ECS 132 Homework 1
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Problem 1. *Let's consider a simple game. Jack and Jill alternate taking turns, rolling a single die. The first one to reach a total of at least d dots wins, with the prize being $d dollars. Jill rolls first.*

1. *Find P(Jill wins).*
2. *Find P(Jill wins, taking 2 turns to do so).*
3. *Find the probability that the difference between the winner's and loser's totals is equal to 1.*
4. *The 10 O'Clock News reports that Jill won, but doesn't say what her prize was. Find the probability that her prize was $6.*
5. *Write a function with call form*
   simjj(d,nreps)
   that will simulate nreps repetitions of the game, for general values of d. The return value will be an R list, with components winner, prize and loserDots, each of which is a vector of length nreps, showing winners, winners' prize money and losers' number of dots. (E.g. if Jack wins and Jill has accumulated 3 dots by then, then loserDots is 3.)

Solution. Let A_i be the roll of Jill and B_i be the roll of Jack where i denotes the turn number.

1. P(Jill wins) = P(Jill wins first turn) + P(Jill wins second turn) + P(Jill wins third turn) + P(Jill wins fourth turn)

$$P(\text{Jill wins first turn}) = P(A_1 \geq 4) = \frac{3}{6}$$

$$\begin{aligned} P(\text{Jill wins second turn}) &= P(\text{Jill and Jack doesn't win on first turn, Jill wins on the second turn}) \\ &= P(\text{Jill doesn't win on 1st turn})P(\text{Jack doesn't win on 1st turn}) \\ &* P(\text{Jill wins on 2nd turn given she doesn't win on first turn}) \\ &= P(A_1 < 4)P(B_1 < 4)P(A_1 + A_2 \geq 4 \mid A_1 < 4) \\ &= P(A_1 < 4)P(B_1 < 4) * \frac{P(A_1 + A_2 \geq 4, A_1 < 4)}{P(A_1 < 4)})(\text{mailing tube 2.7}) \\ &= \frac{3}{6} * \frac{3}{6} * \frac{15}{18} = \frac{5}{24} \end{aligned}$$

$$\begin{aligned} P(\text{Jill wins third turn}) &= P(\text{Jill and Jack doesn't win on 1st 2 turns, Jill wins on the third turn}) \\ &= P(\text{Jill doesn't win on 1st 2 turns})P(\text{Jack doesn't win on 1st 2 turns}) \\ &* P(\text{Jill wins on 3rd turn given she doesn't win on 1st 2 turn}) \\ &= P(A_1 + A_2 < 4)P(B_1 + B_2 < 4)P(A_1 + A_2 + A_3 \geq 4 \mid A_1 + A_2 < 4) \\ &= P(A_1 + A_2 < 4)P(B_1 + B_2 < 4)\frac{P(A_1 + A_2 + A_3 \geq 4, A_1 + A_2 < 4)}{P(A_1 + A_2 < 4)}(\text{mailing tube 2.7}) \\ &= \frac{3}{36} * \frac{3}{36} * \frac{17}{18} = \frac{17}{2592} \end{aligned}$$

$$\begin{aligned}
P(\text{Jill wins fourth turn}) &= P(\text{Jill and Jack doesn't win on 1st 3 turns, Jill wins on the fourth turn}) \\
&= P(\text{Jill doesn't win on 1st 3 turns})P(\text{Jack doesn't win on 1st 3 turns}) \\
&* P(\text{Jill wins on 4th turn given she doesn't win on 1st 3 turn}) \\
&= P(A_1 + A_2 + A_3 < 4)P(B_1 + B_2 + B_3 < 4) \\
&* P(A_1 + A_2 + A_3 + A_4 \geq 4 \mid A_1 + A_2 + A_3 < 4) \\
&= P(A_1 + A_2 + A_3 < 4)P(B_1 + B_2 + B_3 < 4) \\
&* \frac{P(A_1 + A_2 + A_3 + A_4 \geq 4, A_1 + A_2 + A_3 < 4)}{P(A_1 + A_2 + A_3 < 4)} (\text{mailing tube 2.7}) \\
&= \frac{1}{216} * \frac{1}{216} * 1 = \frac{1}{216^2}
\end{aligned}$$

P(Jill wins) $= \frac{3}{6} + \frac{5}{24} + \frac{17}{2592} + \frac{1}{216^2} = 0.715$

2. P(Jill wins in 2 turns)

$$\begin{aligned}
P(\text{Jill wins second turn}) &= P(\text{Jill and Jack doesn't win on first turn, Jill wins on the second turn}) \\
&= P(\text{Jill doesn't win on 1st turn})P(\text{Jack doesn't win on 1st turn}) \\
&* P(\text{Jill wins on 2nd turn given she doesn't win on first turn}) \\
&= P(A_1 < 4)P(B_1 < 4)P(A_1 + A_2 \geq 4 \mid A_1 < 4) \\
&= P(A_1 < 4)P(B_1 < 4) * \frac{P(A_1 + A_2 \geq 4, A_1 < 4)}{P(A_1 < 4)} (\text{mailing tube 2.7}) \\
&= \frac{3}{6} * \frac{3}{6} * \frac{15}{18} = \frac{5}{24}
\end{aligned}$$

3. $P(|\sum A_i - \sum B_i| = 1)$

$$P(|\sum A_i - \sum B_i| = 1) = P(\sum A_i = 4, \sum B_i = 3) + P(\sum A_i = 3, \sum B_i = 4)$$

$$\begin{aligned}
P(\sum A_i = 4, \sum B_i = 3) &= P(A_1 + A_2 = 4, B_1 = 3) + P(A_1 + A_2 + A_3 = 4, B_1 + B_2 = 3) \\
&+ P(A_1 + A_2 + A_3 + A_4 = 4, B_1 + B_2 + B_3 = 3) \\
P(A_1 + A_2 = 4, B_1 = 3) &= \frac{3}{36} * \frac{1}{6} = \frac{18}{36^2} \\
P(A_1 + A_2 + A_3 = 4, B_1 + B_2 = 3) &= \frac{3}{216} * \frac{2}{36} = \frac{12}{216^2} \\
P(A_1 + A_2 + A_3 + A_4 = 4, B_1 + B_2 + B_3 = 3) &= \frac{1}{6^4} * \frac{1}{6^3} = \frac{1}{6^7} \\
P(\sum A_i = 4, \sum B_i = 3) &= \frac{18}{36^2} + \frac{12}{216^2} + \frac{1}{6^7}
\end{aligned}$$

$$\begin{aligned}
P(\sum A_i = 3, \sum B_i = 4) &= P(A_1 = 3, B_1 = 4) + P(A_1 + A_2 = 3, B_1 + B_2 = 4) \\
&+ P(A_1 + A_2 + A_3 = 3, B_1 + B_2 + B_3 = 4) \\
P(A_1 = 3, B_1 = 4) &= \frac{1}{6} * \frac{1}{6} = \frac{1}{36} \\
P(A_1 + A_2 = 3, B_1 + B_2 = 4) &= \frac{2}{36} * \frac{3}{36} = \frac{6}{36^2} \\
P(A_1 + A_2 + A_3 = 3, B_1 + B_2 + B_3 = 4) &= \frac{1}{216} * \frac{3}{216} = \frac{3}{216^2} \\
P(\sum A_i = 3, \sum B_i = 4) &= \frac{1}{36} + \frac{6}{36^2} + \frac{3}{216^2}
\end{aligned}$$

$P(|\sum A_i - \sum B_i| = 1) = \frac{18}{36^2} + \frac{12}{216^2} + \frac{1}{6^7} + \frac{1}{36} + \frac{6}{36^2} + \frac{3}{216^2} = 0.047$

4.

$$
\begin{aligned}
P(\text{Jill's prize was \$6 given she wins}) &= P(\text{Jill's prize was \$6} \mid \text{Jill wins}) \\
&= \frac{P(\text{Jill wins} \mid \text{Jill's prize was \$6})P(\text{Jill's prize was \$6 })}{P(\text{Jill wins})} (\text{Bayes' rule}) \\
P(\text{Jill wins} \mid \text{Jill's prize was \$6}) &= 1 \\
P(\text{Jill wins }) &= 0.715 \\
P(\text{Jill's prize was \$6}) &= P(\text{Jill's prize was \$6} \mid \text{Jill wins first turn})P(\text{Jill wins first turn}) \\
&+ P(\text{Jill's prize was \$6} \mid \text{Jill wins second turn})P(\text{Jill wins second turn}) \\
&+ P(\text{Jill's prize was \$6} \mid \text{Jill wins third turn})P(\text{Jill wins third turn}) \\
&+ P(\text{Jill's prize was \$6} \mid \text{Jill wins fourth turn})P(\text{Jill wins fourth turn}) \\
&= \frac{1}{3} \times \frac{3}{6} + \frac{3}{15} \times \frac{5}{24} + \frac{3}{17} \times \frac{17}{2592} + \frac{1}{6} \times \frac{1}{216^2} \\
&\approx 0.209 \\
P(\text{Jill's prize was \$6 given she wins}) &= \frac{0.209 \times 1}{0.715} \\
&= 0.292
\end{aligned}
$$

5. Coding Problem.

```
roll <- function() return(sample(1:6,1))

simjj <- function(d, nreps) {
  # Creating all the vectors
        w <- vector(length=nreps)
        p <- vector(length=nreps)
        lD <- vector(length=nreps)
        for (rep in 1:nreps) {
          # Initializing the total number of dots
                dots1 <- 0
                dots2 <- 0
                # Simulating the game
                while(dots1 < d && dots2 < d) {
                        dots1 <- dots1 + roll()
                        if(dots1 < d) {
                                dots2 <- dots2 + roll()
                        }
                }
                # Checking the winner
                if(dots1 >= d) {
                        w[rep] <- 'Jill'
                        p[rep] <- dots1
                        lD[rep] <- dots2
                } else {
                        w[rep] <- 'Jack'
                        p[rep] <- dots2
                        lD[rep] <- dots1
                }
        }
        return (list(winner = w, prize = p, loserDots = lD))
}
```

Problem 2. *Coding Problem*

Solution. See Code.

```r
pnk <- function(d,s,k) {
  # Base Cases
  if (d<=0 || k<=0) return (0)
  if (k > d) return (0)
  if (k==1 && d==1) return (1)
  if (d > k*s) return (0)

  # Recursion
  sum <- 0
  for (r in 1:s) {
    sum <- sum + pnk(d-r, s, k-1)
  }
  return (sum / s)
}
```

Problem 3. *Consider the bus ridership example. Say this is a tiny bus, with a limit of 3 passengers.*

1. *Find the probabilities that 0, 1 or 2 waiting passengers at Stop 2 fail to board.*
2. *You plan to go to Stop 2 to take the bus. Find the probability that you are turned away. Assume rejected passengers are chosen randomly, and that you are there in addition to the usual 0, 1 or 2 waiting passengers.*

Solution. Let B_i be the number of people new passengers boarding the bus who board the bus at the ith stop. Let A_i be the number of passengers who left the bus at stop i, independently with probability 0.2. As specified in the Bus Rider example, either 0,1, or 2 passengers get on the bus with probability $0.5, 0.4, 0.1$ respectively.

1. - P(0 people at stop 2 fail to board)

$$\begin{aligned} P(\text{0 people at stop 2 fail to board}) &= 1 - (P(\text{1 people at stop 2 fail to board})) \\ &= 1 - 0.0064 \\ &= 0.9936 \end{aligned}$$

   - P(1 people at stop 2 fail to board)

$$\begin{aligned} P(\text{1 people at stop 2 fail to board}) &= P(B_1 = 2, B_2 = 2, A_2 = 0) \\ &= (.1)(.1)(1 - .2)^2 \\ &= .0064 \end{aligned}$$

   - P(2 people at stop 2 fail to board) = 0

2. Let R_i bus seats open at stop i and Let W_i be the total number of people waiting at stop i (including yourself) . P(rejected) = P(rejected, with 0 waiting passengers) + P(rejected, with 1 waiting passengers) + P(rejected, with 2 waiting passengers)

   - P(rejected, with 0 waiting passengers) = 0
   - P(rejected, with 1 waiting passengers)

$$\begin{aligned} P(\text{rejected, with 1 waiting passengers}) &= P(B_1 = 2, B_2 = 1, A_2 = 0) * P(\text{being rejected}) \\ P(\text{being rejected}) = 1 - \frac{\text{number of seats open on bus}}{\text{number of people waiting at stop}} &= 1 - \frac{R_2 = 1}{W_2 = 2} = .5 \\ &= (.1)(.4)(1 - .2)^2 * (.5) \\ &= 0.0128 \end{aligned}$$

   - P(rejected, with 2 waiting passengers)

$$\begin{aligned} P(\text{rejected, with 2 waiting passengers}) &= P(B_1 = 2, B_2 = 2, A_2 = 0) P(\text{being rejected with } R_2 = 1, W_2 = 3) \\ &+ P(B_1 = 2, B_2 = 2, A_2 = 1) P(\text{being rejected with } R_2 = 2, W_2 = 3) \\ &+ P(B_1 = 1, B_2 = 2, A_2 = 0) P(\text{being rejected with } R_2 = 2, W_2 = 3) \end{aligned}$$

$$P(B_1 = 2, B_2 = 2, A_2 = 0)P(\text{being rejected with } R_2 = 1, W_2 = 3) = (.1)(.1)(1 - .2)^2 * (\frac{2}{3})$$
$$= 0.0042$$

$$P(B_1 = 2, B_2 = 2, A_2 = 1)P(\text{being rejected with } R_2 = 2, W_2 = 3) = (.1)(.1) * 2((1 - .2)(.2)) * (\frac{1}{3})$$
$$= 0.001$$

$$P(B_1 = 1, B_2 = 2, A_2 = 0)P(\text{being rejected with } R_2 = 2, W_2 = 3) = (.4)(.1)(1 - .2)^2 * (\frac{1}{3})$$
$$= 0.009$$

$$P(\text{rejected, with 2 waiting passengers}) = 0.0042 + 0.001 + 0.009 = 0.0142$$

P(rejected) = 0 + 0.0128 + 0.0142 = 0.027

Problem 4. *Coding Problem*

Solution. See Code

```
simvir <- function(c,r,m,n,nreps) {

  counter <- 0
  # Simulating the reps
  for(reps in 1:nreps) {
    # Creating the infected and non-infected
    infected <- 1
    noninfected <- (c - 1)

    # Simulating the epochs
    for (epoch in 1:m) {
      # Temporary storing variables
      k <- infected
      s <- noninfected
      # Simulating all the computers
      for (computer in 1:s) {
        if (runif(1) > ((r)^k) ) {
          noninfected <- noninfected - 1
          infected <- infected + 1
        }
      }
    }
    if(noninfected == n) counter <- counter + 1
  }
  return (counter / nreps)
}
```

Problem 5. *In Problem 1 (but with d = 3), find the expected value and variance of Jill's winnings. If she loses, her winnings are \$0. Note that the expected value is the long-run average of a column labeled "Jill's winnings."*

Solution. E(Prize Money) =

$$\sum_{i=3}^{8} i * P(\text{Jill winning prize} = \$i)$$

Let A_i be the roll of Jill and B_i be the roll of Jack where i denotes the turn number.

$$P(\text{Jill wins \$3}) = P(\text{rolls a 3 on round 1 or rolls a total of 3 on round 2 or rolls a total of 3 on round 3})$$
$$P(\text{Jill wins \$3}) = P(\text{rolls a 3 on round 1}) + P(\text{rolls a total of 3 on round 2}) + P(\text{rolls a total of 3 on round 3})$$

$$P(\text{rolls a 3 on round 1}) = \frac{1}{6}$$

$$\begin{aligned} P(\text{rolls a total of 3 on round 2}) &= P(\text{Jack and Jill did not win round 1}) \\ &* P(\text{Jill rolls add up to 3} \mid \text{she didn't win round 1}) \\ &= \frac{2}{6} * \frac{2}{6} * \frac{2}{12} \end{aligned}$$

$$\begin{aligned} P(\text{rolls a total of 3 on round 3}) &= P(\text{Jack and Jill did not win round 2}) \\ &* P(\text{Jill rolls add up to 3} \mid \text{she didn't win round 1 and 2}) \\ &= \frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6} \end{aligned}$$

P(Jill wins \$3) $= \frac{1}{6} + (\frac{2}{6} * \frac{2}{6} * \frac{2}{12}) + (\frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6}) = 0.1853$

$$P(\text{Jill wins \$4}) = P(\text{rolls a 4 on round 1 or rolls a total of 4 on round 2 or rolls a total of 4 on round 3})$$
$$P(\text{Jill wins \$4}) = P(\text{rolls a 4 on round 1}) + P(\text{rolls a total of 4 on round 2}) + P(\text{rolls a total of 4 on round 3})$$

$$P(\text{rolls a 4 on round 1}) = \frac{1}{6}$$

$$\begin{aligned} P(\text{rolls a total of 4 on round 2}) &= P(\text{Jack and Jill did not win round 1}) P(\text{Jill rolls add up to 4} \mid \text{she didn't win round 1}) \\ &= \frac{2}{6} * \frac{2}{6} * \frac{2}{12} \end{aligned}$$

$$\begin{aligned} P(\text{rolls a total of 4 on round 3}) &= P(\text{Jack and Jill did not win round 1 or 2}) \\ &* P(\text{Jill rolls add up to 4} \mid \text{she didn't win round 1 and 2}) \\ &= \frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6} \end{aligned}$$

P(Jill wins \$4) $= \frac{1}{6} + (\frac{2}{6} * \frac{2}{6} * \frac{2}{12}) + (\frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6}) = 0.1853$

$$P(\text{Jill wins \$5}) = P(\text{rolls a 5 on round 1 or rolls a total of 5 on round 2 or rolls a total of 5 on round 3})$$
$$P(\text{Jill wins \$5}) = P(\text{rolls a 5 on round 1}) + P(\text{rolls a total of 5 on round 2}) + P(\text{rolls a total of 5 on round 3})$$

$$P(\text{rolls a 5 on round 1}) = \frac{1}{6}$$

$$
\begin{aligned}
P(\text{rolls a total of 5 on round 2}) &= P(\text{Jack and Jill did not win round 1})P(\text{Jill rolls add up to 5} \mid \text{she didn't win round 1}) \\
&= \frac{2}{6} * \frac{2}{6} * \frac{2}{12}
\end{aligned}
$$

$$
\begin{aligned}
P(\text{rolls a total of 5 on round 3}) &= P(\text{Jack and Jill did not win round 2}) \\
&\quad * P(\text{Jill rolls add up to 5} \mid \text{she didn't win round 1 and 2}) \\
&= \frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6}
\end{aligned}
$$

P(Jill wins \$5) $= \frac{1}{6} + (\frac{2}{6} * \frac{2}{6} * \frac{2}{12}) + (\frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6}) = 0.1853$

$$
\begin{aligned}
P(\text{Jill wins \$6}) &= P(\text{rolls a 6 on round 1 or rolls a total of 6 on round 2 or rolls a total of 6 on round 3}) \\
P(\text{Jill wins \$3}) &= P(\text{rolls a 6 on round 1}) + P(\text{rolls a total of 6 on round 2}) + P(\text{rolls a total of 6 on round 3})
\end{aligned}
$$

$$
P(\text{rolls a 6 on round 1}) = \frac{1}{6}
$$

$$
\begin{aligned}
P(\text{rolls a total of 6 on round 2}) &= P(\text{Jack and Jill did not win round 1})P(\text{Jill rolls add up to 6} \mid \text{she didn't win round 1}) \\
&= \frac{2}{6} * \frac{2}{6} * \frac{2}{12}
\end{aligned}
$$

$$
\begin{aligned}
P(\text{rolls a total of 6 on round 3}) &= P(\text{Jack and Jill did not win round 2}) \\
&\quad * P(\text{Jill rolls add up to 6} \mid \text{she didn't win round 1 and 2}) \\
&= \frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6}
\end{aligned}
$$

P(Jill wins \$6) $= \frac{1}{6} + (\frac{2}{6} * \frac{2}{6} * \frac{2}{12}) + (\frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6}) = 0.1853$

P(Jill wins \$7)

$$
\begin{aligned}
P(\text{Jill wins \$7}) &= P(\text{rolls a total of 7 on round 2 or rolls a total of 7 on round 3}) \\
&= P(\text{rolls a total of 7 on round 2}) + P(\text{rolls a total of 7 on round 3})
\end{aligned}
$$

$$
\begin{aligned}
P(\text{rolls a total of 7 on round 2}) &= P(\text{Jack and Jill did not win round 1})P(\text{Jill rolls add up to 7} \mid \text{she didn't win round 1}) \\
&= \frac{2}{6} * \frac{2}{6} * \frac{2}{12}
\end{aligned}
$$

$$
\begin{aligned}
P(\text{rolls a total of 7 on round 3}) &= P(\text{Jack and Jill did not win round 2}) \\
&\quad * P(\text{Jill rolls add up to 7} \mid \text{she didn't win round 1 and 2}) \\
&= \frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6}
\end{aligned}
$$

P(Jill wins \$7) $= (\frac{2}{6} * \frac{2}{6} * \frac{2}{12}) + (\frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6}) = 0.01865$

P(Jill wins $8)

$$\begin{aligned} P(\text{Jill wins \$8}) &= P(\text{rolls a total of 8 on round 2 or rolls a total of 8 on round 3}) \\ &= P(\text{rolls a total of 8 on round 2}) + P(\text{rolls a total of 8 on round 3}) \end{aligned}$$

$$\begin{aligned} P(\text{rolls a total of 8 on round 2}) &= P(\text{Jack and Jill did not win round 1}) P(\text{Jill rolls add up to 8} \mid \text{she didn't win round 1}) \\ &= \frac{2}{6} * \frac{2}{6} * \frac{1}{12} \end{aligned}$$

$$\begin{aligned} P(\text{rolls a total of 8 on round 3}) &= P(\text{Jack and Jill did not win round 2}) \\ &\quad * P(\text{Jill rolls add up to 8} \mid \text{she didn't win round 1 and 2}) \\ &= \frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6} \end{aligned}$$

P(Jill wins $8) = $(\frac{2}{6} * \frac{2}{6} * \frac{1}{12}) + (\frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6} * \frac{1}{6}) = 0.00938$

$$\begin{aligned} E(\text{Prize Money}) &= 3(0.1853) + 4(0.1853) \\ &+ 5(0.1853) + 6(0.1853) \\ &+ 7(0.01865) + 8(0.00938) \\ &+ 0 * P(\text{She didn't win}) \\ &= 3.541 \end{aligned}$$

Var(Prize Money) = E((Prize Money)^2) - (E(Prize Money))^2
(E(Prize Money))^2 = $(3.541)^2 = 12.54$

E((Prize Money)^2) =

$$\sum_{i=3}^{8} i^2 * P(\text{Jill winning prize} = \$i)$$

$$\begin{aligned} E(\text{Prize Money}^2) &= 3^2(0.1853) + 4^2(0.1853) \\ &+ 5^2(0.1853) + 6^2(0.1853) \\ &+ 7^2(0.01865) + 8^2(0.00938) \\ &+ 0 * P(\text{She didn't win}) \\ &= 17.45 \end{aligned}$$

Var(Prize Money) = 17.45 - 12.54 = 4.91