ADS2 — Weekplan Randomized I (Exam Notes & Solutions)

Title	Date	Author	Sources used	Week plan filename
ADS2 — Randomized I Notes & Solutions	2025-10-23 (Europe/Copenhagen)	ADS2 Copilot	Weekplan images: weekplan-1.png (p.1), weekplan-2.png (p.2); Slides: slides-1.png…slides-7.png; Textbook: Kleinberg & Tardos,	
Algorithm Design, Ch.13 Randomized Algorithms (notably §§13.1, 13.3, 13.5), pp. 708–750				weekplan-1.png; weekplan-2.png

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Coverage Table

Weekplan ID	Canonical ID	Title/Label (verbatim)	Assignment Source	Text Source	Status
1	—	Randomized print [w]	weekplan-1.png §1	KT §13.3 (expectation), slides-1.png	Solved
2	—	[w] Expected values	weekplan-1.png §2	KT §13.3 pp. 719–723; slides-2.png	Solved
3	KT §13.5	Analysis of Selection (phase redefined)	weekplan-1.png §3	KT §13.5 pp. 727–733; slides-3.png	Solved
4	—	Christmas party at DTU (exam 2015): Find student with most cookies	weekplan-1.png §4 + Alg.2	KT §13.3 (records/indicators); slides-4.png	Solved
5	—	Boxes of beer	weekplan-1.png §5 → weekplan-2.png §5.1–5.4	KT §13.3 (geometric/linearity)	Solved
6*	—	Nuts and bolts (G. J. E. Rawlins)	weekplan-2.png bottom (label shows “5”; treated as 6)	KT §13.5 (quicksort idea); slides-5.png	Solved

> **MISMATCH note (plan numbering):** The week plan lists two items with number “5”. We treat **“Nuts and bolts”** as **Exercise 6** for contiguity. All items 1–6 are covered.

General Methodology and Theory

- **Toolkit:** indicators, linearity $E\left[\sum X_i\right] = \sum E[X_i]$; geometric waiting time ($\mathrm{Geom}(p)$): $E[T] = 1/p$ (failures-before-success variant has $E[F] = (1-p)/p$); union bound; random partition arguments; harmonic numbers $H_n = \sum_{i=1}^{n} 1/i = \Theta(\log n)$.
- **Randomized selection (KT §13.5):** pick a random splitter; expected linear time via phases. With phase threshold ratio $\rho < 1$ and probability $\alpha > 0$ to pick a "central" splitter each iteration, expected iterations/phase $= 1/\alpha$ and total $O\left(\sum_j n\rho^j\right) = O(n)$.
- **Records in a random permutation:** $\Pr[\text{record at position } i] = 1/i$, so $E[\#\text{records}] = H_n$.

Notes

- **Slides-first alignment:** All solutions follow the slide logic for indicators, geometric RVs, selection phases, and randomized quicksort; KT Ch.13 is cited as an alternative view where it differs in constants (e.g., $3/4$ vs $2/3$ phase ratios).
- **Conventions:** Uniform $\mathrm{rand}(1, 10)$ draws are independent; cookies counts are all distinct (as stated); boxes-of-beer has exactly k beer boxes.

Solutions

Exercise 1 — Randomized print [w]

Assignment Source: weekplan-1.png §1 — *Algorithm 1: RandomizedPrint(i)*.
Text Source: KT §13.3 (geometric), slides-1.png.

- Let $p = \Pr[\text{stop on a test}] = \Pr[\text{two equal in } \{1..10\}] = 1/10$. The stars printed equal the number of failures (inequalities) before first success.
- **1.1** $\Pr[\text{exactly 3 stars}] = (1-p)^3 p = (9/10)^3 \cdot (1/10) = 729/10000.$
- **1.2** $E[\text{stars}] = (1-p)/p = (9/10)/(1/10) = 9.$

Verification: independent trials; geometric model applies; boundary case $p = 1/10$ checked digit-by-digit.
Transfer Pattern: *Geometric waiting time* → cue: "repeat until equality"; mapping: trial=condition check, success=equality.
✅**Answer:** $\Pr[3] = 0.0729;\ E = 9.$

Exercise 2 — [w] Expected values

Assignment Source: weekplan-1.png §2.
Text Source: KT §13.3 pp. 719–723; slides-2.png.

- **2.1** $E[X] = 2 \cdot \frac{1}{3} + 5 \cdot \frac{1}{2} + 8 \cdot \frac{1}{6} = \frac{4}{6} + \frac{15}{6} + \frac{8}{6} = \frac{27}{6} = \frac{9}{2} = 4.5.$
- **2.2** Indicator $I \in \{0, 1\} : E[I] = 0 \cdot \Pr[I = 0] + 1 \cdot \Pr[I = 1] = \Pr[I = 1].$

Transfer Pattern: *Linearity of expectation* → cue: mixture of discrete values or indicators.
✅**Answer:** $E[X] = 4.5;\ E[I] = \Pr[I = 1].$

Exercise 3 — KT §13.5 — Analysis of Selection (phase redefined)

Assignment Source: weekplan-1.png §3.
Text Source: KT §13.5 pp. 727–733; slides-3.png.

- **Phase definition (plan):** phase j has size in $\big(n(2/3)^j,\ n(2/3)^{j+1}\big]$. Call a splitter **central** if at least a third of elements lie on each side.
- **Probability of central:** exactly $n/3$ ranks (between $\lceil n/3 \rceil$ and $\lfloor 2n/3 \rfloor$) → $\alpha = 1/3$.
- **Per-phase iterations:** geometric with $p = \alpha \Rightarrow E[\text{iters}/\text{phase}] = 1/p = 3.$
- **Per-iteration work in phase** j **:** $O\big(n(2/3)^j\big).$
- **Total expected time:** $\sum_{j\ge0} 3 \cdot c\,n(2/3)^j = 3cn \cdot \frac{1}{1-2/3} = 9cn = O(n).$

Alternative Approach (KT): Using $3/4$ -central elements gives factor 2 iters/phase; same linear bound.
Pitfalls: redefining “central” inconsistently; forgetting independence across iterations.
Transfer Pattern: *Randomized selection* → cues: “random splitter”, “expected linear”; mapping: choose splitter → partition → recurse on one side.
✅**Answer: Yes.** With the $2/3$ phase definition, expected time remains $O(n)$.

Exercise 4 — Christmas party at DTU (exam 2015)

Assignment Source: weekplan-1.png §4 + Algorithm 2.
Text Source: KT §13.3 (records), slides-4.png.

Algorithm (from plan): scan students in a **random order**; update at line (*) when you see a new maximum.

- **4.1** $\Pr[(*) \text{ executes at last iteration}] = \Pr[\text{last is global max}] = 1/n.$
- **4.2** Let $X_i = \mathbf{1}[(*) \text{ executes at iteration } i]$. Then $\Pr[X_i = 1] = 1/i$ (position i is a record w.p. $1/i$).
- **4.3** $E\big[\#(*)\big] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} 1/i = H_n.$

Verification: distinct cookie counts ⇒ total order; random permutation symmetry.
Transfer Pattern: *Records in permutations* → cues: “random order, count maxima updates”.
✅**Answer:** (4.1) $1/n$; (4.2) $1/i$; (4.3) H_n.

Exercise 5 — Boxes of beer (n boxes, k with beer)

Assignment Source: weekplan-1.png §5 and weekplan-2.png §5.1–5.4.
Text Source: KT §13.3; slides-5.png.

Deterministic baseline (open B1, B2, … until first beer):

- **5.1 Best-case:** 1 (B1 has beer).
- **5.2 Worst-case:** $n - k + 1$ (all $n - k$ empties first, then first beer). Assumes $k \ge 1$.

Randomized v1 (with replacement): *Each trial picks a uniform i∈{1..n}, independent, until a beer is found.*

- **5.3 Expected time:** geometric with success $p = k/n \Rightarrow E[T] = n/k$. Worst case (over randomness): **unbounded** (no finite upper bound on trials).

Randomized v2 (without replacement): *Each round pick a previously unopened box uniformly; stop at first beer.* Let **E** be the set of empties, $|E| = n - k$. For each empty $e \in E$, define indicator $X_e = \mathbf{1}[e \text{ opened before any beer}]$; total opened $X = 1 + \sum_{e \in E} X_e$.

- **5.4.1 Worst-case:** still $n - k + 1$ (open all empties first).
- **(i) Express X :** $X = 1 + \sum_{e \in E} X_e$.
- **(ii) $E[X_e]$:** Among the $k + 1$ boxes (the empty e plus all k beer boxes), the earliest in the random order is uniform ⇒ $E[X_e] = \Pr[X_e = 1] = 1/(k+1)$.
- **(iii) $E[X]$:** $E[X] = 1 + \sum_{e \in E} E[X_e] = 1 + (n-k) \cdot \frac{1}{k+1} = \frac{n+1}{k+1}$.
- **(iv) Expected running time:** $\boxed{(n+1)/(k+1)}$.

Pitfalls: mixing “with” vs “without” replacement; forgetting the +1 for the successful beer opening.
Transfer Pattern: *Search until first marked item* → cues: “k successes in n positions”, “random order”.
✅ **Answer:** Best 1 , worst $n - k + 1$; randomized-with-replacement: $E = n/k$, worst unbounded; randomized-without-replacement: $E = (n+1)/(k+1)$, worst $n - k + 1$.

Exercise 6 — Nuts and bolts (G. J. E. Rawlins)

Assignment Source: weekplan-2.png (bottom).
Text Source: Slides & KT §13.5 (quicksort idea).

Problem. Match each of N nuts to its unique bolt when only *cross* comparisons are allowed (you may compare nut vs bolt; never nut vs nut or bolt vs bolt).

Method (quicksort-style partitioning):

```
Algorithm: match_nuts_and_bolts
Input: sets NUTS={n1..nN}, BOLTS={b1..bN}; comparator cmp(n, b) ∈ {<,=,>}
Output: matched pairs (n_i ↔ b_j) for all i

procedure solve(NUTS, BOLTS):
  if |NUTS| ≤ 1: return NUTS paired with BOLTS
  pick pivot nut n★ uniformly at random
  // partition bolts using n★
  B<, B=, B> ← partition BOLTS by cmp(n★, b)
  let b★ be the unique bolt in B=
  // partition nuts using b★
  N<, N=, N> ← partition NUTS by cmp(n, b★)
  // recurse on corresponding sides
  solve(N<, B<); output (n★, b★); solve(N>, B>)
```

```
return
// Time (expected): O(N log N); Space: O(log N) recursion
```

Why it works: each partition uses only legal cross-comparisons; pivot nut matches exactly one bolt $b_\star$; subproblems are proper; random pivot gives the usual quicksort recurrence.

Verification: at every step, invariants hold—no nut/bolt crosses to the wrong side; sizes of corresponding partitions match.

Variant Drill: choose bolt pivot first; symmetric algorithm, same bound.
Alternative Approach: deterministic linear-time selection for pivots yields $O(N\log N)$ worst-case (more complex).
Transfer Pattern: *Quicksort partition with cross-type comparator* → cues: "two dual sets, only cross comparisons".
✅**Answer:** Expected running time $O(N\log N)$; all pairs matched.

Puzzle

You draw i.i.d. uniform integers in $\{1,...,100\}$ until you see a repeat. What is the expected number of draws? *(Hint: birthday process; coupon-collector with stopping at first collision.)*
Answer sketch: $E\approx\sqrt{\frac{\pi}{2}\cdot 100}\ \approx 12.53$ (Poisson birthday approximation).

Summary

- **Core identities:** indicators + linearity; geometric waiting times; harmonic records H_n .
- **Algorithms:** randomized selection stays **linear** even with $2/3$ phase thresholds; scan-max records problem yields $\Pr[\text{record at } i]=1/i$; nuts-and-bolts via quicksort partitions → **expected** $O(n\log n)$.
- **Beer boxes:** deterministic worst $n-k+1$; random with replacement $E=n/k$ (unbounded worst); without replacement $E=(n+1)/(k+1)$.
- **Notation:** $H_n,\ E[\cdot],\ \mathbf{1}[\cdot],\ \rho,\ \alpha.$

Slides are the primary authority; KT Ch.13 cited for background and constants.