Homework 5

414 sp23

1. $$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right)$$

$$= \underbrace{\int_0^\infty \int_0^\infty}_{\text{this is QI in the xy-plane}} e^{-x^2-y^2}\,dA \qquad x^2+y^2=r^2,\quad dA = r\,dr\,d\theta$$

this is QI in the xy-plane $\longrightarrow 0 \le \theta \le \pi/2,\ r \ge 0.$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{+\infty} e^{-r^2}\, r\,dr\,d\theta$$

$$= \left(\int_0^{\pi/2} d\theta\right)\left(\int_0^\infty e^{-r^2}\, r\,dr\right), \qquad u=r^2,\quad du = 2r\,dr$$

$$= \frac{\pi}{2}\left(\int_0^\infty e^{-u}\left(\frac{1}{2}\,du\right)\right)$$

$$\overset{FTC}{=} \frac{\pi}{2}\left(-\frac{e^{-u}}{2}\Big|_0^\infty\right)$$

$$= \frac{\pi}{2}\left(-\frac{e^{-\infty}}{2} + \frac{e^{0}}{2}\right)$$

$$= \frac{\pi}{4}$$

Hence $I = \dfrac{\sqrt{\pi}}{2}$.

#U5pd

2. a) Here $\phi(x) = \begin{cases} 5, & x>0 \\ 1, & x<0 \end{cases}$.

ϕ clearly not integrable, $\int_{\mathbb{R}} \phi\, dx = \infty$.

But by how ϕ is defined, there are no heat sinks at $\pm\infty$. Here we'd expect total heat profile to level out and flow to the average value as heat is not being allowed to leave the system.
That is, we expect $u(x,t) \to \frac{5+1}{2} = 3$ as $t \to \infty$.

b) $$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{\mathbb{R}} \exp\left[\frac{-(x-y)^2}{4kt}\right] \phi(y)\, dy.$$

$$= \frac{1}{\sqrt{4k\pi t}} \left(\int_{-\infty}^{0} \exp\left[\frac{-(x-y)^2}{4kt}\right] dy + 5 \int_0^{\infty} \exp\left[\frac{-(x-y)^2}{4kt}\right] dy \right)$$

Set $z = \frac{x-y}{\sqrt{4kt}}$, $dz = \frac{-dy}{\sqrt{4kt}} \leftrightarrow dy = -\sqrt{4kt}\, dz$

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \left(\int_{+\infty}^{x/\sqrt{4kt}} e^{-z^2} (-\sqrt{4kt}\, dz) + 5 \int_{x/\sqrt{4kt}}^{-\infty} e^{-z^2} (-\sqrt{4kt}\, dz) \right)$$

$$= \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4kt}}^{\infty} e^{-z^2} dz + \frac{5}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4kt}} e^{-z^2} dz$$

HW 3
p3

$$= \frac{1}{\sqrt{\pi}}\left(\int_0^{\infty} e^{-z^2}dz - \int_0^{x/\sqrt{4kt}} e^{-z^2}dz\right)$$
$$+ \frac{5}{\sqrt{\pi}}\left[\int_0^{x/\sqrt{4kt}} e^{-z^2}dz + \int_{-\infty}^{0} e^{-z^2}dz\right]$$

$$= \frac{1}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{2}\right) - \frac{1}{\sqrt{\pi}}\int_0^{x/\sqrt{4kt}} e^{-z^2}dz$$
$$+ \frac{5}{\sqrt{\pi}}\int_0^{x/\sqrt{4kt}} e^{-z^2}dz + \frac{5}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{2}\right)$$

$$= 3 + \frac{4}{\sqrt{\pi}}\int_0^{x/\sqrt{4kt}} e^{-z^2}dz$$

$$u(x,t) = 3 + 2\,\mathrm{Erf}(x/\sqrt{4kt})$$

c) $$\lim_{t\to\infty} u(x,t) = 3 + 2\lim_{t\to\infty} \mathrm{Erf}(x/\sqrt{4kt})$$
$$= 3 + 2\left(\mathrm{Erf}(0)\right)$$
$$= 3$$

HW5 p4

3 $u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{\mathbb{R}} \exp\left[\frac{-(x-y)^2}{4kt}\right] \phi(y)\, dy$

$= \frac{1}{\sqrt{4k\pi t}} \int_0^{\infty} \exp\left[\frac{-(x-y)^2}{4kt}\right] e^{-y}\, dy.$

$\overset{(**)}{=} \frac{1}{\sqrt{4k\pi t}} \int_0^{\infty} \exp(kt-x) \exp\left[-\left(\frac{y+(2kt-x)}{\sqrt{4kt}}\right)^2\right] dy$

$\underset{\text{by next page}}{\overset{(**)}{=}} \frac{1}{\sqrt{4k\pi t}} \exp(kt-x) \int_0^{\infty} \exp\left[-\left(\frac{y+(2kt-x)}{\sqrt{4kt}}\right)^2\right] dy$

let $z = \frac{y+(2kt-x)}{\sqrt{4kt}}$

$dz = \frac{dy}{\sqrt{4kt}} \leftrightarrow dy = \sqrt{4kt}\, dz$

$= \frac{1}{\sqrt{4k\pi t}} \exp(kt-x) \int_{\frac{2kt-x}{\sqrt{4kt}}}^{+\infty} \exp\left[-z^2\right] \sqrt{4kt}\, dz$

$= \frac{1}{\sqrt{\pi}} \exp(kt-x) \left(\int_0^{\infty} e^{-z^2} dz - \int_0^{\frac{2kt-x}{\sqrt{4kt}}} e^{-z^2} dz \right)$

$= \frac{1}{\sqrt{\pi}} \exp(kt-x) \left(\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \operatorname{Erf}\left(\frac{2kt-x}{\sqrt{4kt}}\right) \right)$

$= \exp(kt-x) \left(\frac{1}{2} - \frac{1}{2} \operatorname{Erf}\left(\frac{2kt-x}{\sqrt{4kt}}\right) \right)$

HW5 p5

$$(*) = \frac{-(x-y)^2}{4kt} - y$$

$$= -\frac{1}{4kt}\left[\underbrace{(x-y)^2 + 4kty}\right]$$

$$x^2 - 2xy + y^2 + 4kty$$

$$= x^2 + (4kt - 2x)y + y^2$$

$$= x^2 + y^2 + (4kt-2x)y + \left(\frac{4kt-2x}{2}\right)^2 - (2kt - x)^2$$

$$= x^2 - (4k^2t^2 - 4ktx + x^2) + (y + (2kt - x))^2$$

$$= -4k^2t^2 + 4ktx + (y + (2kt-x))^2$$

$$(*) = -\frac{1}{4kt}\left[-4k^2t^2 + 4ktx + (y + (2kt-x))^2\right]$$

$$= \left[kt - x - \left(\frac{y + (2kt - x)}{\sqrt{4kt}}\right)^2\right]$$

HW5 p6

4 $u_t + A\sin t\, u = k u_{xx}$

a) $\mu(t) = \exp\left[\int A\sin t\right] = e^{-A\cos t}$

b) $e^{-A\cos t} u_t + A\sin t\, e^{-A\cos t} u - k e^{-A\cos t} u_{xx} = 0$

$$\partial_t (e^{-A\cos t} u) - k(e^{-A\cos t} u_{xx}) = 0.$$

Let $v(x,t) = e^{-A\cos t} u(x,t)$

c) $v_t - k v_{xx} = 0$

$v(x,0) = e^{-A} u(x,0) = e^{-A}\phi(x)$

$$v(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{\mathbb{R}} \exp\left[\frac{-(x-y)^2}{4kt}\right] e^{-A}\phi(y)\,dy.$$

$$= \frac{e^{-A}}{\sqrt{4\pi k t}} \int_{\mathbb{R}} \exp\left[\frac{-(x-y)^2}{4kt}\right] \phi(y)\,dy.$$

d) $u(x,t) = e^{A\cos t} v(x,t)$

$$= \frac{\exp[A\cos t - A]}{\sqrt{4\pi k t}} \int_{\mathbb{R}} \exp\left[\frac{-(x-y)^2}{4kt}\right] \phi(y)\,dy$$