Term Assignment and Categorical Models for Intuitionistic Linear Logic with Subexponentials

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Abstract

In this paper, we present a typed lambda calculus $\mathbf{SILL}(\lambda)_\Sigma$, a type-theoretic version of intuitionistic linear logic with subexponentials, that is, we have many resource comonadic modalities with some interconnections between them given by a subexponential signature. We also give proof normalisation rules and prove the strong normalisation and Church-Rosser properties for β-reduction by adapting the Tait-Girard method to subexponential modalities. Further, we analyse subexponentials from the point of view of categorical logic. We introduce the concepts of a Cocteau category and a Σ-assemblage to characterise models of linear type theories with a single exponential and affine and relevant subexponentials and a more general case respectively. We also generalise several known results from linear logic and show that every Cocteau category and a Σ-assemblage can be viewed as a symmetric monoidal closed category equipped with a family of monoidal adjunctions of a particular kind. In the final section, we give a stronger 2-categorical characterisation of Cocteau categories.

Keywords— Linear logic, symmetric monoidal categories, subexponentials, comonad, Eilenberg-Moore categories, relevant categories, monoidal adjunction

Contents

1 Introduction

The exponential modality ! is a comonadic operator allowing one to introduce the lacking weakening and contraction inference rules in linear logic [Gir87]. ! is given by the so-called promotion and dereliction rules (aka introduction and elimination) rules alongside the weakening and contraction rules for modalised formulas. The standard presentation consists of the below Gentzen-style inference rules:

$$\frac{!\,A_1,\ldots,!\,A_n \vdash A}{!\,A_1,\ldots,!\,A_n \vdash !\,A} \; !_{\mathbf{L}} \qquad\qquad\qquad \frac{\Gamma, A \vdash B}{\Gamma, !\,A \vdash B} \; !_{\mathbf{R}}$$

$$\frac{\Gamma \vdash B}{\Gamma, !\,A \vdash B} \; \mathbf{W} \qquad\qquad\qquad \frac{\Gamma, !\,A, !\,A \vdash B}{\Gamma, !\,A \vdash B} \; \mathbf{C}$$

Subexponentials are operators of the form $!_s$ that may allow (or not) weakening or contraction where s ranges over some finite set of indices. We have a family of non-equivalent modal operators $!_s$ indexed through elements of a finite set I, and each $!_s$ has its own ability of introducing this or that structural rule. Subexponential modalities were initially introduced in [DJS93] as part of "multicolour linear logic" and then developed in [NOP17] to propose a proof-theoretic framework for concurrency theory and linear logic programming. We refer the reader interested in practical uses of subexponential to [NM09], where the authors claim to develop a simple specification language expressive enough to model data structures and iterative procedures. Subexponentials (and, more generally, weaker versions of the exponential modality) in a non-commutative setting are also of interest in mathematical linguistics, see [KKNS19] and [MSWW23]. From a linguistic point of view, structural rules are generally pathologic since they change the structure of a sentence, so they might not stay grammatically correct. Still, the contraction rule allows dealing with such cases, where the localised duplication is unavoidable, for example, in analysing so-called 'parasitic extractions' as discussed in [KKS17].

Linear logic also provides a framework for functional programming with more flexible mechanisms of resource management by adapting ideas of the Curry-Howard correspondence, see [Wad90] and [Gir87]. Some of those ideas have already found their practical implementation (to a certain extent) in such functional languages as Haskell ([BBN+17]) and Idris 2 ([Bra21]). One can transfer the same ideas to other substructural logics, in particular affine logic admitting weakening and relevant logic admitting contractions. Programs well-typed in affine type lambda calculus, for example, are known to terminate in linear time (see [Cur05, §1]), whereas type systems based on relevant logic are of interest in terms of compiler optimisations, see [Wal05].

As regards the questions of semantics, linear logic has a number of semantic frameworks reflecting the intuition that we interpret provability through resource-aware actions, not statements being

valid in the sense of classical or intuitionistic logic, see [Gir11, Part III, Part VI] for a more detailed discussion on geometry of interaction and coherence spaces. Categorical semantics of linear logic, as it is presented, for example, in [AT11] and [Mel09], generalises the formal properties of coherence spaces and allows interpreting proofs as morphisms in symmetric monoidal categories. Therefore, in terms of category-theoretic approach to metamathematics, we have an account of linear logic as the so-called *internal logic* of symmetric monoidal categories.

In this paper, we introduce a type-theoretic formulation of multiplicative intuitionistic linear logic with subexponentials in several versions. We consider a linear type theory called $\mathbf{SILL}(\lambda)_3$ with the three subexponential modal operators $!_\mathbf{i}$, $!_\mathbf{r}$ and $!_\mathbf{a}$ having the following informal semantics:

- $M : !_\mathbf{i} A$ means that M is an element of type A that we can use finitely many times. So $!_\mathbf{i}$ has the behaviour of the usual ! operator from linear logic. That is, M is an element of an *intuitionistic* (or *normal*) type.

- $M : !_\mathbf{r} A$ means that M is an element of A that one should consume *at least* once. In other words, M is an element of a *relevant* type.

- $M : !_\mathbf{a} A$ means that M is an element of A that one should consume *at most* once. In other words, M is an element of an *affine* type.

$\mathbf{SILL}(\lambda)_3$ also contains the following principle introducing a naturally occurring policy of how several modes of resource management are connected with each other. Informally, this principle can be formulated with two (meta)implications (aka symmetric lax monoidal comonad morphisms in category theoretic terms): $!_\mathbf{i} A \Rightarrow !_\mathbf{r} A$ and $!_\mathbf{i} A \Rightarrow !_\mathbf{a} A$. That is, anything that one can duplicate (destroy) an object in a "normal" way, then it can be duplicated (or destroyed) as an object of a relevant (or affine) type. However, this is not the only way of expanding linear logic with modalities since in many cases the presence of many non-equivalent subexponentials with more complicated interaction between them is also of interest, for example, in concurrent programming, see [OPN15]. So we develop the Curry-Howard correspondence for a bigger family of subexponential enrichments of intuitionistic linear logic alongside their categorical semantics.

In categorical semantics of linear logic, the exponential modality corresponds to a comonad associating a cocommutative comonoid with every object of the underlying symmetric monoidal category, see [AT11, §1.8.4], [See87], [Tro92, Chapter 12] and [BBDPH93a]. In computational terms, ! allows relaxing the resource usage restrictions: by default any thing can be used exactly once, but with ! we can locally address to a particular formula as many times as we want. Logically, ! allows embedding intuitionistic logic into linear logic and manipulating with duplications and deletions in a restricted, localised fashion.

Although subexponentials have been already studied quite comprehensively from the aspects of proof theory, linguistics and such topics in applied computer science as epistemic and spatial concurrent constraint programming (see [NOP17, §7-8]), the semantic aspects of subexponentials have remained overlooked. Although such aspects do not play a significant role in developing practical frameworks, the semantic analysis would allow us to equip subexponentials with a rigorous meaning and also provide the framework with several resource policies within a single system with the denotational semantics.

1.1 Map of the paper

In Section 2, we axiomatise a family of natural deduction calculi with proof-terms $\mathbf{SILL}(\lambda)_\Sigma$ with proof normalisation rules. We favoured the Curry-Howard style presentation (see [SU06] to have a more general perspective) of all those calculi for two reasons. First, given such a presentation,

one can think of $\mathbf{SILL}(\lambda)_\Sigma$ (in particular, its instance for a three-element subexponential signature) as a type-theoretic foundation for a strongly linearly typed functional programming language with several policies of how the user consumes data. Secondly, proof-terms in our approach come along with the corresponding proof-conversion rules, so if proof-terms reflect morphisms in syntax, then the conversion rules syntactically represent the axioms of a Cocteau category. Thus, the semantic interpretation is defined for both derivability and proof transformations. We also prove the strong normalisation and confluence properties for β-reduction by developing a standard technique in the fashion of Tait-Girard.

The rest of the sections in the paper develop a semantic analysis for $\mathbf{SILL}(\lambda)_\Sigma$ and $\mathbf{SILL}(\lambda)_3$ and how categorical models of those systems can be represented. In Section 4, we introduce the notions of a Cocteau category, that is, a symmetric monoidal closed category with three lax monoidal comonads: exponential, affine and relevant. We also introduce the concept of a Σ-assemblage which generalises Cocteau categories directly for an arbitrary subexponential signature Σ. We show that every Σ-assemblage is a model of the many-sorted equational theory generated by $\mathbf{SILL}_\Sigma(\lambda)$-proof conversions.

In Section 5, we elaborate on how one can characterise so-called relevant categories, that is, symmetric monoidal categories where each object A can be copied with a natural transformation γ with the components $\gamma_A : A \to A \otimes A$. We treat relevant categories separately because of the important difference between them and Cartesian categories in the following aspect. It is well known that a symmetric monoidal category $\mathcal{C}$ is Cartesian if and only if the forgetful functor $\mathbf{coMon}(\mathcal{C}) \to \mathcal{C}$ is an isomorphism, see, e.g., [Mel09, Corollary 19], where $\mathbf{coMon}(\mathcal{C})$ is the category of cocommutative comonoids in $\mathcal{C}$. As already discussed in [rel], we only have a weaker criterion: a symmetric monoidal category is relevant if and only if the forgetful functor $\mathbf{coSem}(\mathcal{C}) \to \mathcal{C}$ has a strict symmetric monoidal section. The aforementioned nLab article gives neither a complete proof of this fact nor a reference, so we prove this fact in Section 5 as it seems it has not been introduced in such a form earlier in the literature.

In Section 6, we modify Benton's results from [Ben94] connecting linear-non-linear models and linear categories. We introduce Cocteau adjunctions and Σ-boutons to characterise Cocteau categories and Σ-assemblages with symmetric monoidal categories equipped with resource modalities using the characterisation results on relevant categories from the previous section. Generally, we unveil each comonad as a monoidal adjunction with the corresponding Eilenberg-Moore categories of coalgebras. We provide the 2-categorical approach based on the formal theory of comonads to characterising the category of Cocteau categories with monoidal adjunctions, but we also treat functors of Cocteau categories and their natural transformations.

2   System SILL(λ) and its syntactic properties

We first recall the underlying logic called $\mathbf{SILL}_\Sigma$ and then develop a subexponential linear type theory by assigning linear lambda terms to the inference rules of $\mathbf{SILL}_\Sigma$.

Definition 2.1. Let $(I, \preceq)$ be a finite preorder, a *subexponential signature* is a quadruple $\Sigma = (I, \preceq, W, C)$ where W and C are upward closed subsets of I.

The intuitive idea standing behind the notion of a subexponential signature is that we have a preordered finite family of indices of subexponential modal operators $\{!_s \mid s \in I\}$ and W and C are subsets of those indices whose operators admit weakening and contraction respectively. The preorder relation on subexponential indices has the following informal semantics. If $s_1 \preceq s_2$, the operator

$!_1$ is stronger than $!_2$, that is, $!_{s_2} A \Rightarrow !_{s_1} A$. This is why we require W and C to be increasing: the property of admitting weakening or contraction is inherited by stronger modalities from weaker ones. We capture this idea rigorously further.

The definition of a well-formed formula is specified by the corresponding grammar.

Definition 2.2. Let $\Sigma = (I, \preceq, W, C)$ be a subexponential signature and let $\{p_i \mid i < \omega\}$ be a countable set of propositional variables, the set of Σ-formulas is generated by the following grammar:

$$A, B ::= p_i \mid \mathbb{1} \mid (A \multimap B) \mid (A \otimes B) \mid (!_s A)_{s \in \Sigma}$$

By default, subexponentials bind stronger than binary connectives, whereas the tensor product connective has a higher priority than linear implication.

If $s \in W(C)$, then $!_s$ is an *affine (relevant)* subexponential. $!_s$ is *exponential* if $s \in W \cap C$.

The formalism we present below is inspired by [Tro95] and [BBDPH93a]. In fact, we can think of it as the expansion of the linear type theory with additional relevant and affine subexponentials allowing the structural weakening and contraction rules respectively. Let us define linear terms first.

Definition 2.3 (Preterms). Let $\Sigma = (I, \preceq, W, C)$ be a subexponential signature and let Var $= \{x_i \mid i < \omega\}$ be a countable alphabet of variables. The set of preterms is generated by the following BNF-grammar.

$$M, N ::= x_i \mid \mathbf{1} \mid \mathbf{let\ 1} = M\ \mathbf{in}\ N \mid \lambda x : A.M \mid MN \mid M \otimes N \mid \mathbf{let}\ x \otimes y = M\ \mathbf{in}\ N \mid$$
$$(!_s M\ \mathbf{with}\ \vec{x} = \vec{N})_{s \in \Sigma} \mid (\mathbf{der}_s(M))_{s \in \Sigma} \mid (\mathbf{del}_s(M; N))_{s \in W} \mid (\mathbf{let}_s\ x, y = M\ \mathbf{in}\ N)_{s \in W}$$

Let us introduce the following shorthand notation for a tuple of terms $\vec{M} = (M_0, \ldots, M_n)$:

- $\mathbf{del}_{s_0, \ldots, s_n}(\vec{M}; N) \leftcolon= \mathbf{del}_{s_0}(M_0; (\mathbf{del}_{s_1}(M_1; \ldots (\mathbf{del}_{s_n}(M_n; N)) \ldots))),$

- $\mathbf{let}_s\ \vec{y}, \vec{z} = \vec{M}\ \mathbf{in}\ N \leftcolon= \mathbf{let}_s\ y_0, z_0 = M_0\ \mathbf{in}\ (\mathbf{let}_s\ y_1, z_1 = M_1\ \mathbf{in}\ (\ldots (\mathbf{let}_s\ y_n, z_n = M_n\ \mathbf{in}\ N) \ldots)).$

A preterm M is a *linear lambda term* (or simply *term*) if it is well-typed, that is, there is a context Γ and a type A such that $\Gamma \vdash M : A$ is derivable by the typing rules from Figure 1. So we assume that every free variable occurs exactly once as in the term assignment for a multiplicative-exponential fragment intuitionistic linear logic. We do not trouble ourselves with the rigorous definitions of free variables and linear substitution, the reader can transfer them from [Bie94, Definition 3.14, Definition 3.15].

A *context* Γ is a finite multiset of declarations of the form $x : A$, where x is a variable and A is a formula. A *typing judgement* is a typing declaration $M : A$ for a term M. $\Gamma \vdash M : A$ stands for "a typing judgement $M : A$ is provable from Γ" by the inference rules from Figure 1. Such a term assignment directly generalises the linear typed lambda calculus for intuitionistic linear logic from, for example, [Tro92, Chapter 6] for an arbitrary subexponential signature Σ.

Figure 1: System $\mathbf{SILL}(\lambda)_\Sigma$

$$\frac{}{\vdash \mathbf{1} : \mathbb{1}} \; \mathbb{1}\mathbf{I} \qquad\qquad \frac{}{x : A \vdash x : A} \; \mathbf{ax} \qquad\qquad \frac{\Gamma \vdash M : \mathbb{1} \qquad \Delta \vdash N : B}{\Gamma, \Delta \vdash \mathbf{let}\ \mathbf{1} = M\ \mathbf{in}\ N : B} \; \mathbb{1}\mathbf{E}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \multimap B} \; \multimap \mathbf{I} \qquad\qquad \frac{\Gamma \vdash M : A \multimap B \qquad \Delta \vdash N : A}{\Gamma, \Delta \vdash MN : B} \; \multimap \mathbf{E}$$

$$\frac{\Gamma \vdash M : A \qquad \Delta \vdash N : B}{\Gamma, \Delta \vdash M \otimes N : A \otimes B} \; \otimes \mathbf{I} \qquad \frac{\Gamma \vdash M : A \otimes B \qquad \Delta, x : A, y : B \vdash N : C}{\Delta, \Gamma \vdash \mathbf{let}\ x \otimes y\ =\ M\ \mathbf{in}\ N : C} \; \otimes \mathbf{E}$$

$$\frac{\Gamma_1 \vdash M_1 : !_{s_1} A_1, \ldots, \Gamma_n \vdash M_n : !_{s_n} A_n \qquad x_1 : !_{s_1} A_1, \ldots, x_n : !_{s_n} A_n \vdash N : A}{\Gamma_1, \ldots, \Gamma_n \vdash\ !_s\ N\ \mathbf{with}\ \vec{x} = \vec{M} : !_s A} \; !_s \qquad \frac{\Gamma \vdash M : !_s A}{\Gamma \vdash \mathbf{der}_s\ M : A} \; !_{sE}$$

where $s \preceq s_1, \ldots, s_n$. $\hspace{6cm}$ for each $s \in I$.

$$\frac{\Gamma \vdash M : !_s A \qquad \Delta \vdash N : B}{\Gamma, \Delta \vdash \mathbf{del}_s(M; N) : B} \; \mathbf{W} \qquad\qquad \frac{\Gamma \vdash M : !_s A \qquad \Delta, x : !_s A, y : !_s A \vdash N : B}{\Gamma, \Delta \vdash \mathbf{let}_s x, y = M\ \mathbf{in}\ N : B} \; \mathbf{C}$$

for $s \in W$. $\hspace{7cm}$ for $s \in C$.

One can also formulate the sequent-style formulation of $\mathbf{SILL}_\Sigma$ and prove the cut elimination and the subformula properties similarly to [KKNS19, Theorem 2] and [NOP17, Theorem 1].

Lemma 2.4 (Substitution Lemma). *If $\Gamma \vdash M : A$ and $\Delta, x : A \vdash N : B$, then $\Delta, \Gamma \vdash N[x := M] : B$.*

Proof. Induction on the derivation of $\Delta, x : A \vdash N : B$. $\hspace{6cm}$ $\square$

Let Γ be a context of free variable declarations $\{x_1 : A_1, \ldots, x_n : A_n\}$, then let $|\Gamma| = \{A_i \mid i = 1, \ldots, n, x_i : A_i \in \Gamma\}$.

There is a particular subexponential signature that we shall consider separately. Let I be a three-element set $\{\mathbf{i}, \mathbf{r}, \mathbf{a}\}$. The mnemonic is that $\mathbf{i}$, $\mathbf{r}$ and $\mathbf{a}$ stand for "intuitionistic", "relevant" and "affine" respectively. The preorder relation is given by the below graph:

$$\mathbf{a} \xrightarrow{\;\preceq\;} \mathbf{i} \xleftarrow{\;\succeq\;} \mathbf{r}$$

This preorder depicts that idea if we have an object that we can use finitely many times, such an object can be duplicated (and, thus, become relevant) and destroyed (and, thus, become affine). The subsets W and C are $\{\mathbf{a}, \mathbf{i}\}$ and $\{\mathbf{r}, \mathbf{i}\}$ respectively. By $\mathbf{SILL}(\lambda)_3$, we denote $\mathbf{SILL}(\lambda)_\Sigma$ over the above described three-element signature.

Although $\mathbf{SILL}(\lambda)_3$ is obtained from $\mathbf{SILL}(\lambda)_\Sigma$, we would like to consider them as different calculi. $\mathbf{SILL}(\lambda)_3$ describes the natural way of how different policies of resource management interact with one another, whereas $\mathbf{SILL}(\lambda)_\Sigma$ describe a broader class of polymodal intuitionistic linear logics where some of the subexponentials might have nothing to do resource management at all.

2.1 Proof Normalisation

As for linear type theory with the single exponential modality ! considered in [BBDPH93b], one can distinguish the $\beta\eta$-reduction relation and the commuting conversion relation which are conversions between terms occuring from the proof normalisation analysis. Although all those commuting conversions can be transfered to $\mathbf{SILL}(\lambda)$, we consider a simpler approach that combines [Tro95] and [Bie94, §4.3]. First of all we specify the term conversions for $\mathbf{SILL}(\lambda)_\Sigma$.

- The β-conversions specified in Figure 2 describe so-called *detour conversions* [1].

Figure 2: β-conversions

$(\multimap \beta)$ $(\lambda x.M)N \triangleleft M[x := N]$,

$(\mathbb{1}\beta)$ $\mathbf{let\ 1 = 1\ in}\ N \triangleleft N$,

$(\otimes\beta)$ $\mathbf{let}\ x \otimes y = M \otimes N\ \mathbf{in}\ P \triangleleft P[x := M, y := N]$,

$(!_s\beta)$ $\mathbf{der}_s\ (!_s\ N\ \mathbf{with}\ \vec{x} = \vec{M}) \triangleleft N[\vec{x} := \vec{M}]$ for $s \in \Sigma$,

$(\mathbf{W}\beta)$ Let $s \preceq s_1, \ldots, s_n$ and $s \in W$, then $\mathbf{del}_s(!_s N\ \mathbf{with}\ \vec{x} = \vec{M}, N') \triangleleft \mathbf{del}_{s_1,\ldots,s_n}(\vec{M}, N')$

$(\mathbf{C}\beta)$ Let $s \in C$

$\mathbf{let}_s\, y, z = (!_s\, M\, \mathbf{with}\, \vec{x} = \vec{M'})\, \mathbf{in}\, N \triangleleft \mathbf{let}_s\, \vec{x'}, \vec{x''} = \vec{M'}\, \mathbf{in}\, N[y := !_s\, M\, \mathbf{with}\, \vec{x} = \vec{x'}, z := !_s\, M\, \mathbf{with}\, \vec{x} = \vec{x''}]$.

- The extensional η-conversions from Figure 3 can be considered as *simplification conversions* in the derivations where an introduction rule was used after applying the corresponding elimination rule. In type-theoretical terms, simplification conversions correspond to η-reduction.

Figure 3: η-conversions

$(\multimap \eta)$ $(\lambda x.Mx) \triangleleft M$,

$(!_s \eta)$ $!_s(\mathbf{der}_s\ x)\ \mathbf{with}\ x = M \triangleleft M$,

$(\otimes\eta)$ $\mathbf{let}\ x \otimes y = M\ \mathbf{in}\ N[z := x \otimes y] \triangleleft N[z := M]$.

- The conversions from Figure 4 syntactically depict that the tensor product and identity types alongside the weakening and contraction operations satisfy the categorical naturality property. From a proof-theoretic angle, those conversions reflect *permutation conversions* (again, the terminology is due to Troelstra) where we permute minor premises of an elimination rule in a certain way.

- The conversions from rest of the figures elaborate on how copying, deleting and promotion term constructors commute with one another. In particular, the conversions having the form of $(\mathbf{W}_{\mathrm{conv}})$ say that the deletion operation has a coalgebraic nature, but we can also view it as a proof conversion allowing us to avoid some redundant promotions when we combine the promotion and deletion operations.

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[1]Troelstra and Schwichtenberg use the term *detour contractions* in [TS00], but we would reserve the word "contraction" for the corresponding structural rule.

Figure 4: The naturality conversions for $\mathbb{1}$, $\otimes$, contraction and weakening

$(\otimes_{\mathbf{nat}})$ $P[w := \mathbf{let}\ x \otimes y = M\ \mathbf{in}\ N] \lhd \mathbf{let}\ x \otimes y = M\ \mathbf{in}\ P[w := N]$,

$(\mathbb{1}_{\mathbf{nat}})$ $P[w := \mathbf{let}\ \mathbf{1} = M\ \mathbf{in}\ N] \lhd \mathbf{let}\ \mathbf{1} = M\ \mathbf{in}\ P[w := N]$,

$(\mathbf{del}_s)_{\mathbf{nat}}$ Let $s \in W$, then $P[x := \mathbf{del}_s(M; N)] \lhd \mathbf{del}_s(M; P[x := N])$,

$(\mathbf{let}_s)_{\mathbf{nat}}$ Let $s \in C$, then $P[x := \mathbf{let}_s\ x, y = M\ \mathbf{in}\ N] \lhd \mathbf{let}_s\ x, y = M\ \mathbf{in}\ P[x := N]$.

Categorically, the conversions $(\mathbf{C}_{\mathrm{conv}_1})$, $(\mathbf{C}_{\mathrm{conv}_2})$ and $(\mathbf{C}_{\mathrm{conv}_3})$ demonstrate the coalgebraic and commutative comonoid nature of the contraction operation, but, as above, we think of them as a separate sort of prooftree transformation that cannot be reduced to none of the above.

The conversion $(\mathbf{CW}_{\mathrm{conv}})$ specifies how exactly the exponential deletion and copying terms commute with each other. One can think of this conversion as the syntactic encoding of one of the commutative comonoid axioms.

Figure 5: The conversions for weakening and contraction

$(\mathbf{W}_{\mathrm{conv}})$ Let $s, t \in \Sigma$, $s \in W$ and $s \preceq t$, then $!_t(\mathbf{del}_s(x; M))\ \mathbf{with}\ x, \vec{x} = N, \vec{N} \lhd \mathbf{del}_t(N; (!_t\ M\ \mathbf{with}\ \vec{x} = \vec{N}))$,

$(\mathbf{C}_{\mathrm{conv}_1})$ Let $s \preceq s_1$ and $s \in C$
$$!_s\ (\mathbf{let}_s\ y, z = x_1\ \mathbf{in}\ N)\ \mathbf{with}\ x_1, \ldots, x_n = M_1, \ldots, M_n \lhd$$
$$\mathbf{let}_s\ y', z' = M_1\ \mathbf{in}\ (!_s\ N\ \mathbf{with}\ y', z', x_2, \ldots, x_n = y, z, M_2, \ldots, M_n)$$

$(\mathbf{C}_{\mathrm{conv}_2})$ $\mathbf{let}_s\ x, y = M\ \mathbf{in}\ N \lhd \mathbf{let}_s\ y, x = M\ \mathbf{in}\ N$,

$(\mathbf{C}_{\mathrm{conv}_3})$ $\mathbf{let}_s\ x, w = M\ \mathbf{in}\ (\mathbf{let}_s\ y, z = w\ \mathbf{in}\ N) \lhd \mathbf{let}_s\ w, z = M\ \mathbf{in}\ (\mathbf{let}_s\ x, y = w\ \mathbf{in}\ N)$,

$(\mathbf{CW}_{\mathrm{conv}})$ $\mathbf{let}_s\ x, y = M\ \mathbf{in}\ \mathbf{del}_s(x; N) \lhd N[y := M]$ for $s \in W \cap C$.

- The conversion $(!_{s\,\mathbf{conv}})$ was introduced by Troelstra for intuitionistic linear logic, see [Tro95, X1]. We introduce the version of that conversion for all subexponentials in a more general form.

Figure 6: Extra $!_s$-conversion

$(!_t)_{\mathbf{conv}}$ Assume that subexponential indices are agreed with one another reasonably:

$$!_t\ M\ \mathbf{with}\ \vec{y'}, y, \vec{y''} = \vec{M'}, (!_s\ N\ \mathbf{with}\ \vec{x} = \vec{M}), \vec{M''} \lhd$$
$$!_t(M[y := !_s\ N\ \mathbf{with}\ \vec{x'} = \vec{x}])\ \mathbf{with}\ \vec{y'}, \vec{x'}, \vec{y''} = \vec{M'}, \vec{M}, \vec{M''}$$

Definition 2.5. Let M, N be terms, Γ a context and A a type, a *conversion in context* is a judgement of the form $\Gamma \vdash M \lhd N : A$ that we read as M converses to N in a context Γ and both M and N has type A. The conversion rules are specified in Figures 2, 3, 4, 5 and 6. $\unlhd$ (i.e. the *multistep conversion relation*) is the least preorder relation that contains $\lhd$.

An equality in context $\Gamma \vdash M \equiv N : A$ is the symmetric closure of $\trianglelefteq$.

Theorem 2.6 (Normalisation)**.** Let $\Gamma \vdash M : A$ be a typing judgement provable in $\mathbf{SILL}(\lambda)_\Sigma$ and $\Gamma \vdash M \trianglelefteq N : A$, then the $\mathbf{SILL}(\lambda)_\Sigma$-derivation of $\Gamma \vdash M : A$ can be transformed into the $\mathbf{SILL}(\lambda)_\Sigma$-derivation of $\Gamma \vdash N : A$.

Proof. Induction on the generation of $M \triangleleft N$ by using Lemma 2.4. $\qquad\square$

2.2 Strong Normalisation and Confluence for β-reduction

Now let us focus on the conversion rules specified in Figure 2 and show a stronger fact than the subject reduction property for the β-reduction rule. As Benton showed in [Ben95], one can embed the linear term calculus into the second-order lambda calculus, also known as System F, by using the technique of encoding coinductive data types with parametric polymorphism. It does not seem that this idea is transferable indeed to subexponentials, both affine and relevant as well as those ones that do not introduce any structural rules. So we develop the Girard method originally introduced for System F in [Gir72], but our presentation is generally based on [Bie94, Theorem 18]. Although the proof is generally routine, we revisit it in order to check the subexponential-related clauses accurately and explicitly.

We provide the construction based on $\mathcal{P}$-candidates and computability types to deduce both the strong normalisation and confluence properties to refine the presentation of $\mathbf{SILL}(\lambda)_\Sigma$ as a specification of a resource aware functional programming language admitting several resource usage simultaneously. Although the strong normalisation is not transferable to general-purpose languages for well-known reasons coming from the basics of computability theory, those facts still remain important since they demonstrate why the evaluation process is determinate for programs built canonically from constructors and eliminators. So the current goal is to prove the following:

Theorem 2.7. [Strong Normalisation for β-reduction] Every β-reduction path $M_0 \triangleleft_\beta M_1 \triangleleft_\beta \ldots$ terminates at a finite step.

We start by giving some preliminary definitions.

Definition 2.8.

- A *I-term* is a result of applying the introduction rule, that is, a term of one of the following forms.
$$M \otimes N, \lambda x.N, !_s N \textbf{ with } \vec{x} = \vec{M}.$$
  for arbitrary terms M, N and for an arbitrary sequence $\vec{M}$.

- A term is called *simple* if it is not an I-term. A simple term M is *stubborn* if it is either has no redexes amongst its subterms or $M \trianglelefteq_\beta N$ for some simple N.

Let $\mathcal{P} = \{\mathbf{P}_A \mid A \text{ is a type}\}$ denote a family of non-empty sets of terms indexed with the set of well-formed types. Let us define several properties of such families.

(P1) $x \in \mathbf{P}_A$ for every variable of type A,

(P2) $M \triangleleft_\beta N$ and $M \in \mathbf{P}_A$ imply $N \in \mathbf{P}_A$.

(P3) Let M be a simple term and

   – Assume $M \in \mathbf{P}_{A \multimap B}$, $N \in \mathbf{P}_A$ and $(\lambda x.M')N \in \mathbf{P}_B$ for $M \trianglelefteq_\beta \lambda x.M'$. Then $MN \in \mathbf{P}_B$.

- Let $s \in \Sigma$, $M \in \mathbf{P}_{!_s A}$, $\mathbf{der}_s(!_s M' \textbf{ with } \vec{x} = \vec{M}) \in \mathbf{P}_A$ for $M \trianglelefteq_\beta !_s M' \textbf{ with } \vec{x} = \vec{M}$. Then $\mathbf{der}_s M \in \mathbf{P}_A$.

- Let $s \in C$. Assume $M \in \mathbf{P}_{!_s A}$, $N \in \mathbf{P}_B$ and $\mathbf{let}_s\, x, y = (!_s M' \textbf{ with } \vec{x} = \vec{M})$ in $N \in \mathbf{P}_B$ for $M \trianglelefteq_\beta !_s M' \textbf{ with } \vec{x} = \vec{M}$. Then $\mathbf{let}_s\, x, y = M$ in $N \in \mathbf{P}_B$.

- Let $s \in W$. Assume $M \in \mathbf{P}_{!_s A}$ and $N \in \mathbf{P}_B$. Assume that $\mathbf{del}_s(!_s M' \textbf{ with } \vec{x} = \vec{M}; N)$ for $M \trianglelefteq_\beta !_s M' \textbf{ with } \vec{x} = \vec{M}$, then $\mathbf{del}_s(M; N) \in \mathbf{P}_B$.

(P4) 1. If $M \in \mathbf{P}_B$ and $x \in \mathrm{FV}(M)$, then $\lambda x.M \in \mathbf{P}_{A \multimap B}$,

  2. If $M \in \mathbf{P}_A$ and $N \in \mathbf{P}_B$ imply $M \otimes N \in \mathbf{P}_{A \otimes B}$,

  3. $\mathbf{1} \in \mathbf{P}_{\mathbb{1}}$,

  4. Let $M \in \mathbf{P}_A$ and $M_1 \in \mathbf{P}_{!_{s_1} A_1}, \ldots, M_n \in \mathbf{P}_{!_{s_n} A_n}$ for $s \preceq s_1, \ldots, s_n$, then $!_s M \textbf{ with } \vec{x} = \vec{M} \in \mathbf{P}_{!_s A}$.

(P5) 1. Let $N \in \mathbf{P}_A$ and $M[x := N] \in \mathbf{P}_B$, then $(\lambda.M)N \in \mathbf{P}_B$,

  2. Let $M \in \mathbf{P}_{\mathbb{1}}$ and $N \in \mathbf{P}_A$, then $\mathbf{let}\, M = \mathbf{1}$ in $N \in \mathbf{P}_A$,

  3. Let $M \in \mathbf{P}_{A \otimes B}$ and $N[x := M', y := N']$ for $x, y \in \mathbf{FV}(N)$ and $M \triangleleft_\beta M' \otimes N'$,

  4. Let $s \in \Sigma$ and $M \in \mathbf{P}_{!_s A}$ such that $M \triangleleft_\beta !_s M' \textbf{ with } \vec{x} = \vec{M}$, then $\mathbf{der}_s M \in \mathbf{P}_A$.

  5. Let $s \in C$, $M \in \mathbf{P}_{!_s A}$ and $N \in \mathbf{P}_B$ such that $\mathbf{P}_B$ contains the following redex:

$$\mathbf{let}_s\, \vec{u}, \vec{v} = \vec{M'} \text{ in } N[y := \,!_s M' \textbf{ with } \vec{x} = \vec{u}, z := \,!_s M' \textbf{ with } \vec{x} = \vec{v}]$$

for $M \trianglelefteq_\beta !_s M' \textbf{with } \vec{x} = \vec{M}$. Then $\mathbf{let}_s\, y, z = M$ in $N \in \mathbf{P}_B$.

  6. Let $M \in \mathbf{P}_{!_s A}$, $N \in \mathbf{P}_B$ and $\mathbf{del}_{s_1,\ldots,s_n}(M_1, \ldots, M_n; N) \in \mathbf{P}_B$ whenever $M \trianglelefteq_\beta !_s M' \textbf{with} \vec{x} = \vec{M}$ and $s_1, \ldots, s_n \succeq s$, then $\mathbf{del}_s(M; N) \in \mathbf{P}_B$.

A set C_A of terms of type A is called a $\mathcal{P}$-*candidate* if it satisfies the following conditions:

($\mathcal{P}$1) $C_A \subseteq \mathbf{P}_A$,

($\mathcal{P}$2) $M \in C_A$ and $M \triangleleft_\beta N$ imply $N \in C_A$,

($\mathcal{P}$3) If $M \in \mathbf{P}_A$ is simple and $M' \in C_A$ such that $M \trianglelefteq_\beta M'$ and M' is an I-term, then $M \in C$.

Let $\mathcal{P}$ be a family of terms, let us associate the *computability type* $\mathbf{ct}(A)$ for each type A:

- Let p_i be atomic, then $\mathbf{ct}(p_i) = \mathbf{P}_A$ and $\mathbf{ct}(\mathbb{1}) = \mathbf{P}_{\mathbb{1}}$.

- $\mathbf{ct}(A \multimap B) = \{ M \in \mathbf{P}_{A \multimap B} \mid \forall N \in \mathbf{ct}(A)\ MN \in \mathbf{ct}(B) \}$,

- $\mathbf{ct}(A \otimes B) = \{ M \in \mathbf{P}_{A \otimes B} \mid M \trianglelefteq_\beta M_1 \otimes M_2 \Rightarrow M_1 \in \mathbf{ct}(A)\ \&\ M_2 \in \mathbf{ct}(B) \}$,

- Let $s \in \Sigma$, then let $\mathbf{ct}'(!_s A)$ denote:

$$\mathbf{ct}'(!_s A) = \{ M \in \mathbf{P}_{!_s A} \mid \mathbf{der}_s(M) \in \mathbf{ct}(A)\ \&$$
$$M_1 \in \mathbf{P}_{!_{s_1} A_1}, \ldots, M_n \in \mathbf{P}_{!_{s_n} A_n}, s \preceq s_1, \ldots, s_n, M' \in \mathbf{ct}(A) \text{ for } M \trianglelefteq_\beta !_s M' \textbf{with } \vec{x} = \vec{M} \}$$

Then if $s \in C$, then

$$\mathbf{ct}(!_s A) = \mathbf{ct}'(!_s A) \cap \{ M \in \mathbf{P}_{!_s A} \mid M \trianglelefteq_\beta !_s M' \textbf{with } \vec{x} = \vec{M}\ \&\ N \in \mathbf{P}_A \Rightarrow$$
$$\mathbf{let}_s \vec{u}, \vec{v} = \vec{M} \text{ in } N[y := \,!_s M' \textbf{with } \vec{u} = \vec{x}, z := \,!_s M' \textbf{with } \vec{v} = \vec{x}] \}.$$

If $s \in W$, then

$$\mathbf{ct}(!_s A) = \mathbf{ct}'(!_s A) \cap \{M \in \mathbf{P}_{!_s A} \mid$$
$$M \trianglelefteq_\beta !_s M' \text{ with } \vec{x} = \vec{M} \ \& \ s \preceq s_1, \dots, s_n \ \& \ N \in \mathbf{P}_B \Rightarrow \mathbf{del}_{s_1, \dots, s_n}(\vec{M}; N) \in \mathbf{P}_B\}$$

otherwise we let $\mathbf{ct}(!_s A) = \mathbf{ct}'(!_s A)$.

Lemma 2.9. Let $\mathcal{P}$ be a family of sets of terms satisfying $(P1) - (P5)$, then

1. $\mathbf{ct}(A)$ is a $\mathcal{P}$-candidate containing all stubborn terms of $\mathbf{P}_A$.

2. Every $\mathbf{ct}(A)$ is closed under substitution: let $M \in \mathbf{ct}(A)$ and let $S : \text{Var} \to \text{Term}$ be a substitution such that $S(x) \in \mathbf{ct}(B)$ for every $x : B \in \mathbf{ct}(A)$, then $S(M) \in \mathbf{ct}(A)$.

Besides, for each A, $\mathbf{P}_A = \text{Term}_A$.

Proof. The proof of the first part is analogous to [Bie94, Lemma 4, §5.1]. The proof of the second part is by induction on the length of M, let us consider some of the subexponential cases.

1. Consider $!_s M \text{ with } \vec{x} = \vec{M}$. By the induction hypothesis, one has
$S_i(M_i) \in \mathbf{ct}(!_{s_i} A_i)$ for $i \in \{1, \dots, n\}$, $s_1, \dots, s_n \succeq s$ and $S(M') \in \mathbf{ct}(A)$. So by $\mathcal{P}1$, $S_i(M_i) \in \mathbf{P}_{!_{s_i} A_i}$ for $i \in \{1, \dots, n\}$ and $S(M) \in \mathbf{P}_A$. If $!_s M \text{ with } x_1, \dots, x_n = S_1(M_1), \dots, S_n(M_n)$ is stubborn, then $!_s M \text{ with } x_1, \dots, x_n = S_1(M_1), \dots, S_n(M_n) \in \mathbf{ct}(!_s A)$ by $\mathcal{P}3$. Otherwise one has

$$!_s M \text{ with } x_1, \dots, x_n = S_1(M_1), \dots, S_n(M_n) \trianglelefteq_\beta !_s M' \text{ with } x_1, \dots, x_n = M_1', \dots, M_n'$$

By $(\mathcal{P}2)$, we have $M' \in \mathbf{ct}(A)$. Also, we have $M_1' \in \mathbf{P}_{!_{s_1} A_1} \dots, M_n' \in \mathbf{P}_{!_{s_n} A_n}$ by $(P2)$. Thus $!_s M \text{ with } x_1, \dots, x_n = S_1(M_1), \dots, S_n(M_n) \in \mathbf{ct}(!_s A)$. Therefore, we conclude that

$$S_1; \dots; S_n; S(!_s M \text{ with } \vec{x} = \vec{M}) \in \mathbf{ct}(A).$$

where $S_1; \dots; S_n; S$ is the composite substitution.

2. Consider $\mathbf{let}_s \ x, y = M \text{ in } N$. By the induction hypothesis, $S_1(M) \in \mathbf{ct}(!_s A)$ for $s \in C$ and $S_2(N) \in \mathbf{ct}(B)$. Observe that S_2 has the form of $S_2' \cup \{x \mapsto P, y \mapsto Q\}$ for some substitution S_2' and for some $P, Q \in \mathbf{ct}(!_s A)$.

If $S_1(M)$ is stubborn, then $\mathbf{let}_s \ x, y = S_1(M) \text{ in } S_2'(N) \in \mathbf{ct}(B)$. Otherwise, $S_1(M) \trianglelefteq_\beta !_s P \text{ with } \vec{w} = \vec{Q}$. By $(\mathcal{P}2)$, $!_s P \text{ with } \vec{w} = \vec{Q} \in \mathbf{P}_{!_s A}$. Therefore, we have

$$\mathbf{let}_s \ \vec{u}, \vec{v} = \vec{Q} \text{ in } S_2'(N)[x := !_s R \text{ with } \vec{w} = \vec{u}, y := !_s R \text{ with } \vec{w} = \vec{v}] \in \mathbf{P}_B$$

So by $(P5)$, $\mathbf{let}_s \ x, y = S_1(M) \text{ in } S_2(N) \in \mathbf{P}_B$. By the definition, $\mathbf{let}_s \ x, y = S_1(M) \text{ in } S_2(N) \in \mathbf{ct}(B)$. Thus $S_1; S_2'(\mathbf{let}_s \ x, y = M \text{ in } N) \in \mathbf{ct}(B)$.

Now, clearly $\mathbf{P}_A \subseteq \text{Term}_A$. For the converse inclusion, take any term M of type A, then $M \in \mathbf{ct}(A)$ by the applying the identity substitution. $\square$

Proof of Theorem 2.7. Let SN_A stand for the set of all strongly normalising terms of type A. Now let $\mathcal{P}$ be a family $\{\text{SN}_A \mid A \in \text{Type}\}$. The rest is to show that $\mathcal{P}$ satisfies the conditions $(P1)$-$(P5)$.
Let us associate the reduction tree

$$\mathbf{rt}(M) = \{p \mid p \text{ is a reduction path starting at } M\}$$

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with every term $M \in \mathrm{SN}_A$ and every such a tree satisfies König's lemma (see [Hod93, Exercise 5.6.5]): every term has a finite number of redexes and length$(p) < \omega$ for each $p \in \mathbf{rt}(M)$. The proof is by induction on $d(M)$, the depth of $\mathbf{rt}(M)$. So the rest is to check the conditions (P1)-(P5). We consider the most demonstrative cases with subexponentials.

(P3) Let $s \in C$. Assume $M \in \mathrm{SN}_{!_s A}$, $N \in \mathrm{SN}_B$ such that
$\mathbf{let}_s\, x, y = (!_s\, M'\, \mathbf{with}\, \vec{x} = \vec{M})\, \mathbf{in}\, N \in \mathrm{SN}_B$ for $M \trianglelefteq_\beta\, !_s\, M'\, \mathbf{with}\, \vec{x} = \vec{M}$. We need $\mathbf{let}_s\, x, y = M\, \mathbf{in}\, N \in \mathrm{SN}_B$. Take $\mathbf{let}_s\, x, y = M\, \mathbf{in}\, N \vartriangleleft_\beta P$. If $P = \mathbf{let}_s\, x, y = M_1\, \mathbf{in}\, N$, where M_1 is simple and $M \vartriangleleft_\beta M_1$. In this case, we apply the induction hypothesis for $d(M_1) + d(N)$ and conclude $P \in \mathrm{SN}_{!_s A}$.

If, otherwise, $P = \mathbf{let}_s\, x, y = M\, \mathbf{in}\, N_1$, where N_1 is simple and $N \vartriangleleft_\beta N_1$. Similarly, we apply the induction hypothesis for $d(M) + d(N_1)$ and conclude $P \in \mathrm{SN}_{!_s A}$.

(P4) Let $M \in \mathrm{SN}_A$ and let $M_1 \in \mathrm{SN}_{!_{s_1} A}, \ldots, M_n \in \mathrm{SN}_{!_{s_n} A}$. Consider $!_s\, M\, \mathbf{with}\, \vec{x} = \vec{M} \trianglelefteq_\beta N$ for $s \preceq s_1, \ldots, s_n$. Then N has the form of $!_s\, M'\, \mathbf{with}\, \vec{x} = \vec{M}'$ for $M \trianglelefteq_\beta M'$, $M_1 \trianglelefteq_\beta M_1', \ldots, M_n \trianglelefteq_\beta M_n'$. Then we have $!_s\, M\, \mathbf{with}\, \vec{x} = \vec{M} \in \mathrm{SN}_{!_s A}$ by induction on $d(M) + d(M_1) + \ldots + d(M_n)$.

(P5) Let $s \in C$, $M \in \mathrm{SN}_{!_s A}$ and $N \in \mathrm{SN}_B$ such that
$$\mathbf{let}_s\, \vec{u}, \vec{v} = \vec{M}'\, \mathbf{in}\, N[y := !_s\, M\, \mathbf{with}\, \vec{x} = \vec{u}, z := !_s\, M\, \mathbf{with}\, \vec{x} = \vec{v}] \in \mathrm{SN}_B$$
for $M \trianglelefteq_\beta\, !_s\, M'\, \mathbf{with}\, \vec{x} = \vec{M}$. We need $\mathbf{let}_s\, y, z = M\, \mathbf{in}\, N \in \mathrm{SN}_B$. Consider $\mathbf{let}_s\, y, z = M\, \mathbf{in}\, N \vartriangleleft_\beta P$. As above, we have several alternatives: Assume $P = \mathbf{let}_s\, y, z = M_1\, \mathbf{in}\, N$ for $M \vartriangleleft_\beta M_1$, then, as earlier, we apply the induction hypothesis on $d(M_1) + d(N)$ and conclude that $P \in \mathrm{SN}_B$. The case $P = \mathbf{let}_s\, y, z = M\, \mathbf{in}\, N_1$ for $N \vartriangleleft_\beta N_1$ is considered similarly. If $P = \mathbf{let}_s\, y, z = (!_s\, M'\, \mathbf{with}\, \vec{x} = \vec{M})\, \mathbf{in}\, N$, then we are already done by assumption.

$\square$

Theorem 2.10. Let M, N, P be terms such that $M \trianglelefteq_\beta N$ and $M \trianglelefteq_\beta P$, then there is a term P' such that $N \trianglelefteq_\beta P'$ and $P \trianglelefteq_\beta P'$. As a corollary, a normal form is unique up α-equivalence.

Proof. The proof strategy is similar to the strong normalisation proof. Let CR_A be the set of all confluent terms of type A and let $\mathcal{P}$ be $\{\mathrm{CR}_A \mid A$ is a type $\}$. The goal is to show that $\mathcal{P}$ satisfies (P1)-(P5) and then conclude that every term is confluent by Lemma 2.9. Let us consider one of the cases, in particular, $(P5)$ for the polymodal. The rest of the clauses are proved similarly.

Let $s \in C$, $M \in \mathrm{CR}_{!_s A}$ and $N \in \mathrm{CR}_B$ such that CR_B contains the following redex:
$$\mathbf{let}_s\, \vec{u}, \vec{v} = \vec{M}'\, \mathbf{in}\, N[y := !_s\, M'\, \mathbf{with}\, \vec{x} = \vec{u}, z := !_s\, M'\, \mathbf{with}\, \vec{x} = \vec{v}]$$
for $M \trianglelefteq_\beta\, !_s\, M'\, \mathbf{with}\, \vec{x} = \vec{M}'$. We need $\mathbf{let}_s\, y, z = M\, \mathbf{in}\, N \in \mathrm{CR}_B$.

There are the following subcases to show:

1. Assume there are the following independent reductions:
$$\mathbf{let}_s\, y, z = M\, \mathbf{in}\, N \trianglelefteq_\beta \mathbf{let}_s\, y, z = M_1\, \mathbf{in}\, N_1, \mathbf{let}_s\, y, z = M_2\, \mathbf{in}\, N_2$$
for $M \trianglelefteq_\beta M_1, M_2$ and $N \trianglelefteq_\beta N_1, N_2$. Both M and N are confluent by the assumption, so there are M' and N' such that $M_1, M_2 \trianglelefteq_\beta M'$ and $N_1, N_2 \trianglelefteq_\beta N'$. Therefore,
$$\mathbf{let}_s\, y, z = M_1\, \mathbf{in}\, N_1, \mathbf{let}_s\, y, z = M_2\, \mathbf{in}\, N_2 \trianglelefteq_\beta \mathbf{let}_s\, y, z = M'\, \mathbf{in}\, N'$$

2. Assume there are the following top-level reduction:

$$\mathbf{let}_s\, y, z = M \mathbf{\ in\ } N \trianglelefteq_\beta \mathbf{let}_s\, y, z = (!_s M_1 \mathbf{\ with\ } \vec{x} = \vec{M_1}) \mathbf{\ in\ } N \trianglelefteq_\beta$$
$$\mathbf{let}_s\, \vec{u}, \vec{v} = \vec{M_1} \mathbf{\ in\ } N[y := !_s M_1 \mathbf{\ with\ } \vec{x} = \vec{u}, z := !_s M_1 \mathbf{\ with\ } \vec{x} = \vec{v}] \trianglelefteq_\beta P_1,$$
$$\mathbf{let}_s\, y, z = M \mathbf{\ in\ } N \trianglelefteq_\beta \mathbf{let}_s\, y, z = (!_s M_2 \mathbf{\ with\ } \vec{x} = \vec{M_2}) \mathbf{\ in\ } N \trianglelefteq_\beta$$
$$\mathbf{let}_s\, \vec{u}, \vec{v} = \vec{M_2} \mathbf{\ in\ } N[y := !_s M_2 \mathbf{\ with\ } \vec{x} = \vec{u}, z := !_s M_2 \mathbf{\ with\ } \vec{x} = \vec{v}] \trianglelefteq_\beta P_2$$

M and N are already confluent, so there is a term $\mathbf{let}_s\, y, z = M' \mathbf{\ in\ } N'$ such that:

$$\mathbf{let}_s\, y, z = (!_s M_1 \mathbf{\ with\ } \vec{x} = \vec{M_1}) \mathbf{\ in\ } N \trianglelefteq_\beta \mathbf{let}_s\, y, z = M' \mathbf{\ in\ } N'$$

$$\mathbf{let}_s\, y, z = (!_s M_2 \mathbf{\ with\ } \vec{x} = \vec{M_2}) \mathbf{\ in\ } N \trianglelefteq_\beta \mathbf{let}_s\, y, z = M' \mathbf{\ in\ } N'$$

M', in turn, is of the form $!_s P \mathbf{\ with\ } \vec{x} = \vec{P}$. Then we have the following reduction:

$$\mathbf{let}_s\, y, z = (!_s P \mathbf{\ with\ } \vec{x} = \vec{P}) \mathbf{\ in\ } N' \trianglelefteq_\beta$$
$$\mathbf{let}_s\, \vec{u}, \vec{v} = \vec{P} \mathbf{\ with\ } N'[y := !_s P \mathbf{\ with\ } \vec{x} = \vec{u}, z := !_s P \mathbf{\ with\ } \vec{x} = \vec{v}]$$

By assumption, the following terms are confluent:

$$\mathbf{let}_s\, \vec{u}, \vec{v} = \vec{M_1} \mathbf{\ in\ } N[y := !_s M_1 \mathbf{\ with\ } \vec{x} = \vec{u}, z := !_s M_1 \mathbf{\ with\ } \vec{x} = \vec{v}],$$

$$\mathbf{let}_s\, \vec{u}, \vec{v} = \vec{M_2} \mathbf{\ in\ } N[y := !_s M_2 \mathbf{\ with\ } \vec{x} = \vec{u}, z := !_s M_2 \mathbf{\ with\ } \vec{x} = \vec{v}]$$

and we have the following reductions:

$$\mathbf{let}_s\, \vec{u}, \vec{v} = \vec{M_1} \mathbf{\ in\ } N[y := !_s M_1 \mathbf{\ with\ } \vec{x} = \vec{u}, z := !_s M_1 \mathbf{\ with\ } \vec{x} = \vec{v}] \trianglelefteq P_1$$

$$\mathbf{let}_s\, \vec{u}, \vec{v} = \vec{M_2} \mathbf{\ in\ } N[y := !_s M_2 \mathbf{\ with\ } \vec{x} = \vec{u}, z := !_s M_2 \mathbf{\ with\ } \vec{x} = \vec{v}] \trianglelefteq P_2$$

and we also have

$$\mathbf{let}_s\, \vec{u}, \vec{v} = \vec{M_1} \mathbf{\ in\ } N[y := !_s M_1 \mathbf{\ with\ } \vec{x} = \vec{u}, z := !_s M_1 \mathbf{\ with\ } \vec{x} = \vec{v}] \trianglelefteq \mathbf{let}_s\, \vec{u}, \vec{v} = \vec{P} \mathbf{\ with\ } N'[y := !_s P \mathbf{\ with\ } \vec{x} = \vec{u}, z := !_s P \mathbf{\ with\ } \vec{x} = \vec{v}]$$

$$\mathbf{let}_s\, \vec{u}, \vec{v} = \vec{M_2} \mathbf{\ in\ } N[y := !_s M_2 \mathbf{\ with\ } \vec{x} = \vec{u}, z := !_s M_2 \mathbf{\ with\ } \vec{x} = \vec{v}] \trianglelefteq \mathbf{let}_s\, \vec{u}, \vec{v} = \vec{P} \mathbf{\ with\ } N'[y := !_s P \mathbf{\ with\ } \vec{x} = \vec{u}, z := !_s P \mathbf{\ with\ } \vec{x} = \vec{v}]$$

Therefore, we have P' such that $P_1 \trianglelefteq_\beta P'$ and

$$\mathbf{let}_s\, \vec{u}, \vec{v} = \vec{P} \mathbf{\ with\ } N'[y := !_s P \mathbf{\ with\ } \vec{x} = \vec{u}, z := !_s P \mathbf{\ with\ } \vec{x} = \vec{v}] \trianglelefteq_\beta P'. \qquad (1)$$

Further, we have that

$$\mathbf{let}_s\, \vec{u}, \vec{v} = \vec{M_2} \mathbf{\ in\ } N[y := !_s M_2 \mathbf{\ with\ } \vec{x} = \vec{u}, z := !_s M_2 \mathbf{\ with\ } \vec{x} = \vec{v}] \trianglelefteq_\beta P',$$

so we have $P' \trianglelefteq_\beta P''$ and $P_2 \trianglelefteq_\beta P''$.

3. Assume we have the case when the first reduction is top-level, whereas the other one is independent (the symmetric case is considered similarly):

$$\mathbf{let}_s\, y, z = M \mathbf{\ in\ } N \trianglelefteq_\beta \mathbf{let}_s\, y, z = (!_s M_1 \mathbf{with\ } \vec{x} = \vec{M_1}) \mathbf{\ in\ } N \trianglelefteq_\beta$$
$$\mathbf{let}_s\, \vec{u}, \vec{v} = \vec{M_1} \mathbf{\ in\ } N[y := !_s M_1 \mathbf{\ with\ } \vec{x} = \vec{u}, z := !_s M_1 \mathbf{\ with\ } \vec{x} = \vec{v}] \trianglelefteq_\beta P_1,$$
$$\mathbf{let}_s\, y, z = M \mathbf{\ in\ } N \trianglelefteq_\beta \mathbf{let}_s\, y, z = M_2 \mathbf{\ in\ } N_2 = P_2$$

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for $M \trianglelefteq_\beta M_2$ and $N \trianglelefteq_\beta N_2$. By the condition, both M and N are confluent, so there is a term $\mathbf{let}_s\ y, z = M'\ \mathbf{in}\ N'$ such that there are the following reductions:

$$\mathbf{let}_s\ y, z = (!_s\, M_1 \mathbf{with}\ \vec{x} = \vec{M_1})\ \mathbf{in}\ N \trianglelefteq_\beta \mathbf{let}_s\ y, z = M'\ \mathbf{in}\ N'$$
$$\mathbf{let}_s\ y, z = M_2\ \mathbf{in}\ N_2 \trianglelefteq_\beta \mathbf{let}_s\ y, z = M'\ \mathbf{in}\ N'$$

Further, M' has the form of $!_s\, M''\ \mathbf{with}\ \vec{x} = \vec{M}''$, so there is the following reduction:

$$\mathbf{let}_s\ y, z = (!_s\, M''\ \mathbf{with}\ \vec{x} = \vec{M}'')\ \mathbf{in}\ N' \trianglelefteq_\beta$$
$$\mathbf{let}_s\ \vec{u}, \vec{v} = \vec{M}''\ \mathbf{in}\ N'[y := !_s\, M'\ \mathbf{with}\ \vec{x} = \vec{u}, z := !_s\, M'\ \mathbf{with}\ \vec{x} = \vec{v}]$$

By the assumption, the term

$$\mathbf{let}_s\ \vec{u}, \vec{v} = \vec{M_1}\ \mathbf{in}\ N[y := !_s\, M_1\ \mathbf{with}\ \vec{x} = \vec{u}, z := !_s\, M_1\ \mathbf{with}\ \vec{x} = \vec{v}]$$

is confluent. So there is a term P_3 such that $P_1 \trianglelefteq_\beta P_3$ and

$$\mathbf{let}_s\ \vec{u}, \vec{v} = \vec{M}''\ \mathbf{in}\ N'[y := !_s\, M'\ \mathbf{with}\ \vec{x} = \vec{u}, z := !_s\, M'\ \mathbf{with}\ \vec{x} = \vec{v}] \trianglelefteq_\beta P_3.$$

The following term

$$\mathbf{let}_s\ \vec{u}, \vec{v} = \vec{M}''\ \mathbf{in}\ N_2[y := !_s\, M'\ \mathbf{with}\ \vec{x} = \vec{u}, z := !_s\, M'\ \mathbf{with}\ \vec{x} = \vec{v}]$$

is confluent, so there is a term P_4 such that $P_2 \trianglelefteq_\beta P_4$ and $P_3 \trianglelefteq_\beta P_4$.

$\square$

3 Category-theoretic Preliminaries

In this section, we outline the background from 2-category theory and refine the formal theory of comonads to characterise 1-cells and 2-cells in the 2-category of comonads in a 2-category $\mathcal{C}$ with left morphisms of adjunctions, the notion similar to adjunction morphisms in [Zag17]. We assume that the reader is familiar with the underlying notions of category theory such as (1-)categories, functors, natural transformations and adjunctions.

In further sections, we will obtain some results about categories of generally large categories and such objects cannot be defined in the standard set-theoretic foundations of mathematics such as ZFC or NBG, so we accept the foundations based on Grothendieck universes as they are described in [Lur09, §1.2.15] or [Shu08, §8].

3.1 2-categories

We refer the reader to [JY21, Chapter 2], [Lac09] and [Bor94a, Chapter 7] for a more thorough and systematic introduction.

Definition 3.1. A 2-category $\mathcal{C}$ consists of the following data:

- The class of objects (also called *0-cells*) $\mathrm{Ob}(\mathcal{C})$,

- For each pair of 0-cells $A, B \in \mathrm{Ob}(\mathcal{C})$ there is a class of morphisms (also called *1-cells*) $\mathcal{C}(A, B)$ such that for each $A, B, C \in \mathrm{Ob}(\mathcal{C})$ there is a composition map $g \circ f \in \mathcal{C}(A, C)$ for $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$ and there is the identity map $1_A \in \mathcal{C}(A, A)$. The composition operation and identity morphisms satisfy the standard axioms.

- For each 0-cells $A, B \in \mathrm{Ob}(\mathcal{C})$ and for each 1-cells $f, g \in \mathcal{C}(A, B)$, the class $\mathcal{C}(f, g)$ of *2-cells* between f and g is defined. The notation $\alpha : f \Rightarrow g$ stands for $\alpha \in \mathcal{C}(f, g)$. 2-cells are visualised as follows:

$$A \underset{g}{\overset{f}{\Longrightarrow}} B \quad \alpha$$
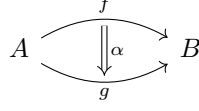
- Let $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$ for $f, g \in \mathcal{C}(A, B)$:

$$A \to B$$
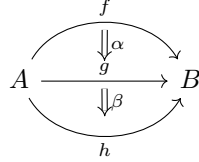
then there is the *vertical composite* of α and β, which is a 2-cell $\beta\alpha$ represented diagrammatically as:

$$A \underset{h}{\overset{f}{\Longrightarrow}} B \quad \beta\alpha$$
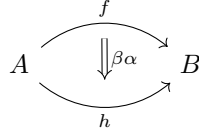
- Let $\alpha_1 : f_1 \Rightarrow f_2$ and $\alpha_2 : g_1 \Rightarrow g_2$ for $f_1, f_2 \in \mathcal{C}(A, B)$ and $g_1, g_2 \in \mathcal{C}(B, C)$:

$$A \underset{f_2}{\overset{f_1}{\Longrightarrow}} B \quad \alpha_1 \qquad B \underset{g_2}{\overset{g_1}{\Longrightarrow}} C \quad \alpha_2$$
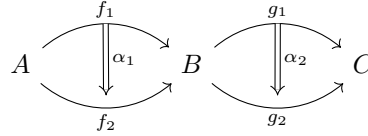
then there is the *horizontal composite* of α_1 and α_2, a 2-cell denoted as $\alpha_2 \circ \alpha_1 : g_1 \circ f_1 \Rightarrow g_2 \circ f_2$ represented diagramatically as:

$$A \underset{g_2 \circ f_2}{\overset{g_1 \circ f_1}{\Longrightarrow}} C \quad \alpha_2 \circ \alpha_1.$$
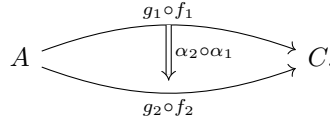
- The vertical and horizontal composites should satisfy the interchange laws that one can find in the aforementioned references and there are two identity two cells: the vertical 2-cell $1_f : f \Rightarrow f$ for any f and the horizontal 2-cell $id_A : 1_A \Rightarrow 1_A$ for any 0-cell A.

In a 2-category $\mathcal{C}$, every $\mathcal{C}(A, B)$ is a category itself for any 0-cells A, B.

Definition 3.2. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories, a *strict 2-functor* $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ consists of the following data:

- A class function $\mathsf{F} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$,

- Given $A, B \in \mathrm{Ob}(\mathcal{C})$, there is a functor $\mathsf{F}_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(\mathsf{F}A, \mathsf{F}B)$.

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- There are identities for each $A, B, C \in \mathrm{Ob}(\mathcal{C})$:

  - $1_{\mathsf{F}A} = \mathsf{F}_{A,A}(\mathbf{id_A})$,
  - $\mathsf{F}_{A,C}(g \circ f) = \mathsf{F}_{B,C}(g) \circ \mathsf{F}_{A,B}(f)$ for each 1-cells $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$.

Definition 3.3. $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ is a 2-category consisting of the following data:

- 0-cells are strict 2-functors,

- $\mathsf{F}, \mathsf{G} : \mathcal{C} \to \mathcal{D}$ be 2-functors, then 1-cells are *strict natural transformations* $\eta : \mathsf{F} \Rightarrow \mathsf{G}$,

- Let $\eta_1, \eta_2 : \mathsf{F} \Rightarrow \mathsf{G}$ be 1-cells, then the class of 2-cells between η_1 and η_2 are *modifications* $\eta_1 \Rightarrow \eta_2$.

Both strict natural transformations and their modifications are given accurately in, for example, [JY21, §4.4].

Let us recall how one treats adjunctions in a 2-categorical setting, the reader can find more details in [KS06, §2] and also [Kal20, Section 3] and [RV16] for more advanced aspects of 2-categorical adjunctions.

Definition 3.4. Let $\mathcal{C}$ be a 2-category and let $A, B \in \mathrm{Ob}(\mathcal{C})$ be 0-cells, an *adjunction* in $\mathcal{C}$, denoted as $\eta, \varepsilon : f_l \dashv f_r : A \to B$ (or just $f_l \dashv f_r$, when it is clear from the context what other components of an adjunction are), consists of a pair of 1-cells $f_r : A \to B$ and $f_l : B \to A$ and a pair of 2-cells $\eta : 1_B \Rightarrow f_r \circ f_l$ and $\varepsilon : f_l \circ f_r \Rightarrow 1_A$ such that the following conditions are satisfied:

We say that f_l is *left adjoint* to f_r and f_r is *right adjoint* to f_l.

Definition 3.5. Let $\mathcal{C}$ be a 2-category, let A, B be 0-cells and let $\eta, \varepsilon : f_l \dashv f_r : A \to B$ and $\eta', \varepsilon' : f_l' \dashv f_r' : A' \to B'$ be adjunctions in $\mathcal{C}$, then a *left morphism of adjunctions* consists a pair of 1-cells $g_1 : A \to A'$ and $g_2 : B \to B'$ such that $g_1 \circ f_l = f_l' \circ g_2$.

Note that there are different non-equivalent definitions of an adjunction morphism. For example, the definition of an adjunction morphism given in [Rie17, Exercise 4.2.v] extends our definition of left adjunction morphism with a similar extra equality of composites with right adjoints. Whereas, the definition of an adjunction morphism from [Zag17, Definition 2.2.2] contains only the equality with right adjoints. So we intended to coin the term "left adjunction morphism" to avoid further termilogical confusion.

Definition 3.6. Given a 2-category $\mathcal{C}$, a 2-category $\mathbf{Adj}^l(\mathcal{C})$ of adjunctions in $\mathcal{C}$ and their left morphisms is defined as the full 1,2-category of $\mathbf{Fun}([1], \mathcal{C})$ spanned by 1-cells in $\mathcal{C}$ having a right adjoint.

[1] denotes a two-element category having two objects 0 and 1 and a single non-identity morphism $0 \to 1$.

Comonads in 2-categories are defined by the dualisation of the corresponding definition of a monad in 2-categories from [Str72]. The formal theory of comonads can be derived by dualising the formal theory of monads. The reader can find the explicit presentation in [Zwa24, §2.3].

Definition 3.7. Let $\mathcal{C}$ be a 2-category, a *comonad* in $\mathcal{C}$ consists of a 0-cell $A \in \mathrm{Ob}(\mathcal{C})$ along with a 1-cell $k : A \to A$ and 2-cells $\delta : k \Rightarrow k \circ k$ (*comultiplication*) and $\varepsilon : k \Rightarrow 1_A$ (*counit*) such that the following diagrams commute:

$$
\begin{array}{ccc}
& k & \\
{}^{1_k}\swarrow \quad \downarrow{}^{\delta} \quad \searrow{}^{1_k} & & \\
k \xleftarrow{k\varepsilon} kk \xrightarrow{\varepsilon k} k & &
\end{array}
\qquad
\begin{array}{ccc}
k & \xrightarrow{\delta} & kk \\
{}^{\delta}\downarrow & & \downarrow{}^{k\delta} \\
kk & \xrightarrow{\delta k} & kkk
\end{array}
$$

The dual notion is the notion of a monad, a triple (j, η, μ), where $\eta : 1 \Rightarrow j$ and $\mu : jj \Rightarrow j$ are 2-cells called *unit* and *multiplication* respectively satisfying the diagrams dual to the above ones.

Most of the time, we will identify a comonad (k, δ, ε) over $A \in \mathrm{Ob}(\mathcal{C})$ with the carrier arrow k. A usual (co)monad over a category $\mathcal{C}$ is a (co)monad in the 2-category $\mathbf{Cat}$ of categories, functors and natural transformations. It is well known that every adjunction in a 2-category gives a comonad. Indeed, let $\eta, \varepsilon : f_l \dashv f_r : A \to B$ be an adjunction in $\mathcal{C}$, then the composite $f_l \circ f_r : A \to A$ forms a comonad on A with the counit 2-cell $\varepsilon : f_l \circ f_r \Rightarrow 1_A$ and the comultiplication 2-cell $f_l \eta f_r : f_l \circ f_r \Rightarrow f_l \circ f_r \circ f_l \circ f_r$.

Definition 3.8. Let (A, k) and (B, l) be comonads in a 2-category $\mathcal{C}$, a *comonad morphism* $(f, \mu) : (A, k) \to (B, l)$ consists of a 1-cell $f : A \to B$ and a 2-cell $\mu : fk \Rightarrow lf$ such that the following conditions are satisfied:

$$
\begin{array}{ccc}
fk & \xrightarrow{\mu} & lf \\
{}^{f\delta}\downarrow & & \downarrow{}^{\delta f} \\
fkk & \xrightarrow[\mu k]{} lfk \xrightarrow[l\mu]{} llf
\end{array}
\qquad
\begin{array}{ccc}
fk & \xrightarrow{\mu} & lf \\
{}_{f\varepsilon}\searrow & & \swarrow{}_{\varepsilon f} \\
& f &
\end{array}
$$

The next is to discuss how left adjunction morphisms induce comonad morphisms. For that, let us recall the notion of a mate. Let $\eta, \varepsilon : f_l \dashv f_r : A \to B$ and $\eta', \varepsilon' : f_l' \dashv f_r' : A' \to B'$ be adjunctions in $\mathcal{C}$. Let $g_1 : A \to A'$ and $g_2 : B \to B'$, then there is a bijection between 2-cells $f_l' \circ g_1 \Rightarrow g_2 \circ f_l$ and $g_1 \circ f_r \Rightarrow f_r' \circ g_2$, so squares of the form

$$
\begin{array}{ccc}
A & \xrightarrow{f_l} & B \\
{}^{g_1}\downarrow & \nearrow & \downarrow{}^{g_2} \\
A' & \xrightarrow[f_l']{} & B'
\end{array}
$$

1-to-1 correspond to squares of the form

$$
\begin{array}{ccc}
A & \xleftarrow{\ f_r\ } & B \\
{\scriptstyle g_1}\downarrow & \Nearrow & \downarrow{\scriptstyle g_2} \\
A' & \xleftarrow{\ f'_r\ } & B'
\end{array}
$$

So those pairs of 2-cells are called *mates*, see [Lac09, §2] and [KS06, §2].

Proposition 3.9. Let $\mathcal{C}$ be a 2-category, and let $\eta, \varepsilon : f_l \dashv f_r : A \to B$ and $\eta', \varepsilon' : f'_l \dashv f'_r : A' \to B'$ be adjunctions in $\mathcal{C}$ and let $g_1 : A \to A'$ and $g_2 : B \to B'$ be 1-cells such that a pair (g_1, g_2) is a left morphism of the corresponding adjunctions. Then (g_1, g_2) induces a morphism of comonads $f_l \circ f_r \Rightarrow f'_l \circ f'_r$.

Proof. The proof is similar to [Zag17, Proposition 2.2.4], but let us just demonstrate how exactly a comonad morphism is given. Let $\beta : g_2 f_r \Rightarrow f'_r g_1$ be the mate of the identity $g_1 f_l = f'_l g_2$, that is, the following composite:

$$
\begin{array}{ccccc}
A & \xrightarrow{\ f_r\ } & B & \xrightarrow{\ g_2\ } & B' \\
 & {\scriptstyle\Swarrow \delta} & \downarrow{\scriptstyle f_l} & \Nearrow & \downarrow{\scriptstyle f'_l} \quad {\scriptstyle 1_{B'}} \\
{\scriptstyle 1_A} & & A & \xrightarrow{\ g_1\ } & A' \xrightarrow{\ f'_r\ } B'
\end{array}
$$

Then a pair $(g_1, f'_l \beta)$ is a comonad morphism $f_l f_r \Rightarrow f'_l f'_r$. $\qquad\square$

Definition 3.10. Let (A, k_1) and (B, k_2) be comonads in a 2-category $\mathcal{C}$ and let $(f_1, \mu_1), (f_2, \mu_2) : k_1 \Rightarrow k_2$ be comonad morphisms, a *comonad morphism transformation* is a 2-cell $\varphi : f_1 \Rightarrow f_2$ such that the following square commutes:

$$
\begin{array}{ccc}
f_1 k_1 & \xrightarrow{\ \varphi k_1\ } & f_2 k_1 \\
{\scriptstyle \mu_1}\downarrow & & \downarrow{\scriptstyle \mu_2} \\
k_2 f_1 & \xrightarrow{\ k_2 \varphi\ } & k_2 f_2
\end{array}
$$

Taking Definition 3.8 and Definition 3.10 together, we define the 2-category **coMnd**$(\mathcal{C})$ consisting of all comonads in a 2-category $\mathcal{C}$, their morphisms and morphism transformations.

Proposition 3.11. Let $\eta, \varepsilon : f_l \dashv f_r$ and $\eta', \varepsilon' : f'_l \dashv f'_r$ be adjunctions between A and B in a 2-category $\mathcal{C}$. Let (g_1, g_2) and (g'_1, g'_2) be left adjunction morphisms from $f_l \dashv f_r$ and $f'_l \dashv f'_g$ inducing comonad morphisms $(g_1, \mu_1) : f_l f_r \Rightarrow f'_l f'_r$ and $(g'_1, \mu_2) : f_l f_r \Rightarrow f'_l f'_r$ respectively and let $\alpha_1 : g_1 \Rightarrow g'_1$ and $\alpha_2 : g_2 \Rightarrow g'_2$ be 2-cells in $\mathcal{C}$ such that $\alpha_1 f_l = f'_l \alpha_2$. Then α_1 and α_2 induce a comonad morphism transformation

$$
f_l f_r \underset{(g'_1, \mu_2)}{\overset{(g_1, \mu_1)}{\Rightarrow}} f'_l f'_r
$$

Proof. Let us show that the following square of 2-cells commutes:

$$
\begin{array}{ccc}
g_1 f_l f_r & \xrightarrow{\ \alpha_1 f_l f_r\ } & g'_1 f_l f_r \\
\mu_1 \downarrow & & \downarrow \mu_2 \\
f'_l f'_r g_1 & \xrightarrow[\ f'_l f'_r \alpha_1\ ]{} & f'_l f'_r g'_1
\end{array}
\tag{2}
$$

By Proposition 3.9, we can unveil μ_1 and μ_2 as $f'_l \beta : f'_l g_2 f_r \Rightarrow f'_l f'_r g_1$ and $f'_l \beta' : f'_l g'_2 f_r \Rightarrow f'_l f'_r g'_1$. So let us rewrite (2) as

$$
\begin{array}{ccc}
g_1 f_l f_r & \xrightarrow{\ \alpha_1 f_l f_r\ } & g'_1 f_l f_r \\
\| & & \| \\
f'_l g_2 f_r & \xrightarrow{\ f'_l \alpha_2 f_r\ } & f'_l g'_2 f_r \\
f'_l \beta \downarrow & & \downarrow f'_l \beta' \\
f'_l f'_r g_1 & \xrightarrow[\ f'_l f'_r \alpha_1\ ]{} & f'_l f'_r g'_1
\end{array}
\tag{3}
$$

Observe that the top square commutes by $\alpha_1 f_l = f'_l \alpha_2$. Whereas the bottom one is unveiled as follows and it holds by the assumption as well:

So we have a 2-functor from $\mathbf{Adj}^l(\mathcal{C})$ to $\mathbf{coMnd}(\mathcal{C})$ by Proposition 3.9 and Proposition 3.11 But there is the other way round 2-functor as well, but for 2-categories admitting the construction of coalgebras, so we need a couple of more definitions.

We assume that the classical notion of a coalgebra over a comonad is already known to the reader, but we refer to [BW00, Chapter 3] to have introduction to algebras over a monad, which is applicable to comonads and coalgebras by duality.

Definition 3.12. Let $\mathcal{C}$ be a 2-category and let (A, k) be a comonad in $\mathcal{C}$. An *object of coalgebras* (also called a *co-Eilenberg-Moore object*) is a 0-cell $A^k \in \mathrm{Ob}(\mathcal{C})$ (if it exists) such that for each 0-cell B:

$$
\mathcal{C}(B, A^k) \cong \mathbf{coMnd}(\mathcal{C})(1_B, k).
\tag{4}
$$

$\mathcal{C}$ *admits the construction of coalgebras* if every comonad has an object of coalgebras.

Therefore, we have (the complete proof can be adapted from [Str72, Theorem 2]):

Theorem 3.13. Let $\mathcal{C}$ be a 2-category that admits the construction of coalgebras, then $\mathbf{coMnd}(\mathcal{C})$ fully faithfully embeds to $\mathbf{Adj}^l(\mathcal{C})$.

3.2 Symmetric Monoidal Categories, Cosemigroups and Comonoids

We also recall underlying definitions and facts related to symmetric monoidal categories. We refer to [EGNO15, Chapter 2] for a more systematic introduction to the topic.

Definition 3.14. A *symmetric monoidal category* (SMC) is a category $\mathcal{C}$ equipped with a bifunctor $\_\otimes\_ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called *tensor product*, the unit object $\mathbb{1}$ along with the following natural isomorphisms for each $A, B, C \in \mathrm{Ob}(\mathcal{C})$:

- $\alpha_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$,

- $\rho_A : \mathbb{1} \otimes A \cong A \cong A \otimes \mathbb{1} : \lambda_A$,

- $\sigma_{A,B} : A \otimes B \cong B \otimes A$.

Those isomorphisms are connected with each other by the well-known axioms that one can find in [ML13, §VII.1, §XI.1].

Definition 3.15. A symmetric monoidal category is *closed* [2] if for each $A \in \mathrm{Ob}(\mathcal{C})$ the functor $B \mapsto A \otimes B$ has a right adjoint $B \mapsto [A, B]$. In other words, for each $A, B \in \mathrm{Ob}(\mathcal{C})$ there is $[A, B] \in \mathrm{Ob}(\mathcal{C})$ (the internal Hom of A and B) along with an arrow $\mathbf{ev}_{A,B} : [A, B] \otimes A \to B$ such that for each $f : C \otimes A \to B$ there exists a unique $\hat{f} : C \to [A, B]$ such that the following triangle commutes:

$$
\begin{array}{ccc}
[A, B] \otimes A & \xrightarrow{\ \mathbf{ev}_{A,B}\ } & B \\
\uparrow{\scriptstyle \hat{f} \otimes 1_A} & \nearrow{\scriptstyle f} & \\
C \otimes A & &
\end{array}
$$

Comonoid objects are rather common in category theory, see [AM10, §1.2]. A cosemigroup has only comultiplication (that we call uniform copying as in [HV19, §4.2]) without counits. Cosemigroups are less studied, but they are of some interest in the study of Banach algebras, see [Poi22].

Definition 3.16. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category, then

- A *cosemigroup object* in $\mathcal{C}$ is an object $A \in \mathrm{Ob}(\mathcal{C})$ with a choice of a morphism $\gamma_A : A \to A \otimes A$ called *uniform copying* such that the coassociativity axiom is satisfied:

$$
\begin{array}{ccc}
A & \xrightarrow{\ \gamma_A\ } & A \otimes A \\
{\scriptstyle \gamma_A}\downarrow & & \downarrow{\scriptstyle 1_A \otimes \gamma_A} \\
A \otimes A & \xrightarrow[\alpha_{A,A,A} \circ (\gamma_A \otimes 1_A)]{} & A \otimes (A \otimes A)
\end{array}
$$

- A *morphism of cosemigroups* $f : (A, \gamma_A) \to (B, \gamma_B)$ is an arrow $f : A \to B$ in $\mathcal{C}$ making the below square commute:

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
{\scriptstyle \gamma_A}\downarrow & & \downarrow{\scriptstyle \gamma_B} \\
A \otimes A & \xrightarrow[f \otimes f]{} & B \otimes B
\end{array}
$$

The coassociativity axiom means that uniform cloning respects the associator canonically.

---

[2]The abbreviation SMCC stands for a "symmetric closed monoidal category".

Definition 3.17. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be an SMC, then a cosemigroup (A, γ_A) is *cocommutative* if the extra axiom is satisfied:

$$
\begin{array}{ccc}
 & A & \\
\gamma_A \swarrow & & \searrow \gamma_A \\
A \otimes A \xrightarrow{\quad \sigma_{A,A} \quad} & & A \otimes A
\end{array}
$$

coSem$(\mathcal{C})$ is the category of cocommutative cosemigroups in $\mathcal{C}$ and their morphisms.

Similarly, the cocommutativity axiom is a requirement making uniform cloning commute with the structure of a symmetric monoidal category.

Definition 3.18. A *comonoid* is a cosemigroup object (A, γ) along with a choice of a morphism $\iota_A : A \to \mathbb{1}$ (*counit*) commuting with formal diagonals the following way:

$$
\begin{array}{ccccc}
 & & A & & \\
\rho_A^{-1} \swarrow & & \downarrow \gamma_A & & \searrow \lambda_A^{-1} \\
\mathbb{1} \otimes A & \xleftarrow{\iota_A \otimes 1_A} & A \otimes A & \xrightarrow{1_A \otimes \iota_A} & A \otimes \mathbb{1}
\end{array}
$$

A *comonoid morphism* is a cosemigroup morphism $f : (A, \gamma_A) \to (B, \gamma_B)$ preserving the counit:

$$
\begin{array}{ccc}
A & \xrightarrow{\quad f \quad} & B \\
\iota_A \searrow & & \swarrow \iota_B \\
 & \mathbb{1} &
\end{array}
$$

A comonoid is *cocommutative* if its uniform cloning is commutative. **coMon**$(\mathcal{C})$ denotes the category of cocommutative comultiplication comonoids and their morphisms.

The following statement is rather well-known:

Theorem 3.19. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal category, then the categories of cocommutative cosemigroups and comonoids **coSem**$(\mathcal{C})$ and **coMon**$(\mathcal{C})$ are symmetric monoidal with the following structure:

- The tensor product of cosemigroups (A, γ_A) and (B, γ_B) is the cosemigroup $(A \otimes B, \gamma_{A \otimes B})$ where $\gamma_{A \otimes B} : A \otimes B \to (A \otimes B) \otimes (A \otimes B)$ is given by the composite:

$$
\begin{array}{ccc}
A \otimes B & \xrightarrow{\quad\quad \gamma_{A \otimes B} \quad\quad} & (A \otimes B) \otimes (A \otimes B) \\
\gamma_A \otimes \gamma_B \downarrow & & \uparrow \alpha_{A,B,A \otimes B}^{-1} \\
(A \otimes A) \otimes (B \otimes B) & & A \otimes (B \otimes (A \otimes B)) \\
\alpha_{A,A,B \otimes B} \downarrow & & \uparrow 1_A \otimes \alpha_{B,A,B} \\
A \otimes (A \otimes (B \otimes B)) & & A \otimes ((B \otimes A) \otimes B) \\
1_A \otimes \alpha_{A,B,B}^{-1} \searrow & & \nearrow 1_A \otimes (\sigma_{A,B} \otimes 1_B) \\
 & A \otimes ((A \otimes B) \otimes B) &
\end{array}
$$

In **coMon**$(\mathcal{C})$, the tensor product of comonoids (A, γ_A, ι_A) and (B, γ_B, ι_B) is the comonoid $(A \otimes B, \gamma_{A \otimes B}, \iota_{A \otimes B})$ where the counit $\iota_{A \otimes B}$ is given by

$$
A \otimes B \xrightarrow{\iota_A \otimes \iota_B} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\lambda = \rho} \mathbb{1}.
$$

- The monoidal unit in **coSem**$(\mathcal{C})$ is the cosemigroup $(\mathbb{1}, \gamma_{\mathbb{1}})$ where $\gamma_{\mathbb{1}}$ is just ρ^{-1}. In the category of comonoids, the cosemigroup $(\mathbb{1}, \gamma_{\mathbb{1}})$ is equipped with the identity map $1_{\mathbb{1}} : \mathbb{1} \to \mathbb{1}$ as the counit morphism.

Moreover, the forgetful functors **coSem**$(\mathcal{C}) \to \mathcal{C}$ and **coMon**$(\mathcal{C}) \to \mathcal{C}$ are strict monoidal and symmetric.

The notions of a monoid and a semigroup are obtained by the dualisation of the above definitions, see [Por08] to have a more thorough introduction to categorical (co)monoids.

Definition 3.20. Let $\mathcal{C}, \mathcal{D}$ be symmetric monoidal categories, then a *symmetric lax monoidal functor* is a functor $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ equipped with

- A morphism $\mathsf{m}_{\mathbb{1}} : \mathbb{1} \to \mathsf{F}\mathbb{1}$,

- A natural transformation m with the components $\mathsf{m}_{A,B} : \mathsf{F}A \otimes \mathsf{F}B \to \mathsf{F}(A \otimes B)$ for $A, B \in \mathrm{Ob}(\mathcal{C})$.

All those natural transformations should satisfy the commutative diagrams from, e.g., [AM10, Chapter 3]. A tuple $(\mathsf{F}, \mathsf{m}) : \mathcal{C} \to \mathcal{D}$ stands for a F is a symmetric lax monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ with the natural transformation m. F is strong if its components are natural isomorphisms and is strict if the components are identity morphisms. F is *strong* if the components $\mathsf{m}_{A,B}$ and $\mathsf{m}_{\mathbb{1}}$ are isomorphisms for each $A, B \in \mathrm{Ob}(\mathcal{C})$. F is *strict* if all those components are identities.

Dually, a functor $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ is *symmetric oplax monoidal* if it is equipped with

- A morphism $\mathsf{n}_{\mathbb{1}} : \mathsf{F}\mathbb{1} \to \mathbb{1}$,

- A natural transformation m with the components $\mathsf{n}_{A,B} : \mathsf{F}(A \otimes B) \to \mathsf{F}A \otimes \mathsf{F}B$. for $A, B \in \mathrm{Ob}(\mathcal{C})$.

such that they satisfy the conditions dual to Definition 3.20.

Definition 3.21. Let $(\mathsf{F}, \mathsf{m}), (\mathsf{G}, \mathsf{n}) : \mathcal{C} \to \mathcal{D}$ be symmetric lax monoidal functors, then a natural transformation $\eta : (\mathsf{F}, \mathsf{m}) \Rightarrow (\mathsf{G}, \mathsf{n})$ is monoidal if the following diagrams commute for each $A, B \in \mathrm{Ob}(\mathcal{C})$:

$$
\begin{array}{ccc}
\mathsf{F}A \otimes \mathsf{F}B & \xrightarrow{\mathsf{m}_{A,B}} & \mathsf{F}(A \otimes B) \\
{\scriptstyle \eta_A \otimes \eta_B} \downarrow & & \downarrow {\scriptstyle \eta_{A \otimes B}} \\
\mathsf{G}A \otimes \mathsf{G}B & \xrightarrow{\mathsf{n}_{A,B}} & \mathsf{G}(A \otimes B)
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{1} & \xrightarrow{\mathsf{m}_{\mathbb{1}}} & \mathsf{F}\mathbb{1} \\
& {\scriptstyle \mathsf{m}_{\mathbb{1}}} \searrow & \downarrow {\scriptstyle \eta_{\mathbb{1}}} \\
& & \mathsf{G}\mathbb{1}
\end{array}
$$

SymmMonCat is a 2-category of symmetric monoidal categories, symmetric lax monoidal natural transformations and symmetric lax monoidal natural transformations. **SymmMonCat**$_{\text{strict}}$ is a 2-subcategory **SymmMonCat** where 1-cells are strict symmetric monoidal functors.

A *symmetric monoidal adjunction* can be defined as an adjunction in the 2-category **SymmMonCat** (see the definition of **SymmMonCat**), but there is an explicit characterisation of symmetric monoidal adjunctions, see [Kel06]:

Proposition 3.22. Let $(\mathcal{C}, \otimes, \mathbb{1})$ and $(\mathcal{D}, \otimes, \mathbb{1})$ be SMCs and let

$$(\mathsf{F}_l, \mathsf{n}) : (\mathcal{C}, \otimes, \mathbb{1}) \to (\mathcal{D}, \otimes, \mathbb{1})$$

be a symmetric lax monoidal functor. Assume that F_l is a left adjoint to $\mathsf{F}_r : \mathcal{D} \to \mathcal{C}$. Then the adjunction $\mathsf{F}_l \dashv \mathsf{F}_r$ lifts to a monoidal adjunction $(\mathsf{F}_l, \mathsf{n}) \dashv (\mathsf{F}_r, \mathsf{o})$ iff F_l is strong. In this case, the components $\mathsf{o}_{A,B} : \mathsf{F}_r A \otimes \mathsf{F}_r B \to \mathsf{F}_r(A \otimes B)$ and $\mathsf{o}_{\mathbb{1}} : \mathbb{1} \to \mathsf{F}_r \mathbb{1}$ are given by:

$$
\begin{array}{ccc}
\mathsf{F}_r A \otimes \mathsf{F}_r B & \xrightarrow{\quad\mathsf{o}_{A,B}\quad} & \mathsf{F}_r(A \otimes B) \\
{\scriptstyle \eta_{\mathsf{F}_r A \otimes \mathsf{F}_r B}}\Big\downarrow & & \Big\uparrow{\scriptstyle \mathsf{F}_r(\varepsilon_A \otimes \varepsilon_B)} \\
\mathsf{F}_r \mathsf{F}_l(\mathsf{F}_r A \otimes \mathsf{F}_r B) & \xrightarrow[\mathsf{F}_r(\mathsf{n}^{-1}_{\mathsf{F}_r A, \mathsf{F}_r B})]{} & \mathsf{F}_r(\mathsf{F}_l \mathsf{F}_r A \otimes \mathsf{F}_l \mathsf{F}_r B)
\end{array}
$$

$$
\mathbb{1} \xrightarrow{\quad\eta_{\mathbb{1}}\quad} \mathsf{F}_r \mathsf{F}_l \mathbb{1} \xrightarrow[\quad\mathsf{n}^{-1}_{\mathbb{1}}\quad]{} \mathsf{F}_r \mathbb{1}
$$

In particular, in any symmetric monoidal adjunction $(\mathsf{F}_l, \mathsf{m}) \dashv (\mathsf{F}_r, \mathsf{n})$, a left adjoint F_l is strong.

Definition 3.23. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal category and let $(\mathsf{K}, \varepsilon, \delta)$ be a comonad on $\mathcal{C}$, then the structure of a *symmetric lax monoidal comonad* on K is given by the structure of a symmetric lax monoidal functor (K, m) such that the following diagrams commute for each $A, B \in \mathrm{Ob}(\mathcal{C})$:

$$
\begin{array}{ccc}
\mathbb{1} \xrightarrow{\mathsf{m}_{\mathbb{1}}} \mathsf{K}\mathbb{1} & \qquad & \mathsf{K}A \otimes \mathsf{K}B \xrightarrow{\quad\mathsf{m}_{A,B}\quad} \mathsf{K}(A \otimes B) \\
{\scriptstyle \mathsf{m}_{\mathbb{1}}}\Big\downarrow \quad \Big\downarrow{\scriptstyle \delta_{\mathbb{1}}} & & {\scriptstyle \delta_A \otimes \delta_B}\Big\downarrow \qquad\qquad\qquad \Big\downarrow{\scriptstyle \delta_{A \otimes B}} \\
\mathsf{K}\mathbb{1} \xrightarrow[\mathsf{K}(\mathsf{m}_{\mathbb{1}})]{} \mathsf{K}^2\mathbb{1} & & \mathsf{K}^2 A \otimes \mathsf{K}^2 B \xrightarrow[\mathsf{m}_{\mathsf{K}A,\mathsf{K}B}]{} \mathsf{K}(\mathsf{K}A \otimes \mathsf{K}B) \xrightarrow[\mathsf{K}(\mathsf{m}_{A,B})]{} \mathsf{K}^2(A \otimes B)
\end{array}
$$

$$
\begin{array}{ccc}
\mathsf{K}A \otimes \mathsf{K}B & \xrightarrow{\mathsf{m}_{A,B}} & \mathsf{K}(A \otimes B) \\
& {\scriptstyle \varepsilon_A \otimes \varepsilon_B}\searrow & \Big\downarrow{\scriptstyle \varepsilon_{A \otimes B}} \\
& & A \otimes B
\end{array}
$$

A *symmetric lax monoidal comonad morphism* of symmetric lax monoidal comonads is a comonad morphism in **Cmd(SymmMonCat)**. Explicitly, the structure of a symmetric lax monoidal comonad morphism given by a comonad morphism with the structure of a symmetric lax monoidal natural transformation.

Let $\mathcal{C}$ be a symmetric monoidal category and let $(\mathsf{K}, \mathsf{m}) : \mathcal{C} \to \mathcal{C}$ be a lax symmetric monoidal functor with the structure of a symmetric lax monoidal comonad. There is a co-free co-forgetful adjunction between the underlying category $\mathcal{C}$ and the Eilenberg-Moore category $\mathcal{C}^{\mathsf{K}}$, which lifts to a symmetric monoidal adjunction if we equip the Eilenberg-Moore category with a symmetric monoidal structure. Let (A, h_A) and (B, h_B) be K-coalgebras from $\mathcal{C}^{\mathsf{K}}$. The tensor product of (A, h_A) and (B, h_B) is defined as a coalgebra $(A \otimes B, h_{A \otimes B})$ where the coalgebra action $h_{A \otimes B}$ is given by the composite:

$$
A \otimes B \xrightarrow{\quad h_A \otimes h_B \quad} \mathsf{K}A \otimes \mathsf{K}B \xrightarrow{\quad\mathsf{m}_{A,B}\quad} \mathsf{K}(A \otimes B)
$$

And the unit coalgebra is defined as $(\mathbb{1}, \mathsf{m}_{\mathbb{1}})$. Therefore, we have the following (see [Lac05] for more details and further generalisations):

Lemma 3.24. Let $\mathcal{C}$ be a symmetric monoidal category and let $\mathsf{K} : \mathcal{C} \to \mathcal{C}$ be a lax symmetric monoidal comonad, then the Eilenberg-Moore category is symmetric monoidal.

4 Assemblages and Semantic Interpretation of SILL$(\lambda)_\Sigma$

In this section, we construct the categorical denotational semantics for the type theory we introduced in the previous section. We introduce Cocteau categories and assemblages to have an adequate semantic interpretation of **SILL**$(\lambda)_3$ and **SILL**$(\lambda)_\Sigma$ respectively.

In categorical semantics of linear logic as it is studied in, for example, [AT11] and [Mel09], one can consider so-called *linear categories*, symmetric monoidal categories equipped with a symmetric lax monoidal comonad that, informally speaking, assign a cocommutative comonoid with each object of an underlying category. The heuristics is that we can associate the data type $!A$ with each A such that elements of $!A$ can be both copied and destroyed. Such comonads allows interpreting the exponential modality semantically. We below suggest the notion of a Cocteau category by expanding linear categories with two extra comonads for modelling those data types that can be destroyed and those that can be copied.

4.1 Cocteau Categories and Assemblages

Definition 4.1. Let $\mathcal{C}$ be a symmetric monoidal category, then:

- An *exponential* comonad is a symmetric monoidal comonad $(\mathsf{K}, \varepsilon, \delta)$ on $\mathcal{C}$ with a pair of symmetric monoidal natural transformations $\mathsf{c} : \mathsf{K} \Rightarrow \mathsf{K}(.) \otimes \mathsf{K}(.)$ and $\mathsf{d} : \mathsf{K} \Rightarrow (\_ \mapsto \mathbb{1})$ such that the following is satisfied:

  - $(\mathsf{K}A, \mathsf{c}_A, \mathsf{d}_A)$ is a cocommutative comonoid for each $A \in \mathrm{Ob}(\mathcal{C})$,
  - δ_A is a comonoid morphism, i.e., the following axioms are satisfied:

$$
\begin{array}{ccc}
\mathsf{K}A & \xrightarrow{\ \ \delta_A\ \ } & \mathsf{K}^2A \\
{\scriptstyle \mathsf{c}_A}\downarrow & & \downarrow{\scriptstyle \mathsf{c}_{\mathsf{K}A}} \\
\mathsf{K}A \otimes \mathsf{K}A & \xrightarrow{\ \delta_A \otimes \delta_A\ } & \mathsf{K}^2A \otimes \mathsf{K}^2A
\end{array}
\qquad
\begin{array}{ccc}
\mathsf{K}A & \xrightarrow{\ \delta_A\ } & \mathsf{K}^2A \\
& {\scriptstyle \mathsf{d}_A}\searrow & \downarrow{\scriptstyle \mathsf{d}_{\mathsf{K}A}} \\
& & \mathbb{1}
\end{array}
$$

  - c_A and d_A are coalgebra morphisms, i.e., the following axioms are satisfied:

$$
\begin{array}{ccc}
\mathsf{K}A & \xrightarrow{\qquad \delta_A \qquad} & \mathsf{K}^2A \\
{\scriptstyle \mathsf{c}_A}\downarrow & & \downarrow{\scriptstyle \mathsf{Kc}_A} \\
\mathsf{K}A \otimes \mathsf{K}A \xrightarrow{\ \delta_A \otimes \delta_A\ } & \mathsf{K}^2A \otimes \mathsf{K}^2A \xrightarrow{\ \mathsf{m}_{\mathsf{K}A,\mathsf{K}A}\ } & \mathsf{K}(\mathsf{K}A \otimes \mathsf{K}A)
\end{array}
\qquad
\begin{array}{ccc}
\mathsf{K}A & \xrightarrow{\ \delta_A\ } & \mathsf{K}^2A \\
{\scriptstyle \mathsf{d}_A}\downarrow & & \downarrow{\scriptstyle \mathsf{K}(\mathsf{d}_A)} \\
\mathbb{1} & \xrightarrow{\ \mathsf{m}_{\mathbb{1}}\ } & \mathsf{K}\mathbb{1}
\end{array}
$$

- A *relevant* comonad is a symmetric monoidal comonad K with a symmetric monoidal natural transformation $\mathsf{c} : \mathsf{K} \Rightarrow \mathsf{K}(.) \otimes \mathsf{K}(.)$ such that a pair $(\mathsf{K}A, \mathsf{c}_A)$ is a cocommutative cosemigroup, δ_A is a cosemigroup morphism and c_A is a coalgebra morphism.

- $((\mathsf{K}, \mathsf{m}), \varepsilon, \delta, \mathsf{d})$ is a *affine* comonad if it has a natural transformation with the components $\mathsf{d}_A : \mathsf{K}A \to \mathbb{1}$ such that the following square commutes:

$$
\begin{array}{ccc}
\mathsf{K}A & \xrightarrow{\ \delta_A\ } & \mathsf{K}^2A \\
{\scriptstyle \mathsf{d}_A}\downarrow & & \downarrow{\scriptstyle \mathsf{K}(\mathsf{d}_A)} \\
\mathbb{1} & \xrightarrow{\ \mathsf{m}_{\mathbb{1}}\ } & \mathsf{K}A
\end{array}
$$

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Definition 4.2. Let $\mathcal{C}$ be a SMCC category and let $(!_\mathbf{i}, \mathsf{m^i}), (!_\mathbf{a}, \mathsf{m^a}), (!_\mathbf{r}, \mathsf{m^r})$ be lax symmetric monoidal endofunctors, then a *Cocteau category*[3] is given by the following data:

- $((!_\mathbf{i}, \mathsf{m^i}), \mathsf{c^i}, \mathsf{d^i})$ is an exponential comonad,

- $((!_\mathbf{a}, \mathsf{m^a}), \mathsf{d^a})$ is an affine comonad,

- $((!_\mathbf{r}, \mathsf{m^r}), \mathsf{c^r})$ is a relevant comonad,

- $\mu_\mathbf{ir} : !_\mathbf{i} \Rightarrow !_\mathbf{r}$ and $\mu_\mathbf{ia} : !_\mathbf{i} \Rightarrow !_\mathbf{a}$ are symmetric lax monoidal comonad morphisms such that the following diagrams commute for each $A \in \mathrm{Ob}(\mathcal{C})$:

$$
\begin{array}{ccc}
!_\mathbf{i}\, A & \xrightarrow{\;\mathsf{c}^\mathbf{i}_A\;} & !_\mathbf{i}\, A \otimes !_\mathbf{i}\, A \\
{\scriptstyle \mu_\mathbf{ir}}\downarrow & & \downarrow{\scriptstyle \mu_\mathbf{ir} \otimes \mu_\mathbf{ir}} \\
!_\mathbf{r}\, A & \xrightarrow{\;\mathsf{c}^\mathbf{r}_A\;} & !_\mathbf{r}\, A \otimes !_\mathbf{r}\, A
\end{array}
\qquad\qquad
\begin{array}{ccc}
!_\mathbf{i}\, A & \xrightarrow{\;\mu_\mathbf{ia}\;} & !_\mathbf{a}\, A \\
& {\scriptstyle \mathsf{d}^\mathbf{i}_A}\searrow & \downarrow{\scriptstyle \mathsf{d}^\mathbf{a}_A} \\
& & \mathbb{1}
\end{array}
$$

Convention 4.3. In order to distinguish similar components of different comonads in a Cocteau category, we will sometimes label them with an upper index as follows: for example, $\varepsilon^\mathbf{a}$ will stand for counit in an affine comonad, whereas $\mathsf{c^r}$ will mean the copying operation in a relevant comonad.

Example 4.4. One can extract a rather natural example of a Cocteau category from quantales, see [EGHK18] for a more general context.

A *quantale* is a structure $\mathcal{Q} = (Q, \cdot, 1, \bigvee)$ where $(Q, \bigvee)$ is a complete sup-semilattice and $(Q, \cdot, 1)$ is a monoid such that the multiplication operation preserves suprema in each coordinate. Any commutative quantale $\mathcal{Q}$ is an SMCC with the internal Hom given by

$$[a, b] = \bigvee \{c \in \mathcal{Q} \mid c \cdot a \le b\}.$$

A *quantic conucleus* on a quantale $\mathcal{Q}$ is a coclosure operator $g : \mathcal{Q} \to \mathcal{Q}$ such that $g(a) \cdot g(b) \le g(a \cdot b)$ and $g(1) = 1$ for each $a, b \in \mathcal{Q}$. This is a known fact that there is a bijection between subquantales of a quantale and quantic conuclei: if $g : \mathcal{Q} \to \mathcal{Q}$ is a quantic conucleus, then the set $\mathcal{Q}_g = \{a \in \mathcal{Q} \mid g(a) = a\}$ forms a subquantale of $\mathcal{Q}$, see, for example, [Ros90, Theorem 3.1.3]. Moreover, if $\mathcal{Q}' \trianglelefteq \mathcal{Q}$ is a subquantale, then there exists a quantic conucleus g such that $\mathcal{Q}' = \mathcal{Q}_g$. Further, one can observe the subsets $\mathcal{Q}_\mathbf{i} = \{a \in \mathcal{Q} \mid a \le 1\ \&\ a \le a \cdot a\}$, $\mathcal{Q}_\mathbf{a} = \{a \in \mathcal{Q} \mid a \le 1\}$ and $\mathcal{Q}_\mathbf{r} = \{a \in \mathcal{Q} \mid a \le a \cdot a\}$ are subquantales of $\mathcal{Q}$. so there are quantic conuclei $g_\mathbf{i}$, $g_\mathbf{a}$ and $g_\mathbf{r}$ such that $\mathcal{Q}_\mathbf{i} = \mathcal{Q}_{g_\mathbf{i}}$, $\mathcal{Q}_\mathbf{a} = \mathcal{Q}_{g_\mathbf{a}}$ and $\mathcal{Q}_\mathbf{r} = \mathcal{Q}_{g_\mathbf{r}}$. For each $a \in \mathcal{Q}$ one also has $g_\mathbf{i}(a) \le 1$ and $g_\mathbf{i}(a) \le g_\mathbf{i}(a) \cdot g_\mathbf{i}(a)$, $g_\mathbf{r}(a) \le g_\mathbf{r}(a) \cdot g_\mathbf{r}(a)$, $g_\mathbf{a}(a) \le 1$. Besides, $g_\mathbf{i}(a) \le g_\mathbf{r}(a)$ and $g_\mathbf{i}(a) \le g_\mathbf{a}(a)$ for each $a \in \mathcal{Q}$. Thus, a commutative quantale $\mathcal{Q}$ along with quantic conuclei $g_\mathbf{i}$, $g_\mathbf{a}$ and $g_\mathbf{r}$ forms a Cocteau category.

One can also extract an example of a Cocteau category of coherence spaces as they are described in [Mel09, Section 8.10].

The notion of a Σ-assemblage is a direct generalisation of Cocteau categories for an arbitrary subexponential signature Σ.

Definition 4.5. Let $\Sigma = (I, \preceq, W, C)$ be a subexponential signature, a Σ-*assemblage*[4] is an SMCC $\mathcal{C}$ (denoted as $(\mathcal{C}, \Sigma)$) equipped with a family of comonads $\{!_s : \mathcal{C} \to \mathcal{C} \mid s \in I\}$ such that:

---

[3]Cocteau categories are named after a French poet and director Jean Cocteau and his Orphic trilogy consisting of the films *The Blood of a Poet*, *Orpheus* and *Testament of Orpheus*.

[4]The term is inspired by the notion of an assemblage from the philosophy of Gilles Deleuze and Felix Guattari.

- The mapping $s \mapsto !_s$ is a contravariant functor from $(I, \preceq)$ to the (1-)category of comonads over $\mathcal{C}$, so if $s_1 \preceq s_2$, then there is a comonad morphism $\mu(s_1, s_2) : !_{s_2} \Rightarrow !_{s_1}$,

- If $s \in W$ ($s \in C$), then $!_s$ is an affine (relevant) comonad,

- If $s \in W \cap C$, then $!_s$ is an exponential comonad,

- If $s_1 \in C$ and $s_1 \preceq s_2$, then for each $A \in \mathcal{C}$, $\mu(s_1, s_2)_A : !_{s_2} A \to !_{s_1} A$ induces a cocommutative cosemigroup morphism.

- If both $s_1, s_2 \in W \cap C$ and $s_1 \preceq s_2$, then $\mu(s_1, s_2)_A : (!_{s_2} A, \mathsf{c}_{s_2 A}, \mathsf{d}_{s_2 A}) \to (!_{s_1} A, \mathsf{c}_{s_1 A}, \mathsf{d}_{s_1 A})$ induces a cocommutative comonoid morphism for each $A \in \mathcal{C}$.

- If $s_1 \in W$ and $s_1 \preceq s_2$, then the following triangle commutes for each $A \in \mathcal{C}$:

$$
\begin{array}{ccc}
!_{s_2} A & \xrightarrow{\ \mu(s_1,s_2)_A\ } & !_{s_1} A \\
& \mathsf{d}_{s_2 A} \searrow \quad \swarrow \mathsf{d}_{s_1 A} & \\
& \mathbb{1} &
\end{array}
$$

4.2 On Soundness and Completeness

Now we formulate the theorem of semantic adequacy of $\mathbf{SILL}(\lambda)_\Sigma$ with respect to Σ-assemblages for a subexponential signature Σ.

Definition 4.6. Let $(\mathcal{C}, \Sigma)$ be a Σ-assemblage over an SMCC $\mathcal{C}$ and let $\mathcal{I}$ be a function mapping each atomic to some object of $\mathcal{C}$. The Cocteau interpretation function $\llbracket . \rrbracket$ assigns an object of $\mathcal{C}$ to a type as follows.

- $\llbracket p_i \rrbracket = \mathcal{I}(p_i)$,

- $\llbracket \mathbb{1} \rrbracket = \mathbb{1}$,

- $\llbracket A \otimes B \rrbracket = \llbracket A \rrbracket \otimes \llbracket B \rrbracket$,

- $\llbracket A \multimap B \rrbracket = [\llbracket A \rrbracket, \llbracket B \rrbracket]$,

- $\llbracket !_s A \rrbracket = !_s \llbracket A \rrbracket$,

Construction 4.7. Given a Σ-assemblage $(\mathcal{C}, \Sigma)$, the interpretation of $\mathbf{SILL}(\lambda)_\Sigma$-promotion rule is given as follows for $s \preceq s_1, \ldots, s_n$:

$$
\frac{\Gamma_1 \xrightarrow{M_1} !_{s_1} A_1, \ldots, \Gamma_n \xrightarrow{M_n} !_{s_n} A_n \qquad !_{s_1} A_1 \otimes \ldots \otimes !_{s_n} A_n \xrightarrow{N} \psi}{\otimes_k \Gamma_k \xrightarrow{!_s(N) \circ \mathsf{m}^s_{!_s A_1, \ldots, !_s A_n} \circ \otimes_k \mu(s,s_k)_{!_{s_k} A_k} \circ \otimes_k \delta^{s_k}_{A_k} \circ \otimes_k M_k} !A}
$$

Let us visualise the above derivation with a diagram for $n = 2$:

$$
\begin{array}{ccc}
\Gamma_1 \otimes \Gamma_2 & \xrightarrow{\hspace{5cm}} & !_s A \\
{\scriptstyle M_1 \otimes M_2} \big\downarrow & & \big\uparrow {\scriptstyle !_s N} \\
!_{s_1} A_1 \otimes !_{s_2} A_2 & & !_s(!_{s_1} A_1 \otimes !_{s_2} A_2) \\
{\scriptstyle \delta^{s_1}_{A_1} \otimes \delta^{s_2}_{A_2}} \big\downarrow & & \big\uparrow {\scriptstyle \mathsf{m}_{!_{s_1} A_1, !_{s_2} A_2}} \\
!^2_{s_1} A_1 \otimes !^2_{s_2} A_2 & \xrightarrow[\mu(s,s_1)_{!_{s_1} A_1} \otimes \mu(s,s_2)_{!_{s_2} A_2}]{} & !_s !_{s_1} A_1 \otimes !_s !_{s_2} A_2
\end{array}
$$

The interpretation of the rest of the inference rules are given in Figure 4.7. The only difference with the syntactic presentation is that we given the explicit interpretation of the exchange rule.

Figure 7: $\mathbf{SILL}(\lambda)_\Sigma$ in Σ-assemblages

$$\frac{\Gamma \otimes A \otimes B \xrightarrow{M} C}{\Gamma \otimes B \otimes A \xrightarrow{M \circ 1_\Gamma \otimes \sigma_{A,B}} C} \textbf{ Exch}$$

$$\frac{}{\mathbb{1} \xrightarrow{1_{\mathbb{1}}} \mathbb{1}} \mathbb{1}\textbf{I} \qquad\qquad \frac{}{A \xrightarrow{1_A} A} \textbf{ ax} \qquad\qquad \frac{\Gamma \xrightarrow{M} \mathbb{1} \qquad \Delta \xrightarrow{N} A}{\Gamma \otimes \Delta \xrightarrow{\lambda_A \circ M \otimes N} A} \mathbb{1}\textbf{E}$$

$$\frac{\Gamma \otimes A \xrightarrow{M} B}{\Gamma \xrightarrow{\widehat{M}} [A,B]} \qquad\qquad\qquad \frac{\Gamma \xrightarrow{M} [A,B] \qquad \Delta \xrightarrow{N} A}{\Gamma \otimes \Delta \xrightarrow{\mathbf{ev}_{A,B} \circ M \otimes N} B}$$

$$\frac{\Gamma \xrightarrow{M} A \qquad \Delta \xrightarrow{N} B}{\Gamma \otimes \Delta \xrightarrow{M \otimes N} A \otimes B} \qquad\qquad \frac{\Gamma \xrightarrow{M} A \otimes B \qquad \Delta \otimes A \otimes B \xrightarrow{N} C}{\Delta \otimes \Gamma \xrightarrow{N \circ \alpha_{\Delta,A,B} \circ 1_\Delta \otimes M} C}$$

$$\frac{\Gamma \xrightarrow{M} \,!_s A \qquad \Delta \xrightarrow{N} B}{\Gamma \otimes \Delta \xrightarrow{(N \circ \rho_\Delta) \circ ((\mathbf{d}_A^s \circ M) \otimes 1_\Delta)} B}\, s \in W \quad \frac{\Gamma \xrightarrow{M} \,!_s A}{\Gamma \xrightarrow{\delta_A \circ M} A}\, s \in \Sigma \quad \frac{\Gamma \xrightarrow{M} \,!_s A \qquad \Delta \otimes \,!_s A \otimes \,!_s A \xrightarrow{N} B}{\Delta \otimes \Gamma \xrightarrow{N \circ (1_\Delta \otimes (\mathbf{c}_A^s \circ M))} B}\, s \in C$$

Theorem 4.8. Every Σ-assemblage is a model of $\mathbf{SILL}(\lambda)_\Sigma$ and its $\mathbf{SILL}(\lambda)_\Sigma$-equalities in context. Moreover, there exists a Σ-assemblage $\mathcal{C}_\Sigma$ with the following property: if $\Gamma \vdash M : A$ and $\Gamma \vdash N : A$ such that $[\![M]\!] = [\![N]\!]$, then the equality in context $\Gamma \vdash M \equiv N : A$ is provable in the typed equational logic generated the proof conversion rules from Figures 2, 3, 4, 5 and 6.

The only difference with the proof-theoretic presentation is that we give the exchange rule explicitly.

Proof. First of all, we observe that any Cocteau interpretation respects substitutions. That is, let $\Gamma \vdash M : A$ and $\Delta_1, x : A, \Delta_2 \vdash N : B$, then

$$[\![\Delta, \Gamma \vdash N[x := M] : B]\!] = [\![N]\!] \circ 1_{[\![\Delta_1]\!]} \otimes [\![M]\!] \otimes 1_{[\![\Delta_2]\!]}.$$

The proof is rather standard and analogous to the proof [AT11, Proposition 99].

Induction on the generation of $\Gamma \vdash M \equiv N : A$. Let us consider one of the cases involving comonad morphisms. Consider the $(\mathbf{C}_{\mathrm{conv}_1})$-rule. Let $s \in C$ and $s \preceq s_1, s_2$. Assume we have the following contraction for:

$$\frac{x_1 : \,!_{s_1} A_1 \vdash x_1 : \,!_{s_1} A_1 \qquad y : \,!_{s_1} A_1, z : \,!_{s_1} A_1, x_2 : \,!_{s_2} A_2 \vdash N : A}{x_1 : \,!_{s_1} A_1, x_2 : \,!_{s_2} A_2 \vdash \mathbf{let}_{s_1} y, z = x_1 \mathbf{\ in\ } N : A}$$

The interpretation is given below:

$$!_{s_1} A_1 \otimes \,!_{s_2} A_2 \xrightarrow{\mathbf{c}_{A_1}^{s_1} \otimes 1_{!_{s_2} A_2}} \,!_{s_1} A_1 \otimes \,!_{s_1} A_1 \otimes \,!_{s_2} A_2 \xrightarrow{N} A$$

And then we apply the promotion rule for $\Gamma_1 \vdash M_1 : !_{s_1} A_1$ and $\Gamma_2 \vdash M_2 : !_{s_2} A_2$ being derivable somehow:

$$\frac{\Gamma_1 \vdash M_1 : !_{s_1} A_1, \Gamma_2 \vdash M_2 : !_{s_2} A_2 \qquad x_1 : !_{s_1} A_1, x_2 : !_{s_2} A_2 \vdash \mathbf{let}_{s_1} y, z = x_1 \mathbf{\ in\ } N : A}{\Gamma_1, \Gamma_2 \vdash !_s(\mathbf{let}_{s_1} y, z = x_1 \mathbf{\ in\ } N) \mathbf{\ with\ } x_1, x_2 = M_1, M_2 : !_s A}$$

Here is the interpretation of the above derivation:

$$\begin{array}{ccc}
\Gamma_1 \otimes \Gamma_2 & \xrightarrow{\hspace{6cm}} & !_s A \\
{\scriptstyle M_1 \otimes M_2}\downarrow & & \uparrow{\scriptstyle !_s N \circ !_s(c^{s_1}_{A_1} \otimes 1_{!_{s_2} A_2})} \\
!_{s_1} A_1 \otimes !_{s_2} A_2 & & !_s(!_{s_1} A_1 \otimes !_{s_2} A_2) \\
{\scriptstyle \delta^{s_1}_{A_1} \otimes \delta^{s_2}_{A_2}}\downarrow & & \uparrow{\scriptstyle \mathsf{m}^s_{!_{s_1} A_1, !_{s_2} A_2}} \\
!^2_{s_1} A_1 \otimes !^2_{s_2} A_2 & \xrightarrow{\mu(s,s_1)_{!_{s_1} A_1} \otimes \mu(s,s_2)_{!_{s_2} A_2}} & !_s !_{s_1} A_1 \otimes !_s !_{s_2} A_2
\end{array}$$

On the other hand, we could make the following promotion first:

$$\frac{y' : !_{s_1} A_1 \vdash y' : !_{s_1} A_1, z' : !_{s_1} A_1 \vdash z' : !_{s_1} A_1, \Gamma_2 \vdash M_2 : !_{s_2} A_2 \qquad y : !_{s_1} A_1, z : !_{s_1} A_1, x_2 : !_{s_2} A_2 \vdash N : A}{y' : !_{s_1} A_1, z' : !_{s_1} A_1, \Gamma_2 \vdash !_s N \mathbf{\ with\ } y, z, x_2 = y', z', M_2 : !_s A}$$

And then we apply the contraction rule

$$\frac{\Gamma_1 \vdash M_1 : !_{s_1} A_1 \qquad y' : !_{s_1} A_1, z' : !_{s_1} A_1, \Gamma_2 \vdash !_s N \mathbf{\ with\ } y, z, x_2 = y', z', M_2 : !_s A}{\Gamma_1, \Gamma_2 \vdash \mathbf{let}_{s_1} y', z' = M_1 \mathbf{\ in\ } (!_s N \mathbf{\ with\ } y, z, x_2 = y', z', M_2) : !_s A}$$

and obtain the following interpretation:

$$\begin{array}{ccc}
\Gamma_1 \otimes \Gamma_2 & \xrightarrow{\hspace{6cm}} & !_s A \\
{\scriptstyle M_1 \otimes 1_{\Gamma_2}}\downarrow & & \uparrow \\
!_{s_1} A_1 \otimes \Gamma_2 & & \hspace{1cm}{\scriptstyle !_s N} \\
{\scriptstyle c^{s_1} \otimes 1_{\Gamma_2}}\downarrow & & \\
!_{s_1} A_1 \otimes !_{s_1} A_1 \otimes \Gamma_2 & & !_s(!_{s_1} A_1 \otimes !_{s_1} A_1 \otimes !_{s_2} A_2) \\
{\scriptstyle 1_{!_{s_1} A_1} \otimes !_{s_1} A_1 \otimes M_2}\downarrow & & \uparrow \\
!_{s_1} A_1 \otimes !_{s_1} A_1 \otimes !_{s_2} A_2 & & \hspace{1cm}{\scriptstyle \mathsf{m}^s_{!_{s_1} A_1, !_{s_1} A_1, !_{s_2} A_2}} \\
{\scriptstyle \delta^{s_1}_{A_1} \otimes \delta^{s_1}_{A_1} \otimes \delta^{s_2}_{A_2}}\downarrow & & \\
!^2_{s_1} A_1 \otimes !^2_{s_1} A_1 \otimes !^2_{s_2} A_2 & \xrightarrow{\mu(s,s_1)_{!_{s_1} A_1} \otimes \mu(s,s_1)_{!_{s_1} A_1} \otimes \mu(s,s_2)_{!_{s_2} A_2}} & !_s !_{s_1} A_1 \otimes !_s !_{s_1} A_1 \otimes !_s !_{s_2} A_2
\end{array}$$

Then one has:

$!_s N \circ !_s(\mathsf{c}^{s_1}_{A_1} \otimes 1_{!_{s_2} A_2}) \circ \mathsf{m}^s_{!_{s_1} A_1, !_{s_2} A_2} \circ \mu(s, s_1)_{!_{s_1} A_1} \otimes \mu(s, s_2)_{!_{s_2} A_2} \circ \delta^{s_1}_{A_1} \otimes \delta^{s_2}_{A_2} \circ M_1 \otimes M_2 =$
    The definition of a lax monoidal functor
$!_s N \circ \mathsf{m}^s_{!_{s_1} A_1 \otimes !_{s_1} A_1, !_{s_2} A_2} \circ !_s(\mathsf{c}^{s_1}_{A_1}) \otimes !_s(1_{!_{s_2} A_2}) \circ \mu(s, s_1)_{!_{s_1} A_1} \otimes \mu(s, s_2)_{!_{s_2} A_2} \circ \delta^{s_1}_{A_1} \otimes \delta^{s_2}_{A_2} \circ M_1 \otimes M_2$

Now we fix the composition $!_s N \circ \mathsf{m}^s_{!_{s_1} A_1 \otimes !_{s_1} A_1, !_{s_2} A_2}$ and proceed with the rest of the big composite without $M_1 \otimes M_2$.

$!_s(\mathsf{c}^{s_1}_{A_1}) \otimes !_s(1_{!_{s_2} A_2}) \circ \mu(s, s_1)_{!_{s_1} A_1} \otimes \mu(s, s_2)_{!_{s_2} A_2} \circ \delta^{s_1}_{A_1} \otimes \delta^{s_2}_{A_2} =$
    The functoriality of $\otimes$
$((!_s(\mathsf{c}^{s_1}_{A_1}) \circ \mu(s, s_1)_{!_{s_1} A_1}) \otimes \mu(s, s_2)_{!_{s_2} A_2}) \circ \delta^{s_1}_{A_1} \otimes \delta^{s_2}_{A_2} =$
    The naturality of μ
$((\mu(s, s_1)_{!_{s_1} A_1 \otimes !_{s_1} A_1} \circ !_{s_1}(\mathsf{c}^{s_1}_{A_1})) \otimes \mu(s, s_2)_{!_{s_2} A_2}) \circ \delta^{s_1}_{A_1} \otimes \delta^{s_2}_{A_2} =$
    The functoriality of $\otimes$
$(\mu(s, s_1)_{!_{s_1} A_1 \otimes !_{s_1} A_1} \circ !_{s_1}(\mathsf{c}^{s_1}_{A_1}) \circ \delta^{s_1}_{A_1}) \otimes (\mu(s, s_2)_{!_{s_2} A_2} \circ \delta^{s_2}_{A_2}) =$
    The definition of a relevant comonad
$(\mu(s, s_1)_{!_{s_1} A_1 \otimes !_{s_1} A_1} \circ \mathsf{m}^{s_1}_{!_{s_1} A_1, !_{s_1} A_1} \circ \delta^{s_1}_{A_1} \otimes \delta^{s_1}_{A_1} \circ \mathsf{c}^{s_1}_{A_1}) \otimes (\mu(s, s_2)_{!_{s_2} A_2} \circ \delta^{s_2}_{A_2}) =$
    μ is a lax monoidal natural transformation
$(\mathsf{m}^{s}_{!_{s_1} A_1, !_{s_1} A_1} \circ \mu(s, s_1)_{!_{s_1} A_1} \otimes \mu(s, s_1)_{!_{s_1} A_1} \circ \delta^{s_1}_{A_1} \otimes \delta^{s_1}_{A_1} \circ \mathsf{c}^{s_1}_{A_1}) \otimes (\mu(s, s_2)_{!_{s_2} A_2} \circ \delta^{s_2}_{A_2}) =$
    Identity
$(\mathsf{m}^{s}_{!_{s_1} A_1, !_{s_1} A_1} \circ \mu(s, s_1)_{!_{s_1} A_1} \otimes \mu(s, s_1)_{!_{s_1} A_1} \circ \delta^{s_1}_{A_1} \otimes \delta^{s_1}_{A_1} \circ \mathsf{c}^{s_1}_{A_1}) \otimes (\mu(s, s_2)_{!_{s_2} A_2} \circ \delta^{s_2}_{A_2} \circ 1_{!_{s_2} A_2}) =$
    The functoriality of $\otimes$
$(\mathsf{m}^{s}_{!_{s_1} A_1, !_{s_1} A_1} \circ \mu(s, s_1)_{!_{s_1} A_1} \otimes \mu(s, s_1)_{!_{s_1} A_1}) \otimes \mu(s, s_2)_{!_{s_2} A_2} \circ \delta^{s_1}_{A_1} \otimes \delta^{s_1}_{A_1} \otimes \delta^{s_2}_{A_2} \circ \mathsf{c}^{s_1}_{A_1} \otimes 1_{!_{s_2} A_2}$

Next, we observe that the following holds

$\mathsf{m}^{s}_{!_{s_1} A_1 \otimes !_{s_1} A_1, !_{s_2} A_2} \circ (\mathsf{m}^{s}_{!_{s_1} A_1, !_{s_1} A_1} \circ \mu(s, s_1)_{!_{s_1} A_1} \otimes \mu(s, s_1)_{!_{s_1} A_1}) \otimes \mu(s, s_2)_{!_{s_2} A_2} =$
    The functoriality of $\otimes$
$\mathsf{m}^{s}_{!_{s_1} A_1 \otimes !_{s_1} A_1, !_{s_2} A_2} \circ \mathsf{m}^{s}_{!_{s_1} A_1, !_{s_1} A_1} \otimes 1_{!_{s_2} !_{s_2} A_2} \circ \mu(s, s_1)_{!_{s_1} A_1} \otimes \mu(s, s_1)_{!_{s_1} A_1} \otimes \mu(s, s_2)_{!_{s_2} A_2} =$
    The definition of a lax monoidal functor
$\mathsf{m}^{s}_{!_{s_1} A_1, !_{s_1} A_1, !_{s_2} A_2} \circ \mu(s, s_1)_{!_{s_1} A_1} \otimes \mu(s, s_1)_{!_{s_1} A_1} \otimes \mu(s, s_2)_{!_{s_2} A_2}$

Let us take $\mathsf{c}^{s_1}_{A_1} \otimes 1_{!_{s_2} A_2}$ and precompose it with $M_1 \otimes M_2$. And then we observe that:

$\mathsf{c}^{s_1}_{A_1} \otimes 1_{!_{s_2} A_2} \circ M_1 \otimes M_2 = (\mathsf{c}^{s_1}_{A_1} \circ M_1) \otimes M_2 =$
    $(1_{!_{s_1} A_1} \otimes 1_{!_{s_1} A_1} \circ \mathsf{c}^{s_1}_{A_1} \circ M_1) \otimes (M_2 \circ 1_{\Gamma_2}) = (1_{!_{s_1} A_1} \otimes 1_{!_{s_1} A_1} \otimes M_2) \circ (\mathsf{c}^{s_1}_{A_1} \circ M_1 \otimes 1_{\Gamma_2})$

The completeness part is proved by construction the free Σ-assemblage from the typing rules and proof conversions from $\mathbf{SILL}(\lambda)_\Sigma$ similarly to [AT11, §1.1.6]. $\qquad\square$

5   Characterising Relevant Categories

This is an auxillary section where we discuss some properties of relevant categories, that is, symmetric monoidal categories where every object can be copied. The name "relevant category" comes from that such categories are models for substructural logics admitting the contaction rule, that is, relevant logics. We will not be going into the categorical analysis of relevant logic and related substructural logics, but we refer the reader to [Pet02] and [Jac94].

Definition 5.1. A symmetric monoidal category $\mathcal{C}$ admits *duplication* (or *copying*) if there is a

symmetric monoidal natural transformation $\mathsf{copy} : 1_{\mathcal{C}} \Rightarrow 1_{\mathcal{C}} \otimes 1_{\mathcal{C}}$ such that the following is satisfied:

$$
\begin{array}{ccc}
A & \xrightarrow{\mathsf{copy}_A} & A \otimes A \\
{\scriptstyle \mathsf{copy}_A}\downarrow & & \downarrow{\scriptstyle 1_A \otimes \mathsf{copy}_A} \\
A \otimes A & \xrightarrow[\alpha_{A,A,A} \circ \mathsf{copy}_A \otimes 1_A]{} & A \otimes (A \otimes A)
\end{array}
\qquad
\begin{array}{ccc}
A \otimes A & \xrightarrow{\sigma_{A,A}} & A \otimes A \\
& \nwarrow{\scriptstyle \mathsf{copy}_A} \quad {\scriptstyle \mathsf{copy}_A}\nearrow & \\
& A &
\end{array}
$$

We also call such categories *relevant* categories.

The following fact about relevant categories is a useful tautology:

Proposition 5.2. An SMC $(\mathcal{C}, \otimes, \mathbb{1})$ is relevant iff there is a symmetric monoidal natural transformation with the components $\gamma_A : A \to A \otimes A$ such that there is a cocommutative cosemigroup (A, γ_A) for each $A \in \mathrm{Ob}(\mathcal{C})$.

One can think of the above fact as the counterpart of Fox's theorem [Fox76] for cocommutative cosemigroups. Moreover, there is a more non-trivial equivalent characterisation of relevant categories that one can think a version of the Eckmann-Hilton argument [EH62] for cocommutative cosemigroups.

Lemma 5.3. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal category.

1. The category $\mathbf{coSem}(\mathcal{C})$ of cocommutative cosemigroups is relevant.

2. The endofunctor $\mathbf{coSem} : \mathbf{SymmMonCat}_{\mathrm{str}} \to \mathbf{SymmMonCat}_{\mathrm{str}}$ forms a comonad over $\mathbf{SymmMonCat}_{\mathrm{str}}$, a category of symmetric monoidal categories and *strict* symmetric monoidal functors.

3. $\mathcal{C}$ is relevant iff the forgetful functor $\mathsf{U}_{\mathcal{C}} : \mathbf{coSem}(\mathcal{C}) \to \mathcal{C}$ has a strict symmetric monoidal section $\mathsf{V}_{\mathcal{C}} : \mathcal{C} \to \mathbf{coSem}(\mathcal{C})$.

Proof.

1. We define the duplication operation $(A, \gamma_A) \xrightarrow{\mathsf{copy}_{(A,\gamma_A)}} (A \otimes A, \gamma_{A \otimes A})$ by $\gamma_A : A \to A \otimes A$. Then $\gamma_{A \otimes A} = \gamma_A \otimes \gamma_A$, which is proved similarly to [Mel09, Corollary 17] for cocommutative comonoids. Further, let us show that $\mathsf{copy}_{(A,\gamma_A)} : (A, \gamma_A) \to (A \otimes A, \gamma_{A \otimes A})$ is natural in (A, γ_A). Indeed, if $f : (A, \gamma_A) \to (B, \gamma_B)$ is a cocommutative cosemigroup morphism, then the below diagram in $\mathbf{coSem}(\mathcal{C})$:

$$
\begin{array}{ccc}
(A, \gamma_A) & \xrightarrow{f} & (B, \gamma_B) \\
{\scriptstyle \mathsf{copy}_{(A,\gamma_A)}}\downarrow & & \downarrow{\scriptstyle \mathsf{copy}_{(B,\gamma_B)}} \\
(A \otimes A, \gamma_{A \otimes A}) & \xrightarrow{f \otimes f} & (B \otimes B, \gamma_{B \otimes B})
\end{array}
$$

translates to the following diagram in $\mathcal{C}$:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle \gamma_A}\downarrow & & \downarrow{\scriptstyle \gamma_B} \\
A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\
{\scriptstyle \gamma_A \otimes \gamma_A}\downarrow & & \downarrow{\scriptstyle \gamma_B \otimes \gamma_B} \\
(A \otimes A) \otimes (A \otimes A) & \xrightarrow[(f \otimes f) \otimes (f \otimes f)]{} & (B \otimes B) \otimes (B \otimes B)
\end{array}
$$

The top diagram commutes by the definition of a cosemigroup morphism, the bottom diagram also commutes by the definition of a cosemigroup morphism combined with the functoriality of tensor product. The axioms of a relevant category follow directly from the definition of a cocommutative cosemigroup.

2. The forgetful functor $U_\mathcal{C}$ is obviously strict and symmetric, so let us check that **coSem** is a comonad. The components of the counit $\varepsilon : \mathbf{coSem} \Rightarrow 1$ are given by the forgetful functors $U_\mathcal{C} : \mathbf{coSem}(\mathcal{C}) \to \mathcal{C}$ for each symmetric monoidal category $\mathcal{C}$. The naturality of ε is immediate. We specify the components of uniform cloning $\delta_\mathcal{C} : \mathbf{coSem}(\mathcal{C}) \to \mathbf{coSem}(\mathbf{coSem}(\mathcal{C}))$ for an SMC $\mathcal{C}$ as follows. Let $(A, \gamma_A) \in \mathbf{coSem}(\mathcal{C})$ be a cocommutative cosemigroup in $\mathcal{C}$, then we put $\delta_\mathcal{C} : (A, \gamma_A) \mapsto ((A, \gamma_A), \gamma_{(A,\gamma_A)})$, where $\gamma_{(A,\gamma_A)} : (A, \gamma_A) \to (A, \gamma_A) \otimes (A, \gamma_A) = (A \otimes A, \gamma_{A \otimes A})$ where the right-hand side is a cocommutative cosemigroup from the previous part of this lemma. Note that (A, γ_A) should be necessarily cocommutative so we can conclude that $(A, \gamma_A) \otimes (A, \gamma_A)$ exists by Theorem 3.19. $\delta_\mathcal{C}$ is defined on morphisms similarly. The rest is check naturality for δ, that is, the following square commutes for SMC's $\mathcal{C}$ and $\mathcal{D}$ and a strict symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$:

$$
\begin{array}{ccc}
\mathbf{coSem}(\mathcal{C}) & \xrightarrow{\ \mathbf{coSem}(F)\ } & \mathbf{coSem}(\mathcal{D}) \\
{\scriptstyle \delta_\mathcal{C}}\downarrow & & \downarrow{\scriptstyle \delta_\mathcal{D}} \\
\mathbf{coSem}^2(\mathcal{C}) & \xrightarrow{\ \mathbf{coSem}^2(F)\ } & \mathbf{coSem}^2(\mathcal{D})
\end{array}
$$

which is routine.

Let us check the axioms of comonad. That is, the claim is that the following squares commute:

$$
\begin{array}{ccc}
\mathbf{coSem}(\mathcal{C}) & \xrightarrow{\ \delta_\mathcal{C}\ } & \mathbf{coSem}^2(\mathcal{C}) \\
{\scriptstyle \delta_\mathcal{C}}\downarrow & & \downarrow{\scriptstyle \delta_{\mathbf{coSem}\mathcal{C}}} \\
\mathbf{coSem}^2(\mathcal{C}) & \xrightarrow[\ \mathbf{coSem}(\delta_\mathcal{C})\ ]{} & \mathbf{coSem}^3(\mathcal{C})
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{coSem}(\mathcal{C}) & \xrightarrow{\ \delta_\mathcal{C}\ } & \mathbf{coSem}^2(\mathcal{C}) \\
{\scriptstyle \delta_\mathcal{C}}\downarrow & {\scriptstyle 1_{\mathbf{coSem}(\mathcal{C})}} & \downarrow{\scriptstyle \varepsilon_{\mathbf{coSem}(\mathcal{C})}} \\
\mathbf{coSem}^2(\mathcal{C}) & \xrightarrow[\ \mathbf{coSem}(\varepsilon_\mathcal{C})\ ]{} & \mathbf{coSem}(\mathcal{C})
\end{array}
$$

Indeed, let $\mathsf{A} = (A, \gamma_A)$ and $\mathsf{A} = (B, \gamma_B)$ be cocommutative cosemigroups in $\mathcal{C}$, then let us show that left-hand side square for cocommutative cosemigroups. The conditions for morphisms are checked similarly.

On the one hand, we have:

$$\delta_{\mathbf{coSem}\mathcal{C}}(\delta_\mathcal{C}(A, \gamma_A)) = \delta_{\mathbf{coSem}\mathcal{C}}(\mathsf{A}, \mathsf{copy}_\mathsf{A}) = ((\mathsf{A}, \mathsf{copy}_\mathsf{A}), \mathsf{copy}_{(\mathsf{A},\mathsf{copy}_\mathsf{A})}) = (\mathbf{coSem}(\delta_\mathcal{C}))(\mathsf{A}, \mathsf{copy}_\mathsf{A})$$

The right-hand side square in a similarly:

$$\varepsilon_{\mathbf{coSem}(\mathcal{C})}(\delta_\mathcal{C}(A, \gamma_A)) = \varepsilon_{\mathbf{coSem}(\mathcal{C})}(\mathsf{A}, \mathsf{copy}_\mathsf{A}) = (A, \gamma_A) = \mathbf{coSem}(\varepsilon_\mathcal{C})(\mathsf{A}, \mathsf{copy}_\mathsf{A}).$$

3. Let $\mathcal{C}$ be relevant, then there is a natural transformation $\mathsf{copy} : 1_\mathcal{C} \Rightarrow 1_\mathcal{C} \otimes 1_\mathcal{C}$ such that its component each $A \in \mathrm{Ob}(\mathcal{C})$ is endowed with the structure of a cocommutative cosemigroup with the uniform cloning operation given by the corresponding component. $\mathsf{copy}_A : A \to A \otimes A$. Put $V_\mathcal{C} : A \mapsto (A, \mathsf{copy}_A)$. Then it is readily checked that $V_\mathcal{C}$ is a required strict monoidal section.

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Now assume that the forgetful functor $\mathsf{U}_{\mathcal{C}} : \mathbf{coSem}(\mathcal{C}) \to \mathcal{C}$ has a strict section $\mathsf{V}_{\mathcal{C}} : \mathcal{C} \to \mathbf{coSem}(\mathcal{C})$. In the previous part of the lemma, we showed that $\mathbf{coSem}(\mathcal{C})$ is relevant, so for each $A \in \mathrm{Ob}(\mathcal{C})$ there is an arrow $\mathsf{copy}_{\mathsf{V}_{\mathcal{C}}A} : \mathsf{V}_{\mathcal{C}}A \to \mathsf{V}_{\mathcal{C}}A \otimes \mathsf{V}_{\mathcal{C}}A$ in $\mathbf{coSem}(\mathcal{C})$ and, therefore, one has $\mathsf{U}_{\mathcal{C}}(\mathsf{copy}_{\mathsf{V}_{\mathcal{C}}A}) : A \to A \otimes A$ in $\mathcal{C}$, so we let $\gamma_A := \mathsf{U}_{\mathcal{C}}(\mathsf{copy}_{\mathsf{V}_{\mathcal{C}}A})$.

$\square$

We can also characterise relevant categories as coalgebras in the Eilenberg-Moore category $\mathbf{SymmMonCat}_{\mathrm{str}}^{\mathbf{coSem}}$ consisting of coalgebras $(\mathcal{C}, h_{\mathcal{C}})$ where $h_{\mathcal{C}} : \mathcal{C} \to \mathbf{coSem}(\mathcal{C})$ is a strict symmetric monoidal functor.

Theorem 5.4. Let $\mathcal{C}$ be an SMC, then the following are equivalent:

1. $\mathcal{C}$ is relevant,

2. $(\mathcal{C}, h_{\mathcal{C}}) \in \mathbf{SymmMonCat}_{\mathrm{str}}^{\mathbf{coSem}}$ for some strict symmetric monoidal functor $h_{\mathcal{C}} : \mathcal{C} \to \mathbf{coSem}(\mathcal{C})$.

Proof. If $\mathcal{C}$ is relevant, then the forgetful functor $\mathsf{U}_{\mathcal{C}} : \mathbf{coSem}(\mathcal{C}) \to \mathcal{C}$ has a strict section $\mathsf{V}_{\mathcal{C}} : \mathcal{C} \to \mathbf{coSem}(\mathcal{C})$ by Lemma 5.3. Let us check that a pair $(\mathcal{C}, \mathsf{V})$ is the required $\mathbf{coSem}$-coalgebra, that is, the below diagrams commute:

$$
\begin{array}{ccc}
\mathcal{C} \xrightarrow{\ \mathsf{V}_{\mathcal{C}}\ } \mathbf{coSem}(\mathcal{C}) & \qquad & \mathcal{C} \xrightarrow{\ \ \ \mathsf{V}_{\mathcal{C}}\ \ \ } \mathbf{coSem}(\mathcal{C}) \\
\searrow_{1_{\mathcal{C}}} \quad \downarrow_{\mathsf{U}_{\mathcal{C}}} & & \downarrow_{\mathsf{V}_{\mathcal{C}}} \qquad\qquad \downarrow_{\delta_{\mathcal{C}}} \\
\mathcal{C} & & \mathbf{coSem}(\mathcal{C}) \xrightarrow[\mathbf{coSem}(\mathsf{V}_{\mathcal{C}})]{} \mathbf{coSem}^2(\mathcal{C})
\end{array}
$$

The left-hand side triangle commutes automatically since $\mathsf{V}_{\mathcal{C}}$ is a strict symmetric section of the forgetful functor. The right-hand side square commutes since if $\mathsf{V}_{\mathcal{C}}(A) = (A, \gamma_A)$, then $\mathbf{coSem}(\mathsf{V}_{\mathcal{C}})(A, \gamma_A) = (A \otimes A, \gamma_{A \otimes A}) = \delta_{\mathcal{C}}(A, \gamma_A)$. But if $(\mathcal{C}, h_{\mathcal{C}})$ is already a coalgebra in the category $\mathbf{SymmMonCat}_{\mathrm{str}}^{\mathbf{coSem}}$, then $h_{\mathcal{C}}$ is already a section of the forgetful functor by the definition of an Eilenberg-Moore category. $\square$

6 Cocteau Categories and Resource Modalities

In this section, we discuss how one can view comonads from Cocteau categories as monoidal adjunctions. To be more precise, we would like to discuss whether such comonads can be materialised as resource modalities, the term coined by Melliès and Tabareau in [MT10] to categorise affine and relevant subexponential modalities in the fashion of Benton's linear-non-linear models [Ben94]. There is a well-known result in categorical semantics of linear logic allowing one to unveil ! as a symmetric monoidal adjunction (see [Ben94]), so in this section we extend this result to $\mathbf{SILL}(\lambda)_3$ and $\mathbf{SILL}(\lambda)_\Sigma$.

Definition 6.1. Let $\mathcal{C}$ be a symmetric monoidal (closed) category, then a *modality* is given by a symmetric monoidal adjunction

$$
\mathcal{D} \underset{\mathsf{F}_r}{\overset{\mathsf{F}_l}{\rightleftarrows}} \mathcal{C}
$$

for a symmetric monoidal category $\mathcal{D}$. The notation is $((\mathsf{F}_l, \mathsf{n}), (\mathsf{F}_r, \mathsf{o}), \mathcal{D}) : \mathcal{C} \to \mathcal{C}$, where n and o are components of F_l and F_r respectively [5].

---

[5] Sometimes we would just write $(\mathsf{F}_l, \mathsf{F}_r, \mathcal{D}) : \mathcal{C} \to \mathcal{C}$ assuming that the letters n and o are already reserved for the components of a left and a right adjoint respectively.

The key observation is that the composite $\mathsf{F}_l \circ \mathsf{F}_r : \mathcal{C} \to \mathcal{C}$ is a symmetric lax monoidal comonad and we shall refer to such composites as modalities on an SMC $\mathcal{C}$. Although the notion of a modality coincides with the notion of a monoidal adjunction, we proceed with the term "modality" inspired by Yetter [Yet90], who used this name for a quantic conucleus over a (commutative) quantale, whose formal properties coincide with the properties of (symmetric) lax monoidal comonads. For comparison, Melliès and Tabareau [MT07, MT10] call such adjunctions "resource modalities", but further we will consider a broader class of comonadic modal operators, so we reserve the term "resource modality" only for the modalities dealing with resource management policies.

As Benton showed in [Ben94, Theorem 3, Theorem 8], every linear category induces a symmetric monoidal adjunction between an SMCC and a Cartesian closed category. Such adjunctions are called *linear-non-linear models*, and they are an instance of the concept of a modality. However, as it is was discussed in [Mel09], one can weaken the definition of a linear-non-linear model and construct the desired adjunction between an SMCC and a Cartesian category, not necessarily admitting exponential objects. We are in favour the latter way since it is less restrictive.

In this section we will show that every Cocteau category can be also viewed as an SMCC equipped with symmetric monoidal adjunctions of a particular kind.

In the previous section, we have already done some preliminary work on how one can characterise relevant categories to use those results further to represent relevant subexponentials as a monoidal adjunction. Let us discuss what monoidal categories we need to obtain further a similar characterisation for affine subexponentials.

Definition 6.2. A symmetric monoidal category $\mathcal{C}$ is *semicartesian* where the unit object is terminal. That is, there is a unique $\top_A : A \to \mathbb{1}$ for each $A \in \mathcal{C}$.

Note that the tensor product operation admits projections in every semicartesian category:

$$\pi_1 : A \otimes B \xrightarrow{1_A \otimes \top_B} A \otimes \mathbb{1} \xrightarrow{\lambda_A} A$$
$$\pi_2 : A \otimes B \xrightarrow{\top_A \otimes 1_B} \mathbb{1} \otimes B \xrightarrow{\rho_B} B$$

Let $(\mathcal{C}, \otimes, \mathbb{1})$ be any monoidal category, then the slice category $\mathcal{C}/\mathbb{1}$ becomes semicartesian with the tensor product operation given by for $f, g \in \mathcal{C}/\mathbb{1}$.

$$A \otimes B \xrightarrow{f \otimes g} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\lambda = \rho} \mathbb{1}$$

$\mathcal{C}/\mathbb{1}$ is semicartesian since the monoidal identity $1_\mathbb{1} : \mathbb{1} \to \mathbb{1}$ is terminal in $\mathcal{C}/\mathbb{1}$. One can also observe that a symmetric monoidal category is semicartesian iff the forgetful functor $\mathsf{U} : \mathcal{C}/\mathbb{1} \to \mathcal{C}$ is an isomorphism.

Now we give the definition of a resource modality.

Definition 6.3. Let $\mathcal{C}$ be an SMCC and let $\mathcal{D}$ be an SMC, then a modality $(\mathsf{F}_l, \mathsf{F}_r, \mathcal{D}) : \mathcal{C} \to \mathcal{C}$ is *affine* if $\mathcal{D}$ is semicartesian, *relevant* if $\mathcal{D}$ is relevant, *exponential* if $\mathcal{D}$ is Cartesian.

A modality is called a *resource modality* if it satisfies any of the above conditions.

Let us modify some well-known results in the categorical semantics of linear logic and show that every Cocteau category can be unveiled as a category equipped with resource modality and vice versa and then transfer this result to Σ-assemblages.

Definition 6.4. A *Cocteau adjunction* on an SMCC $\mathcal{C}$ is given by:

- an affine resource modality $((\mathsf{A}_l, \mathsf{n}^{\mathbf{a}}), (\mathsf{A}_r, \mathsf{o}^{\mathbf{a}}), \mathcal{C}_{\mathbf{a}}) : \mathcal{C} \to \mathcal{C}$,

- a relevant resource modality $((\mathsf{R}_l, \mathsf{n^r}), (\mathsf{R}_r, \mathsf{o^r}), \mathcal{C_r}) : \mathcal{C} \to \mathcal{C}$,

- an exponential resource modality $((\mathsf{I}_l, \mathsf{n^i}), (\mathsf{I}_r, \mathsf{o^i}), \mathcal{C_i}) : \mathcal{C} \to \mathcal{C}$,

- strict symmetric monoidal functors $\mathsf{a} : \mathcal{C_i} \to \mathcal{C_a}$ and $\mathsf{r} : \mathcal{C_i} \to \mathcal{C_r}$ such that the pairs $(\mathsf{a}, 1_\mathcal{C})$ and $(\mathsf{r}, 1_\mathcal{C})$ are left adjunction morphisms $(\mathsf{A}_l, \mathsf{n^a}) \dashv (\mathsf{A}_r, \mathsf{o^a})$ and $(\mathsf{R}_l, \mathsf{n^r}) \dashv (\mathsf{A}_r, \mathsf{o^r})$ respectively, that is, the following diagrams commute

$$
\begin{array}{ccc}
\mathcal{C_i} \xrightarrow{\quad \mathsf{a} \quad} \mathcal{C_a} & \qquad & \mathcal{C_i} \xrightarrow{\quad \mathsf{r} \quad} \mathcal{C_r} \\
\mathsf{I}_l \searrow \quad \swarrow \mathsf{A}_l & & \mathsf{I}_l \searrow \quad \swarrow \mathsf{R}_l \\
\mathcal{C} & & \mathcal{C}
\end{array}
$$

and the following axioms are satisfied for $A \in \mathcal{C_i}$:

$$
\begin{array}{ccc}
\mathsf{r}A \xrightarrow{\ \mathsf{r}(\Delta_A)\ } \mathsf{r}(A \otimes A) & \qquad & \mathsf{a}A \xrightarrow{\ \mathsf{a}(\top_A)\ } \mathsf{a}1 \\
\mathsf{copy}_{\mathsf{r}A} \searrow \quad \| & & \top_{\mathsf{a}A} \searrow \quad \| \\
\mathsf{r}A \otimes \mathsf{r}A & & \mathbb{1}
\end{array}
$$

6.1 Modalities from Adjunctions

This is a folklore fact from basic category theory that the composition of adjoint functors is a comonad. However, the concept of a Cocteau adjunction also has strict functors to reflect the corresponding comonad morphism. Therefore, we need a slightly more complicated proof compared to a standard argument in linear logic showing how one can obtain a linear category from a linear-non-linear model as, for example, in [Mel09, §7.4].

Theorem 6.5. Every Cocteau adjunction is a Cocteau category.

Proof. Let $(\mathsf{R}_l, \mathsf{R}_r, \mathcal{C_r}) : \mathcal{C} \to \mathcal{C}$ be a relevant resource modality on $\mathcal{C}$, let us show that the composite $!_\mathbf{r} = \mathsf{R}_l \circ \mathsf{R}_r : \mathcal{C} \to \mathcal{C}$ forms a relevant comonad on $\mathcal{C}$. The components are given as follows:

- $\mathsf{m}^\mathbf{r}_{A,B} := \mathsf{R}_l\mathsf{R}_r A \otimes \mathsf{R}_l\mathsf{R}_r B \xrightarrow{\mathsf{n}^\mathbf{r}_{\mathsf{R}_r A, \mathsf{R}_r B}} \mathsf{R}_l(\mathsf{R}_r A \otimes \mathsf{R}_r B) \xrightarrow{\mathsf{R}_l(\mathsf{o}^\mathbf{r}_{A,B})} \mathsf{R}_l\mathsf{R}_r(A \otimes B)$,

- $\mathsf{m}^\mathbf{r}_\mathbb{1} := \mathbb{1} \xrightarrow{\mathsf{n}^\mathbf{r}_\mathbb{1}} \mathsf{R}_l\mathbb{1} \xrightarrow{\mathsf{R}_l(\mathsf{o}^\mathbf{r}_\mathbb{1})} \mathsf{R}_l\mathsf{R}_r\mathbb{1}$,

- $\varepsilon^\mathbf{r}_A : \mathsf{R}_l\mathsf{R}_r A \to A$,

- $\delta^\mathbf{r}_A : \mathsf{R}_l\mathsf{R}_r A \xrightarrow{\mathsf{R}_l(\eta^\mathbf{r}_{\mathsf{R}_r A})} \mathsf{R}_l\mathsf{R}_r\mathsf{R}_l\mathsf{R}_r A$.

A similar statement holds for affine and exponential resource modalities. The copying natural transformation c is given by the components $\mathsf{c}_A : !_\mathbf{r} A \to !_\mathbf{r} A \otimes !_\mathbf{r} A$, each of which is defined by the following composite:

$$
!_\mathbf{r} A = \mathsf{R}_l\mathsf{R}_r A \xrightarrow{\mathsf{R}_l(\mathsf{copy}_{\mathsf{R}_r A})} \mathsf{R}_l(\mathsf{R}_r A \otimes \mathsf{R}_r A) \xrightarrow{\mathsf{p}_{\mathsf{R}_r A, \mathsf{R}_r B}} \mathsf{R}_l\mathsf{R}_r A \otimes \mathsf{R}_l\mathsf{R}_r A = !_\mathbf{r} A \otimes !_\mathbf{r} A.
$$

There are the following claims, the complete checking of which is routine.

1. c_A is natural in A, so for any $f : A \to B$ the following squares commutes:

$$
\begin{array}{ccc}
\mathsf{R}_l\mathsf{R}_r A & \xrightarrow{\mathsf{R}_l\mathsf{R}_r f} & \mathsf{R}_l\mathsf{R}_r B \\
{\scriptstyle \mathsf{c}_A^{\mathbf{r}}}\downarrow & & \downarrow{\scriptstyle \mathsf{c}_B^{\mathbf{r}}} \\
\mathsf{R}_l\mathsf{R}_r A \otimes \mathsf{R}_l\mathsf{R}_r A & \xrightarrow[\mathsf{R}_l\mathsf{R}_r f \otimes \mathsf{R}_l\mathsf{R}_r f]{} & \mathsf{R}_l\mathsf{R}_r B \otimes \mathsf{R}_l\mathsf{R}_r B
\end{array}
$$

2. c is a monoidal natural transformation, so the diagrams from Definition 3.21 are satisfied.

3. For each $A \in \mathrm{Ob}(\mathcal{C})$, the axiom of copying are satisfied:

$$
\begin{array}{ccc}
\mathsf{R}_l\mathsf{R}_r A & \xrightarrow{\mathsf{c}_A^{\mathbf{r}}} & \mathsf{R}_l\mathsf{R}_r A \otimes \mathsf{R}_l\mathsf{R}_r A \\
{\scriptstyle \mathsf{cr}_A}\downarrow & & \downarrow{\scriptstyle 1_A \otimes \mathsf{c}_A^{\mathbf{r}}} \\
\mathsf{R}_l\mathsf{R}_r A \otimes \mathsf{R}_l\mathsf{R}_r A & \xrightarrow[\alpha_{A,A,A} \circ (\mathsf{c}_A^{\mathbf{r}} \otimes 1_A)]{} & \mathsf{R}_l\mathsf{R}_r A \otimes (\mathsf{R}_l\mathsf{R}_r A \otimes \mathsf{R}_l\mathsf{R}_r A)
\end{array}
\qquad
\begin{array}{ccc}
\mathsf{R}_l\mathsf{R}_r A \otimes \mathsf{R}_l\mathsf{R}_r A & \xrightarrow{\sigma_{!_{\mathbf{r}} A, !_{\mathbf{r}} A}} & \mathsf{R}_l\mathsf{R}_r A \otimes \mathsf{R}_l\mathsf{R}_r A \\
& {\scriptstyle \mathsf{c}_A^{\mathbf{r}}}\nwarrow & \uparrow{\scriptstyle \mathsf{c}_A^{\mathbf{r}}} \\
& & \mathsf{R}_l\mathsf{R}_r A
\end{array}
$$

But they follow from the structure of $\mathcal{C}_{\mathbf{r}}$ easily.

4. Any free coalgebra morphism $f : (!_{\mathbf{r}} A, \delta_A) \to (!_{\mathbf{r}} B, \delta_B)$ preserves uniform cloning induced by c, that is, the following square commutes:

$$
\begin{array}{ccc}
!_{\mathbf{r}} A & \xrightarrow{\quad f \quad} & !_{\mathbf{r}} B \\
{\scriptstyle \mathsf{c}_A}\downarrow & & \downarrow{\scriptstyle \mathsf{c}_B} \\
!_{\mathbf{r}} A \otimes !_{\mathbf{r}} A & \xrightarrow[f \otimes f]{} & !_{\mathbf{r}} B \otimes !_{\mathbf{r}} B
\end{array}
$$

One can similarly show that $!_{\mathbf{a}}$ is endowed with the structure of an affine comonad, where the deletion is given by the composite $!_{\mathbf{a}} A \xrightarrow{\mathsf{A}_l(\top_{A^* A})} \mathsf{A}_l \mathbb{1} \xrightarrow{\mathsf{p}_{\mathbb{1}}} \mathbb{1}$.

To complete the proof, we need to construct the comonad morphisms $\mu_{\mathbf{ia}} : !_{\mathbf{i}} \Rightarrow !_{\mathbf{a}}$ and $\mu_{\mathbf{ir}} : !_{\mathbf{i}} \Rightarrow !_{\mathbf{r}}$ from the functors $\mathsf{a} : \mathcal{C}_{\mathbf{i}} \to \mathcal{C}_{\mathbf{a}}$ and $\mathsf{r} : \mathcal{C}_{\mathbf{i}} \to \mathcal{C}_{\mathbf{r}}$. The symmetric lax monoidal comonad morphism $\mu_{\mathbf{ir}} : \mathsf{l}_l \mathsf{l}_r \Rightarrow \mathsf{R}_l \mathsf{R}_r$ is given by Proposition 3.22 since we have $\mathsf{l}_l \mathsf{l}_r = \mathsf{R}_l \mathsf{rl}_r$ by the definition of a Cocteau adjunction.

So we let:

$$
\mathsf{l}_l \mathsf{l}_r = \mathsf{R}_l \mathsf{rl}_r \xrightarrow{\mathsf{R}_l(\eta^{\mathbf{r}} \mathsf{rl}_r)} \mathsf{R}_l \mathsf{R}_r \mathsf{R}_r \mathsf{rl}_r = \mathsf{R}_l \mathsf{R}_r \mathsf{l}_l \mathsf{l}_r \xrightarrow{\mathsf{R}_l \mathsf{R}_r \varepsilon^{\mathbf{i}}} \mathsf{R}_l \mathsf{R}_r
$$
$$\underbrace{\phantom{\mathsf{l}_l \mathsf{l}_r = \mathsf{R}_l \mathsf{rl}_r \xrightarrow{\mathsf{R}_l(\eta^{\mathbf{r}} \mathsf{rl}_r)} \mathsf{R}_l \mathsf{R}_r \mathsf{R}_r \mathsf{rl}_r = \mathsf{R}_l \mathsf{R}_r \mathsf{l}_l \mathsf{l}_r \xrightarrow{\mathsf{R}_l \mathsf{R}_r \varepsilon^{\mathbf{i}}} \mathsf{R}_l \mathsf{R}_r}}_{\mu_{\mathbf{ir}}}$$

Further, we note that a pair of functors $(\mathsf{r}, 1_{\mathcal{C}})$, then by Proposition 3.9, $\mu_{\mathbf{ir}}$ is a symmetric lax monoidal comonad morphism. The rest is to check that $\mu_{\mathbf{ir}}$ preserves uniform cloning, that is, we must show that the following diagram commutes:

$$
\begin{array}{ccc}
!_{\mathbf{i}} A & \xrightarrow{\mathsf{c}_A^{\mathbf{i}}} & !_{\mathbf{i}} A \otimes !_{\mathbf{i}} A \\
{\scriptstyle \mu_{\mathbf{ir}} A}\downarrow & & \downarrow{\scriptstyle \mu_{\mathbf{ir}} A \otimes \mu_{\mathbf{ir}} A} \\
!_{\mathbf{r}} A & \xrightarrow{\mathsf{c}_A^{\mathbf{r}}} & !_{\mathbf{r}} A \otimes !_{\mathbf{r}} A
\end{array}
\qquad (5)
$$

Let us unveil the above diagram:

$$
\begin{array}{ccccc}
\mathsf{R}_l\mathsf{rl}_r A & \xrightarrow{\;\mathsf{R}_l\mathsf{r}(\Delta_{\mathsf{l}_r A})\;} & \mathsf{R}_l\mathsf{r}(\mathsf{l}_r A \otimes \mathsf{l}_r A) & \xrightarrow{\;\mathsf{p}^{\mathbf{i}}_{\mathsf{l}_r A, \mathsf{l}_r A}\;} & \mathsf{R}_l\mathsf{rl}_r A \otimes \mathsf{R}_l\mathsf{rl}_r A
\end{array}
$$

(6)

(7)

(8)

(9) and (10)

$\mathsf{R}_l\mathsf{r}(\mathsf{copy}_{\mathsf{rl}_r A})$, $\mathsf{R}_l(\eta^{\mathbf{r}}_{\mathsf{rl}_r A})$, $\mathsf{R}_l(\mathsf{copy}_{\mathsf{R}_r\mathsf{R}_l\mathsf{rl}_r A})$, $\mathsf{R}_l\mathsf{R}_r(\varepsilon^{\mathbf{i}}_A)$, $\mathsf{R}_l(\mathsf{copy}_{\mathsf{R}_r A})$

where:

- (6) holds by the definition of a Cocteau adjunction,

- (7) and (8) hold by the naturality of copy,

- (9) and (10) commute since R_l is symmetric oplax monoidal by Proposition 3.22.

$\square$

6.2 Constructing a Cocteau adjunction from a Cocteau category

To show that every Cocteau category induces a Cocteau adjunction, one needs to show that every constituent comonad of a Cocteau category induces the corresponding adjunction. The construction building an adjunction from an exponential comonad is standard, so we just note that the construction by Benton or Melliès is compeletely preserved since one can think of linear categories as exponential reducts of Cocteau categories. So the rest is build adjunctions from affine and relevant comonads and introduce the corresponding functors to have required comonad morphism.

The simplest part is to unveil an affine comonad.

Lemma 6.6. Let $\mathcal{C}$ be an SMCC equipped with an affine comonad K, then there is an affine resource modality:

$$
\mathcal{C}_{\mathbf{a}} \underset{\mathsf{A}_r}{\overset{\mathsf{A}_l}{\underset{\perp}{\rightleftarrows}}} \mathcal{C}
$$

for some semicartesian $\mathcal{C}_{\mathbf{a}}$, A_l and A_r.

Proof. The Eilenberg-Moore category $\mathcal{C}^{\mathsf{K}}$ is already monoidal by Lemma 3.24. We equip every coalgebra (A, h_A) with the deletion operation $\top_A : A \to \mathbb{1}$ natural in A and unique for each A such that $\top_A : A \to \mathbb{1}$ is a coalgebra morphism and $\top_A$ is given by:

$$
A \xrightarrow{\iota_A} \mathbb{1} = A \xrightarrow{h_A} \mathsf{K}A \xrightarrow{\mathsf{d}_A} \mathbb{1}
$$

The rest of the proof is routine. $\square$

The next is to show that the Eilenberg-Moore category of a relevant comonad is relevant.

Lemma 6.7. Let $\mathcal{C}$ be an SMC and let $\mathsf{K} : \mathcal{C} \to \mathcal{C}$ be a symmetric lax monoidal endofunctor endowed with the structure of a relevant comonad. Then the Eilenberg-Moore category $\mathcal{C}^{\mathsf{K}}$ is relevant.

The proof is adaptation of the construction described in [Mel09, §7.4] with a few modifications, but the structure of the argument is similar. The key idea is that if we have a relevant comonad, then every free coalgebra $(\mathsf{K}A, \delta_A)$ is endowed with a cocommutative cosemigroup structure and we lift this structure to every coalgebra (A, h_A) by showing that every coalgebra of that form is retract of the free coalgebra $(\mathsf{K}A, \delta_A)$. As a result, every coalgebra (A, h_A) is provided with uniform cloning $\gamma_A : A \to A \otimes A$ satisfying the required axioms. One can compose the proof of Lemma 6.7 of the following two propositions.

Proposition 6.8. Every coalgebra (A, h_A) has a retraction $A \xrightarrow{h_A} \mathsf{K}A \xrightarrow{\varepsilon_A} A$ making the below diagram commute:

$$
\begin{array}{ccccccc}
A & \xrightarrow{h_A} & \mathsf{K}A & \xrightarrow{c_A} & \mathsf{K}A \otimes \mathsf{K}A & \xrightarrow{\varepsilon_A \otimes \varepsilon_A} & A \otimes A \\
\downarrow{\scriptstyle h_A} & & & & & & \downarrow{\scriptstyle h_A \otimes h_A} \\
\mathsf{K}A & & & \xrightarrow{\hspace{4cm} c_A \hspace{4cm}} & & & \mathsf{K}A \otimes \mathsf{K}A
\end{array}
$$

Proof. The below claim has the same proof as [Mel09, §7.4, Proposition 26] for comonoids since the argument does not require addressing to counits, but only uniform cloning. $\square$

The below claim is similar to [Mel09, §7.4, Proposition 27], but, however, we elaborate on its proof.

Lemma 6.9. Let $A \in \mathrm{Ob}(\mathcal{C})$ and let (B, γ_B) be a cocommutative cosemigroup in $\mathcal{C}$ such that there is a retraction in $\mathcal{C}$

$$
A \underset{1_A}{\overset{f}{\rightrightarrows}} B \xrightarrow{g} A
$$

Then A is endowed with the structure of a cocommutative cosemigroup (A, γ_A) that lifts f to a cosemigroup morphism $f : (A, \gamma_A) \to (B, \gamma)$ if and only if the below diagram commutes

$$
\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{\gamma_B} & B \otimes B & \xrightarrow{g \otimes g} & A \otimes A \\
\downarrow{\scriptstyle f} & & & & & & \downarrow{\scriptstyle f \otimes f} \\
B & & & \xrightarrow{\hspace{3.5cm} \gamma_B \hspace{3.5cm}} & & & B \otimes B
\end{array}
$$

Moreover, the cosemigroup structure on A is defined where uniform cloning γ_A is given by:

$$
A \xrightarrow{f} B \xrightarrow{\gamma_A} B \otimes B \xrightarrow{g \otimes g} A \otimes A
$$

Proof. If A is endowed with the structure of a cocommutative cosemigroup (A, γ_A) such that a morphism $f : A \to B$ in $\mathcal{C}$ lifts to a cosemigroup morphism $f : (A, \gamma_A) \to (B, \gamma_A)$, so the below square commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\hspace{1cm} f \hspace{1cm}} & B \\
\downarrow{\scriptstyle \gamma_A} & & \downarrow{\scriptstyle \gamma_B} \\
A \otimes A & \xrightarrow{f \otimes f} & B \otimes B
\end{array}
$$

But the following square commutes

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
{\scriptstyle \gamma_A}\downarrow & & \downarrow{\scriptstyle \gamma_B} \\
A \otimes A & \xleftarrow[\ g \otimes g\ ]{} & B \otimes B
\end{array}
$$

since one has $g \circ f = 1_A$. So we define γ_A as the composite $g \otimes g \circ \gamma_B \circ f$, so the desired diagram automatically commutes.

If the diagram from the statement commutes, then one needs to check that the structure (A, γ_A) where $\gamma_A = g \otimes g \circ \gamma_B \circ f$ is a cocommutative cosemigroup indeed, so the cocommutativity and coassociativity axioms are satisfied for uniform cloning. The coassociativity clause is the same as in the proof of [Mel09, §7.4, Proposition 27], so let us consider the cocommutativity axiom. But (B, γ_B) is already a cocommutative cosemigroup, so we have:

$$\sigma_{A,A} \circ g \otimes g \circ \gamma_B \circ f = g \otimes g \circ \sigma_{B,B} \circ \gamma_B \circ f = g \otimes g \circ \gamma_B \circ f = \gamma_A.$$

<div align="right">□</div>

As a corollary from Claim 6.8 and Claim 6.9 below we show that:

Lemma 6.10. Every coalgebra (A, h_A) over a relevant comonad K is endowed with a cocommutative cosemigroup multiplication making the Eilenberg-Moore category $\mathcal{C}^{\mathsf{K}}$ relevant.

Proof. By Proposition 5.2, an SMC $\mathcal{C}$ is relevant if and only if there is a natural natural transformation γ endowing each A with the structure of a cocommutative cosemigroup (A, γ_A).

As we showed in Claim 6.8 and Claim 6.9, we define γ_A as:

$$A \xrightarrow{\ h_A\ } \mathsf{K}A \xrightarrow{\ \mathsf{c}_A\ } \mathsf{K}A \otimes \mathsf{K}A \xrightarrow{\ \varepsilon_A \otimes \varepsilon_A\ } A \otimes A$$

The rest is to check the following clauses:

- γ_A is a coalgebra morphism, that is, we want the below square

$$
\begin{array}{ccccccc}
A & \xrightarrow{\ h_A\ } & \mathsf{K}A & \xrightarrow{\ \mathsf{c}_A\ } & \mathsf{K}A \otimes \mathsf{K}A & \xrightarrow{\ \varepsilon_A \otimes \varepsilon_A\ } & A \otimes A \\
\big\downarrow{\scriptstyle h_A} & & & & & & \big\downarrow{\scriptstyle h_A \otimes h_A} \\
 & & & & & & \mathsf{K}A \otimes \mathsf{K}A \\
 & & & & & & \big\downarrow{\scriptstyle \mathsf{m}_{A,A}} \\
\mathsf{K}A & \xrightarrow[\mathsf{K}(h_A)]{} & \mathsf{K}^2 A & \xrightarrow[\mathsf{K}(\mathsf{c}_A)]{} & \mathsf{K}(\mathsf{K}A \otimes \mathsf{K}A) & \xrightarrow[\mathsf{K}(\varepsilon_A \otimes \varepsilon_A)]{} & \mathsf{K}(A \otimes A)
\end{array}
$$

commute in $\mathcal{C}$. Indeed,

$\mathsf{K}(\gamma_A) \circ h_A = \mathsf{K}(\varepsilon_A \otimes \varepsilon_A) \circ \mathsf{K}(\mathsf{c}_A) \circ \mathsf{K}(h_A) \circ h_A =$
    By the property of coalgebras
$\mathsf{K}(\varepsilon_A \otimes \varepsilon_A) \circ \mathsf{K}(\mathsf{c}_A) \circ \delta_A \circ h_A =$
    By the definition of a relevant comonad and by the definition of lax monoidal functors
$\mathsf{K}(\varepsilon_A \otimes \varepsilon_A) \circ \mathsf{m}_{\mathsf{K}A,\mathsf{K}A} \circ \delta_A \otimes \delta_A \circ \mathsf{c}_A \circ h_A = \mathsf{m}_{A,A} \circ (\mathsf{K}\varepsilon_A \otimes \mathsf{K}\varepsilon_A) \circ \delta_A \otimes \delta_A \circ \mathsf{c}_A \circ h_A =$
    By the definition of a comonad and Claim 6.8
$\mathsf{m}_{A,A} \circ \mathsf{c}_A \circ h_A = \mathsf{m}_{A,A} \circ h_A \otimes h_A \circ \gamma_A$

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- The naturality of γ_A follows immediately from the definition of an Eilenberg-Moore category and the axioms of a relevant comonad.

- The proof that γ_A is a symmetric lax monoidal natural transformation repeats the corresponding clause in the proof of [Mel09, §7.4, Proposition 28].

$\square$

Finally, we conclude that:

Theorem 6.11. Any Cocteau category induces a Cocteau adjunction.

Proof. Fix a Cocteau category $(\mathcal{C}, !_{\mathbf{i}}, !_{\mathbf{r}}, !_{\mathbf{a}}, \mu_{\mathbf{ir}}, \mu_{\mathbf{ia}})$. We are already able to unveil $!_{\mathbf{i}}$, $!_{\mathbf{r}}$ and $!_{\mathbf{a}}$ as resource modalities of the following form

- an affine resource modality $(\mathsf{A}_l, \mathsf{A}_r, \mathcal{C}_{\mathbf{a}}) : \mathcal{C} \to \mathcal{C}$ by Lemma 6.6,

- a relevant resource modality $(\mathsf{R}_l, \mathsf{R}_r, \mathcal{C}_{\mathbf{r}}) : \mathcal{C} \to \mathcal{C}$ by Lemma 6.7,

- an exponential resource modality $(\mathsf{I}_l, \mathsf{I}_r, \mathcal{C}_{\mathbf{i}})$ by [Mel09, Proposition 28—30].

The rest is to treat the comonad morphisms $\mu_{\mathbf{ir}}$ and $\mu_{\mathbf{ia}}$. We consider $\mu_{\mathbf{ir}}$ only, $\mu_{\mathbf{ia}}$ is considered analogously. $\mu_{\mathbf{ir}} : !_{\mathbf{i}} \Rightarrow !_{\mathbf{r}}$ is a comonad morphism, so, by [Bor94b, Proposition 4.5.9], we have a functor $\mathsf{r} : \mathcal{C}^{!_{\mathbf{i}}} \to \mathcal{C}^{!_{\mathbf{r}}}$ such that $\mathsf{r} : (A, h_A) \mapsto (A, \mu_{\mathbf{ir}\,A} \circ h_A)$ and $\mathsf{I}_l = \mathsf{R}_l \mathsf{r}$. r is also strict:

$\mathsf{r}(A, h_A) \otimes \mathsf{r}(A, h_A) = (A, \mu_{\mathbf{ir}\,A} \circ h_A) \otimes (A, \mu_{\mathbf{ir}\,A} \circ h_A) = (A \otimes A, \mathsf{m}^{\mathbf{r}}_{A,A} \circ \mu_{\mathbf{ir}\,A} \otimes \mu_{\mathbf{ir}\,A} \circ h_A \circ h_A) = (A \otimes A, \mu_{\mathbf{ir}\,A} \circ \mathsf{m}^{\mathbf{i}}_{A,A} \circ h_A \circ h_A) = \mathsf{r}(A \otimes A, \mathsf{m}^{\mathbf{i}}_{A,A} \circ h_A \otimes h_A)$

And the corresponding axiom of a Cocteau adjunction is satisfies as follows. First of all, let $\mathsf{copy}^{\mathbf{i}}$ and $\mathsf{copy}^{\mathbf{r}}$ denote the copying operations in $\mathcal{C}^{!_{\mathbf{i}}}$ and $\mathcal{C}^{!_{\mathbf{r}}}$.

Let $(A, h_A) \in \mathcal{C}^{!_{\mathbf{i}}}$ be a coalgebra, so we have uniform cloning $\mathsf{copy}^{\mathbf{i}} : (A, h_A) \to (A \otimes A, \mathsf{m}_{A,A} \circ h_A \otimes h_A)$ realised by the composite:

$$A \xrightarrow{\;h_A\;} !_{\mathbf{i}} A \xrightarrow{\;\mathsf{c}^{\mathbf{i}}_A\;} !_{\mathbf{i}} A \otimes !_{\mathbf{i}} A \xrightarrow{\;\varepsilon^{\mathbf{i}}_A \otimes \varepsilon^{\mathbf{i}}_A\;} A \otimes A \qquad (11)$$

On the one hand, we have $\mathsf{r}(\mathsf{copy}^{\mathbf{i}}) : \mathsf{r}(A, h_A) \to \mathsf{r}(A \otimes A, \mathsf{m}_{A,A} \circ h_A \otimes h_A)$ with the underlying arrow as in (11), but as a morphism of morphism of coalgebras $(A, \mu_{\mathbf{ir}\,A} \circ h_A)$ and $(A \otimes A, \mu_{\mathbf{ir}\,A} \circ \mathsf{m}^{\mathbf{i}}_{A,A} \circ h_A \circ h_A)$.

On the other hand, we have the arrow $\mathsf{copy}^{\mathbf{r}}_{\mathsf{r}(A, h_A)} : (A, \mu_{\mathbf{ir}\,A} \circ h_A) \to (A \otimes A, \mu_{\mathbf{ir}\,A \otimes A} \circ \mathsf{m}^{\mathbf{i}}_{A,A} \circ h_A \otimes h_A)$ in $\mathcal{C}^{!_{\mathbf{r}}}$ with the underlying arrow:

$$A \xrightarrow{\;h_A\;} !_{\mathbf{r}} A \xrightarrow{\;\mathsf{c}^{\mathbf{r}}_A\;} !_{\mathbf{r}} A \otimes !_{\mathbf{r}} A \xrightarrow{\;\varepsilon^{\mathbf{r}}_A \otimes \varepsilon^{\mathbf{r}}_A\;} A \otimes A \qquad (12)$$

So the equality $\mathsf{r}(\mathsf{copy}^{\mathbf{i}}_{(A, h_A)}) = \mathsf{copy}^{\mathbf{r}}_{\mathsf{r}(A, h_A)}$ holds by the following diagram:

$$
\begin{array}{ccccccc}
A & \xrightarrow{\;h_A\;} & !_{\mathbf{i}} A & \xrightarrow{\;\mathsf{c}^{\mathbf{i}}_A\;} & !_{\mathbf{i}} A \otimes !_{\mathbf{i}} A & \xrightarrow{\;\varepsilon^{\mathbf{i}}_A \otimes \varepsilon^{\mathbf{i}}_A\;} & A \otimes A \\
\Big\| & & \Big\downarrow{\scriptstyle \mu_{\mathbf{ir}\,A}} & & \Big\downarrow{\scriptstyle \mu_{\mathbf{ir}\,A} \otimes \mu_{\mathbf{ir}\,A}} & & \Big\| \\
A & \xrightarrow{\;\mu_{\mathbf{ir}\,A} \circ h_A\;} & !_{\mathbf{r}} A & \xrightarrow{\;\mathsf{c}^{\mathbf{r}}_A\;} & !_{\mathbf{r}} A \otimes !_{\mathbf{r}} A & \xrightarrow{\;\varepsilon^{\mathbf{r}}_A \otimes \varepsilon^{\mathbf{r}}_A\;} & A \otimes A
\end{array}
$$

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where the left-hand side square commutes tautologically, the right-hand side square follows from the definition of a symmetric monoidal comonad morphism, whereas the central square is from the definition of a Cocteau category. $\qquad\square$

One can also generalise the above construction in order to represent not only Cocteau categories, but arbitrary assemblages as well. We introduce the notion of a bouton directly generalising Cocteau adjunctions.

Definition 6.12. Let $\Sigma = (I, \preceq, W, C)$ be a subexponential signature and let $\mathcal{C}$ be a symmetric monoidal closed category, then a Σ-*bouton*[6] over $\mathcal{C}$ consists of a contravariant functor $\mathsf{T} : (I, \preceq) \to$ **SymmMonCat**$_{\mathrm{strict}}$ such that:

- For each $s \in I$, there is a modality $(\mathsf{F}_l^s, \mathsf{F}_r^s, \mathsf{T}_s) : \mathcal{C} \to \mathcal{C}$,

- If $s \in W$ ($s \in C$, $c \in W \cap C$), then the resource modality $(\mathsf{F}_l^s, \mathsf{F}_r^s)$ is affine (relevant, exponential),

- If $s_1 \preceq s_2$, then the following triangle commutes:

$$
\begin{array}{ccc}
\mathsf{T}_{s_2} & \xrightarrow{\ \mathsf{T}(s_1, s_2)\ } & \mathsf{T}_{s_1} \\
{\scriptstyle \mathsf{F}_{s_2\, r}} \searrow & & \swarrow {\scriptstyle \mathsf{F}_{s_1\, r}} \\
& \mathcal{C} &
\end{array}
$$

- If $s_1 \preceq s_2$ and $s_1 \in C$, then the following triangle commutes for each $A \in \mathsf{T}_{s_2}$:

$$
\begin{array}{ccc}
\mathsf{T}(s_1, s_2)(A) & \xrightarrow{\ \mathsf{T}(s_1, s_2)(\mathsf{copy}_A^{s_2})\ } & \mathsf{T}(s_1, s_2)(A \otimes A) \\
{\scriptstyle \mathsf{copy}_{\mathsf{T}(s_1, s_2)(A)}^{s_1}} \searrow & & \\
& \mathsf{T}(s_1, s_2)(A) \otimes \mathsf{T}(s_1, s_2)(A) &
\end{array}
$$

- If $s_1 \preceq s_2$ and $s_1 \in W$, then the following triangle commutes for each $A \in \mathsf{T}_{s_2}$:

$$
\begin{array}{ccc}
\mathsf{T}(s_1, s_2)(A) & \xrightarrow{\ \mathsf{T}(s_1, s_2)(\top_A^{s_2})\ } & \mathsf{T}(s_1, s_2)(\mathbb{1}) \\
{\scriptstyle \top_{\mathsf{T}(s_1, s_2)(A)}^{s_1}} \searrow & & \\
& \mathbb{1} &
\end{array}
$$

The following fact directly generalises Theorem 6.5 and Theorem 6.11.

Theorem 6.13. Let $\mathcal{C}$ be an SMCC and let $\Sigma = (I, \preceq, W, C)$ be a subexponential signature, then:

1. Every Σ-bouton on $\mathcal{C}$ induces a Σ-assemblage.

2. Every Σ-assemblage on $\mathcal{C}$ induces a Σ-bouton.

---

[6]The word "bouton" means a flower bud in French. The choice of name is a metaphorical description of a category equipped with some adjunctions viewed as petals.

6.3 Expanding Lafont Categories

The alternative approach to semantic modelling of the exponential modality is to consider it via the free construction, see [Tro92, Chapter 12].

Previously, we have been working with Cocteau categories thinking of them as an expansion of a linear category, but in this section we introduce a subclass of Cocteau categories admitting the representation as Cocteau adjunctions. In particular, we expand Lafont categories initially introduced in [Laf88].

It is not difficult to observe that the axioms of a cocommutative cosemigroup resemble the axioms of a relevant category quite closely, but this coincidence has the following rigorisation.

In the below definition, we overload the terminology and call them Lafont categories too.

Definition 6.14. A *Lafont* category is an SMCC $(\mathcal{C}, \otimes, \mathbb{1})$ such that the forgetful functors $\mathsf{U_a} : \mathcal{C}/\mathbb{1} \to \mathcal{C}$, $\mathsf{U_i} : \mathbf{coMon}(\mathcal{C}) \to \mathcal{C}$ and $\mathsf{U_r} : \mathbf{coSem}(\mathcal{C}) \to \mathcal{C}$ have right adjoints $\mathsf{R_a} : \mathcal{C} \to \mathcal{C}/\mathbb{1}$, $\mathsf{R_i} : \mathcal{C} \to \mathbf{coMon}(\mathcal{C})$ and $\mathsf{R_r} : \mathcal{C} \to \mathbf{coMon}(\mathcal{C})$ respectively. Those right adjoints are also called *the free construction*.

In every Lafont category, $\mathsf{U_a}$, $\mathsf{U_i}$ and $\mathsf{U_r}$ are automatically agreed with one another as follows

$$\mathbf{coMon}(\mathcal{C}) \xrightarrow{\ \mathsf{U_{ir}}\ } \mathbf{coSem}(\mathcal{C}) \qquad\qquad \mathbf{coMon}(\mathcal{C}) \xrightarrow{\ \mathsf{U_{ia}}\ } \mathcal{C}/\mathbb{1}$$

with $\mathsf{U_i}$, $\mathsf{U_r}$ mapping to $\mathcal{C}$, and $\mathsf{U_i}$, $\mathsf{U_a}$ mapping to $\mathcal{C}$.

where $\mathsf{U_{ir}}$ and $\mathsf{U_{ia}}$ are strict monoidal functors realised as $\mathsf{U_{ir}} : (A, \gamma_A, \iota_A) \mapsto (A, \gamma_A)$ and $\mathsf{U_{ia}} : (A, \gamma_A, \iota_A) \mapsto (A, \iota_A)$ respectively.

One can equivalently define Lafont category as an SMCC $(\mathcal{C}, \otimes, \mathbb{1})$ such that the following conditions are satisfied:

- For every $A \in \mathrm{Ob}(\mathcal{C})$ there is a cocommutative comonoid $(A_\mathbf{i}, \gamma_{A_\mathbf{i}}, \iota_{A_\mathbf{i}})$ equipped with a choice of morphism $\varepsilon_A^\mathbf{i} : A_\mathbf{i} \to A$ such that for any other cocommutative comonoid (X, γ_X, ι_X) and any morphism $f : X \to A$ there is a unique a cocommutative comonoid morphism $f^\bullet : (X, \gamma_X, \iota_X) \to (A_\mathbf{i}, \gamma_{A_\mathbf{i}}, \iota_{A_\mathbf{i}})$ making the following triangle commute in $\mathcal{C}$:

$$
\begin{array}{ccc}
 & & A_\mathbf{i} \\
 & \overset{\exists! f^\bullet}{\nearrow} & \big\downarrow {\scriptstyle \varepsilon_A^\mathbf{i}} \\
X & & \\
 & \underset{f}{\searrow} & \\
 & & A
\end{array}
$$

- For each $B \in \mathrm{Ob}(\mathcal{C})$ there is a cocommutative cosemigroup $(B_\mathbf{r}, \gamma_{B_\mathbf{r}})$ such that there is a morphism $\varepsilon_B^\mathbf{r} : B_\mathbf{r} \to B$ in $\mathcal{C}$ such that for any cocommutative cosemigroup (Y, γ_Y) and any morphism $g : Y \to B$ in $\mathcal{C}$ there is a unique a cocommutative cosemigroup morphism $g^\bullet : (Y, \gamma_Y) \to (B_\mathbf{r}, \gamma_{B_\mathbf{r}})$ such that the analogous triangle commutes.

- For every $C \in \mathcal{C}$ there is a counital structure $(C_\mathbf{a}, \iota_{C_\mathbf{a}})$ and a morphism $\varepsilon_C^\mathbf{a} : C_\mathbf{a} \to C$ such that for each counital structure (Z, ι_Z) and for each arrow $h : Z \to C$ in $\mathcal{C}$ there is a unique $h^\bullet : (Z, \iota_Z) \to (C_\mathbf{a}, \iota_{C_\mathbf{a}})$ such that the analogous triangle commutes.

Moreover, the following automatically holds as well. Let $A \in \mathrm{Ob}(\mathcal{C})$ such that there is a cocommutative comonoid $(A_{\mathbf{i}}, \gamma_{A_{\mathbf{i}}}, \iota_{A_{\mathbf{i}}})$, then there are arrows $\mathsf{U}_{\mathbf{ir}\,A} : A_{\mathbf{i}} \to A_{\mathbf{r}}$ and $\mathsf{U}_{\mathbf{ia}\,A} : A_{\mathbf{i}} \to A_{\mathbf{a}}$ such that the following triangle commutes:

$$
\begin{array}{ccc}
& A_{\mathbf{i}} & \\
\mathsf{U}_{\mathbf{ir}\,A} \swarrow & \downarrow \varepsilon_A^{\mathbf{i}} & \searrow \mathsf{U}_{\mathbf{ia}\,A} \\
A_{\mathbf{r}} \xrightarrow{\ \varepsilon_A^{\mathbf{r}}\ } & A & \xleftarrow{\ \varepsilon_A^{\mathbf{a}}\ } A_{\mathbf{a}}
\end{array}
$$

Theorem 6.15. Every Lafont category induces a Cocteau adjunction.

Proof. First of all, as it has been already noted in Theorem 3.19, the forgetful functors $\mathsf{U}_{\mathbf{i}} : \mathbf{coMon}(\mathcal{C}) \to \mathcal{C}$, $\mathsf{U}_{\mathbf{r}} : \mathbf{coSem}(\mathcal{C}) \to \mathcal{C}$ are strict monoidal and symmetric.

Further, one has $\mathsf{U}_{\mathbf{i}} \dashv \mathsf{R}_{\mathbf{i}}$, $\mathsf{U}_{\mathbf{r}} \dashv \mathsf{R}_{\mathbf{r}}$ and $\mathsf{U}_{\mathbf{a}} \dashv \mathsf{R}_{\mathbf{a}}$, so the condition of Proposition 3.22 is satisfied, so those adjunctions are lifted to symmetric monoidal adjunction inducing a Cocteau adjunction by Theorem 6.11. □

Example 6.16. One can extract several curious examples of Lafont categories from presentably symmetric monoidal categories. We refer the reader to [AR94] and [MP89, Chapter 2] for a more systematic study of presentable categories. Let us recall the relevant concepts.

Let $\mathcal{C}$ be a locally small category, then $\mathcal{C}$ is *presentable* if $\mathcal{C}$ has all small colimits κ-directed colimits and there is a *set S* of κ-compact objects that generate $\mathcal{C}$ under κ-filtered colimits for some cardinal κ. A symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ is presentable if $\mathcal{C}$ is presentable and it is an SMC such that the tensor product operation $\otimes$ as a bifunctor preserves all small colimits.

If $(\mathcal{C}, \otimes, \mathbb{1})$ is a presentably symmetric monoidal category, then $\mathcal{C}$ is also closed, which follows from the Adjoint Functor theorem. Moreover, $\mathcal{C}/\mathbb{1}$, $\mathbf{coMon}(\mathcal{C})$ and $\mathbf{coSem}(\mathcal{C})$ are presentable, one can show that similarly to [HM25, Theorem 10]. Besides, the forgetful functors $\mathsf{U}_{\mathbf{i}} : \mathbf{coMon}(\mathcal{C}) \to \mathcal{C}$, $\mathsf{U}_{\mathbf{r}} : \mathbf{coSem}(\mathcal{C}) \to \mathcal{C}$ and $\mathsf{U}_{\mathbf{a}} : \mathcal{C}/\mathbb{1} \to \mathcal{C}$ have right adjoints. Thus any presentably SMC $(\mathcal{C}, \otimes, \mathbb{1})$ satisfies Definition 6.16.

There are quite a few examples of categories satisfying this observation. The first example is the category R-**Mod** of modules over a commutative ring R. Thus, the category Vect_k of vector spaces over a field k and the category of Abelian groups viewed as $\mathbb{Z}$-modules are also Lafont categories. The category of R-modules R-**Mod** is equivalent to the category $\mathbf{QCoh}(\mathcal{O}(\mathrm{Spec}\,R))$ of quasi-coherent $\mathcal{O}(X)$-modules over the Zariski space $\mathrm{Spec}(R)$ [Har13, Corollary II.5.5], so $\mathbf{QCoh}(\mathcal{O}(\mathrm{Spec}\,R))$ is a Lafont category. There are also several examples of presentably SMC's (and thus Lafont) from algebraic topology such as the category $\mathbf{Ch}(\mathcal{A})$ of chain complexes in a Grothendieck Abelian category $\mathcal{A}$ (see [Bek00, Proposition 3.10]). Thus all the aforementioned categories are models of the system $\mathbf{SILL}(\lambda)_3$. There are already some results styding the semantics of (intuitionistic) linear logic in structures from algebraic geometry as in [Mel22], so our example demonstrate how close algebraic geometry and linear logic are once more.

6.4 Representing Cocteau categories 2-categorically

One can strengthen the above result from Theorem 6.11 by using some 2-categorical techniques and the formal theory of comonads. In the previous section, we have proved that any particular Cocteau category induces a Cocteau adjunction, but in this section, we consider the 2-category of *all* Cocteau categories and fully faithfully embed it to the Eilenberg-Moore 2-category over a

2-comonad constructed a particular 2-category of strict functors. This allow us extend the results from the previous section to functors and natural transformations of Cocteau categories.

Definition 6.17. Let **Cocteau** denote the 2-category of Cocteau categories is defined as follows:

- The class of 0-cells is the class of all Cocteau categories,

- Let $(\mathcal{C}, !_{\mathbf{i}}, !_{\mathbf{r}}, !_{\mathbf{a}}, \mu_{\mathbf{ir}}, \mu_{\mathbf{ia}})$ and $(\mathcal{D}, !'_{\mathbf{i}}, !'_{\mathbf{r}}, !'_{\mathbf{a}}, \mu'_{\mathbf{ir}}, \mu'_{\mathbf{ia}})$ be Cocteau categories, the class of 1-cells **Cocteau**$(\mathcal{C}, \mathcal{D})$ consist of strict symmetric monoidal functors $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ such that:

  - F induce symmetric lax monoidal comonad morphisms $(\mathcal{C}, !_{\mathbf{i}}) \Rightarrow (\mathcal{D}, !'_{\mathbf{i}})$, $(\mathcal{C}, !_{\mathbf{r}}) \Rightarrow (\mathcal{D}, !'_{\mathbf{r}})$ and $(\mathcal{C}, !_{\mathbf{a}}) \Rightarrow (\mathcal{D}, !'_{\mathbf{a}})$ with the components $\mathsf{F}_{\mathbf{i}} : \mathsf{F} !_{\mathbf{i}} \Rightarrow !'_{\mathbf{i}} \mathsf{F}$, $\mathsf{F}_{\mathbf{r}} : \mathsf{F} !_{\mathbf{r}} \Rightarrow !'_{\mathbf{r}} \mathsf{F}$ and $\mathsf{F}_{\mathbf{a}} : \mathsf{F} !_{\mathbf{a}} \Rightarrow !'_{\mathbf{a}} \mathsf{F}$ respectively.

  - For each $A \in \mathrm{Ob}(\mathcal{C})$ and for $\mathbf{c} \in \{\mathbf{i}, \mathbf{r}\}$ and $\mathbf{w} \in \{\mathbf{i}, \mathbf{a}\}$, the functor $\widehat{\mathsf{F}_{\mathbf{c}}} : \mathcal{C}^{!_{\mathbf{c}}} \to \mathcal{D}^{!'_{\mathbf{c}}}$ of the induced Eilenberg-Moore categories preserves the copying arrow for each coalgebra $(A, h_A) \in \mathcal{C}^{!_{\mathbf{c}}}$:

$$
\begin{array}{ccc}
\mathsf{F}A & \xrightarrow{\widehat{\mathsf{F}_{\mathbf{c}}}(\gamma_A)} & \mathsf{F}A \otimes \mathsf{F}A \\
\| & & \| \\
\mathsf{F}A & \xrightarrow[\gamma_{\widehat{\mathsf{F}_{\mathbf{c}}}(A)}]{} & \mathsf{F}A \otimes \mathsf{F}A
\end{array}
$$

    and the functor $\widehat{\mathsf{F}_{\mathbf{w}}} : \mathcal{C}^{!_{\mathbf{w}}} \to \mathcal{D}^{!'_{\mathbf{w}}}$ preserves the deletion arrow as follows for each coalgebra $(A, h_A) \in \mathcal{C}^{!_{\mathbf{w}}}$:

$$
\begin{array}{ccc}
\mathsf{F}A & \xrightarrow{\widehat{\mathsf{F}_{\mathbf{w}}}(\iota_A)} & \mathbb{1} \\
\| & & \| \\
\mathsf{F}A & \xrightarrow[\iota_{\widehat{\mathsf{F}_{\mathbf{w}}}(A)}]{} & \mathbb{1}
\end{array}
$$

  - the following squares commute:

$$
\begin{array}{ccc}
\mathsf{F} !_{\mathbf{i}} & \xrightarrow{\mathsf{F}_{\mathbf{i}}} & !'_{\mathbf{i}} \mathsf{F} \\
{\scriptstyle \mathsf{F}\mu_{\mathbf{ir}}} \downarrow & & \downarrow {\scriptstyle \mu'_{\mathbf{ir}}\mathsf{F}} \\
\mathsf{F} !_{\mathbf{r}} & \xrightarrow[\mathsf{F}_{\mathbf{r}}]{} & !'_{\mathbf{r}} \mathsf{F}
\end{array}
\qquad
\begin{array}{ccc}
\mathsf{F} !_{\mathbf{i}} & \xrightarrow{\mathsf{F}_{\mathbf{i}}} & !'_{\mathbf{i}} \mathsf{F} \\
{\scriptstyle \mathsf{F}\mu_{\mathbf{ia}}} \downarrow & & \downarrow {\scriptstyle \mu'_{\mathbf{ia}}\mathsf{F}} \\
\mathsf{F} !_{\mathbf{a}} & \xrightarrow[\mathsf{F}_{\mathbf{a}}]{} & !'_{\mathbf{a}} \mathsf{F}
\end{array}
\tag{13}
$$

- Let $\mathsf{F}, \mathsf{G} : (\mathcal{C}, !_{\mathbf{i}}, !_{\mathbf{r}}, !_{\mathbf{a}}, \mu_{\mathbf{ir}}, \mu_{\mathbf{ia}}) \to (\mathcal{D}, !'_{\mathbf{i}}, !'_{\mathbf{r}}, !'_{\mathbf{a}}, \mu'_{\mathbf{ir}}, \mu'_{\mathbf{ia}})$ be 1-cells, the class **Cocteau**(F, G) of 2-cells from F to G consists of symmetric monoidal natural transformations $\theta : \mathsf{F} \Rightarrow \mathsf{G}$ inducing symmetric lax monoidal comonad morphisms $(\mathsf{F}, !_{\mathbf{i}}) \Rightarrow (\mathsf{G}, !'_{\mathbf{i}})$, $(\mathsf{F}, !_{\mathbf{a}}) \Rightarrow (\mathsf{G}, !'_{\mathbf{a}})$ and $(\mathsf{F}, !_{\mathbf{r}}) \Rightarrow (\mathsf{G}, !'_{\mathbf{r}})$ Besides, the following condition is satisfied

$$
\begin{array}{ccc}
\mathsf{F} !_{\mathbf{i}} & \xrightarrow{\theta_{\mathbf{i}}} & !'_{\mathbf{i}} \mathsf{G} \\
{\scriptstyle \mathsf{F}\mu_{\mathbf{ir}}} \downarrow & & \downarrow {\scriptstyle \mu'_{\mathbf{ir}}\mathsf{G}} \\
\mathsf{F} !_{\mathbf{r}} & \xrightarrow[\theta_{\mathbf{r}}]{} & !'_{\mathbf{r}} \mathsf{G}
\end{array}
\qquad
\begin{array}{ccc}
\mathsf{F} !_{\mathbf{i}} & \xrightarrow{\theta_{\mathbf{i}}} & !'_{\mathbf{i}} \mathsf{G} \\
{\scriptstyle \mathsf{F}\mu_{\mathbf{ia}}} \downarrow & & \downarrow {\scriptstyle \mu'_{\mathbf{ia}}\mathsf{G}} \\
\mathsf{F} !_{\mathbf{a}} & \xrightarrow[\theta_{\mathbf{a}}]{} & !'_{\mathbf{a}} \mathsf{G}
\end{array}
\tag{14}
$$

where $\theta_{\mathbf{i}}$ is defined as

$$
\begin{array}{ccc}
\mathsf{F}\,!_{\mathbf{i}} & \xrightarrow{\;\theta\,!_{\mathbf{i}}\;} & \mathsf{G}\,!_{\mathbf{i}} \\
\downarrow{\scriptstyle \mathsf{F}_{\mathbf{i}}} & \searrow{\scriptstyle \theta_{\mathbf{i}}} & \downarrow{\scriptstyle \mathsf{G}_{\mathbf{i}}} \\
!'_{\mathbf{i}}\,\mathsf{F} & \xrightarrow{\;!'_{\mathbf{i}}\,\theta\;} & !'_{\mathbf{i}}\,\mathsf{G}
\end{array}
$$

and $\theta_{\mathbf{a}}$ and $\theta_{\mathbf{r}}$ are defined similarly.

Prima facie, the reader might find the definition of 2-cells incomplete since 1-cells induce strict functors of Eilenberg-Moore categories that preserve the copying and deletion operations, but it is not reflected at the level of 2-cells. Let $\theta : \mathsf{F} \Rightarrow \mathsf{G}$ be a 2-cell in **Cocteau**. F and G induce comonad morphisms $(\mathcal{C}, !_{\mathbf{r}}) \Rightarrow (\mathcal{D}, !'_{\mathbf{r}})$, so we have the functors of Eilenberg-Moore categories $\widehat{\mathsf{F}_{\mathbf{r}}} : \mathcal{C}^{!_{\mathbf{r}}} \to \mathcal{D}^{!'_{\mathbf{r}}}$ and $\widehat{\mathsf{G}_{\mathbf{r}}} : \mathcal{C}^{!_{\mathbf{r}}} \to \mathcal{D}^{!'_{\mathbf{r}}}$ defined as $\widehat{\mathsf{F}_{\mathbf{r}}} : (A, h_A) \mapsto (\mathsf{F}A, \mathsf{F}_{\mathbf{r}A} \circ \mathsf{F}(h_A))$ and $\widehat{\mathsf{G}_{\mathbf{r}}} : (A, h_A) \mapsto (\mathsf{G}A, \mathsf{G}_{\mathbf{r}A} \circ \mathsf{G}(h_A))$. So $\theta : \mathsf{F} \Rightarrow \mathsf{G}$ extends to $\widehat{\theta} : \widehat{\mathsf{F}_{\mathbf{r}}} \Rightarrow \widehat{\mathsf{G}_{\mathbf{r}}}$ and $\widehat{\theta}$ is a strict symmetric monoidal natural transformation, so the following diagram automatically commutes for each coalgebra $(A, h_A) \in \mathcal{C}^{!_{\mathbf{r}}}$:

$$
\begin{array}{ccccc}
\mathsf{F}A & \xrightarrow{\;\widehat{\mathsf{F}_{\mathbf{r}}}(\gamma_A)\;} & \mathsf{F}A \otimes \mathsf{F}A & =\!=\!=\!= & \mathsf{F}(A \otimes A) \\
\downarrow{\scriptstyle \widehat{\theta}_A} & & \downarrow{\scriptstyle \widehat{\theta}_A \otimes \widehat{\theta}_A} & & \downarrow{\scriptstyle \widehat{\theta}_{A \otimes A}} \\
\mathsf{G}A & \xrightarrow[\;\widehat{\mathsf{G}_{\mathbf{r}}}(\gamma_A)\;]{} & \mathsf{G}A \otimes \mathsf{G}A & =\!=\!=\!= & \mathsf{G}(A \otimes A)
\end{array}
$$

To proceed further, let us instantiate Theorem 3.13 the following way.

Theorem 6.18. **Cmd(SymmMonCat)**, the 2-category of symmetric lax monoidal comonads over SMCs, their symmetric lax monoidal morphisms and transformations, fully faithfully embeds to $\mathbf{Adj}^l(\mathbf{SymmMonCat})$, the category of symmetric monoidal adjunctions in **SymmMonCat**, their left morphisms and modifications.

Now we would like to fully faithfully embed the whole 2-category **Cocteau** to some 2-category of 2-functors, their strict natural transformations and modifications similarly to Theorem 6.18, but we would also like to reflect all the required connections between the constituent symmetric lax monoidal comonads in Cocteau categories. So we have to provide a plausible candidate instead of $\mathbf{Adj}^l(\mathbf{SymmMonCat})$, a subcategory of $\mathbf{Fun}([1], \mathbf{SymmMonCat})$ spanned by those 1-cells in **SymmMonCat** that have a right adjoint. Recall that one can unveil any Cocteau category as a Cocteau adjunction by Theorem 6.11 and left adjoints in those adjunctions are forgetful functors from the Eilenberg-Moore category to the underlying SMC. Recall that those functors are strict symmetric. So we take $\mathbf{SymmMonCat}_{\mathrm{strict}}$ as a 2-subcategory of **SymmMonCat** where left adjoints we are interested in belong to as 1-cells.

Further, we have got to replace [1] with something more appropriate since we have three comonads and transformations between them.

Definition 6.19. The 1-category **FCocteau** (read as "free Cocteau") with four objects $\circledcirc, \circledcirc_0, \circledcirc_1, \circledcirc_2$

and with morphisms generated by the diagram

$$
\begin{array}{ccc}
& \smiley_1 & \\
u_{10} \swarrow & \downarrow f_1 & \searrow u_{12} \\
\smiley_0 \xrightarrow{\;f_0\;} & \smiley & \xleftarrow{\;f_2\;} \smiley_2
\end{array}
$$

Now let us consider the 2-category $\mathbf{Fun}(\mathbf{FCocteau}, \mathbf{SymmMonCat}_{\text{strict}})$, where we choose symmetric monoidal categories $\mathcal{C}$, $\mathcal{D}_0$, $\mathcal{D}_1$ and $\mathcal{D}_2$ and strict symmetric monoidal functors such that the following triangle commutes

$$
\begin{array}{ccc}
& \mathcal{D}_1 & \\
\mathsf{U}_{10} \swarrow & \downarrow \mathsf{F}_1 & \searrow \mathsf{U}_{12} \\
\mathcal{D}_0 \xrightarrow{\;\mathsf{F}_0\;} & \mathcal{C} & \xleftarrow{\;\mathsf{F}_2\;} \mathcal{D}_2
\end{array}
\tag{15}
$$

Further, we shall be using capital Gothic letters $\mathfrak{A}, \mathfrak{B}, \ldots$ to range over elements of $\mathbf{Fun}(\mathbf{FCocteau}, \mathbf{SymmMonCat}_{\text{strict}})$.

Triangles as in (15) reflect how left adjoints commute with one another, but we also expect $\mathcal{D}_0$, $\mathcal{D}_1$ and $\mathcal{D}_2$ to be relevant, Cartesian and semicartesian respectively. But those triangles bear no information about that. $\mathcal{D}_1$ and $\mathcal{D}_2$ are not really problematic to us since an SMC $\mathcal{C}$ is Cartesian (semicartesian) iff the forgetful functor from the category of cocommutative comonoids (from the slice category over $\mathbb{1}$) is an isomorphism. As we know from Theorem 5.4, an SMC is relevant iff it forms a coalgebra in the Eilenberg-Moore category $\mathbf{SymmMonCat}^{\mathbf{coSem}}_{\text{str}}$. So our idea is to consider Cocteau categories as coalgebras from the Eilenberg-Moore 2-category $(\mathbf{Fun}(\mathbf{FCocteau}, \mathbf{SymmMonCat}_{\text{strict}}))^{\mathsf{K}}$ for a strict 2-comonad inheriting the corresponding coalgebra from $\mathbf{SymmMonCat}^{\mathbf{coSem}}_{\text{str}}$ from Theorem 5.4. So recall that:

Definition 6.20. Let $\mathcal{C}$ be a 2-category and let $\mathsf{K} : \mathcal{C} \to \mathcal{C}$ be a strict 2-functor, then a (strict) 2-comonad is a triple $(\mathsf{K}, \delta, \varepsilon)$, where δ and ε are strict 2-natural transformations satisfying the coassociative and coidentity laws.

There is a more detailed description of (lax) 2-monads in [JY21, §6.5], so one can restore the details of the definitions for strict comonads quite easily. If we have a comonad K over a 2-category $\mathcal{C}$, then we can associate the *Eilenberg-Moore 2-category* $\mathcal{C}^{\mathsf{K}}$ of strict coalgebras (A, h_A), morphisms and 2-cells.

The rest is to construct a desired 2-functor K. First of all, recall that the forgetful functors $\mathsf{U}_{\mathbf{r}\mathcal{D}_0} : \mathbf{coSem}(\mathcal{D}_0) \to \mathcal{D}_0$, $\mathsf{U}_{\mathbf{iD}_1} : \mathbf{coMon}(\mathcal{D}_1) \to \mathcal{D}_1$ and $\mathsf{U}_{\mathbf{a}\mathcal{D}_2} : \mathcal{D}_2/\mathbb{1} \to \mathcal{D}_2$ are strict symmetric, so they are 1-cells in the category $\mathbf{SymmMonCat}_{\text{strict}}$. So let us postcompose (15) with the functors as follows

$$
\begin{array}{ccccc}
& & \mathbf{coMon}(\mathcal{D}_1) & & \\
& & \downarrow \mathsf{U}_{\mathbf{iD}_1} & & \\
\mathsf{R}_{\mathcal{D}_1,\mathcal{D}_0} \swarrow & & \mathcal{D}_1 & & \searrow \mathsf{A}_{\mathcal{D}_1,\mathcal{D}_2} \\
& \mathsf{U}_{10} \swarrow & \downarrow \mathsf{F}_1 & \searrow \mathsf{U}_{12} & \\
\mathbf{coSem}(\mathcal{D}_0) \xrightarrow[\mathsf{U}_{\mathbf{r}\mathcal{D}_0}]{} \mathcal{D}_0 & \xrightarrow[\mathsf{F}_0]{} & \mathcal{C} & \xleftarrow[\mathsf{F}_2]{} \mathcal{D}_2 & \xleftarrow[\mathsf{U}_{\mathbf{a}\mathbb{1}}]{} \mathcal{D}_2/\mathbb{1}
\end{array}
\tag{16}
$$

45

where the functors $\mathsf{R}_{\mathcal{D}_1,\mathcal{D}_0}$ and $\mathsf{A}_{\mathcal{D}_1,\mathcal{D}_2}$ are the following composites respectively:

$$\mathbf{coMon}(\mathcal{D}_1) \xrightarrow{\mathsf{U_{ir}}} \mathbf{coSem}(\mathcal{D}_1) \xrightarrow{\mathbf{coSem}(\mathsf{U_{10}})} \mathbf{coSem}(\mathcal{D}_0)$$

$$\mathbf{coMon}(\mathcal{D}_1) \xrightarrow{\mathsf{U_{ia}}} \mathcal{D}_1/\mathbb{1} \xrightarrow{\mathsf{U_{12}}/\mathbb{1}} \mathcal{D}_0/\mathbb{1}$$

where $\mathsf{U_{ir}} : \mathbf{coMon}(\mathcal{D}_1) \to \mathbf{coSem}(\mathcal{D}_1)$ is a strict symmetric monoidal functor dropping a counit from every cocommutative comonoid in $\mathcal{D}_1$ making it a cosemigroup and $\mathsf{U_{ia}} : \mathbf{coMon}(\mathcal{D}_1) \to \mathcal{D}_1/\mathbb{1}$ is a strict symmetric monoidal functor that takes a cocommutative comonoid in $\mathcal{D}_1$ and returns its counit. It is readily checked that $\mathsf{U_{ir}}$ and $\mathsf{U_{ia}}$ are strict as well as $\mathsf{R}_{\mathcal{D}_1,\mathcal{D}_0}$ and $\mathsf{A}_{\mathcal{D}_1,\mathcal{D}_2}$. So all the morphisms in (16) are 1-cells in $\mathbf{SymmMonCat}_{\mathrm{strict}}$.

Lemma 6.21.

1. The mapping K that takes $\mathfrak{A} \in \mathbf{Fun}(\mathbf{FCocteau}, \mathbf{SymmMonCat}_{\mathrm{strict}})$ and returns the triangle of the form of (16) is a 2-endofunctor on $\mathbf{Fun}(\mathbf{FCocteau}, \mathbf{SymmMonCat}_{\mathrm{strict}})$.

2. K is a strict 2-comonad over the 2-category $\mathbf{Fun}(\mathbf{FCocteau}, \mathbf{SymmMonCat}_{\mathrm{strict}})$.

Proof.

1. Let $f : \mathfrak{A} \to \mathfrak{B}$ be a 1-cell in $\mathbf{Fun}(\mathbf{FCocteau}, \mathbf{SymmMonCat}_{\mathrm{strict}})$ given by the following diagram:
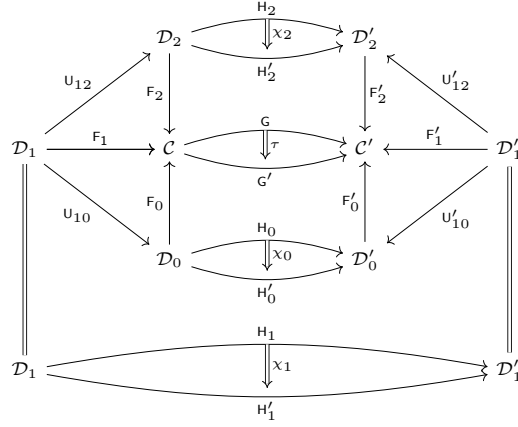

Thus $\mathsf{K}f : \mathsf{K}\mathfrak{A} \to \mathsf{K}\mathfrak{B}$ is given by:

(17)

where $\widehat{\mathsf{H}_1} : \mathbf{coMon}(\mathcal{D}_1) \to \mathbf{coMon}(\mathcal{D}_1')$ is a functor sending a cocommutative comonoid in $\mathcal{D}_1$ to a cocommutative comonoid in $\mathcal{D}_1'$ and this functor does exist since H_1 is strict monoidal, see [AM10, Proposition 3.38]. $\widehat{\mathsf{H}_2} : \mathcal{D}_2/\mathbb{1} \to \mathcal{D}_2'/\mathbb{1}$ is a functor of slice categories extending H_2, which is also strict monoidal. $\widehat{\mathsf{H}_0} : \mathbf{coSem}(\mathcal{D}_0) \to \mathbf{coSem}(\mathcal{D}_0')$ is defined similarly to $\widehat{\mathsf{H}_1}$. Checking functoriality for 1-cells is routine.

2. Let $f_1, f_2 : \mathfrak{A} \to \mathfrak{B}$ be 1-cells and let $\phi : f_1 \Rightarrow f_2$ be a 2-cell given by the following diagram:
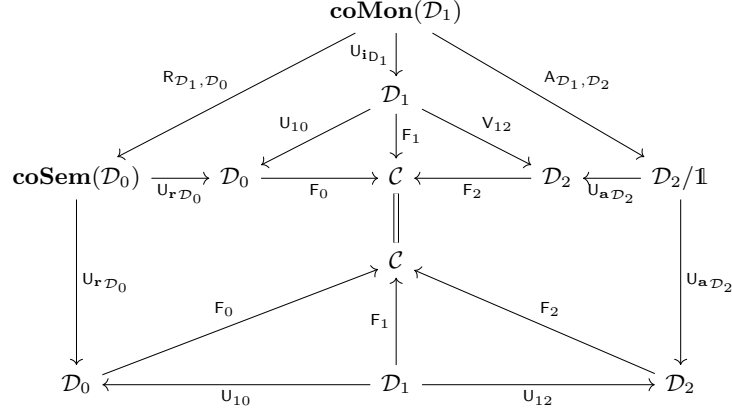
$$
\begin{array}{c}
\mathcal{D}_2 \xrightarrow{\;\;\mathsf{H}_2\;\;} \mathcal{D}_2' \\
\chi_2 \\
\mathsf{H}_2'
\end{array}
$$

$$
\mathsf{U}_{12} \quad \mathsf{F}_2 \quad \mathsf{F}_2' \quad \mathsf{U}_{12}'
$$

$$
\mathcal{D}_1 \xrightarrow{\;\mathsf{F}_1\;} \mathcal{C} \xrightarrow{\;\mathsf{G}\;} \mathcal{C}' \xleftarrow{\;\mathsf{F}_1'\;} \mathcal{D}_1'
$$

$$
\tau \qquad \mathsf{G}'
$$

$$
\mathsf{U}_{10} \quad \mathsf{F}_0 \quad \mathsf{F}_0' \quad \mathsf{U}_{10}'
$$

$$
\mathcal{D}_0 \xrightarrow{\;\mathsf{H}_0\;} \mathcal{D}_0' \qquad \chi_0 \qquad \mathsf{H}_0'
$$

$$
\mathcal{D}_1 \xrightarrow{\;\mathsf{H}_1\;} \mathcal{D}_1' \qquad \chi_1 \qquad \mathsf{H}_1'
$$

so $\mathsf{K}\phi : \mathsf{K}f_1 \Rightarrow \mathsf{K}f_2$ is given by:

$$
\mathcal{D}_2/\mathbb{1} \xrightarrow{\;\widehat{\mathsf{H}_2}\;} \mathcal{D}_2'/\mathbb{1} \qquad \widehat{\chi_2} \qquad \widehat{\mathsf{H}_2'}
$$

$$
\mathsf{U_a}_{\mathcal{D}_2} \qquad \mathsf{U_a}_{\mathcal{D}_2'}
$$

$$
\mathcal{D}_2 \xrightarrow{\;\mathsf{H}_2\;} \mathcal{D}_2' \qquad \chi_2 \qquad \mathsf{H}_2'
$$

$$
\mathsf{U}_{12} \quad \mathsf{F}_2 \qquad \mathsf{F}_2' \quad \mathsf{U}_{12}'
$$

$$
\mathcal{D}_1 \xrightarrow{\;\mathsf{F}_1\;} \mathcal{C} \xrightarrow{\;\mathsf{G}\;} \mathcal{C}' \xleftarrow{\;\mathsf{F}_1'\;} \mathcal{D}_1' \qquad \tau \qquad \mathsf{G}'
$$

$$
\mathsf{U}_{10} \quad \mathsf{F}_0 \qquad \mathsf{F}_0' \quad \mathsf{U}_{10}'
$$

$$
\mathcal{D}_0 \xrightarrow{\;\mathsf{H}_0\;} \mathcal{D}_0' \qquad \chi_0 \qquad \mathsf{H}_0'
$$

$$
\mathsf{A}_{\mathcal{D}_1,\mathcal{D}_2} \qquad \mathsf{A}'_{\mathcal{D}_1',\mathcal{D}_2'}
$$

$$
\mathsf{U_r}_{\mathcal{D}_0} \qquad \mathsf{U_r}_{\mathcal{D}_0'}
$$

$$
\mathbf{coSem}(\mathcal{D}_0) \xrightarrow{\;\widehat{\mathsf{H}_0}\;} \mathbf{coSem}(\mathcal{D}_0') \qquad \widehat{\chi_0} \qquad \widehat{\mathsf{H}_0'}
$$

$$
\mathcal{D}_1 \xrightarrow{\;\mathsf{H}_1\;} \mathcal{D}_1' \qquad \chi_1 \qquad \mathsf{H}_1'
$$

$$
\mathsf{U_i}_{\mathcal{D}_1} \qquad \mathbf{coSem}(\mathcal{D}_0) \qquad \mathbf{coSem}(\mathcal{D}_0') \qquad \mathsf{U_i}_{\mathcal{D}_1'}
$$

$$
\mathsf{R}_{\mathcal{D}_1,\mathcal{D}_0} \qquad \mathsf{R}'_{\mathcal{D}_1',\mathcal{D}_0'}
$$

$$
\mathbf{coMon}(\mathcal{D}_1) \xrightarrow{\;\widehat{\mathsf{H}_1}\;} \mathbf{coMon}(\mathcal{D}_1') \qquad \widehat{\chi_1} \qquad \widehat{\mathsf{H}_1'}
$$
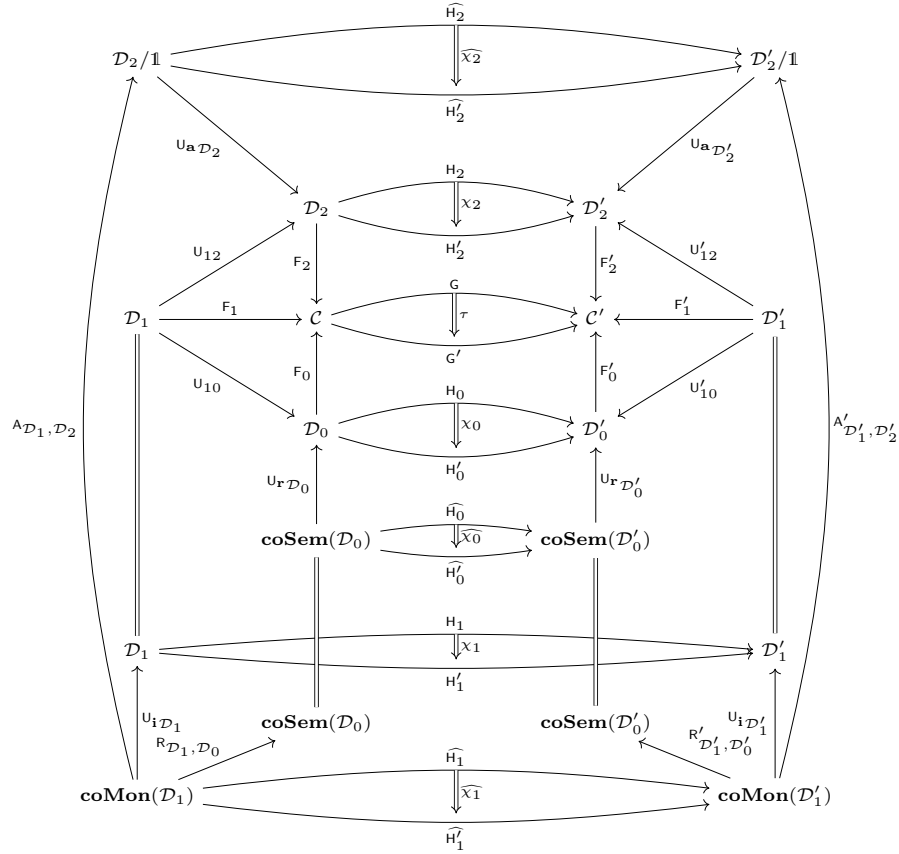
3. We have got to define the counit and comultiplication and check the axioms of 2-comonad.
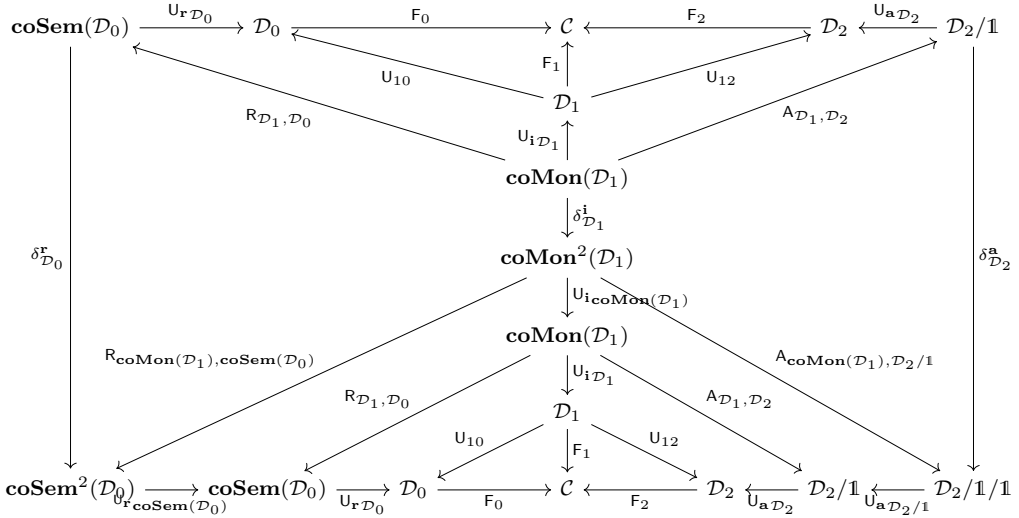
The counit natural transformation $\varepsilon : \mathsf{K} \Rightarrow \mathbf{Id}$ consists of the components $\varepsilon_{\mathfrak{A}} : \mathsf{K}\mathfrak{A} \to \mathfrak{A}$ for $\mathfrak{A} \in \mathbf{Fun}(\mathbf{FCocteau}, \mathbf{SymmMonCat}_{\mathrm{strict}})$ that take a diagram as in (16) and transforms it

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into a diagram as in (15) by the following diagram:

$$
\begin{array}{c}
\mathbf{coMon}(\mathcal{D}_1) \\
\downarrow U_{i_{\mathcal{D}_1}} \\
\end{array}
$$

With edges labeled: $R_{\mathcal{D}_1,\mathcal{D}_0}$, $U_{i_{\mathcal{D}_1}}$, $A_{\mathcal{D}_1,\mathcal{D}_2}$, $\mathcal{D}_1$, U_{10}, F_1, V_{12}, $\mathbf{coSem}(\mathcal{D}_0)$, $U_{\mathbf{r}\mathcal{D}_0}$, $\mathcal{D}_0$, F_0, $\mathcal{C}$, F_2, $\mathcal{D}_2$, $U_{\mathbf{a}\mathcal{D}_2}$, $\mathcal{D}_2/\mathbb{1}$, $U_{\mathbf{r}\mathcal{D}_0}$, $\mathcal{C}$, F_0, F_1, F_2, $U_{\mathbf{a}\mathcal{D}_2}$, $\mathcal{D}_0$, U_{10}, $\mathcal{D}_1$, U_{12}, $\mathcal{D}_2$

We have already noticed above that $\mathsf{K}\mathfrak{A}$ is a 1-cell in $\mathbf{Fun}(\mathbf{FCocteau}, \mathbf{SymmMonCat}_{\mathrm{strict}})$, $\varepsilon_{\mathfrak{A}}$ is well-defined for each $\mathfrak{A}$.

4. The comultiplication natural transformation $\delta : \mathsf{K} \Rightarrow \mathsf{KK}$ with the components $\delta_{\mathfrak{A}} : \mathsf{K}\mathfrak{A} \to \mathsf{KK}\mathfrak{A}$ for each $\mathfrak{A}$ given by a quadruple of functors $(\delta^{\mathbf{r}}_{\mathcal{D}_0}, \delta^{\mathbf{i}}_{\mathcal{D}_1}, \delta^{\mathbf{a}}_{\mathcal{D}_2}, 1_{\mathcal{C}})$ making the below diagram commute:

(large commutative diagram with labels including $U_{\mathbf{r}\mathcal{D}_0}$, F_0, F_2, $U_{\mathbf{a}\mathcal{D}_2}$, $\mathcal{D}_2/\mathbb{1}$, U_{10}, F_1, U_{12}, $\mathcal{D}_1$, $R_{\mathcal{D}_1,\mathcal{D}_0}$, $A_{\mathcal{D}_1,\mathcal{D}_2}$, $U_{i_{\mathcal{D}_1}}$, $\mathbf{coMon}(\mathcal{D}_1)$, $\delta^{\mathbf{i}}_{\mathcal{D}_1}$, $\mathbf{coMon}^2(\mathcal{D}_1)$, $U_{i_{\mathbf{coMon}(\mathcal{D}_1)}}$, $\mathbf{coMon}(\mathcal{D}_1)$, $R_{\mathbf{coMon}(\mathcal{D}_1),\mathbf{coSem}(\mathcal{D}_0)}$, $A_{\mathbf{coMon}(\mathcal{D}_1),\mathcal{D}_2/\mathbb{1}}$, $\delta^{\mathbf{r}}_{\mathcal{D}_0}$, $\delta^{\mathbf{a}}_{\mathcal{D}_2}$, $U_{i_{\mathcal{D}_1}}$, $R_{\mathcal{D}_1,\mathcal{D}_0}$, $A_{\mathcal{D}_1,\mathcal{D}_2}$, $\mathcal{D}_1$, U_{10}, F_1, U_{12}, $\mathbf{coSem}^2(\mathcal{D}_0)$, $U_{\mathbf{r}_{\mathbf{coSem}(\mathcal{D}_0)}}$, $\mathbf{coSem}(\mathcal{D}_0)$, $U_{\mathbf{r}\mathcal{D}_0}$, $\mathcal{D}_0$, F_0, $\mathcal{C}$, F_2, $\mathcal{D}_2$, $U_{\mathbf{a}\mathcal{D}_2}$, $\mathcal{D}_2/\mathbb{1}$, $U_{\mathbf{a}\mathcal{D}_2/\mathbb{1}}$, $\mathcal{D}_2/\mathbb{1}/\mathbb{1}$)

where $\delta^{\mathbf{r}}_{\mathcal{D}_0}$ is comultiplication of the $\mathbf{coSem}$-comonad from Lemma 5.3. $\delta^{\mathbf{i}}_{\mathcal{D}_1} : \mathbf{coMon}(\mathcal{D}_1) \to \mathbf{coMon}^2(\mathcal{D}_1)$ and $\delta^{\mathbf{a}}_{\mathcal{D}_2} : \mathcal{D}_2/\mathbb{1} \to (\mathcal{D}_2/\mathbb{1}/)\mathbb{1}$ are functors defined similarly to $\delta^{\mathbf{r}}_{\mathcal{D}_0}$. Let us show that the above diagram commutes indeed.

The morphisms $\delta^{\mathbf{r}}_{\mathcal{D}_0}$, $\delta^{\mathbf{i}}_{\mathcal{D}_1}$ and $\delta^{\mathbf{a}}_{\mathcal{D}_2}$ are strict indeed: $\delta^{\mathbf{i}}_{\mathcal{D}_1}$ establishes an isomorphism between $\mathbf{coMon}(\mathcal{D}_1)$ and $\mathbf{coMon}^2(\mathcal{D}_1)$ by [Mel09, Corollary 19] (and the same consideration is applicable to $\delta^{\mathbf{a}}_{\mathcal{D}_2}$), whereas $\delta^{\mathbf{r}}_{\mathcal{D}_0}$ is strict symmetric as it follows from Lemma 5.3.

So we take the diagram from (16), which is already commutative and precompose it with the forgetful functors $\mathbf{coSem}^2(\mathcal{D}_0)$ to $\mathbf{coSem}(\mathcal{D}_0)$, from $\mathbf{coMon}^2(\mathcal{D}_1)$ to $\mathbf{coMon}(\mathcal{D}_1)$ and from

$\mathcal{D}_2/\mathbb{1}/\mathbb{1}$ to $\mathcal{D}_2/\mathbb{1}$, so the resulting bigger triangle also commutes. Now consider the following square:

$$
\begin{array}{ccc}
\mathbf{coMon}(\mathcal{D}_1) & \xrightarrow{\quad R_{\mathcal{D}_1,\mathcal{D}_0} \quad} & \mathbf{coSem}(\mathcal{D}_0) \\
\downarrow{\scriptstyle \delta^{\mathbf{i}}_{\mathcal{D}_1}} & & \downarrow{\scriptstyle \delta^{\mathbf{r}}_{\mathcal{D}_0}} \\
\mathbf{coMon}^2(\mathcal{D}_1) & \xrightarrow[R_{\mathbf{coMon}(\mathcal{D}_1),\mathbf{coSem}(\mathcal{D}_0)}]{} & \mathbf{coSem}^2(\mathcal{D}_0)
\end{array}
$$

that we unveil into the following diagram:

$$
\begin{array}{ccccc}
\mathbf{coMon}(\mathcal{D}_1) & \xrightarrow{\ U_{\mathbf{ir}\,\mathcal{D}_1}\ } & \mathbf{coSem}(\mathcal{D}_1) & \xrightarrow{\ \mathbf{coSem}(U_{10})\ } & \mathbf{coSem}(\mathcal{D}_0) \\
\downarrow{\scriptstyle \delta^{\mathbf{i}}_{\mathcal{D}_1}} & & \downarrow{\scriptstyle \delta^{\mathbf{r}}_{\mathcal{D}_1}} & & \downarrow{\scriptstyle \delta^{\mathbf{r}}_{\mathcal{D}_0}} \\
\mathbf{coMon}^2(\mathcal{D}_1) & \xrightarrow[U_{\mathbf{ir}\,\mathcal{D}_1}^2]{} & \mathbf{coSem}^2(\mathcal{D}_1) & \xrightarrow[\mathbf{coSem}^2(U_{10})]{} & \mathbf{coSem}^2(\mathcal{D}_0)
\end{array}
$$

where $U_{\mathbf{ir}\,\mathcal{D}_1}^2$ is

$$
\mathbf{coMon}^2(\mathcal{D}_1) \xrightarrow{\ U_{\mathbf{ir}\,\mathbf{coMon}(\mathcal{D}_1)}\ } \mathbf{coSem}(\mathbf{coMon}(\mathcal{D}_1)) \xrightarrow{\ \mathbf{coSem}(U_{\mathbf{ir}\,\mathcal{D}_1})\ } \mathbf{coSem}^2((\mathcal{D}_1))
$$

The left-hand side square commutes immediately, whereas the right-hand side square commutes by Lemma 5.3. Checking that $U_{\mathbf{ir}\,\mathcal{D}_1}$, $U_{\mathbf{ir}\,\mathcal{D}_1}^2$, $\mathbf{coSem}(U_{10})$ and $\mathbf{coSem}^2(U_{10})$ are 1-cells in $\mathbf{SymmMonCat}_{\mathrm{strict}}$ is routine. The axioms of a 2-comonad follow from Theorem 5.3.

$\square$

Lemma 6.22. Let $(\mathcal{C}, !_{\mathbf{i}}, !_{\mathbf{r}}, !_{\mathbf{a}}, \mu_{\mathbf{ir}}, \mu_{\mathbf{ia}})$ be a Cocteau category. Then there is a K-coalgebra from the Eilenberg-Moore 2-category $\mathbf{Fun}(\mathbf{FCocteau}, \mathbf{SymmMonCat}_{\mathrm{strict}})^{\mathsf{K}}$ associated with $\mathcal{C}$.

Proof. As far as $\mathcal{C}$ is a Cocteau category, so, by Theorem 6.11, we have the following commutative diagram of left adjoints of the corresponding Cocteau adjunction:

$$
\begin{array}{ccccc}
& & \mathcal{C}^{!_{\mathbf{i}}} & & \\
& {\scriptstyle \mathsf{r}}\swarrow & \downarrow{\scriptstyle \mathsf{l}_l} & \searrow{\scriptstyle \mathsf{a}} & \\
\mathcal{C}^{!_{\mathbf{r}}} & \xrightarrow[\mathsf{R}_l]{} & \mathcal{C} & \xleftarrow[\mathsf{A}_l]{} & \mathcal{C}^{!_{\mathbf{a}}}
\end{array}
\tag{18}
$$

where r and a are also strict symmetric. Thus (18) is a 1-cell in $\mathbf{Fun}(\mathbf{FCocteau}, \mathbf{SymmMonCat}_{\mathrm{strict}})$, that we will further denote as $\mathfrak{A}$. To make $\mathfrak{A}$ a coalgebra, we have got to provide the coalgebra action

$h_{\mathfrak{A}} : \mathfrak{A} \to \mathsf{K}\mathfrak{A}$, which is given by:

The above diagram commutes indeed since $\mathcal{C}^{!\mathbf{i}}$ ($\mathcal{C}^{!\mathbf{a}}$) is already Cartesian (semicartesian) isomorphic to the category of its cocommutative comonoids ($\mathcal{C}^{!\mathbf{a}}/\mathbb{1}$) via the corresponding forgetful functor, whereas $\mathcal{C}^{!\mathbf{r}}$ is relevant by Lemma 6.7, so the forgetful functor $\mathsf{U}_{\mathbf{r}\mathcal{C}^{!\mathbf{r}}}$ has a strict section $\mathsf{V}_{\mathbf{r}\mathcal{C}^{!\mathbf{r}}}$.

The coalgebra conditions for $h_{\mathfrak{A}}$ follow from Theorem 5.4. $\qquad\square$

Before formulating the final theorem, let us note that all the functors between Cocteau categories we consider are strict as monoidal functors, whereas in Theorem 6.18 all 1-cells are lax by default. So we formulate the following proposition refining that theorem:

Proposition 6.23. Let $\mathcal{C}, \mathcal{D}$ be SMCs and let S and T be symmetric lax monoidal comonads on $\mathcal{C}$ and $\mathcal{D}$ respectively. Let $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ be a symmetric lax monoidal functor inducing a comonad morphsim $(\mathcal{C}, \mathsf{S}) \to (\mathcal{D}, \mathsf{T})$, then F is strict if and only if $\mathsf{F} : \mathcal{C}^{\mathsf{S}} \to \mathcal{D}^{\mathsf{T}}$ is strict.

In the above lemma, we associated a K-coalgebra with every Cocteau category. However, the 2-category $\mathbf{Fun}(\mathbf{FCocteau}, \mathbf{SymmMonCat}_{\mathrm{strict}})$ has too many triangles, so we take only those coalgebras whose functors F_0, F_1 and F_2 (the choice of names as in 15) admit right adjoints: since they might not exist a priori in a general case, but we need them unveil comonads from Cocteau categories. Let $\mathbf{Fun}(\mathbf{FCocteau}, \mathbf{SymmMonCat}_{\mathrm{strict}})^{\mathsf{K}}_R$ denote such a 2-subcategory. Then we have the following.

Theorem 6.24. $\mathbf{Cocteau}$ fully faithfully embeds to $\mathbf{Fun}(\mathbf{FCocteau}, \mathbf{SymmMonCat}_{\mathrm{strict}})^{\mathsf{K}}_R$.

Proof. Let $(\mathcal{C}, !_{\mathbf{i}}, !_{\mathbf{r}}, !_{\mathbf{a}}, \mu_{\mathbf{ir}}, \mu_{\mathbf{ia}})$ be a Cocteau category. As it has been already discussed in Lemma 6.22, we have a K-coalgebra $(\mathfrak{C}, h_{\mathfrak{C}})$, where $\mathfrak{C}$ is a 0-cell of $\mathbf{Fun}(\mathbf{FCocteau}, \mathbf{SymmMonCat}_{\mathrm{strict}})$ as in (18). Let us fix that $\mathsf{I}^{\mathcal{C}}_l$, $\mathsf{R}^{\mathcal{C}}_l$ and $\mathsf{A}^{\mathcal{C}}_l$ (we will parametrise those functors with the underlying SMC further) already have right adjoints by Theorem 6.11 and this will allow us to stay within $\mathbf{Fun}(\mathbf{FCocteau}, \mathbf{SymmMonCat}_{\mathrm{strict}})^{\mathsf{K}}_R$ all the time, so let us fix it here.

Let us first show that a strict symmetric monoidal functor of Cocteau categories $\mathsf{F} \in \mathbf{Cocteau}(\mathcal{C}, \mathcal{D})$ extends to a coalgebra morphism $\mathsf{F} : (\mathfrak{C}, h_{\mathfrak{C}}) \to (\mathfrak{D}, h_{\mathfrak{D}})$, where $(\mathfrak{D}, h_{\mathfrak{D}})$ is a K-coalgebra obtained by applying Lemma 6.22 to $\mathcal{D}$. In other words, we need to construct a 1-cell $\mathsf{F} : \mathfrak{C} \to \mathfrak{D}$ in

$\mathbf{Fun}(\mathbf{FCocteau}, \mathbf{SymmMonCat}_{\text{strict}})$ making the following square commute:

$$
\begin{array}{ccc}
\mathfrak{C} & \xrightarrow{\ \ F\ \ } & \mathfrak{D} \\
{\scriptstyle h_{\mathfrak{C}}}\downarrow & & \downarrow{\scriptstyle h_{\mathfrak{D}}} \\
\mathsf{K}\mathfrak{C} & \xrightarrow[\ \ \mathsf{K}F\ \ ]{} & \mathsf{K}\mathfrak{D}
\end{array}
\tag{19}
$$

$F : \mathfrak{C} \to \mathfrak{D}$, in turn, is given by the following diagram:

In other words, a 1-cell F is given by a quadruple of functors $(\widehat{F}_{\mathbf{r}}, \widehat{F}_{\mathbf{i}}, \widehat{F}_{\mathbf{a}}, F)$, where the $\widehat{F}_{\mathbf{r}}$, $\widehat{F}_{\mathbf{i}}$ and $\widehat{F}_{\mathbf{a}}$ are functors of the corresponding Eilenberg-Moore categories obtained from the components of comonad morphisms $F_{\mathbf{r}} : F\,!_{\mathbf{r}} \Rightarrow \,!'_{\mathbf{r}}\,F$, $F_{\mathbf{i}} : F\,!_{\mathbf{i}} \Rightarrow \,!'_{\mathbf{i}}\,F$ and $F_{\mathbf{a}} : F\,!_{\mathbf{a}} \Rightarrow \,!'_{\mathbf{a}}\,F$ respectively.

Whereas $\mathsf{K}F : \mathsf{K}\mathfrak{C} \to \mathsf{K}\mathfrak{D}$ is obtained by instantiating parameters properly in (17). However, it is could be useful the reader to have an explicit visualisation:

So we have a 1-cell given by a quadruple:

$$
(\mathbf{coSem}(\widehat{F}_{\mathbf{r}}) \circ \mathsf{V}_{\mathbf{r}\mathcal{C}^{!\mathbf{r}}}, \mathbf{coMon}(\widehat{F}_{\mathbf{i}}) \circ \mathsf{V}_{\mathbf{i}\mathcal{C}^{!\mathbf{i}}}, (\widehat{F}_{\mathbf{a}})/\mathbb{1} \circ \mathsf{V}_{\mathbf{a}\mathcal{C}^{!\mathbf{a}}}, F)
\tag{21}
$$

where $\mathsf{V}_{\mathbf{i}\mathcal{C}^{!\mathbf{i}}}$ and $\mathsf{V}_{\mathbf{a}\mathcal{C}^{!\mathbf{a}}}$ are inverses to the forgetful functors $\mathsf{U}_{\mathbf{i}\mathcal{C}^{!\mathbf{i}}}$ and $\mathsf{U}_{\mathbf{a}\mathcal{C}^{!\mathbf{a}}}$ respectively.

$V_{\mathbf{r}_{\mathcal{C}^{!\mathbf{i}}}}$ is also a coalgebra action, so the following diagrams commute in $\mathbf{SymmMonCat}_{\mathrm{strict}}$:

$$
\begin{array}{ccc}
\mathcal{C}^{!\mathbf{r}} & \xrightarrow{\;\widehat{F_{\mathbf{r}}}\;} & \mathcal{D}^{!'}_{\mathbf{r}} \\
{\scriptstyle V_{\mathbf{r}_{\mathcal{C}^{!\mathbf{i}}}}}\downarrow & & \downarrow{\scriptstyle V_{\mathbf{r}_{\mathcal{D}^{!'}_{\mathbf{i}}}}} \\
\mathbf{coSem}(\mathcal{C}^{!\mathbf{r}}) & \xrightarrow{\;\mathbf{coSem}(\widehat{F_{\mathbf{r}}})\;} & \mathbf{coSem}(\mathcal{D}^{!'}_{\mathbf{r}})
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{C}^{!\mathbf{i}} & \xrightarrow{\;\widehat{F_{\mathbf{i}}}\;} & \mathcal{D}^{!'}_{\mathbf{i}} \\
{\scriptstyle V_{\mathbf{i}_{\mathcal{C}^{!\mathbf{i}}}}}\downarrow & & \downarrow{\scriptstyle V_{\mathbf{i}_{\mathcal{D}^{!'}_{\mathbf{i}}}}} \\
\mathbf{coMon}(\mathcal{C}^{!\mathbf{i}}) & \xrightarrow{\;\mathbf{coMon}(\widehat{F_{\mathbf{i}}})\;} & \mathbf{coMon}(\mathcal{D}^{!'}_{\mathbf{i}})
\end{array}
$$

$$
\begin{array}{ccc}
\mathcal{C}^{!\mathbf{a}} & \xrightarrow{\;\widehat{F_{\mathbf{a}}}\;} & \mathcal{D}^{!'}_{\mathbf{a}} \\
{\scriptstyle V_{\mathbf{a}_{\mathcal{C}^{!\mathbf{a}}}}}\downarrow & & \downarrow{\scriptstyle V_{\mathbf{a}_{\mathcal{D}^{!'}_{\mathbf{a}}}}} \\
\mathcal{C}^{!\mathbf{a}}/\mathbb{1} & \xrightarrow{\;(\widehat{F_{\mathbf{a}}})/\mathbb{1}\;} & \mathcal{D}^{!'}_{\mathbf{a}}/\mathbb{1}
\end{array}
$$

Therefore, we have:

$$
\mathsf{K}\widehat{F}\circ h_{\mathfrak{C}} = (\mathbf{coSem}(\widehat{F_{\mathbf{r}}})\circ V_{\mathbf{r}_{\mathcal{C}^{!\mathbf{r}}}}, \mathbf{coMon}(\widehat{F_{\mathbf{i}}})\circ V_{\mathbf{r}_{\mathcal{C}^{!\mathbf{r}}}}, (\widehat{F_{\mathbf{a}}})/\mathbb{1}\circ V_{\mathbf{a}_{\mathcal{C}^{!\mathbf{a}}}}, F) =
$$
$$
(V_{\mathbf{r}_{\mathcal{D}^{!'}_{\mathbf{r}}}}\circ\widehat{F_{\mathbf{r}}}, V_{\mathbf{i}_{\mathcal{D}^{!'}_{\mathbf{i}}}}\circ\widehat{F_{\mathbf{i}}}, V_{\mathbf{a}_{\mathcal{D}^{!'}_{\mathbf{a}}}}\circ\widehat{F_{\mathbf{a}}}, F) = \widehat{F}\circ h_{\mathfrak{D}}
$$

Checking that the mapping $F \mapsto \widehat{F}$ is functorial is immediate.

Now let $F, G \in \mathbf{Cocteau}(\mathcal{C}, \mathcal{D})$ and let $\theta : F \Rightarrow G$ be a 2-cell in $\mathbf{Cocteau}$. One needs to construct a 2-cell $\hat{\theta} : \widehat{F} \Rightarrow \widehat{G}$ in the 2-category $\mathbf{Fun}(\mathbf{FCocteau}, \mathbf{SymmMonCat}_{\mathrm{strict}})^{\mathsf{K}}_R$. As we have already showed above, $\widehat{F}$ and $\widehat{G}$ are already 1-cells in $\mathbf{Fun}(\mathbf{FCocteau}, \mathbf{SymmMonCat}_{\mathrm{strict}})^{\mathsf{K}}_R$.

So we first need to check that the following diagram commutes in $\mathbf{SymmMonCat}_{\mathrm{strict}}$:

(22)

where $\widehat{F}_s$ and $\widehat{G}_s$ for $s \in \{\mathbf{i}, \mathbf{r}, \mathbf{a}\}$ are the induced functors of the Eilenberg-Moore categories given as follows for $(A, h_A) \in \mathcal{C}^{!_s}$:

$$
\widehat{F}_s : (A, h_A) \mapsto (FA, F_{sA}\circ h_A)
$$
$$
\widehat{G}_s : (A, h_A) \mapsto (GA, G_{sA}\circ h_A)
$$

and both $\widehat{\mathsf{F}}_s$ and $\widehat{\mathsf{G}}_s$ are strict symmetric since F and G are. Further, the following square (and the corresponding ones for $\mathbf{r}$ and $\mathbf{a}$):

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ \mathsf{F}\ \Downarrow\theta\ \mathsf{G}\ } & \mathcal{D} \\
\Big\uparrow{\mathsf{I}_l^{\mathcal{C}}} & & \Big\uparrow{\mathsf{I}_l^{\mathcal{D}}} \\
\mathcal{C}^{!\mathbf{i}} & \xrightarrow{\ \widehat{\mathsf{F}}_{\mathbf{i}}\ \Downarrow\theta\ \widehat{\mathsf{G}}_{\mathbf{i}}\ } & \mathcal{D}^{!'\mathbf{i}}
\end{array}
$$

commutes by Theorem 6.18 and Proposition 6.23, so does (22). Checking that (22) respects the structure of K-coalgebras is routine. The C be a 2-functor mapping each Cocteau category to the coalgebra $(\mathfrak{C}, h_{\mathfrak{C}})$. To complete the proof of the theorem we must show that The functor $\mathsf{C} : \mathcal{C} \mapsto$ is fully faithful on 1-cells and 2-cells. Let $(\mathcal{C}, !_{\mathbf{i}}, !_{\mathbf{r}}, !_{\mathbf{a}}, \mu_{\mathbf{ir}}, \mu_{\mathbf{ia}})$ and $(\mathcal{D}, !'_{\mathbf{i}}, !'_{\mathbf{r}}, !'_{\mathbf{a}}, \mu'_{\mathbf{ir}}, \mu'_{\mathbf{ia}})$ be Cocteau categories and let $\mathsf{F}, \mathsf{G} : \mathcal{C} \to \mathcal{D}$ be 1-cells in **Cocteau** such that $\mathsf{C}(\mathsf{F}) = \mathsf{C}(\mathsf{G})$.

We need $\mathsf{F} = \mathsf{G}$ as the equality of 1-cells in **Cocteau**, that is, the equality of underlying functors $\mathsf{F}, \mathsf{G} : \mathcal{C} \to \mathcal{D}$ as well as the equality of the induced symmetric monoidal comonad morphisms. $\mathsf{C}(\mathsf{F}) = \mathsf{C}(\mathsf{G})$, in turn, means, the following equalities are the case: $\mathsf{F} = \mathsf{G}$, $\widehat{\mathsf{F}}_{\mathbf{i}} = \widehat{\mathsf{G}}_{\mathbf{i}}$, $\widehat{\mathsf{F}}_{\mathbf{r}} = \widehat{\mathsf{G}}_{\mathbf{r}}$ and $\widehat{\mathsf{F}}_{\mathbf{a}} = \widehat{\mathsf{G}}_{\mathbf{a}}$. The equality $\mathsf{F} = \mathsf{G}$ is already satisfied, whereas we have $\mathsf{F}_s = \mathsf{G}_s$ for $s \in \{\mathbf{i}, \mathbf{r}, \mathbf{a}\}$ by Theorem 6.18.

Let $\mathcal{C}$ and $\mathcal{D}$ be a Cocteau categories, so we have coalgebras $(\mathfrak{C}, h_{\mathfrak{C}})$ and $(\mathfrak{D}, h_{\mathfrak{D}})$ as in Lemma 6.22. Let $\mathfrak{F} : \mathfrak{C} \to \mathfrak{D}$ be a coalgebra morphism. As a 1-cell in **Fun**(**FCocteau**, **SymmMonCat**$_{\mathrm{strict}}$), $\mathfrak{F}$ is given as follows.

$$(23)$$

We must show that F provides a 1-cell in **Cocteau**, that is, F satisfies Definition 6.17. Consider the case of exponential and relevant comonads, the affine part is considered similarly.

First of all, the following square commutes:

$$
\begin{array}{ccc}
\mathcal{C}^{!\mathbf{r}} & \xrightarrow{\ \mathsf{F}_0\ } & \mathcal{D}^{!'\mathbf{r}} \\
\Big\downarrow{\mathsf{R}_l} & & \Big\downarrow{\mathsf{R}'_l} \\
\mathcal{C} & \xrightarrow{\ \mathsf{F}\ } & \mathcal{D}
\end{array}
$$

so, by Theorem 6.18, Proposition 3.22 and Proposition 6.23, we have a symmetric lax monoidal comonad morphism $(\mathsf{F}, \widehat{\mathsf{F}_0}) : (\mathcal{C}, !_{\mathbf{r}}) \Rightarrow (\mathcal{D}, !'_{\mathbf{r}})$ with the component $\widehat{\mathsf{F}_0} : \mathsf{F}\, !_{\mathbf{r}} \Rightarrow !'_{\mathbf{r}}\, \mathsf{F}$ induced by F_0. So we let $\mathsf{F}_{\mathbf{r}} := \widehat{\mathsf{F}_0}$. Similarly, we get $\mathsf{F}_{\mathbf{i}} := \widehat{\mathsf{F}_1}$ and $\mathsf{F}_{\mathbf{a}} := \widehat{\mathsf{F}_2}$.

Let us check the rest of the conditions.

1. Let us check that (13) is satisfied.

   On the one hand, we have the following left morphism of adjunctions:

$$
\begin{array}{ccccc}
\mathcal{C}^{!_{\mathbf{i}}} & \xrightarrow{\ \mathsf{F}_1\ } & \mathcal{D}'^{!'_{\mathbf{i}}} & \xrightarrow{\ r'\ } & \mathcal{D}'^{!'_{\mathbf{r}}} \\
\downarrow{\scriptstyle \mathsf{I}_l} & & \downarrow{\scriptstyle \mathsf{I}'_l} & & \downarrow{\scriptstyle \mathsf{R}'_l} \\
\mathcal{C} & \xrightarrow{\ \mathsf{F}\ } & \mathcal{D} & =\!=\!= & \mathcal{D}
\end{array}
$$

   which gives us the comonad morphism $\mu'_{\mathbf{ir}}\mathsf{F}\circ\mathsf{F_i}$ by Theorem 6.18, Proposition 3.22 and Proposition 6.23.

   On the other hand, we have:

$$
\begin{array}{ccccc}
\mathcal{C}^{!_{\mathbf{i}}} & \xrightarrow{\ r\ } & \mathcal{C}^{!_{\mathbf{r}}} & \xrightarrow{\ \mathsf{F}_0\ } & \mathcal{D}'^{!'_{\mathbf{r}}} \\
\downarrow{\scriptstyle \mathsf{I}_l} & & \downarrow{\scriptstyle \mathsf{R}_l} & & \downarrow{\scriptstyle \mathsf{R}'_l} \\
\mathcal{C} & =\!=\!= & \mathcal{C} & \xrightarrow{\ \mathsf{F}\ } & \mathcal{D}
\end{array}
$$

   which, in turn, induces the comonad morphism $\mathsf{F_r}\circ\mathsf{F}\mu_{\mathbf{ir}}$ by the same statements. Finally, the following square

$$
\begin{array}{ccc}
\mathcal{C}^{!_{\mathbf{i}}} & \xrightarrow{\ \mathsf{F}_1\ } & \mathcal{D}'^{!'_{\mathbf{i}}} \\
\downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r'} \\
\mathcal{C}^{!_{\mathbf{r}}} & \xrightarrow{\ \mathsf{F}_0\ } & \mathcal{D}'^{!'_{\mathbf{r}}}
\end{array}
$$

   commutes by (23), and, thus, the composites $\mathsf{F_r}\circ\mathsf{F}\mu_{\mathbf{ir}}$ and $\mu'_{\mathbf{ir}}\mathsf{F}\circ\mathsf{F_i}$ are equal.

2. Now let us check that the functor $\widehat{\mathsf{F}}_{\mathbf{r}}:\mathcal{C}^{!_{\mathbf{r}}}\to\mathcal{D}'^{!'_{\mathbf{r}}}$ (which coincides with F_0) preserves diagonals. The following square already commutes by the condition that $\mathfrak{F}$ is a K-coalgebra morphism:

$$
\begin{array}{ccc}
\mathcal{C}^{!_{\mathbf{r}}} & \xrightarrow{\qquad\ \mathsf{F}\ \qquad} & \mathcal{D}'^{!'_{\mathbf{r}}} \\
\downarrow{\scriptstyle \mathsf{V}_{\mathbf{r}\,\mathcal{C}^{!_{\mathbf{r}}}}} & & \downarrow{\scriptstyle \mathsf{V}_{\mathbf{r}\,\mathcal{D}'^{!'_{\mathbf{r}}}}} \\
\mathbf{coSem}(\mathcal{C}^{!_{\mathbf{r}}}) & \xrightarrow{\ \mathbf{coSem}(\mathsf{F}_0)\ } & \mathbf{coSem}(\mathcal{D}'^{!'_{\mathbf{r}}})
\end{array}
\tag{24}
$$

   Let us discuss what each of the composites does. Let (A,h_A) be a coalgebra in $\mathcal{C}^{!_{\mathbf{r}}}$. Then we have a coalgebra $(\mathsf{F}A,\mathsf{F}_{\mathbf{r}A}\circ\mathsf{F}(h_A))$ in $\mathcal{D}'^{!'_{\mathbf{r}}}$. Finally, we endow the $(\mathsf{F}A,\mathsf{F}_{\mathbf{r}A}\circ\mathsf{F}(h_A))$ with the uniform cloning operation $\gamma_{\mathsf{F}A}:\mathsf{F}A\to\mathsf{F}A\otimes\mathsf{F}A$ as follows:

$$
\mathsf{F}A \xrightarrow{\ \mathsf{F}(h_A)\ } \mathsf{F}(!_{\mathbf{r}}A) \xrightarrow{\ \mathsf{F}_{\mathbf{r}A}\ } !'_{\mathbf{r}}\mathsf{F}A \xrightarrow{\ \mathsf{c}'^{\mathbf{r}}_{\mathsf{F}A}\ } !'_{\mathbf{r}}\mathsf{F}A\otimes !'_{\mathbf{r}}\mathsf{F}A \xrightarrow{\ \varepsilon^{\mathbf{r}}_{\mathsf{F}A}\otimes\varepsilon^{\mathbf{r}}_{\mathsf{F}A}\ } \mathsf{F}A\otimes\mathsf{F}A
\tag{25}
$$

   On the other hand, we take (A,h_A) in $\mathcal{C}^{!_{\mathbf{r}}}$ and endow it with the uniform cloning operation γ_A as follows:

$$
A \xrightarrow{\ h_A\ } !_{\mathbf{r}}A \xrightarrow{\ \mathsf{c}^{\mathbf{r}}_A\ } !_{\mathbf{r}}A\otimes !_{\mathbf{r}}A \xrightarrow{\ \varepsilon^{\mathbf{r}}_A\otimes\varepsilon^{\mathbf{r}}_A\ } A\otimes A
$$

Further, we take a cosemigroup (A, h_A, γ_A) and transform it to a cosemigroup in $\mathcal{D}^{!'_{\mathbf{r}}}$ as follows. The carrier is a coalgebra $(\mathsf{F}A, \mathsf{F}_{\mathbf{r}A} \circ \mathsf{F}(h_A))$ with the uniform cloning operation $\mathsf{F}(\gamma_A)$. But $\mathsf{F}(\gamma_A) = \gamma_{\mathsf{F}A}$ by (24).

3. Let us show that $\mathcal{C}$ is full and faithful on 2-cells. Let $\mathsf{G}, \mathsf{H} : \mathcal{C} \to \mathcal{D}$ be functors of Cocteau categories satisfying Definition 6.17 and let $\theta, \theta' : \mathsf{G} \Rightarrow \mathsf{H}$ be natural transformations such that $\theta, \theta' \in \mathbf{Cocteau}(\mathsf{G}, \mathsf{H})$ and $\mathsf{C}(\theta) = \mathsf{C}(\theta')$. We need θ and θ' to equal as 2-cells in $\mathbf{Cocteau}$.

$\mathsf{C}(\theta) = \mathsf{C}(\theta')$ means that there are the following equalities already satisfied. We have $\theta = \theta'$ as the equality of 2-cells in $\mathbf{SymmMonCat}_{\mathrm{strict}}$. And we also have $\widehat{\theta_{\mathbf{i}}} = \widehat{\theta'_{\mathbf{i}}}$, $\widehat{\theta_{\mathbf{r}}} = \widehat{\theta'_{\mathbf{r}}}$ and $\widehat{\theta_{\mathbf{a}}} = \widehat{\theta'_{\mathbf{a}}}$. But we conclude $\theta_{\mathbf{i}} = \theta'_{\mathbf{i}}$, $\theta_{\mathbf{r}} = \theta'_{\mathbf{r}}$ and $\theta_{\mathbf{a}} = \theta'_{\mathbf{a}}$ from Theorem 6.18 proving that all those θ's are symmetric lax monoidal comonad transformations.

The faithfulness part is slightly more complicated. Let $(\mathcal{C}, !_{\mathbf{i}}, !_{\mathbf{r}}, !_{\mathbf{a}}, \mu_{\mathbf{ir}}, \mu_{\mathbf{ia}})$ and $(\mathcal{D}, !'_{\mathbf{i}}, !'_{\mathbf{r}}, !'_{\mathbf{a}}, \mu'_{\mathbf{ir}}, \mu'_{\mathbf{ia}})$ be Cocteau categories and let $\mathsf{F}, \mathsf{G} \in \mathbf{Cocteau}(\mathcal{C}, \mathcal{D})$, so F and G induce coalgebra morphisms $\mathsf{F}, \mathsf{G} : (\mathfrak{C}, h_{\mathfrak{C}}) \to (\mathfrak{D}, h_{\mathfrak{D}})$ and let $\theta : \mathsf{F} \Rightarrow \mathsf{G}$ be a 2-cell in $\mathbf{Fun}(\mathbf{FCocteau}, \mathbf{SymmMonCat}_{\mathrm{strict}})$ given as follows:

$$(26)$$

Apriori θ is a 2-cell in $\mathbf{SymmMonCat}$, but we have got to show that θ provides a 2-cell in $\mathbf{Cocteau}$. First of all, consider the following diagram:

$$(27)$$

We already know that F and G are equipped with symmetric lax monoidal comonad morphisms $(\mathsf{F}, \mathsf{F}_s) : (\mathcal{C}, !_s) \to (\mathcal{D}, !'_s)$ and $(\mathsf{G}, \mathsf{G}_s) : (\mathcal{C}, !_s) \to (\mathcal{D}, !'_s)$ respectively with the components $\mathsf{F}_s : \mathsf{F}!_s \Rightarrow !'_s \mathsf{F}$ and $\mathsf{G}_s : \mathsf{G}!_s \Rightarrow !'_s \mathsf{G}$ for $s \in \{\mathbf{i}, \mathbf{r}, \mathbf{a}\}$. By Theorem 6.18, Proposition 3.22

and Proposition 6.23, therefore θ induces symmetric lax monoidal comonad morphism transformation $(\mathsf{F}, \mathsf{F}_s) \Rightarrow (\mathsf{G}, \mathsf{G}_s)$. To finally check (14), observe that the following square already commutes by (26):

$$
\begin{array}{ccc}
\mathcal{C}^{!_\mathbf{i}} & \xrightarrow[\widehat{\mathsf{G_i}}]{\overset{\widehat{\mathsf{F_i}}}{\Downarrow \theta_\mathbf{i}}} & \mathcal{D}^{!'_\mathbf{i}} \\
{\scriptstyle r}\Big\downarrow & & \Big\downarrow{\scriptstyle r'} \\
\mathcal{C}^{!_\mathbf{r}} & \xrightarrow[\widehat{\mathsf{G_r}}]{\overset{\widehat{\mathsf{F_r}}}{\Downarrow \theta_\mathbf{r}}} & \mathcal{D}^{!'_\mathbf{r}}
\end{array}
\tag{28}
$$

Now consider the following prism:

$$\tag{29}$$

where laterial triangles commute since $\mathcal{C}$ and $\mathcal{D}$ induce Cocteau adjunctions by Theorem 6.11, the front square already commutes since it is (28), whereas the other two squares commute by the condition as well. Thus, we conclude that (14) by applying Theorem 6.18, Proposition 3.22 and Proposition 6.23.

$\square$

7 Further Work

There are several question that we would like to leave for further investigation.

As regards proof theory, we have proved the strong normalisation property the β-reduction relation only, but the strong normalisation for all proof normalisation rules is a problem requiring separate research. We conjecture that one can develop a technique from [Tro95, §8] or [Tro92, §21].

In Theorem 6.24, we fully faithfully embedded the category of Cocteau categories to the 2-category of functors and thier lax natural transformations, but this technique might require a further generalisation for an arbitrary subexponential signature and the category of all assemblages.

[Hos07], developing some ideas from [AHS02] introduces linear combinatory and *relational* linear combinatory algebras, that is, linear combinatory algebras equipped with a comonadic applicative endomorphism allowing one to construct linear-non-linear adjunctions between assemblies over a BCI-algebra and a partial combinatory algebra and, thus, describe realisability semantics for intuitionistic linear logic. Developing the realisability interpretation for $\mathbf{SILL}(\lambda)_3$ warrants further

research since the realisability semantics would refine the computational motivation standing behind subexponentials.

We have been investigating simply typed linear lambda calculi with subexponential types, but the determining of whether expanding dependent linear type theory in the fashion of, say, [Vák14] with subexponentials is of interest as well, but it also deserves a separate and thorough research.

8   Acknowledgements

References

[AHS02]   Samson Abramsky, Esfandiar Haghverdi, and Philip Scott. Geometry of interaction and linear combinatory algebras. *Mathematical Structures in Computer Science*, 12(5):625–665, 2002.

[AM10]   Marcelo Aguiar and Swapneel Arvind Mahajan. *Monoidal functors, species and Hopf algebras*, volume 29. Citeseer, 2010.

[AR94]   J. Adamek and J. Rosicky. *Locally Presentable and Accessible Categories*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1994.

[AT11]   Samson Abramsky and Nikos Tzevelekos. Introduction to categories and categorical logic. *New structures for physics*, pages 3–94, 2011.

[BBDPH93a]   Nick Benton, Gavin Bierman, Valeria De Paiva, and Martin Hyland. Linear λ-calculus and categorical models revisited. In *Computer Science Logic: 6th Workshop, CSL'92 San Miniato, Italy, September 28–October 2, 1992 Selected Papers 6*, pages 61–84. Springer, 1993.

[BBDPH93b]   Nick Benton, Gavin Bierman, Valeria De Paiva, and Martin Hyland. A term calculus for intuitionistic linear logic. In *Typed Lambda Calculi and Applications: International Conference on Typed Lambda Calculi and Applications TLCA'93 March, 16–18, 1993, Utrech, The Netherlands Proceedings 1*, pages 75–90. Springer, 1993.

[BBN+17]   Jean-Philippe Bernardy, Mathieu Boespflug, Ryan R Newton, Simon Peyton Jones, and Arnaud Spiwack. Linear haskell: practical linearity in a higher-order polymorphic language. *Proceedings of the ACM on Programming Languages*, 2(POPL):1–29, 2017.

[Bek00]   Tibor Beke. Sheafifiable homotopy model categories. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 129, pages 447–475. Cambridge University Press, 2000.

[Ben94]   Nick Benton. A mixed linear and non-linear logic: Proofs, terms and models. In *International Workshop on Computer Science Logic*, pages 121–135. Springer, 1994.

[Ben95]   Peter N Benton. Strong normalisation for the linear term calculus. *Journal of functional programming*, 5(1):65–80, 1995.

[Bie94]   G.M. Bierman. On intuitionistic linear logic. Technical Report UCAM-CL-TR-346, University of Cambridge, Computer Laboratory, August 1994.

[Bor94a]   Francis Borceux. *Handbook of categorical algebra: Categories and structures*, volume 1. Cambridge University Press, 1994.

[Bor94b]   Francis Borceux. *Handbook of categorical algebra: Categories and structures*, volume 2. Cambridge University Press, 1994.

[Bra21]   Edwin Brady. Idris 2: Quantitative type theory in practice. *arXiv preprint arXiv:2104.00480*, 2021.

[BW00]   Michael Barr and Charles Wells. *Toposes, triples, and theories.* Springer-Verlag, 2000.

[Cur05]   Pierre-Louis Curien. Introduction to linear logic and ludics, Part I. *arXiv preprint cs/0501035*, 2005.

[DJS93]   Vincent Danos, Jean-Baptiste Joinet, and Harold Schellinx. The structure of exponentials: Uncovering the dynamics of linear logic proofs. In *Computational Logic and Proof Theory: Third Kurt Gödel Colloquium, KGC'93 Brno, Czech Republic, August 24–27, 1993 Proceedings 3*, pages 159–171. Springer, 1993.

[EGHK18]   Patrik Eklund, J Gutiérrez García, Ulrich Höhle, and Jari Kortelainen. Semigroups in complete lattices. *Quant. Modul. Related Topics*, 54, 2018.

[EGNO15]   Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*, volume 205. American Mathematical Soc., 2015.

[EH62]   Beno Eckmann and Peter J Hilton. Group-like structures in general categories i multiplications and comultiplications. *Mathematische Annalen*, 145(3):227–255, 1962.

[Fox76]   Thomas Fox. Coalgebras and cartesian categories. *Communications in Algebra*, 4(7):665–667, 1976.

[Gir72]   Jean-Yves Girard. Interprétation fonctionnelle et élimination des coupures de l'arithmétique d'ordre supérieur. 1972.

[Gir87]   Jean-Yves Girard. Linear logic. *Theoretical computer science*, 50(1):1–101, 1987.

[Gir11]   Jean-Yves Girard. *The Blind Spot: lectures on logic.* European Mathematical Society, 2011.

[Har13]   Robin Hartshorne. *Algebraic geometry*, volume 52. Springer Science & Business Media, 2013.

[HM25]   Eliès Harington and Samuel Mimram. ∞-categorical models of linear logic. In *10th International Conference on Formal Structures for Computation and Deduction (FSCD 2025)*, pages 23–1. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2025.

[Hod93]   Wilfrid Hodges. *Model theory.* Cambridge University Press, 1993.

[Hos07]   Naohiko Hoshino. Linear realizability. In *International Workshop on Computer Science Logic*, pages 420–434. Springer, 2007.

[HV19]   Chris Heunen and Jamie Vicary. *Categories for Quantum Theory: an introduction.* Oxford University Press, 2019.

[Jac94]   Bart Jacobs. Semantics of weakening and contraction. *Annals of pure and applied logic*, 69(1):73–106, 1994.

[JY21]   Niles Johnson and Donald Yau. *2-dimensional categories.* Oxford University Press, USA, 2021.

[Kal20]   Dmitry Borisovich Kaledin. Adjunction in 2-categories. *Russian Mathematical Surveys*, 75(5):883, 2020.

[Kel06]   G Max Kelly. Doctrinal adjunction. In *Category Seminar: Proceedings Sydney Category Theory Seminar 1972/1973*, pages 257–280. Springer, 2006.

[KKNS19]   Max Kanovich, Stepan Kuznetsov, Vivek Nigam, and Andre Scedrov. Subexponentials in non-commutative linear logic. *Mathematical Structures in Computer Science*, 29(8):1217–1249, 2019.

[KKS17]   Max Kanovich, Stepan Kuznetsov, and Andre Scedrov. Undecidability of the lambek calculus with subexponential and bracket modalities. In *International Symposium on Fundamentals of Computation Theory*, pages 326–340. Springer, 2017.

[KS06]   G Max Kelly and Ross Street. Review of the elements of 2-categories. In *Category Seminar: Proceedings Sydney Category Theory Seminar 1972/1973*, pages 75–103. Springer, 2006.

[Lac05]   Stephen Lack. Limits for lax morphisms. *Applied Categorical Structures*, 13:189–203, 2005.

[Lac09]   Stephen Lack. A 2-categories companion. In *Towards higher categories*, pages 105–191. Springer, 2009.

[Laf88]   Yves Lafont. The linear abstract machine. *Theoretical Computer Science*, 59:157–180, 1988.

[Lur09]   Jacob Lurie. *Higher topos theory.* Princeton University Press, 2009.

[Mel09]   Paul-André Melliès. Categorical semantics of linear logic. *Panoramas et syntheses*, 27:15–215, 2009.

[Mel22]   Paul-André Melliès. A functorial excursion between algebraic geometry and linear logic. In *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 1–13, 2022.

[ML13]   Saunders Mac Lane. *Categories for the working mathematician*, volume 5. Springer Science & Business Media, 2013.

[MP89]   Mihály Makkai and Robert Paré. *Accessible Categories: The Foundations of Categorical Model Theory: The Foundations of Categorical Model Theory*, volume 104. American Mathematical Soc., 1989.

[MSWW23] Lachlan McPheat, Mehrnoosh Sadrzadeh, Hadi Wazni, and Gijs Wijnholds. Categorical vector space semantics for Lambek calculus with a relevant modality. *Compositionality*, 5, 2023.

[MT07] Paul-André Melliès and Nicolas Tabareau. Resource modalities in game semantics. In *22nd Annual IEEE Symposium on Logic in Computer Science (LICS 2007)*, pages 389–398. IEEE, 2007.

[MT10] Paul-André Melliès and Nicolas Tabareau. Resource modalities in tensor logic. *Annals of Pure and Applied Logic*, 161(5):632–653, 2010.

[NM09] Vivek Nigam and Dale Miller. Algorithmic specifications in linear logic with subexponentials. In *Proceedings of the 11th ACM SIGPLAN conference on Principles and practice of declarative programming*, pages 129–140, 2009.

[NOP17] Vivek Nigam, Carlos Olarte, and Elaine Pimentel. On subexponentials, focusing and modalities in concurrent systems. *Theoretical Computer Science*, 693:35–58, 2017.

[OPN15] Carlos Olarte, Elaine Pimentel, and Vivek Nigam. Subexponential concurrent constraint programming. *Theoretical Computer Science*, 606:98–120, 2015.

[Pet02] Zoran Petrić. Coherence in substructural categories. *Studia Logica*, 70(2):271–296, 2002.

[Poi22] Laurent Poinsot. Multipliers of a semigroup object in a monoidal category. *arXiv preprint arXiv:2206.01416*, 2022.

[Por08] Hans-E Porst. On categories of monoids, comonoids, and bimonoids. *Quaestiones Mathematicae*, 31(2):127–139, 2008.

[rel] Relevance monoidal categories. `https://ncatlab.org/nlab/show/relevance+monoidal+category`. Accessed: 05.06.2025.

[Rie17] Emily Riehl. *Category theory in context.* Courier Dover Publications, 2017.

[Ros90] Kimmo I. Rosenthal. *Quantales and their applications.* Longman Scientific & Technical, Essex, 1990.

[RV16] Emily Riehl and Dominic Verity. Homotopy coherent adjunctions and the formal theory of monads. *Advances in Mathematics*, 286:802–888, 2016.

[See87] Robert AG Seely. *Linear logic, *-autonomous categories and cofree coalgebras.* Ste. Anne de Bellevue, Quebec: CEGEP John Abbott College, 1987.

[Shu08] Michael A Shulman. Set theory for category theory. *arXiv preprint arXiv:0810.1279*, 2008.

[Str72] Ross Street. The formal theory of monads. *Journal of Pure and Applied Algebra*, 2(2):149–168, 1972.

[SU06] Morten Heine Sørensen and Pawel Urzyczyn. *Lectures on the Curry-Howard isomorphism.* Elsevier, 2006.

[Tro92]   Anne Sjerp Troelstra. *Lectures on Linear Logic.* Center for the Study of Language and Information Publications, 1992.

[Tro95]   Anne Sjerp Troelstra. Natural deduction for intuitionistic linear logic. *Annals of pure and applied logic*, 73(1):79–108, 1995.

[TS00]   Anne Sjerp Troelstra and Helmut Schwichtenberg. *Basic proof theory.* Number 43. Cambridge University Press, 2000.

[Vák14]   Matthijs Vákár. Syntax and semantics of linear dependent types. *arXiv preprint arXiv:1405.0033*, 2014.

[Wad90]   Philip Wadler. Linear types can change the world! In *Programming concepts and methods*, volume 3, page 5. North-Holland, Amsterdam, 1990.

[Wal05]   David Walker. Substructural type systems. *Advanced topics in types and programming languages*, pages 3–44, 2005.

[Yet90]   David N Yetter. Quantales and (noncommutative) linear logic. *The Journal of Symbolic Logic*, 55(1):41–64, 1990.

[Zag17]   Dimitri Zaganidis. Classifiers for monad morphisms and adjunction morphisms. *Theory and Applications of Categories*, 32(31):1050–1097, 2017.

[Zwa24]   Colin Zwanziger. The natural display topos of coalgebras. *arXiv preprint arXiv:2405.00498*, 2024.