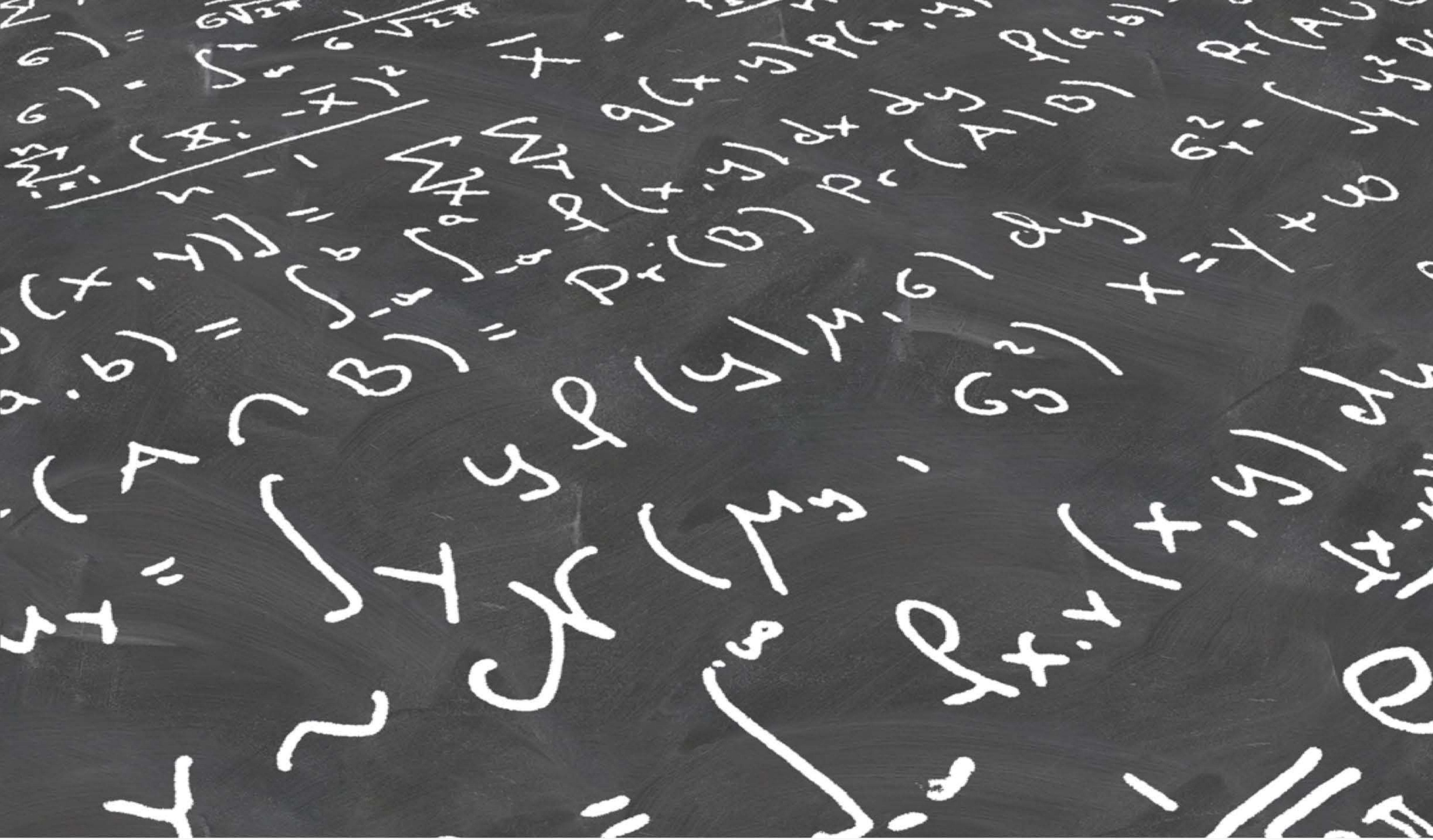

3. Discrete Random Variables

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Lars Mandrup

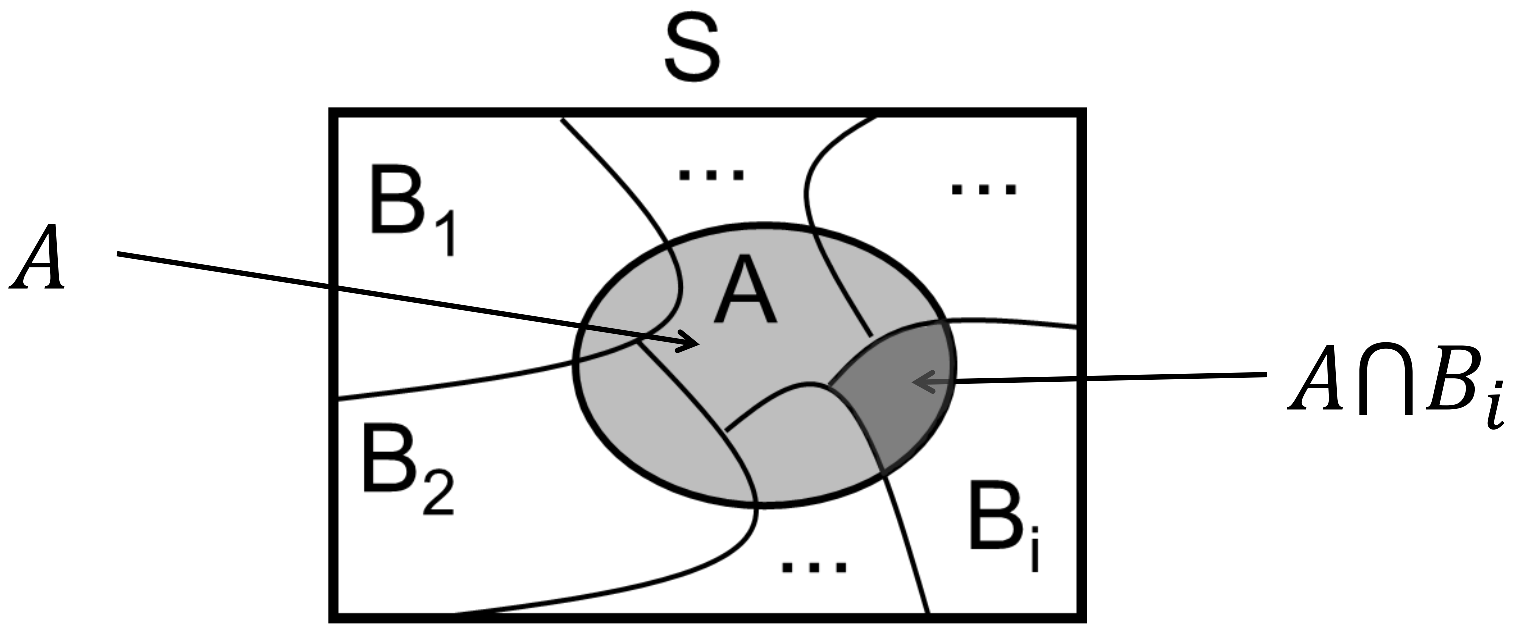
Agenda for Today

- Repetition from last time
- Definition of a Stochastic Random Variable
- Discrete Stochastic Variables

Total Probability

We sometime call it the marginal

- Pr(A) of an event is the total probability of that event.



$$Pr(A) = Pr(A \cap B_1) + Pr(A \cap B_2) + \cdots + Pr(A \cap B_i) + \cdots$$
$$= Pr(A|B_1) \cdot Pr(B_1) + Pr(A|B_2) \cdot Pr(B_2) + \cdots$$

where the B_i's are mutually exclusive ($B_i \cap B_j = \varnothing$ for $i \neq j$)
and $S = B_1 \cup B_2 \cup \ldots \cup B_i \cup \ldots$

Bayesian Terms

- **Prior**: What are the overall probability of an event E?

$$Pr(E)$$

- **Likelihood**: What are the probability of a test T given event E?

$$Pr(T|E) = \frac{Pr(T \cap E)}{Pr(E)} = \frac{Pr(E|T) \cdot Pr(T)}{Pr(E)}$$

- **Total Probability**: What is the total probability of the test?

$$Pr(T) = Pr(T|E) \cdot Pr(E) + Pr\left(T|\bar{E}\right) \cdot Pr(\bar{E})$$

- **Posterior**: What are the probability the event given the test T?

$$Pr(E|T) = \frac{Pr(T \cap E)}{Pr(T)} = \frac{Pr(T|E) \cdot Pr(E)}{Pr(T)}$$

Combinatorics

- The number of possible outcomes of k trials, sampled from a set of n objects.

Types of Experiments:

- With or without replacement
- Ordered or unordered

		Replacement	
		With	Without
Sam-pling	Ordered	n^k	$P_k^n = \frac{n!}{(n-k)!}$
	Unordered	$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!\,(n-1)!}$	$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

The Binomial Distribution

- We have n repeated trials.
- Each trial has two possible outcomes
  - **Success** — probability p
  - **Failure** — probability q=1-p

Bernoulli trial

- What is the probability of having k successes out of n trials?
- We write this question as:

$$\boxed{Pr_n(k) = \frac{n!}{k!\,(n-k)!} p^k q^{n-k} = \binom{n}{k} p^k q^{n-k}}$$

- Faculty: $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$

  $0! = 1$

Also called a random experiment

Stochastic Experiment

- An experiment in which you can not predict the outcome

Examples:

- Rolling a dice
- Sample space for the experiment is: {1, 2, 3, 4, 5, 6}

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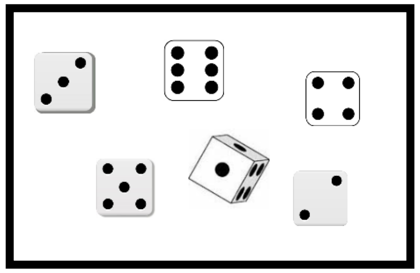

- Flip a coin

- Sample space for the experiment is: {head, tail}

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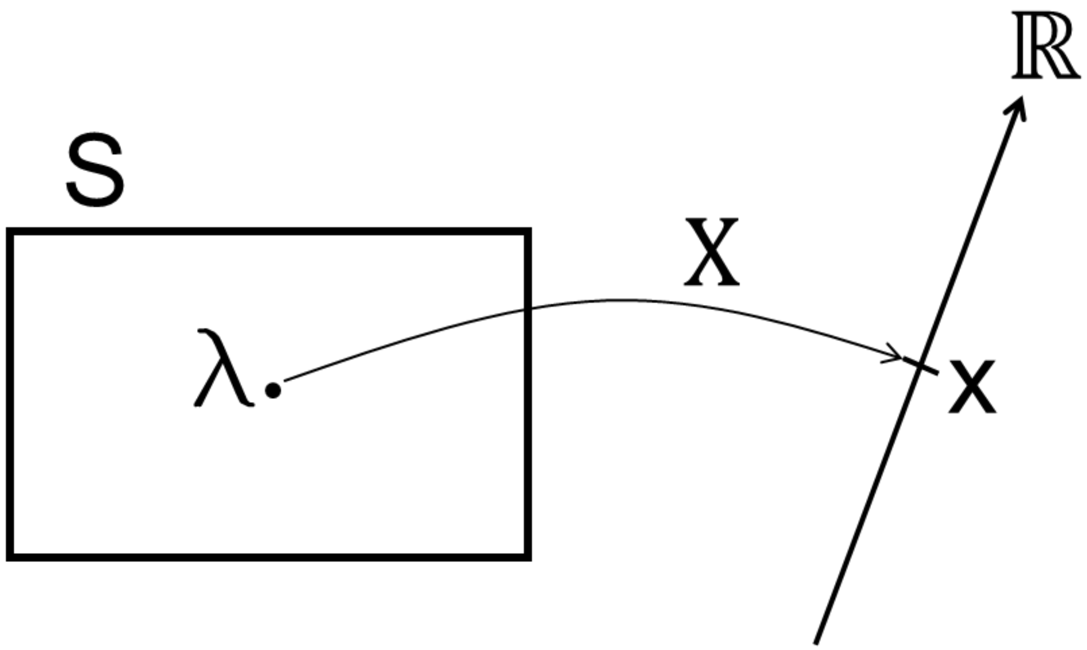
Also just called a random variables

Stochastic Random Variables

- A random variable tells something important about a stochastic experiment.
- Can be discrete or continous



Examples:

- The numbers on a dice (discrete):
  - Sample space for variable X is : $\{1, 2, 3, 4, 5, 6\}$
  - Sample space for variable Y “Even (1)/Uneven (-1)”: $\{1, -1\}$
- The hight of students at IHA (continous):
  - Sample space for variable H is all real numbers: [100;250] cm.

Probability Mass Function (PMF)

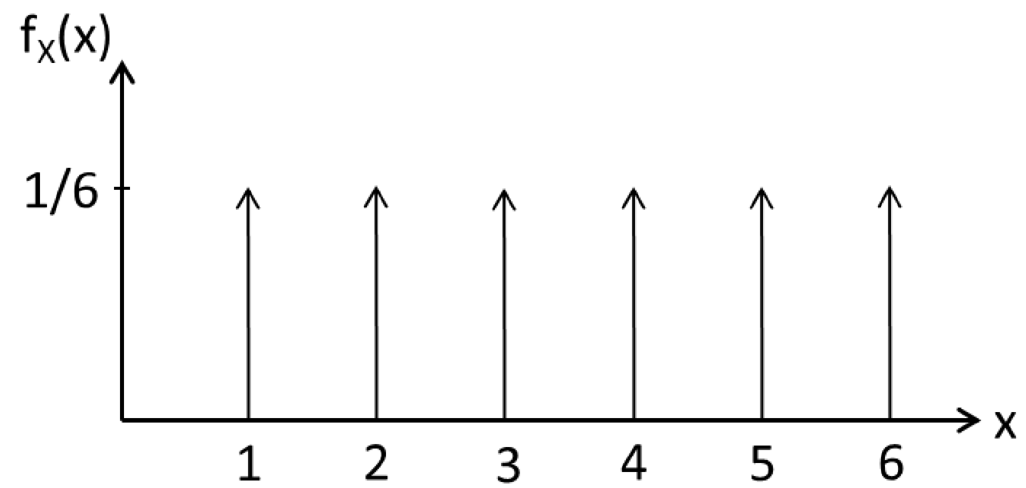
- Sample space for X.
- X is a discreet stochastic variable.

$$\boxed{f_X(x) = \begin{cases} Pr(X = x_i) & for\ X = x_i \\ 0 & otherwise \end{cases}}$$

$$0 \leq f_X(x) \leq 1$$

- We have that: $\sum_{i=1}^{n} f_X(x_i) = \sum_{i=1}^{n} Pr(X = x_i) = 1$

Example: Laplace Dice
(perfect dice)



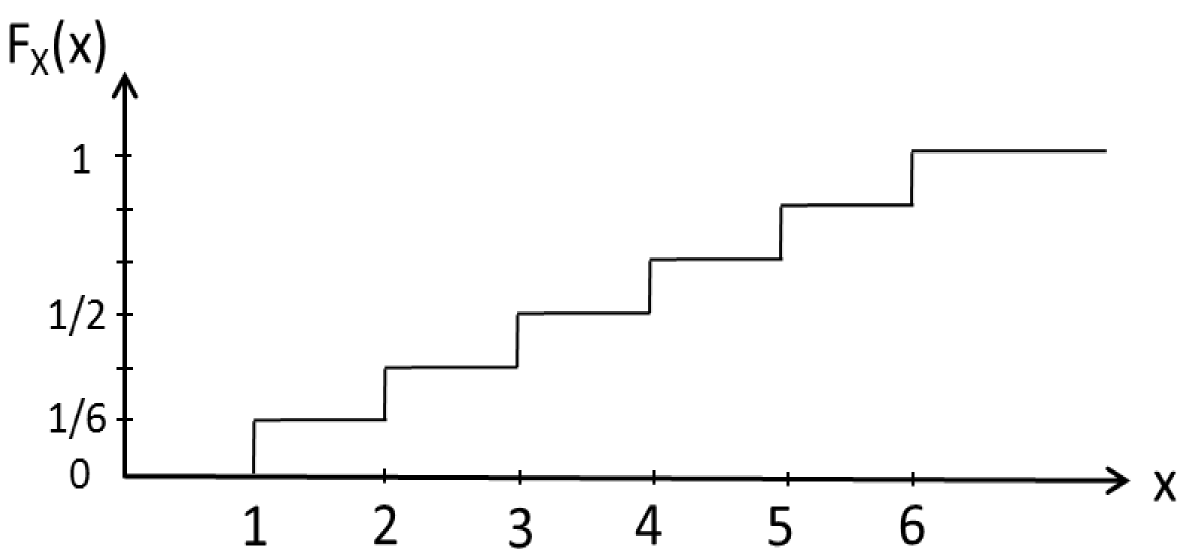
Cumulative Distribution Function (CDF)

- Sample space for X.
- X is a discreet stochastic variable.
- $F_X(x)$ is a non-decreasing step-function.

$$\boxed{F_X(x) = Pr(X \leq x)} \qquad 0 \leq F_X(x) \leq 1$$

- We have that: $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$

Example: Laplace Dice
(perfect dice)



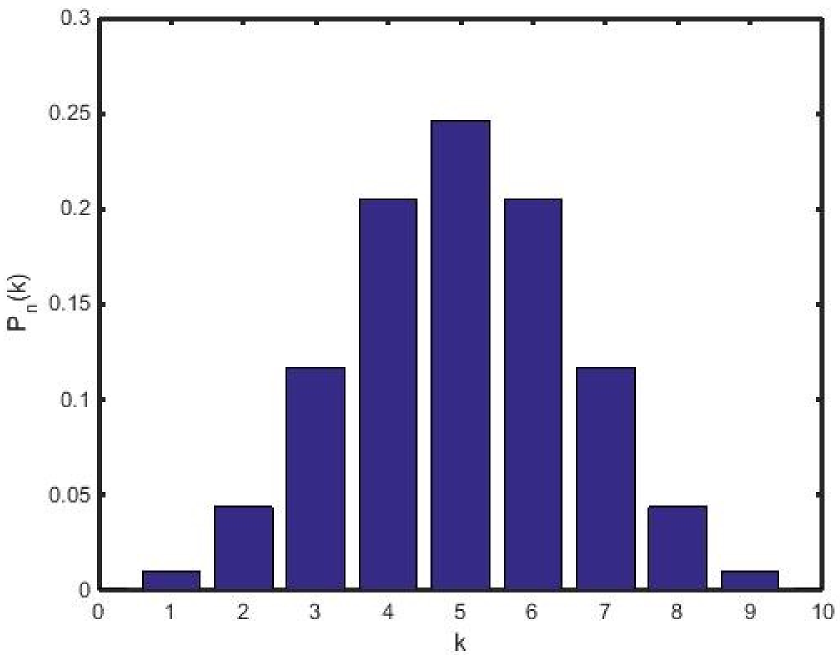
The Binomial Mass Function

- We have n repeated trials.
- Each trial has two possible outcomes
  - **Success** — probability p
  - **Failure** — probability 1-p

Also called a Bernoulli trial

- We write the mass function as:

$$\boxed{f(k|n,p) = \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k}}$$



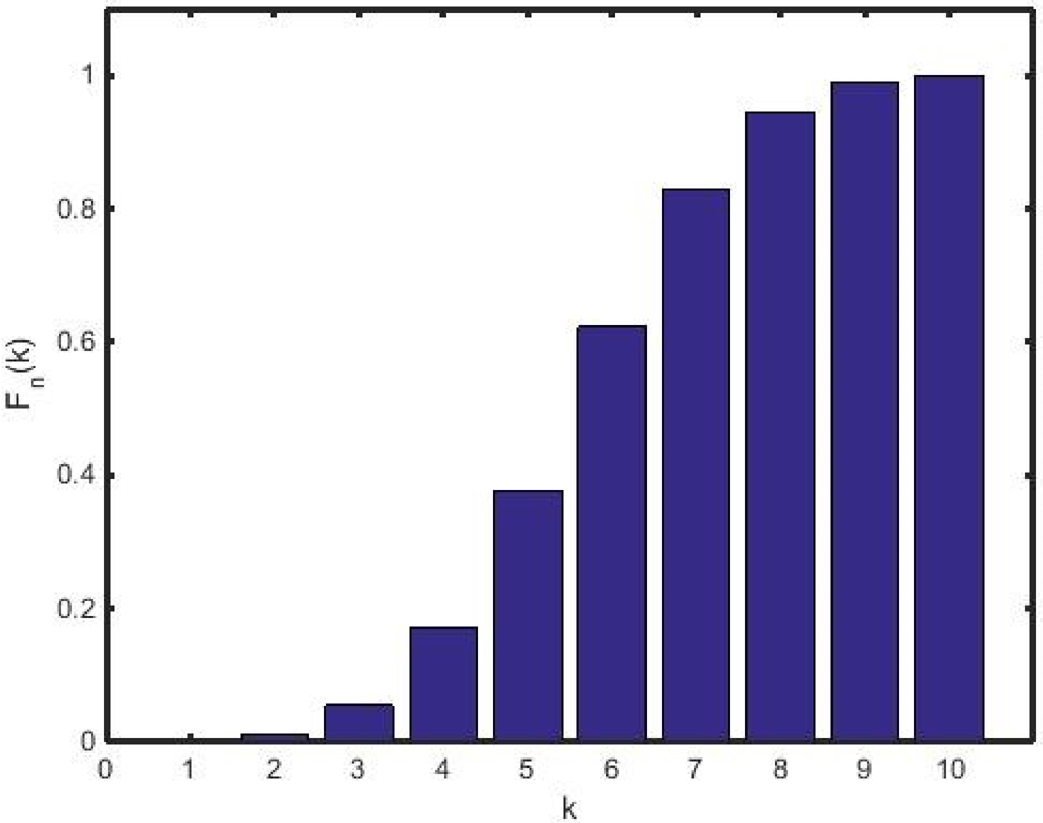
The Binomial Distribution

- The probability mass function is given as:

$$f(k|n,p) = \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}$$

- We write the distribution as the sum:

$$F(k|n,p) = \sum_{i=0}^{k} f(i|n,p)$$



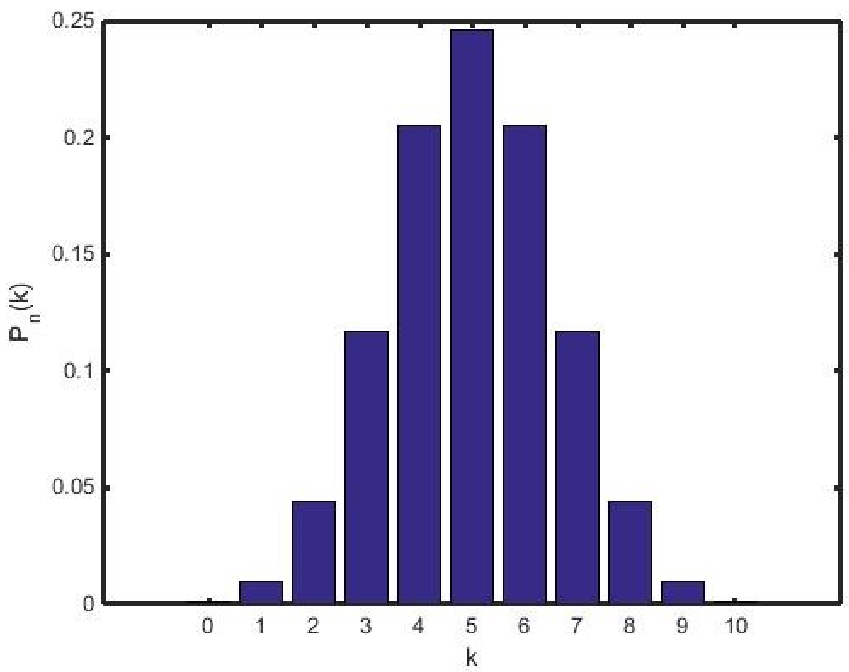
Expectation of a Discrete Random Variable

Example: If I want ten children, how many girls can I expect to get?

Answer: I assume a Binomial distribution with p=0.5:

$$f(k|10{,}0.5) = \binom{10}{k} \cdot 0.5^{k} \cdot 0.5^{10-k} = \binom{10}{k} \cdot 0.5^{10}$$

$$where \ \binom{10}{k} = \frac{10!}{k!\,(10-k)!}$$



$$\begin{aligned} E[k] &= 0 \cdot f(0|10{,}0.5) + 1 \cdot f(1|10{,}0.5) + \cdots + 10 \cdot f(10|10{,}0.5) \\ &= \left(0 + 1 \cdot \binom{10}{1} + 2 \cdot \binom{10}{2} \ldots + 10 \cdot \binom{10}{10}\right) \cdot 0.5^{10} \\ &= (0 + 1 \cdot 10 + 2 \cdot 45 + \cdots + 10 \cdot 1) \cdot 0.5^{10} = 10 \cdot 0.5 = 5 \end{aligned}$$

Expectation of a Discrete Random Variable

- We define the <u>mean</u> or the <u>expectation</u> of a discreet random variable as:

n is the number of outcomes

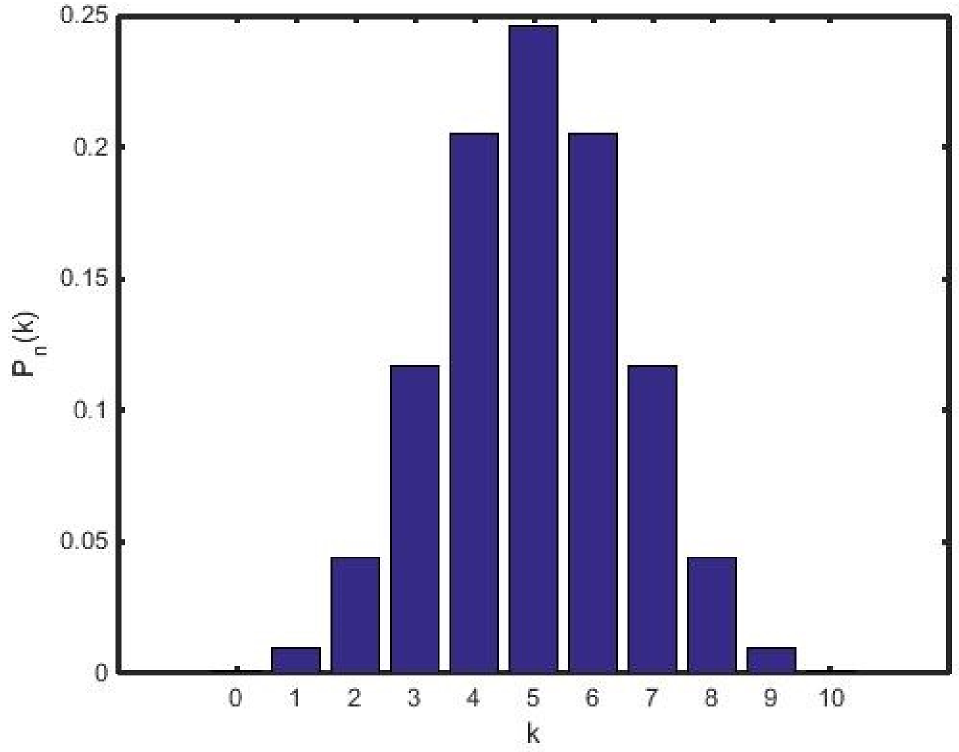
$$\boxed{\bar{X} = E[X] = \sum_{i=1}^{n} x_i \, f_X(x_i)}$$

x_i is its outcome

The Binomial Distribution (cont'd)

- For the Binomial distribution, we have:

$$\boxed{\begin{aligned} E[k] &= n \cdot p \\ Var(X) &= n \cdot p \cdot (1-p) \end{aligned}}$$



- Where the variance is defined as:

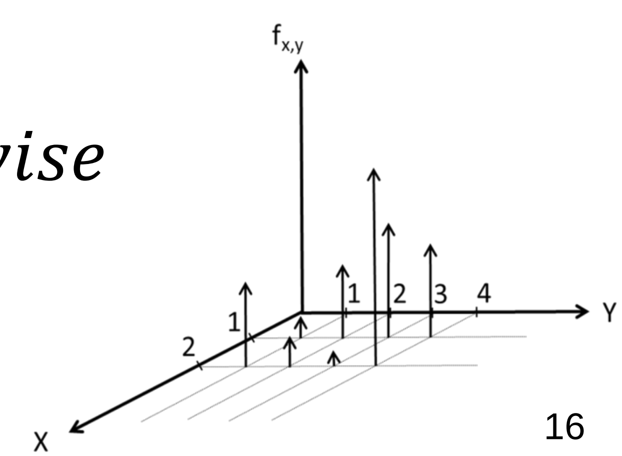
$$Var(X) = \sigma^2 = E\left[X^2\right] - E[X]^2$$

Two Simultaneous Discreet Random Variables

- Two (or more) discreet random variables X and Y
- We can discribe the two probabilities as a simultaneous pmf:

Joint (Simultaneous) pmfs:

$$f_{X,Y}(x,y) = \begin{cases} Pr\Big((X = x_i) \cap (Y = y_j)\Big) & for\ \ X = x_i \wedge Y = y_j \\ 0 & otherwise \end{cases}$$

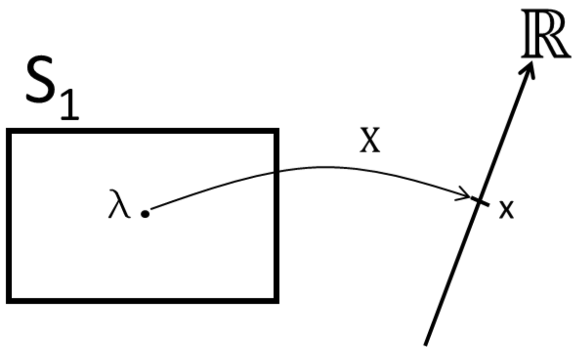


Fx.: X = The number of bicycles in front of IHA
Y = The number of people inside IHA

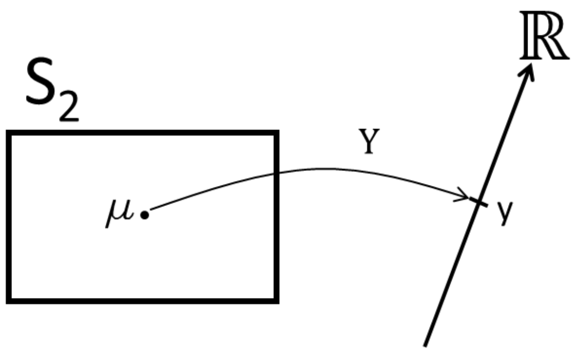
Two Simultaneous Discrete Random Variables

Marginal pmfs:

$$f_X(x) = \sum_{y} f_{X,Y}(x,y) \qquad f_Y(y) = \sum_{x} f_{X,Y}(x,y)$$



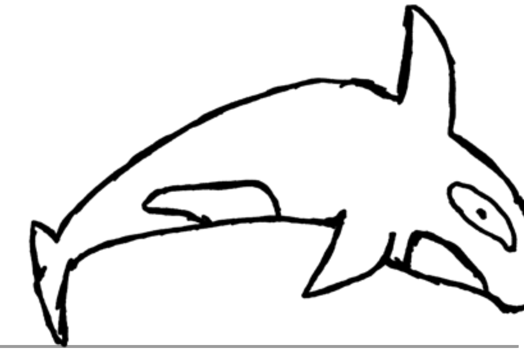
Conditional pmfs / Bayes Rule:



$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \Pr(X = x|\mathrm{Y} = \mathrm{y})$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \Pr(Y = y|\mathrm{X} = \mathrm{x})$$

Orca Example



- Let us assume that the discreet simultaneous mass function (pmf) for observing a orca at a specific ocean and its gender is

$f_X(x)$

$f_{X,Y}(x,y)$

Gender (X)\ Location (Y)	Atlantic (1)	Antartica (2)	Pacific (3)	Seaworld (4)	Total
female (1)	2/60	7/60	11/60	9/60	29/60
male (2)	8/60	3/60	1/60	19/60	31/60
Total	10/60	10(60	12/60	28/60	1

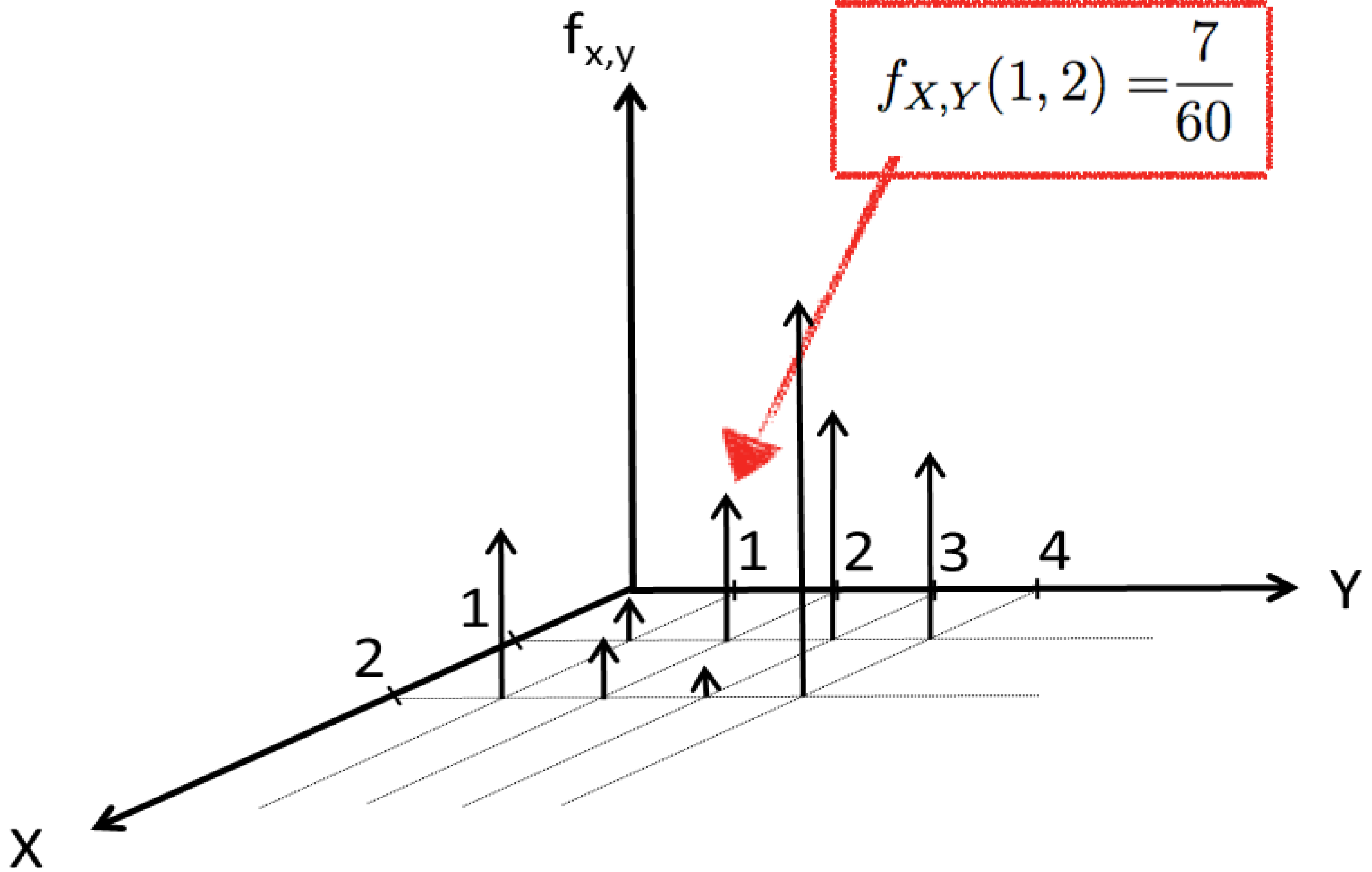
$f_Y(y)$

Fx.: $Pr(Male|Atlantica) = f_{X|Y}(2|1) = \frac{f_{X,Y}(2,1)}{f_Y(1)} = \frac{8/_{60}}{10/_{60}} = \frac{8}{10} = 0{,}8$

Orca Example - Joint pmf

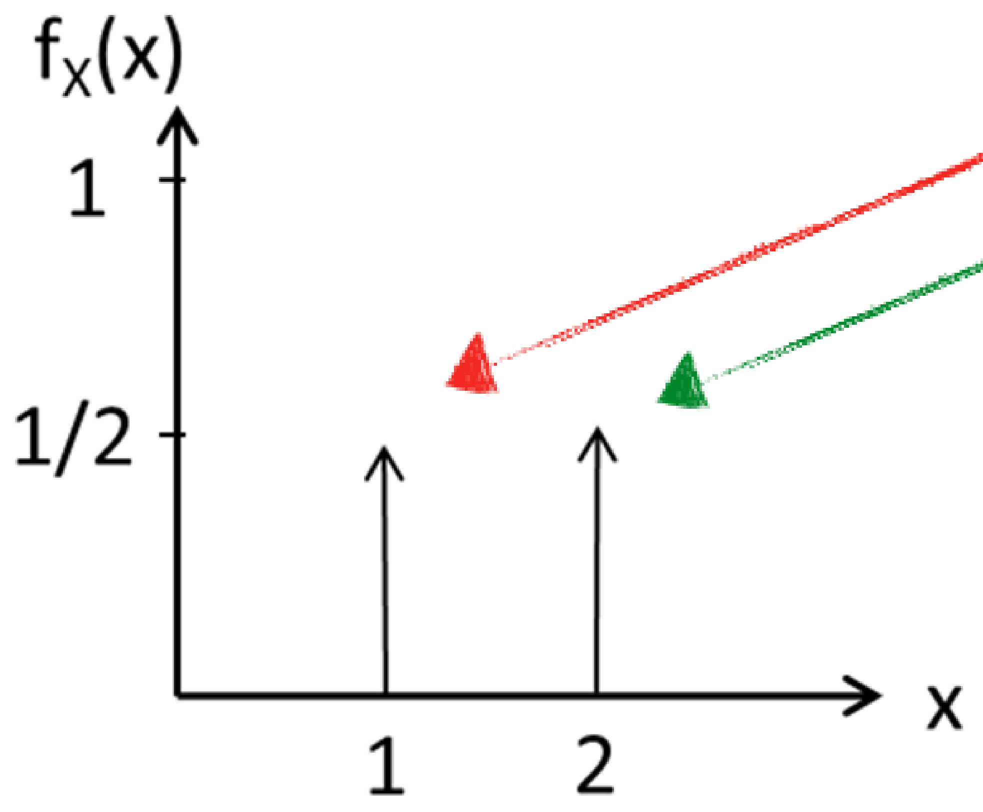
Gender (X)\ Location (Y)	Atlantic (1)	Antartica (2)	Pacific (3)	Seaworld (4)	Total
female (1)	2/60	7/60	11/60	9/60	29/60
male (2)	8/60	3/60	1/60	19/60	31/60
Total	10/60	10(60	12/60	28/60	1

$$f_{X,Y}(1,2) = \frac{7}{60}$$



Orca Example – Marginal pmf

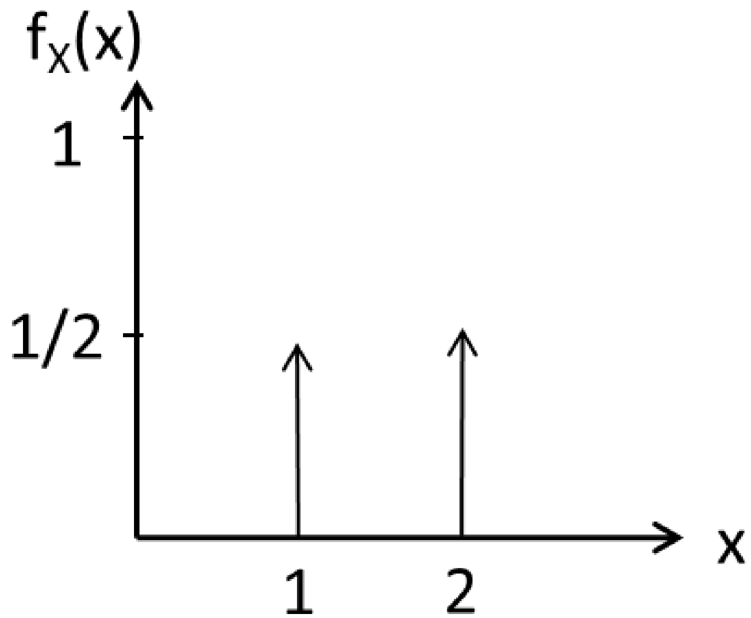
Gender (X)\ location (Y)	Atlantic (1)	Antartica (2)	Pacific (3)	Seaworld (4)	Total
female (1)	2/60	7/60	11/60	9/60	29/60
male (2)	8/60	3/60	1/60	19/60	31/60
Total	10/60	10(60	12/60	28/60	1



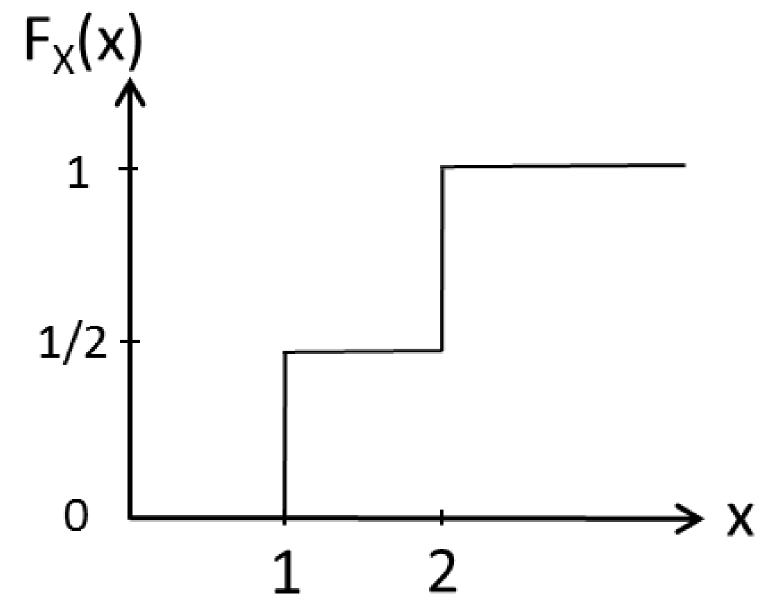
$$
\begin{aligned}
f_X(1) = & f_{X,Y}(1,1) + f_{X,Y}(1,2) + f_{X,Y}(1,3) + f_{X,Y}(1,4) \\
= & \frac{2}{60} + \frac{7}{60} + \frac{11}{60} + \frac{9}{60} = \frac{29}{60} \\
f_X(2) = & f_{X,Y}(2,1) + f_{X,Y}(2,2) + f_{X,Y}(2,3) + f_{X,Y}(2,4) \\
= & \frac{8}{60} + \frac{3}{60} + \frac{1}{60} + \frac{19}{60} = \frac{31}{60}
\end{aligned}
$$

Orca Example – Quick Rewrite to cdf

- We can rewrite the pmf to the cdf



Marginal pmf



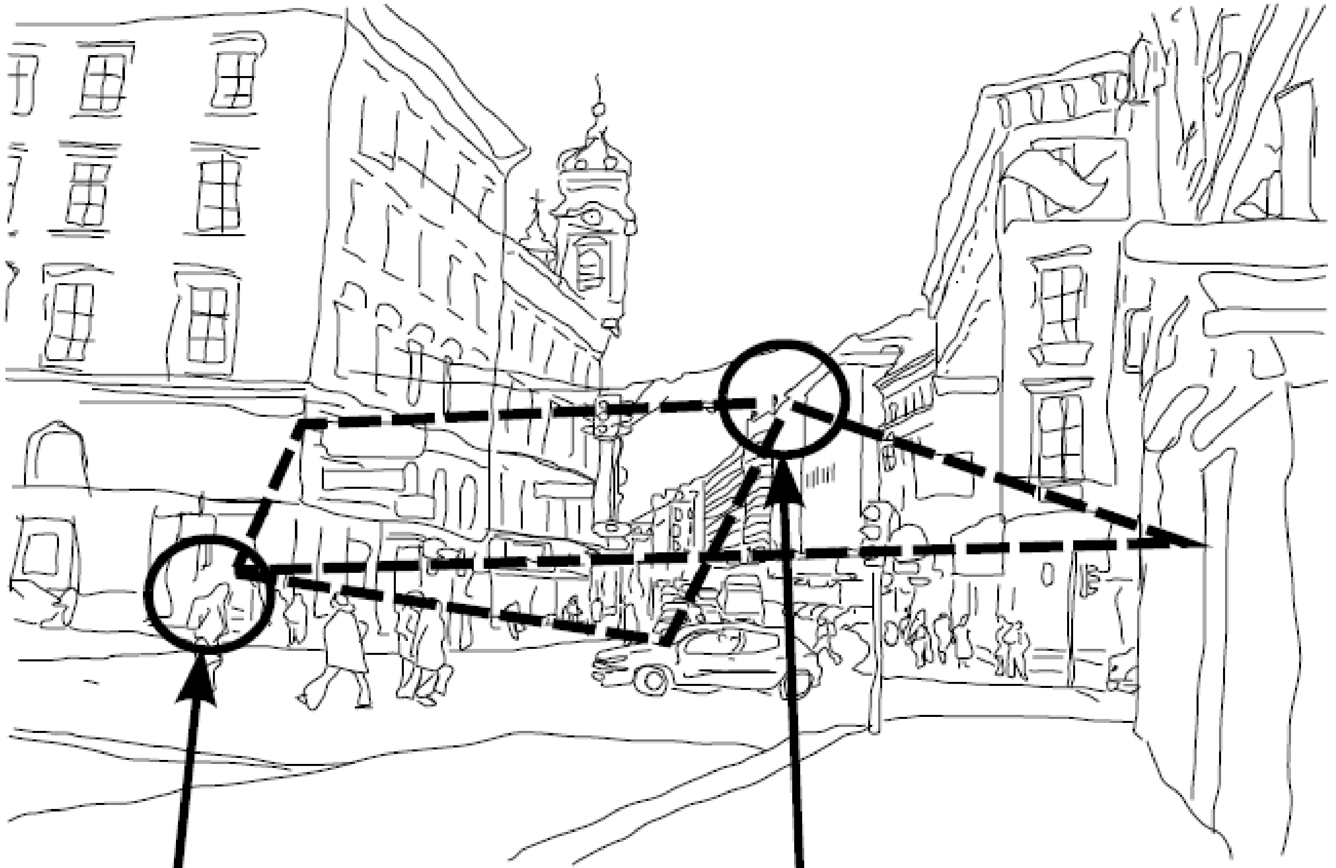
Marginal cdf

$$f_X(1) = \frac{29}{60}$$
$$f_X(2) = \frac{31}{60}$$

$$F_X(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{29}{60} & \text{for } 1 \leq x < 2 \\ 1 & \text{for } 2 \leq x \end{cases}$$

Example - Wireless Channel

- A signal in a wireless channel travels with equal probability of three different path from transmitter to receiver



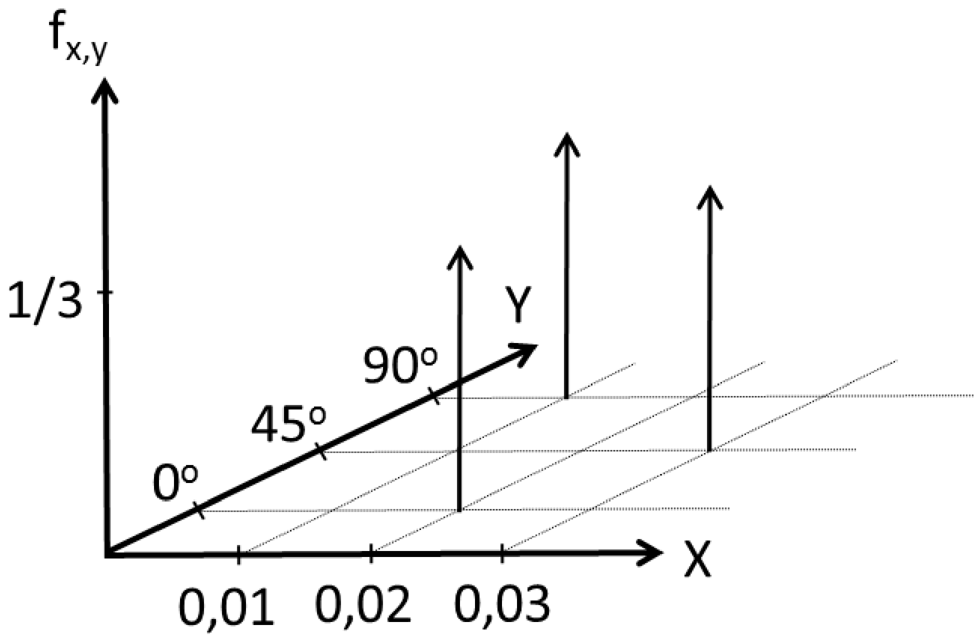
Amplitude\ Phase	0^o	45^o	90^o	**Total**
0.01	0	0	$\frac{1}{3}$	$\frac{1}{3}$
0.02	$\frac{1}{3}$	0	0	$\frac{1}{3}$
0.03	0	$\frac{1}{3}$	0	$\frac{1}{3}$
Total	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	**1**

Example - Wireless Channel: Assignment

- Plot the pmf for the wireless channel.
- What is the Expected Amplitude and Phase?

X Y

Amplitude\ Phase	0^o	45^o	90^o	**Total**
0.01	0	0	$\frac{1}{3}$	$\frac{1}{3}$
0.02	$\frac{1}{3}$	0	0	$\frac{1}{3}$
0.03	0	$\frac{1}{3}$	0	$\frac{1}{3}$
Total	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	**1**



$$E[X] = (0{,}01 + 0{,}02 + 0{,}03) \cdot \frac{1}{3} = 0{,}02$$

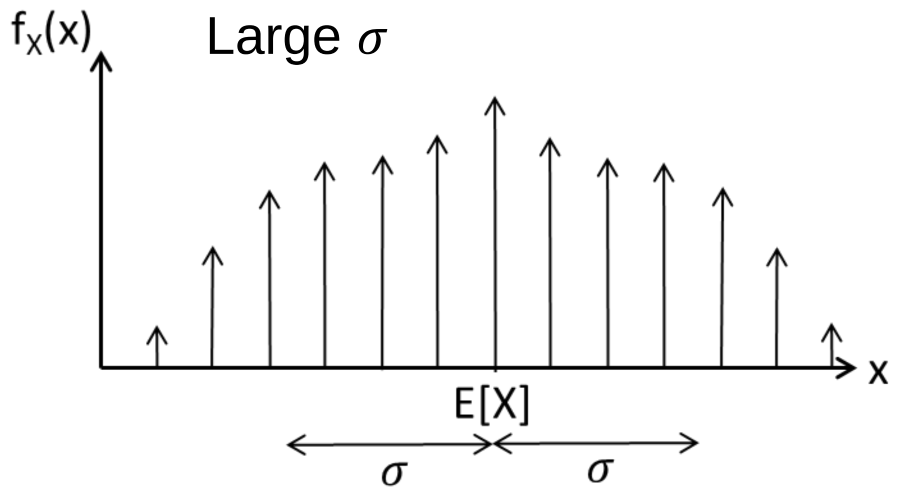
$$E[Y] = (0^o + 45^o + 90^o) \cdot \frac{1}{3} = 45^o$$

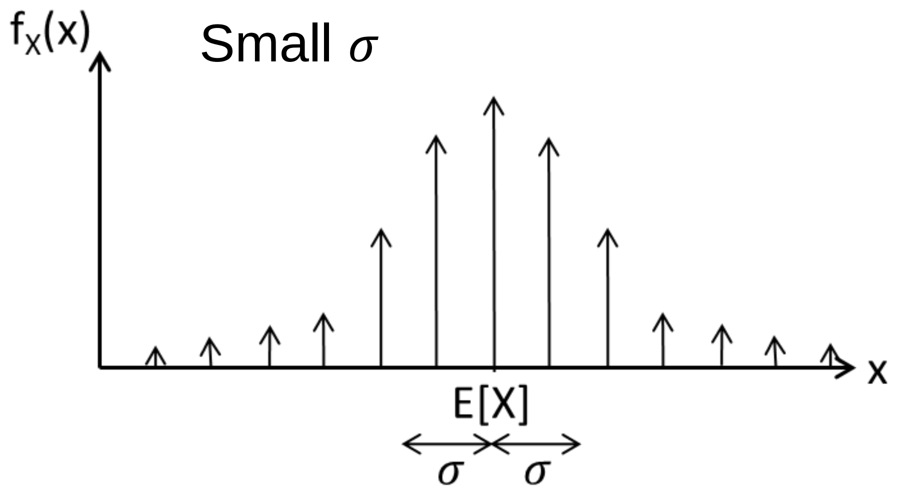
Variance and standard deviation

Variance and standard deviation tells of the spreading of the data

- The variance is an indicator on how much the values of a random variable X are spread around (deviates from) the expectation value.
- The standard deviation σ is the square root of the variance.

$$\boxed{Var(X) = {\sigma_X}^2 = E[X^2] - E[X]^2}$$





Correlation Coefficient

Correlation tells of the coupling between variables

- The correlation coefficient, is an indicator on how much two random variables X and Y are correlated.

$$\boxed{\rho = E\left[\frac{X-\bar{X}}{\sigma_X}\cdot\frac{Y-\bar{Y}}{\sigma_Y}\right] = \frac{E[XY]-E[X]E[Y]}{\sigma_X\sigma_Y}}$$

- We have that: $-1 \leq \rho \leq 1$

Independence

- We have independence between X and Y if and only if:

$$\boxed{f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)}$$

Example of independent random variables:

- A persons height and the current exact distance from the earth to the moon.

Example of dependent random variables:

- The time of day and the amount of bicycles parked the at the engineering college.
- The energy of a mobile signal and the length in meters to a basestation.

Independence

Independence: $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

- Bayes Rule: $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$

  gives that if X and Y are independent, then:

$$\boxed{f_{X|Y}(x|y) = f_X(x)}$$

- Also:

  $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \Rightarrow E[XY] = E[X]E[Y] \Rightarrow \ \rho = 0$

  but the opposite is not allways true!

Dependant Variables – Simple Example

- Given a random variable X
- We define a new random variable Y=X

$$f_{X,Y}(1,1) = \frac{1}{2}$$

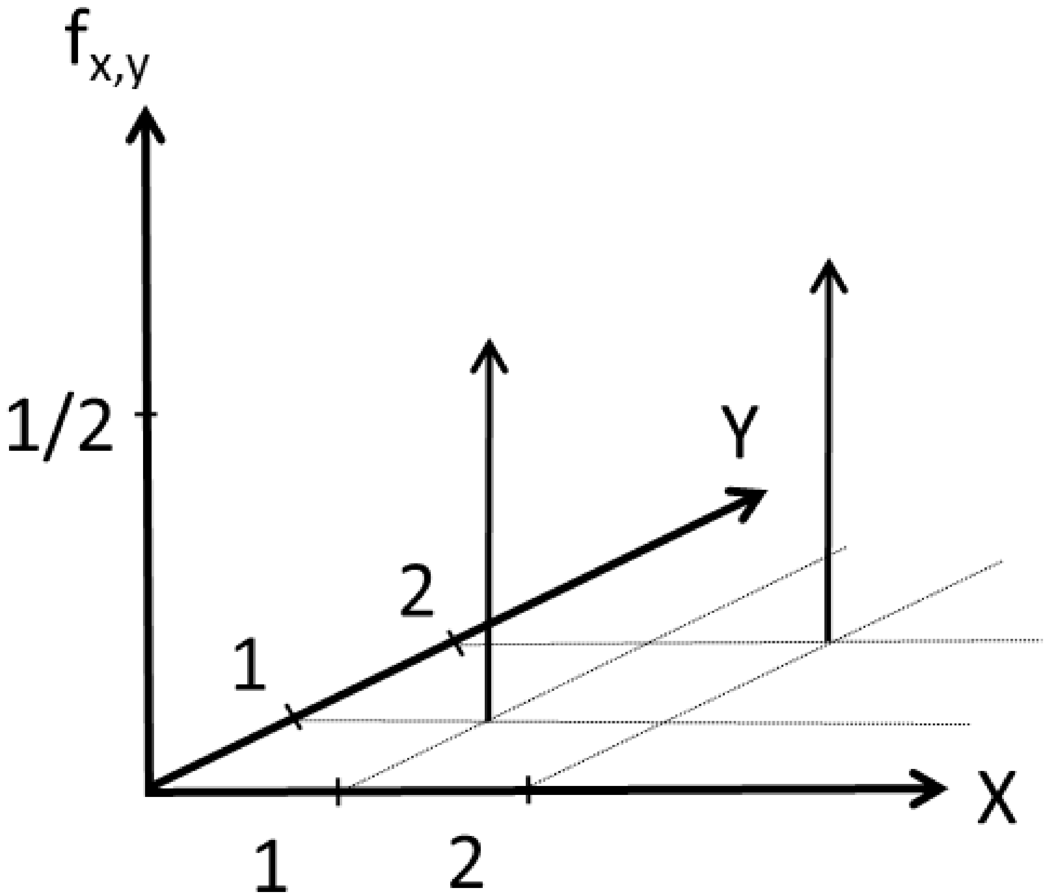
$$f_{X,Y}(2,2) = \frac{1}{2}$$

$$f_{X,Y}(1,2) = 0$$

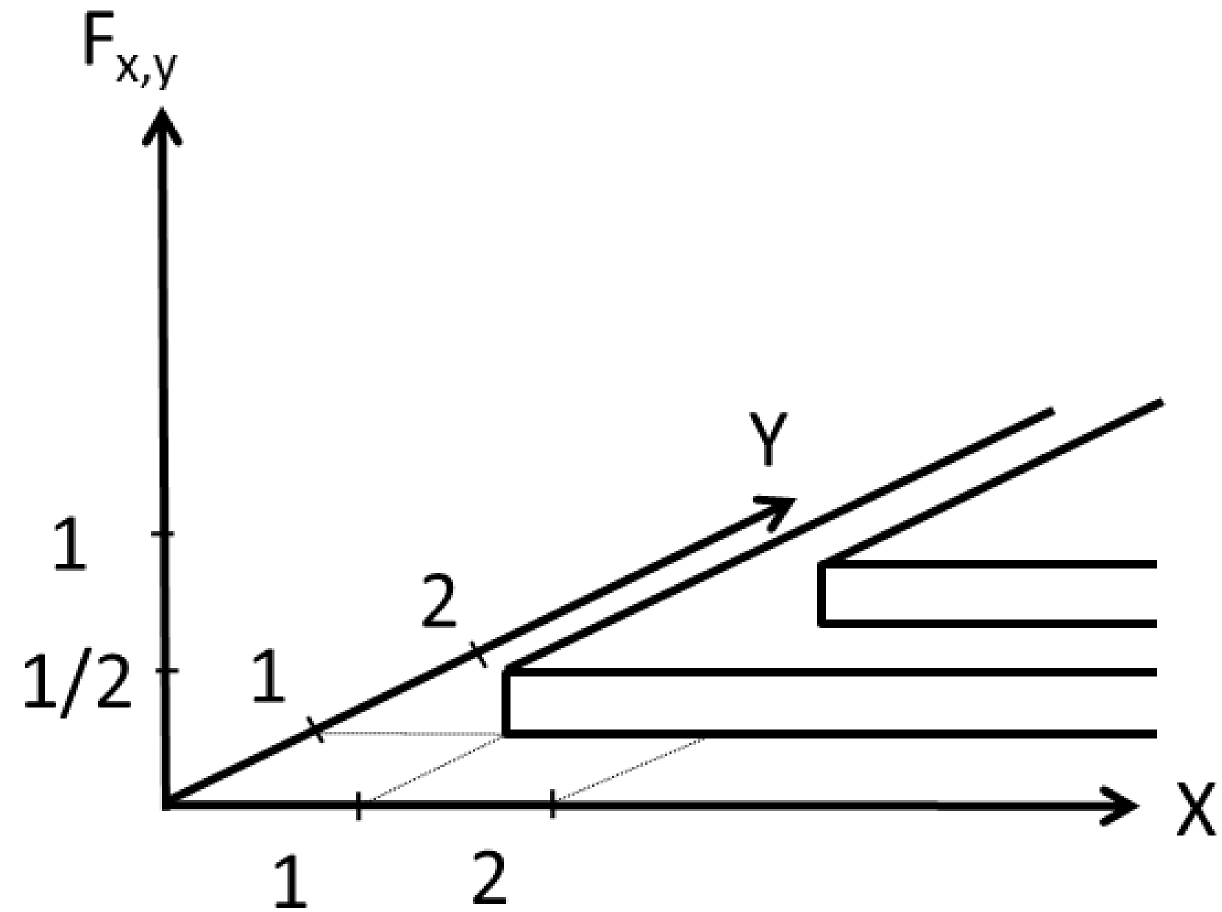
$$f_{X,Y}(2,1) = 0$$

Simple Example - Simultaneous pmf

Plots of the pmf and the cdf:



pmf



cdf

Simple Example – Marginal pmf

$$f_Y(y) = \sum_x f_{X,Y}(x,y)$$

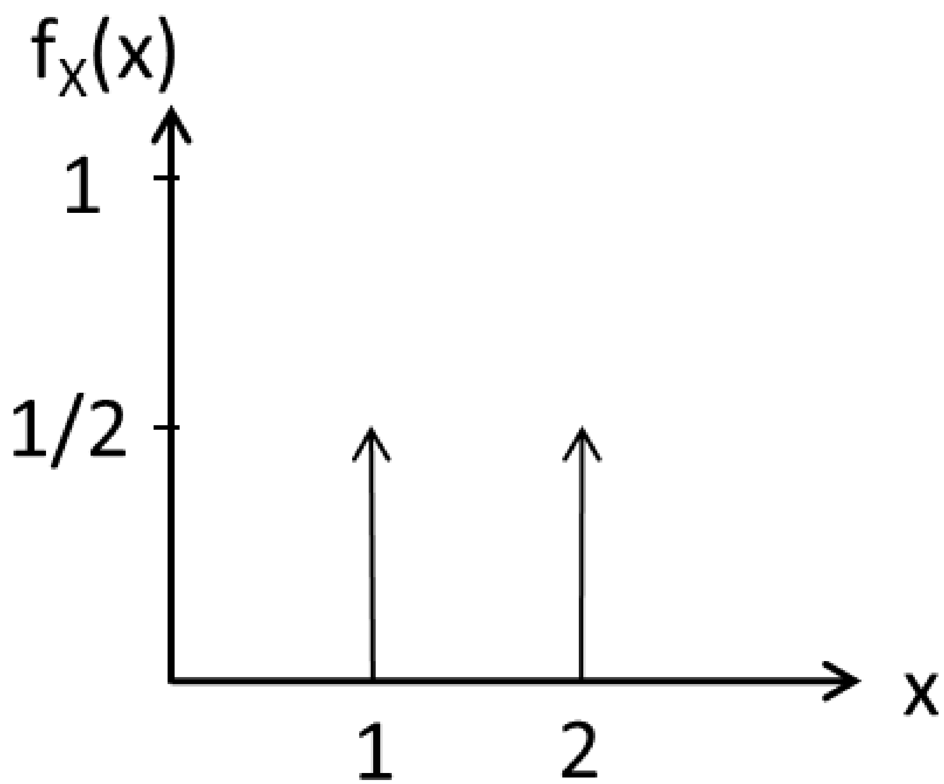
$$f_Y(1) = f_{X,Y}(1,1) + f_{X,Y}(2,1) = \frac{1}{2}$$

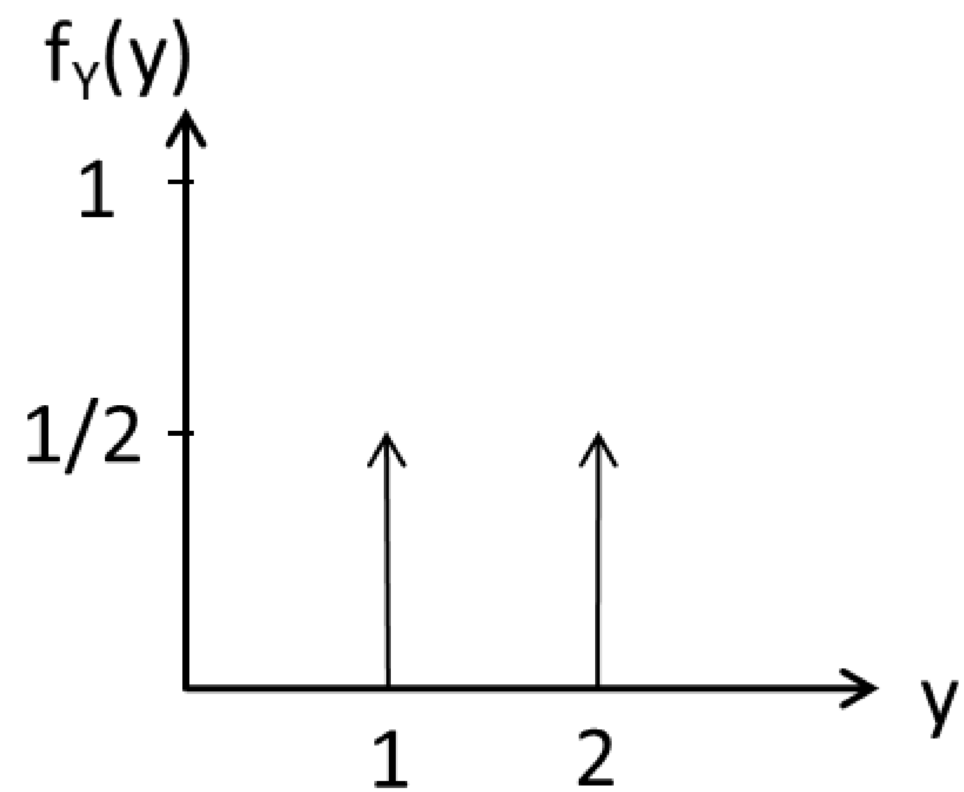
$$f_Y(2) = f_{X,Y}(1,2) + f_{X,Y}(2,2) = \frac{1}{2}$$

$$f_X(x) = \sum_y f_{X,Y}(x,y)$$

$$f_X(1) = f_{X,Y}(1,1) + f_{X,Y}(1,2) = \frac{1}{2}$$

$$f_X(2) = f_{X,Y}(2,1) + f_{X,Y}(2,2) = \frac{1}{2}$$





Dependant Variables – Simple Example

- Are X and Y independent?

$$f_{X,Y}(1,1) = \frac{1}{2} \neq \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = f_X(1) \cdot f_Y(1)$$

$$f_{X,Y}(1,2) = 0 \neq \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = f_X(1) \cdot f_Y(2)$$

...

- No, X and Y are not independent!

Words and Concepts to Know

Stochastic

Cumulative Distribution Function

Marginal

Probability Mass Function

Correlation coefficient

Simultanious pmf

cdf

Joint pmf

pmf

Standard deviation

Binomial Mass Function

Mean

Variance

Expectation