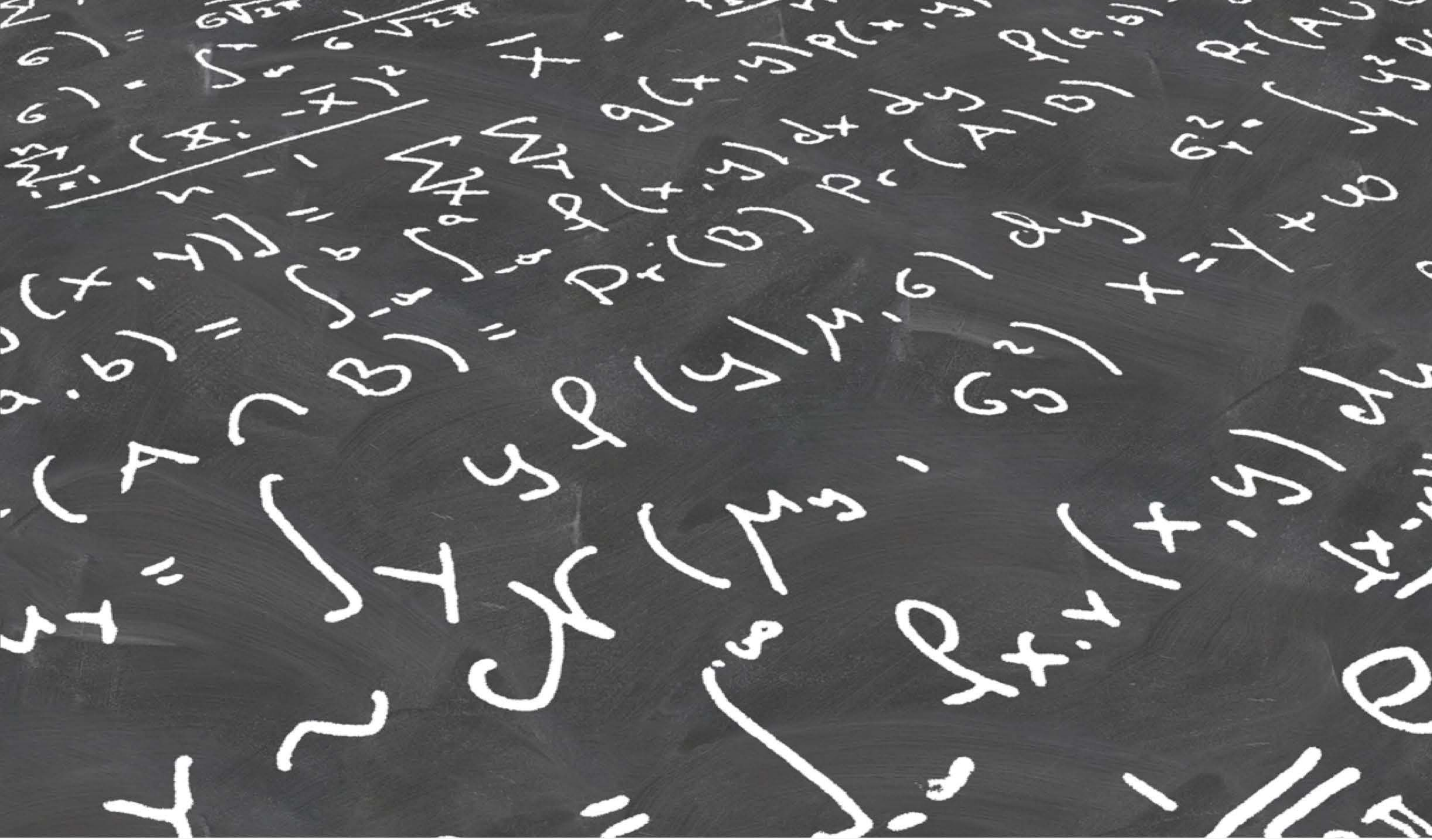

8. Resume of Probability and Stochastic Processes

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Lars Mandrup

Agenda for Today

Resume of stochastic processes:

- Probability
  - Bayes rule
  - Conditional
  - Total
- Stochastic variables
  - pmf/pdf/cdf
  - Joint/marginal/conditional
  - Mean/Variance/Correlation
- Stochastic Processes
  - Ensemble/Sample functions
  - Stationarity and Ergodic Processes
  - Auto- and Cross-correlation functions
  - Power Spectrum Density

Basic Probability

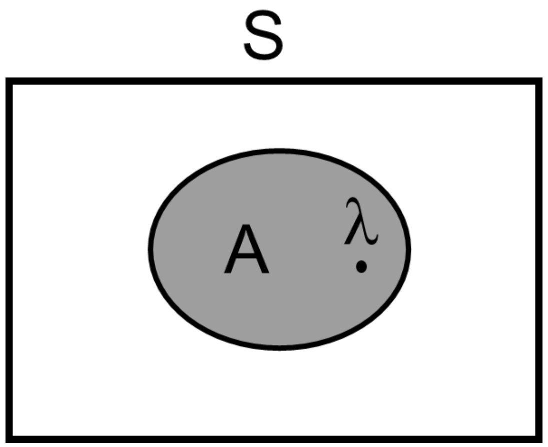
- Probability theory tells us what is in the sample given nature.

- Basic Axions:

Axion 1: $\mathbf{0 \leq \boldsymbol{Pr}(\boldsymbol{A}) \leq 1}$

Axion 2: $\boldsymbol{Pr}(\boldsymbol{S}) = \mathbf{1}$

S: Sample space
A: Event
λ: Sample point

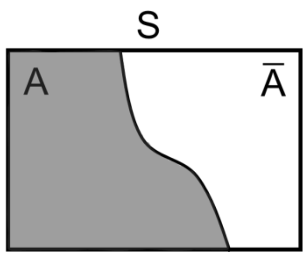


- Often (but not always) we use the relative frequency:

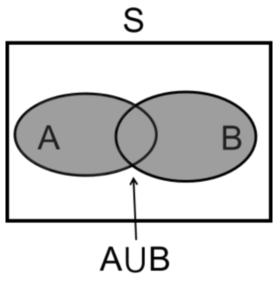
$$\Pr(A) = \frac{N_A}{N}$$

Basic Probability

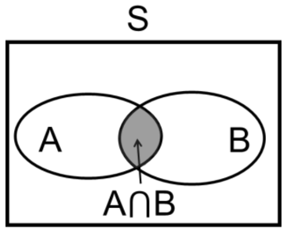
- Complement: $Pr(A) = 1 - Pr(\bar{A})$



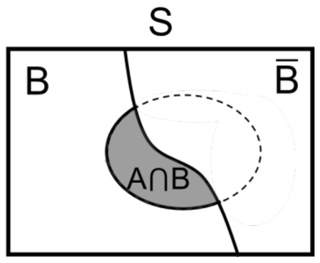
- Union: $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$



- Joint: $Pr(A \cap B) = Pr(A|B) \cdot Pr(B) = Pr(B|A) \cdot Pr(A)$



- Conditional: $Pr(A|B)$



Bayes Rule and Independence

- Bayes Rule:

$$\boxed{Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)} = \frac{Pr(B|A) \cdot Pr(A)}{Pr(B)}}$$

- A and B independent:

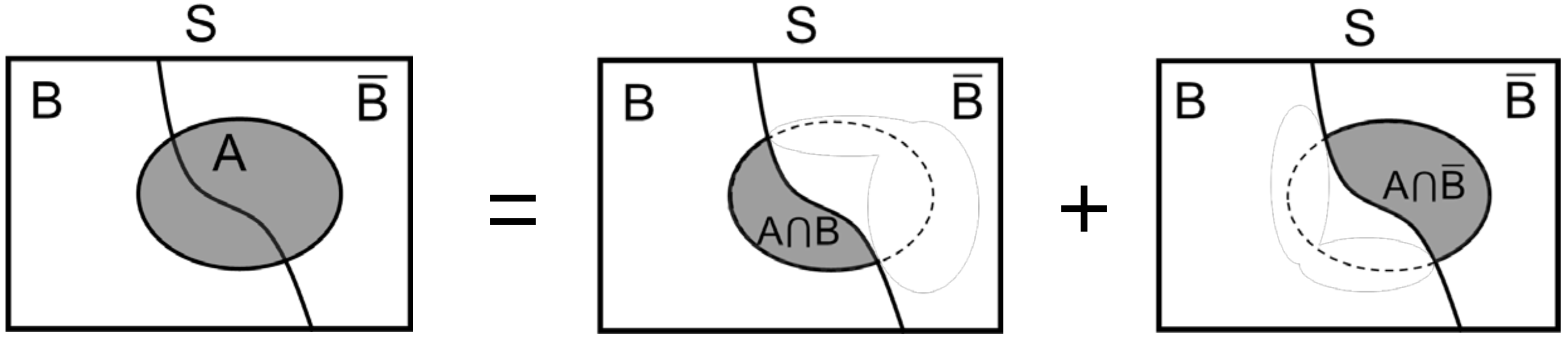
$$\boxed{Pr(A \cap B) = Pr(A) \cdot Pr(B)}$$

$$\boxed{Pr(B|A) = Pr(B) \quad \text{and} \quad Pr(A|B) = Pr(A)}$$

Total Probability

We sometime call it the marginal

- Pr(A) of an event is the total probability of that event.

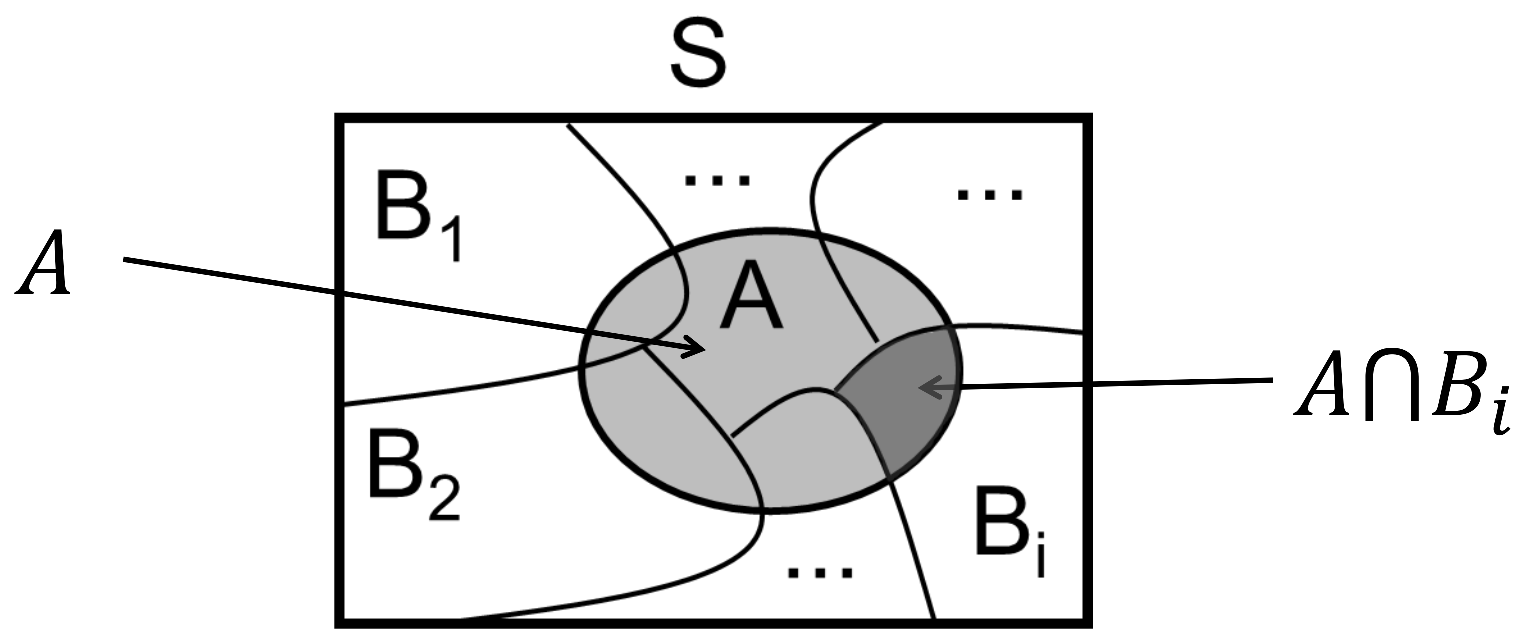


$$\Pr(A) = \Pr(A \cap B) + \Pr(A \cap \bar{B})$$
$$= \Pr(A|B) \cdot \Pr(B) + \Pr(A|\bar{B}) \cdot \Pr(\bar{B})$$

Total Probability

We sometime call it the marginal

- Pr(A) of an event is the total probability of that event.



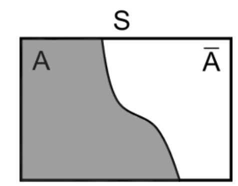
$$Pr(A) = Pr(A \cap B_1) + Pr(A \cap B_2) + \cdots + Pr(A \cap B_i) + \cdots$$
$$= Pr(A|B_1) \cdot Pr(B_1) + Pr(A|B_2) \cdot Pr(B_2) + \cdots$$

where the B_i's are mutually exclusive ($B_i \cap B_j = \varnothing$ for $i \neq j$)
and $S = B_1 \cup B_2 \cup \ldots \cup B_i \cup \ldots$

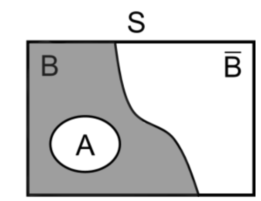
Summary of Probability

Relative frequency: $Pr(A) = \frac{N_A}{N_S}$

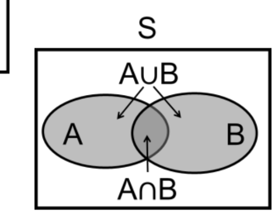
Complement: $Pr(\bar{A}) = 1 - Pr(A)$



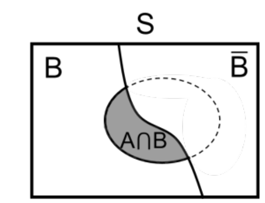
Exclusive: $Pr(\bar{A} \cap B) = Pr(B) - Pr(A)$ if $A \subset B$



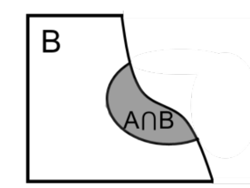
Union: $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$



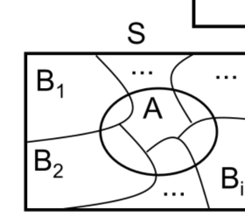
Joint: $Pr(A \cap B) = Pr(A|B) \cdot Pr(B) = Pr(B|A) \cdot Pr(A)$



Conditional: $Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$ if $Pr(B) \neq 0$



Total probability: $Pr(A) = \sum_{i=1}^{n} Pr(A|B_i) \cdot Pr(B_i)$



Bayes rule: $Pr(B|A) = \frac{Pr(A|B) \cdot Pr(B)}{Pr(A)}$

Bayes formula: $Pr(B_i|A) = \frac{Pr(A|B_i) \cdot Pr(B_i)}{\sum_{i=1}^{n} Pr(A|B_i) \cdot Pr(B_i)}$

Independence: $Pr(A \cap B) = Pr(A) \cdot Pr(B)$

Combinatorics

- The number of possible outcomes of k trials, sampled from a set of n objects.

Types of Experiments:

- With or without replacement
- Ordered or unordered

		Replacement	
		With	Without
Sam-pling	Ordered	n^k	$P_k^n = \frac{n!}{(n-k)!}$
	Unordered	$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!\,(n-1)!}$	$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

The Binomial Distribution

- We have n repeated trials.
- Each trial has two possible outcomes

Bernoulli trial

  - **Success** — probability p
  - **Failure** — probability $q = 1 - p$
- What is the probability of having k successes out of n trials?
- We write this question as:

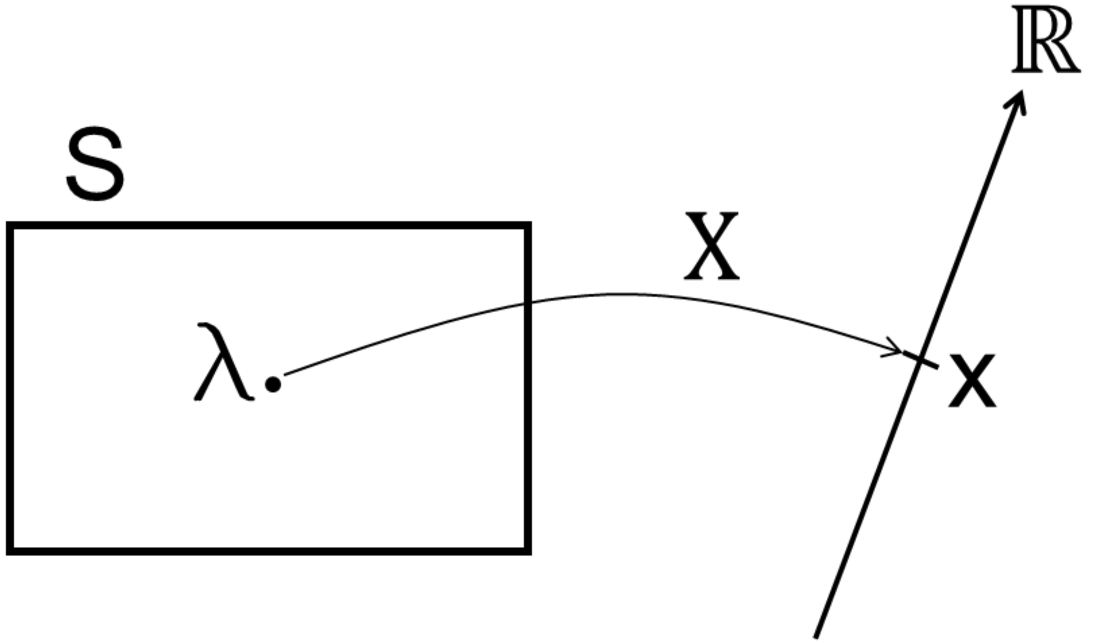
$$\boxed{Pr_n(k) = \frac{n!}{k!\,(n-k)!} p^k q^{n-k} = \binom{n}{k} p^k q^{n-k}}$$

- Faculty: $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$
  $0! = 1$

Also just called a random variables

Stochastic Random Variables

- A random variable tells something important about a stochastic experiment.
- Can be discrete or continous



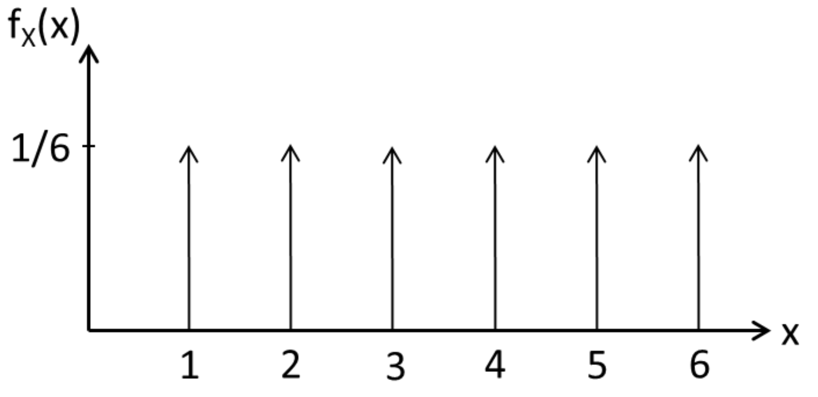
Examples:

- The numbers on a dice (discrete):
  - Sample space for variable X is : $\{1, 2, 3, 4, 5, 6\}$
  - Sample space for variable Y "Even (1)/Uneven (-1)": $\{1, -1\}$
- The hight of students at IHA (continous):
  - Sample space for variable H is all real numbers: [100;250] cm.

One Stochastic Variable – Discrete

- Probability mass function (pmf):

$$f_X(x) = \begin{cases} Pr(X = x_i) & for\ X = x_i \\ 0 & otherwise \end{cases}$$

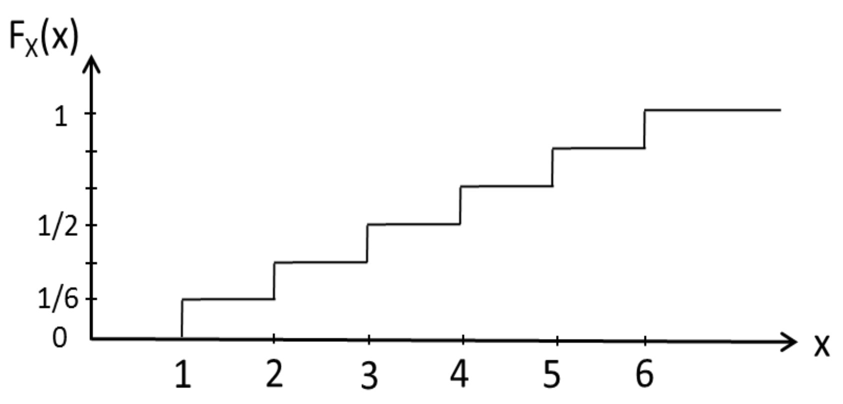


$$0 \le f_X(x) \le 1$$

$$\sum_{i=1}^{n} f_X(x_i) = \sum_{i=1}^{n} Pr(X = x_i) = 1$$

- Cumulative distribution function (cdf):

$$F_X(x) = P\,r(X \le x) = \sum_{i=1}^{n_x} f_X(x_i)$$



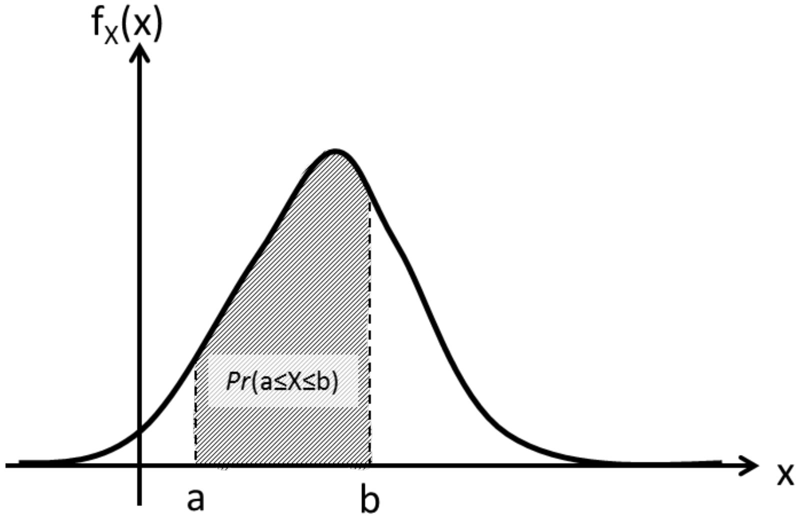
$$0 \le F_X(x) \le 1$$

$$\lim_{x \to -\infty} F_X(x) = 0$$

$$\lim_{x \to \infty} F_X(x) = 1$$

One Stochastic Variable – Continuous

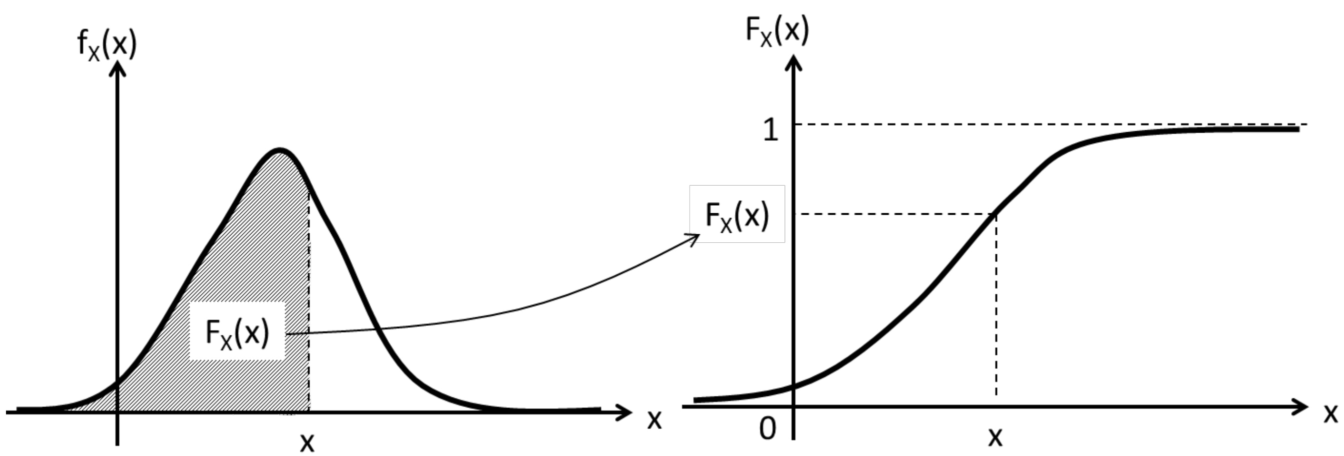
- Probability density function (pdf): $$Pr(a \leq X \leq b) = \int_a^b f_X(x)\, dx$$



$$f_X(x) \geq 0$$

$$\int_{-\infty}^{\infty} f_X(x) dx = 1$$

- Cumulative distribution function (cdf): $$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du = Pr(X \leq x)$$



$$0 \leq F_X(x) \leq 1$$

$$\lim_{x \to -\infty} F_X(x) = 0$$

$$\lim_{x \to \infty} F_X(x) = 1$$

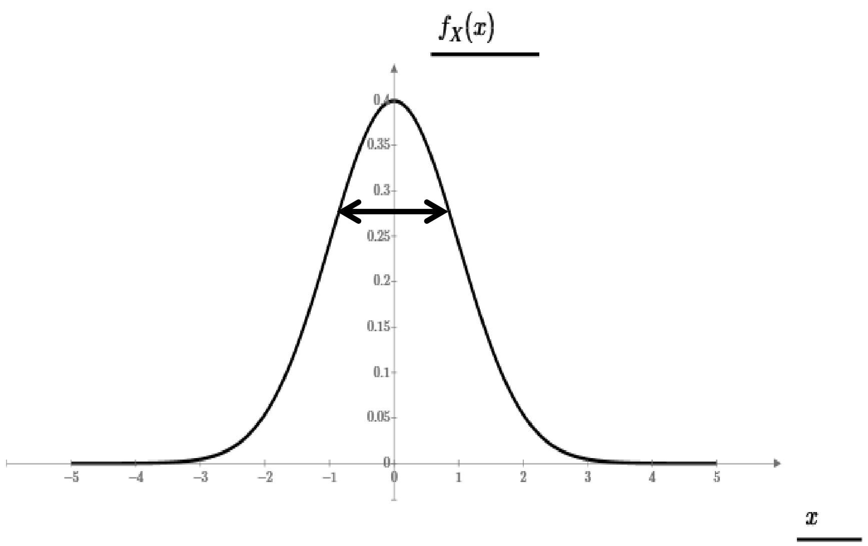
Transformation of Variable X to Y

- Given:
  - Pdf: $f_X(x)$
  - Function/Transformation: $\mathrm{Y} = g(X)$
  - Limits: $\mathrm{a} \leq X \leq b$

- Find new pdf: $f_Y(y)$:
  1. Inverse: $x = g^{-1}(y)$
  2. Differentiate: $\frac{dg^{-1}(y)}{dy} = \frac{dx(y)}{dy} = \frac{1}{\frac{dg(x)}{dx}}$
  3. Limits: Find $g(a) = a_Y \leq Y \leq b_Y = g(b)$ based on $\mathrm{a} \leq X \leq b$
  4. New pdf: $f_Y(y) = \sum \left|\frac{dx(y)}{dy}\right| f_X\left(g^{-1}(y)\ \right) = \sum \frac{f_X(x)}{\left|\frac{dy}{dx}\right|}$

Expectations

- Mean value: $E[X] = \bar{X} = \mu_X = \int_{-\infty}^{\infty} x \cdot f_X\,(x)dx \quad \left(\sum_{i=1}^{n} x_i\, f_X(x_i)\right)$
- Variance: $Var(\mathrm{X}) = {\sigma_X}^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 \cdot f_X\,(x)dx = E\left[X^2\right] - E[X]^2$



- Standard deviation: $\sigma_X = \sqrt{Var(\mathrm{X})}$
- Linear function: $E[aX + b] = a \cdot E[X] + b$

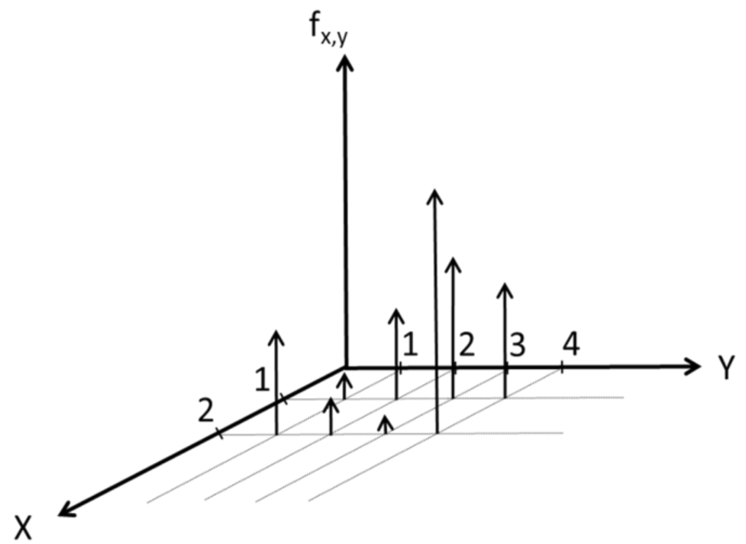
$$Var[aX + b] = a^2\left(E[X^2] - E[X]^2\right) = a^2 \cdot Var(X)$$

Two Stochastic Variables X, Y – Discrete

Joint (Simultaneous) pmf:

$$f_{X,Y}(x,y) = \begin{cases} Pr\Big((X = x_i\,) \cap (Y = y_j)\Big) & for\;\; X = x_i \;\wedge\; Y = y_j \\ \\ 0 & otherwise \end{cases}$$

$$0 \leq f_{X,Y}(x,y) \leq 1 \qquad\qquad \sum_{i=1}^{m}\sum_{j=1}^{n} f_{X,Y}(x_i, x_j) = 1$$



Marginal pmfs:

$$f_X(x) = \sum_y f_{X,Y}(x,y) \qquad f_Y(y) = \sum_x f_{X,Y}(x,y)$$

Cumulative Distribution Function cdf:

$$F_X(x_i, y_j) = Pr\Big((X \leq x_i\,) \cap (Y \leq y_j)\Big) = \sum_{m=1}^{i}\sum_{n=1}^{j} f_{X,Y}(x_m, y_n)$$

Two Stochastic Variables X, Y – Continuous

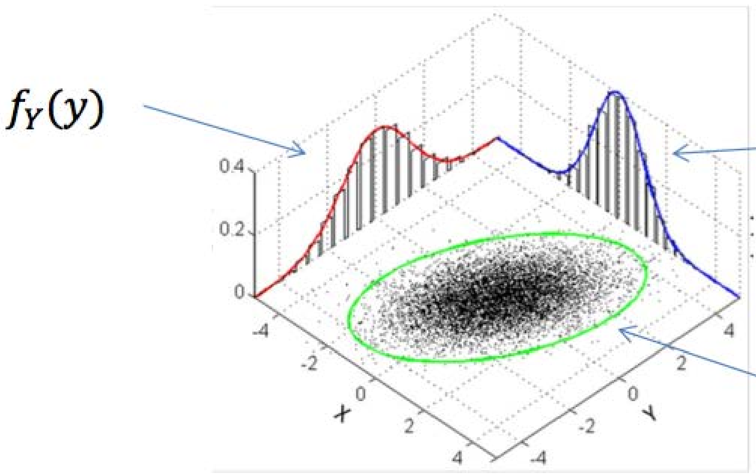
Joint (Simultaneous) pdf: $f_{X,Y}(x,y) \geq 0$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$$

Marginals:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$



Cumulative Distribution Function cdf:

cdf $F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(x,y)dxdy = Pr(X \leq x \wedge Y \leq y)$

pdf $f_{X,Y}(x,y) = \dfrac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$

Bayes Rule, Conditional PDF and Independence

Bayes rule:

- The joint/simultaneous pmf/pdf for two stochastic variables:

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y) = f_{Y|X}(y|x)f_X(x)$$

Conditional pdf:

- For a two dimensional pmf/pdf $f_{X,Y}(x,y)$, we can find the conditional pdf with Bayes rule:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Independence:

- X and Y are independent if and only if:

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \quad \text{or} \quad f_{X|Y}(x|y) = f_X(x) \quad \text{for all x and y}$$

Correlation and Covariance

Correlation tells of the (biased) coupling between variables

- Correlation: $corr(X,Y) = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot y \cdot f_{X,Y}(x,y)dxdy$

Covariance is without bias from the mean

- Covariance: $cov(X,Y) = E\left[(X-\bar{X})(Y-\bar{Y})\right] = E[XY] - E[X] \cdot E[Y]$

Correlation Coefficient is the normalized Covariance

- Correlation coefficient: $\rho = E\left[\frac{X-\bar{X}}{\sigma_X} \cdot \frac{Y-\bar{Y}}{\sigma_Y}\right] = \frac{E[XY] - E[X]E[Y]}{\sigma_X \cdot \sigma_Y}$

$$-1 \le \rho \le 1$$

- If X and Y are independent:

$E[XY] = E[X] \cdot E[Y]$ and $cov(X,Y) = \rho = 0$

Important Rules

- $E[aX + b\ ] = a \cdot E[X] + b$
- $Var[aX + b\ ] = a^2 \cdot Var(X)$

- $E[aX + bY] = a \cdot E[X] + b \cdot E[Y]$ → Linearity of the mean
- $Var[aX + bY] = a^2 \cdot Var[X] + b^2 \cdot Var[Y] + 2ab \cdot Cov(X, Y)$

Correlation

- $Corr(X, Y) = E[XY]$ $(= E[X] \cdot \mathrm{E}[\mathrm{Y}]$ if X and Y are independent)
- $Cov(X, Y) = E\left[(X - \bar{X})(Y - \bar{Y})\right] = E[XY] - E[X] \cdot E[Y]$
- $\rho = E\left[\frac{X - \bar{X}}{\sigma_X} \cdot \frac{Y - \bar{Y}}{\sigma_Y}\right] = \frac{E[XY] - E[X]E[Y]}{\sigma_X \cdot \sigma_Y}$

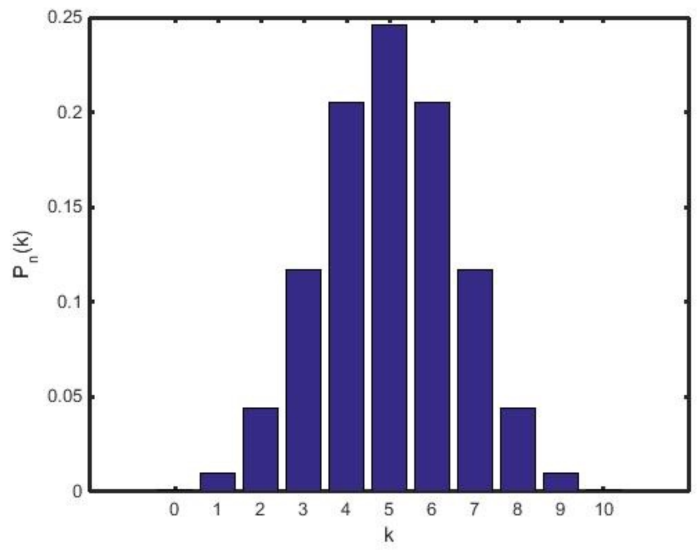
Correlation coefficient

Notice that correlation and correlation coefficient are different, but can have same name and same notation!!

The Binomial Distribution

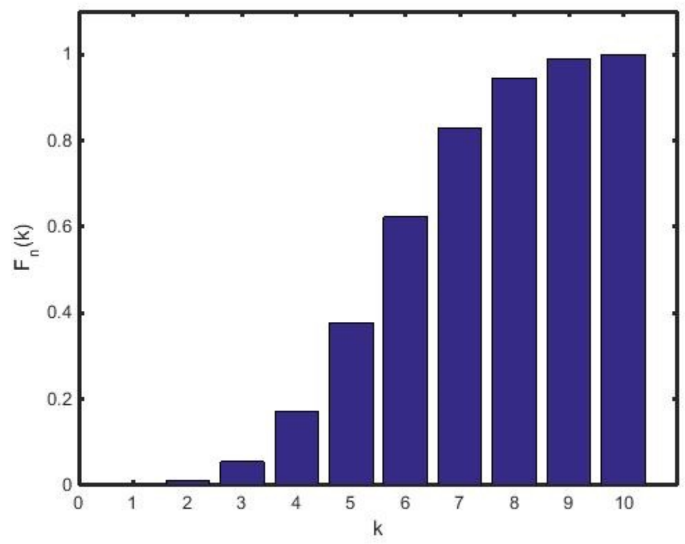
Also called a Bernoulli trial

- n repeated trials – each with two possible outcomes
  - **Success** — probability p
  - **Failure** — probability $q = 1 - p$



- Probability mass function (pmf):

$$f(k|n,p) = \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k}$$



- Cumulative distribution function (cdf):

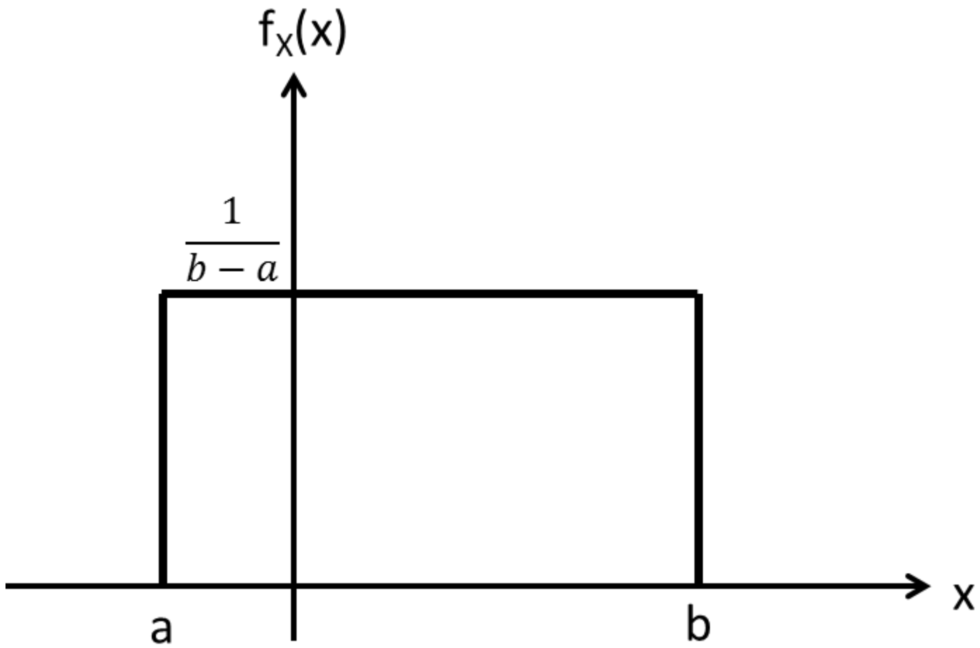
$$F(k|n,p) = \sum_{i=0}^{k} f(i|n,p)$$

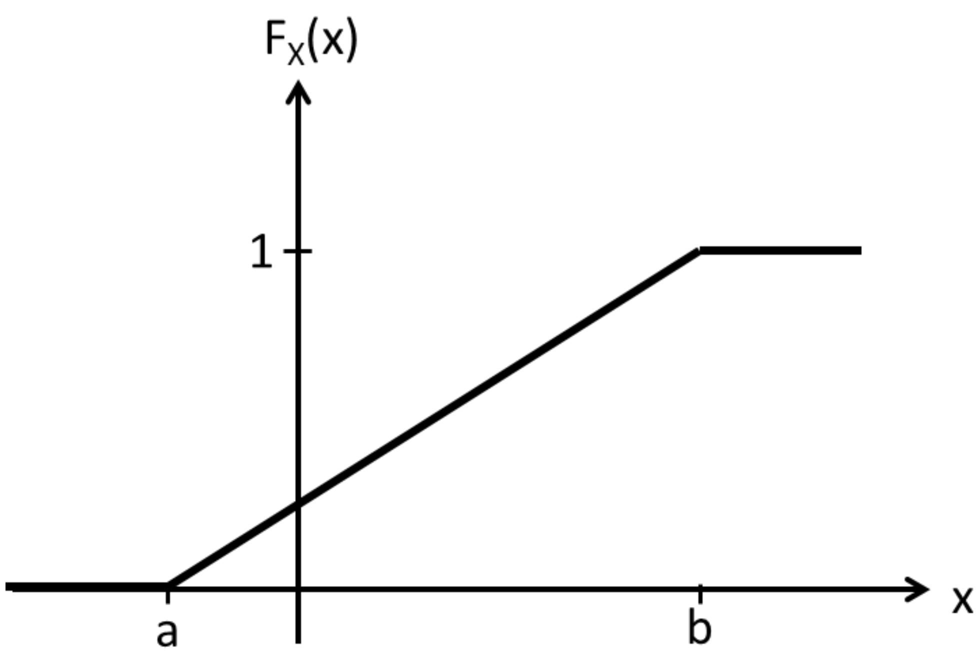
- Mean and variance:

$$E[X] = n \cdot p$$
$$Var(X) = n \cdot p \cdot (1-p)$$

Uniform Distribution (continuous)

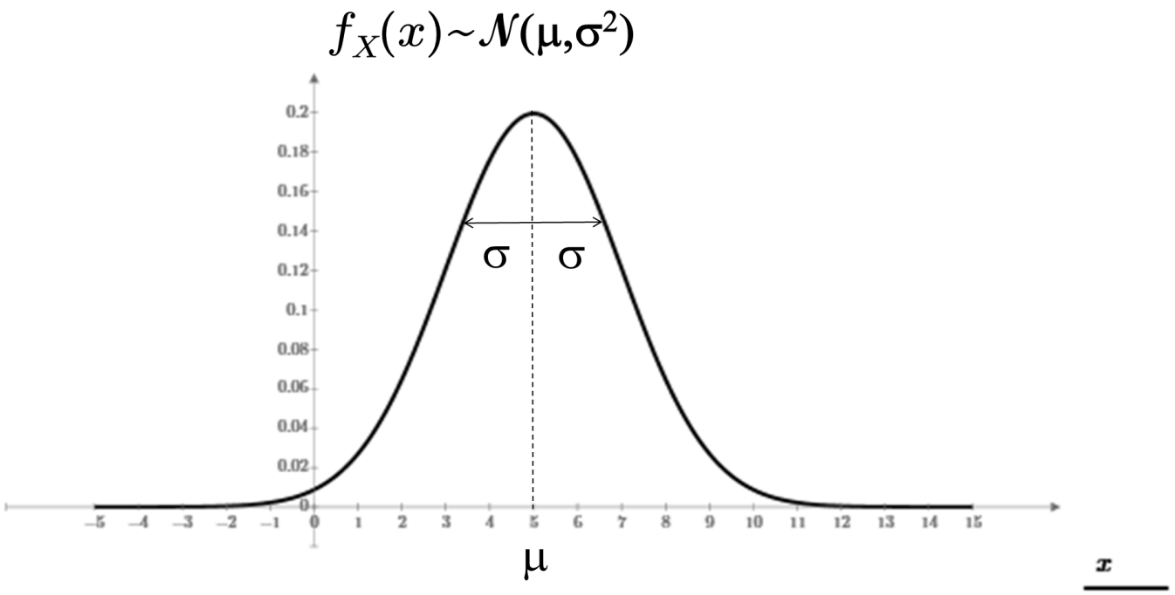
- $\mathcal{U}$(a,b)
- Mean value: $\mu = \frac{a+b}{2}$
- Variance: $\sigma^2 = \frac{1}{12}(b-a)^2$
- pdf: $f_X(x) = \begin{cases} \frac{1}{b-a} & for\ a \le x \le b \\ 0 & otherwise \end{cases}$
- cdf: $F_X(x) = \begin{cases} 0 & for\ x \le a \\ \frac{x-a}{b-a} & for\ a \le x \le b \\ 1 & for\ x \ge b \end{cases}$



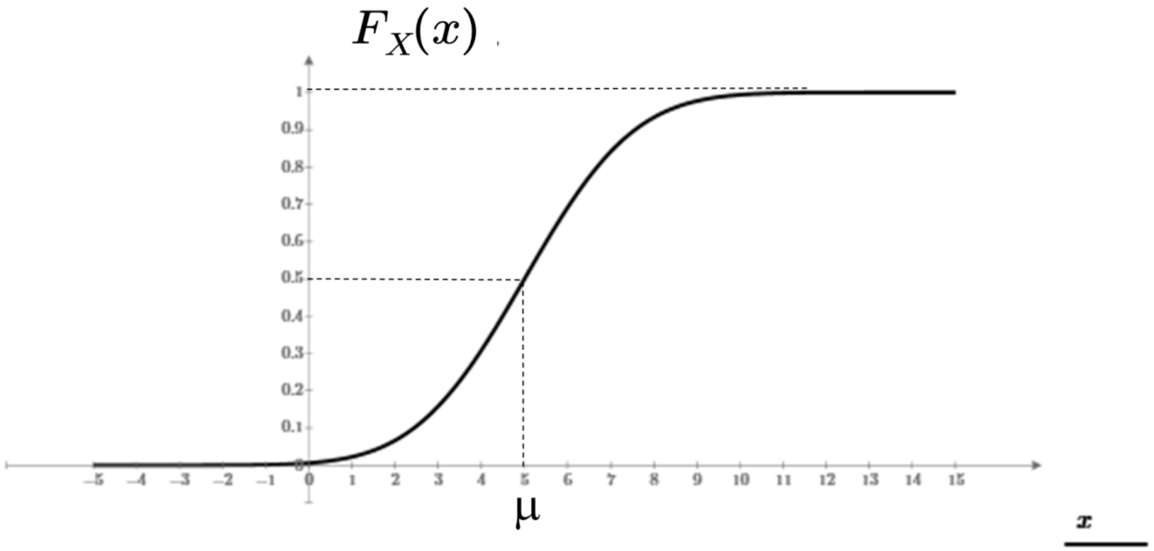


Gaussian Distribution = Normal Distribution

- $\mathcal{N}(\mu,\sigma^2)$
- Mean value: μ
- Variance: σ^2



- pdf: $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$



- cdf: $F_X(x) = \frac{1}{2}\left[1 + \mathrm{erf}(\frac{x-\mu}{\sigma\sqrt{2}})\right]$

No closed expression for the cdf

erf= error-function: $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$

Gaussian Distribution = Normal Distribution

- Beregninger med normalfordelinger: Tabelopslag og Matlab:
- $X \sim \mathcal{N}(\mu, \sigma^2) \;\rightarrow\; Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$ (Standard Normal Distribution)
- $F_X(x) = Pr(X \leq x) = Pr\left(Z \leq \frac{x-\mu}{\sigma}\right) = F_Z(z) \qquad \text{hvor } z = \frac{x-\mu}{\sigma}$

$$= \begin{cases} \Phi(z) & \textit{Tabel } 1 \text{ ("}\textit{Statistik og Sandsynlighedsregning}\text{")} \\ 1 - \mathrm{Q}(z) & \textit{App. D} \text{ ("}\textit{Random Signals}\text{")} \end{cases}$$

- $\Phi(z) = Pr(Z \leq z)$
- $\Phi(-z) = 1 - \Phi(z)$
- $\mathrm{Q}(z) = Pr(Z \geq z) = 1 - Pr(Z \leq z) = 1 - \Phi(z)$
- $\mathrm{Q}(-z) = 1 - \mathrm{Q}(z)$

- Matlab:
  - $\Pr(X \leq x) = F_X(x) = normcdf(x, \mu, \sigma)$
  - $\Pr(Z \leq z) = F_Z(z) = normcdf(z, 0, 1) = normcdf(z)$

Very important!

i.i.d.: Independent and Identically distributed

- We define that for series of random variables that is taken from the same distribution (identically distributed), and are sampled independent of each other, that they are i.i.d.

$$\boxed{\text{i.i.d. = Independent and Identically distributed}}$$

- i.i.d. is a very important characteristic in stochastic variable processing and statistics

Example:

- Quantisation noise.

Very important!

Central Limit Theorem

- Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with mean μ and variance σ^2
- Let $\bar{X}$ be the random variable (average):

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

- Then in the limit: $n \to \infty$ we have that: $\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$

  i.e. in the limit $\bar{X}$ will be normally distributed with mean = μ and variance = $\frac{\sigma^2}{n}$.

The variance is reduced with a factor 1/n

Very important!

Central Limit Theorem

- Let $X_1, X_2, \dots, X_n$ be i.i.d. random variables with mean μ and variance σ^2
- Let X be the random variable:

$$X = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}} = \frac{\sum_{i=1}^{n} \frac{1}{n} X_i - \mu}{\sqrt{\sigma^2/_n}} = \frac{\bar{x} - \mu}{\sqrt{\sigma^2/_n}}$$

- Then in the limit: $n \to \infty$ we have that: $X \sim \mathcal{N}(0,1)$

  i.e. in the limit X will be normally distributed with mean = 0 and variance = 1 (standard normal distributed).

Sampling From Any Distribution

For test or simulation you need testdata (”measurements”) randomly sampled from a given distribution:

- Find the cdf of the distribution: $F_X(x)$
- Find the inverse of the cdf: $y = F_X(x) \Rightarrow x = F_X^{-1}(y)$
- Draw a ramdom sample: $y \sim \mathcal{U}[0;1]$
- Insert into the inverse cdf: $x = F_X^{-1}(y)$
- The samples $X = x$ is distributed according to: $F_X(x)$

Stochastic Processes

Definitions:

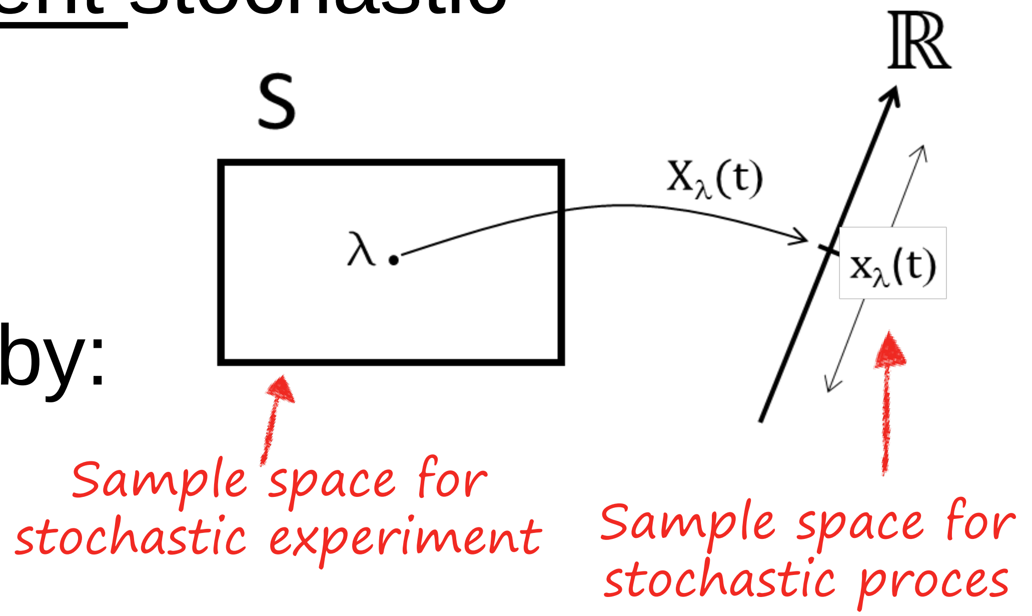
- A stochastic process is a <u>time dependent</u> stochastic variable:

$$X(t)$$

- A discrete stochastic process is given by:

$$X[n] = X(nT)$$

where n is an integer.



- Random events that develops in time

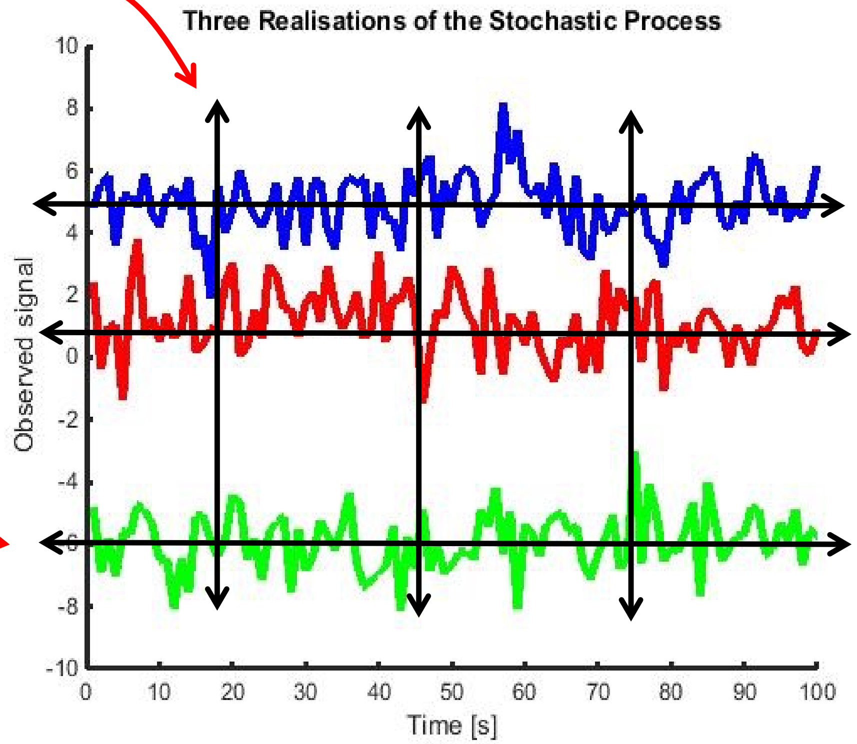
- A sample function (observed signal) is a realization of a stochastic process $x(t)$

The Mean Functions

- Ensemple mean:

The mean of all possible realizations to time t

$$\mu_{X(t)}(t) = E[X(t)] = \int_{-\infty}^{\infty} x(t) f_{X(t)}(x(t))\, dx(t)$$



The time average for one realization of the stochastic process

- Temporal mean:

$$\hat{\mu}_{X_i} = \langle X_i \rangle_T = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_i(t)\, dt$$

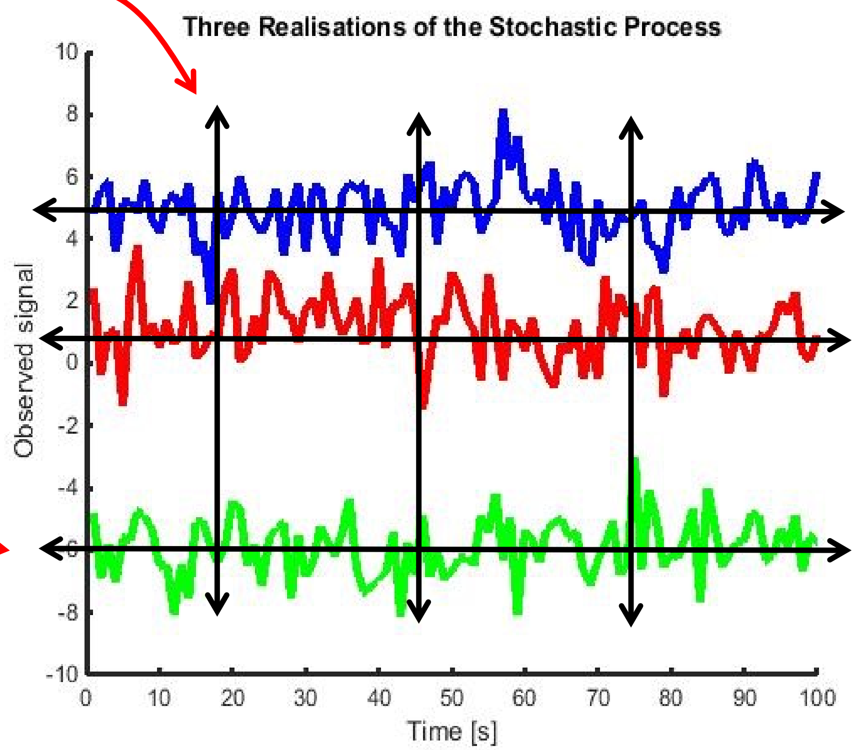
$$\left(\lim_{T\to\infty} \frac{1}{T} \int_{0}^{T} x_i(t)\, dt \right)$$

The Variance Functions

The variance of all possible realizations to time t

- Ensemple variance:

$$Var\big(X(t)\big) = \sigma^2_{X(t)}(t) = E\left[\big(X(t) - \mu_{X(t)}(t)\big)^2\right]$$



The variance over time for one realization of the stochastic process

- Temporal variance:

$$\hat{\sigma}^2_{X_i} = \langle X_i^2 \rangle_T - {\langle X_i \rangle_T}^2 = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left(x_i(t)^2 - {\hat{\mu}_{X_i}}^2\right) dt = Var(X_i)$$

$$\left(\lim_{T\to\infty} \frac{1}{T}\int_{0}^{T} \left(x_i(t)^2 - {\hat{\mu}_{X_i}}^2\right) dt\right)$$

Stationarity in the Wide Sense (WSS)

- Ensemble mean is a constant

Can be tested.

$$\mu_X(t) = E[X(t)] = \mu_X$$ - independent of time

- Ensemble variance is a constant

$$\sigma_X^2(t) = E\left[X(t)^2\right] - E[X(t)]^2 = \sigma_X^2$$ - independent of time

Stationarity in the Strict Sense (SSS):

- The density function $f_{X(t)}\big(x(t)\big)$ do not change with time

Difficult to test in reality.

Ergodicity

- We can say something about the properties of the stochastic process in general <u>based on one sample function</u>, as long as we have observed it for long enough.

- If ensemble averaging is equivalent to temporal averaging:

$$\mu_X(t) = \bar{X}(t) = \int_{-\infty}^{\infty} x f_X(x)\, dx = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_i(t)\, dt = \langle X_i \rangle_T = \hat{\mu}_{X_i}$$

- For any moment:

In practice: n=2 (Variance)

$$\overline{X^n} = \int_{-\infty}^{\infty} x^n f_X(x)\, dx = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_i^n\,(t)\, dt$$

One realization

Ensemple (WSS)

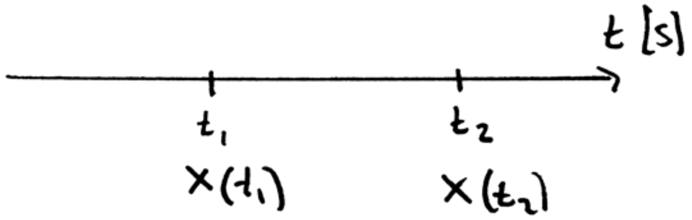
$$\left.\begin{matrix} \langle X_i \rangle_T = \mu_X \\ \hat{\sigma}_{X_i}^2 = \sigma_X^2 \end{matrix}\right\} \rightarrow Ergodic$$

All information is achieved with one measurement (realization)

Comparing realizations

Correlations

- We compare the process at two different times



Correlation of a realization with itself

- <u>Autocorrelation</u>: $R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)^*]$
  - Says something about how much the signal $X(t_1)$ resembles itself at time t_2

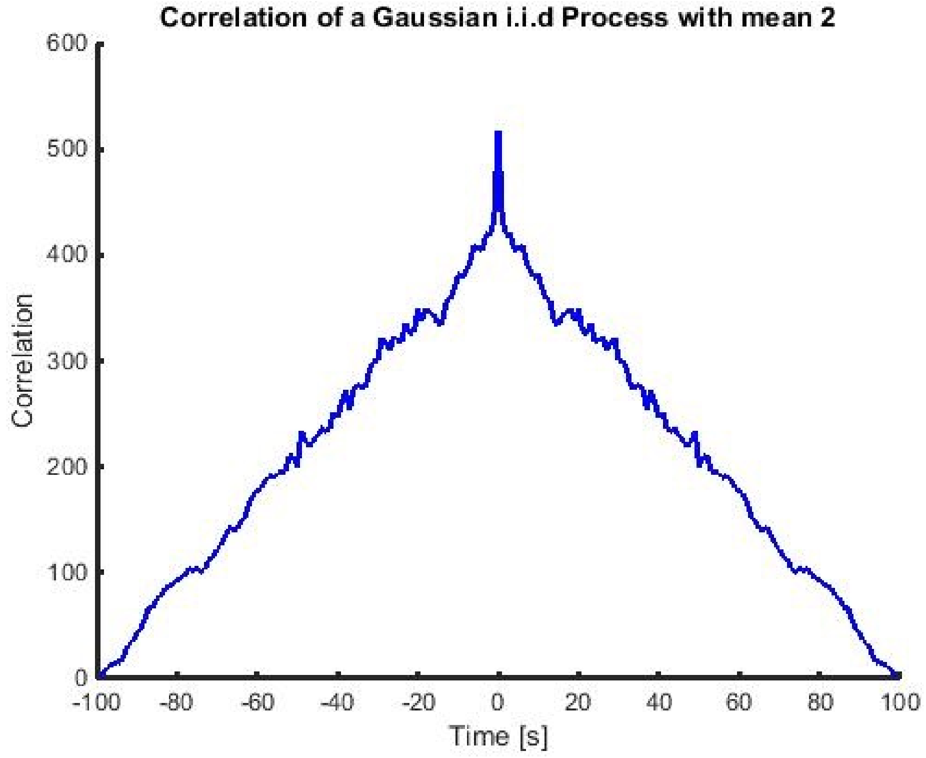
Correlation of two realizations

- <u>Crosscorrelation</u>: $R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)^*]$
  - Can be used to look for places where the signal $X(t)$ is similar to the signal $Y(t)$

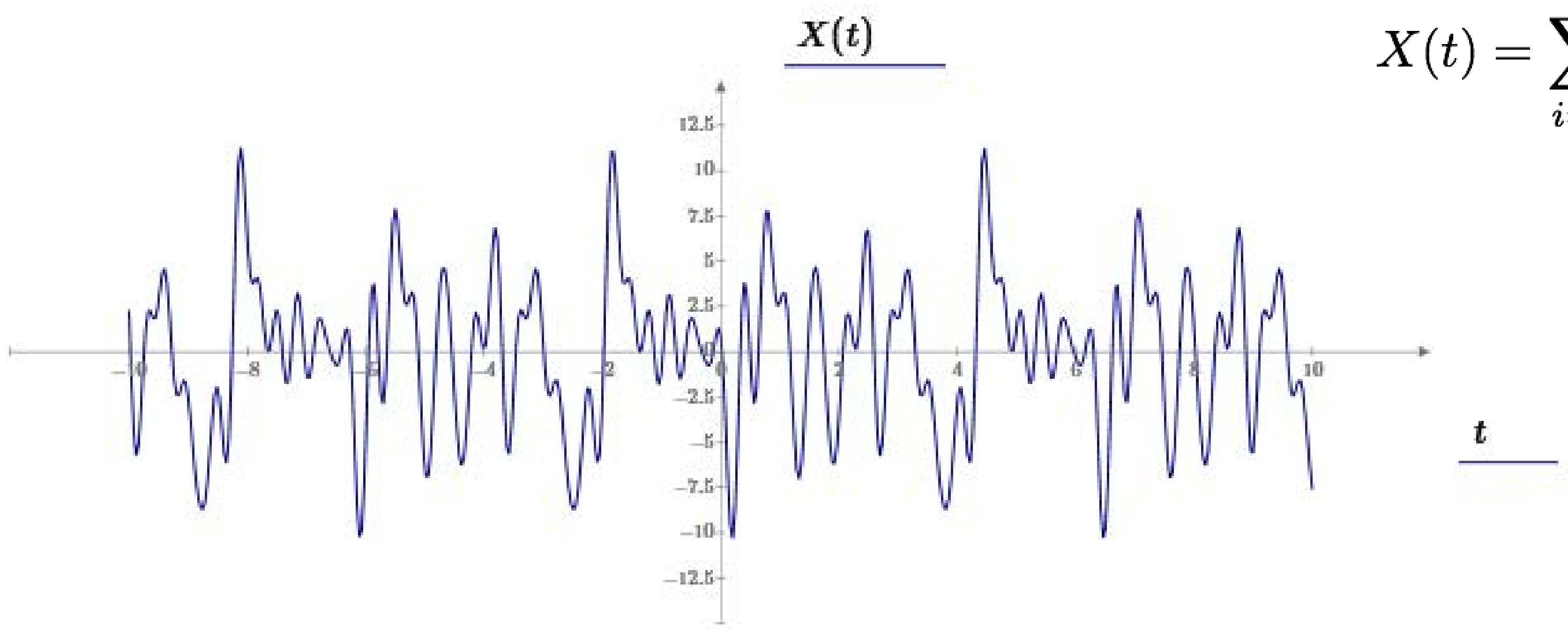
Autocorrelation

- For Real WSS: $R_{XX}(\tau) = E[X(t)X(t+\tau)]$

- Properties of the autocorrelation function $R_{XX}(\tau)$:
  - ➢ An even function of τ $(R_{XX}(\tau) = R_{XX}(-\tau))$
  - ➢ Bounded by: $|R_{XX}(\tau)| \leq R_{XX}(0) = E\left[X^2\right]$ (max. in $\tau = 0$)
  - ➢ If X(t) changes fast, then $R_{XX}(\tau)$ decreases fast from $\tau = 0$
  - ➢ If X(t) changes slowly, then $R_{XX}(\tau)$ decreases slowly from $\tau = 0$
  - ➢ if X(t) is periodic, then $R_{XX}(\tau)$ is also periodic



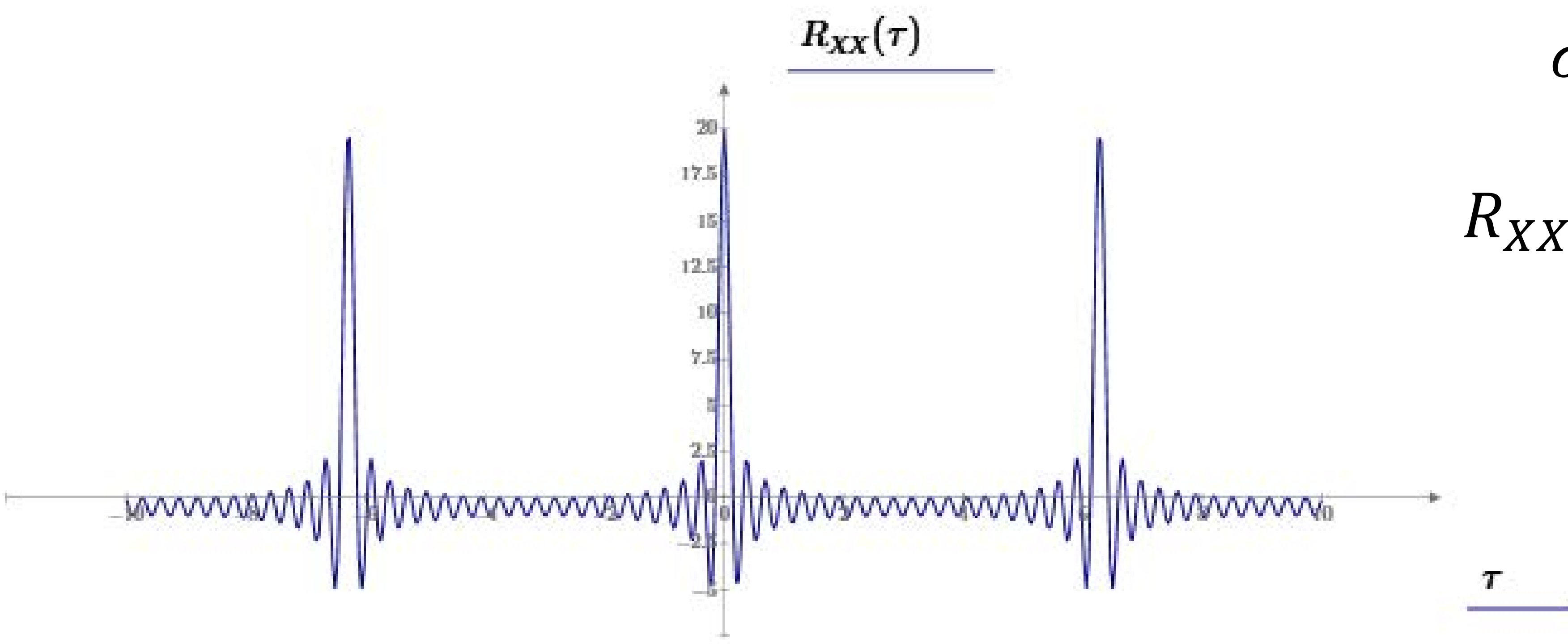
Uncalibrated Noisy Signal



$$X(t) = \sum_{i=1}^{n} A_i \cos \omega_i t + B_i \sin \omega_i t$$

$$A_i, B_i \sim \mathcal{N}(0, \sigma^2)$$

$$\omega_i = i \cdot \omega_0$$
$$\omega_0 = 1$$



$$\sigma = 1, n = 20$$

$$R_{XX}(0) = n\sigma^2 = 20$$

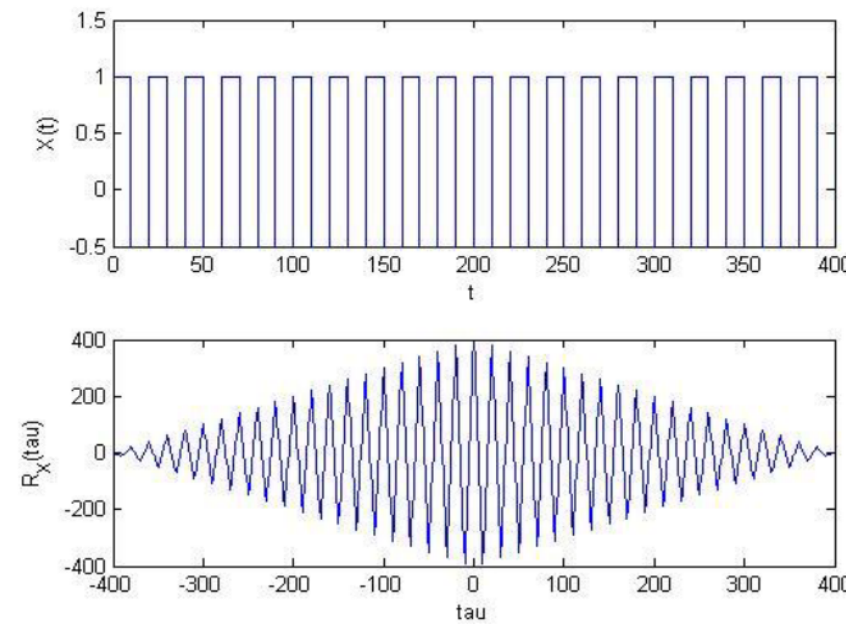
Random Binary (Digital) Signal

Deterministic:

Periodic signal

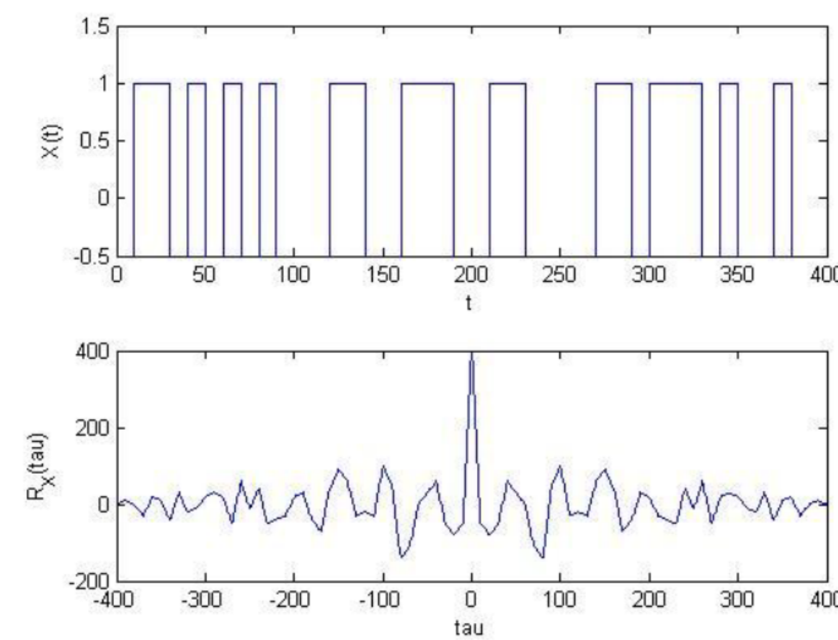
↓

R_{XX} periodic




```
Rx = conv(x,fliplr(x));
```

Non-deterministic (Stochastic)




```
Rx = conv(x,fliplr(x));
```

Tells about how much we can predict the future

Autocovariances

- Autocovariance function:

$$C_{XX}(t_1, t_2) = E\left[\left(X(t_1) - \mu_X(t_1)\right)(X(t_2) - \mu_X(t_2))^*\right]$$
$$= R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$

Especially: $C_{XX}(t, t\ ) = E\left[(X(t) - \mu_X(t))^2\right] = E\left[X(t)^2\right] - E[X(t)]^2 = \sigma_X^2(t)$

- Autocorrelation coefficient:

$$r_{XX}(t_1, t_2) = \frac{C_{XX}(t_1,t_2)}{\sqrt{C_{XX}(t_1,t_1)C_{XX}(t_2,t_2)}}; \qquad 0 \leq r_{XX}(t_1, t_2) \leq 1$$

Especially: $r_{XX}(t, t) = 1$ $\quad$ ($X(\mathrm{t})$ is totally dependent of itself!)

Two Stochastic Processes

- If we have two stochastic processes $X(t)$ and $Y(t)$
  - We can compare them by looking at the cross-correlation and cross-covariance:

Cross-correlation $$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)^*]$$

Cross-covariance $$C_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)^*] - E[X(t_1)]E[Y(t_2)]$$

Cross-Correlation Functions

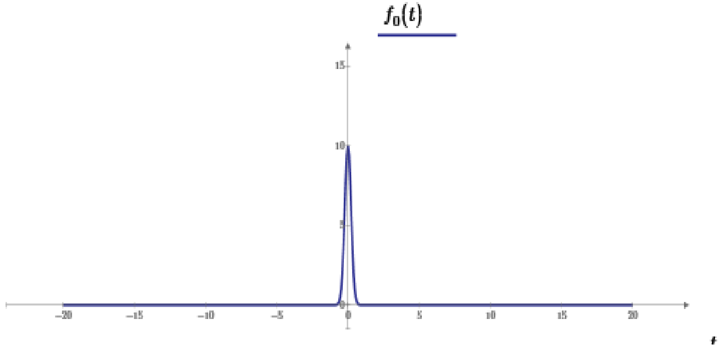
- For Real WSS processes $X(t)$ and $Y(t)$:

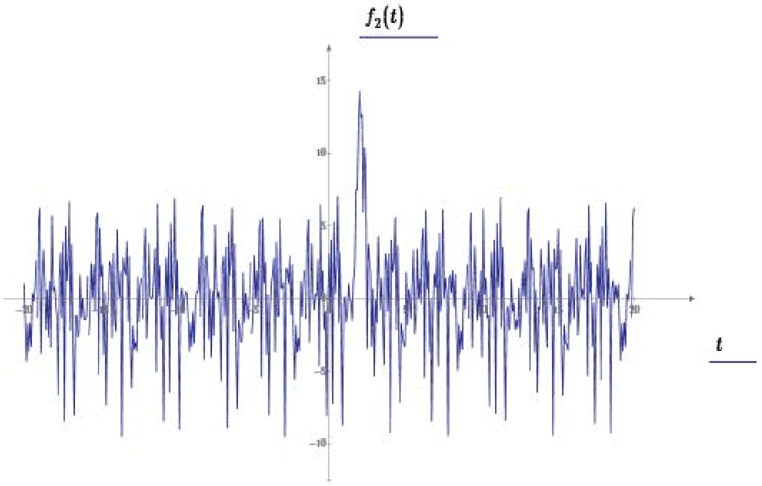
$$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$

- Properties of the cross-correlation function $R_{XY}(\tau)$:
  - $R_{XY}(\tau) = R_{YX}(-\tau))$
  - $|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)R_{YY}(0)} = \sqrt{E[X^2]E[Y^2]}$ (max. in $\tau = 0$)
  - $|R_{XY}(\tau)| \leq \frac{1}{2}(R_{XX}(0) + R_{YY}(0))$
  - If $X(t)$ and $\mathrm{Y}(t)$ are orthogonal, then $R_{XY}(\tau) = 0$
  - If $X(t)$ and $\mathrm{Y}(t)$ are independant, then $R_{XY}(\tau) = \mu_X \cdot \mu_Y$

Cross-correlation – Uncalibrated noisy signal

- Comparing two signals:
  - An uncalibrated and noisy signal $f_2(t)$
  - Reference signal $f_0(t) = 10 \cdot e^{-10t^2}$

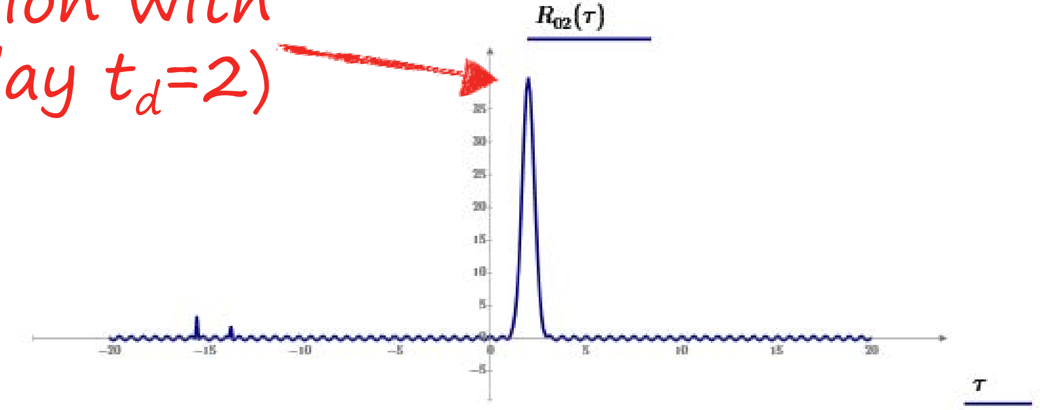




- Cross-correlation:

$$R_{02}(\tau) = \int_{-\infty}^{\infty} f_0(t) \cdot f_2(t+\tau)dt$$

Correlation with time delay t_d=2)



Power Spectral Density (psd)

- WSS random signals X(t):
- Power Spectral Density Function (psd):

Fourier-transform

$$S_{XX}(f) = \mathcal{F}(\langle R_{XX}(\tau)\rangle_{T_0}) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\cdot 2\pi f\cdot\tau}d\tau$$

Invers Fourier-transform

$$\Rightarrow R_{XX}(\tau) = \mathcal{F}^{-1}(\langle R_{XX}(\tau)\rangle = \int_{-\infty}^{\infty} S_{XX}(f)e^{j\cdot 2\pi f\cdot\tau}df$$

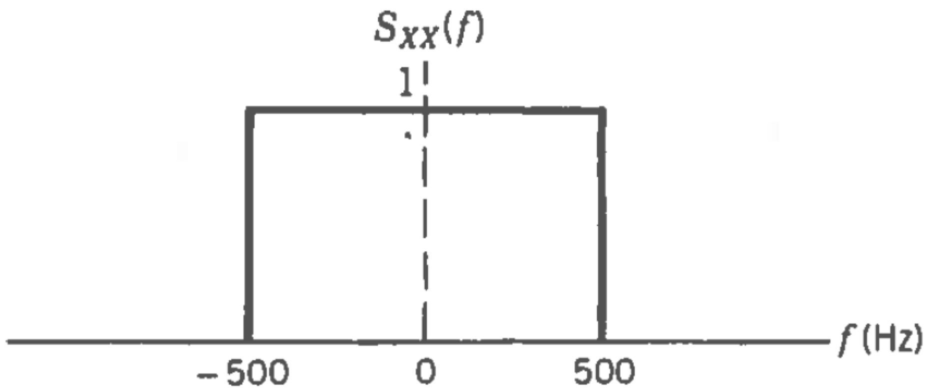


Figure 3.19a Psd of a lowpass random process $X(t)$.

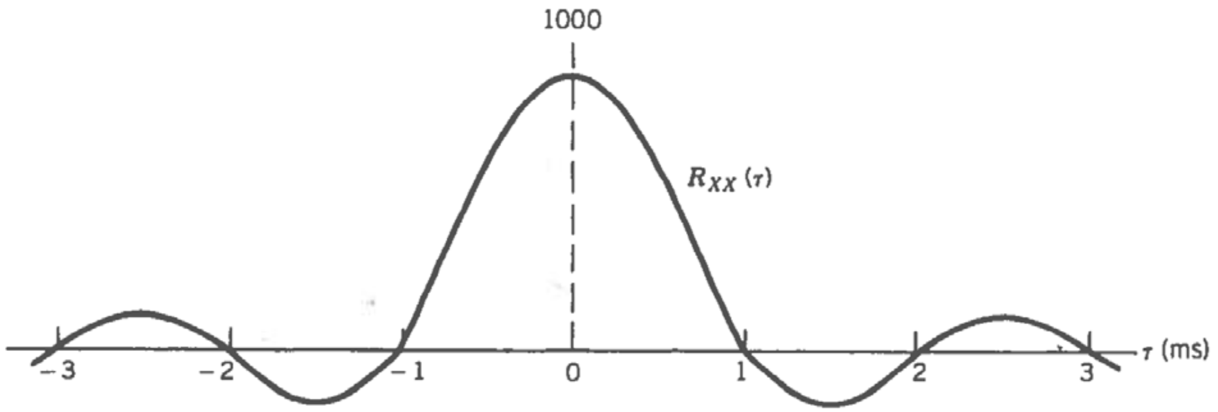


Figure 3.19b Autocorrelation function of $X(t)$.

Figures from ”Random Signals”

Power Spectral Density (psd)

- Properties of psd $S_{XX}(f)$ (spectrum of $X(t)$):
  - ➢ $S_{XX}(f) \in \mathbb{R}$
  - ➢ $S_{XX}(f) \geq 0$
  - ➢ If $X(t) \in \mathbb{R}$: $R_{XX}(-\tau) = R_{XX}(f)$ and $S_{XX}(-f) = S_{XX}(f)$ → even functions
  - ➢ If $X(t)$ periodic components: $S_{XX}(f)$ will have impulses (δ−functions)
  - ➢ $[S_{XX}(f)] = \frac{W}{Hz}$ → Distribution of power with frequency (power spectral density of the stationary random process X(t)
  - ➢ $P_X = E\left[X(t)^2\right] = R_{XX}(0) = \int_{-\infty}^{\infty} S_{XX}(f)df$

    i.e. if $X(t) = V(t)$ (voltage signal)

    $\rightarrow P_X =$ power in 1Ω-resistor

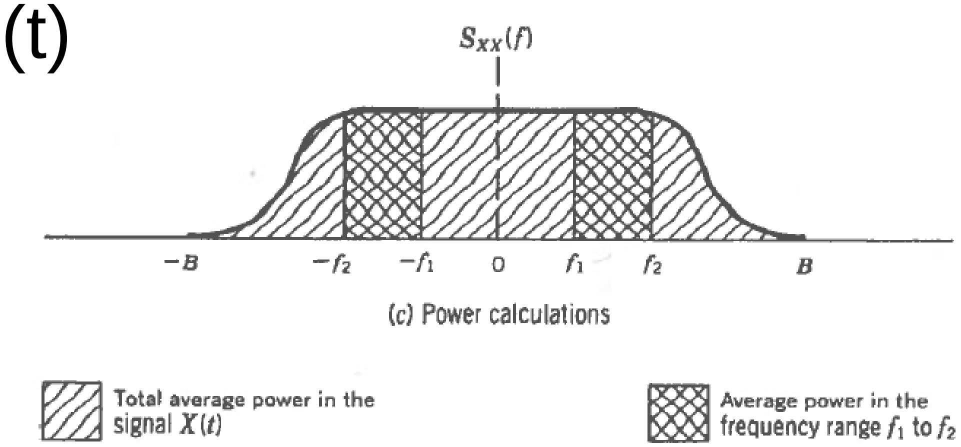


Figure from ”Random Signals”

  - ➢ $P_X[f_1, f_2] = 2\int_{f_1}^{f_2} S_{XX}(f)df$ → Power in the frequency-interval $[f_1, f_2]$

Words and Concepts to Know

Probability density function Binomial coefficient Cross-covariance Convolution
Deterministic Rayleigh Distribution Deterministic Intersection Type I Error SSS
pdf Temporal cross-correlation Cross-correlation Correlation Markov chain
Probability Mass Function i.i.d. Temporal mean Continuous random variable
Randomly Sampled Data Temporal variance Marginal Correlation coefficient
Stochastic Processes Unordered Mutually Exclusive/Disjoint Ensemple variance
Uniform distribution Replacement Sampling Non-deterministic Ergodicity
Sample point Specificity Stationarity Gaussian distribution Sample space
Central Limit Theorem Experiment/Trial cdf Complement/not Joint pmf WSS
Covariance
Likelihood Simultanious pmf
Independent and Identically Distributed Event
Relative frequency Realization Independence Union Correlation coefficient
Normal distribution Sensitivity Combinatorics
Bivariate Normal Distribution
Transformation of stochastic variables Binomial distribution
Joint events
Empty set/Null set Binomial Mass Function Standard deviation
Total probability
Strict Sense Stationary Ordered Set Conditional probability
Mean Simultaneous density function Variance Bayes Rule pmf Ensemple mean
Autocovariance Type II Error Autocorrelation Coefficient Joint density function
Power Spectral Density Non-deterministic Stochastic Posterior Autocorrelation
Wide Sense Stationary Bernoulli Trial Prior Expectation Subset
Cumulative Distribution Function psd Marginal probability density function

Assignment 8

- Find a stochastic process in your area

  (discharge of a capacitor, bitrate, failure, hight, weight, …)
- Make a signal model: $X(t) = \cdots$
- Make three realizations
- Determine the ensemble mean and variance
- Determine the temporal mean and variance
- Determine stationarity and ergodicity