Calibration of Heston Model

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Abstract

In this article we explore the calibration of stochastic volatility model as proposed by Heston [Heston, 1993]. We use least squares method to find the parameters of the model for the call options on S&P 500 using weekly data from 2005 to 2010. Except a few dates, we observe that the estimates are stable over the course of the estimation period and estimated prices show skewness as observed in real markets. We also observe that average pricing error as a proportion of mean option price varies around 1.57% for the whole period and has dropped down to around 1.00% since 2008.

I. Introduction

Heston stochastic volatility framework has now become a baseline model in realm of stochastic volatility and option pricing theory. Along with it's nice theoretical properties and closed form solution, it very well captures leverage effect and mean reversion in volatility. Moreover, it very well matches the prices observed in the market. In this article we will try to calibrate the model for Call Options on S&P 500 from 2005 to 2010 using least square estimation framework. We will follow very closely the methodology as proposed in Crisostomo (2014) however, we also look at much richer dataset of option which includes one of the toughest times in the modern finance. The results represent the qualitiave behavior in the market and also very closely matches the observed market price. In section 2 we will describe the model setup, followed by the estimation methodology in section 3. Section 4 comprises the results, followed by conclusion in section 5.

II. Model Setup

Heston model in the risk neutral measure has the setup given by the following equations -

$$
\begin{aligned}
dS_t &= rS_t dt + \sqrt{V_t} S_t dW_t^1 \\
dV_t &= a(\bar{V} - V_t)dt + \eta \sqrt{V_t} dW_t^2 \\
dW_t^2 dW_t^1 &= \rho dt
\end{aligned} \tag{1}
$$

r is the risk free rate, V_t represents the volatility, η represents the volatility of volatility and ρ represents the correlation between the two stochastic processes. ρ represents the leverage effect and is expected to be negative. The option prices in the Heston model has nice closed form solution that is obtained using the characteristic functions.
The price of a call option can be written using characteristic function given by -

$$
\begin{aligned}
C_0 &= S_0 \Pi_1 - e^{-rT} K \Pi_2 \\
\Pi_1 &= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left\{ \frac{e^{-iw \ln K} \phi_{\ln S_T}(w-i)}{iwF} \right\} dw \\
\Pi_2 &= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left\{ \frac{e^{-iw \ln K} \phi_{\ln S_T}(w)}{iw} \right\} dw \\
F &= S_0 e^{rT}
\end{aligned}
$$

It is important to realize that $\phi_{\ln S_T}$ is the characteristic function of the $\ln S_T$. In the Heston model, this is given by the following equations:

$$
\begin{aligned}
\phi_{\ln S_T}(w) &= exp\{C(t,w)\bar{V} + D(t,w)V_0 + iw \ln(S_0 e^{rT})\} \\
C(t,w) &= a \left\{ r_1 t - \frac{2}{\eta^2} \ln\left(\frac{1 - g e^{-hT}}{1 - g}\right) \right\} \\
D(t,w) &= r_1 \frac{1 - e^{-hT}}{1 - g e^{-hT}}
\end{aligned}
$$

$$r_1 = \frac{\beta + h}{\eta^2} \; ; r_2 = \frac{\beta - h}{\eta^2}$$
$$h = \sqrt{\beta^2 - 4\alpha\gamma} \; ; g = \frac{r_1}{r_2}$$
$$\alpha = \frac{-(w^2 + iw)}{2} \; ; \beta = a - \rho\eta i w$$
$$\gamma = \frac{\eta^2}{2}$$

So, the parameters to be estimated are - $[V_0, \bar{V}, \rho, a, \eta]$.

III. Data Description and Estimation Methodology

For this work, we use data for Call options on S&P 500 from 2005-2010. We use data at 5 day intervals starting from Dec 31, 2004 to August 2, 2010. We consider only call options that are within 20% range of the spot for the day. It has been observed that deep in-the-money options or deep out-of-the-money options are very sensitive to the price changes. So, we remove those options from the study. We also take option which have atleast 20 calendar days to maturity and up untill 400 days. So, we have on an average around 75 options daily from 2005-2007 and around 100 options daily from 2008-2010. We use least squares method to estimate the parameters of the model described in the prevous section. Thus the objective function is given by equation 2

$$\min \sum_{i,j} (C_{MP}(K_i, T_j) - C(K_i, T_j, a, \rho, V_0, \eta, \bar{V}))^2 \tag{2}$$

We used MATLAB's lsqnonlin function to do the optimization. We constraint our estimation space with the following conditions: $0 \leq V_0 \leq 1$, $0 \leq \bar{V} \leq 1$, $0 \leq \eta \leq 5$, $-1 \leq \rho \leq 1$ and $0 \leq 2a\bar{V} - \eta^2 \leq 20$. The final condition is to make sure that the volatility in the CIR process is non-negative.

IV. Results

We will now present the results of our estimation methodology. We use three main measures (equation 3) to check how close the model is to observed prices. RMSE represents the root mean square error, MAE represents the mean absolute error and APE represents the absolute error as a percent of mean price.

$$\begin{aligned}
RMSE &= \sqrt{\sum_{i=1}^{N} \frac{(MarketPrice_i - ModelPrice_i)^2}{N}} \\
MAE &= \sum_{i=1}^{N} \frac{|MarketPrice_i - ModelPrice_i|}{N} \\
APE &= \sum_{i=1}^{N} \frac{|MarketPrice_i - ModelPrice_i|}{N(\text{Mean Price})}
\end{aligned} \tag{3}$$

In Table 1 we can see that the mean absolute error is at around \$0.9360 and standard deviation of around \$0.5321. Mean absolute error as proportion to the mean price stands at around 1.57% with standard deviation of 0.79%. However, the error is much smaller from 2008-2010 compared to 2004-2008. This is clearly evident in the figure 1. One of the reasons for this could be increased volume and liquidity since 2008.

Table 1: *Estimation error*

	RMSE	MAE	APE
Average	1.3167	0.9360	0.0157
Standard Dev	0.7532	0.5321	0.0079

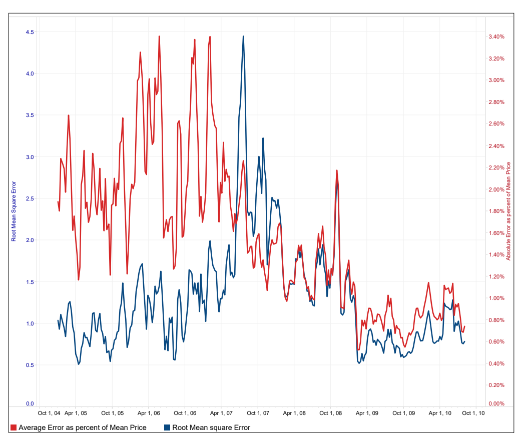

Figure 1: *Root mean square error and average error as percent of mean price*

Next, we present the time series of the estimates of the model. Figure 3 represents the estimates of $\bar{V}$, the long term volatility at different time points. It's very evident that the volatility jumped up as we entered 2008 when the markets crashed brutally. We aslo see spike in the summers of 2010.
The initial variance V_0 (figure 2) also shows a very similar behavior with a strong jump in 2008 and summers of 2010. However, it also shows a spike in the summer of 2007 when the long term volatility was changing its regime.
The correlation coefficient is very close to -1 and remained so for most part of the estimated time series while ranging between -0.75 to -1.
As mentioned by Crisostomo(2014) that volatility of volatility can change very quickly, we observe in figure 4 that the graph achieves local peaks way more than other parameters in the model. Finally, for the mean reversion parameter, we observe that the estimates are not stable especially in the time period before 2007. For the period after 2008, a is observed to be around 5 and very stable. Figure 5 shows the time series of the estimates along with the average value within the period.

In figure 6, 7 and 8, we present the market price and model estimates as of August 2, 2010. It's evident that the model fits the data very well and also very well captures the skew observed in the market. The spot price for the day was 1125.86. Thus, we had roughly 54 in-the-money and 53 out-of-the money call options.

V. Conclusion

Through this project, we implemented and tested for the accuracy and stability of the parameters of the Heston Stochastic volatility model. We observed that although a very simple and elegant model, it very well captures the key features observed in the market. The model has very small estimation error in the observed and estimated prices and presents the market skew nicely.

However, one of the most important draw-back of the model is constant coefficients. We have seen from various graphs that the long term volatility, mean reversion speed and volatility of variance changes very dramatically over time and especially during market crash. This clearly violates the assumptions of the model and should be addressed for sound calibration. Moreover, we observed that the model produces negative prices sometimes for deep-out-of money options that have short expiry. We can also improve the estimates and also make it more stable by using weighted least squares and penalty functions in the objective function.

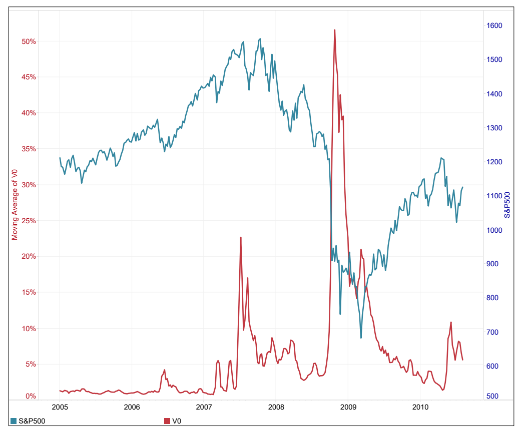

Figure 2: *Estimates of V_0 compared to S&P500*

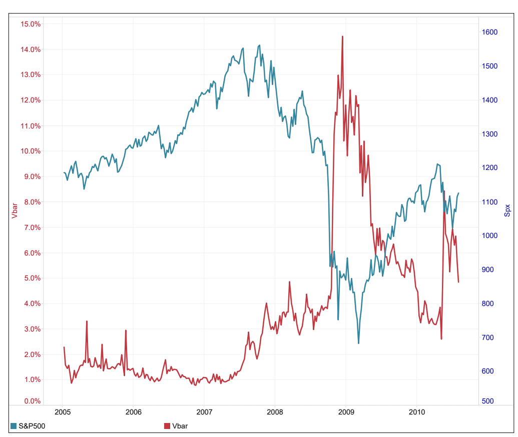

Figure 3: *Estimates of $\bar{V}$ compared to S&P500*

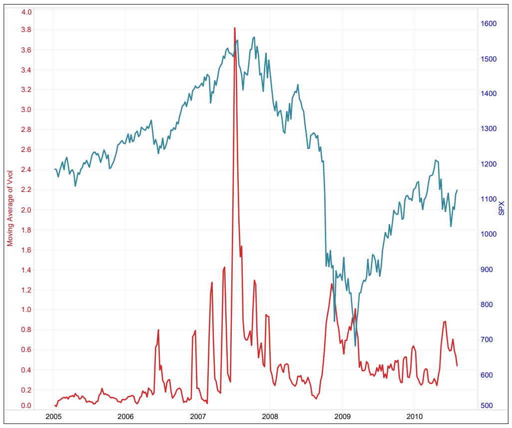

Figure 4: *Estimates of η compared to S&P500*

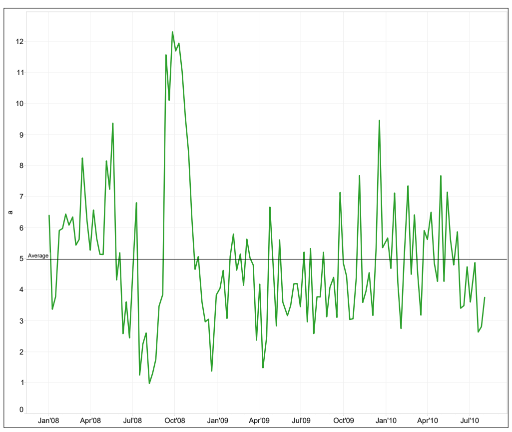

Figure 5: *Estimates of a compared to S&P500*

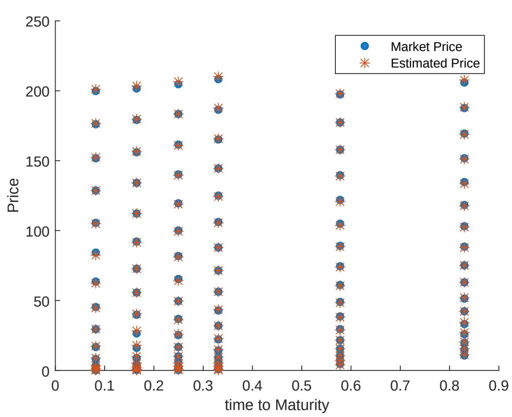

Figure 6: *Market Price and Model Estimate for different time to maturity as of August 2, 2010*

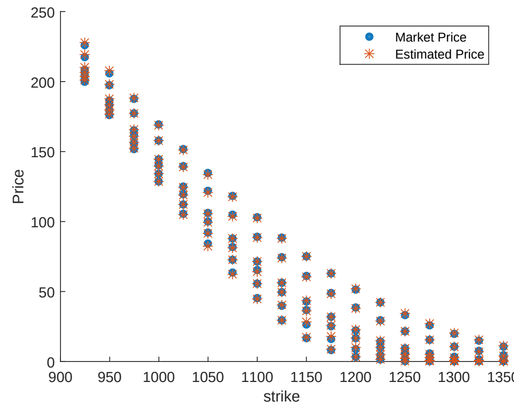

Figure 7: *Market Price and Model Estimates for different strike as of August 2, 2010*

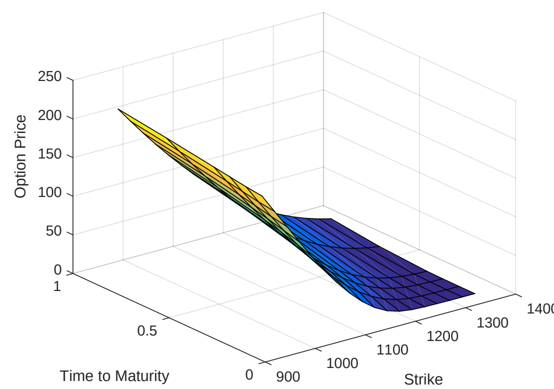

Figure 8: *Estimated Price of the options for various strike and maturity as of August 2, 2010*

References

[Heston, 1993] A closed-form solutions for options with stochastic volatility. *Review of financial studies*, 6, 327-343

[Crisostomo, 2014] An Analysis of the Heston Stochastic Volatility Model: Implementation and Calibration using Matlab. *http://arxiv.org/abs/1502.02963*,