Math 414 **Exam 1**
Professor Carty **March 10, 2023**
50 points possible

Name: ______________________________

1. No hats or dark sunglasses. All hats are to be removed.
2. All book bags are to be closed and placed in a way that makes them inaccessible. Do not reach into your bag for anything during the exam. If you need extra pencils, pull them out now.
3. Be sure to print your proper name clearly.
4. *No calculators are allowed.* Watches with recording, internet, communication or calculator capabilities (e.g. a smart watch) are prohibited.
5. All electronic devices, including cell phones and other wearable devices, must be powered off and stored out of sight for the entirety of the exam.
6. If you have a question, raise your hand and I will come to you. Once you stand up, you are done with the exam. If you have to use the facilities, do so now. You will not be permitted to leave the room and return during the exam.
7. Every exam is worth a total of **50 points**. Including the cover sheet, each exam has 9 pages.
8. At 2:50, you will be instructed to put down your writing utensil. You must stop writing the exam at this time.
9. If you finish early, quietly and respectfully and in your exam. You may leave early.
10. You will hand in the paper copy of the exam on your way out of the classroom.
11. You have fifty minutes to complete the exam. I hope you do well.

1. (5 points) The one-dimensional wave equation over the line $\mathbb{R}$.

(a) State the general IVP associated for the wave equation over the line. Describe, in words, what the unknown function u and the initial data represent.

$u_{tt} = c^2 u_{xx}$, $x \in \mathbb{R}$, $t \geq 0$

$u(x,0) = \phi(x)$, initial wave profile

$u_t(x,0) = \psi(x)$, initial velocity (vertical)

$u(x,t)$ - displacement from equilibrium ($u(x,t)=0$) at position x and time t.

(b) State D'Alembert's solution to the wave equation over the line $\mathbb{R}$.

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

2. (15 points) Consider the first-order PDE

$$u_t + xu_x = 3u + 1.$$

(a) Find the characteristic curves for the this equation.

$$\frac{dx}{dt} = x \Rightarrow \frac{dx}{x} = dt \Rightarrow \ln|x| = t + C$$

$$x = Ce^{t} \quad \text{or} \quad C = xe^{-t}$$

(b) Find the general solution to the PDE.

$$u(t, x(t)) = u(t, Ce^{t}) = v(t, C)$$

$$v_t = u_t + u_x \frac{dC}{dt} = u_t + u_x Ce^{t} = u_t + u_x x.$$

$$\text{So } v_t = 3v + 1, \quad v_t - 3v = 1 \qquad \mu(t) = \exp\left[\int -3dt\right] = e^{-3t}$$

$$\text{So } \partial_t (e^{-3t} v) = e^{-3t}$$

$$e^{-3t} v = -\frac{1}{3} e^{-3t} + f(C)$$

$$v(t, C) = -\frac{1}{3} + e^{3t} f(C)$$

$$u(x,t) = -\frac{1}{3} + e^{3t} f(xe^{-t}).$$

Prob 2.

a) Or $t = \ln|x| + C$

b) $u(t(x,c), x) = v(C,x)$

$$v_x = u_t \frac{dt}{dx} + u_x = \frac{1}{x} u_t + u_x$$

$$\Rightarrow x v_x = u_t + x u_x$$

$$\Rightarrow x v_x = 3v + 1 \ , \quad v_x - \frac{3}{x} v = \frac{1}{x}$$

$$\mu(x) = \exp\left[\int \frac{-3}{x} dx\right] = x^{-3}$$

$$x^{3} v_x - \frac{3}{x^4} v = \frac{1}{x^4}$$

$$\partial_x \left(\frac{1}{x^3} v \right) = \frac{1}{x^4}$$

$$\frac{1}{x^3} v = \frac{-1}{3x^3} + f(c)$$

$$v(x,c) = \frac{-1}{3} + x^3 f(c)$$

$$u(x,t) = -\frac{1}{3} + x^3 f(\ln|x| - t) .$$

(c) Find the unique solution to the IVP given the data $u(x, 0) = x^2$.

$$u(x,0) = -\frac{1}{3} + f(x) = x^2$$

$$f(x) = x^2 + \frac{1}{3}$$

$$\Rightarrow u(x,t) = -\frac{1}{3} + e^{3t}\left[(xe^{-t})^2 + \frac{1}{3}\right]$$

$$= -\frac{1}{3} + \frac{1}{3}e^{3t} + x^2 e^{t}$$

(d) What is the purpose of using characteristics. How do they help you solve a PDE?

Along these curves the directional derivative of $u(t,x)$ is zero. Restricting to these curves allows our 2^{nd}-order PDE to be reinterpreted as a 1^{st}-order ODE (temporarily).

3. (10 points) Consider the second-order linear PDE

$$2u_{tt} - 5u_{tx} - 3u_{xx} = 0.$$

(a) Define the linear operator L such that the PDE can be written as $Lu = 0$.

$$L = 2\frac{\partial^2}{\partial t^2} - 5\frac{\partial}{\partial t}\frac{\partial}{\partial x} - 3\frac{\partial^2}{\partial x^2}$$

(b) Via a factorization of L, find two characteristic curves for the PDE and state the general solution to the PDE.

$$L = 2\partial_t^2 - 5\partial_t\partial_x - 3\partial_x^2 = \underbrace{(2\partial_t + \partial_x)}_{(*)}\underbrace{(\partial_t - 3\partial_x)}_{(**)}$$

$$(*) \Rightarrow 2u_t + u_x = 0 \Rightarrow \frac{dx}{dt} = \frac{1}{2} \Rightarrow x = \frac{1}{2}t + C$$

or $C = 2x - t$.

$$(**) \Rightarrow u_t - 3u_x = 0 \Rightarrow \frac{dx}{dt} = -3 \Rightarrow x = -3t + C$$

or $C = x + 3t$.

$$u(x,t) = f(2x - t) + g(x + 3t)$$

for any twice differentiable f or g.

(c) Given the initial data $u(x,0) = 0$, $u_t(x,0) = \sin x$, find solve the IVP.

$u(x,0) = 0$: $f(2x) + g(x) = 0$ ①

$u_t(x,t) = -f'(2x-t) + 3g'(x+3t)$

$u_t(x,0) = \sin x$: $-f'(2x) + 3g'(x) = \sin x$ ②

① $\Rightarrow$ ①' : $2f'(2x) + g'(x) = 0$

$-2f'(2x) + 6g'(x) = 2\sin x$

$7g'(x) = 2\sin x \Rightarrow g'(x) = \frac{2}{7}\sin x$

$g(x) = \frac{-2}{7}\cos x$

$f(2x) = -g(x) = \frac{2}{7}\cos x$

let $u = 2x \leftrightarrow u/2 = x \Rightarrow f(u) = \frac{2}{7}\cos(u/2)$

i.e $f(x) = \frac{2}{7}\cos(x/2)$

Then $u(x,t) = \frac{2}{7}\cos\left(\frac{2x-t}{2}\right) - \frac{2}{7}\cos(x+3t)$

4. (8 points) Consider the one-dimensional diffusion equation

$$\begin{cases} u_t = ku_{xx}, & 0 \le x \le L,\ t > 0, \\ u(x,0) = \phi(x), & 0 \le x \le L \\ u(0,t) = f(x), & t > 0 \\ u(L,t) = g(x), & t > 0 \end{cases}$$

(a) State the maximum principle for the diffusion equation.

Given the rectangle $(x,t) \in [0,L] \times [0,T]$, the maximum value of $u(x,t)$ occurs on the boundary lines
i) $x=0,\ 0 \le t \le T$, ii) $x=L,\ 0 \le t \le T$, or iii) $t=0,\ 0 \le x \le L$.

(b) Use the maximum principle to prove uniqueness. That is, show that if there are two solutions u and v that solve the above IVP, then $u(x,t) = v(x,t)$ for all x and t.

Consider $w = u - v$.
Note $w(x,0) = u(x,0) - v(x,0) = \phi(x) - \phi(x) = 0$
and $w(0,t) = 0$ and $w(L,t) = 0$.
This yields the IVP in w: $w_t - kw_{xx} = 0$
$w(x,0) = 0$,
$w(0,t) = 0$
$w(L,t) = 0$

By max principle, $\max w = 0$ on $(x,y) \in R$
i.e. $w(x,t) \le 0$.
Similarly $-w = v - u$ solves same IVP.
Hence by max principle $\max(-w) = 0$ on R
i.e. $-w(x,t) \le 0$
$\Rightarrow w(x,t) \ge 0$

$\therefore w(x,t) = 0$.

5. (7 points) Find the solution to the IVP

$$\begin{cases} u_t = k u_{xx}, & -\infty < x < \infty,\ t > 0, \\ u(x,0) = \phi(x), & \end{cases}$$

where $\phi(x) = 3$ for $0 < x < 4$ and $\phi(x) = 0$ otherwise. Be sure to show the details of your calculation.
Note: For full credit, you must write your solution in terms of the error function

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz.$$

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_0^4 \exp\left[\frac{-(x-y)^2}{4kt}\right] (3)\, dy$$

$$u = \frac{x-y}{\sqrt{4kt}} \longleftrightarrow du = \frac{-dy}{\sqrt{4kt}} \longleftrightarrow dy = -\sqrt{4kt}\, du$$

$$u(x,t) = \frac{-3}{\sqrt{\pi}} \int_{x/\sqrt{4kt}}^{\frac{x-4}{\sqrt{4kt}}} e^{-u^2}\, du$$

$$= \frac{-3}{\sqrt{\pi}} \left[\int_0^{\frac{x-4}{\sqrt{4kt}}} e^{-u^2}\, du + \int_{x/\sqrt{4kt}}^{0} e^{-u^2}\, du \right]$$

$$= \frac{3}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-u^2}\, du - \frac{3}{\sqrt{\pi}} \int_0^{\frac{x-4}{\sqrt{4kt}}} e^{-u^2}\, du$$

$$= \frac{3}{2} \mathrm{Erf}\left(x/\sqrt{4kt}\right) - \frac{3}{2} \mathrm{Erf}\left(\frac{x-4}{\sqrt{4kt}}\right).$$

6. (5 points) Consider a partial differential equation $Lu = 0$ along with initial and/or boundary conditions. Clearly define what it means for the problem to be *well-posed*.

(1) existence - there exists a fcn u with enough derivative such that $Lu=0$

(2) uniqueness - if u and v are two sol'ns to ~~a~~ the same IVP, we must have $u=v$.

(3) stability - small changes in data lead only to small changes in the sol'n.