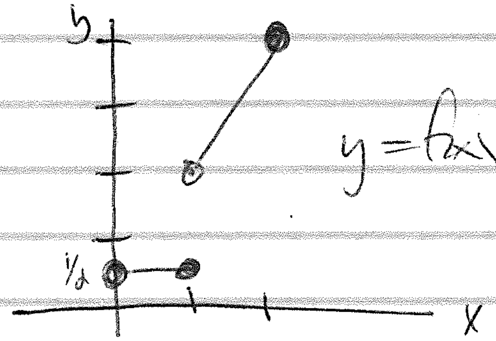
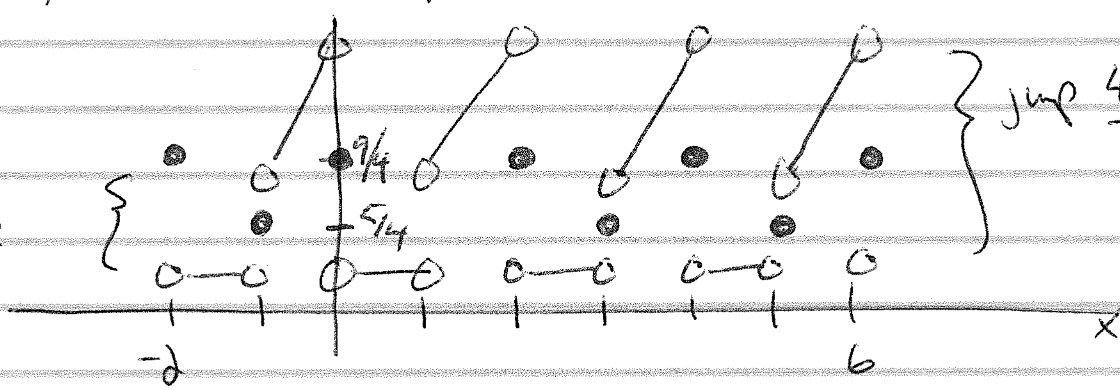
MTH 414 Homework 6 p1

1. $y=f(x)$



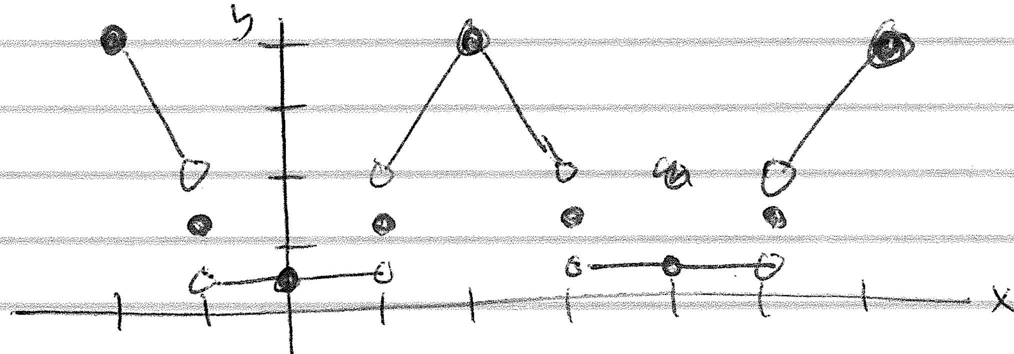
a) period of 2, repeated 4x w/ FCT points

jump $\frac{2+\frac{1}{2}}{2}$

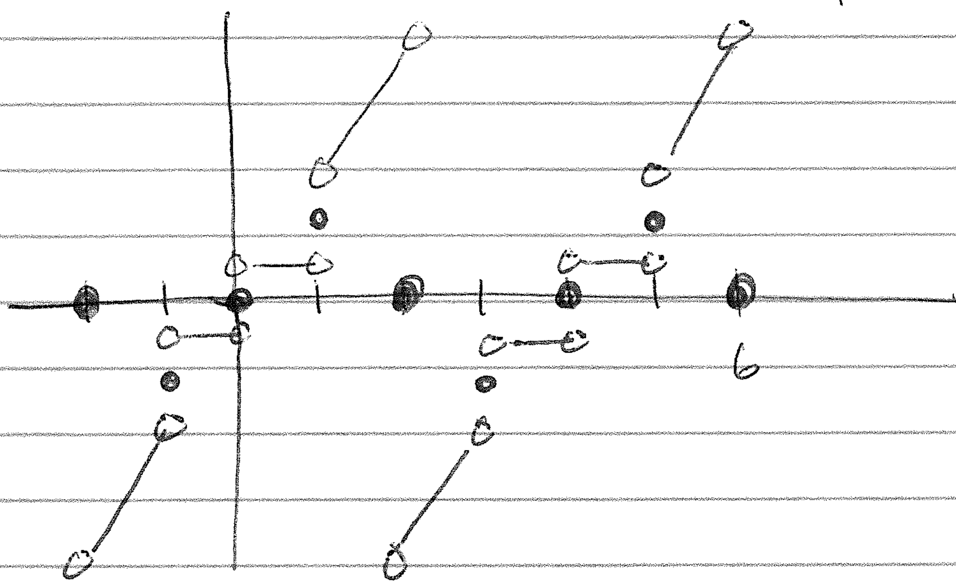


jump $\frac{4+\frac{1}{2}}{2} = 9/4$

b) period of 4, repeated 2x w/ FCT points.
even extension.



c) period 4, odd extension, w/ FCT point



d) $c_0 = \frac{\langle f,1\rangle}{\langle 1,1\rangle} = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}\left(3+\frac{1}{2}\right) = \frac{7}{4}$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\int_0^1 \cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 2x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$u = \frac{n\pi x}{2} \longleftrightarrow \frac{2}{n\pi}u = x, \quad \frac{2}{n\pi}du = dx$$

$$= \frac{1}{2}\int_0^{n\pi/2}\cos(u)\left(\frac{2}{n\pi}\right)du + 2\int_{n\pi/2}^{n\pi}\frac{2}{n\pi}u\cos(u)\left(\frac{2}{n\pi}\right)du$$

$$= \frac{1}{n\pi}\int_0^{n\pi/2}\cos u\,du + \frac{8}{n^2\pi^2}\int_{n\pi/2}^{n\pi}u\cos u\,du$$

$$= \frac{1}{n\pi}(\sin u)\Big|_0^{n\pi/2} + \frac{8}{n^2\pi^2}(u\sin u + \cos u)\Big|_{n\pi/2}^{n\pi}$$

$$= \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{8}{n^2\pi^2}\Big[n\pi\sin(n\pi) + \cos n\pi - \left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{2}\right)$$

HW 6p3

$$= \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{8}{n\pi}\sin(n\pi) + \frac{8}{n^2\pi^2}\cos(n\pi)$$

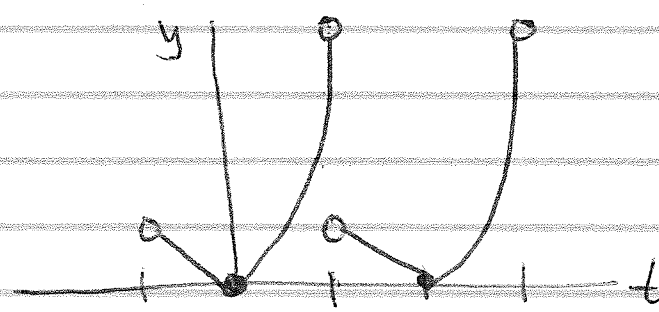
$$- \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{8}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right)$$

$$C_n = \frac{-3}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{8}{n^2\pi^2}\left[\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right]$$

$$f(x) \sim \hat{f}(x) = \frac{7}{4} + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{2}\right).$$

HW6p4

2) a)



b) $f(x) \sim \hat{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{3}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{3}\right)$

$$a_0 = \frac{2}{3}\int_{-3}^{3} f(t)\,dt = \frac{2}{3}\left[\int_{-3}^{0} -t\,dt + \int_{0}^{3} t^2\,dt\right]$$

$$= \frac{2}{3}\left[\frac{-t^2}{2}\Big|_{-3}^{0} + \frac{t^3}{3}\Big|_{0}^{3}\right] = \frac{2}{3}\left[\frac{9}{2} + 9\right]$$

$$= 3 + 6 = 9.$$

$$a_n = \frac{2}{3}\left[\int_{-3}^{0} -t\cos\left(\frac{n\pi t}{3}\right) + \int_{0}^{3} t^2 \cos\left(\frac{n\pi t}{3}\right)\right]$$

$$u = \frac{n\pi t}{3} \leftrightarrow \frac{3}{n\pi}\,du = dt$$

$$= \frac{2}{3}\left[\int_{-n\pi}^{0} \frac{3}{n\pi} u \cos u \, du + \int_{0}^{n\pi} \frac{9}{n^2\pi^2} u^2 \cos u \, du\right]$$

$$= \frac{2}{n\pi}(u\sin u + \cos u)\Big|_{-n\pi}^{0}$$

$$+ \frac{6}{n^2\pi^2}(u^2 \sin u - 2\sin u + 2u\cos u)\Big|_{0}^{n\pi}$$

HW6 p5

$$= \frac{2}{3}\left[\int_{-n\pi}^{0} -\frac{3}{n\pi} u\cos u \left(\frac{3}{n\pi}\right) du + \int_{0}^{n\pi} \frac{9}{n^2\pi^2} u^2 \cos u \left(\frac{3}{n\pi}\right) du\right]$$

$$= \frac{-6}{n^2\pi^2}\left(u\sin u + \cos u\right)\Big|_{-n\pi}^{0} + \frac{18}{n^3\pi^3}\left(u^2\sin u - 2\sin u + 2u\cos u\right)\Big|_{0}^{n\pi}$$

$$= \frac{-6}{n^2\pi^2}\left(0+1-(0+\cos(-n\pi))\right) + \frac{18}{n^3\pi^3}\left(2n\pi\cos n\pi - 0\right)$$

$$= \frac{-6}{n^2\pi^2}\left(1-\cos(n\pi)\right) + \frac{36}{n^2\pi^2}\cos n\pi$$

$$= \frac{-6}{n^2\pi^2} + \frac{\cancel{42}}{n^2\pi^2}\cos(n\pi) = \cancel{\begin{cases} \frac{-24}{n^2\pi^2}, & n \text{ odd} \\ \frac{36}{n^2\pi^2}, & n \text{ even} \end{cases}}$$

$$= \begin{cases} \frac{-48}{n^2\pi^2}, & n \text{ odd} \\ \frac{36}{n^2\pi^2}, & n \text{ even} \end{cases}$$

HUp6

$$b_n = \frac{2}{3}\left[\int_{-3}^{0} -t\sin\left(\frac{n\pi t}{3}\right) + \int_{0}^{3} t^2 \sin\left(\frac{n\pi t}{3}\right)\right]$$

$$u = \frac{n\pi t}{3} \leftrightarrow \frac{3}{n\pi}u = t \leftrightarrow \frac{3}{n\pi}du = dt$$

$$= \frac{2}{3}\left[\int_{-n\pi}^{0} \frac{-3}{n\pi} u\sin(u)\left(\frac{3}{n\pi}du\right) + \int_{0}^{n\pi} \frac{9}{n^2\pi^2} u^2 \sin u \left(\frac{3}{n\pi}du\right)\right]$$

$$= \frac{-6}{n^2\pi^2}\int_{-n\pi}^{0} u\sin u + \frac{18}{n^3\pi^3}\int_{0}^{n\pi} u^2 \sin u$$

$$= \frac{-6}{n^2\pi^2}\left(\sin u - u\cos u\right)\Big|_{-n\pi}^{0} + \frac{18}{n^3\pi^3}\left(-u^2\cos u + 2\cos u + 2u\sin u\right)\Big|_{0}^{n\pi}$$

$$= \frac{-6}{n^2\pi^2}\left(0 - \left(0 + n\pi\cos(n\pi)\right)\right)$$

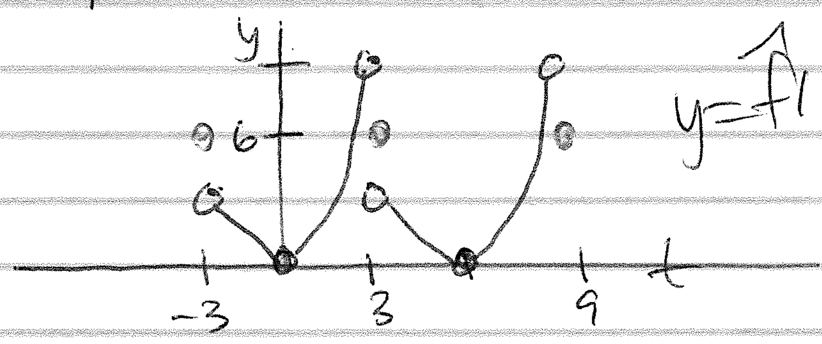
$$+ \frac{18}{n^3\pi^3}\left(-n^2\pi^2\cos n\pi + 2\cos n\pi - 2\right)$$

$$= \frac{6}{n\pi}\cos(n\pi) - \frac{18}{n\pi}\cos(n\pi) + \frac{36}{n^3\pi^3}\cos n\pi - \frac{36}{n^3\pi^3}$$

$$= \frac{-12}{n\pi}\cos(n\pi) + \frac{36}{n^3\pi^3}\left(\cos n\pi - 1\right)$$

$$b_n = \begin{cases} \dfrac{12n^2\pi^2 - 72}{n^3\pi^3}, & n \text{ odd} \\[2ex] \dfrac{-12}{n\pi}, & n \text{ even} \end{cases}$$

c)



$y = \hat{f}(t)$

HW6p7

3 a) heat is not allowed to escape the system thru the ends.

b) $u_t = k u_{xx}$ ← should have had a k here!

i) $u = X(x)T(t) \Rightarrow XT' = kX''T$

$$\Rightarrow \frac{T'}{kT} = \frac{X''}{X}.$$

ii) $\frac{T'}{kT}(t) = \frac{X''}{X}(x) = -\lambda(x,t).$

Since $\partial_t\left(\frac{T'}{kT} = \frac{X''}{X}\right) \Rightarrow \partial_t\left(\frac{T'}{kT}\right) = 0$

and $\partial_x\left(\frac{T'}{kT} = \frac{X''}{X}\right) \Rightarrow \partial_x\left(\frac{X''}{X}\right) = 0$

we get $\partial_x\lambda = \partial_t\lambda = 0$. i.e λ constant.

iii) $\lambda \in \mathbb{R}$, $X'' + \lambda X = 0$

$T' + \lambda k T = 0$

iv) $u_x(x,t) = X'(x)T(t)$

$u_x(0,t) = X'(0)T(t) = 0 \quad \forall t \Rightarrow X'(0) = 0$

$u_x(L,t) = X'(L)T(t) = 0 \Rightarrow X'(L) = 0$

BVP $X'' + \lambda X = 0$

$X'(0) = X'(L) = 0$

Know soln is a cosine series $X_0(x)=1$, $\lambda_0=0$

$X_n(x)=\cos\left(\frac{n\pi x}{L}\right)$, $\lambda_n=\left(\frac{n\pi}{L}\right)^2$.

v) $n=0$, $T_0'(t)=0 \Rightarrow T(t)=1$

$n\geq 1$, $T_n'(t)=-\lambda_n k T \Rightarrow T_n(t)=\exp\left[\frac{-kn^2\pi^2 t}{L^2}\right]$

vi) $u(x,t)=\sum_{n=0}^{\infty} C_n X_n(x) T_n(t)$

$= C_0 + \sum_{n=1}^{\infty} C_n \exp\left[\frac{-kn^2\pi^2 t}{L^2}\right]\cos\left(\frac{n\pi x}{L}\right)$

vii) @ $t=0$, $u(x,0)=C_0+\sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right)=f(x)$

We need the C_n's to be the Fourier Cosine series coef. of $f(x)$:

$$C_0=\frac{1}{L}\int_0^L f(x)\,dx$$

$$C_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

HW6 p9

4 $u_t = \frac{1}{5} u_{xx} \Rightarrow k = 1/5, L = 10, \lambda_n = \left(\frac{n\pi}{L}\right)^2 = \frac{n^2\pi^2}{100}$

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} C_n e^{-\frac{n^2\pi^2 t}{500}} \cos\left(\frac{n\pi x}{10}\right)$$

$$C_0 = \frac{1}{10}\int_0^{10} 4x\,dx = \frac{1}{10}(2x^2)\Big|_0^{10} = 20.$$

$$C_n = \frac{2}{10}\int_0^{10} 4x\cos\left(\frac{n\pi x}{10}\right)dx$$

$$= \frac{4}{5}\int_0^{10} x\cos\left(\frac{n\pi x}{10}\right)dx \quad u = \frac{n\pi x}{10}$$

$$= \frac{4}{5}\int_0^{n\pi} \frac{10}{n\pi} u\cos u\left(\frac{10}{n\pi}\right)du$$

$$= \frac{80}{n^2\pi^2}\int_0^{n\pi} u\cos u\,du$$

$$= \frac{80}{n^2\pi^2}(u\sin u + \cos u)\Big|_0^{n\pi}$$

$$= \frac{80}{n^2\pi^2}(0 + \cos n\pi - \cos 0)$$

$$= \frac{80}{n^2\pi^2}(\cos(n\pi) - 1) = \begin{cases} \frac{-160}{n^2\pi^2} &, n \text{ odd} \\ 0 &, n \text{ even} \end{cases}$$

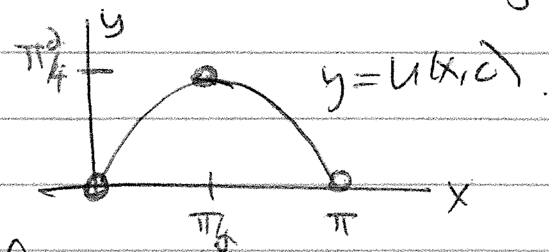
$$u(x,t) = 20 + \sum_{\substack{n=2, \\ n \text{ even}}}^{\infty} \frac{-160}{n^2\pi^2} e^{-\frac{n^2\pi^2 t}{500}} \cos\left(\frac{n\pi x}{10}\right)$$

HW6 p10

5. $u(0,t)=u(\pi,t)=0$ — anchored ends of string/wave

$u(x,0) = x(\pi - x)$

the initial wave form



$u_t(x,0)=0$ — released from rest

From Class Notes (Here $c=10$, $L=\pi$)

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{10n\pi t}{\pi}\right)+B_n\sin\left(\frac{10n\pi t}{\pi}\right)\right)\sin\left(\frac{n\pi x}{\pi}\right)$$

$$= \sum_{n=1}^{\infty}\left(A_n\cos(10nt)+B_n\sin(10nt)\right)\sin(nx)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} 10n\left[-A_n\sin(10nt)+B_n\cos(10nt)\right]\sin(nx)$$

$$u_t(x,0)=0 \leftrightarrow \sum_{n=1}^{\infty} 10nB_n\sin(nx) = 0$$

$$\Rightarrow B_n = 0 \ \forall n.$$

$$\therefore \ u(x,t) = \sum_{n=1}^{\infty} A_n\cos(10nt)\sin(nx).$$

$$u(x,0) = \pi x - x^2 = \sum_{n=1}^{\infty} A_n\sin(nx)$$

Need $u(x,0)$'s sine series expansion.

406p11

$$A_n = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin nx\,dx$$

$$= 2\int_0^{\pi} x\sin nx\,dx - \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\,dx$$

$$u = nx \leftrightarrow \frac{u}{n} = x \leftrightarrow \frac{du}{n} = dx$$

$$= \frac{2}{n^2}\int_0^{n\pi} u\sin u\,du - \frac{2}{\pi n^3}\int_0^{n\pi} u^2 \sin u\,du$$

$$= \frac{2}{n^2}\left(-u\cos u + \sin u\right)\Big|_0^{n\pi} - \frac{2}{\pi n^3}\left(-u^2\cos u + 2\cos u + 2u\sin u\right)\Big|_0^{n\pi}$$

$$= \frac{2}{n^2}\left(-n\pi\cos(n\pi) - 0\right) - \frac{2}{\pi n^3}\left(-n^2\pi^2\cos n\pi + 2\cos n\pi - 2\right)$$

$$= -\frac{2\pi}{n}\cos(n\pi) + \frac{2\pi}{n}\cos(n\pi) - \frac{4}{\pi n^3}\cos(n\pi) + \frac{4}{\pi n^3}$$

$$A_n = \frac{4}{\pi n^3}\left(1 - \cos(n\pi)\right)$$

$$= \begin{cases} \frac{8}{\pi n^3}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$$\text{So } u(x,t) = \sum_{\substack{n=1, \\ n \text{ odd}}}^{\infty} \frac{8}{\pi n^3}\sin(nx)\cos(10nt)$$

HW6p12

6 a) $\lambda = 0 \Rightarrow y'' = 0 \Rightarrow y(x) = Ax + B$

$y(0) = 0 \Rightarrow B = 0$, $y(x) = Ax$, and $y'(x) = A$

$hy(L) - y'(L) = 0 \Rightarrow hAL - A = 0$
$A(hL - 1) = 0$

Want $A \neq 0$, so only a sol'n if $hL = 1$.

If $hL = 1$, $\lambda = 0$ is an eigenvalue and $y_0(x) = x$.

b) If $\lambda < 0$, $y(x) = A\cosh(\sqrt{\lambda}\, x) + B\sinh(\sqrt{\lambda}\, x)$

$y(0) = 0 \Rightarrow A + 0 = 0 \Rightarrow A = 0$
and $y(x) = B\sinh(\sqrt{\lambda}\, x)$ w/ $y'(x) = B\cosh(\sqrt{\lambda} x)\sqrt{\lambda}$

$hy(L) - y'(L) = 0 \Rightarrow hB\sinh(\sqrt{\lambda}\, L) - B\cosh(\sqrt{\lambda}\, L)\sqrt{\lambda} = 0$

Want $B \neq 0$, $h\tanh(\sqrt{\lambda}\, L) = \sqrt{\lambda}$
or $\tanh(\sqrt{\lambda}\, L) = \dfrac{\sqrt{\lambda}}{h}$

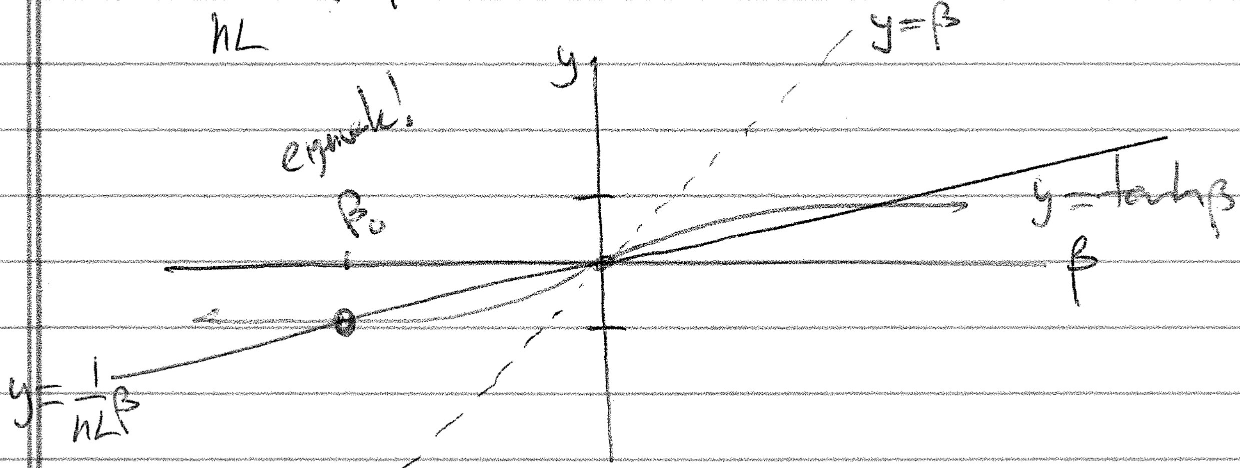
For any λ that solves this eqn, $y(x) = \sinh(\sqrt{\lambda}\, x)$ is an eigenfn.

Let $\beta = \sqrt{\lambda}\, L$, eqn is $\tanh(\beta) = \dfrac{\beta}{hL}$

Look for sol'n w/ $\beta<0$ by looking at for intersections of $y=\tanh(\beta)$ and $y=\frac{\beta}{hL}$.

Note slope $\frac{1}{hL}>0$.

But $y'=(\tanh(\beta))' = 1-\tanh^2\beta \le 1$ for all β.
Thus the curves will only have non-origin intersections if $\frac{1}{hL}<1$.



For β_0 above, $y_0(x)=\sinh(\sqrt{\lambda_0}\,x)$ is eigenfn.

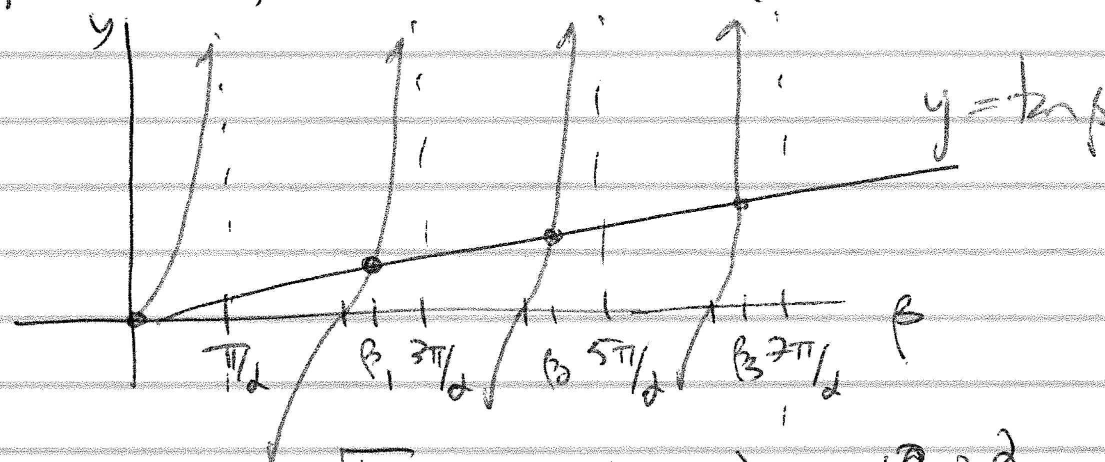
c) If $\lambda>0$, $y(x)=A\cos(\sqrt{\lambda}\,x)+B\sin(\sqrt{\lambda}\,x)$

$y(0)=0 \Rightarrow A+0=0 \Rightarrow A=0$
$\therefore\ y(x)=B\sin(\sqrt{\lambda}\,x)$, $y'(x)=B\cos(\sqrt{\lambda}\,x)\sqrt{\lambda}$.

$hy(L)-y'(L)=0 \Rightarrow hB\sin(\sqrt{\lambda}L)-B\cos(\sqrt{\lambda}L)\sqrt{\lambda}=0$
$\Rightarrow \tan(\sqrt{\lambda}L)=\frac{\sqrt{\lambda}}{h}$

For $\beta = \sqrt{\lambda} L$, $\beta > 0$, seek solns to
$$\tan\beta = \frac{\beta}{hL}.$$

For any $h, L > 0$, there are a countably infinite # of roots.



For each β_n, $\sqrt{\lambda_n} = \frac{\beta_n}{L} \Rightarrow \lambda_n = \left(\frac{\beta_n}{L}\right)^2$

w/ associated eigenfn $X_n(x) = \sin\left(\frac{\beta_n x}{L}\right)$.

d) If $hL = 1$, by (a) and (c), $f(x) = c_0 x + \sum_{n=1}^{\infty} c_n \sin\left(\frac{\beta_n x}{L}\right)$.

To solve for coefficients, use the L^2 inner-product.

In class, we proved these eigenfns are orthogonal.
But we should be able to show this directly.

HW pK

$n=0$: $\langle f(x), x\rangle = C_0\langle x,x\rangle + \sum_{n=1}^{\infty} C_n\langle x, \sin\left(\frac{\beta_n x}{L}\right)\rangle$

$$\langle f(x), x\rangle = \int_0^L x f(x)\,dx.$$

$$\langle x,x\rangle = \int_0^L x^2\,dx = \frac{L^3}{3}$$

Need $\perp$, i.e. $\langle x, \sin\left(\frac{\beta_n x}{L}\right)\rangle = 0$.

$$\langle x, \sin\left(\frac{\beta_n x}{L}\right)\rangle = \int_0^L x\sin\left(\frac{\beta_n x}{L}\right)dx$$

$$u = \frac{\beta_n x}{L} \Leftrightarrow \frac{L}{\beta_n}u = x \Leftrightarrow \frac{L}{\beta_n}du = dx$$

$$= \int_0^{\beta_n} \frac{L}{\beta_n} u\sin u\left(\frac{L}{\beta_n}\right)du = \frac{L^2}{\beta_n^2}\int_0^{\beta_n} u\sin u\,du$$

$$= \frac{L^2}{\beta_n^2}\left(-u\cos u + \sin u\right)\Big|_0^{\beta_n}$$

$$= \frac{L^2}{\beta_n^2}\left(\underbrace{-\beta_n\cos\beta_n + \sin\beta_n}\right)$$

But this equals 0 as it is the eigenvalue constraint $\tan\beta = \frac{\beta}{hL}$ w/ $hL = 1$.

$= 0$.

Thus $C_0 = \frac{\langle f(x), x\rangle}{\langle x,x\rangle} = \frac{3}{L^3}\int_0^L x f(x)\,dx$.

HW p16

$k \geq 1$ $\left\langle f(x), \sin\left(\frac{\beta_k x}{L}\right)\right\rangle = C_0 \left\langle x, \sin\left(\frac{\beta_k x}{L}\right)\right\rangle$ ← Not $=0$ by Kst.

$$+ \sum_{n=1}^{\infty} C_n \left\langle \sin\left(\frac{\beta_n x}{L}\right), \sin\left(\frac{\beta_k x}{L}\right)\right\rangle$$

$$= \underbrace{C_k \left\langle \sin\left(\frac{\beta_k x}{L}\right), \sin\left(\frac{\beta_k x}{L}\right)\right\rangle}_{\text{computable}} + \sum_{n \neq k} C_n \underbrace{\left\langle \sin\left(\frac{\beta_n x}{L}\right), \sin\left(\frac{\beta_k x}{L}\right)\right\rangle}_{\text{need } \perp \text{, i.e. } = 0.}$$

$$\langle X_n, X_k \rangle = \int_0^L \sin\left(\frac{\beta_k x}{L}\right) \sin\left(\frac{\beta_n x}{L}\right) dx$$

$$= \int_0^L \frac{1}{2}\left[\cos\left(\frac{\beta_k - \beta_n}{L} x\right) - \cos\left(\frac{\beta_k + \beta_n}{L} x\right)\right] dx$$

$$= \frac{1}{2}\left[\frac{L}{\beta_k - \beta_n} \sin\left(\frac{\beta_k - \beta_n}{L} x\right) - \frac{L}{\beta_k + \beta_n} \sin\left(\frac{\beta_k + \beta_n}{L} x\right)\right]\Bigg|_0^L$$

$$= \frac{L}{2}\left[\frac{1}{\beta_k - \beta_n} \sin(\beta_k - \beta_n) - \frac{1}{\beta_k + \beta_n} \sin(\beta_k + \beta_n)\right]$$

$$= \frac{L}{2(\beta_k^2 - \beta_n^2)}\left[(\beta_k + \beta_n)\sin(\beta_k - \beta_n) - (\beta_k - \beta_n)\sin(\beta_k + \beta_n)\right]$$

$$= \frac{L}{2(\beta_k^2 - \beta_n^2)}\Big[(\beta_k + \beta_n)(\sin\beta_k \cos\beta_n - \cos\beta_k \sin\beta_n)$$
$$- (\beta_k - \beta_n)(\sin\beta_k \cos\beta_n + \cos\beta_k \sin\beta_n)\Big]$$

$$= \frac{L}{2(\beta_k^2 - \beta_n^2)}\left[(\beta_k - \beta_k)\sin\beta_k \cos\beta_n + (\beta_n - \beta_n)\cos\beta_k \sin\beta_n\right] = 0$$

✓

$$\langle X_k, X_k \rangle = \int_0^L \sin^2\left(\frac{\beta_k x}{L}\right) dx$$

$$= \frac{L}{\beta_k}\int_0^{\beta_k} \sin^2 u \, du \quad , \; u = \frac{\beta_k x}{L}$$

$$= \frac{L}{2\beta_k}\int_0^{\beta_k} (1-\cos 2u)\,du = \frac{L}{2\beta_k}\left(u - \frac{\sin 2u}{2}\right)\Big|_0^{\beta_k}$$

$$= \frac{L}{2\beta_k}\left(\frac{2\beta_k - \sin 2\beta_k}{2}\right) = \frac{L(2\beta_k - \sin 2\beta_k)}{4\beta_k}$$

Then $\langle f(x), X_k \rangle = C_k \langle X_k, X_k \rangle$

$$\Rightarrow C_k = \frac{4\beta_k}{L(2\beta_k - \sin 2\beta_k)}\int_0^L f(x)\sin\left(\frac{\beta_k x}{L}\right)dx$$

e) Let $f(x) = A$. Compute its series representation.

$$C_0 = \frac{3}{L^3}\int_0^L A x \, dx = \frac{3A}{2L}$$

$$C_k = \frac{4\beta_k}{L(2\beta_k - \sin 2\beta_k)}\int_0^L A\sin\left(\frac{\beta_k x}{L}\right)dx$$

$$= \frac{4\beta_k}{L(2\beta_k - \sin 2\beta_k)}\left[\frac{AL}{\beta_k}\left(-\cos\left(\frac{\beta_k x}{L}\right)\Big|_0^L\right)\right]$$

$$= \frac{4A}{2\beta_k - \sin 2\beta_k}\left(1 - \cos(\beta_k)\right)$$

$\therefore \quad f(x) = A \sim \hat{f}(x) = \frac{3A}{2L}x + \sum_{n=1}^{\infty} \frac{4A(1-\cos\beta_n)}{2\beta_n - \sin 2\beta_n} \sin\left(\frac{\beta_n x}{L}\right).$

Crazy Equality - Identity.

Note $x = \frac{2}{3}L \in (0, L)$ and

$$\hat{f}(2L/3) = A + \sum_{n=1}^{\infty} \frac{4A(1-\cos\beta_n)}{2\beta_n - \sin 2\beta_n} \sin\left(\frac{2\beta_n}{3}\right)$$

$= A$ for all x.

So $\displaystyle\sum_{n=1}^{\infty} \frac{4(1-\cos\beta_n)}{2\beta_n - \sin 2\beta_n} \sin\left(\frac{2\beta_n}{3}\right) = 0$

where β_n are the roots of $\tan\beta_n = \beta_n$.

f) feels redundant.