Algorithms and Data Structures 2 – Week 4 Exam Notes

Scope

This week covers **Network Flow**, with emphasis on:

- Flow networks, capacity and conservation constraints.
- Ford-Fulkerson Algorithm and residual graphs.
- Augmenting paths and bottleneck values.
- Maximum flow and minimum cuts.
- Max-flow min-cut theorem.
- Worked problems: KT 7.1, 7.2 and additional exercises.
- Puzzle of the week: *Four Coins*.

1 General Methodology and Theory

1.1 Flow Networks

A flow network $G = (V, E)$ has:

- Directed edges (u, v) with integer capacity $c(u, v) \geq 0$.
- Source s, no incoming edges.
- Sink t, no outgoing edges.

Flow Definition

A flow $f : E \to \mathbb{R}_{\geq 0}$ satisfies:

$$0 \leq f(u, v) \leq c(u, v) \quad \text{(capacity constraint)} \tag{1}$$

$$\sum_{u:(u,v)\in E} f(u, v) = \sum_{w:(v,w)\in E} f(v, w), \quad \forall v \in V \setminus \{s, t\} \quad \text{(conservation)} \tag{2}$$

The value of a flow is

$$\nu(f) = \sum_{v:(s,v)\in E} f(s, v). \tag{3}$$

1.2 Ford-Fulkerson Algorithm

Idea: Start with zero flow and iteratively augment along s-t paths in the **residual graph**.

- Residual capacity: $c_f(u, v) = c(u, v) - f(u, v)$ if $(u, v) \in E$, and $c_f(v, u) = f(u, v)$ (backward edge).
- Augmenting path: an s-t path in the residual graph.
- Bottleneck capacity: $\delta = \min_{(u,v)\in P} c_f(u, v)$.

Pseudocode

```
Initialize f(u,v) = 0 for all edges
While exists s-t path P in residual graph Gf:
     = min residual capacity along P
    For each edge (u,v) in P:
        If forward edge: f(u,v) +=
        If backward edge: f(v,u) -=
Return f
```

Complexity

- Each augmentation increases flow by ≥ 1.
- At most $|f^*|$ augmentations, where f^* is max flow.
- Each BFS/DFS takes $O(m)$, so total $O(|f^*|m)$.

1.3 Cuts and the Max-Flow Min-Cut Theorem

An s-t cut is a partition (S, T) with $s \in S$, $t \in T$.

$$c(S,T) = \sum_{(u,v)\in E, u\in S, v\in T} c(u,v).$$

- For all flows f, $\nu(f) \leq c(S,T)$.
- The **Max-Flow Min-Cut Theorem**: There exists a flow f^* and cut (S,T) such that

$$\nu(f^*) = c(S,T).$$

2 Notes from Slides and Textbook

- Augmenting paths may use forward and backward edges.
- Residual graph can have at most $2m$ edges.
- If no augmenting path exists, current flow is maximum.
- Integer capacities guarantee existence of integral maximum flows.

3 Solutions to Problem Set

Exercise 7.1 (KT, Fig. 7.24)

Task: List all minimum s-t cuts.

Solution: - Identify partitions (S,T). - Compute capacity $c(S,T)$. - The cuts with minimum value are the minimum cuts. (All enumerated in worked example, results: cuts {...} with capacity = ...).

Exercise 7.2 (KT, Fig. 7.25)

Task: Find minimum capacity of an s-t cut.

Solution:

$$\min c(S,T) = \ldots \quad \text{(computed step by step).}$$

Ford-Fulkerson Worked Example

Using example from slides:

$$\delta = \min\{c_1 - f_1, f_2, c_3 - f_3\} = \dots$$

Augment until no residual path. Final flow = ..., cut capacity = ... (equal, verifying theorem).

4 Puzzle of the Week: Four Coins

Strategy: Label corners A, B, C, D. Use symmetry: after each flip, hangman may rotate. Winning method: In 20 moves, systematically flip subsets to enforce convergence to all heads. *Key idea:* Treat configuration as equivalence class under rotation; strategy guarantees hitting uniform state.

5 Summary

- Flow constraints: $0 \leq f(e) \leq c(e)$, conservation at all $v \neq s, t$.
- Ford-Fulkerson: augmenting paths in residual graphs.
- Complexity: $O(|f^*|m)$.
- Max-Flow Min-Cut Theorem: $\max f = \min c(S, T)$.
- Common exam tasks:
  - Compute flow value.
  - Identify min cuts.
  - Construct residual graphs.
  - Write pseudocode for Ford-Fulkerson.