MTH 414 - Spring 2023
Assignment #2
Due: Monday, January 30th 2023 (2:00PM)

1. For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous; provide reasons;

   (a) $u_t - u_{xx} + 1 = 0$

   Solution: Second order linear inhomogenous since $\mathcal{L} = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$ and $g = -1$.

   (b) $u_t - u_{xx} + xu = 0$

   Solution: Second order linear homogenous since $\mathcal{L} = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + x$ and $g = 0$.

   (c) $u_t - u_{xxt} + uu_x = 0$

   Solution: Third order nonlinear since

$$\begin{aligned}(u+v)_t - (u+v)_{xxt} + (u+v)(u+v)_x &= u_t + v_t - u_{xxt} - v_{xxt} + uu_x + vu_x + uv_x + vv_x \\ &\neq u_t + v_t - u_{xxt} - v_{xxt} + uu_x + vv_x\end{aligned}$$

   (d) $u_{tt} - u_{xx} + \frac{u}{x} = 0$

   Solution: Second order linear homogenous since $\mathcal{L} = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + \frac{1}{x}$ and $g = 0$.

   (e) $\frac{u_x}{\sqrt{1+{u_x}^2}} - u_{xy} = 0$

   Solution: Second order nonlinear since

$$\frac{u_x}{\sqrt{1+{u_x}^2+{v_x}^2+2u_xv_x}} + \frac{v_x}{\sqrt{1+{u_x}^2+{v_x}^2+2u_xv_x}} - u_{xy} - v_{xy} \neq \frac{u_x}{\sqrt{1+{u_x}^2}} + \frac{v_x}{\sqrt{1+{v_x}^2}} - u_{xy} - v_{xy}$$

   (f) $u_t + u_{xxxx} + \sqrt{1+u} = 0$

   Solution: Fourth order nonlinear since

$$u_t + u_{xxxx} + v_t + v_{xxxx} + \sqrt{1+u+v} \neq u_t + u_{xxxx} + v_t + v_{xxxx} + \sqrt{1+u} + \sqrt{1+v}$$

   (g) $u_x - e^y u_y = 0$

   Solution: First order linear homogenous since $\mathcal{L} = \frac{\partial}{\partial x} - e^y \frac{\partial}{\partial y}$ and $g = 0$.

2. Solve the equation $2u_t + 3u_x = 0$ with the auxiliary condition $u = \sin x$ when $t = 0$.

   Solution: Using the general solution $u(t,x) = f(bt-ax)$ then $u(t,x) = f(3t-2x)$. Evaluated at $t = 0$ yields $u(0,x) = f(-2x) = \sin x$. Letting $w = -2x \iff x = -\frac{1}{2}w$ gives $f(w) = \sin\left(-\frac{1}{2}w\right)$ and substituting w back gives the final solution $u(t,x) = \sin\left(-\frac{1}{2}(3t-2x)\right)$

3. (a) Solve the equation $yu_x + xu_y = 0$ with the condition $u(0,y) = e^{-y^2}$.

   Solution: $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} = \frac{x}{y} \iff \int y\,dy = \int x\,dx \iff \frac{y^2}{2} = \frac{x^2}{2} + C \implies C = y^2 - x^2$. Then $u(x,y) = f(C) = f(y^2 - x^2)$. Evaluated at $u(0,y)$ gives $f(y^2) = e^{-y^2}$. Let $w := y^2$ then $f(w) = e^{-w}$ so $u(x,y) = e^{-(y^2-x^2)} = e^{x^2-y^2}$.

   (b) In which region of the xy-plane is the solution the to IVP in (a) uniquely determined? (Hint: Recall that data is transported along characteristic curves. Look at the characteristic curves and think about what happens to the data of our initial condition.)

   Solution: The region where a unique solution exists will be at the intersection of the characteristic curves and the auxiliary condition. That is, $x = 0 \cap \mathbb{R}^2 \equiv \{(x,y) \mid x = 0\}$.

4. Solve $au_x + bu_y + cu = 0$. (Assume $a, b \in \mathbb{R}^*$)

   Solution: $au_x + bu_y = -cu$ then $f(K) = f(bx - ay)$. Hence $y = \dfrac{bx-k}{a}$ and $x = \dfrac{k+ay}{b}$.

   For $v(x,C) = v(x,y(x))$, $v_x = u_x + \frac{b}{a}u_y$ by the multivariable chain rule.

   Then $av_x = au_x + bu_y = -cv$. Solving,

$$
\begin{aligned}
& av_x = -cv \\
\iff\quad & v_x = -\frac{c}{a}v \\
\iff\quad & \frac{\partial v}{\partial x} = -\frac{c}{a}v \\
\iff\quad & \frac{1}{v}\,dv = -\frac{c}{a}\,dx \\
\iff\quad & \ln|v| = -\frac{c}{a}x + g(bx - ay) \\
\iff\quad & v(x,y) = e^{-cx/a}G(bx - ay)
\end{aligned}
$$

5. Solve the equation

$$u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2.$$

6. Consider the equation

$$u_x + yu_y = 0$$

with the boundary condition $u(x, 0) = \phi(x)$.

(a) Find the general solution to the PDE.

Solution: $\frac{\partial y}{\partial x} = \frac{y}{1} \iff \frac{1}{y}\,\mathrm{d}y = \mathrm{d}x \iff \ln|y| = x + C \iff y = Ce^x \iff C = e^{-x}y.$

Then $\frac{\partial}{\partial x}u(x, Ce^x) = u_x + Ce^x u_y = u_x + yu_y = 0$. Letting $x = 0$ then $u(0, Ce^0) = u(0, C) = u(0, e^{-x}y)$. Thus

$$u(x, y) = f(e^{-x}y)$$

(b) (BVP without a solution) For $\phi(x) \equiv x$, show that no solution exists.

Solution: $u(x, y) = f(ye^{-x})$ and $u(x, 0) = f(0) = \phi(x) = x$

$f(0) = k$ for some constant $k \in \mathbb{R}$, but $\phi(x) = x \; \forall \; x \in \mathbb{R}$.

The alleged solution curve $y_1 = k$ and BVP of $y_2 = \phi(x) = x$ Clearly $y_1 \neq y_2 \; \forall \; x \in \mathbb{R}$ since the intersection is at (k, k).

(c) (BVP without uniqueness) For $\phi(x) \equiv 1$, show that there are many solutions.