ADS2 — Week 6 Notes & Solutions (Randomized I)

Metadata

Field	Value
Title	Week 6 — Randomized Algorithms: Expectation, Records, Selection, Quicksort, Boxes, Nuts & Bolts
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Sources used	weekplan-1.png (p.1), weekplan-2.png (p.2); slides-1.png…slides-7.png; kt_chap12.pdf (background only)
Week plan filename	weekplan-1.png • weekplan-2.png

Coverage Table (from the week plan images only)

Weekplan ID	Canonical ID	Title/Label (verbatim or inferred)	Assignment Source	Text Source	Status
1.1	RandomizedPrint	Probability exactly 3 stars are printed	weekplan-1.png p.1	slides-5.png (Geom./Expectation)	Solved
1.2	RandomizedPrint	Expected number of stars printed	weekplan-1.png p.1	slides-5.png	Solved
2.1	Expectation	E[X] for values {2,5,8} with probs {1/3,1/2,1/6}	weekplan-1.png p.1	slides-7.png (Expected values)	Solved
2.2	Indicator RV	Prove E[X]=Pr[X=1] for indicator X	weekplan-1.png p.1	slides-5.png/7.png (Linearity)	Solved
3	Select-Phase	Analysis of Selection with redefined phases (≤ n·(3/4)^j)	weekplan-1.png p.1	slides-2.png (Select: Running time)	Solved

Weekplan ID	Canonical ID	Title/Label (verbatim or inferred)	Assignment Source	Text Source	Status
4.1	Records	Cookies game: Pr[(*) runs in last iteration]	weekplan-1.png p.1	slides-5.png (Record-broker)	Solved
4.2	Records	Pr[X_i=1] where X_i indicates line (*) at iter i	weekplan-1.png p.1	slides-5.png	Solved
4.3	Records	E[# executions of (*))]	weekplan-1.png p.1	slides-5.png	Solved
5.1	Boxes (det.)	Best-case running time of deterministic scan	weekplan-1.png p.1→p.2	slides-5.png (Linearity)	Solved
5.2	Boxes (det.)	Worst-case running time of deterministic scan	weekplan-1.png p.1→p.2	slides-5.png	Solved
5.3a	Boxes (rand)	Expected running time if pick one random box and open it	weekplan-2.png p.2	slides-5.png	Solved
5.3b	Boxes (rand)	Worst-case running time of that one-shot random algorithm	weekplan-2.png p.2	slides-5.png	Solved
5.4a	Boxes (no repeat)	Worst-case time when sampling unopened boxes uniformly	weekplan-2.png p.2	slides-5.png	Solved
5.4.1	Boxes (no repeat)	Express X via indicators X_i over empty boxes	weekplan-2.png p.2	slides-5.png	Solved
5.4.2	Boxes (no repeat)	Compute E[X_i]	weekplan-2.png p.2	slides-5.png	Solved
5.4.3	Boxes (no repeat)	Compute E[X]	weekplan-2.png p.2	slides-5.png	Solved
5.4.4	Boxes (no repeat)	Expected running time of the algorithm	weekplan-2.png p.2	slides-5.png	Solved

Weekplan ID	Canonical ID	Title/Label (verbatim or inferred)	Assignment Source	Text Source	Status
5 (Nuts)	Nuts & Bolts	Give an efficient method (hint: customize quicksort)	weekplan-2.png p.2	slides-1.png/ 2.png/3.png (Quicksort)	Solved

No blockers: every enumerated line from the plan images has a solution stub and is solved below.

General Methodology and Theory

- **Linearity of expectation (LOE):** for any RVs X,Y, $E[X+Y]=E[X]+E[Y]$ (no independence needed).
- **Geometric waiting time:** first success with prob p each round ⇒ $E[\text{failures}]=(1-p)/p$; $\Pr[\text{exactly } r \text{ failures then success}]=(1-p)^r p$.
- **Record-breaker fact:** in a random order of distinct keys, $\Pr[\text{record at } i]=1/i$ ⇒ expected #records is $H_n=\sum_{i=1}^{n} 1/i$.
- **Selection (randomized):** a pivot is *central* (at least 1/4 smaller and 1/4 larger) with prob $\geq 1/2$; phases shrink by factor $\leq 3/4$ ⇒ expected linear time by summing a geometric series over phases (slides-2.png).
- **Quicksort pair-exposure:** keys $a_i < a_j$ are compared iff one is the first pivot chosen from $\{a_i,\ldots,a_j\}$ ⇒ $\Pr[\text{compare}]=2/(j-i+1)$ ⇒ $E[\text{comparisons}]=2nH_n=O(n\log n)$ (slides-3.png).

Notes (slides-first)

- **Probability & RVs:** probability spaces; independence; expectation as theoretical average (slides-7.png).
- **Geometric/Waiting:** $E[X]=1/p$ for round of first success; use power series identity or LOE on indicators (slides-5.png).
- **Guessing/Records:** indicators over positions yield harmonic sums; same pattern drives cookies-record questions (slides-5.png).
- **Select:** phase analysis with central pivots ⇒ $E[T(n)]\leq 8cn$ for partition cost constant c (slides-2.png).
- **Quicksort:** partition + recurse; analysis via indicator pair trick (slides-1.png, slides-3.png).

Solutions

Exercise 1.1 — RandomizedPrint

Assignment Source: weekplan-1.png p.1
Text Source: slides-5.png (waiting for first success)

Snapshot. Loop prints a star while two fresh uniforms in {1,...,10} differ. Success (stop) when they match with prob $p = 1/10$.

Solution. Exactly 3 stars ⇔ 3 failures then success: $(1-p)^3 p = (9/10)^3 \cdot (1/10) = 0.0729$.
✅**Answer:** $0.0729 = 729/10{,}000$.

Exercise 1.2 — RandomizedPrint

Assignment Source: weekplan-1.png p.1
Text Source: slides-5.png

Solution. Stars printed = #failures before first success with prob $p = 1/10$. Expectation $(1-p)/p = 9$.

Mini-Checklist. Identify geometric with failures-count; use $E[\text{failures}] = (1-p)/p$.
✅**Answer:** 9.

Exercise 2.1 — Expectation

Assignment Source: weekplan-1.png p.1
Text Source: slides-7.png

Solution. $E[X] = 2 \cdot \frac{1}{3} + 5 \cdot \frac{1}{2} + 8 \cdot \frac{1}{6} = \frac{2}{3} + \frac{5}{2} + \frac{4}{3} = 2 + \frac{5}{2} = \frac{9}{2} = 4.5$.
✅**Answer:** $E[X] = 9/2$.

Exercise 2.2 — Indicator RV

Assignment Source: weekplan-1.png p.1
Text Source: slides-5.png/7.png

Proof. For indicator X∈{0,1}, $E[X] = 0 \cdot \Pr[X=0] + 1 \cdot \Pr[X=1] = \Pr[X=1]$.
✅**Answer:** $E[X] = \Pr[X=1]$.

Exercise 3 — Select-Phase (expected time)

Assignment Source: weekplan-1.png p.1
Text Source: slides-2.png (Select: Running time)

Concept. Phase j has subproblem size in $\left(n(3/4)^{j+1},\, n(3/4)^{j}\right]$. A pivot is central with prob $\geq 1/2$; expected trials per phase ≤ 2.

Derivation. Work per partition on size m is $\leq cm$. Expected work in phase j $\leq 2c\,n(3/4)^{j}$. Summing: $E[T(n)] \leq \sum_{j\geq 0} 2c\,n\,(3/4)^{j} = 2cn \cdot \frac{1}{1-3/4} = 8cn = O(n).$
✅**Answer:** Yes—expected linear time; constant $\leq 8c$.

Exercise 4.1 — Cookies record at last iteration

Assignment Source: weekplan-1.png p.1
Text Source: slides-5.png (Record analysis)

Solution. With distinct counts and random order, each position equally likely to contain the global maximum. $\Pr[\text{record at } n] = 1/n.$
✅**Answer:** $1/n.$

Exercise 4.2 — Pr[X_i=1]

Assignment Source: weekplan-1.png p.1
Text Source: slides-5.png

Solution. Line (*) executes at i iff s_i is larger than all first i−1 values ⇒ a record. $\Pr[X_i = 1] = 1/i.$
✅**Answer:** $1/i.$

Exercise 4.3 — E[# of (*)]

Assignment Source: weekplan-1.png p.1
Text Source: slides-5.png

Solution. Let X=ΣX_i. By LOE, $E[X] = \sum_{i=1}^{n} \Pr[X_i = 1] = \sum_{i=1}^{n} 1/i = H_n.$
✅**Answer:** $H_n.$

Exercise 5.1 — Boxes (deterministic) best case

Assignment Source: weekplan-1.png p.1→p.2
Text Source: slides-5.png

Solution. Beer in first box ⇒ open 1.
✅**Answer:** $1.$

Exercise 5.2 — Boxes (deterministic) worst case

Assignment Source: weekplan-1.png p.1→p.2
Text Source: slides-5.png

Solution. Place beers in the last k positions. Earliest beer at index $n-k+1$.
✅**Answer:** $n-k+1.$

Exercise 5.3a — One random pick: expected time

Assignment Source: weekplan-2.png p.2
Text Source: slides-5.png

Solution. Algorithm opens exactly one box.
✅**Answer:** Expected steps $=1$ (success probability k/n , but time is 1 regardless).

Exercise 5.3b — One random pick: worst case

Assignment Source: weekplan-2.png p.2
Text Source: slides-5.png

Solution. Always 1 opening.
✅**Answer:** Worst-case steps $=1$ (but it may fail to find beer with prob $1-k/n$).

Exercise 5.4a — No-repeat random sampling: worst case

Assignment Source: weekplan-2.png p.2
Text Source: slides-5.png

Solution. Adversary places all k beers at the end of the (random) order ⇒ earliest beer rank $n-k+1$.
✅**Answer:** $n-k+1.$

Exercise 5.4.1 — Express X via indicators

Assignment Source: weekplan-2.png p.2
Text Source: slides-5.png

Setup. There are $n-k$ empty boxes. For each empty box i, let $X_i=1$ if it is opened before the first beer, else 0.

Expression. $X=\left(\sum_{\text{empty } i} X_i\right)+1$ (the +1 is for the first beer box).

✅**Answer:** $X=\sum_{i\in\text{empties}} X_i+1.$

Exercise 5.4.2 — Compute E[X_i]

Assignment Source: weekplan-2.png p.2
Text Source: slides-5.png

Reasoning. Consider the relative order among empty i and the k beer boxes: (k+1) items in uniform random order. i is before all beers with prob $1/(k+1)$.

✅**Answer:** $E[X_i] = 1/(k+1)$.

Exercise 5.4.3 — Compute E[X]

Assignment Source: weekplan-2.png p.2
Text Source: slides-5.png

By LOE. $E[X] = \sum E[X_i] + 1 = (n-k) \cdot \frac{1}{k+1} + 1 = \frac{n+1}{k+1}$.

✅**Answer:** $E[X] = (n+1)/(k+1)$.

Exercise 5.4.4 — Expected running time

Assignment Source: weekplan-2.png p.2
Text Source: slides-5.png

Conclusion. The expected number of opened boxes (steps) equals $E[X] = (n+1)/(k+1)$.
✅**Answer:** $\boxed{\text{Expected time} = (n+1)/(k+1)}$.

Exercise 5 (Nuts & Bolts) — Quicksort customization

Assignment Source: weekplan-2.png p.2
Text Source: slides-1.png/2.png/3.png

Problem. N nuts, N bolts; you can only compare a nut to a bolt (tell <,>,=). Find matching pairs and sort.

Algorithm (quicksort-style).

```
Algorithm: nuts_and_bolts_quicksort
Input: sets NUTS, BOLTS (|NUTS| = |BOLTS| = n), comparator cmp(nut, bolt) ∈
{<, =, >}
Output: matched pairs in sorted order by size

if n ≤ 1: return the (trivial) set
pick a random pivot nut ν ∈ NUTS
partition BOLTS into {B_<, b*, B_>} using cmp(ν, ·) // find matching bolt b*
partition NUTS into {N_<, ν, N_>} using cmp(·, b*)  // only bolt↔nut
comparisons
return nuts_and_bolts_quicksort(N_<, B_<)
       ⊕ {(ν, b*)}
       ⊕ nuts_and_bolts_quicksort(N_>, B_>)
// Time (expected): O(n log n); Space: O(log n) recursion depth
```

Why it works. The unique matching partner b* of pivot ν is discovered during the first partition; using b* we cleanly split nuts. Subproblems are independent by comparability restrictions.

Pitfalls. Never compare nut↔nut or bolt↔bolt; ensure equality case isolates the exact match.

Alternative Approach. Deterministic O(n log n) exists but is intricate; the above gives expected O(n log n) via random pivot (slides show standard quicksort analysis).

✅**Answer:** Expected O(n log n) with the two-way partition driven by a matched pivot pair.

Puzzle

You draw boxes uniformly **without replacement** until the **t-th** beer appears (t≤k). What is $E[X_t]$, the expected opens?
Hint: generalize the indicator trick: each empty contributes $1/(k-t+1)$ to expectation before the t-th beer; result $E[X_t]=(n+1)/(k-t+1)-(t-1)$.

Summary

- Use **indicators + LOE** to convert counts into easy sums; independence usually not needed.
- **Geometric** reasoning: stop-on-match loops ⇒ $E=(1-p)/p$ failures.
- **Records** in random orders give **harmonic** expectations H_n .
- **Selection** with central-pivot phases ⇒ **expected linear** time; **Quicksort** analysis via pair exposure ⇒ $2nH_n$ comparisons.
- **Boxes**: deterministic worst case $n-k+1$; no-repeat randomized has $E[(n+1)/(k+1)]$ steps.
- **Nuts & Bolts:** quicksort with matched pivots; only nut↔bolt comparisons; expected **O(n log n)**.

Notation recap. $H_n=\sum_{i=1}^{n} 1/i;\ p=\Pr[\text{success}];\ n=\text{items};\ k=\text{beer boxes}.$