Homework 4

(1) $E(t) = \frac{\rho}{2} \int_{\mathbb{R}} (u_t^2 + c^2 u_x^2)\,dx$

$$E'(t) = \rho \int_{\mathbb{R}} (u_t u_{tt} + c^2 \underbrace{u_x u_{xt}}_{\text{move the deriv. by IBP}})\,dx$$

$$= \rho \int_{\mathbb{R}} u_t u_{tt}\,dx + c^2 \Big[-\int_{\mathbb{R}} u_{xx} u_t\,dx + \underbrace{u_x u_t \Big|_{x=-\infty}^{\infty}}_{=0 \text{ by compact supp.}} \Big]$$

$$= \rho \int_{\mathbb{R}} (u_t u_{tt} - c^2 u_t u_{xx})\,dx$$

$$= \rho \int_{\mathbb{R}} u_t [\underbrace{u_{tt} - c^2 u_{xx}}_{\text{now use PDE}}]\,dx$$

$$= \rho \int_{\mathbb{R}} u_t [-r u_t]\,dx$$

$$= -r\rho \int_{\mathbb{R}} u_t^2\,dx$$

< 0 (provided $r < 0$).

If $r < 0$, energy decreasing.

(If $r > 0$, energy increasing.)

2 a) $u_t = -2k$, $u_x = -2x$, $u_{xx} = -2$.

$u_t = k u_{xx}$ as $-2k = k(-2)$.

b) By max-min principle, the max and mins will occur on the boundary lines

① $t=0$, $-1 \le x \le 1$
② $x=-1$, $0 \le t \le T$
③ $x=1$, $0 \le t \le T$

On ①, $u(x,0) = 1-x^2$
$u_x(x,0) = -2x$, $u_x(x,0)=0$ @ $x=0$.
EVT : $u(-1,0) = u(1,0) = 0$
$u(0,0) = 1$

On ②, $u(-1,t) = -2kt$
$u_t(-1,t) = -2k < 0$ for all t, decreasing.
$u(-1,T) = -2kT$

On ③, same analysis as $u(-1,t) = u(1,t)$.

Absolute max @ $u(0,0) = 1$
Absolute mins @ $u(-1,T) = u(1,T) = -2kT$.

4B

3a) $u(x,0) = 4x - 4x^2$

$u_x(x,0) = 4 - 8x$, $u_x(x,0) = 0$ when $x = 1/2$.

$u(1/2, 0) = 1$ and $u(0,0) = u(1,0) = 0$.

By the strong max. principle, we get

$0 < u(x,t) < 1 \quad \forall\, t > 0,\ 0 < x < 1.$

b) Note $u(x,t)$ solves
$$(*)\ \begin{cases} u_t = ku_{xx}, & x \in [0,1],\ t > 0 \\ u(x,0) = 4x(1-x), & x \in [0,1] \\ u(0,t) = u(1,t) = 0, & t \geq 0 \end{cases}$$

~~Let~~ Set $v(x,t) = u(1-x,t)$

$v_t = u_t(1-x,t)$, $v_{xx} = u_{xx}(1-x,t)$

and v solves the same eqn as

$$v_t - ku_{xx} = u_t(1-x,t) - ku_{xx}(1-x,t) = 0.$$

Converting the IVP (*)
$$(\square)\ \begin{cases} v_t = ku_{xx}, & x \in [0,1],\ t \geq 0 \\ v(x,0) = u(1-x,0) = 4(1-x)x \\ v(0,t) = u(1,0) = 0, & t \geq 0 \\ v(1,t) = u(0,t) = 0, & t \geq 0 \end{cases}$$

This is the same IVP.

By uniqueness, $u(x,t) = v(x,t) \quad \forall\, x \in [0,1],\ t \geq 0.$

3c) $u_t = u_{xx}$

$u u_t = u u_{xx}$ and $\int_0^1 u u_t \, dx = \int_0^1 u u_{xx} \, dx$ $\quad \alpha = u \quad d\beta = u_{xx}$

$\Rightarrow \int_0^1 \partial_t \left(\frac{u^2}{2} \right) dx = \underbrace{u u_x \Big|_{x=0}^{1}}_{=0 \text{ as } u(0,t) = u(1,t) = 0.} - \int_0^1 u_x^2 \, dx$ $\quad d\alpha = u_x dx \quad \beta = u_x$

and $\frac{d}{dt} \int_0^1 \frac{1}{2} u^2 dx = - \int_0^1 u_x^2 dx < 0.$

(this is basically what is in our notes)

4p5

4. $u \le v \Rightarrow v - u \ge 0$.

Set $w = v - u$. Clearly w solve $u_t = k u_{xx}$.
and we have the IVP
$$\begin{cases} w_t = k w_{xx} & \\ w(x,0) \ge 0 & , 0 \le x \le L \\ w(0,t) \ge 0 & t > 0 \\ w(L,t) \ge 0 & \end{cases}$$
By max-min principle, we get $w(x,t) \ge 0$
for all $x \in [0, L]$, $t > 0$.
i.e. $v(x,t) \ge u(x,t)$