Math 414 – Homework 1 **Due Monday, January 23, 2023**

Calculus/ODE Review

1. (partial differentiation)

   For the function $f(x, y) = \sin(xy) - x^3y + xy^4 - 12$:

   (a) compute f_x and f_y,

   (b) compute f_{xx}, f_{xy}, f_{yy} and f_{yx}.

2. (differential operators)

   If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(z) = \cos(z) + z^2$ and $u : \mathbb{R}^2 \to \mathbb{R}$ is defined by $u(x, y) = x^2y + 2x + y^3$, compute

   (a) $\partial_x f(u(x, y))$,

   (b) $\nabla f(u(x, y))$.

3. (the Laplacian)

   Consider the function $f(x, y) = \log(x^2 + y^2)$. Show that $f_{xx} + f_{yy} = 0$ and determine the domain in $\mathbb{R}^2$ for which your calculation is valid.

4. (first-order linear) Let $y(t)$ and $r \in \mathbb{R}$, $r \neq 0$. Find the general solution to the ODE

$$ry' + 2y = e^{t^2}.$$

5. (initial value problem)

   (a) Given any $\alpha \in \mathbb{R}$, solve the initial value problem

$$y' = y^2 \cos(t), y(0) = \alpha.$$

   (b) For what values of α is the solution defined for all time? (Hint: You may need to treat $\alpha = 0$ and $\alpha \neq 0$ separately.)

6. (second-order constant coefficient)

   Find the general solution to the differential equation

$$y'' + 2y' + 5y = 0$$

7. (solution spaces)

Consider the second order linear homogeneous ODE of the form

$$y'' + P(x)y' + Q(x)y = 0$$

where P and Q are defined on some interval I in $\mathbb{R}$. Show that the set of all solutions to this ODE forms a vector space. That is, verify each of the following:

(a) $y = 0$ is a solution

(b) Given any two solutions y_1 and y_2 and scalars $\alpha, \beta \in \mathbb{R}$, the function $\alpha y_1 + \beta y_2$ is also a solution of the ODE.

In fact, you can show (you don't need to do this here, although it might be good to look up) that the vector space of all solutions to the above ODE has dimension 2 and a basis can be found by finding two linearly independent solutions on the interval I.