Math 414 – Homework 5 **Due Monday, March 6, 2023**

more heat: Section 2.4

1. Prove the following fact that we used in lecture:

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

Hint: Define I equal to the integral and consider the product

$$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right).$$

Combine this product into an double integral and use polar coordinates to evaluate.

2. Consider the heat equation with the initial condition $\phi(x) = 5$ for $x > 0$ and $\phi(x) = 1$ for $x < 0$.

   (a) While this initial data is physically impossible, qualitatively describe the long-term behavior of the solution to this IVP. Briefly explain your answer.

   (b) Solve the IVP and write your final answer in terms of the error function. You should also find Problem 1 useful.

   (c) Use a limit to prove your conjecture in part (a).

3. Solve the heat equation if $\phi(x) = e^{-x}$ for $x > 0$ and $\phi(x) = 0$ for $x < 0$. You answer will be simplified and in terms of $\mathrm{Erf}(x)$, if necessary.

4. Solve the heat equation with variable dissipation:

$$u_t - ku_{xx} + A\sin(t)u = 0,\ \ x \in \mathbb{R}, t > 0, A > 0.$$

with initial condition $u(x, 0) = \phi(x)$.

   (a) We need to transform this into the basic heat equation $v_t + \hat{k}v_{xx} = 0$. To do so, we construct an integrating factor. For a fixed x, treat the PDE as a first-order ODE in t and determine the integrating factor $\mu(t)$ that would be required to solve the ODE.

(b) Using your integrating factor, multiply the PDE by $\mu(t)$ and reinterpret the PDE as basic diffusion equation $v_t + \hat{k}v_{xx} = 0$. Clearly define $v(x, t)$.

(c) Determine the associated IVP in v and solve the IVP for the general solution.

(d) Determine the general solution to the original IVP in u.