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FWC22025

# OPTIMIZATION-ADVANCED

Contents



## 1 Problem

Show that the rectangle of maximum area that can be inscribed in a circle is a square.

## 2 Solution

Let the radius of circle be r = 5cm with center as origin and let the coordinates of the rectangle be A,B,C and D such that,

$$\mathbf{A} = \begin{pmatrix} rcos(\theta_1) \\ rsin(\theta_1) \end{pmatrix} \quad , \quad \mathbf{B} = \begin{pmatrix} -rcos(\theta_2) \\ rsin(\theta_2) \end{pmatrix} \quad , \quad \mathbf{C} = -\mathbf{A} \quad , \quad \mathbf{D} = -\mathbf{B}$$
$$\theta_1 = 45^\circ$$
$$\theta_2 = 60^\circ$$

The equation of circle is represented as,

$$\mathbf{x}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{1}$$

where, f =-25 As the coordinates of rectangle lies on circumference of circle, they satisfy the above equation.

$$\begin{aligned} \mathbf{A}^{\mathbf{T}}\mathbf{A} + f &= \begin{pmatrix} \frac{5}{\sqrt{2}} & \frac{5}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \end{pmatrix} - 25 \\ &= 25 - 25 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{B}^{\mathbf{T}}\mathbf{B} + f &= \begin{pmatrix} \frac{5}{2} & \frac{5\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{5}{2} \\ \frac{5\sqrt{3}}{2} \end{pmatrix} - 25 \\ &= \frac{25}{4} + \frac{75}{4} - 25 \\ &= 0 \end{aligned}$$

Similarly, C and D also satisfies the above equation And,

$$(\mathbf{A} - \mathbf{B})^{\mathbf{T}}(\mathbf{C} - \mathbf{B}) = \begin{pmatrix} \frac{5}{\sqrt{2}} - \frac{5}{2} & \frac{5}{\sqrt{2}} - \frac{5\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{-5}{\sqrt{2}} - \frac{-5}{2} \\ \frac{-5}{\sqrt{2}} - \frac{-5\sqrt{3}}{2} \end{pmatrix} \tag{2}$$

$$\implies (\mathbf{A} - \mathbf{B})^{\mathbf{T}}(\mathbf{C} - \mathbf{B}) = 0 \tag{3}$$

And,

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} \frac{-5}{\sqrt{2}} \\ \frac{-5}{\sqrt{2}} \end{pmatrix} - \begin{pmatrix} \frac{5}{2} \\ \frac{5\sqrt{3}}{2} \end{pmatrix}$$
$$\implies \mathbf{C} - \mathbf{B} = \begin{pmatrix} \frac{-5}{\sqrt{2}} - \frac{5}{2} \\ \frac{-5}{\sqrt{2}} - \frac{5\sqrt{3}}{2} \end{pmatrix}$$
$$\mathbf{D} - \mathbf{A} = \begin{pmatrix} \frac{-5}{2} \\ \frac{-5\sqrt{3}}{2} \end{pmatrix} - \begin{pmatrix} \frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \end{pmatrix}$$
$$\implies \mathbf{D} - \mathbf{A} = \begin{pmatrix} \frac{-5}{2} - \frac{5}{\sqrt{2}} \\ \frac{-5\sqrt{3}}{2} - \frac{-5}{\sqrt{2}} \end{pmatrix}$$

Hence,

$$(\mathbf{C} - \mathbf{B}) = (\mathbf{D} - \mathbf{A}) \tag{4}$$

As the diagonals of the rectangle bisect each other, let the intersection point of the diagonals be point P, such that it is equidistant from the coordinates A,B,C and D.

$$\mathbf{P} = \frac{(\mathbf{A} + \mathbf{C})}{2}$$
$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence, point P is center of circle.

$$\|P - A\| = \|P - C\| = 5$$

Hence, the distance from P to A and C is equal to the radius. Therefore, the diagonal of the rectangle is equal to the diameter of the circle.

Let the sides of the rectangle be x and y, by pythagoras theorem,

$$x^2 + y^2 = (2r)^2 \tag{5}$$
$$\implies y = \sqrt{4r^2 - x^2} \tag{6}$$

Area of rectangle = xy

Area of rectangle = x$\sqrt{4r^2 - x^r}$

Solving using Gradient ascent, to get maximum area

$$x_{n+1} = x_n + \alpha \nabla f(x_n)$$
$$\implies x_{n+1} = x_n + \alpha \nabla f(x\sqrt{4r^2 - x^2})$$

Taking $x_0 = 1, \alpha = 0.001$ and $precision = 0.00000001$, values obtained using python are:

$$\boxed{\text{Maxima} = 49.99} \tag{7}$$
$$\boxed{\text{Maxima Point} = 7.07} \tag{8}$$

As,the maximum point is x, when substituted in eq(2), we get
y = 7.07
Hence, the sides of the rectangle are equal and forms a square of maximum area.

https://github.com/madind5668/FWC
/blob/main/optimization/advanced
/codes/main.py