ADS2 — DP2 Exam Notes & Worked Solutions

Meta	Value
Title	ADS2 — Dynamic Programming II (Knapsack & Sequence Alignment)
Date	2025-10-18
Author	Generated from **system_prompt** + provided PDFs/images
Sources used	Weekplan **weekplan.pdf** (pp.1–2); Slides **DP2-4x1.pdf** (Knapsack & Sequence Alignment); Kleinberg–Tardos **§6.4** (pp.266–272) and **§6.6** (pp.278–284); Exercise sheet **exercise_6_8.pdf** (KT 6.8); images: *Pasted image.png*, *Pasted image (2).png*, *Pasted image (3).png*
Week plan filename	weekplan.pdf

General Methodology and Theory

- **DP recipe**: define subproblem → recurrence → base cases → evaluation order → table fill → witness recovery.
- **0/1 Knapsack** (slides-first): subproblem $$OPT(i,w)$$ = best value using first $$i$$ items within capacity $$w$$. Recurrence (slides): if $$w_i\le w$$ then $$OPT(i,w)=\max(OPT(i-1,w),\ v_i+OPT(i-1, w-w_i))$$; else $$OPT(i,w)=OPT(i-1,w)$$. Time/space $$O(nW)$$ (pseudo-polynomial).
- **Global sequence alignment** (Needleman–Wunsch, linear gap $$\delta$$): $$D(i,j)=\min{\alpha(x_i,y_j)+D(i-1,j-1),\ \delta+D(i-1,j),\ \delta+D(i,j-1)}$$ with $$D(i,0)=i\delta,\ D(0,j)=j\delta$$. Time/space $$O(mn)$$.
- **Design hygiene**: verify with small tables; state tie-breaks for determinism; trace back to produce a certificate.

Notes

- Slides are primary; textbook variants noted under *Alternative Approach* per exercise.
- Enumeration **must** follow the week plan; images supplement statements only.
- Keep tables concise: include DP values or tracebacks only where it clarifies the witness.

Coverage Table

Weekplan ID	Canonical ID	Title/Label (verbatim)	Assignment Source	Text Source	Status
1	KT §6.4 (Knapsack)	**[w] Knapsack** (items (5,7), (2,6), (3,3), (2,1), W=6)	weekplan.pdf p.1	weekplan.pdf p. 1; slides DP2-4x1 (Knapsack); KT §6.4	Solved
2	KT §6.6 (Alignment)	**[w] Sequence alignment** (APPLE vs PAPE, gap $$ \delta=2$$, matrix on {A,E,L,P})	weekplan.pdf p.1	weekplan.pdf p. 1; slides DP2-4x1 (Alignment); KT §6.6	Solved
3.1	CSES 1158	**Book Shop 3.1** — compute max pages	weekplan.pdf p.2	slides DP2-4x1 (Knapsack pattern)	Solved
3.2	CSES 1158	**Book Shop 3.2** — $$O(x)$$ space	weekplan.pdf p.2	slides DP2-4x1 (1-row DP)	Solved
3.3	CSES 1158	**Book Shop 3.3** — implement (site ref)	weekplan.pdf p.2	problem statement on CSES; pattern here	Solved (pseudocode)
4	—	**Longest palindrome subsequence**	weekplan.pdf p.2	standard LPS DP (course canon)	Solved
5	KT 6.8	**Defending Zion** (EMP scheduling)	weekplan.pdf p.2	exercise_6_8.pdf	Solved

Solutions

Exercise 1 — KT §6.4 — Knapsack

Assignment Source: weekplan.pdf p.1. **Text Source**: weekplan.pdf p.1; slides DP2-4x1; KT §6.4.

Instance: items $$i_1!(w{=}5,v{=}7),\ i_2!(2,6),\ i_3!(3,3),\ i_4!(2,1);\ W{=}6$$.

Recurrence (slides): as in *General Methodology*. Base row/col 0.

DP table (values) $$i\times w$$:

i\w	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	7	7
2	0	0	6	6	6	7	7
3	0	0	6	6	6	9	9
4	0	0	6	6	7	9	9

Witness (traceback): from $$(4,6)$$ → take $$i_3$$ (to $$(3,3)$$) → take $$i_2$$ (to $$(2,1)$$) → stop. **Picked** $${i_2,i_3}$$, weight 5, value 9.

Pitfalls: mixing 0/1 with unbounded; forgetting base row/col.

Variant Drill: If $$W=7$$, best is $${i_1,i_2}$$ with value 13.

Alternative Approach (KT): identical recurrence; value/weight emphasis differs.

Transfer Pattern: 0/1 knapsack. *Cues*: capacity $$W$$, each item at most once, additive value. *Mapping*: nouns → $$(w_i,v_i)$$, limit → $$W$$. *Certificate*: subset with total weight $\leq$ $$W$$ achieving $$OPT(n,W)$$. *Anti-cues*: fractional or unlimited copies.

Pseudocode

```
Algorithm: knapsack_01
Input: n, arrays w[1..n], v[1..n], capacity W
Output: best value and one witness via keep[][]
for c=0..W: dp[0,c]←0
for i=1..n:
  for c=0..W:
    dp[i,c]←dp[i-1,c]
    if w[i]≤c and dp[i-1,c-w[i]]+v[i] > dp[i,c]:
      dp[i,c]←dp[i-1,c-w[i]]+v[i]; keep[i,c]←true
// traceback from (n,W)
// Time: O(nW); Space: O(nW)
```

✅**Answer**: $$OPT=9$$ with items $${(2,6),(3,3)}$$.

Exercise 2 — KT §6.6 — Sequence Alignment

Assignment Source: weekplan.pdf p.1. **Text Source**: weekplan.pdf p.1; slides DP2-4x1; KT §6.6.

Instance: $$X=\text{APPLE}$$ (columns), $$Y=\text{PAPE}$$ (rows). Gap $$\delta=2$$. Penalty matrix over $${A,E,L,P}$$ with 0 on diagonal and off-diagonals as in plan.

Recurrence: as in *General Methodology*. Init $$D(i,0)=2i, D(0,j)=2j$$. Tie-break: diag < up < left.

DP values (compact):

i\j	0	1	2	3	4	5
0	0	2	4	6	8	10
1	2	1	2	4	6	8
2	4	2	2	3	5	7
3	6	4	2	2	4	6
4	8	6	4	3	4	**4**

Traceback (one optimum):

```
X: A P P L E
Y: P A P - E
```

Cost: $$1+1+0+2+0=4$$.

Pitfalls: swapping X/Y; forgetting linear gap init.

Variant Drill: Lower $$\delta$$ to 1 → cheaper gaps; recompute to expect cost $$\le 3$$.

Alternative Approach: shortest path on grid graph with diagonal weights $$\alpha(·,·)$$ and horizontal/vertical $$\delta$$.

Transfer Pattern: global alignment (linear gap). *Cues*: substitution matrix + uniform gap. *Mapping*: chars → nodes; operations → DP moves. *Certificate*: paired strings with per-column costs summing to $$D(n,m)$$. *Anti-cues*: local alignment (resets to 0); affine gaps (three matrices).

Pseudocode

```
Algorithm: needleman_wunsch_linear
Input: strings X[1..m], Y[1..n], penalties α(·,·), gap δ
Output: cost D[n,m] and an alignment via backpointers
for j=0..m: D[0,j]←j·δ
for i=0..n: D[i,0]←i·δ
for i=1..n:
  for j=1..m:
    D[i,j]←min{ α(Y[i],X[j])+D[i-1,j-1], δ+D[i-1,j], δ+D[i,j-1] }
// traceback from (n,m)
// Time: O(mn); Space: O(mn)
```

✅**Answer**: minimum cost **4** with alignment shown.

Exercise 3.1 — CSES 1158 — Book Shop (max pages)

Assignment Source: weekplan.pdf p.2. **Text Source**: slides DP2-4x1 (knapsack pattern); CSES statement.

Mapping: price $$h_i$$ → weight; pages $$s_i$$ → value; budget $$x$$ → capacity. 0/1 knapsack.

Recurrence: $$DP(i,c)=\max(DP(i-1,c),\ s_i+DP(i-1,c-h_i))$$ for $$h_i\le c$$, else $$DP(i,c)=DP(i-1,c)$$. Base row 0.

Pseudocode

```
Algorithm: book_shop_max_pages
Input: n, price h[1..n], pages s[1..n], budget x
Output: maximum pages
for c=0..x: dp[0,c]←0
for i=1..n:
  for c=0..x:
    dp[i,c]←dp[i-1,c]
    if h[i]≤c and dp[i-1,c-h[i]]+s[i] > dp[i,c]:
      dp[i,c]←dp[i-1,c-h[i]]+s[i]
return dp[n,x]
// Time: O(nx); Space: O(nx)
```

Pitfalls: confusing price vs pages; exceeding budget.

✅**Answer**: Reduces exactly to 0/1 knapsack; table yields optimum pages within budget.

Exercise 3.2 — CSES 1158 — Book Shop in $$O(x)$$ space

Idea: 1-row DP scanning capacities **descending** to preserve 0/1 semantics.

Pseudocode

```
Algorithm: book_shop_one_row
Input: n, h[1..n], s[1..n], budget x
Output: maximum pages
for c=0..x: dp[c]←0
for i=1..n:
  for c=x down to h[i]:
    dp[c]←max(dp[c], s[i]+dp[c-h[i]])
return dp[x]
// Time: O(nx); Space: O(x)
```

Why descending? Prevents reusing the same book multiple times.

✅**Answer**: Achieves $$O(x)$$ space with identical optimal value.

Exercise 3.3 — CSES 1158 — Implementation note

I/O micro-card: read $$n, x$$; arrays $$h[1..n], s[1..n]$$; print $$\text{book_shop_one_row}(...)$$ result.

Testing tips: include edge cases $$x{=}0$$, a single book equal to $$x$$, and many books with identical prices.

✅**Answer**: Use the one-row DP above; outputs the maximum total pages.

Exercise 4 — Longest Palindrome Subsequence (LPS)

Assignment Source: weekplan.pdf p.2.

Subproblem: $$L(i,j)$$ = length of LPS in substring $$s[i..j]$$.

Recurrence: - If $$i>j$$: 0; if $$i=j$$: 1. - If $$s[i]=s[j]$$: $$L(i,j)=2+L(i+1,j-1)$$. - Else: $$L(i,j)=\max(L(i+1,j), L(i,j-1))$$.

Pseudocode

```
Algorithm: lps_length
Input: string s[1..n]
Output: length of an LPS
for i=1..n: dp[i,i]←1
for len=2..n:
  for i=1..n-len+1:
    j←i+len-1
    if s[i]=s[j]: dp[i,j]←2 + (dp[i+1,j-1] if i+1≤j-1 else 0)
    else: dp[i,j]←max(dp[i+1,j], dp[i,j-1])
return dp[1,n]
// Time: O(n^2); Space: O(n^2) → can be reduced to O(n) for length only on
diagonals
```

Correctness sketch: last pair either matches (use both ends) or not (drop one end). Standard interval-DP.

Witness: keep a parent direction to reconstruct one palindrome.

✅**Answer**: Recurrence and algorithm as above; length in $$O(n^2)$$.

Exercise 5 — KT 6.8 — Defending Zion (EMP scheduling)

Assignment Source: weekplan.pdf p.2. **Text Source**: exercise_6_8.pdf.

Model: arrivals $$x_1..x_n$$; recharge function $$f(j)$$ (power after $$j$$ seconds since last use). Using at time $$t$$ after $$j$$ seconds since previous use destroys $$\min(x_t, f(j))$$.

Counterexample to greedy (part a): Let $$x=[0,10,10,0]$$ and $$f=[_,1,3,8]$$ (i.e., $$f(1){=}1,f(2){=}3,f(3){=}8$$). Greedy "fire at $$t{=}4$$ with smallest $$j$$ s.t. $$f(j)\ge x_4$$" fires at $$t{=}4$$ with $$j{=}1$$ (kills 0), then recurses on $$[0,10,10]$$, missing the optimal plan: fire at $$t{=}3$$ (kills 8) and **don't** fire at 4 → total 8.

DP (part b) — *slides-style formulation over last-fire time*: - Let $$G[t]$$ = best robots destroyed up to time $$t$$ **with last activation exactly at $$t$$**. - Let $$F[t]$$ = best up to $$t$$ with last activation at $$\le t$$ (overall optimum prefix). - Base: $$G[0]=0,\ F[0]=0$$. - Transition for $$t\ge1$$: $$G[t]=\max_{0\le k < t}\big(G[k] + \min(x_t,\ f(t-k)) \big)$$ (previous last fire at time $$k$$; if $$k=0$$, it's the first fire with $$j=t$$). Then $$F[t]=\max(F[t-1],\ G[t])$$.

Pseudocode

```
Algorithm: emp_max_destroyed
Input: n, arrivals x[1..n], recharge f[1..n]
Output: max robots destroyed
G[0]←0; F[0]←0
for t=1..n:
  best←0
  for k=0..t-1:
    j←t-k
    best←max(best, G[k] + min(x[t], f[j]))
  G[t]←best
  F[t]←max(F[t-1], G[t])
return F[n]
// Time: O(n^2); Space: O(n)
```

Correctness: last activation time partitions the schedule; the gap length $$j$$ is determined; subproblems do not overlap in time.

Variant Drill: if $$f$$ is nondecreasing and concave, consider Knuth-/divide-&-conquer-style optimizations; otherwise stick to $$O(n^2)$$.

Transfer Pattern: segmented scheduling with state = time since last use. *Cues*: recharge curve, "since last" wording. *Mapping*: choose activation times; reward per activation depends on gap. *Certificate*: list of activation timestamps and per-use kills summing to optimum.

✅**Answer**: DP above returns the maximum robots destroyed; greedy fails.

Puzzle

Blind Man's Deck (10 face-up among 52): take **any 10 cards** as pile A; let the rest be pile B. Flip pile A. Now both piles have the **same** number of face-up cards. *Reason*: if pile A initially had $$k$$ face-up, B had $$10-k$$; flipping A makes exactly $$10-k$$ face-up in A.

Summary

- **Knapsack**: 0/1 DP with $$OPT(i,w)$$; pseudo-polynomial $$O(nW)$$; 1-row $$O(W)$$ space when only value is needed.
- **Alignment**: Needleman–Wunsch with linear gaps; initialize borders with cumulative gaps; traceback yields a certified alignment.
- **Book Shop**: direct 0/1 mapping (price→weight, pages→value); descending 1-row DP avoids reuse.
- **LPS**: interval DP over substrings; elegant two-case recurrence.
- **Zion (KT 6.8)**: schedule by DP on last-fire time; greedy counterexample; $$O(n^2)$$ time, $$O(n)$$ space.
- **Notation**: $$W, v_i, w_i,\ \delta,\ \alpha(·,·),\ D(i,j),\ L(i,j)$$. Always show a witness (subset, alignment, or schedule) to certify optimality.

Dynamic Programming II

Inge Li Gørtz

KT section 6.4 and 6.6

Thank you to Kevin Wayne for inspiration to slides

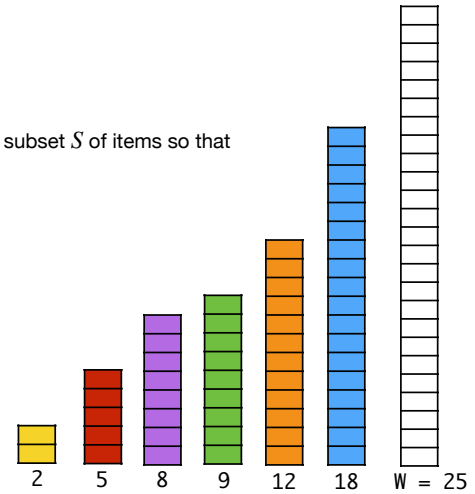
Dynamic Programming

- Optimal substructure
- Last time
  - Weighted interval scheduling
- Today
  - Knapsack
  - Sequence alignment

Subset Sum and Knapsack

Subset Sum

- Subset Sum
  - Given n items $\{1,\ldots,n\}$
  - Item i has weight w_i
  - Bound W
  - Goal: Select maximum weight subset S of items so that
    $$\sum_{i \in S} w_i \leq W$$
- Example
  - {2, 5, 8, 9, 12, 18} and W = 25.
  - Solution: 5 + 8 + 12 = 25.



Subset Sum

- Subset Sum
  - Given n items $\{1,\ldots,n\}$
  - Item i has weight w_i
  - Bound W
  - Goal: Select maximum weight subset S of items so that

$$\sum_{i \in S} w_i \leq W$$

- Example
  - {2, 5, 8, 9, 12, 18} and W = 25.
  - Solution: 5 + 8 + 12 = 25.

Subset Sum

- $\mathcal{O}$ = optimal solution
- Consider element n.
  - Either in $\mathcal{O}$ or not.
    - $n \notin \mathcal{O}$: Optimal solution using items $\{1,\ldots,n-1\}$ is equal to $\mathcal{O}$.
    - $n \in \mathcal{O}$: Value of $\mathcal{O}$ = w_n + weight of optimal solution on $\{1,\ldots,n-1\}$ with capacity $W - w_n$.
- Recurrence
  - OPT(i, w) = optimal solution on $\{1,\ldots,i\}$ with capacity w.
  - From above:

$$\text{OPT}(n, W) = \max(\text{OPT}(n-1, W), w_n + \text{OPT}(n-1, W-w_n))$$

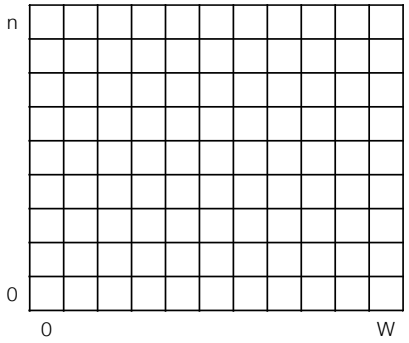
  - If $w_n > W$:

$$\text{OPT}(n, W) = \text{OPT}(n-1, W)$$

Subset Sum

- Recurrence:

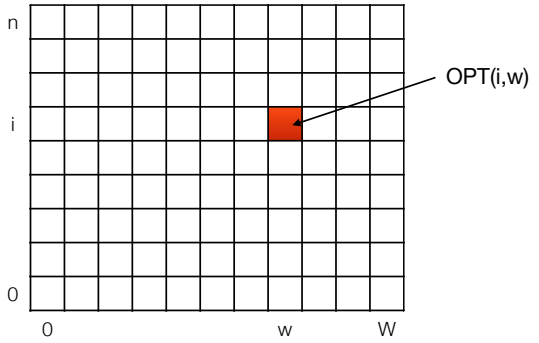
$$\text{OPT}(i, w) = \begin{cases} \text{OPT}(i-1, w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1, w), w_i + \text{OPT}(i-1, w-w_i)) & \text{otherwise} \end{cases}$$



Subset Sum

- Recurrence:

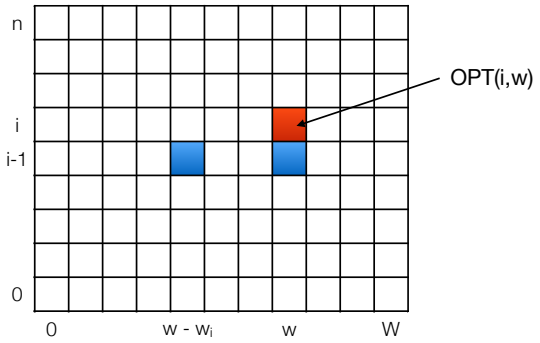
$$\text{OPT}(i, w) = \begin{cases} \text{OPT}(i-1, w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1, w), w_i + \text{OPT}(i-1, w-w_i)) & \text{otherwise} \end{cases}$$



Subset Sum

- Recurrence:

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w), w_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$



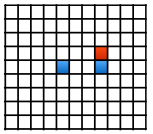
Subset Sum

- Recurrence:

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w), w_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

```
Array M[0…n][0…W]
Initialize M[0][w] = 0 for each w = 0,1,…,W
Subset-Sum(n,W)

Subset-Sum(i,w)
  if M[i][w] empty
    if w < wᵢ
      M[i][w] = Subset-Sum(i-1,w)
    else
      M[i][w] = max(Subset-Sum(i-1,w), wᵢ +
      Subsetsum(i-1,w-wᵢ))
  return M[i][w]
```



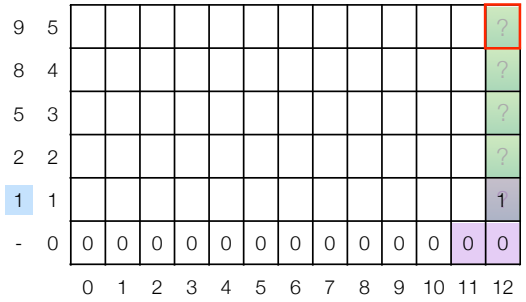
Subset Sum

- Recurrence:

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w), w_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

0 1 + 0

- Example
  - {1, 2, 5, 8, 9} and W = 12



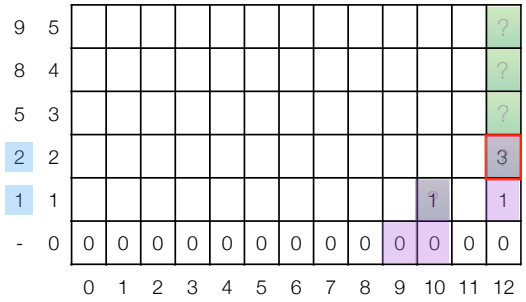
Subset Sum

- Recurrence:

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w), w_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

1 2 + 1

- Example
  - {1, 2, 5, 8, 9} and W = 12

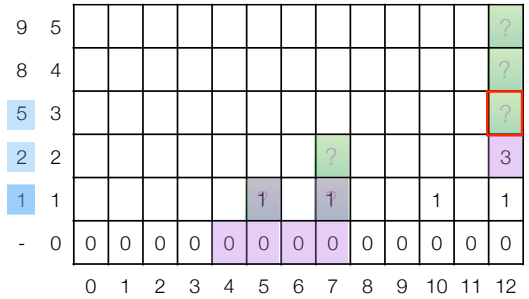


Subset Sum

- Recurrence:

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w), w_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

- Example
  - {1, 2, 5, 8, 9} and W = 12



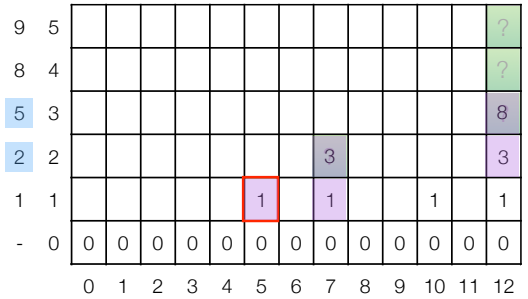
Subset Sum

- Recurrence:

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w), w_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

3 8 + 3

- Example
  - {1, 2, 5, 8, 9} and W = 12

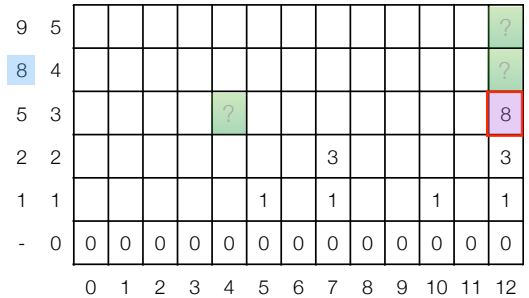


Subset Sum

- Recurrence:

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w), w_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

- Example
  - {1, 2, 5, 8, 9} and W = 12

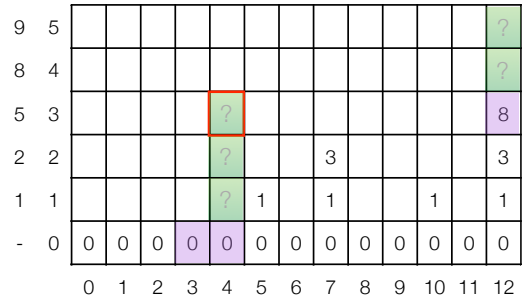


Subset Sum

- Recurrence:

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w), w_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

- Example
  - {1, 2, 5, 8, 9} and W = 12

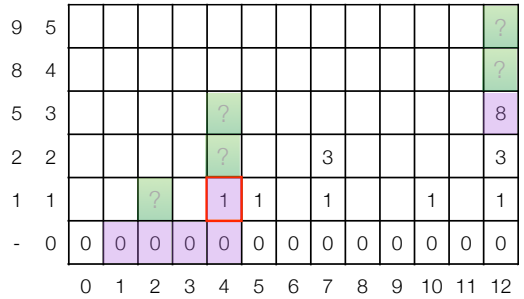


Subset Sum

- Recurrence:

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w), w_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

- Example
  - {1, 2, 5, 8, 9} and W = 12

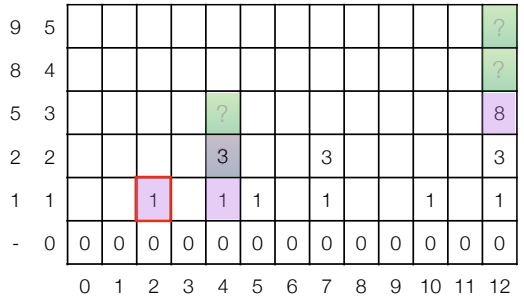


Subset Sum

- Recurrence:

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w), w_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

- Example
  - {1, 2, 5, 8, 9} and W = 12

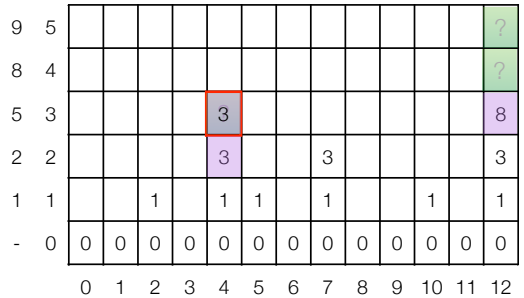


Subset Sum

- Recurrence:

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w), w_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

- Example
  - {1, 2, 5, 8, 9} and W = 12

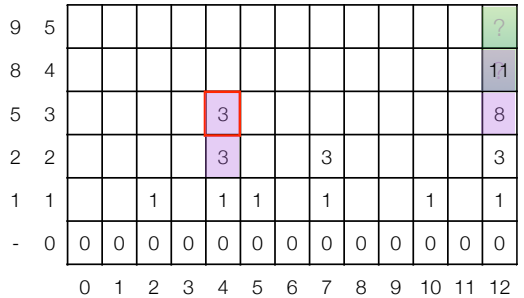


Subset Sum

- Recurrence:

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w), w_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

- Example
  - {1, 2, 5, 8, 9} and W = 12

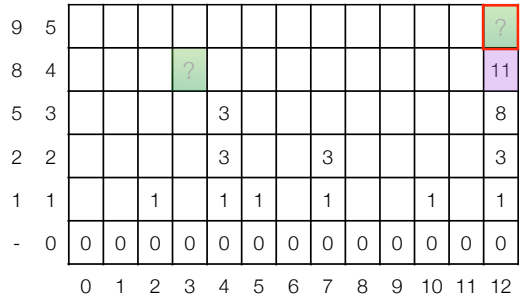


Subset Sum

- Recurrence:

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w), w_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

- Example
  - {1, 2, 5, 8, 9} and W = 12

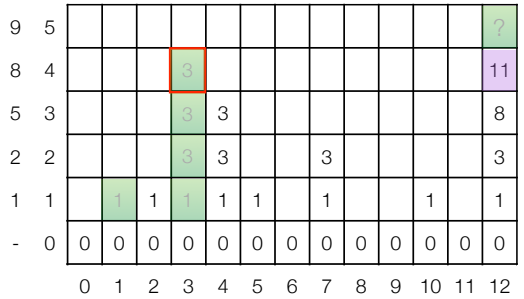


Subset Sum

- Recurrence:

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w), w_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

- Example
  - {1, 2, 5, 8, 9} and W = 12

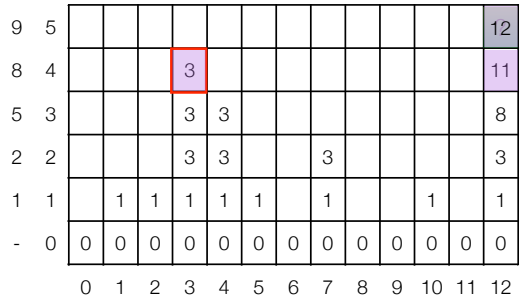


Subset Sum

- Recurrence:

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w), w_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

- Example
  - {1, 2, 5, 8, 9} and W = 12

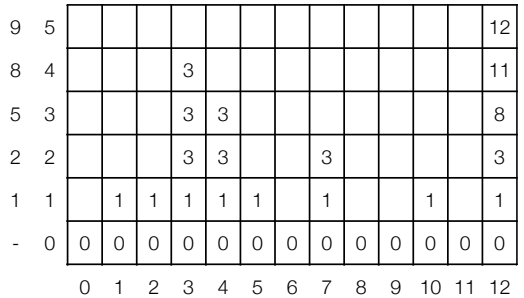


Subset Sum

- Recurrence:

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w), w_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

- Example
  - {1, 2, 5, 8, 9} and W = 12

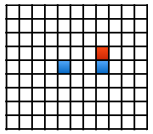


Subset Sum

- Recurrence:

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w), w_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

```
Subset-Sum(n,W)
  Array M[0…n][0…W]
  Initialize M[0][w] = 0 for each w = 0,1,…,W
  for i = 1 to n
    for w = 0 to W
      if w < wᵢ
        M[i][w] = M[i-1][w]
      else
        M[i][w] = max(M[i-1][w], wᵢ + M[i-1][w-wᵢ])
  return M[n,W]
```

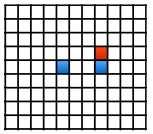


Subset Sum

- Recurrence:

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w), w_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

- Running time:
  - Number of subproblems = nW
  - Constant time on each entry $\Rightarrow O(nW)$
  - *Pseudo-polynomial time.*
    - Not polynomial in input size:
      - whole input can be described in O(n log n + n log w) bits, where w is the maximum weight (including W) in the instance.



Knapsack

- Knapsack
  - Given n items $\{1,\ldots,n\}$
  - Item i has weight w_i and value v_i
  - Bound W
  - Goal: Select maximum *value* subset S of items so that

$$\sum_{i \in S} w_i \le W$$

- Example

Capacity 12

Optimal solution: {vol 2, vol 3} has value 40

Knapsack

Knapsack

- $\mathcal{O}$ = optimal solution
- Consider element n.
  - Either in $\mathcal{O}$ or not.
    - $n \notin \mathcal{O}$: Optimal solution using items $\{1,\ldots,n-1\}$ is equal to $\mathcal{O}$.
    - $n \in \mathcal{O}$: Value of $\mathcal{O}$ = v_n + value on optimal solution on $\{1,\ldots,n-1\}$ with capacity $W - w_n$.
- Recurrence
  - OPT(i, w) = optimal solution on $\{1,\ldots,i\}$ with capacity w.

$$\text{OPT}(i,w) = \begin{cases} \text{OPT}(i-1,w) & \text{if } w < w_i \\ \max(\text{OPT}(i-1,w), v_i + \text{OPT}(i-1,w-w_i)) & \text{otherwise} \end{cases}$$

- Running time $O(nW)$

Dynamic programming

- **First formulate the problem recursively.**
  - Describe the *problem* recursively in a clear and precise way.
  - Give a recursive formula for the problem.
- **Bottom-up**
  - Identify all the subproblems.
  - Choose a memoization data structure.
  - Identify dependencies.
  - Find a good evaluation order.
- **Top-down**
  - Identify all the subproblems.
  - Choose a memoization data structure.
  - Identify base cases.
  - Remember to save results and check before computing.

Sequence Alignment

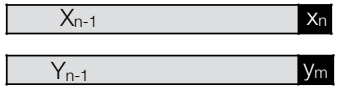
Sequence alignment

- How similar are ACAAGTC and CATGT.
- Align them such that
  - all items occurs in at most one pair.
  - no crossing pairs.
- Cost of alignment
  - gap penalty δ
  - mismatch cost for each pair of letters α(p,q).
- Goal: find minimum cost alignment.
- Input to problem: 2 strings X nd Y, gap penalty δ, and penalty matrix α(p,q).

```
A C A A G T C        A C A A - G T C
- C A T G T -        - C A - T G T -
1 mismatch, 2 gaps    0 mismatches, 4 gaps
```

Sequence Alignment

- Subproblem property.



- In the optimal alignment either:
  - x_n and y_m are aligned.
    - OPT = price of aligning x_n and y_m + minimum cost of aligning X_{i-1} and Y_{j-1}.
  - x_n and y_m are not aligned.
    - Either x_n and y_m (or both) is unaligned in OPT. Why?
    - OPT = δ + min(min cost of aligning X_{n-1} and Y_m,
      min cost of aligning X_n and Y_{m-1})

Sequence Alignment

- Subproblem property.

- SA(X_i,Y_j) = min cost of aligning strings X[1…i] and Y[1…j].

- Case 1. Align x_i and y_j.
  - Pay mismatch cost for x_i and y_j + min cost of aligning X_{i-1} and Y_{j-1}.
- Case 2. Leave x_i unaligned.
  - Pay gap cost + min cost of aligning X_{i-1} and Y_j.
- Case 3. Leave y_j unaligned.
  - Pay gap cost + min cost of aligning X_i and Y_{j-1}.

Sequence alignment

$$SA(X_i, Y_j) = \begin{cases} j\delta & \text{if } i = 0 \\ i\delta & \text{if } j = 0 \\ \min \begin{cases} \alpha(x_i, y_j) + SA(X_{i-1}, Y_{j-1}), \\ \delta + SA(X_i, Y_{j-1}), \\ \delta + SA(X_{i-1}, Y_j) \end{cases} & \text{otherwise} \end{cases}$$

		A	C	A	A	G	T	C
C								
A								
T						←		
G								
T								

$SA(X_5, Y_3)$

δ = 1

Penalty matrix

	A	C	G	T
A	0	1	2	2
C	1	0	2	3
G	2	2	0	1
T	2	3	1	0

Sequence alignment

$$SA(X_i, Y_j) = \begin{cases} j\delta & \text{if } i = 0 \\ i\delta & \text{if } j = 0 \\ \min \begin{cases} \alpha(x_i, y_j) + SA(X_{i-1}, Y_{j-1}), \\ \delta + SA(X_i, Y_{j-1}), \\ \delta + SA(X_{i-1}, Y_j) \end{cases} & \text{otherwise} \end{cases}$$

		A	C	A	A	G	T	C
C								
A								
T								
G								
T								

$SA(X_5, Y_3)$

Depends on ?

δ = 1

Penalty matrix

	A	C	G	T
A	0	1	2	2
C	1	0	2	3
G	2	2	0	1
T	2	3	1	0

Sequence alignment

$$SA(X_i, Y_j) = \begin{cases} j\delta & \text{if } i = 0 \\ i\delta & \text{if } j = 0 \\ \min \begin{cases} \alpha(x_i, y_j) + SA(X_{i-1}, Y_{j-1}), \\ \delta + SA(X_i, Y_{j-1}), \\ \delta + SA(X_{i-1}, Y_j) \end{cases} & \text{otherwise} \end{cases}$$

		A	C	A	A	G	T	C
C								
A								
T								
G								
T								

$SA(X_5, Y_3)$
Depends on ?

δ = 1

Penalty matrix

	A	C	G	T
A	0	1	2	2
C	1	0	2	3
G	2	2	0	1
T	2	3	1	0

Sequence alignment

$$SA(X_i, Y_j) = \begin{cases} j\delta & \text{if } i = 0 \\ i\delta & \text{if } j = 0 \\ \min \begin{cases} \alpha(x_i, y_j) + SA(X_{i-1}, Y_{j-1}), \\ \delta + SA(X_i, Y_{j-1}), \\ \delta + SA(X_{i-1}, Y_j) \end{cases} & \text{otherwise} \end{cases}$$

		A	C	A	A	G	T	C
	0	1	2	3	4	5	6	7
C	1							
A	2							
T	3							
G	4							
T	5							

δ = 1

Penalty matrix

	A	C	G	T
A	0	1	2	2
C	1	0	2	3
G	2	2	0	1
T	2	3	1	0

Sequence alignment

$$SA(X_i, Y_j) = \begin{cases} j\delta & \text{if } i = 0 \\ i\delta & \text{if } j = 0 \\ \min \begin{cases} \alpha(x_i, y_j) + SA(X_{i-1}, Y_{j-1}), \\ \delta + SA(X_i, Y_{j-1}), \\ \delta + SA(X_{i-1}, Y_j) \end{cases} & \text{otherwise} \end{cases}$$

min(1+0, 1+1, 1+1)

		A	C	A	A	G	T	C
	0	1	2	3	4	5	6	7
C	1							
A	2							
T	3							
G	4							
T	5							

δ = 1

Penalty matrix

	A	C	G	T
A	0	1	2	2
C	1	0	2	3
G	2	2	0	1
T	2	3	1	0

Sequence alignment

$$SA(X_i, Y_j) = \begin{cases} j\delta & \text{if } i = 0 \\ i\delta & \text{if } j = 0 \\ \min \begin{cases} \alpha(x_i, y_j) + SA(X_{i-1}, Y_{j-1}), \\ \delta + SA(X_i, Y_{j-1}), \\ \delta + SA(X_{i-1}, Y_j) \end{cases} & \text{otherwise} \end{cases}$$

min(1+0, 1+1, 1+1)

		A	C	A	A	G	T	C
	0	1	2	3	4	5	6	7
C	1	1						
A	2							
T	3							
G	4							
T	5							

δ = 1

Penalty matrix

	A	C	G	T
A	0	1	2	2
C	1	0	2	3
G	2	2	0	1
T	2	3	1	0

Sequence alignment

$$SA(X_i, Y_j) = \begin{cases} j\delta & \text{if } i = 0 \\ i\delta & \text{if } j = 0 \\ \min \begin{cases} \alpha(x_i, y_j) + SA(X_{i-1}, Y_{j-1}), \\ \delta + SA(X_i, Y_{j-1}), \\ \delta + SA(X_{i-1}, Y_j) \end{cases} & \text{otherwise} \end{cases}$$

min(0+1, 1+2, 1+1)

		A	C	A	A	G	T	C
	0	1	2	3	4	5	6	7
C	1	1						
A	2							
T	3							
G	4							
T	5							

δ = 1

Penalty matrix

	A	C	G	T
A	0	1	2	2
C	1	0	2	3
G	2	2	0	1
T	2	3	1	0

Sequence alignment

$$SA(X_i, Y_j) = \begin{cases} j\delta & \text{if } i = 0 \\ i\delta & \text{if } j = 0 \\ \min \begin{cases} \alpha(x_i, y_j) + SA(X_{i-1}, Y_{j-1}), \\ \delta + SA(X_i, Y_{j-1}), \\ \delta + SA(X_{i-1}, Y_j) \end{cases} & \text{otherwise} \end{cases}$$

min(0+1, 1+2, 1+1)

		A	C	A	A	G	T	C
	0	1	2	3	4	5	6	7
C	1	1	1					
A	2							
T	3							
G	4							
T	5							

δ = 1

Penalty matrix

	A	C	G	T
A	0	1	2	2
C	1	0	2	3
G	2	2	0	1
T	2	3	1	0

Sequence alignment

$$SA(X_i, Y_j) = \begin{cases} j\delta & \text{if } i = 0 \\ i\delta & \text{if } j = 0 \\ \min \begin{cases} \alpha(x_i, y_j) + SA(X_{i-1}, Y_{j-1}), \\ \delta + SA(X_i, Y_{j-1}), \\ \delta + SA(X_{i-1}, Y_j) \end{cases} & \text{otherwise} \end{cases}$$

min(1+2, 1+3, 1+1)

		A	C	A	A	G	T	C
	0	1	2	3	4	5	6	7
C	1	1	1					
A	2							
T	3							
G	4							
T	5							

δ = 1

Penalty matrix

	A	C	G	T
A	0	1	2	2
C	1	0	2	3
G	2	2	0	1
T	2	3	1	0

Sequence alignment

$$SA(X_i, Y_j) = \begin{cases} j\delta & \text{if } i = 0 \\ i\delta & \text{if } j = 0 \\ \min \begin{cases} \alpha(x_i, y_j) + SA(X_{i-1}, Y_{j-1}), \\ \delta + SA(X_i, Y_{j-1}), \\ \delta + SA(X_{i-1}, Y_j) \end{cases} & \text{otherwise} \end{cases}$$

min(1+2, 1+3, 1+1)

		A	C	A	A	G	T	C
	0	1	2	3	4	5	6	7
C	1	1	1	2				
A	2							
T	3							
G	4							
T	5							

δ = 1

Penalty matrix

	A	C	G	T
A	0	1	2	2
C	1	0	2	3
G	2	2	0	1
T	2	3	1	0

Sequence alignment

$$SA(X_i, Y_j) = \begin{cases} j\delta & \text{if } i = 0 \\ i\delta & \text{if } j = 0 \\ \min \begin{cases} \alpha(x_i, y_j) + SA(X_{i-1}, Y_{j-1}), \\ \delta + SA(X_i, Y_{j-1}), \\ \delta + SA(X_{i-1}, Y_j)\} \end{cases} & \text{otherwise} \end{cases}$$

min(1+3, 1+4, 1+2)

		A	C	A	A	G	T	C
	0	1	2	3	4	5	6	7
C	1	1	1	2				
A	2							
T	3							
G	4							
T	5							

δ = 1

Penalty matrix

	A	C	G	T
A	0	1	2	2
C	1	0	2	3
G	2	2	0	1
T	2	3	1	0

Sequence alignment

$$SA(X_i, Y_j) = \begin{cases} j\delta & \text{if } i = 0 \\ i\delta & \text{if } j = 0 \\ \min \begin{cases} \alpha(x_i, y_j) + SA(X_{i-1}, Y_{j-1}), \\ \delta + SA(X_i, Y_{j-1}), \\ \delta + SA(X_{i-1}, Y_j)\} \end{cases} & \text{otherwise} \end{cases}$$

min(1+3, 1+4, 1+2)

		A	C	A	A	G	T	C
	0	1	2	3	4	5	6	7
C	1	1	1	2	3			
A	2							
T	3							
G	4							
T	5							

δ = 1

Penalty matrix

	A	C	G	T
A	0	1	2	2
C	1	0	2	3
G	2	2	0	1
T	2	3	1	0

Sequence alignment

$$SA(X_i, Y_j) = \begin{cases} j\delta & \text{if } i = 0 \\ i\delta & \text{if } j = 0 \\ \min \begin{cases} \alpha(x_i, y_j) + SA(X_{i-1}, Y_{j-1}), \\ \delta + SA(X_i, Y_{j-1}), \\ \delta + SA(X_{i-1}, Y_j)\} \end{cases} & \text{otherwise} \end{cases}$$

min(2+4, 1+5, 1+3)

		A	C	A	A	G	T	C
	0	1	2	3	4	5	6	7
C	1	1	1	2	3	4		
A	2							
T	3							
G	4							
T	5							

δ = 1

Penalty matrix

	A	C	G	T
A	0	1	2	2
C	1	0	2	3
G	2	2	0	1
T	2	3	1	0

Sequence alignment

$$SA(X_i, Y_j) = \begin{cases} j\delta & \text{if } i = 0 \\ i\delta & \text{if } j = 0 \\ \min \begin{cases} \alpha(x_i, y_j) + SA(X_{i-1}, Y_{j-1}), \\ \delta + SA(X_i, Y_{j-1}), \\ \delta + SA(X_{i-1}, Y_j)\} \end{cases} & \text{otherwise} \end{cases}$$

		A	C	A	A	G	T	C
	0	1	2	3	4	5	6	7
C	1	1	1	2	3	4	5	6
A	2	1	2	1	2	3	4	5
T	3	2	3	2	3	3	3	4
G	4	3	4	3	4	3	4	5
T	5	4	5	4	5	4	3	4

δ = 1

Penalty matrix

	A	C	G	T
A	0	1	2	2
C	1	0	2	3
G	2	2	0	1
T	2	3	1	0

Sequence alignment

```
SA(X[1…m],Y[1…n],δ,A){
  for i=0 to m
    M[i,0] := iδ

  for j=0 to n
    M[0,j] := jδ

  for i=1 to m
    for j = 1 to n
      M[i,j] := min{ A[i,j] + M[i-1,j-1],
                     δ + M[i-1,j],
                     δ + M[i,j-1]}

  Return M[m,n]
}
```

- Time: Θ(mn)
- Space: Θ(mn)

Sequence alignment: Finding the solution

$$SA(X_i, Y_j) = \begin{cases} j\delta & \text{if } i = 0 \\ i\delta & \text{if } j = 0 \\ \min \begin{cases} \alpha(x_i, y_j) + SA(X_{i-1}, Y_{j-1}), \\ \delta + SA(X_i, Y_{j-1}), \\ \delta + SA(X_{i-1}, Y_j) \end{cases} & \text{otherwise} \end{cases}$$

Penalty matrix

	A	C	G	T
A	0	1	2	2
C	1	0	2	3
G	2	2	0	1
T	2	3	1	0

δ = 1

		A	C	A	A	G	T	C
	0	1	2	3	4	5	6	7
C	1	1	1	2	3	4	5	6
A	2	1	2	1	2	3	4	5
T	3	2	3	2	3	3	3	4
G	4	3	4	3	4	3	4	5
T	5	4	5	4	5	4	3	4

		A	C	A	A	G	T	C
		←	←	←	←	←	←	←
C	↑	↖	↖	←	←	←	←	↖
A	↑	↖	↖	↖	↖	←	←	←
T	↑	↑	↑	↑	↖	↖	↖	←
G	↑	↑	↖	↑	↖	↖	↖	↖
T	↑	↑	↑	↑	↖	↑	↖	←

Sequence alignment

- Use dynamic programming to compute an optimal alignment.
  - Time: Θ(mn)
  - Space: Θ(mn)
- Find actual alignment by backtracking (or saving information in another matrix).
- Linear space?
  - Easy to compute value (save last and current row)
  - How to compute alignment? Hirschberg. (not part of the curriculum).

Reading material

At the lecture we will continue with dynamic programming. We will talk about the knapsack problem and sequence alignment. You should read KT section 6.4 and 6.6.

Exercises

1 $[w]$ **Knapsack** Solve the following knapsack problem by filling out the table below. The items are givens as pairs (w_i, v_i): $(5,7),(2,6),(3,3),(2,1)$. The capacity $W = 6$.

4							
3							
2							
1							
0							
$i \setminus w$	0	1	2	3	4	5	6

2 $[w]$ **Sequence alignment** Consider the strings `APPLE` and `PAPE` over the alphabet $\Sigma = \{\texttt{A},\texttt{E},\texttt{L},\texttt{P}\}$ and a penalty matrix P:

	A	E	L	P
A	0	1	3	1
E	1	0	2	1
L	3	2	0	2
P	1	1	2	0

Compute the sequence alignment of the two strings when the penalty for a gap $\delta = 2$. Fill the dynamic programming table below, and explain how the minimum cost sequence alignment is found in it.

	j	0	1	2	3	4	5
i			A	P	P	L	E
0							
1	P						
2	A						
3	P						
4	E						

3 Book Shop You are in a book shop which sells n different books. You know the price h_i and number of pages s_i of each book $i \in \{1,\ldots,n\}$. You have decided that the total price of your purchases will be at most x and you will buy each book at most once.

3.1 Give an algorithm that computes the maximum number of pages you can buy. Analyse the time and space usage of your algorithm.

3.2 Modify your algorithm to only use $O(x)$ space (if it doesn't already). It is possible to solve the problem using only a single 1-dimensional array D.

3.3 Implement your linear space algorithm on CSES: https://cses.fi/problemset/task/1158

4 Longest palindrome subsequence A *palindrome* is a (nonempty) string over an alphabet Σ that reads the same forward and backward. For example are `abba` and `racecar` palindromes. A string P is a *subsequence* of string T if we can obtain P from T by removing 0 or more characters in T. For instance, `abba` is a subsequence of `bcadfbbba`.

Give an efficient algorithm to find the longest palindrome that is a subsequence of a given input string. To do this first give a recurrence for the problem and then write pseudo code for an algorithm based on your recurrence and dynamic programming. Argue that your recurrence is correct and analyse the running time and space usage of your algorithm.

5 Defending Zion Solve KT 6.8

Puzzle of the week: The Blind Man A blind man was handed a deck of 52 cards with exactly 10 cards facing up. How can he divide it into two piles, each of which having the same number of cards facing up?