HW 2 PDE Sp'23

1 a) 2nd order, linear, inhomogeneous
b) 2nd order, linear, homogeneous.
c) 3rd order, non linear
d) 2nd order, linear homogeneous
e) 2nd order, non linear
f) 4th order, non linear
g) 1st order, linear homogeneous.

2. $\frac{dx}{dt} = \frac{3}{2} \Rightarrow x = \frac{3}{2}t + C$ or $-3t + 2x = C$

$u(x,t) = f(-3t+2x)$, $f \in C^1(\mathbb{R})$

$u(0,x) = \sin x$. So $f(2x) = \sin x$
$f(w) = \sin(w/2)$

Then $u(t,x) = \sin\left(\frac{-3t+2x}{2}\right)$

HW2 p2

3a) $\frac{dy}{dx} = \frac{x}{y} \Rightarrow$ $y\,dy = x\,dx$

$y^2 = x^2 + C$

$y^2 - x^2 = C$

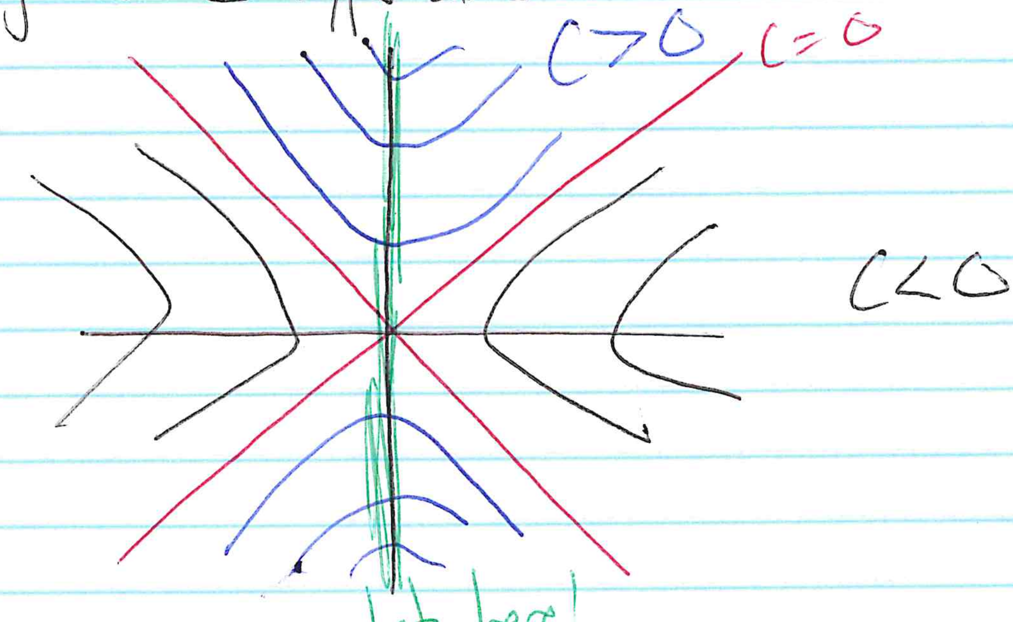
$u(x,y) = f(y^2 - x^2), \quad f \in C^1(\mathbb{R})$

$u(0,y) = f(y^2) = e^{-y^2}$

$u = y^2 \Rightarrow e^{-u}$

So $u(x,y) = \exp[x^2 - y^2]$.

b) $y^2 - x^2 = C$ hyperbolas



The data is only transported along the red and blue lines. We don't know exactly what happens in the domain

$y^2 - x^2 < 0$.

Sol'n uniquely determined in $\mathbb{R}^2$ where $y^2 - x^2 \geq 0$.

HW3 p3

4. $\frac{dy}{dx} = \frac{b}{a} \Rightarrow y = \frac{b}{a}x + K$

or $ay - bx = K$.

Let $u(x, y(x)) = v(x, K)$

$v_x = u_x + u_y\left(\frac{b}{a}\right)$ or $av_x = au_x + u_y(b)$.

PDE becomes ~~[illegible]~~ $av_x + cv = 0$

separable $\frac{v_x}{v} = \frac{-c}{a}$

$\ln|v(x)| = -\frac{c}{a}x + f(K)$

$v(x, c) = \alpha \exp\left[-\frac{c}{a}x + f(K)\right]$, $\alpha \in \mathbb{R}$.

$u(x,y) = \alpha \exp\left[-\frac{cx}{a}\right] \exp\left[f(ay-bx)\right]$

$= F(ay-bx)\, e^{-cx/a}$, F arbitrary C^1 fzn.

HW3p4

5. $u_x + 2u_y = 0 \quad \frac{dy}{dx} = 2 \quad , y = 2x + C$
or $C = y - 2x$

$u(x, y(x)) = u(x, 2x+C) = v(x, C)$
$v_x = u_x + u_y \cdot 2$.

Sub into original PDE:

$$v_x - Cv = 2x^2 + 3x(2x+C) - 2(2x+C)^2$$
$$= -5Cx - 2C^2$$

1^{st}-order linear in v: $\mu(x) = e^{-Cx}$

$$(e^{-Cx} v)_x = -5Cx e^{-Cx} - 2C^2 e^{-Cx}$$

$$e^{-Cx} v(x, C) = -5C \int x e^{-Cx} dx + 2C e^{-Cx} + f(C)$$

$\alpha = x \quad d\beta = e^{-Cx} dx$
$d\alpha = dx \quad \beta = \frac{-1}{C} e^{-Cx}$

$$= -5C\left[-\frac{x}{C} e^{-Cx} + \frac{1}{C}\int e^{-Cx} dx\right] + 2C e^{-Cx} + f(C)$$
$$= 5x e^{-Cx} + \frac{5}{C} e^{-Cx} + 2C e^{-Cx} + f(C)$$

Solve for v:
$$v(x, C) = 5x + \frac{5}{C} + 2C + f(C) e^{Cx}$$

Then
$$u(x, y) = 5x + \frac{5}{y - 2x} + 2(y - 2x) + f(y - 2x) e^{xy - 2x^2}$$

5. $u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2$

$y = 2x + C \longleftrightarrow C = y - 2x$

Let $\xi = x$

$\phi = y - 2x \longrightarrow \cancel{2x = \phi} \quad y = \phi + 2x$
$= \phi + 2\xi$

$u(\xi, \phi)$, $\quad u_x = u_\xi + u_\phi(-2)$
$u_y = u_\xi(0) + u_\phi(1)$

$$(u_\xi - 2u_\phi) + 2(u_\phi) - \phi u = 2\xi^2 + 3\xi(\phi + 2\xi) - 2(\phi + 2\xi)^2$$

$$u_\xi - \phi u = 2\xi^2 + 3\xi\phi + 6\xi^2 - 2\left[\phi^2 + 4\phi\xi + 4\xi^2\right]$$

$$= \underline{2\xi^2} + 3\xi\phi + \underline{6\xi^2} - 2\phi^2 - 8\phi\xi - \underline{8\xi^2}$$

$$= -5\phi\xi - 2\phi^2$$

Same "ODE" as on last.

#4 3p6

6. $\frac{dy}{dx} = y \qquad \frac{dy}{y} = dx \qquad y = Ce^{x}$ or $C = ye^{-x}$.

a) $u(x,y) = f(ye^{-x}) \quad , \quad f \in C^1(\mathbb{R})$

b) $u(x,0) = \phi(x) \Rightarrow f(0) = \phi(x)$

$\cancel{0}$ constant $\neq x$ for all x.

no sol'n exists

c) $u(x,0) = 1 \quad \Rightarrow \quad f(0) = 1$.

Here any differential fn $f(\alpha)$ s.t. $f(0) = 1$ solves the BVP.

∞ set of sol'ns.