MTH 427 - Spring 2023
Assignment #7
Due: Monday, 4 17 2023

1 Exercise text book, Shumway and Stoffer

Problem #2.6 Consider a process consisting of a linear trend with an additive noise term consisting of independent random variables wt with zero means and variances σ_w^2, that is,

$$x_t = \beta_0 + \beta_1 t + w_t$$

where β_0, β_1 are fixed constants.

(a) Prove x_t is nonstationary

Solution: Since $\mathrm{E}[x_t] = \mathrm{E}[\beta_0 + \beta_1 t + w_t] = \beta_0 + \beta_1 t$ depends on t, x_t is non stationary.

(b) Prove that the first difference series $\nabla x_t = x_t - x_{t-1}$ is stationary by finding its mean and autocovariance function.

Solution: $\nabla x_t = [\beta_0 + \beta_1 t + w_t] - [\beta_0 + \beta_1(t-1) + w_{t-1}] = \beta_1 + w_t - w_{t-1}$. Then

$$\mathrm{E}[\nabla x_t] = \mathrm{E}[\beta_1 + w_t - w_{t-1}] = \beta_1.$$

Next,

$$\begin{aligned}\text{ACVF} &= \mathrm{Cov}(\nabla x_t, \nabla x_{t+h}) \\ &= \mathrm{Cov}(w_t - w_{t-1},\ w_{t+h} - w_{t+h-1})\end{aligned}$$

$$\begin{aligned} h = 0 &\implies \mathrm{Cov}(w_t - w_{t-1}, w_t - w_{t-1}) = 2\sigma_w^2 \\ |h| = 1 &\implies \mathrm{Cov}(w_t - w_{t-1}, w_{t+1} - w_t) = -\sigma_w^2 \\ |h| \geq 2 &\implies \mathrm{Cov}(w_t - w_{t-1}, w_{t+2}, w_{t+1}) = 0 \end{aligned}$$

Since the ACVF doesnt depend on t and the mean is constant, ∇x_t is stationary.

(c) Repeat part (b) if w_t is replaced by a general stationary process, say y_t, with mean function μ_y and autocovariance function $\gamma_y(h)$.

Solution: $\nabla x_t = [\beta_0 + \beta_1 t + y_t] - [\beta_0 + \beta_1(t-1) + y_{t-1}] = \beta_1 + y_t - y_{t-1}$. Then

$$\mathrm{E}[\nabla x_t] = \mathrm{E}[\beta_1 + y_t - y_{t-1}] = \beta_1 + \mu_y - \mu_y = \beta_1.$$

Next,

$$\begin{aligned}\text{ACVF} &= \text{Cov}(\nabla x_t, \nabla x_{t+h}) \\ &= \text{Cov}(y_t - y_{t-1},\ y_{t+h} - y_{t+h-1})\end{aligned}$$

$$\begin{aligned}h = 0 &\implies \text{Cov}(y_t - y_{t-1}, y_t - y_{t-1}) = 2\gamma_y(0) \\ |h| = 1 &\implies \text{Cov}(y_t - y_{t-1}, y_{t+1} - y_t) = -\gamma_y(1) \\ |h| \geq 2 &\implies \text{Cov}(y_t - y_{t-1}, y_{t+2}, y_{t+1}) = 0\end{aligned}$$

Since the ACVF doesnt depend on t, and only on $\gamma_y(h)$ (which is stationary by hypothesis) and the mean is constant at β_1, ∇x_t is stationary.

2 Additional Exercises

2.1 Exercise 1 Use R

(a) Detrend the series Airline Passengers (airpass) from the package "itsmr" by fitting a linear regression of the series on time t. Is there a significant trend in the series? comment.

Solution:

```
library(itsmr)
model = lm(airpass~time(airpass))
summary(model)
```

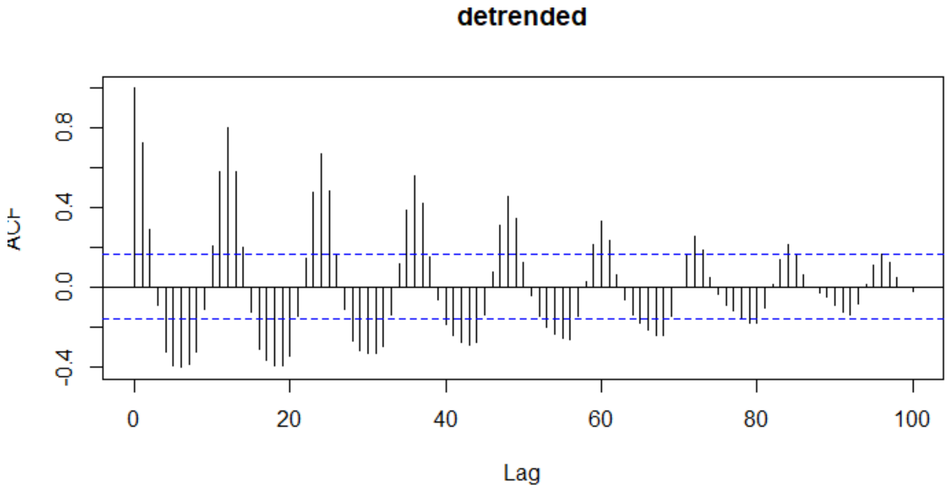
```
            Estimate Std Error t value Pr(>|t|)
(Intercept) 87.65278 7.71635 11.36 <2e-16 ***
time(airpass) 2.65718 0.09233 28.78 <2e-16 ***
```

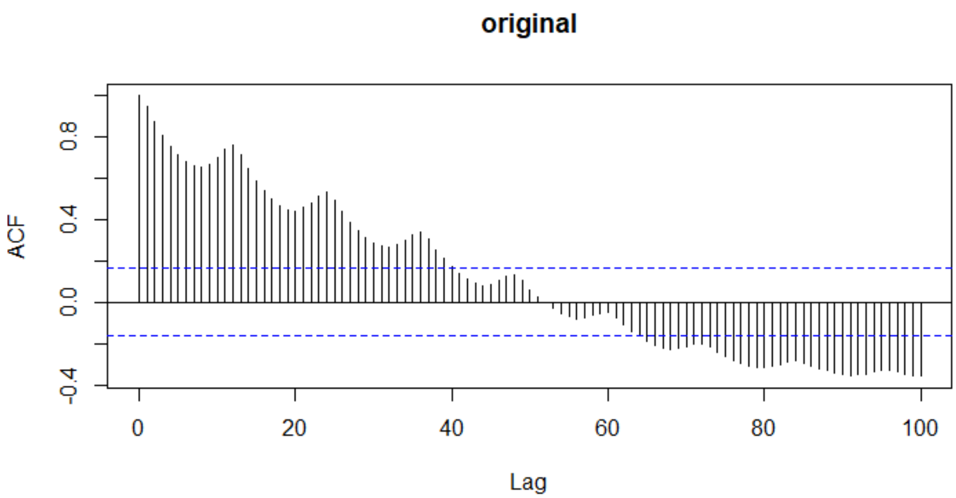
Theres a 99.9% significance level in the time coefficient, so there is definitely a trend in the series.

(b) Plot a sample **acf** of the residuals from part (a) and compare it a sample **acf** of the original series.

Solution: `acf(resid(model), 100, main="detrended")`

```
acf(airpass, 100, main="original")
```



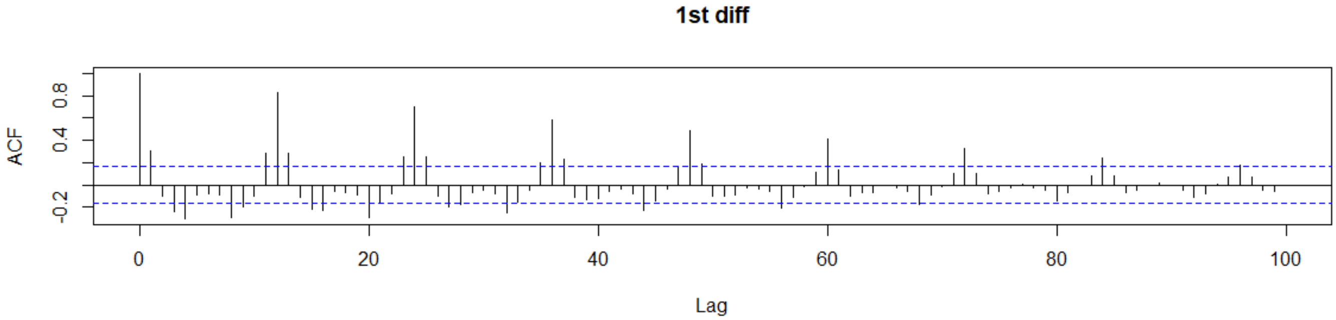


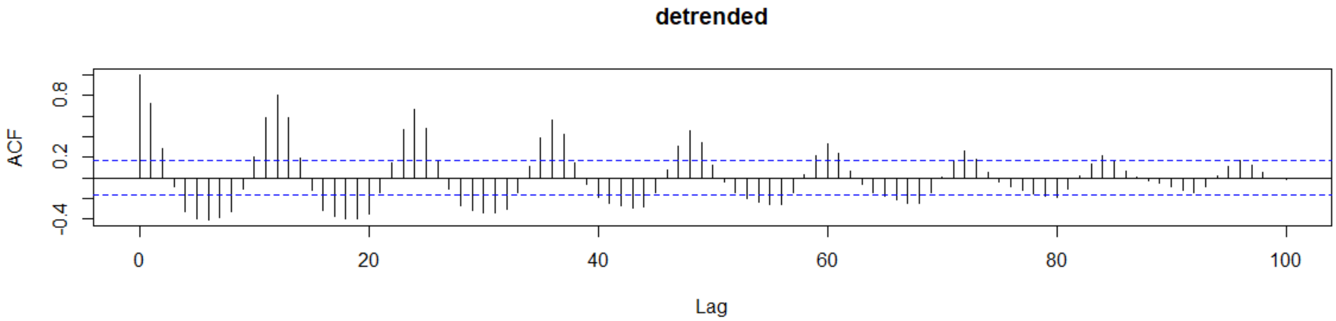
In the detrended model, the residuals are decreasing in magnitude with time. In the original model, the residuals are simply decreasing with time. Thus the detrended model should converge to 0 but the original will diverge to $-\infty$

(c) Use "diff(airpass)" to compute the first difference of the original series. Examine its sample **acf** plot and compare it with the **acf** plot of the residuals in part (b)

```
acf(diff(airpass), 100, main="1st diff")
acf(resid(model), 100, main="detrended")
```

The ACF plots look very similar between the two. The first diff seems to spikes with less magnitude that the detrended model, but not by much. The 1st difference also has more variance between its spikes, aka it's less smooth.





2.2 Exercise 2

Show your work in order to earn credit.

Consider the model $x_t = w_t - 0.3w_{t-1}$

(a) Find its ACVF (Autocovariance function).

Solution: $\text{Cov}(x_s, x_t) = \text{Cov}(w_s - 0.3w_{s-1}, w_t - 0.3w_{t-1})$. When $s = t$ then $\text{Cov}(w_t, w_t) + \text{Cov}(-0.3w_{t-1}, -0.3w_{t-1}) = 1.09\sigma_w^2$. For $|t-s| = 1$ then $\text{Cov}(w_{t+1} - 0.3w_{t+1-1}, w_t - 0.3w_{t-1}) = -0.3\sigma_w^2$. Everything else is zero. That is,

$$\text{ACVF} = \begin{cases} 1.09\sigma_w^2 & s = t \\ -0.3\sigma_w^2 & |s-t| = 1 \\ 0 & |s-t| \geq 2 \end{cases}$$

(b) Find its ACF (Auto correlation function).

Solution: Since x_t is stationary ($\text{E}[x_t] = 0$ and ACVF independent of t), then

$$\text{ACF} = \frac{\gamma(h)}{\gamma(0)} = \frac{\gamma(h)}{1.09\sigma_w^2}.$$

When $h = 0$ $\gamma(0) = 1.09\sigma_w^2$, when $h = 1$ $\gamma(1) = -0.3$, and when $h \geq 2$ $\gamma(h) = 0$. Thus

$$\text{ACF} = \begin{cases} 1 & h = 0 \\ -0.275 & |h| = 1 \\ 0 & |h| \geq 2 \end{cases}$$

2.3 Exercise 3

Identify the following models as ARMA(p, q) models.

(a) $x_t = 0.5x_{t-1} - 0.7x_{t-2} + w_t - 0.3w_{t-1}$ ARMA$(2, 1)$

(b) $x_t + 0.6x_{t-1} - 0.2x_{t-2} = w_t$ ARMA$(2, 0)$

(c) $x_t = w_t + 0.2w_{t-1}$ ARMA$(0, 1)$