Lecture 6: Time Diversification

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Autumn 2022

FINM 36700: Portfolio Management

Notation

notation	description	
r_t		return rate, $(t-1 \text{ to } t)$
$r_{t,t+h}$	$\left(\prod_{\tau=1}^{h} R_{t+\tau}\right) - 1$	cumulative return rate, from t to $t+h$
$\mathtt{r}_t$	$\log\left(1+r_t\right)$	log return from $t-1$ to t
$\mathtt{r}_{t,t+h}$	$\log\left(1+r_{t,t+h}\right)$	log cumulative return from t to $t+h$.

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Cumulative return risk and autocorrelation

Log cumulative returns are simply a portfolio of single-period returns.

- Define μ and σ^2 as

$$\mathbb{E}\left[\mathtt{r}_{t,t+1}\right] = \mu, \qquad \text{var}\left[\mathtt{r}_{t,t+1}\right] = \sigma^2$$

- For the h-period log return,

$$\begin{aligned}\mathbb{E}\left[\mathtt{r}_{t,t+h}\right] &= h\mu \\ \text{var}\left[\mathtt{r}_{t,t+h}\right] &= \sum_{j=1}^{h}\sum_{i=1}^{h} \text{cov}\left[\mathtt{r}_{t+i}, \mathtt{r}_{t+j}\right]\end{aligned}$$

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Autocorrelation models

As seen in the previous slide,

- The variance of cumulative returns, $\mathtt{r}_{t,t+h}$, depends critically on the auto-covariance of the return series, denoted as

$$\text{cov}\left[\mathtt{r}_t, \mathtt{r}_{t+i}\right], \quad \text{or} \quad \sigma_{t,t+i}$$

- Specifying a form for the autocorrelations, $\text{corr}\left[\mathtt{r}_t, \mathtt{r}_{t+i}\right]$, is equivalent.
- A model of autocorrelations uniquely specifies a linear time series model.

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AR(1) models

Autoregressive (AR) models are among the most popular in time-series statistics.

Consider the AR(1) model,

$$\operatorname{cov}\left[\mathtt{r}_t, \mathtt{r}_{t+i}\right] = \rho^i \sigma^2$$
$$\operatorname{corr}\left[\mathtt{r}_t, \mathtt{r}_{t+i}\right] = \rho^i$$

With AR models, covariances are easy to scale over time.

Diversification and cumulative returns

Mean returns scale linearly in horizon h,

$$\mathbb{E}\left[\mathtt{r}_{t,t+h}\right] = h\mu$$

But scaling of variance depends on correlation,

- $\rho = 1.$ Std.Dev. is linear in cumulation: $\text{std}\left[\mathtt{r}_{t,t+h}\right] = h\sigma$
- $\rho = 0.$ Variance is linear in cumulation: $\text{var}\left[\mathtt{r}_{t,t+h}\right] = h\sigma^2$
- $\rho = -1.$ The return is riskless: $\text{var}\left[\mathtt{r}_{t,t+h}\right] = 0$

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Example: Riskless bond

At time $t = 0$, consider a bond with riskless payout at $t = 10$. The yield of the bond at $t = 0$ is 5%.

- The 10-year cumulative return, $\mathtt{r}_{0,10}$ is riskless, and equals $5\% \times 10$.
- At any intermediate time, (t, such that $0 < t < 10$,) the bond price is uncertain.
- Thus, the intermediate returns, $\mathtt{r}_{0,t}$ and $\mathtt{r}_{t,10}$ are uncertain.
- However, $\mathtt{r}_{0,10} = \mathtt{r}_{0,t} + \mathtt{r}_{t,10}$.
- So if $\mathtt{r}_{0,t}$ is unexpectedly high, then $\mathtt{r}_{t,10}$ must be relatively low, such that the riskless return $\mathtt{r}_{0,10}$ ends up at $5\% \times 10$.

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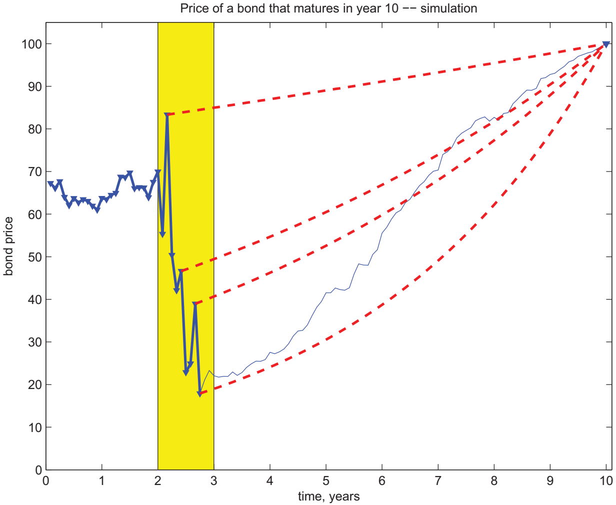

Figure: A ten-year risk-free bond has a 10-year return with perfect mean reversion. Source: Cochrane (2011).

Negative serial correlation in bonds

Thus, riskless bonds should have negatively autocorrelated returns:

$$\text{corr}\left(\mathtt{r}_t, \mathtt{r}_{t+1}\right) < 0$$

And if the bond matures at T, then for any $0 < h < T - t$,

$$\text{corr}\left(\mathtt{r}_{t,t+h}, \mathtt{r}_{t+h,T}\right) = -1$$

Default-free bonds are safer in the long-run, with $\text{var}\left[\mathtt{r}_{t,T}\right] = 0$.

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Cumulative Sharpe ratios in AR(1) model

Consider again the cumulative return, $\mathtt{r}_{t,t+h}$.

- For $\rho = 1$,

$$\mathsf{SR}\left(\mathtt{r}_{t,t+h}\right) = \mathsf{SR}\left(\mathtt{r}_{t}\right)$$

- For $|\rho| < 1$,

$$\mathsf{SR}\left(\mathtt{r}_{t,t+h}\right) > \mathsf{SR}\left(\mathtt{r}_{t}\right)$$

- For $\rho = 0$,

$$\mathsf{SR}\left(\mathtt{r}_{t,t+h}\right) = \sqrt{h}\,\mathsf{SR}\left(\mathtt{r}_{t}\right)$$

Mean annualized returns

The annualized mean (log) return on the h-period investment, $\texttt{r}_{t,t+h}$ is

$$\frac{\texttt{r}_{t,t+h}}{h} = \frac{\sum_{i=1}^{h} \texttt{r}_{t+i}}{h}$$

This is just the usual sample estimate of the mean of one-period returns, $\bar{r}$, based on a sample-size of h!

For any ρ,	For $\rho = 0$,
$\mathbb{E}\left[\bar{r}\right] = \mu$	$\text{var}\left[\bar{r}\right] = \frac{\sigma^2}{h}$

- So as the investment horizons gets large, the mean annualized return, $\bar{r}$, converges to the true annual mean return, μ.
- Even if $\rho \neq 0$ we can still conclude that $\bar{r} \to \mu$. (Law of large numbers still holds.)

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Time diversification

Time-diversification refers to this idea that mean annualized return becomes riskless for large investment horizons.

- True, as horizon increases the variance of the annualized return goes to zero.
- However, the variance of the cumulative return, $\texttt{r}_{t,t+h}$ is still growing.

So-called time diversification depends on the risk one is measuring.

Evidence: Are equity returns serially correlated?

Table: Auto-regression estimates for market returns, risk-free rate, and excess market returns. Regression estimates of $y_{t+1} = a + \rho y_t + \epsilon_{t+1}$

	Monthly			Annual		
$y = \dots$	r^m	r^f	$\tilde{r}^m$	r^m	r^f	$\tilde{r}^m$
$\hat{\rho}$	0.11	0.89	0.12	0.01	0.92	0.02
$t(\hat{\rho})$	2.02	30.38	2.05	0.09	13.31	0.15
R^2	0.01	0.80	0.01	0.00	0.83	0.00

Source: CRSP value-weighted equity markets, 1927-2010. CRSP 3-month U.S. treasury bill. GMM standard errors.

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Serial correlation of equities

- The table shows that empirically the serial correlation of excess equity returns is small.
- Not surprisingly, the serial correlation of the risk-free rate is very high.
- The serial correlation is much smaller for annual returns.

Does estimating the autocorrelation of longer-horizon returns tell a different story?

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Evidence: Cumulative equity returns

Table: Std.dev., Sharpe-ratios, and serial correlation, of excess cumulative returns.

	Horizon h (years)				
	1	3	5	7	10
$\sigma\left(\tilde{r}^{m}_{t,t+h}\right)/\sqrt{h}$	0.21	0.24	0.28	0.26	0.32
$SR/\sqrt{h}$	0.36	0.36	0.32	0.34	0.30
$\hat{\rho}$	0.01	-0.29	-0.30	0.40	0.06

- $\tilde{r}^{m}_{t,t+h}$ denotes the h-year excess return of the CRSP U.S. equity index over the 90-day t-bill.
- $\hat{\rho}$ is the serial correlation of long-horizon excess returns estimated by regressing $\tilde{r}^{m}_{t,t+h}$ on $\tilde{r}^{m}_{t-h,t}$.

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Long-run uncertainty

- The results in the table above show that normalized by $\sqrt{h}$, the Sharpe ratios remain almost constant across return horizon, or slightly decrease.
- This would be consistent with market equity excess returns having small serial correlation.
- Unfortunately, the basic estimates in the previous table are not statistically conclusive.
- Not surprising, since there are not a lot of long-horizon data points available.

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