MTH 427 - Spring 2023
Assignment #8
Due: Wednesday 5-3-2023

1 Exercise text book, Shumway and Stoffer

Problem #3.4
Identify the following models as ARMA(p, q) models (watch out for parameter redundancy), and determine whether they are causal and/or invertible:

(a) $x_t = 0.8x_{t-1} - 0.15x_{t-2} + w_t - 0.3w_{t-1}$

Solution:

Rewriting the equation with the backshift operator,

$$(1 - 0.8B + 0.15B^2)x_t = (1 - 0.3B)w_t.$$

Therefore,

$$\phi(z) = 1 - 0.8z + 0.15z^2 \qquad \text{and} \qquad \theta(z) = 1 - 0.3z.$$

Factoring ϕ and equating it to θ, we derive the simplified models

$$\cancel{(1 - 0.3z)}(1 - 0.5z) = \cancel{(1 - 0.3z)} \quad \Longrightarrow \quad \underbrace{1 - 0.5z}_{\phi} = \underbrace{\;1\;}_{\theta}.$$

Therefore the simplified time series is

$$x_t - 0.5x_{t-1} = w_t \quad \Longrightarrow \quad \text{ARMA}(1, 0).$$

Since $\phi(z)$ has only 1 root $z = 2$ and $|2| > 1$, the model is causal.

Since $\theta(z)$ has no roots, the model is invertible. ($\theta(z) \neq 0 \; \forall \; z \in \mathbb{C} \ni |z| \leq 1$.)

(b) $x_t = x_{t-1} - 0.5x_{t-2} + w_t - w_{t-1}$

Solution:

Rewriting the equation with the backshift operator,

$$(1 - B + 0.5B^2)x_t = (1 - B)w_t.$$

Therefore

$$\phi(z) = 1 - z + 0.5z^2 \qquad \text{and} \qquad \theta(z) = 1 - z.$$

They have no common factors, so we can conclude it is an ARMA$(2, 1)$ model.

The model is causal because $\phi(z) \neq 0 \; \forall \; z \leq 1$. In particular, the roots are

$$z = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(0.5)(1)}}{2(0.5)} \implies z = 1 \pm i$$

And $|1 \pm i| = \sqrt{2} > 1$.

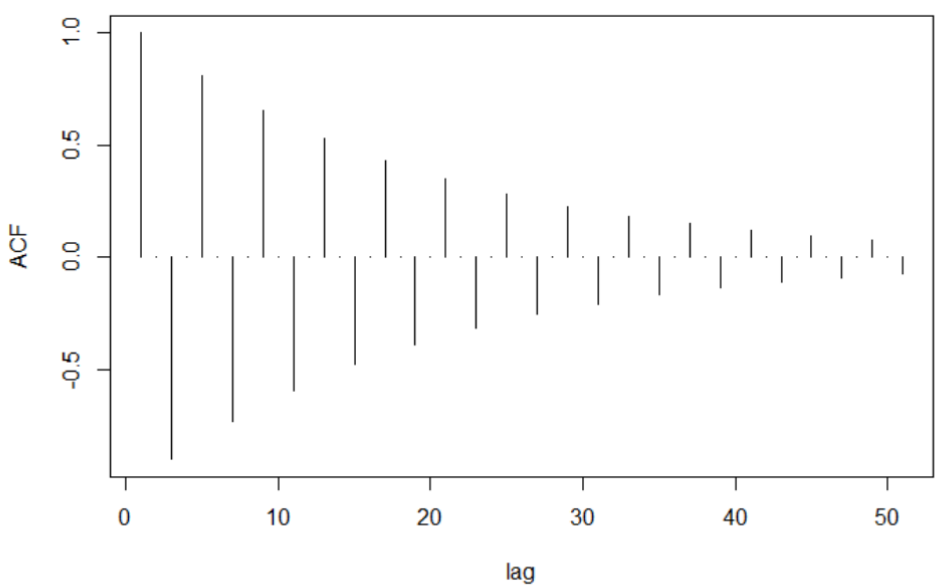
The model is not invertible because $\theta(1) = 0$ and $|1| \leq 1$.

Problem #3.6
For the AR(2) model given by $x_t = -0.9x_{t-2} + w_t$, find the roots of the autoregressive polynomial, and then sketch the ACF, $\rho(h)$.

Solution:

This can be rewritten using the backshift operator to $(1 + 0.9B^2)x_t = w_t$ which is AR(2) with polynomial $\phi(z) = 1 + 0.9z^2$ and roots $\pm\frac{\sqrt{10}}{3}$.

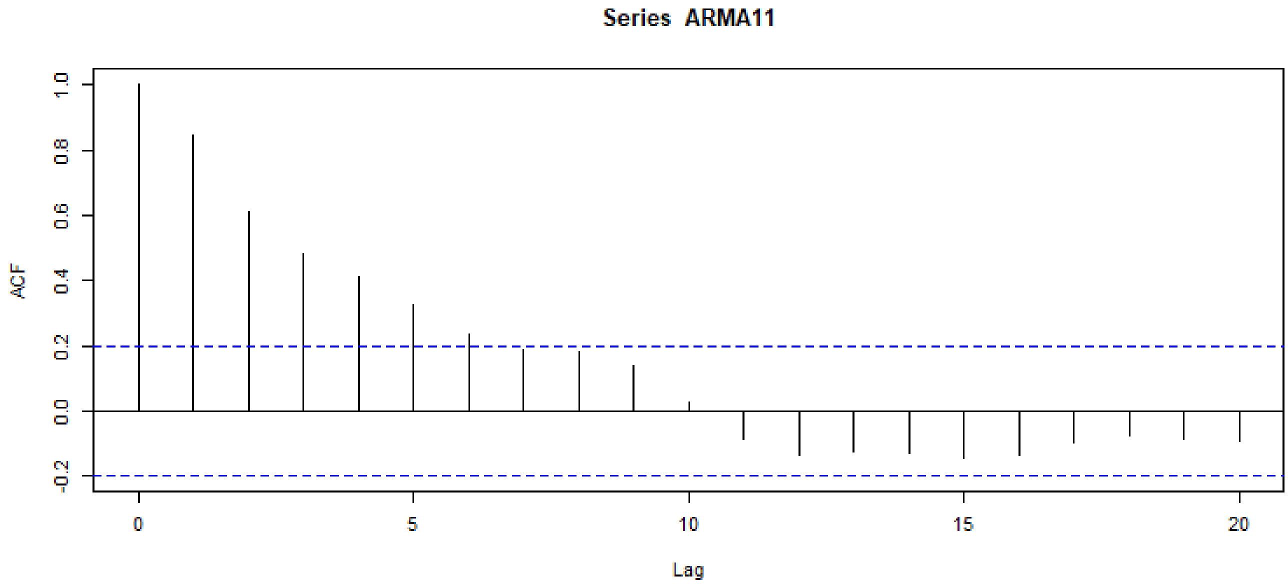
```
ACF = ARMAacf(ar=c(0, -0.9), ma=0, 50)
plot(ACF, type="h", xlab="lag")
```

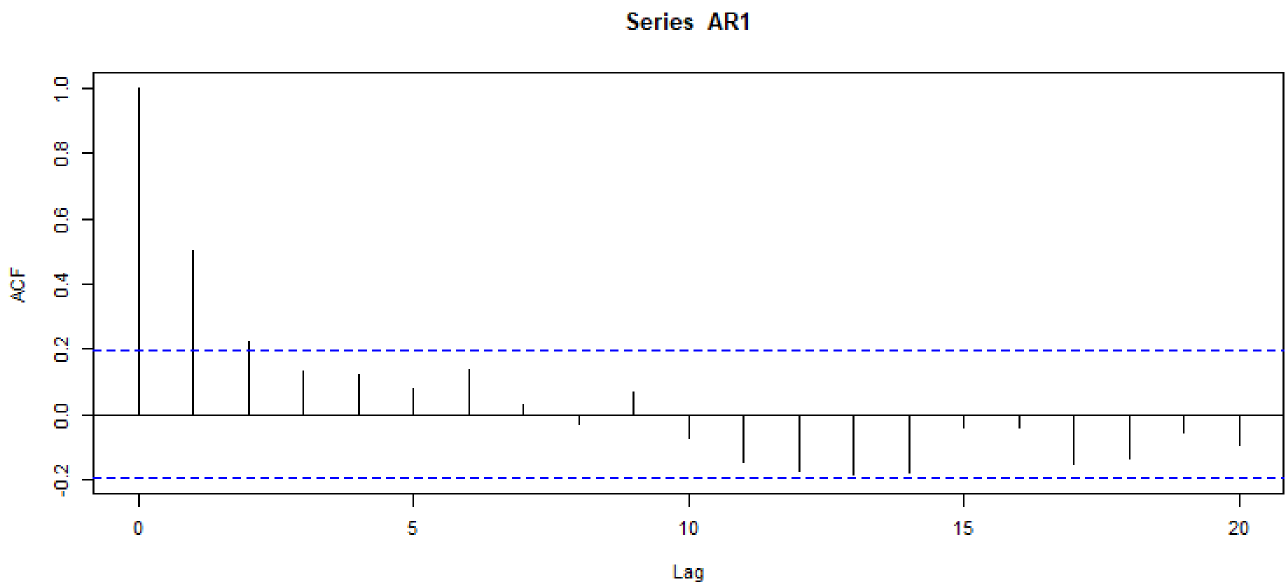


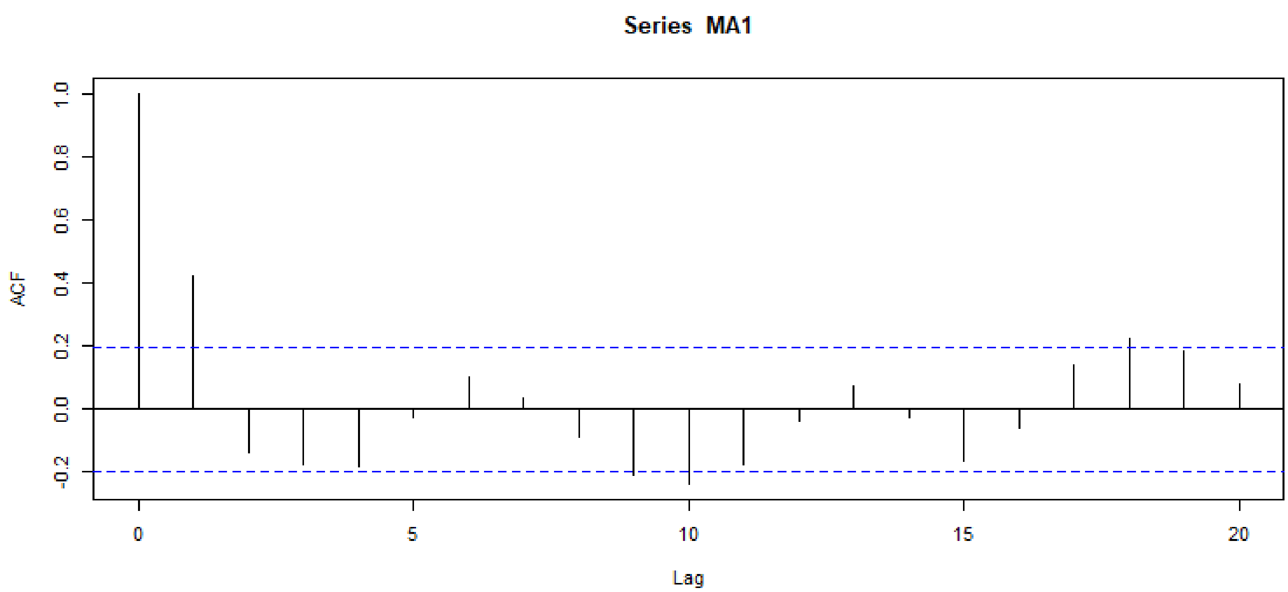
Problem #3.8
Verify the calculations for the autocorrelation function of an ARMA$(1,1)$ process given in Example 3.13. Compare the form with that of the ACF for the ARMA$(1,0)$ and the ARMA$(0,1)$ series. Plot (or sketch) the ACFs of the three series on the same graph for $\phi = 0.6, \theta = 0.9$, and comment on the diagnostic capabilities of the ACF in this case
Solution:

```
par(mfrow = c(3,1))

ARMA11=arima.sim(list(order=c(1,0,1), ar=.6, ma=0.9), n=100)
acf(ARMA11)

AR1=arima.sim(list(order=c(1,0,0), ar=.6), n=100)
acf(AR1)

MA1=arima.sim(list(order=c(0,0,1), ma=0.9), n=100)
acf(MA1)
```

After lag 2 and lag 1 for AR and MA respectively, the series becomes mostly stationary. With the full ARMA it is only below the threshold after lag 6. Thus the MA and AR alone are better than the combination of them. Between themselves, AR is better than MA because it is easier to work with and doesnt have as many spikes outside the dashed lines.


Series ARMA11
ACF
Lag
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
0
5
10
15
20


Series AR1
ACF
Lag
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
0
5
10
15
20


Series MA1
ACF
Lag
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
0
5
10
15
20

Problem #3.10

Let x_t represent the cardiovascular mortality series (`cmort`) discussed in Chapter 2, Example 2.2.

(a) Fit an AR(2) to x_t using linear regression as in Example 3.17.

Solution:

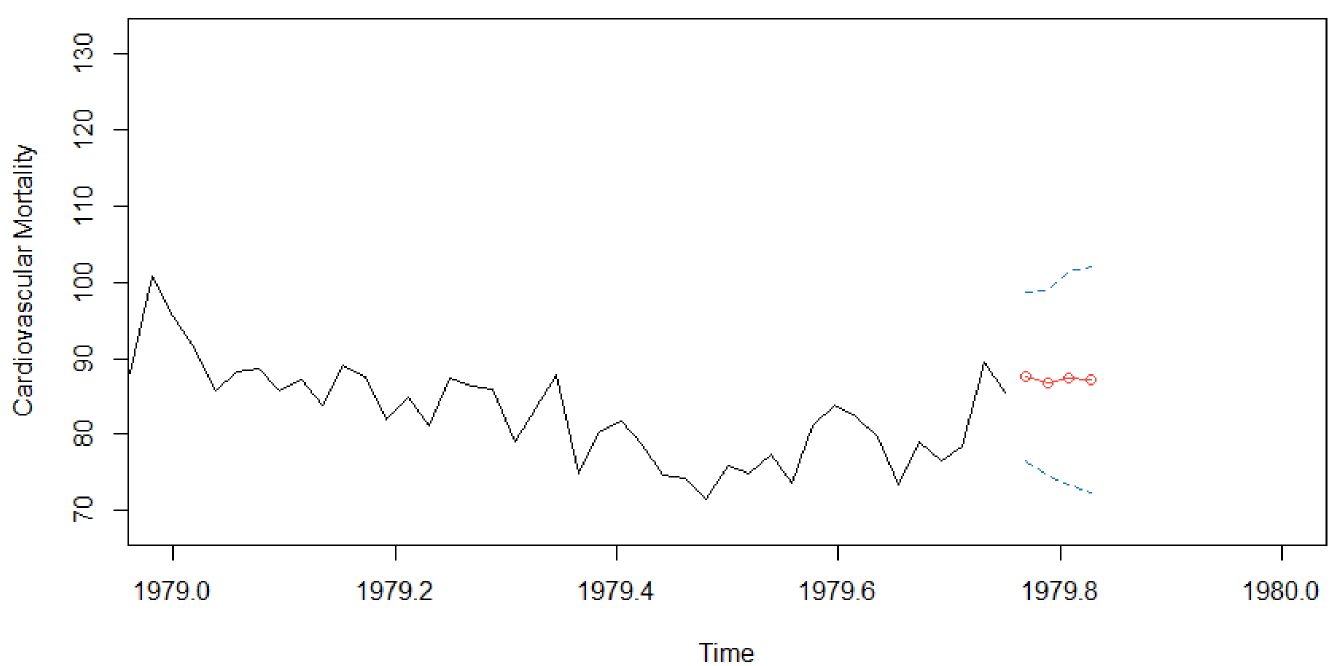
```
library(astsa)
regr = ar.ols(cmort, order=2, demean=FALSE, intercept=TRUE)
```

```
Coefficients:
     1       2
0.4286  0.4418

Intercept: 11.45 (2.394)

Order selected 2  sigma^2 estimated as  32.32
```

(b) Assuming the fitted model in (a) is the true model, find the forecasts over a four-week horizon, x^n_{n+m}, for $m = 1, 2, 3, 4$, and the corresponding 95% prediction intervals.

Solution:

```
fore=predict(regr, n.ahead=4)
ts.plot(cmort, fore$pred, col=1:2, xlim=c(1979,1980), ylab="Cardiovascular Mortality")
lines(fore$pred, type="p", col=2)
lines(fore$pred+1.96*fore$se, lty="dashed", col=4)
lines(fore$pred-1.96*fore$se, lty="dashed", col=4)
```



For the exact values of the prediction intervals,

```
ciUpper = fore$pred+1.96*fore$se
ciLower = fore$pred-1.96*fore$se

for (m in 1:4) {
  cat("m =", m, "(", ciLower[m], ",", ciUpper[m], ")\n")
}
```

yields

```
m = 1 ( 76.45756 , 98.74217 )
m = 2 ( 74.64094 , 98.88604 )
m = 3 ( 73.35405 , 101.3202 )
m = 4 ( 72.33052 , 102.0965 )
```

2 Additional Exercises

2.1 Exercise (linear process representation of ARMA)

For those models of Problem #3.4 (text book) that are causal, compute the first four coefficients $\psi_0, \psi_1, \ldots, \psi_3$ in the causal linear process representation $x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}$

Solution:

Since both models were causal, will complete in 2 parts.

(a) $x_t - 0.5x_{t-1} = w_t$

$$\begin{aligned} & \psi_0 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \cdots = \frac{\theta(z)}{\phi(z)} = \frac{1}{1-0.5z} \\ \iff \quad & (1-0.5z)(\psi_0 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \cdots) = 1 \\ \iff \quad & \psi_0 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \cdots - 0.5\psi_0 z - 0.5\psi_1 z^2 - 0.5\psi_2 z^3 - 0.5\psi_3 z^4 - \cdots = 1 \end{aligned}$$

Computing the coefficients,

$$\begin{cases} \psi_0 = 1 & \implies \psi_0 = 1 \\ \psi_1 z - 0.5(1)z = 0 & \implies \psi_1 = 0.5 \\ \psi_2 z^2 - 0.5(0.5)z^2 & \implies \psi_2 = 0.25 \\ \psi_3 z^3 - 0.5(0.25)z^3 & \implies \psi_3 = 0.125 \end{cases}$$

(b) $x_t = x_{t-1} - 0.5x_{t-2} + w_t - w_{t-1}$ with $\phi(z) = 1 - z + 0.5z^2$ and $\theta(z) = 1 - z$.

$$\begin{aligned}
& \psi_0 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \cdots = \frac{\theta(z)}{\phi(z)} = \frac{1 - z}{1 - z + 0.5z^2} \\
\iff \quad & (1 - z + 0.5z^2)(\psi_0 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \cdots) = 1 - z \\
\iff \quad & \psi_0 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \cdots - \psi_0 z - \psi_1 z^2 - \psi_2 z^3 - \psi_3 z^4 - \cdots \\
& \qquad + 0.5\psi_0 z^2 + 0.5\psi_1 z^3 + 0.5\psi_2 z^4 + 0.5\psi_3 z^5 = 1 - z
\end{aligned}$$

Computing the coefficients,

$$\begin{cases}
\psi_0 = 1 & \implies \psi_0 = 1 \\
\psi_1 z - (1)z = -z & \implies \psi_1 = 0 \\
\psi_2 z^2 - (0)z^2 + 0.5(1)z^2 = 0 & \implies \psi_2 = -0.5 \\
\psi_3 z^3 - (-0.5)z^3 + 0.5(0)z^3 = 0 & \implies \psi_3 = -0.5
\end{cases}$$