# Regularisation of real emissions with distributed soft counterterms

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## 1 Process $e^+e^- \to u\bar{u}d\bar{d}$

In this section we consider the process $e^+e^- \to u_1\bar{u}_2d_3\bar{d}_4$ as a double-real correction to $e^+e^- \to jj+X$, and we attempt to construct a subtraction procedure which features a minimal number of reduced kinematic configurations.

The elementary unresolved limits that need to be regulated are

$$C_{12}, \quad C_{34}, \quad C_{134}, \quad C_{234}, \quad C_{123}, \quad C_{124}, \quad S_{12}, \quad S_{34}. \tag{1}$$

At most three of these limits have a common overlap and need to be considered simultaneously. Due to the lack of a tree-level diagram for $e^+e^- \to gg$, the limit of two collinear pairs $C_{12}C_{34}$ is regular. The maximal overlaps that we need to consider are therefore

$$C_{123}S_{12}C_{12}, \quad C_{124}S_{12}C_{12}, \quad C_{134}S_{34}C_{34}, \quad C_{234}S_{34}C_{34}. \tag{2}$$

These are all of the same type, so it will be sufficient to consider the representative $C_{123}S_{12}C_{12}$.

In order to have a minimal number of kinematic configurations per evaluation, all measurement functions and matrix elements within counterterms will be evaluated for collinear reduced kinematics, where a set of particles have been merged into a single on-shell parent without violating overall momentum conservation. The momentum mapping that we use to achieve this is the rescaling mapping of ref. [3].[1] Counterterms that are associated to soft configurations will be split into multiple contributions and distributed among collinear reduced kinematics. Within each counterterm we let the limit operation act on the phase space volume, i.e. we divide each counterterm by the jacobian $\mathcal{J}$ defined by

$$\mathrm{d}\Phi = \mathcal{J} \times \mathrm{d}\tilde{\Phi} \prod_C \frac{\mathrm{d}s_C}{2\pi} \,\mathrm{d}\Phi_C\,, \tag{3}$$

where $\Phi$ is the full phase space, $\tilde{\Phi}$ the reduced one, and $C$ each merged set of particles.

In the expression for local currents we make extensive use of the momentum fractions $z_{A,B}$ and the transverse momenta $k_{A,B}$. Since these variables which describe unresolved kinematics are shared by multiple counterterms, we find it useful to discuss them here. Given the light-like vector $n$ in the direction of $\vec{p}_{AB}$ (where $p_{AB} = p_A + p_B$) and an arbitrary reference vector $\bar{n}$, momentum fractions are defined by

$$z_{A,B} \equiv \frac{\bar{n}\cdot p_A}{\bar{n}\cdot p_{AB}}, \tag{4}$$

and transverse momenta are determined via

$$k_A^\mu \equiv p_A^\mu - \frac{\bar{n}\cdot p_A}{\bar{n}\cdot n} n^\mu - \frac{n\cdot p_A}{n\cdot\bar{n}}\bar{n}^\mu, \qquad k_{A,B}^\mu \equiv k_A^\mu - z_{A,B}k_{AB}^\mu. \tag{5}$$

### 1.1 Counterterm $C_{12}$

We subtract the limit of a $q\bar{q}$ pair going collinear using the current

$$C_{12}^{\mu\nu} = \frac{T_R}{s_{12}}\left[-g^{\mu\nu} + 4z_{1,2}z_{2,1}\frac{k_{1,2}^\mu k_{1,2}^\nu}{k_{1,2}^2}\right], \tag{6}$$

which is associated to 12-collinear kinematics.

[1] A generalisation of the rescaling mapping that applies to massive particles was presented in [].

### 1.2 Counterterm $C_{123}$

The triple-collinear counterterm for $q'\bar{q}'q$ is determined using the current [1]

$$C_{123} = \frac{C_F T_R}{2s_{123}^2}\left[-\frac{t_{12,3}^2}{s_{12}^2} + \frac{s_{123}}{s_{12}}\left(\frac{4z_{3,12} + (z_{1,23} - z_{2,13})^2}{z_{12,3}} + (1-2\epsilon)z_{12,3}\right) - (1-2\epsilon)\right], \tag{7}$$

where

$$t_{12,3} \equiv 2\frac{z_{1,23}s_{23} - z_{2,13}s_{13}}{z_{12,3}} + \frac{z_{1,23} - z_{2,13}}{z_{12,3}}s_{12}. \tag{8}$$

Clearly in this case we use 123-collinear kinematics.

### 1.3 Counterterm $S_{12}$

In order to construct the $q\bar{q}$ soft counterterm, we start from the form of the current used in [2] which reads

$$\frac{T_R}{s_{12}^2}\sum_i\sum_j \frac{s_{1i}s_{2j} + s_{1j}s_{2i} - s_{12}s_{ij}}{s_{(12)i}s_{(12)j}}\mathbf{T}_i\cdot\mathbf{T}_j, \tag{9}$$

where the sum runs over all coloured partons of the reduced process and includes the case $i = j$.

Before discussing partial fractioning, we observe that the global factor $s_{12}^{-2}$ may cause the counterterm to diverge in limits where neither $s_{12i}$ nor $s_{12j}$ go to zero. More concretely, the contribution from the terms with $i = j$ reads

$$\frac{T_R}{s_{12}^2}\sum_i \frac{2s_{1i}s_{2i}}{s_{(12)i}^2}\mathbf{T}_i^2, \tag{10}$$

and *all* terms are divergent in the triple-collinear limit $12j$ for *any* $j$. Thus, although from eq. (10) one might be tempted to assign the $i$-th term to $12i$-collinear kinematics, every term needs to be distributed among all $12j$-collinear kinematics. To this end, we use colour conservation to move all terms off the colour diagonal

$$\sum_i \frac{2s_{1i}s_{2i}}{s_{(12)i}^2}\mathbf{T}_i^2 = -\sum_i\sum_{j\neq i}\frac{2s_{1i}s_{2i}}{s_{(12)i}^2}\mathbf{T}_i\cdot\mathbf{T}_j = -\sum_i\sum_{j\neq i}\left[\frac{s_{1i}s_{2i}}{s_{(12)i}^2} + \frac{s_{1j}s_{2j}}{s_{(12)j}^2}\right]\mathbf{T}_i\cdot\mathbf{T}_j. \tag{11}$$

In this sense, the kinematics that we assign do not follow the divergent structure of the invariant poles but rather the colour, in a similar way as proposed for geometric subtraction [4].

The complete off-diagonal soft current reads

$$\frac{T_R}{s_{12}^2}\sum_i\sum_{j\neq i}\left[\frac{s_{1i}s_{2j} + s_{1j}s_{2i} - s_{12}s_{ij}}{s_{(12)i}s_{(12)j}} - \frac{s_{1i}s_{2i}}{s_{(12)i}^2} - \frac{s_{1j}s_{2j}}{s_{(12)j}^2}\right]\mathbf{T}_i\cdot\mathbf{T}_j. \tag{12}$$

At this stage, we observe that we may replace the invariants $s_{(12)i} = 2p_{12}\cdot p_i$ and $s_{(12)j} = 2p_{12}\cdot p_j$ in the denominator with $s_{12i}$ and $s_{12j}$ at our leisure. Indeed, this operation modifies the counterterm by terms which are of higher order in the double-soft limit $S_{12}$, and therefore does not spoil the cancellation in the counterterm's defining limit. Using the triple invariant $s_{12i}$ seems convenient because it makes denominators naturally match the ones of the collinear counterterm (which cannot be changed because the modification $s_{123} \to s_{(12)3}$ is *not* higher-order in the triple-collinear limit). We thus use

$$S_{12} = \frac{T_R}{s_{12}^2}\sum_{j\neq i}\left[\frac{s_{1i}s_{2j} + s_{1j}s_{2i} - s_{12}s_{ij}}{s_{12i}s_{(12)j}} - \frac{s_{1i}s_{2i}}{s_{12i}^2} - \frac{s_{1j}s_{2j}}{s_{(12)j}^2}\right]\mathbf{T}_i\cdot\mathbf{T}_j. \tag{13}$$

Whether the choice of using $s_{(12)j}$ instead of $s_{12j}$ is important elsewhere in the subtraction was not documented and needs investigation.

It is easy to partial-fraction eq. (13) into collinear kinematics. In the present implementation we use

$$1 = \frac{s_{12i}}{s_{12i} + s_{12j}} + \frac{s_{12j}}{s_{12i} + s_{12j}}, \tag{14}$$

for each term in the dipole sum which leads to

$$S_{12}^{(i)} = \frac{T_R}{s_{12}^2} \sum_{j \neq i} \frac{s_{12j}}{s_{12i} + s_{12j}} \left[ \frac{s_{1i}s_{2j} + s_{1j}s_{2i} - s_{12}s_{ij}}{s_{12i}s_{(12)j}} - \frac{s_{1i}s_{2i}}{s_{12i}^2} - \frac{s_{1j}s_{2j}}{s_{(12)j}^2} \right] \mathbf{T}_i \cdot \mathbf{T}_j. \tag{15}$$

Possibly in the future we may want to change the partial fractions to be dependent only on angles and not on energies: in the case of two collinear pairs with distributed single-soft limits, this has been noted to be essential for disjoint collinear limits to work in combination with distributed soft subtraction.

### 1.4 Counterterm $C_{123}S_{12}$

Shifting out of the diagonal the sum over colour dipoles for the $q\bar{q}$ soft limit turns out to be extremely practical also to take its $C_{123}$ triple-collinear limit. We start with either eq. (13) or eq. (15) (the partial fraction makes no difference in the $C_{123}$ limit), and observe that for a given term to contribute one of $i$ or $j$ needs to be equal to 3, and in the collinear limit the ratio of scalar products with another leg is equal to a ratio of momentum fractions. After this replacement, colour conservation can be used and we find

$$C_{123}S_{12} = -\frac{2T_R}{s_{12}^2}\mathbf{T}_3^2 \left[ \frac{s_{13}z_{2,13} + s_{23}z_{1,23} - s_{12}z_{3,12}}{s_{123}z_{12,3}} - \frac{s_{13}s_{23}}{s_{123}^2} - \frac{z_{1,23}z_{2,13}}{z_{12,3}^2} \right]. \tag{16}$$

Note that, if the contributions on the diagonal have not been reshuffled, some effort is needed to see that the latter two terms are needed. We also note that subtracting this sub-limit from $C_{123}$ many terms simplify and we are left with

$$C_{123} - C_{123}S_{12} = \frac{C_F T_R}{s_{123}^2} \left[ \frac{s_{123}}{s_{12}} \frac{z_{1,23}^2 + z_{2,13}^2}{z_{12,3}} - 1 + \epsilon \left( 1 + \frac{s_{123}}{s_{12}} z_{12,3} \right) \right], \tag{17}$$

which is what is currently implemented in the code (for $\epsilon = 0$). This hard triple-collinear counterterm is clearly associated to 123-collinear kinematics, and for the simplifications to occur the momentum fractions have to be computed as in $C_{123}$.

### 1.5 Counterterm $C_{123}C_{12}$

The strong-ordered collinear limit $C_{123}C_{12}$ is the first nested limit that we encounter whose counterterm we implement in an iterated fashion. To this end we follow the steps of [3]. Starting from the counterterm $C_{12}$, we take the extra collinear limit of the parent gluon $\widehat{12}$ of the quark-antiquark pair going collinear to the mapped, different-species quark $\widehat{3}$. This involves taking the collinear limit of a spin-correlated matrix element, which gives the splitting function

$$C_{\widehat{\widehat{12}3}}^{\alpha\beta,ss'} = \frac{C_F}{s_{\widehat{123}}} \delta_{ss'} \left[ \frac{z_{\widehat{12},\widehat{3}}}{2} d^{\alpha\beta} - 2 \frac{z_{\widehat{3},\widehat{12}}}{z_{\widehat{12},\widehat{3}}} \frac{k_{\widehat{3},\widehat{12}}^{\alpha} k_{\widehat{3},\widehat{12}}^{\beta}}{k_{\widehat{3},\widehat{12}}^2} \right]. \tag{18}$$

The sum over physical polarisations is given by the transverse tensor with respect to the light-cone vector in the collinear direction $p$ and a reference null vector $n$,

$$d^{\alpha\beta}(p,n) \equiv -g^{\alpha\beta} + \frac{p^\alpha n^\beta + p^\beta n^\alpha}{p \cdot n}. \tag{19}$$

We have indicated with a hat the variables which are computed after merging particles 1 and 2. Performing the Lorentz algebra we find

$$C_{\widehat{\widehat{12}3}}^{\alpha\beta,ss'} C_{12,\alpha\beta} = \frac{C_F T_R}{s_{12}s_{\widehat{123}}} \delta^{ss'} \left[ \left( 2\frac{z_{\widehat{3},\widehat{12}}}{z_{\widehat{12},\widehat{3}}} + z_{\widehat{12},\widehat{3}}(1-\epsilon) \right) - 2z_{1,2}z_{2,1} \left( z_{\widehat{12},\widehat{3}} + \frac{z_{\widehat{3},\widehat{12}}}{z_{\widehat{12},\widehat{3}}} \frac{(2k_{1,2} \cdot k_{\widehat{3},\widehat{12}})^2}{k_{1,2}^2 k_{\widehat{3},\widehat{12}}^2} \right) \right]. \tag{20}$$

The reduced matrix element is evaluated for momenta that have been obtained merging particles 1, 2 and 3 with a generalised rescaling mapping. This is equivalent to merging 1 and 2 into $\widehat{12}$ recoiling against all other legs, and later merging $\widehat{12}$ with $\widehat{3}$ recoiling against all remaining momenta.

### 1.6 Counterterm $S_{12}C_{12}$

The limit where the $q\bar{q}$ pair is both collinear and soft is over-subtracted and needs to be added back. The corresponding counter-counterterm $S_{12}C_{12}$ may also be constructed iteratively as done in [2]. After the $C_{12}$ limit has been taken the reduced, spin-correlated matrix element which contains the single parent $\widehat{12}$ in the limit of soft $\widehat{12}$ factorises with the current

$$S^{\mu\nu}_{\widehat{12}} = \sum_{i,j} \frac{\widehat{p}_i^\mu \widehat{p}_j^\nu + \widehat{p}_i^\nu \widehat{p}_j^\mu}{s_{\widehat{12}\widehat{i}} s_{\widehat{12}\widehat{j}}} 2\mathbf{T}_i \cdot \mathbf{T}_j. \tag{21}$$

Since this current multiplies the splitting function for the 12-collinear limit which features a factor $s_{12}^{-1}$, the counterterm is divergent in all triple-collinear configurations and not just in the $12i$- or $12j$-collinear limits. Similarly to the case of $S_{12}$, it is thus convenient to shift away the elements on the colour diagonal using colour conservation, which gives

$$S^{\mu\nu}_{\widehat{12}} = \sum_{i\neq j} \left[ \frac{\widehat{p}_i^\mu \widehat{p}_j^\nu + \widehat{p}_i^\nu \widehat{p}_j^\mu}{s_{\widehat{12}\widehat{i}} s_{\widehat{12}\widehat{j}}} - \frac{\widehat{p}_i^\mu \widehat{p}_i^\nu}{s_{\widehat{12}\widehat{i}}^2} - \frac{\widehat{p}_j^\mu \widehat{p}_j^\nu}{s_{\widehat{12}\widehat{j}}^2} \right] 2\mathbf{T}_i \cdot \mathbf{T}_j. \tag{22}$$

Contracting this expression with eq. (6), with the assumption that $\widehat{p}_i$ and $\widehat{p}_j$ be massless, we find

$$\begin{aligned} S^{\mu\nu}_{12} C_{12,\mu\nu} = \frac{T_R}{s_{12}} \sum_{i\neq j} \Bigg[ &- \frac{s_{\widehat{i}\widehat{j}}}{s_{\widehat{12}\widehat{i}} s_{\widehat{12}\widehat{j}}} + 2 z_{1,2} z_{2,1} \frac{(2k_{1,2}\cdot\widehat{p}_i)(2k_{1,2}\cdot\widehat{p}_j)}{s_{\widehat{12}\widehat{i}} s_{\widehat{12}\widehat{j}} k_{1,2}^2} \\ &\times \left( 1 - \frac{1}{2}\frac{s_{\widehat{12}\widehat{j}}}{s_{\widehat{12}\widehat{i}}} \frac{2k_{1,2}\cdot\widehat{p}_i}{2k_{1,2}\cdot\widehat{p}_j} - \frac{1}{2}\frac{s_{\widehat{12}\widehat{i}}}{s_{\widehat{12}\widehat{j}}} \frac{2k_{1,2}\cdot\widehat{p}_j}{2k_{1,2}\cdot\widehat{p}_i} \right) \Bigg] \mathbf{T}_i \cdot \mathbf{T}_j. \end{aligned} \tag{23}$$

At this stage, the $S_{12}C_{12}$ counterterm may be partial-fractioned into triple-collinear sectors following the colour structure. We choose to perform this operation *after* the limit $C_{12}$ has been taken, as if we were dealing with a single-soft gluon $\widehat{12}$.[2] We thus use the current

$$S^{(i),\mu\nu}_{12} C_{12,\mu\nu} = \frac{s_{\widehat{12}\widehat{j}}}{s_{\widehat{12}\widehat{i}} + s_{\widehat{12}\widehat{j}}} S^{\mu\nu}_{12} C_{12,\mu\nu}, \tag{24}$$

within $12i$ triple-collinear kinematics.

This, however, is not sufficient to achieve the desired cancellation pattern. Indeed, when the counterterm built from the current (23) is included with $12k$ collinear kinematics for some $k$, which clearly reduce to no momentum mapping for soft $p_{12}$, it does *not* automatically match the counterterm $C_{12}$ from eq. (6) in the $S_{12}$ limit. This is due to our choice of dividing by the jacobian factor of eq. (3) which, in the $S_{12}$ limit, is different between the mappings that merge the sets $\{1,2\}$ and $\{1,2,3\}$. To fix this mismatch we have two options. The first one, which essentially follows [3], is to forfeit eq. (3) and divide by jacobians which are defined by the alternative canonical factorisation of phase space

$$\mathrm{d}\Phi = \mathcal{J} \times \mathrm{d}\tilde{\Phi} \prod_C \frac{\mathrm{d}\alpha_C}{2\pi}\, \mathrm{d}\Phi_C\,, \tag{25}$$

where $\alpha_C$ are the parameters of the rescaling mapping defined in [3]. Given that $\alpha_C$ is essentially the light-cone component of the momentum of the collinear set in the anti-collinear direction, i.e. its 'minus' component, this choice amounts to parametrise the virtuality variables with the anti-collinear components of the collinear momenta. The second possibility is to reabsorb the mismatch factor into the definition of the counterterm, replacing for instance

$$S^{(i),\mu\nu}_{12} C_{12,\mu\nu} \to \frac{Q\cdot p_{12}}{Q\cdot p_{\widehat{12}}} S^{(i),\mu\nu}_{12} C_{12,\mu\nu}. \tag{26}$$

[2] This choice might match better the structure of iterated limits, such that $S_{12}C_{12}$ regulates the $C_{12}$ integrated counterterm in the $S_{\widehat{12}}$ single-unresolved limit within the single-unresolved phase space. However the specific form of partial fraction employed may have interplay with other subtraction elements and deserves further investigation.

Note that the factor is only defined up to terms which are higher-order in the $S_{12}$ limit, with the constraint that it goes to 1 in the $C_{12}$ and $C_{123}$ limits. At the moment it does not seem clear which option is the most convenient, but it is relatively easy to switch between the two.[3]

### 1.7 Counterterm $S_{12}C_{123}C_{12}$

The last piece that is missing for the three limits $S_{12}$, $C_{123}$ and $C_{12}$ to be regulated is the triple overlap, which has been subtracted three times in the simple counterterms and added back three times in the double overlaps, so that it still needs to be subtracted from the original integrand. The singular current, which is constructed to match [3], can be obtained from eq. (20) dropping terms which do not have a $z_{12,3}$ denominator[4]

$$S_{12}C_{123}C_{12} = \frac{2C_F T_R}{s_{12}s_{123}} \frac{z_{3,12}}{z_{12,3}} \left[ 1 - z_{1,2}z_{2,1} \frac{(2k_{1,2} \cdot k_{3,12})^2}{k_{1,2}^2 k_{3,12}^2} \right]. \tag{27}$$

This counterterm is associated to 123-collinear kinematics.

## References


[1] S. Catani and M. Grazzini, Nucl. Phys. B **570**, 287 (2000) doi:10.1016/S0550-3213(99)00778-6 [hep-ph/9908523].

[2] G. Somogyi, Z. Trocsanyi and V. Del Duca, JHEP **0506**, 024 (2005) doi:10.1088/1126-6708/2005/06/024 [hep-ph/0502226].

[3] G. Somogyi, Z. Trocsanyi and V. Del Duca, JHEP **0701**, 070 (2007) doi:10.1088/1126-6708/2007/01/070 [hep-ph/0609042].

[4] F. Herzog, Geometric subtraction for real radiation at NNLO, `https://www.ggi.infn.it/talkfiles/slides/slides4304.pdf`


[3] Within `MadNkLO`, the first option can be chosen by selecting `alpha_jacobian` as `factor` for the currents, either individually or globally in `QCD_local_currents.py` (beware that this might affect other schemes). The mismatch factor is instead implemented within the `ee2qqgg-NNLO-IR-limits` branch in the `cataniseymour` NNLO local current `QCD_final_collinear_0_QQxq`, and more precisely in the method `S12C12_kernel` with the variable `fix`.

[4] In performing this step we somewhat arbitrarily decide to keep the $z_{3,12}$ terms at the numerator which theoretically could or should be set to one. Very roughly speaking this is equivalent to taking the $C_{123}$ limit of the soft counterterm for $S_{12}$, as it is sometimes useful to do for single-unresolved nested soft-collinear limits.