Math 414 – Homework 8 **Due Monday, April 17, 2023**

Chapter 6 & 7: Laplace' Equation

1. Consider the general Neumann problem for the non-homogeneous Laplace's equation

$$\Delta u(x,y,z) = f(x,y,z) \text{ in } D \text{ with } \frac{\partial u}{\partial n} = g(x,y,z) \text{ on } \partial D.$$

   (a) Show that there is no solution of BVP in three dimensions, unless

$$\iiint_D f\,dV = \iint_{\partial D} g\,dS.$$

   (Hint: Consider the Divergence Theorem.)

   (b) Consider the related homogeneous BVP ($f = 0$). For solutions to $\Delta u = 0$ to exist, what must be true about the average value of $g(x,y,z)$ on the boundary ∂D?

2. Suppose that u is a harmonic function in the disk $x^2 + y^2 < 4$ and that $u(x,y) = \frac{3}{2}xy + 1$ on the circle $x^2 + y^2 = 4$.

   (a) Determine the maximum and minimum values of u on the closure $\bar{D} = D \cup \partial D$.

   (b) Calculate the value of $u(0,0)$.

   (Hint: You do not need the solution to answer either of these questions.)

3. Prove the uniqueness of solutions to Laplace's equation with Robin boundary conditions:

$$\Delta u = 0 \text{ in } D \text{ with } \frac{\partial u}{\partial n} + a(\mathbf{x})u(\mathbf{x}) = 0 \text{ on } \partial D$$

   provided that $a(\mathbf{x}) > 0$ on the ∂D.

   (Hint: Find a BVP satisfied by the difference v of two solutions. Now multiply the PDE by v, integrate over D, and use Green's First.)

4. (*energy methods for the Neumann boundary conditions*) Consider the energy function

$$E[w] = \frac{1}{2}\iiint_D |\nabla w|^2\,dV - \iint_{\partial D} hw\,dS.$$

Show that if a given function u satisfies the variational problem

$$E[u] = \min_{w : \bar{D} \to \mathbb{R}} E[w]$$

then u satisfies Laplace's equation with the Neumann boundary conditions

$$\Delta u = 0 \text{ in } D \text{ with } \frac{\partial u}{\partial n} = h(\mathbf{x}) \text{ on } \partial D.$$

(Hint: Adapt the notes from class where we proved this for the Dirichlet problem with boundary conditions $u(\mathbf{x}) = h(\mathbf{x})$.)