**Math 414 – Homework 7** **Due Monday, April 10, 2023**

## Chapter 6: Laplace' Equation

1. Find the harmonic function $u(x, y)$ in the square $D = \{(x, y) \ : \ 0 < x < \pi, 0 < y < \pi\}$ with the boundary conditions

$$\begin{aligned} u_y = 0 \quad &\text{for} \quad y = 0 \text{ and } y = \pi, \\ u = 0 \quad &\text{for} \quad x = 0, \\ u(\pi, y) \quad &= \quad \frac{1}{2} + \frac{1}{2}\cos 2y. \end{aligned}$$

2. (a) Find the solution to $\Delta u = 0$ in the semi-infinite strip $0 \leq x \leq \pi$, $y \geq 0$ that satisfies the boundary conditions

$$\begin{aligned} u(0, y) &= u(\pi, y) = 0, \\ u(x, 0) &= h(x), \text{ and} \\ \lim_{y \to \infty} u(x, y) &= 0. \end{aligned}$$

   (b) What would go awry if we omitted the condition at infinity?

3. Show that the polar form of Laplace's equation $u_{xx} + u_{yy} = 0$ is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

   Do this by direct substitution. Using the change of variables $x(r, \theta) = r\cos\theta$ and $y(r, \theta) = r\sin\theta$, consider the solution $u(x(r, \theta), y(r, \theta))$ and use the chain rule to show that polar form is equivalent to the Cartesian form.

4. Solve $u_{xx} + u_{yy} = 0$ in the disk $r < a$ with the boundary condition

$$u = 1 + 3\sin\theta \text{ on } r = a.$$

   In the end, convert your answer to Cartesian coordinates, if possible.

5. (a) Solve $u_{xx} + u_{yy} = 0$ in the exterior of the disk $r < a$ with the boundary condition

$$u = 1 + 3\sin\theta \text{ on } r = a,$$

and the condition that $u$ must be bounded at $r \to \infty$.

(b) Convert your answer in (a) to Cartesian coordinates and demonstrate the solution $u(x, y)$ solves Laplace's equation.

6. Find the harmonic function $u$ in the semidisk $r < 1$, $0 < \theta < \pi$ with $u$ vanishing on the diameter $y = 0$ (i.e. $\theta = 0, \theta = \pi$) and

$$u = \pi \sin\theta - \sin 2\theta \text{ on } r = 1.$$

In the end, convert your answer to Cartesian coordinates, if possible.

7. Solve the $u_{xx}+u_{yy} = 1$ in the annulus $a < r < b$ with $u(x, y)$ zero on the circle $x^2+y^2 = a^2$ and $x^2 + y^2 = b^2$ without appealing to a Poisson's formula. In the end, convert your answer to Cartesian coordinates, if possible.

Hint: The trick problem is to recognize the Laplace operator $\Delta$ is invariant under rotations. (This is proved on page 150 of the text, but makes sense. Given a solution to the above BVP, and "spin" of the solution will still satisfy $\Delta u = 0$ and be zero on the boundary. Note, I am not asking you to prove this fact.) Given this invariance, assume the solution $u$ is independent of $\theta$ and use the polar form of Laplace's equation to solve for the solution to the PDE of the form $u(r)$. (Recall, by uniqueness, if this guess yields a solution, it is *the* solution.)