MAE/MSE 502, Spring 2021 Homework # 2

Please include a statement on collaboration in the report. Please follow the rules on collaboration as given in the first page of the problem statement for Homework #1.

Problem 1 (2.5 points)
For $u(x,y)$ defined on the square domain of $0 \le x \le 1$ and $0 \le y \le 1$, solve the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

with the boundary conditions,

(i) $u(0, y) = 0$ (ii) $u(1, y) = \sin(\pi \sqrt{y})$ (iii) $u(x, 0) = 0$ (iv) $u(x, 1) = 4\,(x^3 - x^6)$.

Make a contour plot of the solution, $u(x, y)$. See Additional Note in the last page for an example of using Matlab to make a contour plot. The recommended contour interval for the plot is 0.05. For this problem, we expect the solution to be expressed as an infinite series. Please see the remark below HW1-Q1 on how to truncate the series and numerically compute the expansion coefficients.

Problem 2 (2.5 points)
For $u(x,y)$ defined on the square domain of $0 \le x \le 1$ and $0 \le y \le 1$, consider the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

with the boundary conditions (u_x and u_y denote $\partial u/\partial x$ and $\partial u/\partial y$, respectively)

(i) $u_x(0, y) = 0$ (ii) $u_x(1, y) = 0$ (iii) $u_y(x, 0) = 100$ (iv) $u_y(x, 1) = 100 + \cos(100\pi x)$

(a) Test the solvability condition on the system. Based on it, which of the following is true?
(I) The system has no solution. (II) The system has a unique solution. (III) The system has multiple solutions. If your answer is (I), no need to proceed further. Otherwise, proceed to Part (b).
(b) Find the solution(s) of the system. If your answer for Part (a) is (III), please clearly write out what those multiple solutions are.

Problem 3 (2.5 points)
For $u(x, y)$ defined on the domain of $0 \le x \le \pi$ and $0 \le y \le 1$, solve the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 16\,u = 0,$$

with the boundary conditions

(i) $u_x(0, y) = 0$ (ii) $u_x(\pi, y) = 0$ (iii) $u(x, 0) = 1 + \cos(4x)$ (iv) $u(x, 1) = \cos(5x)$

For this problem, we expect a closed-form analytic solution that consists of only a finite number of terms and without any unevaluated integral. Expect a deduction if the requirement is not met.

Problem 4 (2.5 points)
For $u(x,y)$ defined on the rectangular domain of $0 \le x \le ln(2)$ and $0 \le y \le \pi$, solve the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 2\frac{\partial u}{\partial x} + u = 0 ,$$

with the boundary conditions

(i) $u(0, y) = 15 \sin(2y)$ (ii) $u(ln(2), y) = 0$ (iii) $u(x, 0) = 0$ (iv) $u(x, \pi) = 0$.

[Note: $ln(2)$ is the natural logarithm of 2. We assign this value in the problem to help simplify the computation. Note that $A\, ln(B) = ln(B^A)$.]
For this problem, we expect a closed-form analytic solution that consists of only a finite number of terms and without any unevaluated integral. A deduction will be assessed on the solution otherwise.

Problem 5 (1 point)
For $u(x,y)$ defined on the square domain of $0 \le x \le 1$ and $0 \le y \le 1$, consider the PDE

$$\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial y^2} + A\,x^3 = 0 ,$$

where A is a given constant, with the boundary conditions

(i) $u_x(0, y) = 1$ (ii) $u_x(1, y) = 2$ (iii) $u_y(x, 0) = 3$ (iv) $u_y(x, 1) = 4$.

For what value of A may a solution (or solutions) exist for the system? Must provide the correct reasoning to receive credit. (Otherwise, no need to find the full solution for the system.)

Additional Note: Using Matlab to make a contour plot

The following Matlab code makes a contour plot for $u(x,y) = \sin(2\pi x) + x \cos(3\pi y)$ over the domain of $0 \le x \le 1$ and $0 \le y \le 1$, using the contour interval of 0.2 (and over the range of $u = -2$ to 2). Selected contours with $u = -1.6 + 0.4p$, $p = 1, 2, 3, \dots$, are labeled. The contour plot is shown in the next page. It is essential to provide the coordinates of the grid (x2d and y2d in this example) as the input for the contour function. Without this piece of information, Matlab would not know the grid spacing and the correct directions of x and y. Be aware that the setting used in this example might not be what is recommended for the actual contour plot in Prob 1. For that problem, please adjust the contour interval as needed.

```
x = [0:0.01:1];y = [0:0.01:1];
for m = 1:length(x)
    for n = 1:length(y)
        u(m,n) = sin(2*pi*x(m))+x(m)*cos(3*pi*y(n));
        x2d(m,n) = x(m); y2d(m,n) = y(n);
    end
end
[C h] = contour(x2d,y2d,u,[-2:0.2:2],'LineWidth',2)
clabel(C,h,[-1.6:0.4:1.6]); xlabel('x'); ylabel('y')
```