MTH 414 - Spring 2023
Assignment #6
Due: Wednesday, 3 29 2023 (2:00 PM)

The following formulas are useful:

$$\int_a^b x\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \left[\frac{Lx}{n\pi}\sin\left(\frac{n\pi x}{L}\right)+\frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)\right]_{x=a}^{x=b} \tag{MW-1}$$

$$\int_a^b x\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \left[\frac{-Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right)+\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\right]_{x=a}^{x=b} \tag{MW-2}$$

$$\int_a^b x^2\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \left[\frac{Lx^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right)+\frac{2L^2x}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)-\frac{2L^3}{n^3\pi^3}\sin\left(\frac{n\pi x}{L}\right)\right]_{x=a}^{x=b} \tag{MW-3}$$

$$\int_a^b x^2\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \left[\frac{-Lx^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)+\frac{2L^2x}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)+\frac{2L^3}{n^3\pi^3}\cos\left(\frac{n\pi x}{L}\right)\right]_{x=a}^{x=b} \tag{MW-4}$$

Proof. (MW-1)

Let $u = x$, $\mathrm{d}v = \cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x$, $\mathrm{d}u = \mathrm{d}x$, $v = \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$. Then by IBP,

$$\begin{aligned}\int_a^b x\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x &= \frac{Lx}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_a^b - \int_a^b \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\ &= \frac{Lx}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_a^b + \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_a^b \\ &= \left[\frac{Lx}{n\pi}\sin\left(\frac{n\pi x}{L}\right)+\frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)\right]_{x=a}^{x=b}\end{aligned}$$

□

Proof. (MW-2) This follows from some sign changes in MW-1. □

Proof. (MW-3)

Let $u = x^2$, $\mathrm{d}v = \cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x$, $\mathrm{d}u = 2x\mathrm{d}x$, $v = \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$. Then by IBP,

$$\begin{aligned}\int_a^b x^2\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x &= \frac{Lx^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_a^b - \frac{2L}{n\pi}\underbrace{\int_a^b x\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x}_{MW-2} \\ &= \frac{Lx^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_a^b - \frac{2L}{n\pi}\left[\frac{-Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right)+\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\right]_{x=a}^{x=b} \\ &= \left[\frac{Lx^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right)+\frac{2L^2x}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)-\frac{2L^3}{n^3\pi^3}\sin\left(\frac{n\pi x}{L}\right)\right]_{x=a}^{x=b}\end{aligned}$$

□

Proof. (MW-4)

Let $u = x^2$, $dv = \sin\left(\frac{n\pi x}{L}\right) dx$, $du = 2x dx$, $v = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$. Then by IBP,

$$\int_a^b x^2 \cos\left(\frac{n\pi x}{L}\right) dx = \frac{-Lx^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_a^b + \frac{2L}{n\pi}\underbrace{\int_a^b x\cos\left(\frac{n\pi x}{L}\right) dx}_{MW-1}$$

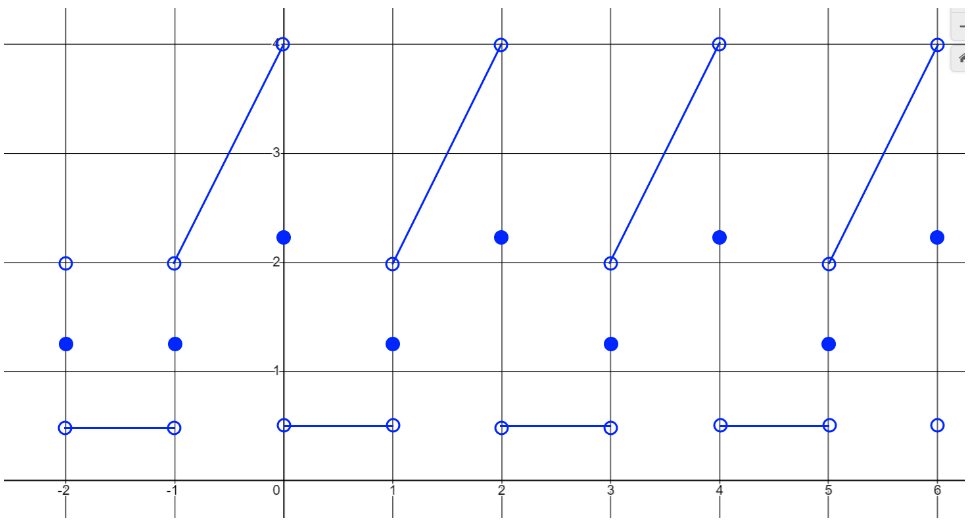
$$= \frac{-Lx^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_a^b + \frac{2L}{n\pi}\left[\frac{Lx}{n\pi}\sin\left(\frac{n\pi x}{L}\right) + \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)\right]_{x=a}^{x=b}$$

$$= \left[\frac{-Lx^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{2L^2x}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right) + \frac{2L^3}{n^3\pi^3}\cos\left(\frac{n\pi x}{L}\right)\right]_{x=a}^{x=b}$$

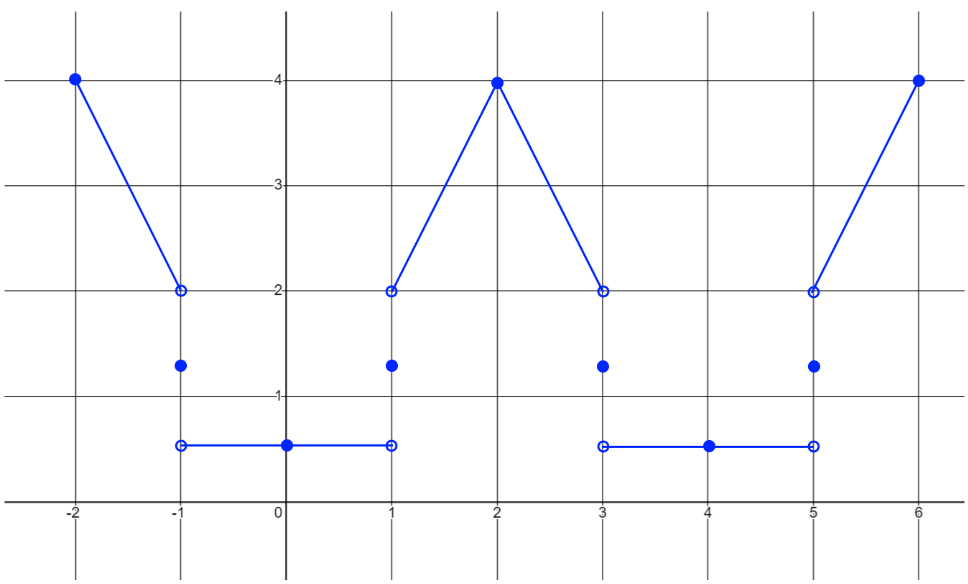
$\square$

1. Let $f(x) = \begin{cases} 1/2 & \text{if } x \in [0,1] \\ 2x & \text{if } x \in (1,2] \end{cases}$

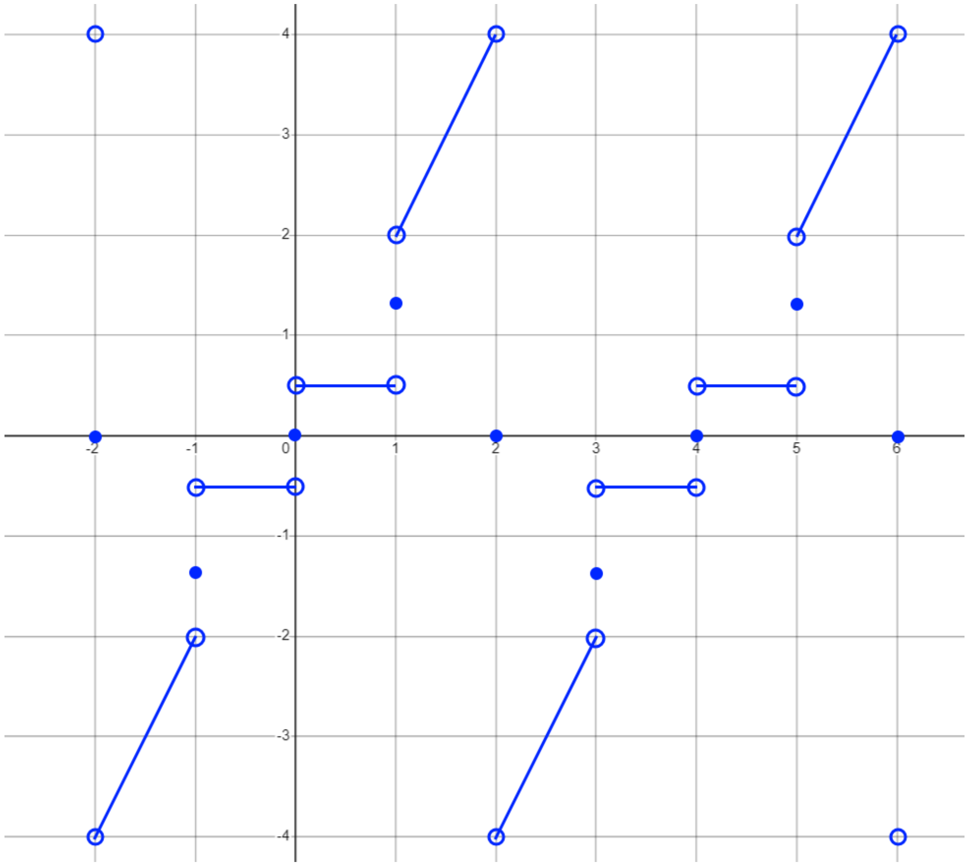
   (a) Sketch the graph of the Fourier series expansion of $f(x)$ over $[-2, 6]$.



   (b) Sketch the graph of the Fourier cosine series expansion of $f(x)$ over $[-2, 6]$.



(c) Sketch the graph of the Fourier sine series expansion of $f(x)$ over $[-2, 6]$.



(d) Compute the cosine series expansion of $f(x)$.

Solution: Goal: $\widehat{f}(x) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right)$. Here $L = 2$.

$$\begin{aligned} c_0 &= \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^2 f(x)\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^1 \frac{1}{2}\,\mathrm{d}x + \frac{1}{2}\int_1^2 2x\,\mathrm{d}x \\ &= \frac{7}{4} \end{aligned}$$

Next,

$$\begin{aligned}
c_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) \\
&= \int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right) \\
&= \underbrace{\frac{1}{2}\int_0^1 \cos\left(\frac{n\pi x}{2}\right)}_{*} + \underbrace{2\int_1^2 x\cos\left(\frac{n\pi x}{2}\right)}_{**}
\end{aligned}$$

$$* = \frac{1}{2}\cdot\frac{2}{n\pi}\left[\sin\left(\frac{n\pi x}{2}\right)\right]_0^1 = \frac{1}{n\pi}\left(\sin\left(\frac{n\pi}{2}\right) - \sin(0)\right) = \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$\begin{aligned}
** &= 2\int_1^2 x\cos\left(\frac{n\pi x}{2}\right)\mathrm{d}x \\
&\text{with } u = x,\quad \mathrm{d}v = \cos\left(\frac{n\pi x}{2}\right)\mathrm{d}x,\quad \mathrm{d}u = \mathrm{d}x,\quad v = \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right) \\
&= 2\left([uv]_1^2 - \int_1^2 v\,\mathrm{d}u\right) \\
&= 2\left(\left[\frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]_1^2 - \int_1^2 \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\mathrm{d}x\right) \\
&= \frac{4}{n\pi}\left(\left[x\sin\left(\frac{n\pi x}{2}\right)\right]_1^2 - \int_1^2 \sin\left(\frac{n\pi x}{2}\right)\mathrm{d}x\right) \\
&= \frac{4}{n\pi}\left(\left[x\sin\left(\frac{n\pi x}{2}\right)\right]_1^2 + \left[\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_1^2\right) \\
&= \frac{4}{n\pi}\left(\left[x\sin\left(\frac{n\pi x}{2}\right) + \frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_1^2\right) \\
&= \frac{4}{n\pi}\left(\left(0 + \frac{2}{n\pi}\cos(n\pi)\right) - \left(\sin\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right)\right) \\
&= \frac{4}{n\pi}\left(\frac{2}{n\pi}\cos(n\pi) - \sin\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right) \\
&= \frac{8}{n^2\pi^2}\left(\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right) - \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}$$

Therefore

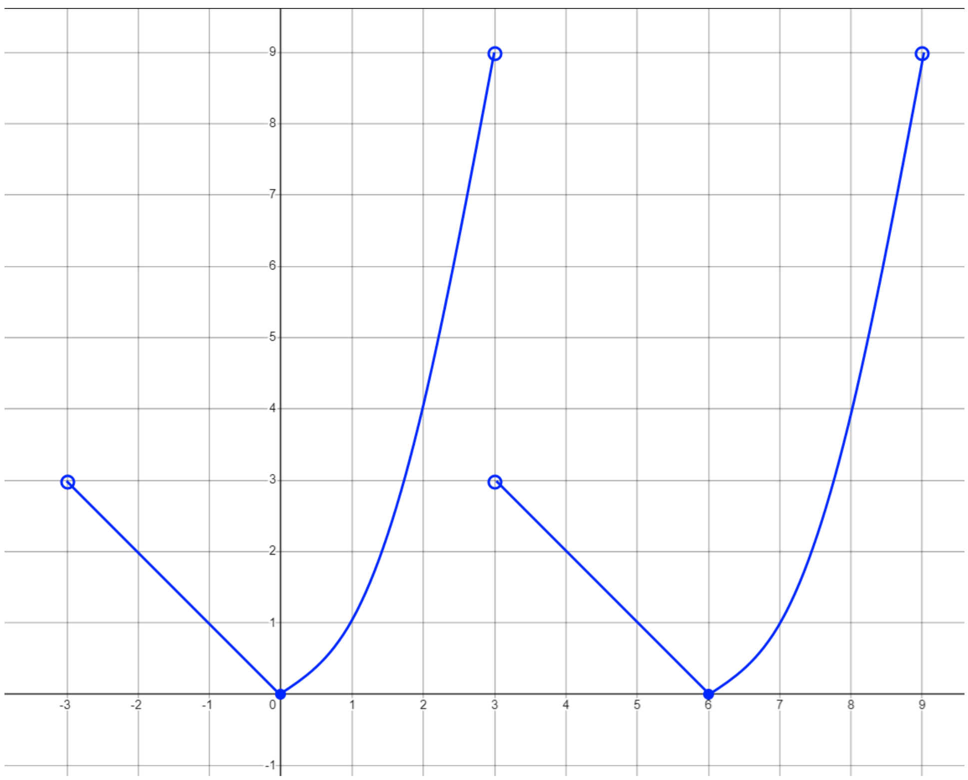
$$\begin{aligned}
c_n &= * + ** \\
&= \frac{8}{n^2\pi^2}\left(\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right) - \frac{3}{n\pi}\sin\left(\frac{n\pi}{2}\right) \\
&= \begin{cases} \frac{8}{n^2\pi^2}\cos(n\pi) - \frac{3}{n\pi}\sin\left(\frac{n\pi}{2}\right) & n \text{ odd} \\ \frac{8}{n^2\pi^2}\left(\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right) & n \text{ even} \end{cases}
\end{aligned}$$

Thus

$$\begin{aligned}\widehat{f}(x) &= c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) \\ &= \frac{7}{4} + \sum_{n=1}^{\infty} \left[\cos\left(\frac{n\pi x}{L}\right)\left(\frac{8}{n^2\pi^2}\left(\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right) - \frac{3}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)\right]\end{aligned}$$

2. Let $f(t) = \begin{cases} -t & \text{if } t \in (-3, 0) \\ t^2 & \text{if } t \in [0, 3) \end{cases} \qquad f(t+6) = f(t),\ t \in \mathbb{R}$

(a) Graph two periods of $f(t)$



(b) Find the Fourier Series of f.

Solution: Goal: $\widehat{f}(t) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$. Here, since the period is length 6, $6 = 2L \implies L = 3$.

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} f(t)\mathrm{d}t \\
&= \frac{1}{3}\int_{-3}^{3} f(t)\mathrm{d}t \\
&= \frac{1}{3}\left(\int_{-3}^{0} -t\mathrm{d}t + \int_{0}^{3} t^2\mathrm{d}t\right) \\
&= \frac{1}{3}\left(\left[\frac{-t^2}{2}\right]_{-3}^{0} + \left[\frac{t^3}{3}\right]_{0}^{3}\right) \\
&= \frac{1}{3}\left(0 + \frac{9}{2} + \frac{27}{3} - 0\right) \\
&= \frac{9}{2}
\end{aligned}$$

For the odd series coefficients...

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)\mathrm{d}t \\ &= \frac{1}{3}\int_{-3}^{3} f(t)\cos\left(\frac{n\pi t}{3}\right)\mathrm{d}t \\ &= \frac{1}{3}\left(\underbrace{\int_{-3}^{0} -t\cos\left(\frac{n\pi t}{3}\right)\mathrm{d}t}_{*} + \underbrace{\int_{0}^{3} t^2\cos\left(\frac{n\pi t}{3}\right)\mathrm{d}t}_{**}\right) \end{aligned}$$

$$\begin{aligned} * &= -\int_{-3}^{0} t\cos\left(\frac{n\pi t}{3}\right)\mathrm{d}t \\ &= -\left[\frac{3t}{n\pi}\sin\left(\frac{n\pi t}{3}\right) + \frac{9}{n^2\pi^2}\cos\left(\frac{n\pi t}{3}\right)\right]_{t=-3}^{t=0} && \text{(by MW-1)} \\ &= -\left[\left(0 + \frac{9}{n^2\pi^2}\cos(0)\right) - \left(\frac{-9}{n\pi}\sin(-n\pi) + \frac{9}{n^2\pi^2}\cos(-n\pi)\right)\right] \\ &= -\left[\frac{9}{n^2\pi^2} - \left(\frac{9}{n\pi}\sin(n\pi) + \frac{9}{n^2\pi^2}\cos(n\pi)\right)\right] && \text{(even and odd props)} \\ &= -\left[\frac{9}{n^2\pi^2} - \frac{9}{n^2\pi^2}\cos(n\pi)\right] && (\sin n\pi = 0) \\ &= \frac{9}{n^2\pi^2}\left(\cos(n\pi) - 1\right) \end{aligned}$$

$$\begin{aligned} ** &= \int_{0}^{3} t^2\cos\left(\frac{n\pi t}{3}\right)\mathrm{d}t \\ &= \left[\frac{3t^2}{n\pi}\sin\left(\frac{n\pi t}{3}\right) + \frac{18t}{n^2\pi^2}\cos\left(\frac{n\pi t}{3}\right) - \frac{54}{n^3\pi^3}\sin\left(\frac{n\pi t}{3}\right)\right]_{t=0}^{t=3} && \text{(by MW-3)} \\ &= \left(\frac{27}{n\pi}\sin(n\pi) + \frac{54}{n^2\pi^2}\cos(n\pi) - \frac{54}{n^3\pi^3}\sin(n\pi)\right) \\ &\quad - \left(\frac{0}{n\pi}\sin(0) + \frac{0}{n^2\pi^2}\cos(0) - \frac{54}{n^3\pi^3}\sin(0)\right) \\ &= \frac{54}{n^2\pi^2}\cos(n\pi) && (\sin n\pi = 0) \end{aligned}$$

Thus

$$\begin{aligned} a_n &= \frac{1}{3}\left(* + **\right) \\ &= \frac{1}{3}\left(\frac{9}{n^2\pi^2}\left(1 - \cos(n\pi)\right) + \frac{54}{n^2\pi^2}\cos(n\pi)\right) \\ &= \frac{27\cos(n\pi) - 3}{n^2\pi^2} \end{aligned}$$

Next for the odd series,

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)\mathrm{d}t \\ &= \frac{1}{3}\Bigg(-\underbrace{\int_{-3}^{0} t\sin\left(\frac{n\pi t}{3}\right)\mathrm{d}t}_{*} + \underbrace{\int_{0}^{3} t^2\sin\left(\frac{n\pi t}{3}\right)\mathrm{d}t}_{**}\Bigg) \end{aligned}$$

First,

$$\begin{aligned} * &= \int_{-3}^{0} t\sin\left(\frac{n\pi t}{3}\right)\mathrm{d}t \\ &= \left[\frac{-3t}{n\pi}\cos\left(\frac{n\pi t}{3}\right) + \frac{9}{n^2\pi^2}\sin\left(\frac{n\pi t}{3}\right)\right]_{t=-3}^{t=0} && \text{(by MW-2)} \\ &= \left(\frac{0}{n\pi}\cos\left(\frac{n\pi t}{3}\right) + \frac{9}{n^2\pi^2}\sin(0)\right) - \left(\frac{9}{n\pi}\cos(-n\pi) + \frac{9}{n^2\pi^2}\sin(-n\pi)\right) \\ &= -\frac{9}{n\pi}\cos(n\pi) && \text{(by cosine evenness and } \sin n\pi = 0\text{)} \end{aligned}$$

Then

$$\begin{aligned} ** &= \int_{0}^{3} t^2\sin\left(\frac{n\pi t}{3}\right)\mathrm{d}t \\ &= \left[\frac{-3t^2}{n\pi}\cos\left(\frac{n\pi t}{3}\right) + \frac{18t}{n^2\pi^2}\sin\left(\frac{n\pi t}{3}\right) + \frac{54}{n^3\pi^3}\cos\left(\frac{n\pi t}{3}\right)\right]_{t=0}^{t=3} && \text{(by MW-4)} \\ &= \left(\frac{-27}{n\pi}\cos(n\pi) + 0 + \frac{54}{n^3\pi^3}\cos(n\pi)\right) - \left(0 + 0 + \frac{54}{n^3\pi^3}\cos(0)\right) \\ &= \frac{-27}{n\pi}\cos(n\pi) + \frac{54}{n^3\pi^3}(\cos(n\pi) - 1) \end{aligned}$$

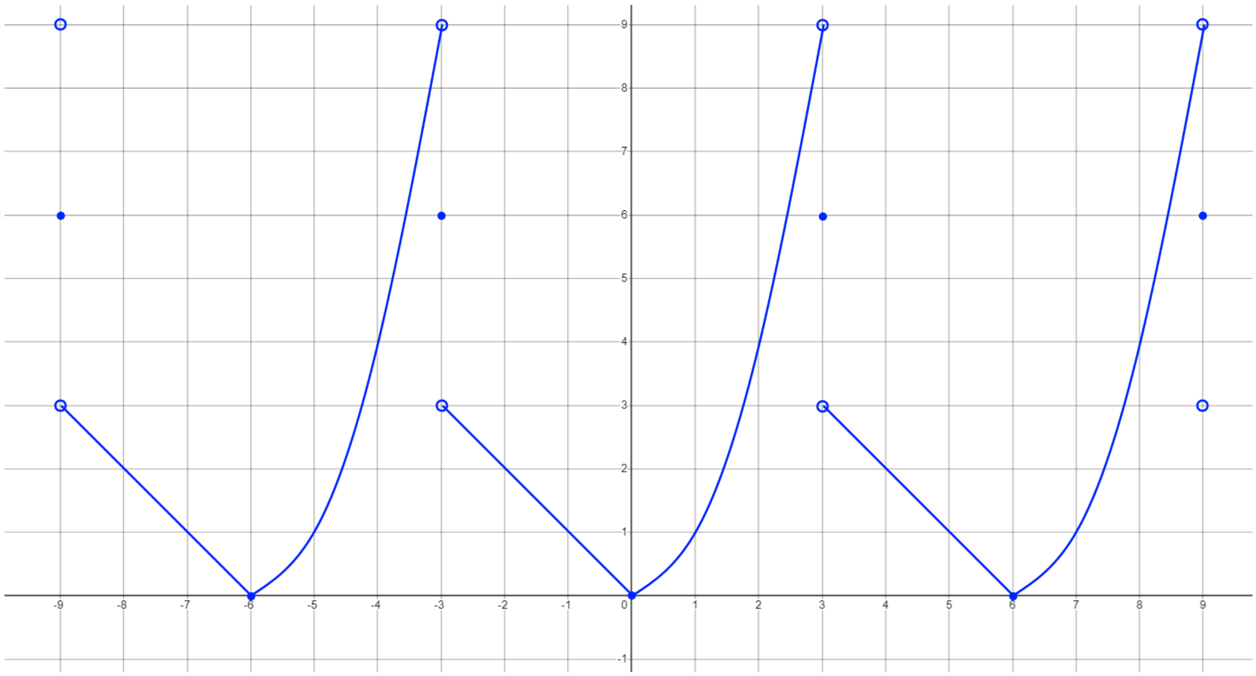
Thus

$$\begin{aligned} b_n &= \frac{1}{3}(** - *) \\ &= \frac{54}{n^3\pi^3}(\cos(n\pi) - 1) - \frac{18}{n\pi}\cos(n\pi) \end{aligned}$$

Finally, substituting all of this into $\widehat{f}(t) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$ yields

$$\begin{aligned} \widehat{f}(t) = \frac{9}{4} &+ \sum_{n=0}^{\infty}\left(\cos\left(\frac{n\pi t}{L}\right)\left(\frac{27\cos(n\pi) - 3}{n^2\pi^2}\right)\right) \\ &+ \sum_{n=1}^{\infty}\left(\sin\left(\frac{n\pi t}{L}\right)\left(\frac{54}{n^3\pi^3}(\cos(n\pi) - 1) - \frac{18}{n\pi}\cos(n\pi)\right)\right) \end{aligned}$$

(c) Use the Fourier Convergence Theorem and sketch a the graph of the function to which the Fourier series in (a) converges for $-9 \leq t \leq 9$.



3. As we did in class with the wave equation, you will construct the general form of solution to the heat equation with Neumann boundary conditions.

Consider the IBVP $\begin{cases} u_t(x,t) = u_{xx}(x,t) & \text{if } x \in (0,L) \text{ and } t \in \mathbb{R} \\ u'(0,t) = u'(L,t) = 0 & \text{if } t \in [0,\infty) \\ u(x,0) = f(x) & \text{if } x \in [0,L] \end{cases}$

(a) Interpret the initial conditions $u'(0,t) = u'(L,t) = 0$ physically.

Solution:
The ends/boundaries are insulated (heat does not enter nor leave the system).

(b) Look for a nontrivial solution to the PDE of the form $u(x,t) = X(x)T(t)$ as follows.

i. Substitute into the PDE u_t and u_{xx} and algebraically manipulate it into the form $T'/T = X''/X$.

Solution:

$$u_t = XT' \qquad u_{xx} = X''T$$

Substituting into the heat equation $u_t = ku_{xx}$, we get $XT' = kX''T$. Then

$$\begin{aligned} XT' = kX''T &\iff T' = \frac{kX''T}{X} \\ &\iff \frac{T'}{T} = k\frac{X''}{X} \end{aligned}$$

ii. Set $\lambda(x,t) = \dfrac{T'(t)}{T(t)} = \dfrac{X''(x)}{X(x)}$. Explain why $\lambda(x,t)$ must be a constant.

Solution:

$$\frac{\partial\lambda}{\partial x} = \left(\frac{T'(t)}{T(t)}\right)_x = 0 \text{ and } \frac{\partial\lambda}{\partial t} = \left(\frac{X''(x)}{X(x)}\right)_t = 0$$

Hence $\nabla\lambda = 0 \ \forall\, x, t$ and is thus a constant.

iii. Using $\lambda = \dfrac{T'(t)}{T(t)} = \dfrac{X''(x)}{X(x)}$, "separate" the PDE into two ODEs

$$X'' = \lambda X \tag{1}$$
$$T' = \lambda T \tag{2}$$

Solution:

(1)

$$\lambda = \frac{X''}{X} \iff \lambda X = X''$$

(2)

$$\lambda = \frac{T'}{T} \iff \lambda T = T'$$

iv. To solve the second-order ODE in the spacial coordinate, we need to interpret the initial condition $u'(0,t) = u'(L,t) = 0$. Do so. Then find and solve the associated eigenvalue problem.

Solution: $u'(0,t) = u'(L,t) = 0$ means via substitution that $X'(0) = X'(L) = 0$ and we can solve an eigenvalue problem $X'' + \lambda X = 0$, hence $\mathcal{L} = D^2 - \lambda$.

Case $\lambda = 0$:
Yields $D^2 = 0$ and thus $X(x) = A + Bx$ and $X' = B$.

Substituting in boundary conditions, $X'(0) = B = 0$. Thus $X = A$, and since $X' = 0$ then A is a free variable. $A = 1$ is a nice number so $X_0(x) = A = 1$ is an associated eigenfunction.

Case $\lambda < 0$:
Yields $D^2 - |\lambda| = 0$ gives

$$X(x) = A\sin\left(\sqrt{|\lambda|}x\right) + B\cos\left(\sqrt{|\lambda|}x\right)$$
$$X'(x) = \sqrt{|\lambda|}A\cos\left(\sqrt{|\lambda|}x\right) - \sqrt{|\lambda|}B\sin\left(\sqrt{|\lambda|}x\right)$$

$X'(0) = 0 \iff \sqrt{|\lambda|}A\cos\left(\sqrt{|\lambda|}0\right) = \sqrt{|\lambda|}B\sin\left(\sqrt{|\lambda|}0\right) \implies |\lambda|A = 0 \implies A = 0$

$X'(L) = 0 \iff -\sqrt{|\lambda|}B\sin\left(\sqrt{|\lambda|}L\right) = 0 \implies \sqrt{|\lambda|}L = n\pi \implies \lambda_n = \left(\frac{n\pi}{L}\right)^2$

Thus the associated eigenfunction is $X_n(x) = \cos\left(\frac{n\pi x}{L}\right)$.

Case $\lambda > 0$:
Yields $D^2 + |\lambda| = 0$ giving

$$X(x) = A\sinh\left(\sqrt{|\lambda|}x\right) + B\cosh\left(\sqrt{|\lambda|}x\right)$$
$$X'(x) = \sqrt{|\lambda|}A\cosh\left(\sqrt{|\lambda|}x\right) + \sqrt{|\lambda|}B\sinh\left(\sqrt{|\lambda|}x\right)$$

$X'(0) = 0 \iff \sqrt{|\lambda|}A\cosh(0) + 0 = \sqrt{|\lambda|}A \iff A = 0$

$$X(x) = B\cosh\left(\sqrt{|\lambda|}x\right)$$

$X'(L) = 0 \iff \sqrt{|\lambda|}B\sinh\left(\sqrt{|\lambda|}L\right) = 0 \implies B = 0$

Hence the trivial solution $X(x) = 0$

v. Use your eigenvalues from the last part and solve the first-order ODE(s) in t.

Solution:

$$\lambda T = T' \iff T' - \lambda T = 0 = T' - \left(\frac{n\pi}{L}\right)^2 \implies D - k\lambda_n = 0$$

Hence $T_n(t) = \exp\left\{\left(\frac{n\pi}{L}\right)^2 kt\right\}$

vi. Use your previous work to find the infinite set of solutions to the problem $u_t(x,t) = u_{xx}(x,t), \qquad u'(0,t) = u'(L,t) = 0$

Solution:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \exp\left\{\left(\frac{n\pi}{L}\right)^2 kt\right\} \cos\left(\frac{n\pi x}{L}\right)$$

vii. We still haven't satisfied the final initial condition $u(x,0) = f(x)$. How do we do so?

Solution: This will by satisfied by solving for the coefficients a_0 and a_n in the Fourier cosine series. I.e.

$$a_0 = \frac{1}{L}\int_0^L f(x)\mathrm{d}x \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

4. Solve the IBVP $\begin{cases} 5u_t(x,t) = u_{xx}(x,t) & \text{if } x \in (0,10) \text{ and } t \in \mathbb{R} \\ u'(0,t) = u'(10,t) = 0 & \text{if } t \in [0,\infty) \\ u(x,0) = f(x) = 4x & \text{if } x \in [0,10] \end{cases}$

Note here that $u_t = ku_{xx}$ implies $k = \frac{1}{5}$.

$$a_0 = \frac{1}{10}\int_0^{10} 4x\mathrm{d}x = \frac{1}{5}\left[x^2\right]_0^{10} = 20$$

$$\begin{aligned} a_n &= \frac{4}{5}\int_0^{10} x\cos\left(\frac{n\pi x}{10}\right)\mathrm{d}x \\ &= \frac{4}{5}\left[\frac{10x}{n\pi}\sin\left(\frac{n\pi x}{10}\right) + \frac{100}{n^2\pi^2}\cos\left(\frac{n\pi x}{10}\right)\right]_{x=0}^{x=10} \\ &= \frac{4}{5}\left(\left(0 + \frac{100}{n^2\pi^2}\cos(n\pi)\right) - \left(0 + \frac{100}{n^2\pi^2}\right)\right) \\ &= \frac{80}{n^2\pi^2}(\cos(n\pi) - 1) \end{aligned} \tag{MW-1}$$

Therefore

$$u(x,t) = 20 + \sum_{n=1}^{\infty}\left(\frac{80}{n^2\pi^2}(\cos(n\pi) - 1)\exp\left\{\left(\frac{n\pi}{10}\right)^2 \frac{t}{5}\right\}\cos\left(\frac{n\pi x}{10}\right)\right)$$

5. Consider the IBVP $\begin{cases} u_{tt}(x,t) = 100u_{xx}(x,t) & \text{if } x \in (0,\pi) \text{ and } t \in \mathbb{R} \\ u(0,t) = u(\pi,t) = 0 & \text{if } t \in [0,\infty) \\ u(x,0) = x\pi - x^2 & \text{if } x \in [0,\pi] \\ u_t(x,0) = 0 & \text{if } x \in [0,\pi] \end{cases}$

   Give a physical interpretation to the initial conditions of this problem and then solve the IBVP.

   Solution: It is a string of length π fixed at its ends. There is no initial velocity (yet to be released) and an initial displacement of $f(x) = \pi x - x^2$. The string has a wavespeed of 10 units.

   Separating into $u(x,t) = X(x)T(t)$ then $u_{tt} = XT''$ and $u_{xx} = X''T$. Substituting into the PDE,

$$-\lambda = \frac{T''}{T} = c^2\frac{X''}{X}$$

   yielding the ODEs

$$c^2X'' + \lambda X = 0 \qquad \text{and} \qquad T'' + \lambda T = 0 \qquad \text{with} \qquad X(0) = X(L) = 0$$

   Case $\lambda < 0$**:** $X(x) = A\cosh\left(\sqrt{|\lambda|}x\right) + B\sinh\left(\sqrt{|\lambda|}x\right) = 0$ implies $A = B = 0$, trivial solution.

   Case $\lambda = 0$**:** $X(x) = A + Bx = 0$ at $X(0) = A = 0$ implies $A = 0$, so $X(x) = Bx$, but $X(L) = B \cdot L = 0$ implies $B = 0$. Trivial solution.

   Case $\lambda > 0$**:** $X(x) = A\cos\left(\sqrt{|\lambda|}x\right) + B\sin\left(\sqrt{|\lambda|}x\right)$ at $X(0) = A = 0$ implies $A = 0$. So $X(x) = B\sin\left(\sqrt{|\lambda|}x\right)$. At $X(L) = B\sin\left(\sqrt{|\lambda|}L\right) = 0$ implies $\sqrt{|\lambda|}L = n\pi$ and hence $\lambda_n = \left(\frac{n\pi}{L}\right)^2$. Therefore the associated eigenfunction is

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

   Solving the ODE in T,

$$T(t) = A\cos\left(\sqrt{|\lambda|}ct\right) + B\sin\left(\sqrt{|\lambda|}ct\right).$$

   Then

$$u_n(x,t) = T(t)\sin\left(\frac{n\pi x}{L}\right) = \left(A\cos\left(\sqrt{|\lambda|}ct\right) + B\sin\left(\sqrt{|\lambda|}ct\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

   and

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{cn\pi t}{L}\right) + b_n\sin\left(\frac{cn\pi t}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right).$$

   Where the initial conditions are in the coefficients,

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \qquad \text{and} \qquad b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)$$

Now to solve the specific problem,

$$\begin{aligned}
a_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= \frac{2}{L}\int_0^L (\pi x - x^2)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= \frac{2}{L}\left(\pi\left(\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x\right) - \left(\int_0^L x^2\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x\right)\right) \\
&= \frac{2\pi}{L}\left(\left[\frac{-Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\right]_{x=0}^{x=L}\right) && \text{(MW-2)} \\
&\quad - \frac{2}{L}\left(\left[\frac{-Lx^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{2L^2x}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right) + \frac{2L^3}{n^3\pi^3}\cos\left(\frac{n\pi x}{L}\right)\right]_{x=0}^{x=L}\right) && \text{(MW-4)} \\
&= \frac{-2L\cos(n\pi)}{n} + \left(\frac{2L^2}{n\pi} - \frac{4L^2}{n^3\pi^3}(\cos(n\pi)+1)\right) \\
&= \frac{2L^2 - 2\pi L\cos(n\pi)}{n\pi} - \frac{4L^2}{n^3\pi^3}(\cos(n\pi)+1)
\end{aligned}$$

In this particular IC, $L = \pi$, so

$$\begin{aligned}
\frac{2L^2 - 2\pi L\cos(n\pi)}{n\pi} - \frac{4L^2}{n^3\pi^3}(\cos(n\pi)+1) &= \frac{2\pi^2 - 2\pi\cdot\pi\cos(n\pi)}{n\pi} - \frac{4\pi^2}{n^3\pi^3}(\cos(n\pi)+1) \\
&= -\frac{4}{n^3\pi}(\cos n\pi + 1)
\end{aligned}$$

Since $g(x) \equiv 0$, so too is b_n. So the specific solution is

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{cn\pi t}{L}\right)\right) \\
&= \sum_{n=1}^{\infty}\left(-\frac{4}{n^3\pi}(\cos n\pi + 1)\cos(10nt)\right)
\end{aligned}$$

6. (an eigenvalue problem) Consider the regular Sturm-Liouville problem

$$y'' - \lambda y = 0 \quad (0 < x < L)$$

$$y(0) = 0, \quad hy(L) - y'(L) = 0 \quad \text{(Robin BC)}$$

where $h > 0$.

(a) Show that $\lambda_0 = 0$ is an eigenvalue of the problem if and only if $hL = 1$, in which case the associated eigenfunction is $y_0(x) = x$.

Proof. ($\Longrightarrow$)

Suppose $\lambda_0 = 0$ is an eigenvalue. Then $X(x) = A + Bx$. By the first boundary condition, $y(0) = A = 0$, so $y(x) = Bx$. Then the second boundary condition, after computing $y'(x) = B$, gives $hy(L) - y'(L) = 0 = hBL - B = B(hL - 1)$. Since $B \neq 0$ to avoid the trivial solution, $hL - 1 = 0$ which implies $hL = 1$. □

Proof. ($\Longleftarrow$)

Suppose $hL = 1$. □

(b) Show that the problem has a single negative eigenvalue λ_0 if and only if $hL > 1$, in which case $\lambda_0 = -\beta_0^2/L^2$ and $y_0 = \sinh \beta_0 x/L$, where β_0 is the positive root of the equation $\tanh x = x/hL$. (Suggestion: Sketch the graphs of $y = \tanh$ and $y = x/hL$)

Proof. ($\Longrightarrow$)

$\square$

Proof. ($\Longleftarrow$)

$\square$

(c) Show that the positive eigenvalues and associated eigenfunctions of the problem are $\lambda_n = \beta_n^2/L^2$ and $y_n(x) = \sin \beta_n x/L (n \geq 1)$, where β_n is the nth positive root of $\tan x = x/hL$

Solution: For positive eigenvalues, $D^2 + \lambda$ implies $y(x) = A \sin\left(\sqrt{|\lambda|}x\right) + B \cos\left(\sqrt{|\lambda|}x\right)$. At the boundary conditions, $y(0) = 0 \implies B = 0$, so $y(x) = A \sin\left(\sqrt{|\lambda|}x\right)$

At the Robin BC, $hy(L) = y'(L)$ thus

$$h \sin\left(\sqrt{|\lambda|}L\right) = \sqrt{|\lambda|} \cos\left(\sqrt{|\lambda|}L\right) \quad \iff \quad \frac{\sqrt{|\lambda|}}{h} = \tan\left(\sqrt{|\lambda|}L\right).$$

Let $x := \sqrt{|\lambda|}L$, then $\sqrt{|\lambda|} = \dfrac{x}{L}$. Substituting into the above yields

$$\tan(x) = \frac{x}{hL}$$

(d) Suppose that $hL = 1$ and that $f(x)$ is piecewise smooth. Show that

$$f(x) = c_0 x + \sum_{n=1}^{\infty} c_n \sin\left(\frac{\beta_n x}{L}\right),$$

where $\{\beta_b\}_1^\infty$ are the positive roots of $\tan x = x$, and

$$c_0 = \frac{3}{L^3}\int_0^L x f(x)\mathrm{d}x$$

$$c_n = \frac{2\beta_n}{L(\beta_n - \cos(\beta_n)\sin(\beta_n))}\int_0^L f(x)\sin\left(\frac{\beta_n x}{L}\right)$$

Solution: From part (a), $hL = 1 \iff \lambda_0 = 0 \implies y = A + Bx$. This gives an infinite family of solutions

$$y = c_0 y_0 + \sum_{n=1}^{\infty}(A_n + B_n x).$$

Using the initial conditions,

$$\begin{aligned} y(0) &= 0 \\ &= 0 + \sum_{n=1}^{\infty}(A_n + B_n \cdot 0) \\ &= \sum_{n=1}^{\infty} A_n \end{aligned}$$

In order to have the same form as a sine series, $A_n = c_n \sin\left(\frac{n\pi x}{L}\right)$, thus

$$y(x) = c_0 x + \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right).$$

Then $c_0 = \dfrac{\langle f(x), x\rangle}{\langle x, x\rangle}$.

$$\langle f(x), x\rangle = \int_0^L x f(x)\mathrm{d}x \qquad \langle x, x\rangle = \int_0^L x^2 \mathrm{d}x = \frac{L^3}{3}$$

Therefore $c_0 = \dfrac{3}{L^3}\displaystyle\int_0^L x f(x)\mathrm{d}x$.

Next $c_n = \dfrac{\left\langle f(x), \sin\left(\frac{\beta_n x}{L}\right)\right\rangle}{\left\langle \sin\left(\frac{\beta_n x}{L}\right), \sin\left(\frac{\beta_n x}{L}\right)\right\rangle} = \dfrac{\int_0^L f(x)\sin\left(\frac{\beta_n x}{L}\right)\mathrm{d}x}{\left\langle \sin\left(\frac{\beta_n x}{L}\right), \sin\left(\frac{\beta_n x}{L}\right)\right\rangle}$.

$$\begin{aligned}
\left\langle \sin\left(\frac{\beta_n x}{L}\right), \sin\left(\frac{\beta_n x}{L}\right)\right\rangle &= \int_0^L \sin^2\left(\frac{\beta_n x}{L}\right) \mathrm{d}x \\
&= \frac{1}{2}\int_0^L 1 - \cos\left(2\beta_n x/L\right) \mathrm{d}x && \text{(trig identity)} \\
&= \frac{1}{2}\left(L - \left[\sin\left(\frac{2\beta_n x}{L}\right)\frac{L}{2\beta_n}\right]_0^L\right) \\
&= \frac{1}{2}\left(L - \sin\left(2\beta_n\right)\frac{L}{2\beta_n}\right) \\
&= \frac{L}{2\beta_n}\left(\beta_n - \sin\left(2\beta_n\right)\frac{1}{2}\right) \\
&= \frac{L}{2\beta_n}\left[\beta_n - \cos\left(\beta_n\right)\sin\left(\beta_n\right)\right] && \text{(trig identity)}
\end{aligned}$$

Thus

$$c_n = \frac{2\beta_n}{L\left(\beta_n - \cos\left(\beta_n\right)\sin\left(\beta_n\right)\right)}\int_0^L f(x)\sin\left(\frac{\beta_n x}{L}\right)\mathrm{d}x$$

(e) Suppose the $hL = 1$. Represent the function $f(x) = A$, A a constant, as a series of eigenfunctions of the above Sturm-Liouville problem.

Solution:

$$\begin{aligned}
c_0 &= \frac{3A}{L^3}\int_0^L x\mathrm{d}x \\
&= \frac{3A}{2L^3}\left[x^2\right]_0^L \\
&= \frac{3A}{2L}
\end{aligned}$$

$$\begin{aligned}
c_n &= \frac{2A\beta_n}{L(\beta_n - \cos(\beta_n)\sin(\beta_n))}\int_0^L \sin\left(\frac{\beta_n x}{L}\right) \\
&= \frac{2A\beta_n}{L(\beta_n - \cos(\beta_n)\sin(\beta_n))}\cdot\frac{L}{\beta_n}\left[-\cos\left(\frac{\beta_n x}{L}\right)\right]_0^L \\
&= \frac{2A\beta_n}{L(\beta_n - \cos(\beta_n)\sin(\beta_n))}\cdot\frac{L}{\beta_n}(1-\cos(\beta_n)) \\
&= \frac{2A}{\beta_n - \cos(\beta_n)\sin(\beta_n)}(1-\cos(\beta_n))
\end{aligned}$$

$$\widehat{f}(x) = \frac{3Ax}{2L} + \sum_{n=1}^{\infty}\frac{2A}{\beta_n - \cos(\beta_n)\sin(\beta_n)}(1-\cos(\beta_n))\sin\left(\frac{\beta_n x}{L}\right)$$

(f) Suppose the $hL = 1$. Represent the function $f(x) = x$ as a series of eigenfunctions of the above Sturm-Liouville problem.

Solution:

$$\begin{aligned} c_0 &= \frac{3}{L^3}\int_0^L x^2 \mathrm{d}x \\ &= \frac{1}{L^3}\left[x^3\right]_0^L \mathrm{d}x \\ &= 1 \end{aligned}$$

$$\begin{aligned} c_n &= \frac{2\beta_n}{L(\beta_n - \cos(\beta_n)\sin(\beta_n))}\int_0^L x\sin\left(\frac{\beta_n x}{L}\right) \\ &= \frac{2\beta_n}{L(\beta_n - \cos(\beta_n)\sin(\beta_n))}\left[\frac{-Lx}{\beta_n}\cos\left(\frac{\beta_n x}{L}\right) + \frac{L^2}{\beta_n^2}\sin\left(\frac{\beta_n x}{L}\right)\right]_{x=0}^{x=L} && \text{(MW-2)} \\ &= \frac{2}{\beta_n - \cos(\beta_n)\sin(\beta_n)}\left[-x\cos\left(\frac{\beta_n x}{L}\right) + \frac{L}{\beta_n}\sin\left(\frac{\beta_n x}{L}\right)\right]_{x=0}^{x=L} && \text{(MW-2)} \\ &= \frac{2}{\beta_n - \cos(\beta_n)\sin(\beta_n)}\left[-L\cos(\beta_n) + \frac{L}{\beta_n}\sin(\beta_n)\right] \end{aligned}$$

$$\widehat{f}(x) = x + \sum_{n=1}^{\infty}\frac{2}{\beta_n - \cos(\beta_n)\sin(\beta_n)}\left[-L\cos(\beta_n) + \frac{L}{\beta_n}\sin(\beta_n)\right]\sin\left(\frac{\beta_n x}{L}\right)$$