Course MM529, Matematiske metoder

Plan for week 37 (lecture) and 38 (examinatorier)

General information

Next week, the lecture will be given by Søren Möller. Below, you find the first series of mandatory exercises. Please read the instructions carefully!

Topics of the lecture

Special Functions (P.6, P.7):
Multiplication of polynomials, rational functions, exponential function, injective/surjective/bijective functions, inverse functions, trigonometric functions

Sequences and convergence (9.1):
Sequences, limits of sequences, convergent and divergent sequences, limit laws for convergent sequences

Mandatory exercises

Here is the first series of mandatory exercises. You have to submit your solutions on blackboard until next Thursday, September 19, 9am. The electronic submission will not be possible after this deadline. For every group there will be an extra page for the submission, please upload your solution on the page of your group. Your solutions will be corrected by the teaching assistents. Each exercise is worth 20 points for a full and correct solution. For successfully passing the course you need to receive 50% of the possible points in each of the three quarters.

Mandatory exercises are meant as an individual work. You are welcome to work together with other students, but you must write up the solutions in your own words. It is not allowed just to copy the solutions of other students.

In your solution, you also have to document the way in which you obtained the solution, it is not sufficient just to write down the final result. Now the first two mandatory exercises follow:

(1) Let $S = \{a, \clubsuit, 1\}$ and $T = \{1, 2\}$.

a) Determine $S \cap T$, $S \cup T$, $S \setminus T$, and $T \setminus S$.

b) Determine $S \times S$ and $S \times T$.

c) Let $[0, 10) = \{x \in \mathbb{R} \ : \ 0 \leq x < 10\}$ and $S = [0, 10) \times \mathbb{N}$. Which of the following six objects are elements of S (and why/why not)? The empty set $\emptyset$, $(\pi, 4)$, $(\sqrt{2}, 10)$, 0, $(4, \pi)$, $\{1, 2\}$.

(2) Let $p(x) = x^n + x^{n-1} + x^{n-2} + \ldots + x^2 + x + 1$ the polynomial of degree n with all coefficients equal to one and $q(x) = x - 1$.

a) Calculate $p \cdot q$ and determine the degree and all coefficients of $p \cdot q$.

b) For $n = 3$, calculate the polynomials $p \circ q$, $q \circ p$, their respective degrees and all their coefficients.

Topics and exercises for examinatorier

(1) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{3x}$.

a) Restrict the codomain so that f is bijective.

b) Determine the inverse function f^{-1} and sketch the graphs of f and f^{-1}.

c) Justify the choice of f^{-1} by determining the composition $f^{-1} \circ f$ and $f \circ f^{-1}$.

(2) The following equality holds for every $b, x \in \mathbb{R}$, $b > 0$: $b^x = e^{x \cdot \ln b}$.

a) Verify the equality using the logarithm rules and that e^x and $\ln x$ are inverse functions.

b) Sketch the graphs of the functions $f_1(x) = x^{1000}$ and $f_2(x) = (1.1)^x$ and the graphs of the functions $g_1(x) = \ln x$, $g_2(x) = x$ and $h(x) = \frac{x}{\ln x}$, taking the largest possible domain. Which of the functions are injective, surjective, or bijective?

c) Find a large positive number k for which $f_1(k) < f_2(k)$ and explain, why $f_1(x) < f_2(x)$ for all $x \geq k$ (exponential growth beats polynomial growth!).

d) For the occurring functions determine a maximal domain and codomain for which the function is bijective and sketch the graph of the inverse function. Which functions are the inverses of f_1, f_2, g_1, g_2. Determine their domain and range.

(3) The Addition Theorems for trigonometric functions are:

$$\sin(\theta + \varphi) = \sin\theta\ \cos\varphi + \sin\varphi\ \cos\theta$$

$$\cos(\theta + \varphi) = \cos\theta\ \cos\varphi - \sin\theta\ \sin\varphi$$

a) Derive from the two Addition Theorems the corresponding statements for $\sin(\theta + \varphi)$ and $\cos(\theta + \varphi)$.

b) Exercises P.7: 8, 10.

(4) For the sequence $(a_n)_{n\in\mathbb{N}} = (\frac{1}{5}, \frac{4}{10}, \frac{9}{17}, \frac{16}{26}, \frac{25}{37}, \ldots)$ determine the construction rule and calculate the next elements and find a general description for a_n. What is

$$\lim_{n\to\infty} a_n?$$

(5) Consider the sequence $(a_n)_{n\in\mathbb{N}_0}$ where $a_n = 2^{-n} = \frac{1}{2^n}$.

a) Show that $a_n < \varepsilon$ holds for $\varepsilon > 0$ whenever $n > \log_2 \frac{1}{\varepsilon}$.

b) Use a) to justify that $\lim\limits_{n\to\infty} a_n = 0$.

c) Show that the limit of the sequence $(0, 9, 0, 99, 0, 999, 0.9999, \ldots)$ is 1.

d) Exercises 9.1: 14, 16, 20 (note that the textbook uses curly brackets to denote sequences)

Topics for studiegrupper (Studiecafé)

(1) Rational functions

Exercises P.6: 13, 15, 17, 19.

(2) Composition and inverse functions

Solve exercise P.5: 7. For the two functions f and g find a maximal domain and codomain such that the functions are bijective and determine the inverse functions f^{-1} and g^{-1}. Determine $g^{-1} \circ g$ and $g \circ g^{-1}$ and the domain and range of the two functions.

(3) Trigonometry review

Exercises P.7: 31, 35, 37.

(4) Sequences (note that the textbook uses curly brackets to denote sequences)

a) Discuss the definition of the limit of a sequence (p. 499 in the textbook), the illustration in Figure 9.1 and the proof in Example 4. Discuss the concepts of convergent sequences, divergent sequences and divergent sequences diverging to a limit $\pm\infty$.

b) Check the examples and definitions in the Wikipedia articles ”Sequence” and ”Limit of a sequence”.

c) Exercises 9.1: 15, 17. Compare your results with the solutions in the back of the textbook.

Adress any problems, questions, that you could not solve or remained unclear in the next exercise courses.