Subtraction of $e^+e^- \to u\bar{u}d\bar{d}$ with distributed soft counterterms

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1 Counterterms

For the process $e^+e^- \to u_1\bar{u}_2d_3\bar{d}_4$ as a real correction to $e^+e^- \to jj+X$, the elementary limits that need to be regulated are

$$C_{12}, \quad C_{34}, \quad C_{134}, \quad C_{234}, \quad C_{123}, \quad C_{124}, \quad S_{12}, \quad S_{34}. \tag{1}$$

At most three of these limits have a common overlap and need to be considered simultaneously. Due to the lack of a tree-level diagram for $e^+e^- \to gg$, the limit of two collinear pairs $C_{12}C_{34}$ is regular. The maximal overlaps that we need to consider are therefore

$$C_{123}S_{12}C_{12}, \quad C_{124}S_{12}C_{12}, \quad C_{134}S_{34}C_{34}, \quad C_{234}S_{34}C_{34}. \tag{2}$$

These are all of the same type, so it will be sufficient to consider the representative $C_{123}S_{12}C_{12}$.

1.1 C(1,2)

We subtract the limit of a $q\bar{q}$ pair going collinear using the current

$$C_{12}^{\mu\nu} = \frac{T_R}{s_{12}}\left[-g^{\mu\nu} + 4z_{1,2}z_{2,1}\frac{k_{1,2}^{\mu}k_{1,2}^{\nu}}{k_{1,2}^2}\right], \tag{3}$$

in combination with the generalised rescaling mapping and ? variables.

1.2 C(1,2,3)

The triple-collinear counterterm for $q'\bar{q}'q$ is determined using the current [1]

$$C_{123} = \frac{C_F T_R}{2s_{123}^2}\left[-\frac{t_{12,3}^2}{s_{12}^2} + \frac{s_{123}}{s_{12}}\left(\frac{4z_{3,12} + (z_{1,23} - z_{2,13})^2}{z_{12,3}} + (1-2\epsilon)z_{12,3}\right) - (1-2\epsilon)\right], \tag{4}$$

where

$$t_{12,3} \equiv 2\frac{z_{1,23}s_{23} - z_{2,13}s_{13}}{z_{12,3}} + \frac{z_{1,23} - z_{2,13}}{z_{12,3}}s_{12}. \tag{5}$$

Once more we use ? variables and the generalised rescaling mapping.

1.3 S(1,2)

In order to construct the $q\bar{q}$ soft counterterm, we start from the form of the current used in [2] which reads

$$\frac{T_R}{s_{12}^2}\sum_i\sum_j \frac{s_{1i}s_{2j} + s_{1j}s_{2i} - s_{12}s_{ij}}{s_{(12)i}s_{(12)j}}\mathbf{T}_i \cdot \mathbf{T}_j, \tag{6}$$

where the sum runs over all coloured partons of the reduced process and includes the case $i = j$.

Before discussing partial fractioning, we observe that the global factor s_{12}^{-2} may cause the counterterm to diverge in limits where neither s_{12i} nor s_{12j} go to zero. More concretely, the contribution from the terms with $i = j$ reads

$$\frac{T_R}{s_{12}^2} \sum_i \frac{2 s_{1i} s_{2i}}{s_{(12)i}^2} \mathbf{T}_i^2, \tag{7}$$

and *all* terms are divergent in the triple-collinear limit $12j$ for *any* j. Thus, although from eq. (7) one might be tempted to assign the i-th term to $12i$-collinear kinematics, every term needs to be distributed among all $12j$-collinear kinematics. To this end, we use colour conservation to move all terms off the colour diagonal

$$\sum_i \frac{2 s_{1i} s_{2i}}{s_{(12)i}^2} \mathbf{T}_i^2 = -\sum_i \sum_{j \neq i} \frac{2 s_{1i} s_{2i}}{s_{(12)i}^2} \mathbf{T}_i \cdot \mathbf{T}_j = -\sum_i \sum_{j \neq i} \left[\frac{s_{1i} s_{2i}}{s_{(12)i}^2} + \frac{s_{1j} s_{2j}}{s_{(12)j}^2} \right] \mathbf{T}_i \cdot \mathbf{T}_j. \tag{8}$$

In this sense, the kinematics that we assign do not follow the divergent structure of the invariant poles but rather the colour, in a similar way as proposed for geometric subtraction [4].

The complete off-diagonal soft current reads

$$\frac{T_R}{s_{12}^2} \sum_i \sum_{j \neq i} \left[\frac{s_{1i} s_{2j} + s_{1j} s_{2i} - s_{12} s_{ij}}{s_{(12)i} s_{(12)j}} - \frac{s_{1i} s_{2i}}{s_{(12)i}^2} - \frac{s_{1j} s_{2j}}{s_{(12)j}^2} \right] \mathbf{T}_i \cdot \mathbf{T}_j. \tag{9}$$

At this stage, we observe that we may replace the invariants $s_{(12)i} = 2 p_{12} \cdot p_i$ and $s_{(12)j} = 2 p_{12} \cdot p_j$ in the denominator with s_{12i} and s_{12j} at our leisure. Indeed, this operation modifies the counterterm by terms which are of higher order in the double-soft limit S_{12}, and therefore does not spoil the cancellation in the counterterm's defining limit. Using the triple invariant s_{12i} seems convenient because it makes denominators naturally match the ones of the collinear counterterm (which cannot be changed because the modification $s_{123} \to s_{(12)3}$ is *not* higher-order in the triple-collinear limit). We thus use

$$S_{12} = \frac{T_R}{s_{12}^2} \sum_{j \neq i} \left[\frac{s_{1i} s_{2j} + s_{1j} s_{2i} - s_{12} s_{ij}}{s_{12i} s_{(12)j}} - \frac{s_{1i} s_{2i}}{s_{12i}^2} - \frac{s_{1j} s_{2j}}{s_{(12)j}^2} \right] \mathbf{T}_i \cdot \mathbf{T}_j. \tag{10}$$

Whether the choice of using $s_{(12)j}$ instead of s_{12j} is important elsewhere in the subtraction was not documented and needs investigation.

It is easy to partial-fraction eq. (10) into collinear kinematics. In the present implementation we use

$$1 = \frac{s_{12i}}{s_{12i} + s_{12j}} + \frac{s_{12j}}{s_{12i} + s_{12j}}, \tag{11}$$

for each term in the dipole sum which leads to

$$S_{12}^{(i)} = \frac{T_R}{s_{12}^2} \sum_{j \neq i} \frac{s_{12j}}{s_{12i} + s_{12j}} \left[\frac{s_{1i} s_{2j} + s_{1j} s_{2i} - s_{12} s_{ij}}{s_{12i} s_{(12)j}} - \frac{s_{1i} s_{2i}}{s_{12i}^2} - \frac{s_{1j} s_{2j}}{s_{(12)j}^2} \right] \mathbf{T}_i \cdot \mathbf{T}_j. \tag{12}$$

Possibly in the future we may want to change the partial fractions to be dependent only on angles and not on energies: in the case of two collinear pairs with distributed single-soft limits, this has been noted to be essential for disjoint collinear limits to work in combination with distributed soft subtraction.

1.4 C(S(1,2),3)

Shifting out of the diagonal the sum over colour dipoles for the $q\bar{q}$ soft limit turns out to be extremely practical also to take its C_{123} triple-collinear limit. We start with either eq. (10) or eq. (12) (the partial fraction makes no difference in the C_{123} limit), and observe that for a given term to contribute one of i or j needs to be equal to 3, and in the collinear limit the ratio of scalar products with another leg is equal to a ratio of momentum fractions. After this replacement, colour conservation can be used and we find

$$C_{123} S_{12} = -\frac{2 T_R}{s_{12}^2} \mathbf{T}_3^2 \left[\frac{s_{13} z_{2,13} + s_{23} z_{1,23} - s_{12} z_{3,12}}{s_{123} z_{12,3}} - \frac{s_{13} s_{23}}{s_{123}^2} - \frac{z_{1,23} z_{2,13}}{z_{12,3}^2} \right]. \tag{13}$$

Note that, if the contributions on the diagonal have not been reshuffled, some effort is needed to see that the latter two terms are needed. We also note that subtracting this sub-limit from C_{123} many terms simplify and we are left with

$$C_{123} - C_{123}S_{12} = \frac{C_F T_R}{s_{123}^2}\left[\frac{s_{123}}{s_{12}}\frac{z_{1,23}^2 + z_{2,13}^2}{z_{12,3}} - 1 + \epsilon\left(1 + \frac{s_{123}}{s_{12}} z_{12,3}\right)\right], \tag{14}$$

which is what is currently implemented in the code (for $\epsilon = 0$). This hard triple-collinear counterterm is clearly associated to 123-collinear kinematics, and for the simplifications to occur the momentum fractions have to be computed as in C_{123}.

1.5 C(C(1,2),3)

The strong-ordered collinear limit $C(C(1,2),3)$ is the first nested limit that we encounter whose counterterm we implement in an iterated fashion. To this end we follow the steps of [3]. Starting from the counterterm $C(1,2)$, we take the extra collinear limit of the parent gluon $\widehat{12}$ of the quark-antiquark pair going collinear to the mapped, different-species quark $\widehat{3}$. This involves taking the collinear limit of a spin-correlated matrix element, which gives the splitting function

$$C_{\widehat{12}\widehat{3}}^{\alpha\beta,ss'} = \frac{C_F}{s_{\widehat{12}\widehat{3}}}\delta_{ss'}\left[\frac{z_{\widehat{12},\widehat{3}}}{2} d^{\alpha\beta} - 2\frac{z_{\widehat{3},\widehat{12}}}{z_{\widehat{12},\widehat{3}}}\frac{k_{\widehat{3},\widehat{12}}^{\alpha} k_{\widehat{3},\widehat{12}}^{\beta}}{k_{\widehat{3},\widehat{12}}^2}\right]. \tag{15}$$

The sum over physical polarisations is given by the transverse tensor with respect to the light-cone vector in the collinear direction p and a reference null vector n,

$$d^{\alpha\beta}(p,n) \equiv -g^{\alpha\beta} + \frac{p^\alpha n^\beta + p^\beta n^\alpha}{p\cdot n}. \tag{16}$$

We have indicated with a hat the variables which are computed after merging particles 1 and 2. Performing the Lorentz algebra we find

$$C_{\widehat{12}\widehat{3}}^{\alpha\beta,ss'} C_{12,\alpha\beta} = \frac{C_F T_R}{s_{12} s_{\widehat{12}\widehat{3}}}\left[\left(2\frac{z_{\widehat{3},\widehat{12}}}{z_{\widehat{12},\widehat{3}}} + z_{\widehat{12},\widehat{3}}(1-\epsilon)\right) - 2 z_{1,2} z_{2,1}\left(z_{\widehat{12},\widehat{3}} + \frac{z_{\widehat{3},\widehat{12}}}{z_{\widehat{12},\widehat{3}}}\frac{(2k_{1,2}\cdot k_{\widehat{3},\widehat{12}})^2}{k_{1,2}^2 k_{\widehat{3},\widehat{12}}^2}\right)\right]. \tag{17}$$

The reduced matrix element is evaluated for momenta that have been obtained merging particles 1, 2 and 3 with a generalised rescaling mapping. This is equivalent to merging 1 and 2 into $\widehat{12}$ recoiling against all other legs, and later merging $\widehat{12}$ with $\widehat{3}$ recoiling against all remaining momenta.

1.6 S(C(1,2))

The limit where the $q\bar{q}$ pair is both collinear and soft is over-subtracted and needs to be added back. The corresponding counter-counterterm $S(C(1,2))$ may also be constructed iteratively as done in [2]. After the C_{12} limit has been taken the reduced, spin-correlated matrix element which contains the single parent $\widehat{12}$ in the limit of soft $\widehat{12}$ factorises with the current

$$S_{\widehat{12}}^{\mu\nu} = \sum_{i,j}\frac{\widehat{p}_i^\mu \widehat{p}_j^\nu + \widehat{p}_i^\nu \widehat{p}_j^\mu}{s_{\widehat{12}\widehat{i}} s_{\widehat{12}\widehat{j}}} 2\mathbf{T}_i\cdot\mathbf{T}_j. \tag{18}$$

Since the current multiplies the splitting function for the 12-collinear limit, hich features a factor s_{12}^{-1},

the counterterm is divergent in all triple-collinear configurations and not just in the $12i$- or $12j$-collinear limits. Similarly to the case of S_{12}, it is thus convenient to shift away the elements on the colour diagonal using colour conservation, which gives

$$S_{\widehat{12}}^{\mu\nu} = \sum_{j\neq k}\left[\frac{\widehat{p}_i^\mu \widehat{p}_j^\nu + \widehat{p}_i^\nu \widehat{p}_j^\mu}{s_{\widehat{12}\widehat{i}} s_{\widehat{12}\widehat{j}}} - \frac{\widehat{p}_i^\mu \widehat{p}_i^\nu}{s_{\widehat{12}\widehat{i}}^2} - \frac{\widehat{p}_j^\mu \widehat{p}_j^\nu}{s_{\widehat{12}\widehat{j}}^2}\right] 2\mathbf{T}_i\cdot\mathbf{T}_j. \tag{19}$$

$$S_{12}C_{12} = \frac{T_R}{s_{12}}\sum_{j\neq k}\left[\frac{s_{jk}}{s_{12j}s_{12k}} - z(1-z)\left(\frac{2s_{j\perp}s_{k\perp}}{s_{12j}s_{12k}n_\perp^2} - \frac{s_{j\perp}^2}{s_{12j}^2 n_\perp^2} - \frac{s_{k\perp}^2}{s_{12k}^2 n_\perp^2}\right)\right]\mathbf{T}_j\cdot\mathbf{T}_k, \tag{20}$$

1.7 C(S(C(1,2)),3)

References

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