Homework 9

1a. $\mathcal{U}(f(x+h)) = \int_{\mathbb{R}} e^{-2\pi i \gamma x} f(x+h)\,dx$

$u = x+h \rightarrow x = u-h$

$$= \int_{\mathbb{R}} e^{-2\pi i \gamma (u-h)} f(u)\,du$$

$$= e^{+2\pi i \gamma h} \int_{\mathbb{R}} e^{-2\pi i \gamma u} f(u)\,du$$

$$= e^{2\pi i \gamma h}\,\mathcal{U}(f(x))$$

b. $(f*g)(x) = \int_{\mathbb{R}} f(x-t)g(t)\,dt$ $\quad u = x-t$, $du = -dt$

$$= \int_{+\infty}^{-\infty} f(u)\,g(x-u)(-du)$$

$$= \int_{\mathbb{R}} g(x-u)\,f(u)\,du$$

$$= (g*f)(x)$$

c. $\frac{d}{dx}\int_{\mathbb{R}} f(x-t)g(t)\,dt$

$= \int_{\mathbb{R}} \frac{df}{dx}(x-t)g(t)\,dt$ by $f \in S(\mathbb{R})$ and Leibniz Rule

d. $\frac{d}{dx}((f*g)(x)) = \frac{d}{dx}((g*f)(x))$ by (b)

$= \left(\frac{dg}{dx} * f\right)(x)$ by (c)

$= \left(f * \frac{dg}{dx}\right)(x)$ by (b).

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e) $\mathcal{U}(e^{2\pi i a x} f(x)) = \int_{\mathbb{R}} e^{-2\pi i \gamma x} e^{2\pi i a x} f(x)\, dx$
$= \int_{\mathbb{R}} e^{-2\pi i(\gamma - a)x} f(x)\, dx$
$= \mathcal{U}(f(\psi))$ where $\psi = \gamma - a$.
$= \mathcal{U}(f(\gamma - a))$.

f) $\mathcal{U}(2\pi i x f(x)) = \int_{\mathbb{R}} e^{-2\pi i \gamma x} (2\pi i x) f(x)\, dx$

and

$\frac{d}{d\gamma} \hat{f}(\gamma) = \frac{d}{d\gamma}\left(\int_{\mathbb{R}} e^{-2\pi i \gamma x} f(x) dx\right)$
$= \int_{\mathbb{R}} (-2\pi i x) e^{-2\pi i \gamma x} f(x) dx$
$= -\int_{\mathbb{R}} e^{-2\pi i \gamma x} (2\pi i x f(x)) dx$
$= -\mathcal{U}(2\pi i x f(x))$.

g) $\int_{\mathbb{R}} \hat{f}(x) g(x)\, dx$
$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-2\pi i x y} f(y)\, dy\right) g(x)\, dx$
$= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-2\pi i x y} f(y) g(x)\, dx\, dy$ (Fubini)
$= \int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} e^{-2\pi i x y} g(x)\, dx\right) dy = \int_{\mathbb{R}} f(x) \hat{g}(x) dx$

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2. By #1f, $\mathcal{U}(2\pi i x f(x)) = -\frac{d}{d\gamma}\hat{f}(\gamma)$

$\Rightarrow \mathcal{U}(x f(x)) = \frac{-1}{2\pi i}\frac{d}{d\gamma}\hat{f}(\gamma) = \frac{i}{2\pi}\frac{d}{d\gamma}\hat{f}(\gamma)$.

Here $f(x) = e^{-kx^2}$ ... which looks a lot like a Gaussian.

Recall $G_1(x) = e^{-\pi x^2}$ and $\hat{G}_1(\gamma) = e^{-\pi\gamma^2}$.

$f(x) = \exp\left(-\frac{\pi k x^2}{\pi}\right) = \exp\left(-\pi\left(\sqrt{\frac{k}{\pi}}x\right)^2\right)$, $k>0$

$= G_1(bx)$ where $b = \sqrt{k/\pi}$.

By change of scale corollary, $\mathcal{U}(f(bx)) = \frac{1}{b}\hat{f}(\gamma/b)$,

we get $\hat{f}(\gamma) = \mathcal{U}(G_1(bx)) = \frac{1}{b}\hat{G}_1(\gamma/b)$

$= \frac{1}{\sqrt{k/\pi}}\exp\left[-\pi\left(\gamma/\sqrt{k/\pi}\right)^2\right]$ or

$\hat{f}(\gamma) = \sqrt{\pi/k}\,\exp\left[-\pi^2\gamma^2/k\right]$

Then $\mathcal{U}(x f(x)) = \frac{i}{2\pi}\frac{d}{d\gamma}\left(\sqrt{\frac{\pi}{k}}\exp\left(-\pi^2\gamma^2/k\right)\right)$

$= \frac{i}{2\pi}\sqrt{\frac{\pi}{k}}\left(\frac{-2\pi^2\gamma}{k}\right)\exp\left(-\pi^2\gamma^2/k\right) = -i\left(\frac{\pi}{k}\right)^{3/2}\gamma e^{-\pi^2\gamma^2/k}$

3. F-transform $\widehat{u_t} = -\widehat{u_{xxxx}}$
$$= -(2\pi i\gamma)^4 \hat{u}$$
$$\hat{u}_t = -16\pi^4\gamma^4 \hat{u}$$

Note $\hat{u}_t = C(\gamma) \exp[-16\pi^4\gamma^4 t]$

For I.C. $\hat{u}(x,0) = \hat{f}(x)$

$\hat{u}_t(x,0) = C(\gamma) = \hat{f}(x)$

So $\hat{u}_t(x,t) = \hat{f}(x) \exp[-16\pi^4\gamma^4 t]$

Note $\hat{K}_t(\gamma) = \exp[-16\pi^4\gamma^4 t] \in S(\mathbb{R})$.

Clearly invertible. define $k(x,t) = U^{-1}(\hat{K}_t(\gamma))$.

Then the convolution sol'n to the PDE is

$$u(x,t) = (k(\cdot,t) * f(\cdot))(x)$$
$$= \int_{\mathbb{R}} k(x-y,t) f(y)\, dy.$$

4) $u_{xx} + u_{yy} = 0 \quad , 0 < x < L \, , y \in \mathbb{R}$

F-transform in y: $\hat{u}_{xx} + (2\pi i\gamma)^2 \hat{u} = 0$

$$\hat{u}_{xx} - 4\pi^2\gamma^2 \hat{u} = 0$$

ODE sol'n: $\hat{u}(x,\gamma) = C_1(\gamma)\cosh(2\pi|\gamma|x) + C_2(\gamma)\sinh(2\pi|\gamma|x)$

Using I.C.s $\hat{u}(0,\gamma) = \hat{g}_1(\gamma)$, $\hat{u}(L,\gamma) = \hat{g}_2(\gamma)$

$\hat{u}(0,\gamma) = C_1(\gamma) = \hat{g}_1(\gamma)$.

$\hat{u}(L,\gamma) = \hat{g}_1(\gamma)\cosh(2\pi|\gamma|L) + C_2(\gamma)\sinh(2\pi|\gamma|L) = \hat{g}_2(\gamma)$

$$\Rightarrow C_2(\gamma) = \frac{\hat{g}_2(\gamma) - \hat{g}_1(\gamma)\cosh(2\pi|\gamma|L)}{\sinh(2\pi|\gamma|L)} \quad (*)$$

We have $\hat{u}(x,\gamma) = \hat{g}_1(\gamma)\cosh(2\pi|\gamma|x) + C_2(\gamma)\sinh(2\pi|\gamma|x)$

where $C_2(\gamma)$ defined $(*)$.

While we don't currently have inverse transforms of $\cosh\theta$ and $\sinh\theta$ in our table, via our work w/ D'Alembert and sine and cosine, conceivable using exponential defn's.

Not as obvious as Problem 3 that a convolution sol'n works.

$\hat{g}_1(\gamma)\cosh(2\pi\gamma x)$ should be fine.

It is the sinh term in denom. of $G_2(\gamma)$ that causes a pause.

However, consider $\left(\hat{g}_2(\gamma)-\hat{g}_1(\gamma)\cosh(2\pi\gamma L)\right)\left(\frac{\sinh(2\pi|\gamma|x)}{\sinh(2\pi|\gamma|L)}\right)$

By L'Hôpital's rule, as $\gamma\to 0$, the limit exists as x/L if $x>0$ and $-x/L$ if $x<0$.

Using a Heaviside function, this complicated thing should exist.

For convolution form, we like the Duhamel's representation

Want $\hat{u}(x,\gamma)=\hat{K}_1(x,\gamma)\hat{g}_1(\gamma)+\hat{K}_2(x,\gamma)\hat{g}_2(\gamma)$

$$=\left(\cosh(2\pi\gamma x)-\tanh(2\pi|\gamma|L)\sinh(2\pi|\gamma|L)\right)\hat{g}_1(\gamma)$$

$$+\frac{\sin(2\pi|\gamma|x)}{\sin(2\pi|\gamma|L)}\hat{g}_2(\gamma)$$

and

$$u(x,y)=\int_{\mathbb{R}}\left(K_1(x,y-s)g_1(s)+K_2(x,y-s)g_2(s)\right)ds.$$