Homework 7

1. Recall a harmonic fcn $u(x,y)$ satisfies $\Delta u = 0$.

We need a sol'n to the Laplacian $\Delta u = 0$ on $[0,\pi] \times [0,\pi]$ with indicated boundary.

As D is a rectangle, seek separated sol'ns $u(x,y) = X(x)Y(y)$.

$$\Longrightarrow \quad \frac{X''}{X} + \frac{Y''}{Y} = 0.$$

Before "λ"-ing this eqn, check boundary and hope for known problem / sol'n.

$$u_y = X(x)\,Y'(y) \quad \dots \quad \begin{array}{l} u_y(x,0) = 0 \\ u_y(x,\pi) = 0 \end{array} \quad \begin{array}{l} \Rightarrow Y'(0) = 0 \\ \Rightarrow Y'(\pi) = 0 \end{array}$$

This yields a Neumann problem in Y.

$$\frac{Y''}{Y} = -\frac{X''}{X} = -\lambda$$

and PDE decouples

$$(*)\begin{cases} Y'' + \lambda Y = 0 \\ Y'(0) = Y'(\pi) = 0 \end{cases} \quad \text{and} \quad \begin{cases} X'' - \lambda X = 0 \\ X(0) = 0 \end{cases}$$

For $(*)$, $L = \pi$, $\lambda_n = \left(\frac{n\pi}{\pi}\right)^2 = n^2$, $n = 0, 1, 2, \dots$

and $Y_n = \cos(n\pi)$.

Then $X_n'' - n^2 X_n = 0 \Rightarrow X_n(x) = C_1 \sinh(nx) + C_2\cosh(nx)$ if $n \neq 0$

and $X_0(x) = C_1 x + C_2$ if $\lambda = 0$.

For $n>0$, $X(0)=0 \Rightarrow X_n(0) = C_2 = 0$
and $X_n(x) = C_n \sinh x$.

For $n=0$, $X_0(0)=0 \Rightarrow C_2 = 0$ and $X_0(x) = C_1 x$

So $u(x,y) = C_0 X_0(x) Y_0(y) + \sum_{n=1}^{\infty} C_n X_0(x) Y_0(y)$

$$= C_0 x + \sum_{n=1}^{\infty} C_n \sinh(nx) \cos(ny)$$

Want to satisfy the last BC $u(\pi,y) = \frac{1}{2} + \frac{1}{2}\cos 2y$

$u(\pi,y) = C_0\pi + \sum_{n=1}^{\infty} C_n \sinh(n\pi)\cos(ny)$

$$= C_0\pi + C_1 \sinh(\pi)\cos y + C_2 \sinh(2\pi)\cos(2y) + \sum_{3}^{\infty} C_n X_n Y_n.$$

For equality, need $C_0\pi = 1/2 \Rightarrow C_0 = 1/2\pi$,
$C_2 \sinh(2\pi) = 1/2 \Rightarrow C_2 = \dfrac{1}{2\sinh(2\pi)}$

and $C_n = 0$ else.

$$u(x,y) = \frac{1}{2\pi} x + \frac{1}{2\sinh(2\pi)} \sinh(2x)\cos(2y).$$

7p3

~~23~~/24

2. a) this is a lot like the example in lecture.

$u(x,y) = X(x)Y(y) \Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = 0$

and $u(0,y) = u(\pi,y) = 0$ yields Dirichlet BC

$X(0) = X(\pi) = 0$.

Hence, use λ-eqn, $\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$ and

this yields $\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(\pi) = 0 \end{cases}$ and $Y'' - \lambda Y = 0$

BVP has ~~eigen~~ eigenpairs $\lambda_n = n^2$ and $X_n(x) = \sin(nx)$, $n = 1,2,3,\ldots$

For $Y_n'' - \lambda_n Y_n = 0$, we could try $\sinh(ny)$ and $\cosh(ny)$, but note that for fixed n, both functions blow up to ∞ due to the $+e^{ny}$ terms in the numerators of their definitions.

As an alternative, we try the form more common in an ODE class.

let $Y_n = A_n e^{-ny} + B_n e^{ny}$.

Requiring $\lim_{y \to \infty} Y_n = 0 \Rightarrow B_n = 0$ for all n.

(we need to omit the unbounded fns from $u_n(x,y)$).

So $Y_n(y) = e^{-ny}$ and $u(x,y) = \sum_{n=1}^{\infty} C_n e^{-ny} \sin(nx)$

Now satisfying the other BC is straight forward

Note $u(x,0) = \sum_{n=1}^{\infty} C_n \sin(nx)$ ... a classic sine series

Thus $C_n = \frac{2}{\pi}\int_0^{\pi} h(x)\sin(nx)\,dx$, $n = 1, 2, 3, \ldots$

b) Series constructed out of unbounded fns diverge (generally).

7pts5

3. Let $u(x(r,\theta), y(r,\theta))$, $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial u}{\partial \theta} = u_x x_\theta + u_y y_\theta$$

$$= u_x(-r\sin\theta) + u_y(r\cos\theta)$$

$$= u_x(-y) + u_y(x) = -yu_x + xu_y$$

$$\frac{\partial^2 u}{\partial \theta^2} = (u_x)_\theta(-y) + u_x(-y)_\theta + (u_y)_\theta x + u_y(x_\theta)$$

$$= (u_{xx}x_\theta + u_{yx}y_\theta)(-y) + u_x(-x) + (u_{yx}x_\theta + u_{yy}y_\theta)x + u_y(-y)$$

$$= (u_{xx}(-y) + u_{yx}(x))(-y) - xu_x + (u_{yx}(-y) + u_{yy}(x))x - yu_y$$

$$= y^2 u_{xx} - 2xy\,u_{xy} + x^2 u_{yy} - xu_x - yu_y.$$

$$\frac{\partial u}{\partial r} = u_x x_r + u_y y_r = u_x\cos\theta + u_y\sin\theta.$$

Note: $r\frac{\partial u}{\partial r} = u_x r\cos\theta + u_y r\sin\theta$

$$= xu_x + yu_y$$

$$\frac{\partial^2 u}{\partial r^2} = (u_x)_r\cos\theta + (u_y)_r\sin\theta$$

$$= (u_{xx}x_r + u_{xy}y_r)\cos\theta + (u_{yx}x_r + u_{yy}y_r)\sin\theta$$

$$= (u_{xx}\cos\theta + u_{xy}\sin\theta)\cos\theta + (u_{yx}\cos\theta + u_{yy}\sin\theta)\sin\theta$$

$$= u_{xx}\cos^2\theta + 2u_{xy}\cos\theta\sin\theta + u_{yy}\sin^2\theta.$$

again note: $r^2 \dfrac{\partial^2 u}{\partial r^2} = u_{xx}(x^2) + 2u_{xy}(xy) + u_{yy}(y^2)$

$$= x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy}.$$

Consider the sum

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2}$$

$$= (x^2+y^2)u_{xx} + (x^2+y^2)u_{yy}$$

$$= r^2 u_{xx} + r^2 u_{yy}$$

$$\Rightarrow u_{xx} + u_{yy} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

equivalently, the polar form of the Laplace operator (in 2D) is

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

4. We have two representations to the sol'n from the notes ① the series sol'n and ② Poisson's formula.

Here Poisson's would be a nightmare.

Try $u(r,\theta) = \frac{A_0}{2} + \sum_1^\infty r^n (A_n \cos n\theta + B_n \sin n\theta)$

$u(a,\theta) = 1 + 3\sin\theta$ (BC)

$$u(a,\theta) = \frac{A_0}{2} + \sum_1^\infty a^n (A_n \cos n\theta + B_n \sin n\theta)$$
$$= \frac{A_0}{2} + a(A_1 \cos\theta + B\sin\theta) + \sum_2^\infty a^n (A_n \cos n\theta + B_n \sin n\theta)$$

Equating coefficients, $A_0 = 2$, $A_1 = 0$, $B_1 = 3/a$ and $A_n = B_n$ all other n.

So $u(r,\theta) = 1 + r(3/a \sin\theta)$
$$= 1 + \frac{3}{a} r\sin\theta.$$

Note, in Cartesian Coordinates $u(x,y) = 1 + \frac{3}{a} y$

5. In class, we showed sol'n on the exterior of the disk take the form

$$u(r,\theta) = \frac{A_0}{2} + \sum_1^\infty \frac{1}{r^n}(A_n \cos n\theta + B_n \sin n\theta)$$

Like the last,

$$u(a,\theta) = \frac{A_0}{2} + \frac{1}{a}(A_1 \cos\theta + B_1 \sin\theta) + \sum_2^\infty \frac{1}{r^n}(A_n \cos n\theta + B_n \sin n\theta)$$

require $= \frac{1}{2} + 3\sin\theta$ yields

$$A_0 = 2,\quad B_1 = 3a,\quad A_1 = 0,\quad A_n = B_n = 0 \ \forall n > 1.$$

So $u(r,\theta) = 1 + \frac{1}{r}(3a\sin\theta)$

$= 1 + \frac{3a}{r}\sin\theta$ in polar form.

This is actually quite a different sol'n. Note

$$u(r,\theta) = 1 + \frac{3ar\sin\theta}{r^2} = 1 + \frac{3ay}{x^2+y^2} = u(x,y)$$

Note: $u(x,y)$ solves the Cartesian Laplacian

$$u_x = \frac{-6axy}{(x^2+y^2)^2},\quad u_{xx} = \frac{-6ay}{(x^2+y^2)^2} + \frac{24ax^2y(x^2+y^2)}{(x^2+y^2)^4}$$

$$u_{xx} = \frac{6ay(3x^2-y^2)}{(x^2+y^2)^3}$$

$$u_y = \frac{3a}{x^2+y^2} - \frac{6ay^2}{(x^2+y^2)^2}$$

$$u_{yy} = \frac{-6ay}{(x^2+y^2)^2} - \left[\frac{12ay}{(x^2+y^2)^2} - \frac{24ay^3}{(x^2+y^2)^3}\right]$$

$$= \frac{-18ay}{(x^2+y^2)^2} + \frac{24ay^3}{(x^2+y^2)^3}$$

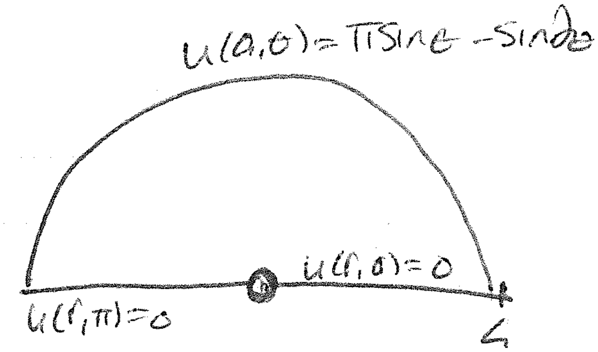
$$= \frac{-18ax^2y + 6ay^3}{(x^2+y^2)^3}$$

$$= -\frac{6ay(3x^2-y^2)}{(x^2+y^2)^3} = -u_{xx}\ !$$

Hence $u_{xx} + u_{yy} = 0$.

7p6

6. From class,



yields the Dirichlet problem in $\Theta(\theta)$: $\Theta'' + \lambda\Theta = 0$
$\Theta(0) = \Theta(\pi) = 0$

Hence $\lambda_n = n^2$ and $\Theta_n(\theta) = \sin(n\theta)$.
The R ODE is the same as before and our series sol'n has the form
$$u(r,\theta) = \sum_1^\infty A_n r^n \sin(n\theta).$$

On $r=1$, $u(r,\theta) = \sum_1^\infty A_n \sin(n\theta)$

$$= A_1\sin\theta + A_2\sin 2\theta + \sum_3^\infty A_n\sin(n\theta)$$

$$\stackrel{BC}{=} \pi\sin\theta - \sin 2\theta$$

$\Rightarrow A_1 = \pi,\ A_2 = -1,\ A_n = 0,\ n > 2.$

So sol'n in polar form is
$$u(r,\theta) = \pi r\sin\theta - r^2\sin 2\theta$$

To convert to Cartesian, use double-angle identity
$$u(r,\theta) = \pi r\sin\theta - r^2(2\sin\theta\cos\theta)$$
$$= \pi r\sin\theta - 2 r\cos\theta r\sin\theta$$

Hence $u(x,y) = \pi y - 2xy.$

7. Note $u_{xx}+u_{yy}=1$ in non-homogeneous.
We can't necessarily use our prior solns. (Series, Poisson).

Have $u_{xx}+u_{yy}=1$ in $a<r<b$,
$u(a,\theta)=0$, $u(b,\theta)=0$.

Convert to polar form $u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=1$

Assume $u(r)$ alone.
Reason for this is that the boundary conditions are independent of θ. Then $u_{\theta\theta}=0$ and we have the ODE / BVP

$$\begin{cases} u_{rr}+\frac{1}{r}u_r=1 \\ u(a)=u(b)=0 \end{cases}$$

Note the ODE is 1st order in u_r.

$$\mu(r)=\exp\left[\int\frac{1}{r}\right]=r$$

$$\text{ODE} \Rightarrow \quad r u_{rr}+u_r = r$$
$$(r u_r)' = r$$
$$r u_r = \frac{r^2}{2}+C$$
$$u_r = \frac{r}{2}+\frac{C}{r}$$

$$u(r)=\frac{r^2}{4}+C\ln r+D$$

$$u(a)=0 \Rightarrow \frac{a^2}{4}+C\ln a+D=0$$

$$u(b)=0 \Rightarrow \frac{b^2}{4}+C\ln b+D=0$$

$$-: \quad \frac{a^2-b^2}{4}+C(\ln a-\ln b)=0$$

$$C=\frac{b^2-a^2}{4(\ln a-\ln b)}$$

Then

$$D=-C\ln a-\frac{a^2}{4}$$

$$=-\frac{(b^2-a^2)\ln a}{4(\ln a-\ln b)}-\frac{a^2}{4}\cdot\frac{(\ln a-\ln b)}{(\ln a-\ln b)}$$

$$=\frac{-b^2\ln a+a^2\ln a-a^2\ln a+a^2\ln b}{4(\ln a-\ln b)}$$

$$=\frac{a^2\ln b-b^2\ln a}{4(\ln a-\ln b)}$$

To convert to cartesian,

$$u(r)=\frac{r^2}{4}+C\ln r+D \qquad (r>0)$$

$$=\frac{r^2}{4}+C\left(\frac{1}{2}\cdot 2\right)\ln r+D$$

$$=\frac{r^2}{4}+\frac{C}{2}\ln r^2+D$$

$$\Rightarrow u(x,y)=\frac{x^2+y^2}{4}+\frac{C}{2}\ln(x^2+y^2)+D.$$