Binomial Asset Pricing Model

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Introduction

- used to price derivative securities stocks/bonds in the risk-neutral world
- introduced by Cox, Ross and Rubenstein in 1979
- desirable because it is possible to check at any point in an option's life the possibility of early exercise
- does not depend on certain outcome probabilities, meaning the model is independent of investors that have subjective probabilities about an upward/downward movement in the underlying asset

- Determining the potential price of an asset is necessary due to the fact that short selling is a common practice: "an investor borrows the stock, sells it, and uses the proceed to make some other investments and then repurchase the stock and return it to the owner with any dividends due".

Binomial Distribution

The binomial model follows a discrete binomial distribution where a random variable X has a binomial distribution with parameters n and p if X has a discrete distribution for which the probability function is

$$f(x|n,p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{for } x = 0, 1, 2, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

where n is some positive integer, and p is between 0 and 1.

p is referred to as the risk-neutral probability

Definitions

derivative security: financial agreement in which a buyer is given the right to purchase assets at a predetermined price, which can be exercised at any point before the expiration of the security

security: a "negotiable financial instrument that represents a type of financial value"

traders/investors: includes both speculators and financial managers

speculators: typically attracted to the options market because of the potential for high profits

financial managers: typically participate in the options market because it requires less capital than the stock market, and they can hedge risk in their portfolios

options analysis: "allows one to make better investment decisions because it can incorporate the value of flexibility of an investment into the initial evaluation of that investment"

risk-neutral: "an investor is indifferent between a certain return and an uncertain return with the same expected value" In this state, investors require no compensation for bearing risk, and as a result, the expected return on all securities is the risk-free interest rate.

interest rate: a "quantified property of the money market that yields $1 + r$ dollars at a time for one dollar invested in the money market at time 0"

Upstate and Downstate

two mutually exclusive outcomes:

upstate or upward movement - meaning the given variable rises in value
downstate or downward movement - meaning the given variable drops in value

To denote the factor by which an asset rises, we use u.
To denote the factor by which an asset drops in value, we use d.

An option takes on an upstate or downstate in every trial. Each trial is referred to as a period, and we assume these periods are independent of one another.

The Binomial Asset Pricing Model is “simple” due to the fact that the number of nodes of options increases linearly with time.

Arbitrage

no arbitrage means . . .

- no possibility to begin with zero wealth
- have a zero-possibility of losing money
- positive probability of earning money through investments

to rule out arbitrage, we assume $0 < d < 1 + r < u$ where r represents the fixed interest rate of the money market

$0 < d$: ensures that the value of a stock price remains positive, regardless of the factor of the downstate

$d < 1 + r$: rules out arbitrage

If on the other hand we assume d > 1+ r, there is a possibility to begin with no initial wealth and to borrow an initial amount x to invest in, and in the event of a failure, the downstate price, xd, would be larger than the amount owed to the bank, $x(1 + r)$. This would lead to zero possibility of losing money because we would keep a profit after repaying the loan including interest.

$1 + r < u$: incentivizes asset investment

If this inequality were not the case, the money market would yield greater returns with less risk.

History

Bernoulli's contributions

Arrow: showed that "by using the temporal structure of the economy, equilibrium can be attained with a more limited number of markets," which is more commonly known as Arrow's theory of general equilibrium with incomplete asset markets

Markowitz: revolutionized the ways investors view assets when he demonstrated that "the portfolio with maximum expected return is not necessarily the one with minimum variance. There is a rate at which the investor can gain expected return by taking on variance, or reduce variance by giving up expected return." He also points out that "in trying to make variance small it is not enough to invest in many securities. It is necessary to avoid investing in securities with high covariances among themselves"

Tobin: takes Markowitz's analysis one step further and shows how to identify which portfolio should be held by an investor, and considers how an investor should distribute their funds between safe liquid assets and risky assets (Dimson)

Sharpe: determined that stocks are likely to co-move in the market (Dimson).

Black Scholes Model

1973 - Black and Scholes modeled the distinction between American and European options with a closed-form solution

$$c = S_t N(d_1) - Ke^{-rT} N(d_2) - - - - [7]$$

$$p = Ke^{-rT} N(-d_2) - S_t N(-d_1) - - - - [8]$$

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

the most commonly known and utilized option pricing models -
Binomial Asset Pricing Model: simple statistical method
Black Scholes Model: “requires a solution of a stochastic differential equation”
used exclusively for pricing European options

assumptions:

- the assets follow a lognormal distribution
- there are no taxes
- the risk-free rate of interest, *r*, is constant
- the volatility (σ) of the underlying asset is constant and known and uses risk neutral possibilities (*p*) rather than subjective probabilities

volatility: the “instantaneous standard deviation” of the asset

paired t-test: no significant difference between the asset price calculated
Tukey-pairwise comparison: no significant difference between the asset price calculated

It is generally said that the binomial model requires a less intensive mathematical background and skill set in comparison to the Black-Scholes model.

Background

The payoff and price of a derivative depends on the type of security and differs for the call-options.
A call option specifically is an investor as a buyer, and a put option is an investor from the perspective of a seller.

An option gives its owner the choice to trade a given risky asset (stock) at a certain price by a certain date.
However, the holder is not obliged to exercise the right at the option's maturity; it is only done so if considered economically advantageous.

Pricing derivatives requires the use of "risk-neutral measures".
These are similar to traditional probability measures but are adjusted for the risk taken on when purchasing assets.
This enables us to price derivatives with consideration to a buyer's risk aversion and aids in pricing derivatives for stock options and bond securities.

American options can be exercised at any point up to maturity, while European options can only be acted on at maturity.
Note that this distinction is not geographical.

The intrinsic value of an option is the value of an option if it were to be exercised immediately; however, the time value of an option is the value an option can take if it is left to mature.
Hence, the intrinsic value is the minimum that an option will be worth at any given time, and the total price of an option with time left to maturity consists of its intrinsic value in addition to its time value.

Single-Period Binomial Model

1. This begins with an initial stock price, S_0, which takes on the value of the upstate or downstate depending on the outcome of a trial. This trial is represented with a coin-toss. The outcome of the coin-toss is random, and each outcome is mutually exclusive.

2. If the toss lands on heads, the price of the upstate, $S_1(H)$, is the product of the original price and the up-factor. In other words, $S_1(H) = uS_0$. The downstate can be modeled similarly.

3. The call option can be bought for a price, V_0, and allows the buyer to purchase stock for an agreed upon stock price, K. At any time before the expiration of this call option, this price may be exercised for a payoff, V_t.

4. For a single period model, the payoff of the option can be represented by the maximum of $S_1 - K$.. If $S_1 > K$, the option is "in the money" and can be exercised for a profit at time t. If $S_1 < K$, the option is "out of money" and cannot be immediately exercised for profit.

5. Under this model, an agent's initial wealth, X_0, is held either in stocks, the money market, or a combination of both. The buyer invests in Δ_0 shares of stock at time 0 for a stock price of S_0. There is a total of $\Delta_0 S_0$ invested in stocks at time 0 which will generate a profit of $\Delta_0 S_1$ at time 1.

   Δ_0 is referred to as the hedge ratio or delta. Hedging as a financial method is the overall attempt to minimize risk by investing in a way that controls market fluctuations. This method has caused considerable increases in derivative investments.

6. The remaining money, $X_0 - \Delta_0 S_0$ is invested into the money market and makes a return of $1 + r$.

7. The wealth of the agent at time 1 can be represented by $X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)I$. The goal when pricing an option is to choose an initial stock investment and wealth such that $X_1(H)=V_1(H)$ and $X_1(T)=V_1(T)$ in order to price our call option

8. After completing the derivation and utilizing past assumptions of the model, given our initial investment totals X_0 and our investment into stock totals Δ_0, the portfolio is worth either $V_1(H)$ or $V_1(T)$.

Multi-Period Binomial Model

The single period model must be extended to include multiple finite periods.

The Binomial Asset Pricing Model assumes that prices can only move up and down at each step by predetermined amounts, and this may seem unrealistic at first glance. However, it is especially important when moving to a multi-period model to remember that this assumption remains true due to the fact that steps remain very small and are combined together. This is what makes the overall presumption reasonable, along with the idea that the model converges and can therefore be estimated in this fashion.

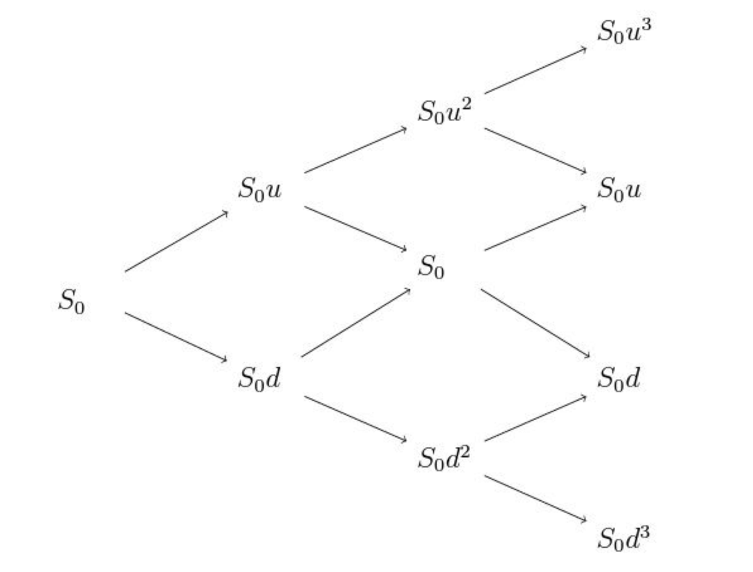
The multi-period model can be visualized with a tree model that shows the possible outcomes. The tree represents all possible paths an asset value could take during the life of the option, and at the expiration of the option, “all the terminal values for each of the final possible asset values are known, as they simply equal their intrinsic values”. As a result, the value of the option at each step is calculated working from the expiration to the beginning timestamp. Because the model follows a binomial distribution, the stock price can adhere to 2^n possible paths throughout the model, where n is the number of periods.

The wealth equation for a general time n is given by $X_n = \Delta_{n-1}S_n + (1 + r)(X_{n-1} - \Delta_{n-1}S_{n-1})$, and the payoff of a call-option at time n is given by the maximum of $S_n - K$.

Lastly, the price of a call option a time n based on the expected payoff of the option in the time $n + 1$ is where the sequence $\omega_1,\omega_2 \ldots \omega_n$ denotes the result of the first n coin tosses. The theorem that governs this pricing model states: “Let V_n be a random variable derivative security paying off at time N, which depends on the first N coin tosses $\omega_1,\omega_2 \ldots \omega_n$. Define a sequence of random variables $V_{n-1}, V_{n-2} \ldots V_0$ recursively by the equation above so that each V_n depends on the first n coin tosses $\omega_1,\omega_2 \ldots \omega_n$ where n falls in the interval [0, N-1]”.

$$V_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)].$$

$$\tilde{p} = \frac{1+r-d}{u-d} \text{ and } \tilde{q} = \frac{u-1-r}{u-d}$$



$$V_n(\omega_1\omega_2\ldots\omega_n) = \frac{1}{1+r}[\widetilde{p}V_{n+1}(\omega_1\omega_2\ldots\omega_nH) + \widetilde{q}V_{n+1}(\omega_1\omega_2\ldots\omega_nT)]$$

$$\Delta_n(\omega_1\omega_2\ldots\omega_n) = \frac{V_{n+1}(\omega_1\omega_2\ldots\omega_nH)-V_{n+1}(\omega_1\omega_2\ldots\omega_nT)}{S_{n+1}(\omega_1\omega_2\ldots\omega_nH)-S_{n+1}(\omega_1\omega_2\ldots\omega_nT)}$$

Real-World Probabilities vs. Risk-Neutral Probabilities

benefits:

- direct inference
- practitioners using the model at corporations avoid anxiety that comes with using risk-neutral probabilities on inherently risky cash flows
- the model simplifies pricing when skewness and kurtosis appear
- aims to safeguard against conceptual disparities that arise when computing risk-neutral probabilities: "it is difficult to understand why we need event probabilities from an event economy that does not compensate risk bearing [in risk-neutral models], even though we are pricing assets from a real-world economy that does compensate risk bearing"

real-world probability Binomial Asset Pricing Model: requires a "stochastic risk-adjusted discount rate"
(accomplished with discretization)

In this model, S_0 and S_T are the asset price at time $t = 0$. Assume the asset pays no dividends, and R_S, the ratio of S_T to S_0, is the total discretely-compounded return on the asset from time 0 to time T. R_F is the total risk-free rate from time 0 to time T. Let r_F be the annualized continuously-compounded risk-free rate, and k_S be the annualized continuously-compounded expected return on the stock.

One-Period and Multi-Period Real World Option Pricing Models

one-period option pricing formula:

Although these equations entail discounting at the risk-free rate, they are not risk-neutral pricing. There is no change of probability measure, and the expected cash flow ($E(V_T)$) is in the real-world, not a risk-neutral world. This risk-adjusted cash flow is the real-world expected cash flow minus a risk premium.

$$
\begin{aligned}
V_0 &= \frac{1}{R_F}\left[E(V_T) - \left(\frac{V_u - V_d}{u - d}\right)\left(E(R_S) - R_F\right)\right] \\
V_0 &= e^{-r_F T}\left[E(V_T) - \left(\frac{V_u - V_d}{e^{\sigma\sqrt{T}} - e^{-\sigma\sqrt{T}}}\right)\left(e^{k_S T} - e^{r_F T}\right)\right]
\end{aligned}
$$

multi-period option pricing formula:

The real-world underlying security discount rate is used in place of the risk-free rate. This allows the underlying security price to increase and decrease at a given stage in the binomial tree. Then, option prices can be calculated recursively. We let *i* be the number of upward price movements and *j* be the number of downward price movements. For stage *(i,j)* where *i+j* is less than the terminal stage, the option price *V(i,j)* follows the equation.

$$
\begin{aligned}
V(i,j) &= e^{-r_F T}\Big\{\left[pV(i+1,j) + (1-p)V(i,j+1)\right] \\
&\quad - \left(\frac{V(i+1,j) - V(i,j+1)}{e^{\sigma\sqrt{T}} - e^{-\sigma\sqrt{T}}}\right)\left(e^{k_S T} - e^{r_F T}\right)\Big\}
\end{aligned}
$$

Explanation and Differences Between Real-World and Risk-Neutral

If the model were to be generated using the risk free rate, r_F, rather than the underlying security's discount rate, k_S, the model becomes risk-neutral.

This model is set to match European style options.
To adapt this model to American-style options, much like the risk-neutral model, the maximum between the solution $V(i,j)$ and the option's immediate exercise value must be taken.

This model can then give traders "real-time, real-world probabilities that individual American-style options will finish in the money".

This model "allows parameter inference from the real-world probability density function of the underlying security," and "allows real-world statistical information (e.g. historical or forecast volatility) to be incorporated into option pricing".

Lastly, the model allows stochastic parameters throughout the option's life (Arnold).

Fuzzy-Stochastic Models

fuzzy: the level of uncertainty present in input data

this would affect the probability *p* parameter in our model, and a fuzzy measure would be introduced instead

Wide breadth of applications under the Binomial Asset Pricing Model.

- generally only used to evaluate American options
- all input parameters are given fuzzy volatility such as the up index, down index, risk-free rate, growth rate, initial underlying asset price, and exercise price as these parameters are often difficult to concretely define
- a fuzzy random variable is introduced that utilizes random intervals and fuzzy sets to define a range of possibilities for the inputs
- the rest of the model functions the same; however, the outputs take into account that there may be slight perturbations in the input parameters that are fed to the model
- “fuzzy” parameters are marked with a tilde
- advantageous as it ultimately make the model more risk-comprehensive

Following model evaluation, when it is desirable to reduce the derivative price to a crisp value for decision purposes, “defuzzyfication” methods such as center of gravity area, first of maxima, last of maxima, mean of maxima, center of maxima (median), centroid method, or the bisector method can be performed.

Trinomial Model

A Binomial Asset Pricing Model can also be extended into a Trinomial Asset Pricing Model using subordination. The classic trinomial pricing model is defined in the risk-neutral world and takes the following forms:

$$S(\mathrm{E}^{(TR,t+\Delta t)},\ t+\Delta t) = S(\mathrm{E}^{(TR,t)},\ t)\begin{cases} 1+\frac{3}{2}\sigma^2\Delta t+\sigma\sqrt{3\Delta t}, & \textit{if}\ \varepsilon^{(TR,t+\Delta t,\Delta t)}=1, \\ 1, & \textit{if}\ \varepsilon^{(TR,t+\Delta t,\Delta t)}=0, \\ 1+\frac{3}{2}\sigma^2\Delta t-\sigma\sqrt{3\Delta t}, & \textit{if}\ \varepsilon^{(TR,t+\Delta t,\Delta t)}=-1. \end{cases}$$

$$S(\mathrm{E}^{(U,t+\Delta t)},\ t+\Delta t) = S(\mathrm{E}^{(U,t)},\ t)\begin{cases} 1+\left(\mu+\frac{\sigma^2}{4}\right)\Delta t+\sqrt{\frac{3}{2}}\sigma\sqrt{\Delta t}, & \textit{if}\ \varepsilon^{(U,t+\Delta t,\Delta t)}=1 \\ 1+\left(\mu-\frac{\sigma^2}{2}\right)\Delta t, & \textit{if}\ \varepsilon^{(U,t+\Delta t,\Delta t)}=0 \\ 1+\left(\mu+\frac{\sigma^2}{4}\right)\Delta t-\sqrt{\frac{3}{2}}\sigma\sqrt{\Delta t}, & \textit{if}\ \varepsilon^{(U,t+\Delta t,\Delta t)}=-1. \end{cases}$$

$$S^{\mathbf{Q}}(\mathrm{E}^{(U,t+\Delta t)},\ t+\Delta t) = S^{\mathbf{Q}}(\mathrm{E}^{(U,t)},\ t)\begin{cases} 1+\left(r+\frac{\sigma^2}{4}\right)\Delta t+\sqrt{\frac{3}{2}}\sigma\sqrt{\Delta t},\ w.\,p.\ \ q^{(u)}=\frac{1}{3} \\ 1+\left(r-\frac{\sigma^2}{2}\right)\Delta t,\ 1,\ w.\,p.\ \ q^{(n)}=\frac{1}{3} \\ 1+\left(r+\frac{\sigma^2}{4}\right)\Delta t-\sqrt{\frac{3}{2}}\sigma\sqrt{\Delta t}\ w.\,p.\ \ q^{(d)}=\frac{1}{3}\ . \end{cases}$$

Multivariate Case

- "the payoff on the option depends on the outcome of two or more random variables"
- frequently used for American options considering that the intermediate value of the option depends on the underlying price of the asset and other variables such as interest rate, *r*

We must model the covariance of the asset price and the interest rate, as well as the time series properties of each.

bivariate case (X, Y): compute

- the up and down movements for each random variable(X_1,X_2,Y_1,Y_2)
- the conditional probability of an increase in the second variable given an increase in the first (Y_1/X_1)
- the conditional probability of an increase in the first variable at the second time step (X_2/X_1)
- the conditional probability of the second variable at the second time stamp given its value at the first time and the value of the first variable at the second time stamp (Y_2/Y_1,X_2)

It is important to consider the limitations on accuracy when natural limits are placed on the conditional probabilities.

Summary

option pricing using the Binomial Asset Pricing Model can be distinguished by the following steps:

- modeling an evolution of underlying asset in accordance with the observed volatility
- computation of intrinsic value (payoff function)
- at maturity day T, option price is equal to intrinsic value
- the American option can be exercised at any point during the pre-allocated time period, and the asset's price can be defined by the Bellman dynamic programming optimal equation

A seller is thought to be satisfied if they have a hedging portfolio that has enough value to pay off the options contract when it is exercised and enough value to match the value of the options contract at any given time.

To pay off a call when exercised, the portfolio must be valued at the maximum of $(S_n - K)$. For a put, the portfolio should be valued at a maximum of $(K - S_n)$.

A binomial model is generally thought of as the benchmark for more complex models, particularly the American call option, while the standard for the European call option is the Black Scholes Model.

The binomial model has many variations, and no particular form seems to dominate the rest. However, the necessary principles include prohibiting arbitrage for a finite number of time steps and recovering the correct volatility.

The choice of the actual risk-neutral probability is meaningless when it comes to the limit of the function; however, a risk-neutral probability of ½ offers the fastest convergence.