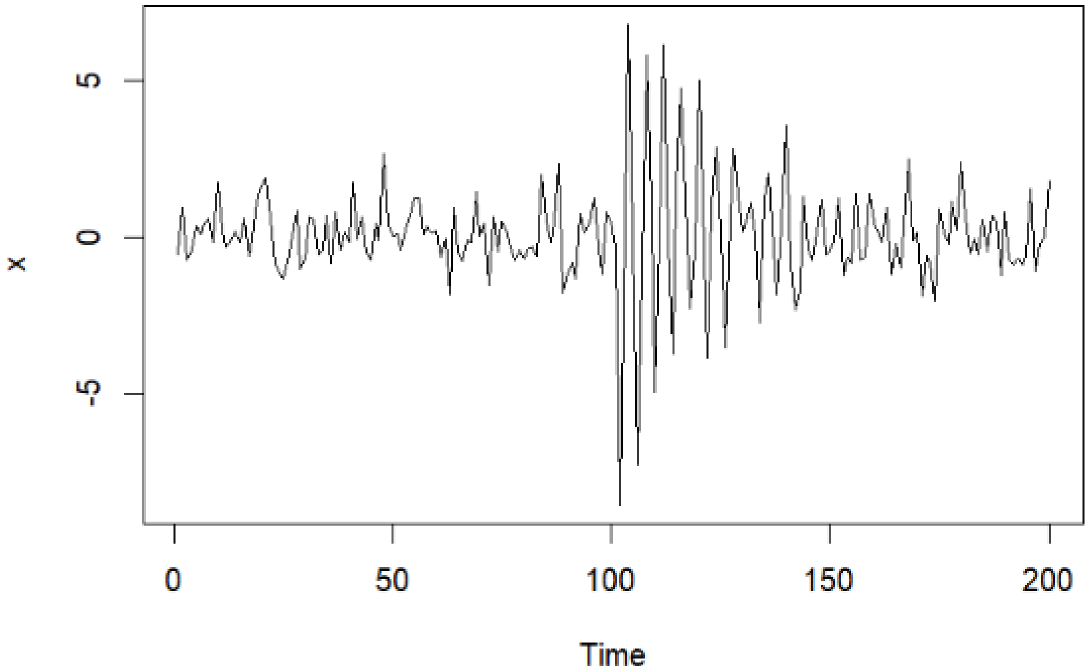
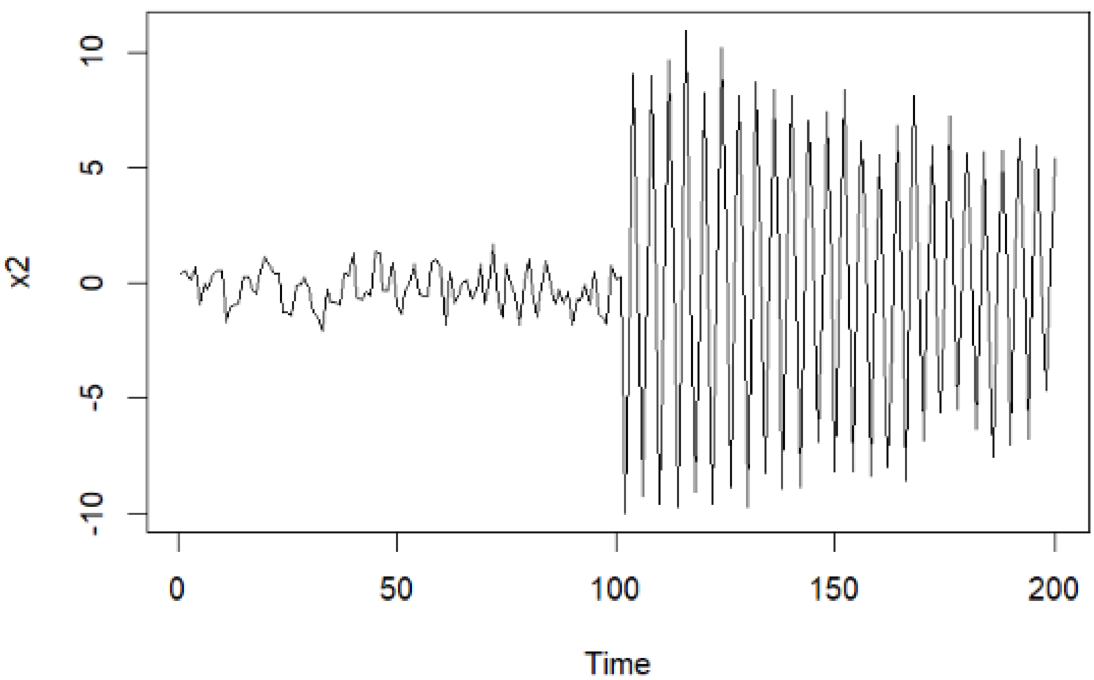
MTH 427 - Spring 2023
Assignment #6
Due: Friday, 4 7 2023

Problem 1.2 (a, b)



(a)



(b)

Problem 1.4 (a,b)

$$\begin{aligned}
\gamma(s,t) &= \mathrm{Cov}(x_s, x_t) \\
&= \mathrm{E}\left[(x_s - \mu_s)(x_t - \mu_t)\right] \\
&= \mathrm{E}\left[x_s x_t - x_s\mu_t - \mu_s x_t + \mu_t\mu_s\right] \\
&= \mathrm{E}\left[x_s x_t\right] - \mathrm{E}\left[x_s\mu_t\right] - \mathrm{E}\left[\mu_s x_t\right] + \mathrm{E}\left[\mu_t\mu_s\right] \\
&= \mathrm{E}\left[x_s x_t\right] - \mu_t\,\mathrm{E}\left[x_s\right] - \mu_s\,\mathrm{E}\left[x_t\right] + \mu_t\mu_s \\
&= \mathrm{E}\left[x_s x_t\right] - \mu_t\mu_s - \mu_s\mu_t + \mu_t\mu_s \\
&= \mathrm{E}\left[x_s x_t\right] - \mu_s\mu_t
\end{aligned}$$

Problem 1.6 (a, b)

(a) $\mathrm{E}\left[x_t\right] = \mathrm{E}\left[\beta_1\right] + \mathrm{E}\left[\beta_2 t\right] + \mathrm{E}\left[w_t\right] = \beta_1 + \beta_2 t + 0$. Since this is dependent on t, it is non-stationary.

(b) Need to show 2 conditions. First, $\mathrm{E}\left[x_t\right]$ is constant, and second $\gamma(t+h,\ t)$ depends only on h.

(1)

$$\begin{aligned}
\mathrm{E}\left[y_t\right] &= \mathrm{E}\left[x_t - x_{t-1}\right] \\
&= \mathrm{E}\left[x_t\right] - \mathrm{E}\left[x_{t-1}\right] \\
&= \beta_1 + \beta_2 t - \left[\beta_1 + \beta_2(t-1)\right] \\
&= \beta_2 && \text{(Constant)}
\end{aligned}$$

(2) $y_t = x_t - x_{t-1}$ and $x_t = \beta_1 + \beta_2 t + w_t$

$$\begin{aligned}
&\gamma(t+h,\ t) \\
&= \mathrm{Cov}(y_{t+h},\ y_t) \\
&= \mathrm{Cov}(x_{t+h} - x_{t+h-1},\ x_t - x_{t-1}) \\
&= \mathrm{Cov}(\beta_2(t+h) + w_{t+h} \ - \ (\beta_2(t+h-1) + w_{t+h-1}), \quad \beta_2(t) + w_t \ - \ (\beta_2(t-1) + w_{t-1})) \\
&= \mathrm{Cov}(w_{t+h} + w_{t+h-1},\ w_t + w_{t-1}) \\
&= \mathrm{Cov}(w_{t+h},\ w_t) + \mathrm{Cov}(w_{t+h},\ w_{t-1}) + \mathrm{Cov}(w_{t+h-1},\ w_t) + \mathrm{Cov}(w_{t+h-1},\ w_{t-1})
\end{aligned}$$

If $h = 0$ then

$$\gamma(t+0,\ t) = \underbrace{\mathrm{Cov}(w_t,\ w_t)}_{\sigma_w^2} + \underbrace{\mathrm{Cov}(w_t,\ w_{t-1})}_{0} + \underbrace{\mathrm{Cov}(w_{t-1},\ w_t)}_{0} + \underbrace{\mathrm{Cov}(w_{t-1},\ w_{t-1})}_{\sigma_w^2} = 2\sigma_w^2$$

If $h = \pm 1$ then

$$\gamma(t+1,\ t) = \underbrace{\mathrm{Cov}(w_{t+1},\ w_t)}_{0} + \underbrace{\mathrm{Cov}(w_{t+1},\ w_{t-1})}_{0} + \underbrace{\mathrm{Cov}(w_t,\ w_t)}_{\sigma_w^2} + \underbrace{\mathrm{Cov}(w_t,\ w_{t-1})}_{0} = \sigma_w^2$$

If $h = \pm 2$ then

$$\gamma(t+2,\ t) = \mathrm{Cov}(w_{t+2} + w_{t+1},\ w_t + w_{t-1}) = 0$$

Similarly for $|h| > 2,\ \gamma(t+h, t) \equiv 0$.
In none of these cases does γ depend on t.

1.8 (a,b,c)

(a)

$$\begin{aligned}
x_t &= \delta + w_t + x_{t_1} \\
&= \delta + w_t + \delta + w_{t-1} + x_{t-2} \\
&\vdots \\
&= (\delta + w_t) + (\delta + w_{t-1}) + \cdots + (\delta + w_1) + \underbrace{x_0}_{0} \\
&= \sum_{n=1}^{t} (\delta + w_n) \\
&= t\delta + \sum_{n=1}^{t} w_n
\end{aligned}$$

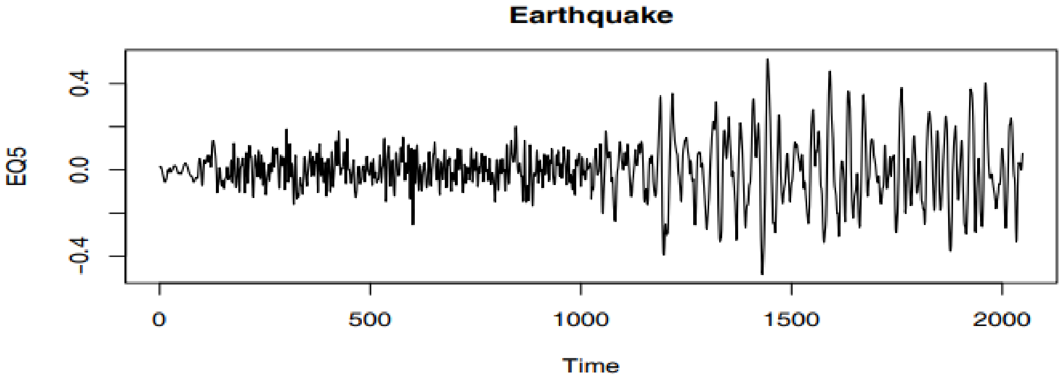
(b)

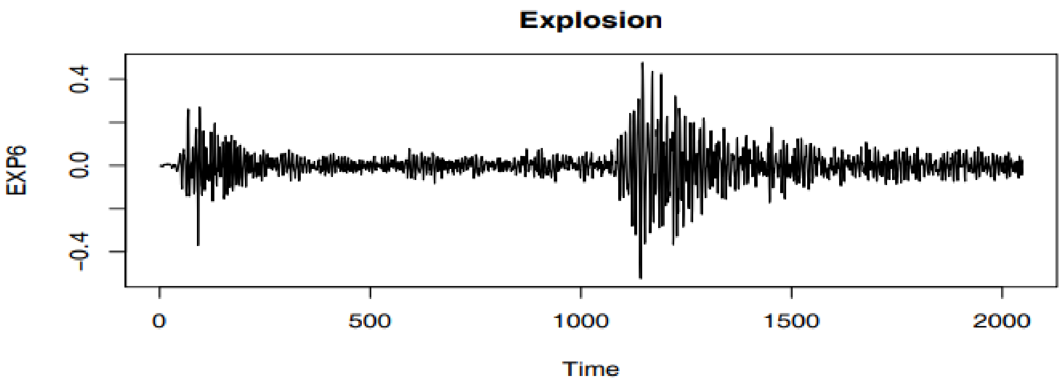
$$\begin{aligned}
\mathrm{E}\left[x_t\right] &= \mathrm{E}\left[t\delta + \sum_{n=1}^{t} w_n\right] \\
&= t\delta + \sum_{n=1}^{t} \mathrm{E}\left[w_n\right] \\
&= t\delta + \sum_{n=1}^{t} 0 \\
&= t\delta
\end{aligned}$$

And the ACVF is $t\sigma_w^2$ by the recursive nature.

(c) It is non stationary because the mean isnt constant and the ACVF depends on t.

2.1 Exercise 1





1.2a looks like the explosion, where it quickly decays near 0, and 1.2b looks like the earthquake where it slowly decays back near 0.

2.2 Exercise 2

(a) **Accidental deaths in USA:** Not stationary because the value depends on t. (More deaths when t=July and less deaths when t=Feb)

(b) **USA population** Not stationary because it grows in time.

(c) **International Airline Passengers** Not stationary because it depends on the time of year, as well as growing linearly.

2.3 Exercise 3

(a) $\mathrm{E}[X_t] = \mathrm{E}[W_2] = \mu = 0$ constant therefore stationary (with mean 0).
Then $\gamma(t+h,t) = \mathrm{Cov}(X_{t+h}, X_t) = \mathrm{Cov}(W_2, W_2) = \mathrm{Var}[W_2] = \sigma^2 = 1$

(b) $\mathrm{E}[X_t] = \mathrm{E}[t] + \mathrm{E}[W_2] = \mathrm{E}[t]$ depends on t, not stationary

(c) Since $\mathrm{Var}[X] = \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2$ then $\mathrm{E}\left[X^2\right] = \mathrm{Var}[X] + \mathrm{E}[X]^2 = \mathrm{Cov}(X,X) + \mathrm{E}[X]^2$. It follows that $\mathrm{E}[X_t] = \mathrm{E}\left[W_t^2\right] = \mathrm{Cov}(W_t, W_t) + \mathrm{E}[W_t]^2 = \sigma^2 + \mu^2 = 1 + 0^2 = 1$. Stationary because constant value.
Then $\gamma(h) = \mathrm{E}\left[(W_{t+h}^2 - \mu)(W_t^2 - \mu)\right] = \mathrm{E}\left[W_{t+h}^2 W_t^2\right]$. Then $\gamma(0) = \mathrm{E}\left[W_t^2 W_t^2\right] = \mathrm{E}\left[W_t^4\right]$. Letting $Z := W_t^2$, then $\gamma(h) = \mathrm{E}\left[Z^2\right] = \mathrm{Var}[Z] + \mathrm{E}[Z]^2$. Using how we computed the mean, $\mathrm{E}[Z]^2 = \mathrm{E}\left[W_t^2\right]^2 = 1^1 = 1$. For the variance, this is chi-square with 1 degree of freedom. Then by properties of chi-square, $\mathrm{Var}[Z] = 2$. Thus $\gamma(0) = \mathrm{E}\left[W_t^4\right] = \mathrm{E}\left[Z^2\right] = 2 + 1 = 3$. For $h \neq 0$ then W_{t+h^*} and W_t and independent so for $h^* \in \mathbb{R}^*$ then $\gamma(h^*) = \mathrm{E}\left[W_{t+h^*}^2\right] \mathrm{E}\left[W_t^2\right] = 1 \cdot 1 = 1$