Math 414 – Homework 6 **Due Friday, March 24, 2023**

Chapter 4 and 5: Fourier Series

1. Let $f(x) = \begin{cases} 1/2, & 0 \le x \le 1; \\ 2x, & 1 < x \le 2. \end{cases}$

   (a) Sketch the graph of the Fourier series expansion of $f(x)$ over $[-2, 6]$.

   (b) Sketch the graph of the Fourier cosine series expansion of $f(x)$ over $[-2, 6]$.

   (c) Sketch the graph of the Fourier sine series expansion of $f(x)$ over $[-2, 6]$.

   (d) Compute the cosine series expansion of $f(x)$.

2. Let $f(t) = \begin{cases} -t, & -3 < t < 0 \\ t^2, & 0 \le t < 3. \end{cases}$ $, f(t+6) = f(t), -\infty < t < \infty$

   (a) Graph two periods of $f(t)$.

   (b) Find the Fourier Series of f.

   (c) Use the Fourier Convergence Theorem and sketch a the graph of the function to which the Fourier series in (a) converges for $-9 \le t \le 9$.

3. As we did in class with the wave equation, you will construct the general form of solution to the heat equation with Neumann boundary conditions.

   Consider the IBVP $\begin{cases} u_t(x,t) = k u_{xx}(x,t), & 0 < x < L, \quad 0 < t < \infty, k > 0 \\ u_x(0,t) = u_x(L,t) = 0, & 0 \le t < \infty \\ u(x,0) = f(x), & 0 \le x \le L. \end{cases}$

   (a) Interpret the initial conditions $u'(0,t) = u'(L,t) = 0$ physically.

   (b) Look for a nontrivial solution to the PDE of the form $u(x,t) = X(x)T(t)$ as follows.

      i. Substitute into the PDE u_t and u_{xx} and algebraically manipulate it into the form $T'/T = X''/X$.

      ii. Set $-\lambda(x,t) = \dfrac{T'(t)}{T(t)} = \dfrac{X''(x)}{X(x)}$. Explain why $\lambda(x,t)$ must be a constant.

      iii. Using $-\lambda = \dfrac{T'(t)}{T(t)} = \dfrac{X''(x)}{X(x)}$, "separate" the PDE into the two ODEs

$$X'' + \lambda X = 0 \tag{1}$$

$$T' + \lambda T = 0 \tag{2}$$

iv. To solve the second-order ODE in the spacial coordinate, we need to interpret the initial condition $u_x(0,t) = u_x(L,t) = 0$. Do so. Then find and solve the associated eigenvalue problem.

v. Use your eigenvalues from the last part and solve the first-order ODE(s) in t.

vi. Use your previous work to find the infinite set of solutions to the problem $u_t(x,t) = u_{xx}(x,t)$, $u'(0,t) = u'(L,t) = 0$.

vii. We still haven't satisfied the final initial condition $u(x,0) = f(x)$. How do we do so?

4. Solve the IBVP

$$\begin{array}{ll} 5u_t(x,t) = u_{xx}(x,t), & 0 < x < 10, \quad 0 < t < \infty \\ u_x(0,t) = u_x(10,t) = 0, & 0 \le t < \infty \\ u(x,0) = 4x, & 0 \le x \le 10. \end{array} .$$

5. Consider the IBVP

$$\begin{array}{ll} u_{tt}(x,t) = 100u_{xx}(x,t), & 0 < x < \pi, \quad 0 < t < \infty \\ u(0,t) = u(\pi,t) = 0, & 0 \le t < \infty \\ u(x,0) = x(\pi - x), & 0 \le x \le \pi. \\ u_t(x,0) = 0, & 0 \le x \le \pi. \end{array} .$$

Give a physical interpretation to the initial conditions of this problem and then solve the IBVP.

6. (an eigenvalue problem) Consider the regular Sturm-Liouville problem

$$y'' + \lambda y = 0 \quad (0 < x < L);$$
$$y(0) = 0, \quad hy(L) - y'(L) = 0 \text{ (Robin BC)}$$

where $h > 0$.

(a) Show that $\lambda_0 = 0$ is an eigenvalue of the problem if and only if $hL = 1$, in which case the associated eigenfunction is $y_0(x) = x$.

(b) Show that the problem has a single negative eigenvalue λ_0 if and only if $hL > 1$, in which case $\lambda_0 = -\beta_0^2/L^2$ and $y_0 = \sinh \beta_0 x/L$, where β_0 is the positive root of the equation $\tanh x = x/hL$. (*Suggestion*: Sketch the graphs of $y = \tanh x$ and $y = x/hL$.)

(c) Show that the positive eigenvalues and associated eigenfunctions of the problem are $\lambda_n = \beta_n^2/L^2$ and $y_n(x) = \sin \beta_n x/L$ $(n \geq 1)$, where β_n is the nth positive root of $\tan x = x/hL$.

(d) Suppose that $hL = 1$ and that $f(x)$ is piecewise smooth. Show that

$$f(x) = c_0 x + \sum_{n=1}^{\infty} c_n \sin \frac{\beta_n x}{L},$$

where $\{\beta_n\}_1^\infty$ are the positive roots of $\tan x = x$, and

$$c_0 = \frac{3}{L^3} \int_0^L x f(x)\, dx,$$

$$c_n = \frac{2\beta_n}{L(\beta_n - \cos \beta_n \sin \beta_n)} \int_0^L f(x) \sin \frac{\beta_n x}{L}\, dx.$$

(e) Suppose the $hL = 1$. Represent the function $f(x) = A$, A a constant, as a series of eigenfunctions of the above Sturm-Liouville problem.

(f) Suppose the $hL = 1$. Represent the function $f(x) = x$ as a series of eigenfunctions of the above Sturm-Liouville problem.