Institut for Matematik og Datalogi
Syddansk Universitet, Odense

2 October 2015
JBJ

DM551 – 1. Exam assignment

Hand in by Friday Oct 30 14:15.

Rules

This is the first of two sets of problems which together with the oral exam in january constitute the exam in DM551. This first set of problems must be solved **individualy**; any collaboration will be considered exam fraud. Thus you are not allowed to show your solutions to fellow students and you may not discuss the solutions with others. On the other hand, you can learn a lot from discussing the problems with each other so you may do this to some extend, such as which methods can be used or similar problems from the book or excersise classes.

Remember that this (and the second set of assignment to follow later) counts as part of your exam, so do a good job and try to answer all questions carefully. It is important that you **argue so that the reader can follow your calculations and explanations**.

How to hand in your report

Your report must be handed in by Friday October 30 at 14.15 **both**

- as a **paper** copy to the instructor **and**
- as a **pdf** file via Blackboard. Formats other than PDF will not be accpted.

On the first page you must write your **name** and **CPR-number**.

Handin via Blackboard is done as follows:

- Choose “DM551, Algoritmer og Sandsynlighed, efterår 15”.
- click on “SDU Assignment” (in the left menu).
- Fill in the form and attach your PDF file. Finish by clicking ’submit’
- Blackboard will send you a receipt by email.

1 Exam problems

Solve the following problems. **Remember to justify all answers.**

Problem 1

Consider a round robin soccer tournament between n teams. That is, every team plays every other team once and we assume there are no ties possible so one of the two teams wins.

(a) How many matches are played in total when there are n teams?

(b) What is the maximum $k = k(n)$ (meaning k depends on n) such that, no matter what the outcome of the individual games is, there is always some team has won at least k games? Give an example for odd n which shows that this value of k is indeed maximum.

(c) Suppose no team scores more than 5 goals in any match. Determine the minimum value of n such that two matches must have exactly the same score.

(d) Suppose that the tournament is played over 4 months. How many teams must there be in the tournament to ensure that there are 3 teams such that all the 3 matches between these were played in the first 2 months or all were played in the last 2 months. Explain how you reach your answer.

Problem 2

This problem is about generating permutations, combinations and bit strings as in Rosen Section 6.6. If you did not do it already, you should implement (in Java, python or similar) the algorithms 1,2 and 3 from Rosen as well as algorithms that use these to generate the following.

- All permutations of a given set containing n elements $a_1, a_2, \ldots, a_n$
- All bit strings of length n
- All r combinations of an n set

Let $S = \{a, b, c, d, e\}$. Use the algorithms above to do the following:

(a) List all permutations of the first 4 elements of S in lexicographic order.

(b) List all non-empty subsets of S in lexicographic order.

(c) List all 3-combinations of S in lexicographic order.

(d) Which is the lexicographically next larger 4-combination after $\{a, b, d, e\}$?

You must hand in a printout of your programs and their output.

Problem 3

Prove that for all non-negative integers n we have

$$\sum_{k=0}^{n} \binom{n}{k} 3^k 2^{n-k} = \sum_{k=0}^{n} \binom{n}{k} 4^n$$

Problem 4

Some definitions:

- A **2-partition** of a set C is any collection of disjoint sets A, B such that $A \cup B = C$. We say that a pair of 2-partitions A_1, B_1 and A_2, B_2 of the same set C are identical if $A_1 = A_2$ or $A_1 = B_2$ holds.
- Let $G = (V, E)$ be a graph. A **spanning** subgraph of G is a graph $H = (V, E')$ on the same vertex set V as G and a subset $E' \subseteq E$ of the edges of G.
- A graph $B = (V, E)$ is **bipartite** is its vertex set V has a 2-partition V_1, V_2, such that every edge $uv \in E$ has $u \in V_1, v \in V_2$ or $u \in V_2, v \in V_1$.
- A spanning bipartite subgraph $H = (V, E')$ of a graph $G = (V, E)$ corresponds to a 2-partition of V into disjoint sets $U, W = V - U$ such that every edge $e \in E'$ has precisely one vertex in U and the other in W. Note that there may be several choices for the sets U, W if H is not connected.

The main goal of the assignment is to use the probabilistic method to prove the following.

Theorem 1 *Every graph $G = (V, E)$ contains a spanning bipartite subgraph $H^* = (V, E^*)$ such that $|E^*| \geq |E|/2$*

You should prove this by following the steps (a)-(e) below and answering each question on the way.

Consider the following randomized algorithm $\mathcal{A}$: for each vertex $v \in V$ put v in U with probability 1/2 (e.g. by tossing a fair coin once for each vertex in V and if it comes up 'head' in the toss for v, we put v in U, otherwise we put v in W).

Before proving the theorem, you should first answer a few questions deal with 2-partitions.

(i) Let S be the set of all possible distinct 2-partitions of V. Show that $|S| = 2^{|V|-1}$.

(ii) Prove that $\mathcal{A}$ generates a random 2-partition of V, that is, the probability that a certain 2-partition (U, W) is generated is $1/|S|$.

(iii) Prove that the expected size of each set U, W is $|V|/2$. Hint: use indicator random variables.

We now go to the proof of the theorem and for this we study the resulting bipartite graph H whose vertex partition is the output U, W from $\mathcal{A}$ and whose edge set E' consists of those edges $e \in E$ for which precisely one end vertex of e is in U. Below $s \in S$ refers to the random 2-partition U, W that we generated above.

(a) Argue that the probability that an edge uv belongs to E' is precisely 1/2.

(b) Define, for each edge e, the indicator random variable $X_e = X_e(s)$ so that $X_e(s) = 1$ if e has one end in U and the other in W and $X_e(s) = 0$ if both end vertices of e are in U or both are in W. Prove that the expected value of X_e is 1/2.

(c) Define the random variable $X = X(s)$ to be $X(s) = |E'|$. Show that $X(s) = \sum_{e \in E} X_e(s)$.

(d) Use (c) to prove that the expected value of X is given by $E(X) = |E|/2$.

(e) We saw above that the expected value (size of $|E'|$) of a random 2-partition is $|E|/2$. Use this and the general formula for the expected value of $X(s)$ to prove that there is at least one 2-partition with $|E'| \geq |E|/2$. Hint: suppose all values are less than $|E|/2$ and derive a contradiction to (d).

Problem 5

(a) Find the number of non-negative integer solutions to $x_1 + x_2 + x_3 = 10$

(b) Solve the problem above with the extra condition that $x_1 \geq 3, x_3 \geq 2$.

(c) In how many ways can one distribute 10 identical balls into 3 distinct boxes such that no box contains more than 5 balls?

Problem 6

This problem is about distribution of distinct balls in distinct boxes

(a) In how many ways can we distribute 12 distinct balls in 4 distinct boxes so that no box is empty?

(b) In how many ways can we distribute 12 distinct balls in 4 distinct boxes so that precisely one box is empty?

(c) Suppose now that we have 12 red and 12 blue balls so that all 24 balls are distinct (can be recognized from each other). Consider an experiment where the 24 balls are distributed randomly into 4 boxes (each ball has probability 1/4 of landing in each box). What is the probability that all boxes contain both a red and a blue ball?