' + Q(x)\alpha y_1\right) + \left(\beta y_2'' + P(x)\beta y_2' + Q(x)\beta y_2\right) \\ &= \alpha\left(y_1'' + P(x)y_1' + Q(x)y_1\right) + \beta\left(y_2'' + P(x)y_2' + Q(x)y_2\right) \\ &= 0\alpha + 0\beta && \text{(by hypothesis)} \\ &= 0. \end{aligned}$$

Hence $f := \alpha y_1 + \beta y_2$ is a also solution. $\square$

In fact, you can show (you don't need to do this here, although it might be good to look up), that a vector space of all solutions to the above ODE has dimension 2 and a basis can be found by finding two linearly independent solutions on the interval I.

$$
\begin{aligned}
\Pr\left(x > 10\right) &= 1 - \Pr\left(x \leq 10\right) \\
&= 1 - \sum_{k=0}^{10} \Pr\left(x = k\right) \\
&= 1 - \sum_{k=0}^{10} {}^{n}C_{k} p^{k} q^{n-k} \\
&= 1 - \sum_{k=0}^{10} {}^{35}C_{k} (0.1)^{k} (0.9)^{35-k} \\
&= 1 - \sum_{k=0}^{10} \frac{35!}{k!(35-k)!} (0.1)^{k} (0.9)^{35-k} \\
&= 1 - \frac{4997878512022745871397452694244043}{5000000000000000000000000000000000} \\
&= \frac{2121487977254128602547305755957}{5000000000000000000000000000000000} \\
&\approx 0.00042429759545082572050946115119 14 \\
&\approx 0.0424\%
\end{aligned}
$$