Guaranteeing a physically realizable battery dispatch without charge-discharge complementarity constraints

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***Abstract*— The non-convex complementarity constraints present a fundamental computational challenge in energy constrained optimization problems. In this work, we present a new, linear, and robust battery optimization formulation that sidesteps the need for battery complementarity constraints and integers and prove analytically that the formulation guarantees that all energy constraints are satisfied which ensures that the optimized battery dispatch is physically realizable. In addition, we bound the worst-case model mismatch and discuss conservativeness. Simulation results further illustrate the effectiveness of this approach.**

I. Introduction

Due to the increasing penetration of variable renewable generation, widespread concerns over reliability of power systems are being raised. Deploying battery storage systems is widely considered a solution to improve grid operations and reliability [1]. However, optimizing battery storage requires being cognizant of the dynamics of the state of charge (SoC), limits on SoC, limits on the rate of change of the SoC (i.e., power input/output), and the physical operating modes of the battery: it can either charge (i.e., consume energy) or discharge (i.e., produce energy), but not both at the same time. Previous works in literature have employed binary variables (i.e., $1=$charging and $0=$discharging) in the optimization to overcome this issue [2]. However, solving optimization problems with binary variables is computationally challenging.

To sidestep this problem, several works in literature have proposed different battery models as a way to overcome the non-convex complementarity constraints [1], [3]–[5]. The authors in [3] relaxed the complementarity constraints in a bulk transmission economic dispatch problem. Then, through KKT analysis, they provide sufficient conditions under which the relaxation holds. However, these conditions do not hold under negative locational marginal prices (LMPs). Similar to the work in [3], the authors in [1], [4] extend the formulation to distribution networks and provide methods to avoid simultaneous charging and discharging by modifying the objective function. However, these methods do not hold under high renewable penetration, specifically under reverse power flow. The work in [5] quantifies the effects of simultaneous charging and discharging and also provides a heuristic approach to avoid this phenomenon. Recently [6] reiterated that many of these approaches fail in practical settings and engender simultaneous charging and discharging.

From the above literature, battery models based on relaxing complementarity constraints fail under practical conditions which then leads to violation of battery SoC constraints when the said models are employed in optimization problems. This is particularly true with reverse power flow from increasing number of vehicle to grid (V2G) systems. Hence, there is a need for models that respect the SoC constraints without having to resort to non-linear complementarity constraints. Another factor is a shift towards real-time control of power systems [7], which motivates a need to avoid mixed-integer formulations. To tackle this critical problem, in this work, we propose a method that respects the battery SoC constraints under practical conditions and, at the same time, avoids the need for non-linear complementarity constraints or binary variables. We augment the battery model with a linear term that utilizes a simplified battery model using only the net battery power exchanges. This simplified linear term results in tightening of the SoC upper limit in the battery model. The contribution is a new battery dispatch formulation whose optimal solution predicts a physically realizable dispatch, i.e., one that respects the SoC limits. We provide analysis that proves the feasibility of this technique and also provide bounds on its conservativeness with regards to the tightening of the SoC limits. Bounds on the worst-case tightening of SoC limits can be calculated *a priori* based on parameters such as the optimization horizon and time-step and the battery specs.

In the rest of the manuscript we develop a novel, linear formulation of the optimal battery dispatch problem that respects the SoC limits without using complementarity constraints. We analyze the approach and provide simulation results that demonstrate its effectiveness.

II. Standard battery model

Consider a battery with SoC at (discrete) time-step k, $E[k]\in[0,E_{\max}]$, where each time-step represents a duration $\Delta t>0$. The battery also has charging and discharging inputs that can be applied over time-step k defined as $P_{\mathrm{c}}[k], P_{\mathrm{d}}[k] \in [0, P_{\max}]$, respectively, and charging and discharging efficiencies $\eta_{\mathrm{c}},\eta_{\mathrm{d}}\in(0,1]$, respectively. In addition, the battery can either charge or discharge but not both at time k, which yields non-convex complementarity condition $P_{\mathrm{c}}[k]P_{\mathrm{d}}[k]=0$. Then, starting with a given initial SoC E_0 and a sequence of inputs over period $\mathcal{T}=\{0,1,...,T-1\}$, the battery SoC dynamics evolve along a admissible trajectory described by the following set of equalities and inequalities:

$$E[k+1]=E[k]+\Delta t\eta_{\mathrm{c}}P_{\mathrm{c}}[k]-\frac{\Delta t}{\eta_{\mathrm{d}}}P_{\mathrm{d}}[k], \qquad \forall k\in\mathcal{T} \tag{1a}$$

$$E[0]=E_0 \tag{1b}$$

$$0\le P_{\mathrm{c}}[k]\le P_{\max}, \qquad \forall k\in\mathcal{T} \tag{1c}$$

$$0\le P_{\mathrm{d}}[k]\le P_{\max}, \qquad \forall k\in\mathcal{T} \tag{1d}$$

$$0\le E[k+1]\le E_{\max}, \qquad \forall k\in\mathcal{T} \tag{1e}$$

$$P_{\mathrm{c}}[k]P_{\mathrm{d}}[k]=0 \qquad \forall k\in\mathcal{T}. \tag{1f}$$

The resulting SoC trajectory can be expressed as

$$\mathbf{E}(\mathbf{P}_{\mathrm{c}},\mathbf{P}_{\mathrm{d}})=\mathbf{1}_T E_0+\eta_{\mathrm{c}}\mathbf{A}\mathbf{P}_{\mathrm{c}}-\frac{1}{\eta_{\mathrm{d}}}\mathbf{A}\mathbf{P}_{\mathrm{d}}, \tag{2}$$

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where $\mathbf{E} = \mathrm{col}\{E[k+1]\}_{k\in\mathcal{T}}$, $\mathbf{P}_\mathrm{c} = \mathrm{col}\{P_\mathrm{c}[k]\}_{k\in\mathcal{T}}$, and $\mathbf{P}_\mathrm{d} = \mathrm{col}\{P_\mathrm{d}[k]\}_{k\in\mathcal{T}}$ and $\mathbf{A}$ is a lower triangular matrix that relates the input at time k to $\Delta t E[l]$ at time $l \geq k$, and $\mathbf{1}_T := [1,...,1]^\top \in \mathbb{R}^T$.

Past work has relaxed the battery model in (1) by removing complementarity condition (1f) [1], [3]–[5]. In doing so, the relaxed model allows *simultaneous charging and discharging*, i.e., $P_\mathrm{c}[k]P_\mathrm{d}[k] \geq 0$.

In the next section, we present the relaxed model and a simplified 1-input model that considers only the net-charging input, i.e., $\mathbf{P}_\mathrm{b} := \mathbf{P}_\mathrm{c} - \mathbf{P}_\mathrm{d}$. Then, we analytically show how these two models together present necessary and sufficient bounds on the SoC in the standard model. This informs a novel battery optimization formulation that is both convex and whose optimal open-loop dispatch schedule is guaranteed to be physically realizable.

III. Relaxed and simplified battery models

A. Relaxed model

The relaxed model is obtained by removing (1f) from the standard battery model (1) and is, thus, convex. It defines a relaxed SoC trajectory $\mathbf{E}^\mathrm{r} := \mathrm{col}\{E^\mathrm{r}[k]\}_{k=1}^T$ that $\forall k \in \mathcal{T}$ satisfies

$$E^r[k+1] = E^r[k] + \Delta t \eta_\mathrm{c} P^r_\mathrm{c}[k] - \frac{\Delta t}{\eta_\mathrm{d}} P^r_\mathrm{d}[k], \tag{3a}$$
$$E^r[0] = E_0 \tag{3b}$$
$$0 \leq P^r_\mathrm{c}[k] \leq P_\mathrm{max}, \tag{3c}$$
$$0 \leq P^r_\mathrm{d}[k] \leq P_\mathrm{max}, \tag{3d}$$
$$0 \leq E^r[k+1] \leq E_\mathrm{max}, \tag{3e}$$

Note that with complementarity conditions relaxed in (3) we have new inputs $P^r_\mathrm{c}[k]$ and $P^r_\mathrm{d}[k]$ that are different from those in (1). The variables are related

$$\mathbf{P}_\mathrm{c} = \max\{\mathbf{0}, \mathbf{P}^r_\mathrm{c} - \mathbf{P}^r_\mathrm{d}\}, \quad \mathbf{P}_\mathrm{d} = \max\{\mathbf{0}, -(\mathbf{P}^r_\mathrm{c} - \mathbf{P}^r_\mathrm{d})\}, \tag{4}$$

which implies that $\mathbf{P}_\mathrm{c} - \mathbf{P}_\mathrm{d} = \mathbf{P}^r_\mathrm{c} - \mathbf{P}^r_\mathrm{d}$. The relaxed model's SoC trajectory is then defined by

$$\mathbf{E}^r(\mathbf{P}^r_\mathrm{c}, \mathbf{P}^r_\mathrm{d}) = \mathbf{1}_T E_0 + \eta_\mathrm{c} \mathbf{A} \mathbf{P}^r_\mathrm{c} - \frac{1}{\eta_\mathrm{d}} \mathbf{A} \mathbf{P}^r_\mathrm{d}. \tag{5}$$

B. Simplified 1-input model

For the simplified battery model, we approximate the battery efficiencies, η_c and $\frac{1}{\eta_\mathrm{d}}$ by a single net-charge efficiency $\eta \in [\eta_\mathrm{c}, \frac{1}{\eta_\mathrm{d}}]$ and replace the two inputs in (1) $P_\mathrm{c}[k]$ and $P_\mathrm{d}[k]$ with a single net-charging input $P_\mathrm{b}[k] = P_\mathrm{c}[k] - P_\mathrm{d}[k] \in [-P_\mathrm{max}, P_\mathrm{max}]$, which yields the simplified 1-input model:

$$E^s[k+1] := E^s[k] + \eta \Delta t P_\mathrm{b}[k], \qquad \forall k \in \mathcal{T} \tag{6a}$$
$$E^s[0] = E_0 \tag{6b}$$
$$-P_\mathrm{max} \leq P_\mathrm{b}[k] \leq P_\mathrm{max} \qquad \forall k \in \mathcal{T} \tag{6c}$$
$$0 \leq E^s[k+1] \leq E_\mathrm{max}. \qquad \forall k \in \mathcal{T} \tag{6d}$$

The simplified model's SoC trajectory is then

$$\mathbf{E}^\mathrm{s}(\mathbf{P}_\mathrm{b}) = \mathbf{1}_T E_0 + \eta \mathbf{A} \mathbf{P}_\mathrm{b}. \tag{7}$$

C. Analyzing model mismatch

Clearly, by relaxing complementarity conditions and simplifying efficiencies, the corresponding open-loop SoC trajectories may not match the actual trajectory in (1). However, we will next show that the models are ordered in that $\mathbf{E}^r \leq \mathbf{E} \leq \mathbf{E}^\mathrm{s}$ for $\eta_\mathrm{c}, \eta_\mathrm{d} \in (0,1]$.

Lemma III.1. *If inputs $\mathbf{P}_b = \mathbf{P}_c - \mathbf{P}_d = \mathbf{P}^r_c - \mathbf{P}^r_d$ satisfy $\mathbf{P}_c \cdot \mathbf{P}_d = \mathbf{0}$ and $\mathbf{P}^r_c \cdot \mathbf{P}^r_d \geq \mathbf{0}$, then $\mathbf{E}^r(\mathbf{P}^r_c, \mathbf{P}^r_d) \leq \mathbf{E}(\mathbf{P}_c, \mathbf{P}_d) \leq \mathbf{E}^s(\mathbf{P}_b)$.*

Proof. First we shall prove that $\mathbf{E}^r \leq \mathbf{E}$. Subtracting (5) from (2) and substituting the values of $\mathbf{P}_\mathrm{c}$ and $\mathbf{P}_\mathrm{d}$ from (4) and applying basic algebraic operations, we get:

$$\Delta \mathbf{E}^r := \mathbf{E} - \mathbf{E}^r = \mathbf{A}\left(\frac{1}{\eta_\mathrm{d}} - \eta_\mathrm{c}\right) \min\{\mathbf{P}^r_\mathrm{c}, \mathbf{P}^r_\mathrm{d}\} \geq 0 \tag{8}$$

This proves the first part of the lemma. To prove $\mathbf{E} \leq \mathbf{E}^\mathrm{s}$, we subtract (2) from (7) and substitute $\mathbf{P}_\mathrm{b} = \mathbf{P}_\mathrm{c} - \mathbf{P}_\mathrm{d}$, which gives

$$\Delta \mathbf{E}^\mathrm{s} := \mathbf{E}^\mathrm{s} - \mathbf{E} = \mathbf{A}\left[\mathbf{P}_\mathrm{c}(\eta - \eta_\mathrm{c}) + \mathbf{P}_\mathrm{d}\left(\frac{1}{\eta_\mathrm{d}} - \eta\right)\right] \geq 0 \tag{9}$$

due to $\eta_\mathrm{c} \leq \eta \leq \frac{1}{\eta_\mathrm{d}}$. Thus, $\mathbf{E}^\mathrm{r} \leq \mathbf{E} \leq \mathbf{E}^\mathrm{s}$. □

Lemma III.1 shows that the simplified model overestimates the actual SoC, while the relaxed model underestimates the SoC. Furthermore, from the proof of Lemma III.1, we can analyze the worst-case SoC model mismatches $\Delta \mathbf{E}^r, \Delta \mathbf{E}^s$. The bounds on the mismatches represent the conservativeness of the two battery models. For the relaxed model, the worst-case mismatch over the trajectory is given by:

$$\Delta \mathbf{E}^\mathrm{r} = \mathbf{A}(\frac{1}{\eta_\mathrm{d}} - \eta_\mathrm{c}) \min\{\mathbf{P}^\mathrm{r}_\mathrm{c}, \mathbf{P}^\mathrm{r}_\mathrm{d}\} \leq (\frac{1}{\eta_\mathrm{d}} - \eta_\mathrm{c}) \mathbf{A} \mathbf{1}_T P_\mathrm{max} \tag{10}$$

This worst-case error can further be reduced by including the cutting-plane $\mathbf{P}^r_\mathrm{c} + \mathbf{P}^r_\mathrm{d} \leq \mathbf{1}_T P_\mathrm{max}$ in (3), which gives:

$$\Delta \mathbf{E}^\mathrm{r} \leq (\frac{1}{\eta_\mathrm{d}} - \eta_\mathrm{c}) \mathbf{A} \mathbf{1}_T \frac{P_\mathrm{max}}{2}. \tag{11}$$

Similarly, the simplified model's mismatch can be written:

$$\Delta \mathbf{E}^\mathrm{s} = \mathbf{A}\left[(\eta - \eta_\mathrm{c}) \max\{\mathbf{0}, \mathbf{P}_\mathrm{b}\} + \left(\eta - \frac{1}{\eta_\mathrm{d}}\right) \min\{\mathbf{0}, \mathbf{P}_\mathrm{b}\}\right]. \tag{12}$$

Note that $\Delta \mathbf{E}^\mathrm{s}$ depends on choice of η. Consider the choice of η such that $\eta - \eta_\mathrm{c} = \frac{1}{\eta_\mathrm{d}} - \eta = \frac{1}{2}(\frac{1}{\eta_\mathrm{d}} - \eta_\mathrm{c}) =: \alpha$. Based on this η, the worst-case simplified model mismatch is

$$\Delta \mathbf{E}^\mathrm{s} = \alpha \mathbf{A}[\max\{\mathbf{0}, \mathbf{P}_\mathrm{b}\} - \min\{\mathbf{0}, \mathbf{P}_\mathrm{b}\}] = \alpha \mathbf{A} |\mathbf{P}_\mathrm{b}| \tag{13}$$
$$\implies \Delta \mathbf{E}^\mathrm{s} \leq \alpha \mathbf{A} \mathbf{1}_T P_\mathrm{max} = (\frac{1}{\eta_d} - \eta_c) \mathbf{A} \mathbf{1}_T \frac{P_\mathrm{max}}{2}. \tag{14}$$

From (11) and (14), it can be seen that these worst-case model mismatch bound are equivalent for the given choice of η. Clearly, for $\eta_\mathrm{c} = 1 = \eta_\mathrm{d}$, both battery models are exact (as is known). However, in practice, $(\frac{1}{\eta_d} - \eta_c) \leq 0.2$ for most lithium-based and lead-acid battery technologies (with round-trip efficiencies $> 80\%$), which yields model mismatches (well) below $\frac{P_\mathrm{max}}{10} \mathbf{A} \mathbf{1}_T$.

Since errors are reasonable, we can employ the relaxed and simplified models as lower and upper bounds, respectively, in a linear battery optimization formulation that ensures the actual SoC is persistently within SoC limits.

IV. Optimal battery dispatch formulation

Based on the analysis in Lemma III.1, the two battery models bound the actual SoC. The linear robust battery dispatch (RBD) problem can then be formulated as follows:

$$\textbf{(RBD)} \qquad \min_{\mathbf{P}_\mathrm{c}-\mathbf{P}_\mathrm{d}} \quad f(\mathbf{P}_\mathrm{c}-\mathbf{P}_\mathrm{d}) \tag{15a}$$

$$\text{s.t} \quad \mathbf{0} \leq \mathbf{1}_T E_0 + \eta_\mathrm{c}\mathbf{A}\mathbf{P}_\mathrm{c} - \frac{1}{\eta_\mathrm{d}}\mathbf{A}\mathbf{P}_\mathrm{d} \tag{15b}$$

$$\mathbf{E}_\mathrm{max} \geq \mathbf{1}_T E_0 + \eta\mathbf{A}(\mathbf{P}_\mathrm{c}-\mathbf{P}_\mathrm{d}) \tag{15c}$$

$$0 \leq \mathbf{P}_c \leq \mathbf{1}_T P_\mathrm{max} \tag{15d}$$

$$0 \leq \mathbf{P}_d \leq \mathbf{1}_T P_\mathrm{max} \tag{15e}$$

$$\mathbf{P}_\mathrm{c} + \mathbf{P}_\mathrm{d} \leq \mathbf{1}_T P_\mathrm{max} \tag{15f}$$

Remark. *We can easily adapt* (15) *to power systems with* N *batteries and modify the objective to* $f(\sum_{i=1}^{N}(\mathbf{P}_{c,i} - \mathbf{P}_{d,i}))$*. The formulation can also be augmented by coupling the batteries inputs via power flow equations, e.g., [4].*

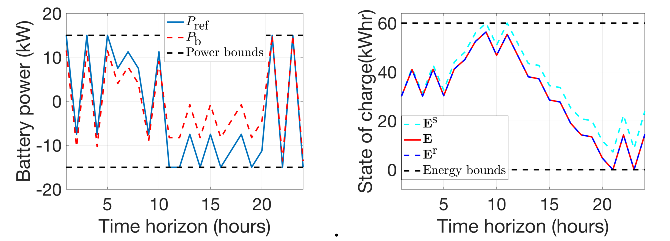
The robust optimization problem in (15) leads to a conservative battery dispatch. However, the conservativeness is with respect to the objective function. That is, by guaranteeing that the actual SoC trajectory is within its energy limits, the optimization problem always ensures that an optimized power dispatch is realizable. In fact, the conservativeness in the objective depends on the time step width Δt and the horizon T (i.e., $\mathbf{A}$) and battery specs (η_c,η_d,P_max).

Next, we illustrate the effectiveness of the proposed approach in (15) with simulation results.

V. Simulation Results

Consider a battery with $P_\mathrm{max} = 15$kW and $E_\mathrm{max} = 60$kWh. Let $\eta_c = 0.95 = \eta_d$ and choose $\eta = (\eta_n + \eta_d)/2 = 1.0013$, which results in $\alpha = 0.0513$. The time-step Δt is 1 hour and the control and prediction horizon length T is 24 hours. The objective in (15a) is chosen as $\sum_k (P_\mathrm{ref}[k] - (P_\mathrm{c}[k] - P_\mathrm{d}[k]))^2$. In Fig. 1a, the results shows one battery tracking a reference power signal while Fig. 1b compares the predicted (upper and lower bounds) SoC resulting from (15) to the actual battery SoC obtained from (2). Fig. 1b illustrates that trajectory $\mathbf{E}$ is within its energy limits, which means the optimized power dispatch $\mathbf{P}_\mathrm{b}$ is guaranteed to be realizable.

Furthermore, to highlight computational efficiency, Table I compares the RBD in (15) to exact mixed-integer (MIP) and non-linear (NLP) formulations as the number of batteries N increases and we track NP_ref. The RBD and MIP are solved using Gurobi 9.1, while the NLP uses IPOPT on a standard laptop. The table shows that the RBD method is 10-200 times faster than MIP for $N \leq 200$ batteries. For $N \geq 500$, MIP does not find a solution with MIP-gap $< 10\%$ within 3600s. The RBD approach is also 5-50 times faster than the NLP, which only achieves local optimum. Note also that the RBD outperforms the NLP with respect to open-loop tracking performance (RMSE) and is still within 10% of the globally optimal, exact MIP. The RBD's fast solve time enables a receding-horizon implementation that should greatly reduce RMSE. Thus, with (15), we sidestep the challenges with non-convex or integer-based complementarity constraints and provide a linear formulation that guarantees a realizable dispatch.



(a) (b)

Fig. 1. (a) Tracking a battery reference power signal P_ref with the net battery output $P_\mathrm{b} \in [-P_\mathrm{max}, P_\mathrm{max}]$. (b) Comparison between predicted SoC ($\mathbf{E}^s$,$\mathbf{E}^r$) and actual SoC $\mathbf{E}$ resulting from optimized dispatch with the energy limits [0,60]. Clearly, the actual SoC trajectory $\mathbf{E}$ satisfies energy limits.

TABLE I
Solve time (sec) and power tracking RMSE (kW) comparison with increasing batteries for RBD vs MIP vs NLP

	RBD		MIP		NLP	
Batteries	Time	RMSE	Time	RMSE	Time	RMSE
10	1.7	47.8	16.3	43.7	5.1	54
100	3.1	478.7	271.8	437.8	50.5	478.7
200	6.3	957.4	1114	866	133.2	1190.2
500	11.5	2327.4	–	–	351.6	2415.2
1000	22.6	4787.1	–	–	1115	4787.1

VI. Conclusions and Future work

This paper presented a new formulation to optimally dispatch batteries while guaranteeing satisfaction of SoC constraints, without having to resort to a non-convex or mixed-integer battery formulation. Through mathematical analysis, we prove that two simple (linear) battery models provide upper and lower bounds on the actual SoC, which enables their use as proxy variables in optimization formulation. Furthermore, we provide worst-case bounds on the conservativeness of this approach. This formulation has the potential to greatly reduce the complexity of energy constrained battery optimization problems, while guaranteeing satisfaction of actual SoC constraints.

Future work will incorporate the battery constraints from (15) into optimal power flow formulation and study the impact of conservativeness on practical case-studies.

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