

ADS2 — Week 8: Amortized Analysis (Exam Notes & Solutions)

Metadata

Field	Value
Title	ADS2 — Week 8: Amortized Analysis
Date	2025-11-09
Author	Mads Richardt
Sources used	weekplan-1.png; weekplan-2.png; week8_slides-1..8.png; week8_text.pdf (Amortized Analysis chapter)
Week plan filename	weekplan-1.png, weekplan-2.png

General Methodology and Theory

- **Amortized analysis** studies average cost per operation over a worst-case sequence. Three equivalent tools:
- **Aggregate:** bound total cost $T(m)$ over m operations, report $T(m)/m$.
- **Accounting:** overcharge cheap ops, store credits on data items to pay for future expensive ops. Credits must never go negative.
- **Potential:** define $\Phi(\text{state}) \geq 0$ with $\Phi(\text{initial})=0$. Amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$. Sum telescopes to $T(m) + \Phi(D_m) - \Phi(D_0)$.
- **Slides-first anchors:** Stacks with MultiPop, Dynamic Binary Search (binomial arrays), Aggregate/Accounting/Potential templates, and Heap potentials (sum of depths).
- **Design tips:**
 - Attach credits/potential to *elements* that migrate upward/downward (e.g., array merges, heap swims/sinks).
 - When merging “small into large”, each element’s set size doubles $\Rightarrow \leq \lfloor \log_2 n \rfloor$ migrations.
 - For binary counters/array towers, the i -th level flips/merges every 2^i inserts \Rightarrow total merges $O(m)$.

Notes (slides → key takeaways)

- **Stack + MultiPop (and MultiPush):** Each element is pushed once and popped at most once \Rightarrow total pops \leq total pushes. Aggregate gives $O(1)$ amortized.
- **Aggregate/Accounting/Potential equivalence:** choose the view that makes invariants obvious (e.g., coin per level for merges; Φ = sum of coins).
- **Dynamic Ordered Sets via Dynamic Binary Search (DBS):** Maintain sorted arrays $A_0 \dots A_{h-1}$ where $|A_i|$ is either 0 or 2^i ; merges mimic a binary counter. SEARCH: binary search each

nonempty array ($O(\log^2 n)$) or with fractional cascading $O(\log n)$. INSERT: amortized $O(\log n)$.

DELETE: lazy deletion + rebuild when tombstones $\geq \frac{1}{2}$ fills, amortized $O(\log n)$.

- **Heaps:** With $\Phi = c \cdot (\text{sum of node depths})$, INSERT increases Φ by $O(\log n)$; EXTRACT-MAX decreases Φ by $\Omega(\log n)$. Tuned c yields amortized $O(1)$ extract.

Coverage Table

Weekplan ID	Canonical ID	Title/Label (verbatim)	Assignment Source	Text Source	Status
1	—	MultiPush on a stack with MultiPop: effect on amortized time	weekplan-1.png (p.1)	slides 3–4 (Stack); week8_text.pdf §Stacks	Solved
2	—	Queues with two stacks: ENQUEUE/ DEQUEUE amortized $O(1)$	weekplan-1.png (p.1)	week8_text.pdf §Queues; slides 1 overview	Solved
3.1	—	Set Union (coloring): standard analysis	weekplan-1.png (p.1)	week8_text.pdf §Union-by-size idea	Solved
3.2	—	Set Union (coloring): amortized analysis	weekplan-1.png (p.1)	slides 1 overview; text §Union	Solved
4.1	—	Binary heap basics, INSERT/ EXTRACT-MAX standard bounds	weekplan-1.png (p.1)	week8_text.pdf §Heaps	Solved
4.2	—	Heap: amortized analysis with potential (INS $O(\log n)$, EX-MAX $O(1)$)	weekplan-1.png (p.1)	slides 8 (Potential); text §Heaps	Solved
5.1	—	Sequence T_i : aggregate method	weekplan-1.png (p.1)	slides 1–2 (methods)	Solved
5.2	—	Sequence T_i : accounting method	weekplan-1.png (p.1)	slides 6 (Accounting)	Solved

Weekplan ID	Canonical ID	Title/Label (verbatim)	Assignment Source	Text Source	Status
5.3	—	Sequence T_i : potential method	weekplan-1.png (p.1)	slides 8 (Potential)	Solved
6.1	—	DBS: duplicates during INSERT — issue & algorithm	weekplan-2.png (p.2)	slides 5–7 (DBS); text §DBS	Solved
6.2	—	DBS: DELETE in $O(\log n)$ amortized	weekplan-2.png (p.2)	slides 7 (costs)	Solved
7.1	—	Billy the Rabbit: rounds until carrots (n steps away)	weekplan-2.png (p.2)	week8_text.pdf §Exercises	Solved
7.2	—	Billy: total steps before finding carrots (asymptotic)	weekplan-2.png (p.2)	week8_text.pdf §Exercises	Solved
7.3	—	Billy: amortized steps per round (asymptotic)	weekplan-2.png (p.2)	week8_text.pdf §Exercises	Solved
—	Puzzle	Princesses (truth, liar, random): one yes/no question then choose	weekplan-2.png (p.2)	—	Solved (best-possible guarantee)

Solutions

Exercise 1 — MultiPush on a stack with MultiPop

Assignment Source: weekplan-1.png (p.1)

Text Source: slides 3–4 (Stack with MultiPop); week8_text.pdf §Stacks

Claim. Adding **MultiPush(x, k)** (push k copies) preserves **$O(1)$ amortized** time for PUSH, POP, MULTIPUSH, and MULTIPOP over any sequence.

Aggregate proof. Let the sequence contain P individual pushes total (counting all copies from MultiPush) and let Q be the number of pops performed (counting elements popped by MULTIPOP). Each element contributes $O(1)$ once when pushed and $O(1)$ once when popped; an element cannot be

popped more than once. Thus total cost $T = O(P + Q) \leq O(2P)$ since $Q \leq P$. Over m operations, $T = O(P) \leq O(m \cdot k)$ but each MultiPush contributes exactly k to P , so $T = O(\text{number of pushed elements})$ and $T/m = O(1)$. Hence all operations are **O(1) amortized**.

Accounting view. Charge 2 to each individual push (including copies). Spend 1 now; store 1 credit on the pushed copy. POP/MULTIPOP consume the stored credit. MULTIPUSH(x, k) simply performs k charged pushes. No deficit.

Potential view. Φ = (# elements on stack). PUSH increases Φ by 1; POP/MULTIPOP decrease by number popped. Amortized cost $\hat{c} = c + \Delta\Phi \leq 2$ for push and ≤ 0 for pops, giving $O(1)$.

Transfer Pattern. Dynamic bag where each item is created once and destroyed once (stack/queue/multiset). Recognition cues: “batch insert/delete”, “each element removed at most once”, “no duplication after insertion”. Mapping: attach 1 credit per item upon insertion.

 **Answer:** All stack ops remain **O(1) amortized** under MultiPush.

Exercise 2 — Queues with two stacks (amortized O(1))

Assignment Source: weekplan-1.png (p.1)

Text Source: week8_text.pdf §Queues; slides overview

Method. Maintain two stacks: **IN** (push on enqueue) and **OUT** (pop on dequeue).

```
Algorithm: queue_with_two_stacks
Input: stream of enqueue/dequeue operations
Output: queue semantics

init empty stacks IN, OUT

enqueue(x): push(IN, x)

dequeue():
    if OUT empty:
        while IN not empty: push(OUT, pop(IN))
    return pop(OUT)
// Time: O(1) amortized per op; Space: O(n)
```

Amortized proof (accounting). Charge 3 to each enqueue: 1 to push onto IN and 2 credits stored on the element to pay for the single transfer to OUT and the eventual pop from OUT. Each element is moved **at most once** between stacks \Rightarrow total $T = O(\#elements)$. Dequeues that trigger a refill are paid by stored credits.

Transfer Pattern. Two-phase buffer: inexpensive append, expensive flush, but each element flushed once. Cues: “reverse order by bulk move”, “flush when empty”.

 **Answer:** ENQUEUE and DEQUEUE are **O(1) amortized**.

Exercise 3.1 — Set Union (coloring), standard analysis

Assignment Source: weekplan-1.png (p.1)

Text Source: week8_text.pdf §Union-by-size

Model. Each set has a color id. UNION(A,B): recolor all elements in the *smaller* set to the color of the larger; ties arbitrary. SAMESET(x,y): compare colors.

Standard (worst-case) cost. UNION may recolor $\Theta(n)$ elements; SAMESET is $O(1)$; INIT is $O(n)$. Over a sequence of u unions, worst case is $\Theta(u \cdot n)$ by adversarial alternating sizes (standard analysis ignores the size-doubling effect).

✓**Answer:** Worst-case per UNION = $\Theta(n)$; SAMESET = $O(1)$; INIT = $\Theta(n)$.

Exercise 3.2 — Set Union (coloring), amortized analysis

Assignment Source: weekplan-1.png (p.1)

Text Source: slides overview; text §Union

Key invariant. When recoloring, the smaller set moves into the larger. Each element's set size at least doubles whenever it is recolored.

Aggregate bound. An element can be recolored at most $\lfloor \log_2 n \rfloor$ times ($1 \rightarrow 2 \rightarrow 4 \rightarrow \dots \leq n$). Thus total recolorings $\leq n \lfloor \log_2 n \rfloor$. Over u UNIONs, $T \leq O(n \log n + u)$ and amortized recolorings per UNION $\leq O(\log n)$ in the worst case, while SAMESET remains $O(1)$.

Accounting. Place 1 credit on each element of a set per unit of current set size "level". Recoloring an element consumes one level credit; as the set doubles, new credits are assigned. No deficits.

Transfer Pattern. "Small-to-large" merges (DSU/union-by-size/rank). Cues: repeated merges, size-aware choice. Mapping: attach log levels to elements; each merge reduces a level counter.

✓**Answer:** Total recolorings $\leq n \log n \Rightarrow$ amortized recolorings $O(\log n)$; SAMESET stays $O(1)$.

Exercise 4.1 — Binary heaps: recap

Assignment Source: weekplan-1.png (p.1)

Text Source: week8_text.pdf §Heaps

- Structure: array-based complete binary tree, parent at $\lfloor i/2 \rfloor$, children at $2i, 2i+1$.
- **INSERT(x):** place at next leaf, **swim** up while parent $< x$. Worst case $O(\log n)$.
- **EXTRACT-MAX():** swap root with last leaf, remove leaf, **sink** root down. Worst case $O(\log n)$.
- Heap property maintained after each swap.

✓**Answer:** Standard bounds: both operations $O(\log n)$ worst-case.

Exercise 4.2 — Heap: amortized analysis (Φ via depths)

Assignment Source: weekplan-1.png (p.1)

Text Source: slides 8 (Potential); text §Heaps

Potential. Let $\Phi(D) = c \cdot (\text{sum of depths of all nodes in the heap})$, $\text{depth}(\text{root})=0$. Choose constant $c \geq 1$ so one swap changes Φ by at least the work saved.

- **INSERT.** Actual cost $c_{\text{ins}} = \# \text{swims} \leq h = \lfloor \log_2(n+1) \rfloor$. Each swim decreases depth by 1 $\Rightarrow \Delta\Phi = -c$ per swap, but the inserted leaf initially increases Φ by its initial depth $d_0 \leq h$. Net $\hat{c}_{\text{ins}} = c_{\text{ins}} + \Delta\Phi \leq h - c \cdot h + c \cdot h = O(\log n)$.
- **EXTRACT-MAX.** Actual cost $c_{\text{ext}} = \# \text{sinks} \leq h$. Removing last leaf (moved to root) decreases Φ by its old depth ($\approx h$). Each sink increases depth by 1 $\Rightarrow \Delta\Phi \leq +c$ per level, but the removal contributes $-c \cdot h$. With c large enough (e.g., $c=2$), $\hat{c}_{\text{ext}} \leq c_{\text{ext}} - c \cdot h + c \cdot h / 2 \leq O(1)$.

Rigorous tuning: pick c so that each **sink level's** $+c$ is offset by the removed leaf's depth budget ($\approx h$). Summed over a sequence, Φ never negative.

Transfer Pattern. Tree of bounded height with symmetric up/down adjustments; potential = sum of node depths.

 **Answer:** INSERT **O(log n)** amortized; EXTRACT-MAX **O(1)** amortized (with Φ as above).

Exercise 5.1 — Sequence T_i (aggregate)

Assignment Source: weekplan-1.png (p.1)

Text Source: slides 1–2

Let

```
$$ T_i = \begin{cases} 2i & \text{if } i \text{ is a power of two}, \\ 1 & \text{otherwise.} \end{cases} $$
```

Over m operations, expensive indices are $i = 1, 2, 4, 8, \dots, 2^{\lfloor \log_2 m \rfloor}$. Their total is $\sum_{k=0}^{\lfloor \log_2 m \rfloor} 2 \cdot 2^k = 2(2^{\lfloor \log_2 m \rfloor + 1} - 1) < 4m$. All other (at most m) operations cost 1 each. Hence total $T(m) < 5m$ and amortized cost is **O(1)**.

 **Answer:** Aggregate $\Rightarrow \mathbf{O(1)}$ amortized.

Exercise 5.2 — Sequence T_i (accounting)

Assignment Source: weekplan-1.png (p.1)

Text Source: slides 6

Charge 5 to every operation. For non-power-of-two i , spend 1 and bank 4. At $i=2^k$, actual cost is $2 \cdot 2^k$; the preceding block of $2^k - 1$ operations contributed $\geq 4(2^k - 1) > 2 \cdot 2^k - 4$ credits, and earlier blocks cover the remaining constant slack. Credits never negative.

 **Answer:** With charge 5, bank covers spikes $\Rightarrow \mathbf{O(1)}$ amortized.

Exercise 5.3 — Sequence T_i (potential)

Assignment Source: weekplan-1.png (p.1)

Text Source: slides 8

Let i in binary have t trailing zeros ($\text{tz}(i)$). Define Φ after i operations as $\Phi_i = 2 \cdot (2^{\text{tz}(i)} - 1)$. Bit-flip view: moving from $i-1$ to i flips t low bits from $1 \rightarrow 0$ and one bit from $0 \rightarrow 1$.

- For non-power-of-two ($t = \text{tz}(i) < k$): actual cost 1, potential change $\Delta\Phi \in [-2, +2] \Rightarrow \hat{c} = \mathbf{O(1)}$.
- For $i = 2^k$: actual cost $2 \cdot 2^k$, but $\text{tz}(i)=k$ and $\Delta\Phi = 2(2^k - 1) - (2(2^{\text{tz}(i-1)} - 1)) \leq 2 \cdot 2^k$. Choose constants so $\hat{c} \leq \text{constant}$.

(Any equivalent binary-counter potential suffices.)

 **Answer:** $\mathbf{O(1)}$ amortized.

Exercise 6.1 — Dynamic Binary Search (DBS): duplicates on INSERT

Assignment Source: weekplan-2.png (p.2)

Text Source: slides 5–7 (DBS); week8_text.pdf §DBS

Issue. DBS keeps each nonempty level as a distinct **sorted array** of size 2^i . Duplicate keys across levels waste space and complicate DELETE.

Efficient fix. Store **(key, count)** pairs in arrays. During binary search, compare keys; INSERT(x): create $(x, 1)$ at A_0 and cascade merges as usual, but when merging arrays, **deduplicate runs**: if the same key appears in both inputs, output one $(\text{key}, \text{count}_a + \text{count}_b)$. SEARCH unchanged.

Complexity. Merging still linear in total run length; run-compression is linear in the number of duplicate occurrences. Asymptotic costs unchanged: INSERT amortized $\mathbf{O(\log n)}$; SEARCH $\mathbf{O(\log n)}$ (with cascading) as assumed.

Transfer Pattern. From set to multiset: replace element unit weight with multiplicity counters; maintain invariant at merge points.

 **Answer:** Use **(key, count)** with deduplication during merges; INSERT remains amortized $\mathbf{O(\log n)}$; SEARCH stays $\mathbf{O(\log n)}$ under the given assumption.

Exercise 6.2 — DBS: DELETE in $\mathbf{O(\log n)}$ amortized

Assignment Source: weekplan-2.png (p.2)

Text Source: slides 7 (Costs)

Strategy (lazy deletion + rebuild).

1. **SEARCH** x across $O(\log n)$ arrays to find a (key, count). If count > 1: decrement and stop ($O(\log n)$).
2. If count = 1: mark as a **tombstone**. Maintain invariant per array A_i : when tombstones $\geq \frac{1}{2}$ of live items, **rebuild** A_i by compacting to a new sorted array (and possibly cascading merges downward to restore the $0/2^i$ fullness invariant).

Amortized analysis (accounting). Attach 2 credits to each newly inserted *live* item in A_i : 1 to pay for its eventual deletion mark, 1 to pay for its share of a future rebuild. Each element is compacted at most once before leaving the level; rebuild work $O(2^i)$ is charged to $\Omega(2^i)$ tombstones.

Complexity. SEARCH $O(\log n)$. DELETE worst-case $O(2^i)$ inside a rebuild, but **amortized $O(\log n)$** over sequences.

Transfer Pattern. Maintain density by **lazy delete + repack when half empty**; classic in hash tables and ordered arrays.

 **Answer:** DELETE can be implemented in **$O(\log n)$ amortized** using tombstones and thresholded rebuilds.

Exercise 7.1 — Billy the Rabbit: rounds

Assignment Source: weekplan-2.png (p.2)

Text Source: week8_text.pdf §Exercises

Round r moves forward 2^{r-1} steps before returning. He reaches the carrots in the first r with $2^{r-1} \geq n$, i.e., $r = \lceil \log_2 n \rceil$.

 **Answer:** $\lceil \log_2 n \rceil$ rounds.

Exercise 7.2 — Billy: total steps (asymptotic)

Assignment Source: weekplan-2.png (p.2)

Text Source: week8_text.pdf §Exercises

Total before the final round: $2(1 + 2 + \dots + 2^{r-2}) = 2^r - 2$. In the final round he walks exactly n steps forward and stops upon finding the carrots. Since $2^{r-2} < n \leq 2^{r-1}$, we have $2^r < 4n$. Thus total steps $< (4n - 2) + n = O(n)$ and also $> n$.

 **Answer:** $\Theta(n)$ total steps.

Exercise 7.3 — Billy: amortized steps per round

Assignment Source: weekplan-2.png (p.2)

Text Source: week8_text.pdf §Exercises

With $r = \lceil \log_2 n \rceil$ rounds and $\Theta(n)$ total steps, the amortized steps/round are $\Theta\left(\frac{n}{\log n}\right)$.

 **Answer:** $\Theta(n / \log n)$.

Puzzle — Princesses (truth, liar, random)

Assignment Source: weekplan-2.png (p.2)

Goal. Ask one yes/no question to one daughter, then choose whom to marry, preferring an extreme (oldest truth-teller or youngest liar) and avoiding the middle (random).

Fact (impossibility). With exactly one yes/no question to a possibly random respondent, it is **impossible** to deterministically identify an extreme in all cases: if you question the random daughter, Nature can force any answer, making any deterministic selection rule fail on some labeling.

Best-possible strategy (avoid asking the one you might pick). Pick two daughters B and C in front of you; ask A (the third):

“If I asked you ‘Is B the middle?’, would you say yes?”

- If A is an extreme, this self-reference neutralizes truth/lie and yields a *reliable* answer to whether B is middle. Then **marry the non-middle among B and C** accordingly.
- If A is the middle, the answer is unreliable, but you still **never marry A**. Your choice between B and C succeeds with probability $\geq 1/2$; no one-question strategy can guarantee 1.

 **Answer:** No deterministic guarantee exists with one question if you might query the random sister. The above question maximizes reliability; never choose the respondent.

Summary

- **Stacks (with MultiPush/MultiPop):** O(1) amortized via “pushed once, popped once”.
- **Queues via two stacks:** O(1) amortized; each element moves at most once.
- **Set Union (coloring small→large):** total recolorings $\leq n \log n$; SAMESET O(1).
- **Heaps:** With $\Phi = \text{sum of depths}$, EXTRACT-MAX becomes amortized O(1), INSERT stays O($\log n$).
- **DBS:** INSERT O($\log n$) amortized; handle duplicates with (key, count); DELETE O($\log n$) amortized via lazy deletion + rebuild.
- **Spike sequence T_i:** O(1) amortized by aggregate/accounting/potential.
- **Billy:** rounds $\lceil \log_2 n \rceil$; total steps $\Theta(n)$; amortized per round $\Theta(n/\log n)$.

Notation blurb. n = number of elements; m = number of operations; Φ = potential; “amortized” means per-operation cost averaged over sequences in the worst case.