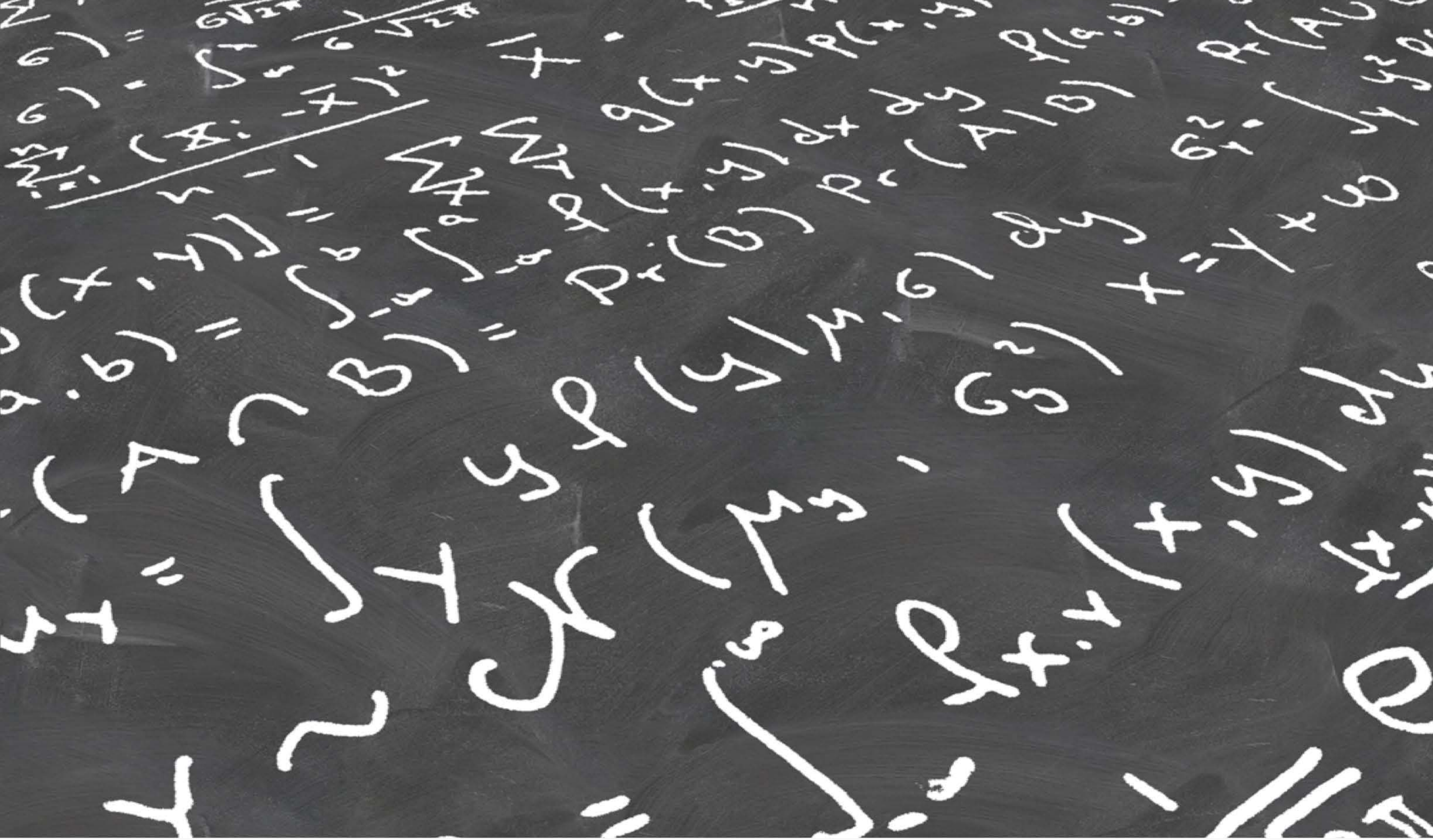

7. Stochastic Processes and Correlation Functions

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Agenda for Today

- Stochastic Processes (repetition)
  - Mean and variance
  - Stationarity (WSS, SSS)
  - Ergodic Processes
- Correlation functions
  - Autocorrelation functions
  - Cross-correlation functions
- Power spectrum density

Stochastic Processes

Definitions:

- A stochastic process is a time dependent stochastic variable:

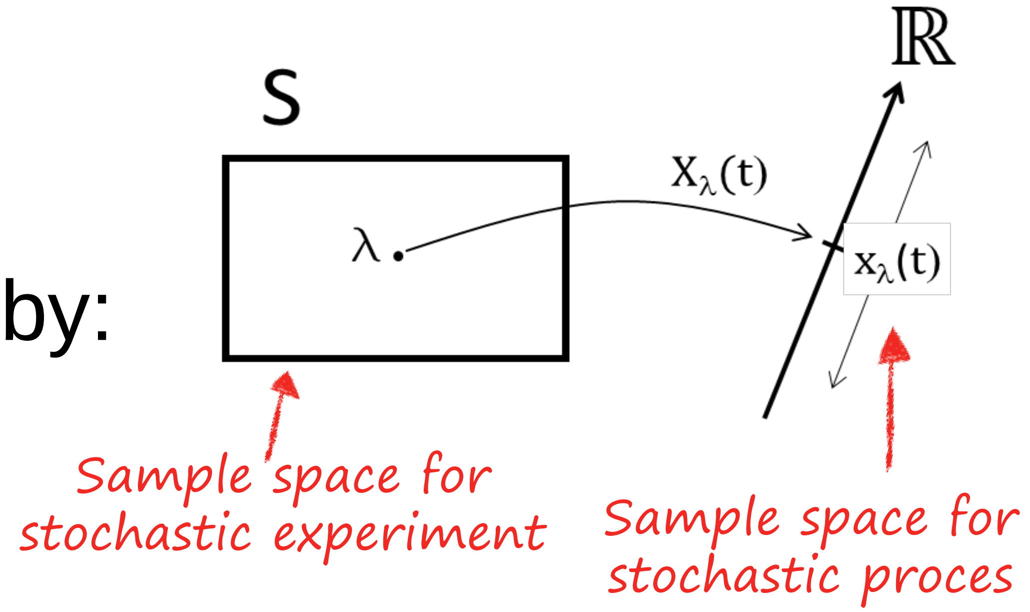
$$X(t)$$

- A discrete stochastic process is given by:

$$X[n] = X(nT)$$

  where n is an integer.

- Random events that develops in time



Notice:

- When we sample a signal from a stochastic process, we observe only one realization of the process

Sample Function / Ensemble

Definitions:

- A sample function is a realization of a stochastic process $x(t)$
- The ensemble of a stochastic process is the collection of all possible realizations $x(t)$ of the stochastic process X

Example:

- A coin is thrown every minute: H = head, T = tail
- One realization of the stochastic signal is: $HTHT$ ...
- The ensemble of the stochastic signals is:

$$HTHT, HHTT, TTHH, THTH, THHT, TTHT, HHHH\ldots$$

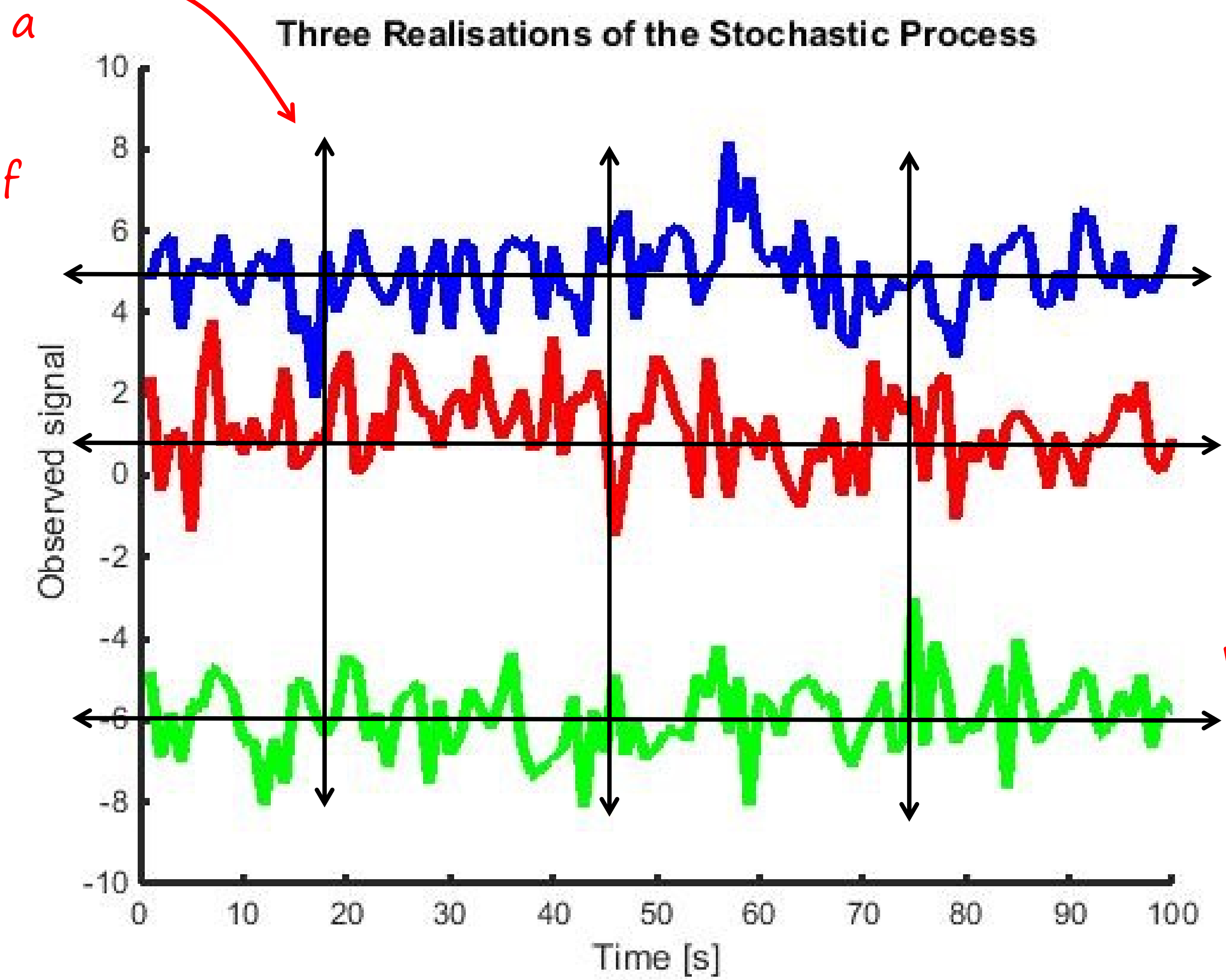
Stochastic Processes (signals)

Additive Noisemodel

$$observed\ signal = signal\ +\ noise$$

Ensemble mean and variance (to a specific time).

If indepentent of time: WSS



Time average and variance of each realization.

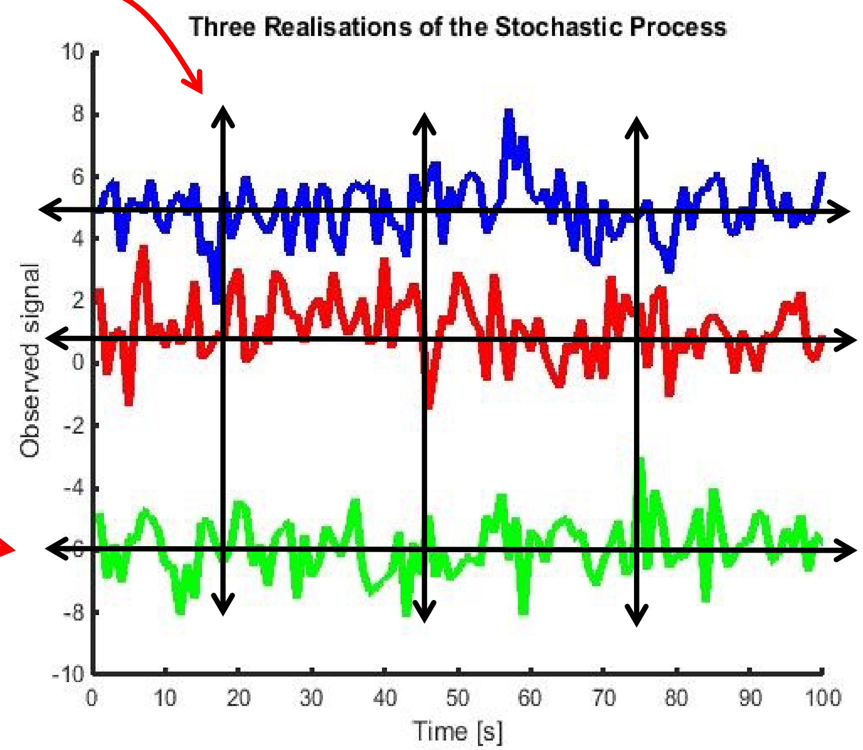
If equal (for all realizations): Ergodic

The Mean Functions

- Ensemple mean:

The mean of all possible realizations to time t

$$\mu_{X(t)}(t) = E[X(t)] = \int_{-\infty}^{\infty} x(t) f_{X(t)}(x(t))\, dx(t)$$



The time average for one realization of the stochastic process

- Temporal mean:

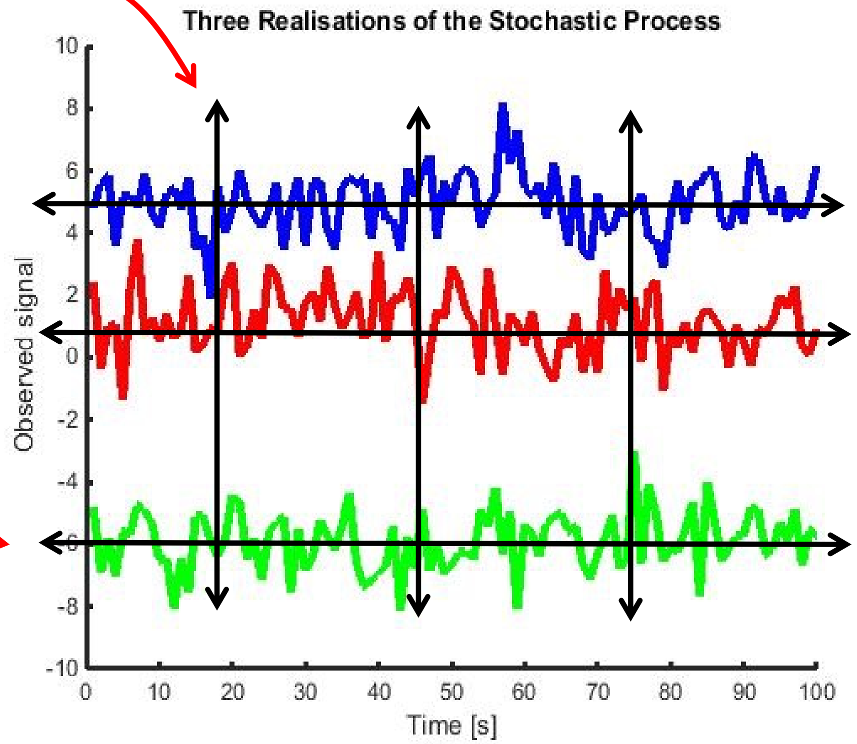
$$\hat{\mu}_{X_i} = \langle X_i \rangle_T = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_i(t)\, dt$$

The Variance Functions

The variance of all possible realizations to time t

- Ensemple variance:

$$Var\big(X(t)\big) = \sigma_{X(t)}^2(t) = E\Big[\big(X(t) - \mu_{X(t)}(t)\big)^2\Big]$$



The variance over time for one realization of the stochastic process

- Temporal variance:

$$\hat{\sigma}_{X_i}^2 = \langle X_i^2 \rangle_T - \langle X_i \rangle_T{}^2 = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left(x_i(t)^2 - \hat{\mu}_{X_i}{}^2\right) dt = Var(X_i)$$

Stationarity in the Wide Sense (WSS)

- Ensemble mean is a constant

Can be tested.

$$\mu_X(t) = E[X(t)] = \mu_X$$ - independent of time

- Ensemble variance is a constant

$$\sigma_X^2(t) = E\left[X(t)^2\right] - E[X(t)]^2 = \sigma_X^2$$ - independent of time

Stationarity in the Strict Sense (SSS):

- The density function $f_{X(t)}\big(x(t)\big)$ do not change with time

Difficult to test in reality.

Ergodicity

- We can say something about the properties of the stochastic process in general based on one sample function, as long as we have observed it for long enough.

- If ensemble averaging is equivalent to temporal averaging:

Any realization

Ensemble (WSS)

$$\left.\begin{aligned} \langle X_i \rangle_T &= \mu_X \\ \hat{\sigma}^2_{X_i} &= \sigma^2_X \end{aligned}\right\} \rightarrow Ergodic$$

All information is achieved with one measurement (realization)

Realizations / Samples - Example

Discrete stochastic process:

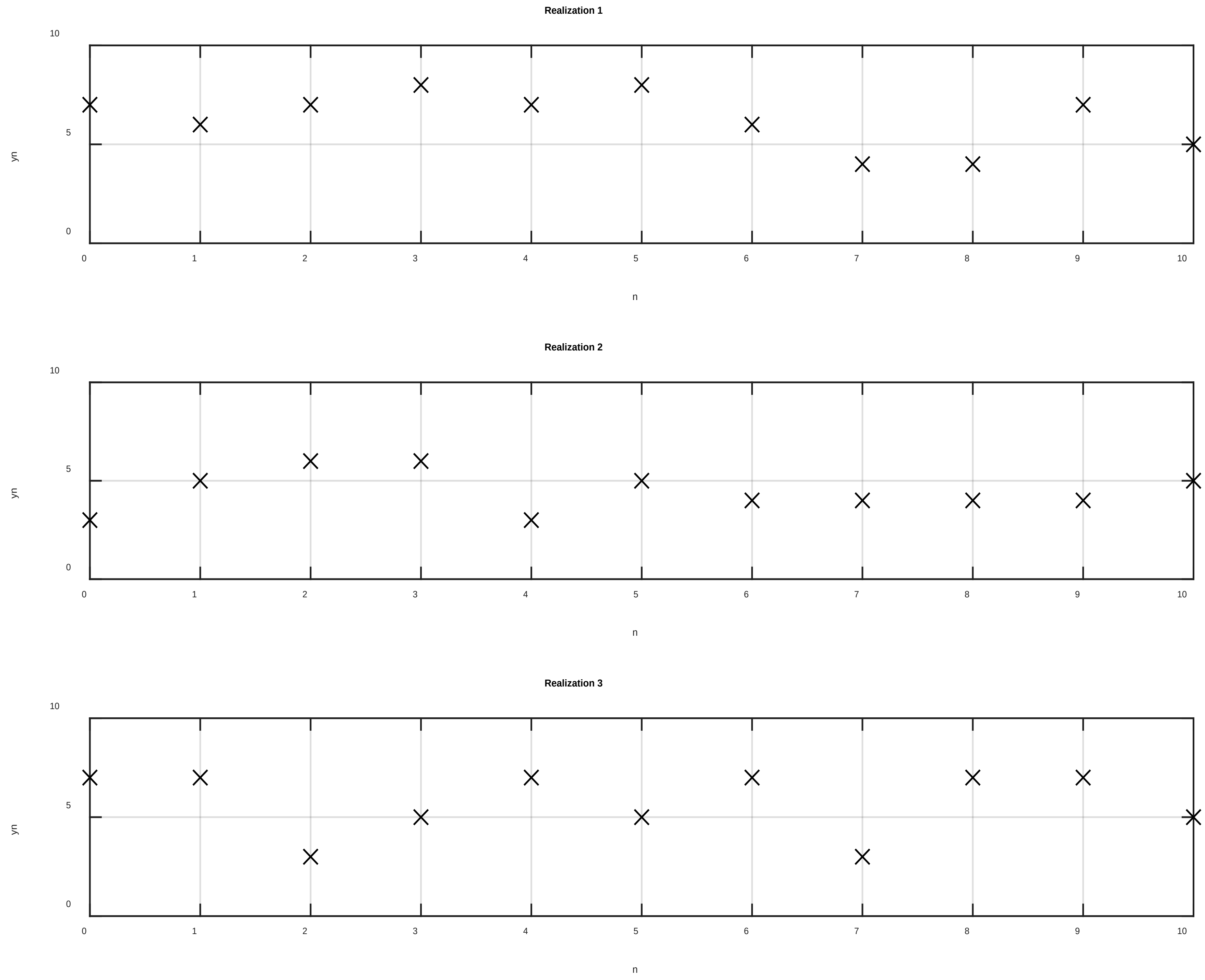
$$Y(n) = X + W(n);$$

$$X \sim \mathcal{B}(10{,}0.5)$$
$$W(n) \sim \mathcal{U}_i[-2{,}2]$$

3 realizations
11 samples
(n=0,..,10)

WWS ✓

Ergodic ÷



Realizations / Samples - Example

Discrete stochastic process:

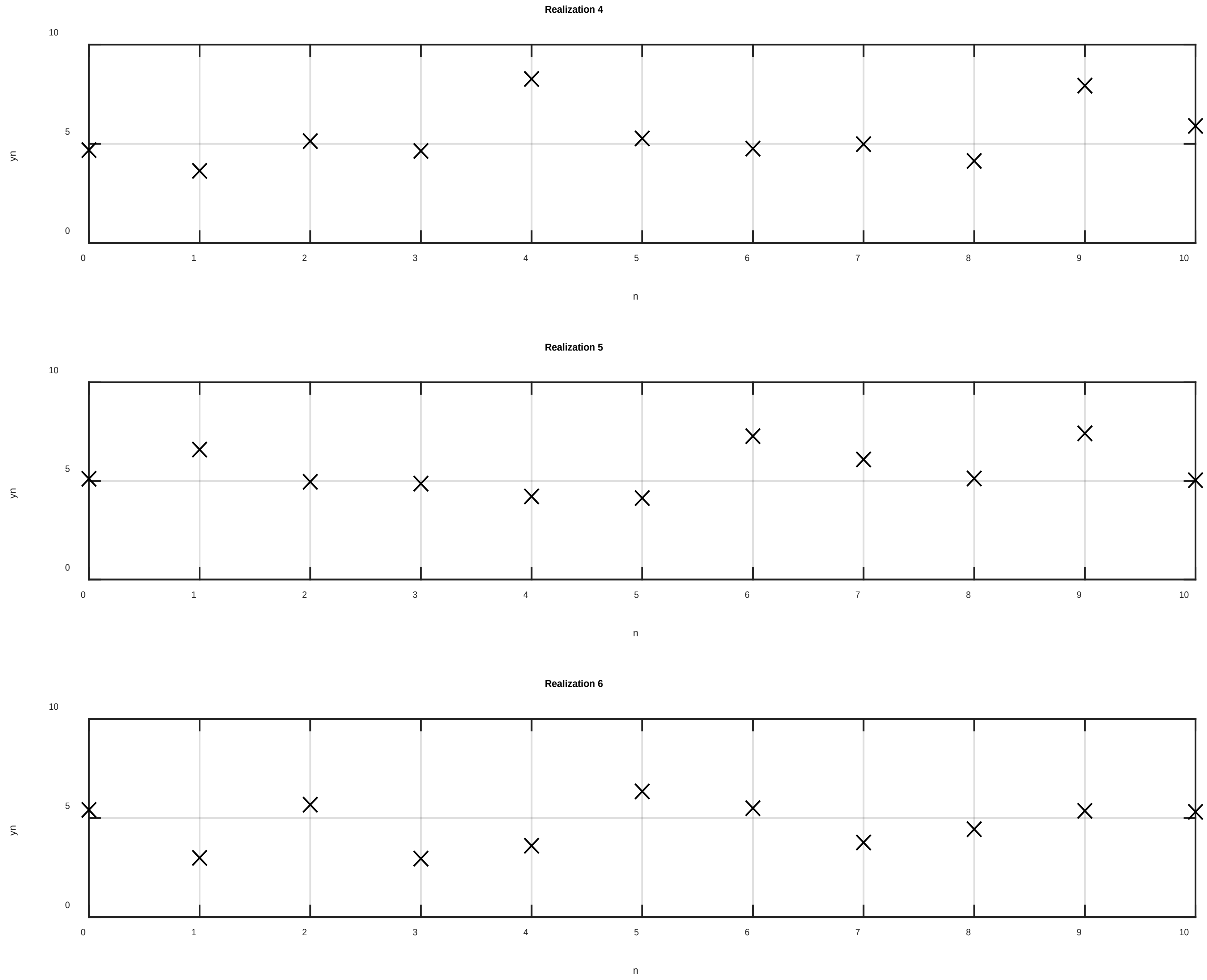
$Y(n) = W(n);$

$W(n) \sim \mathcal{N}(5,2)$

3 realizations
11 samples
(n=0,..,10)

WWS ✓

Ergodic ✓



Realizations / Samples - Example

Continous
stochastic process:

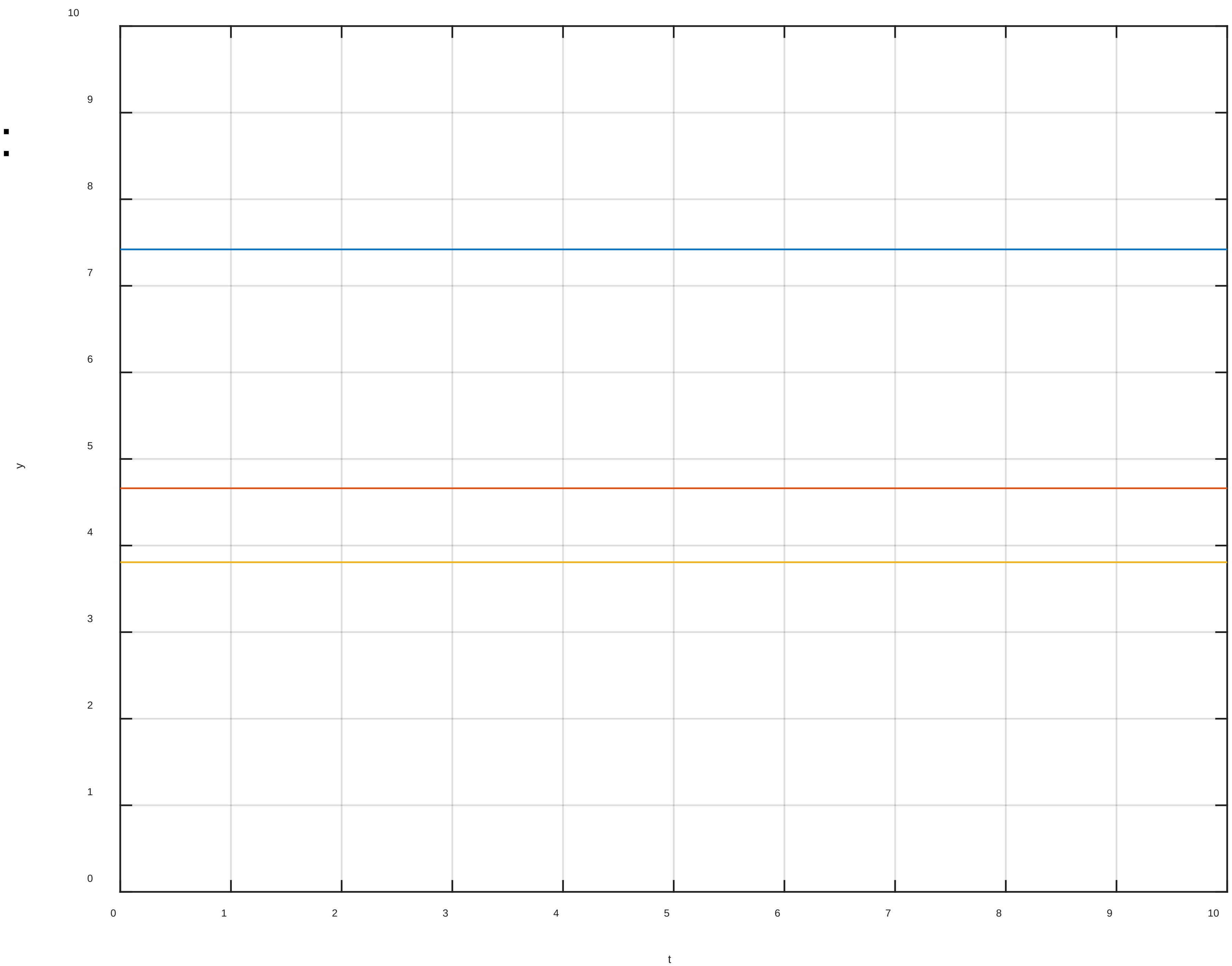
$Y(t) = W;$

$W \sim \mathcal{N}(5,2)$

3 realizations
$0 \leq \mathrm{t} \leq 10$

WWS ✓

Ergodic ÷



Realizations / Samples - Example

Continous stochastic process:

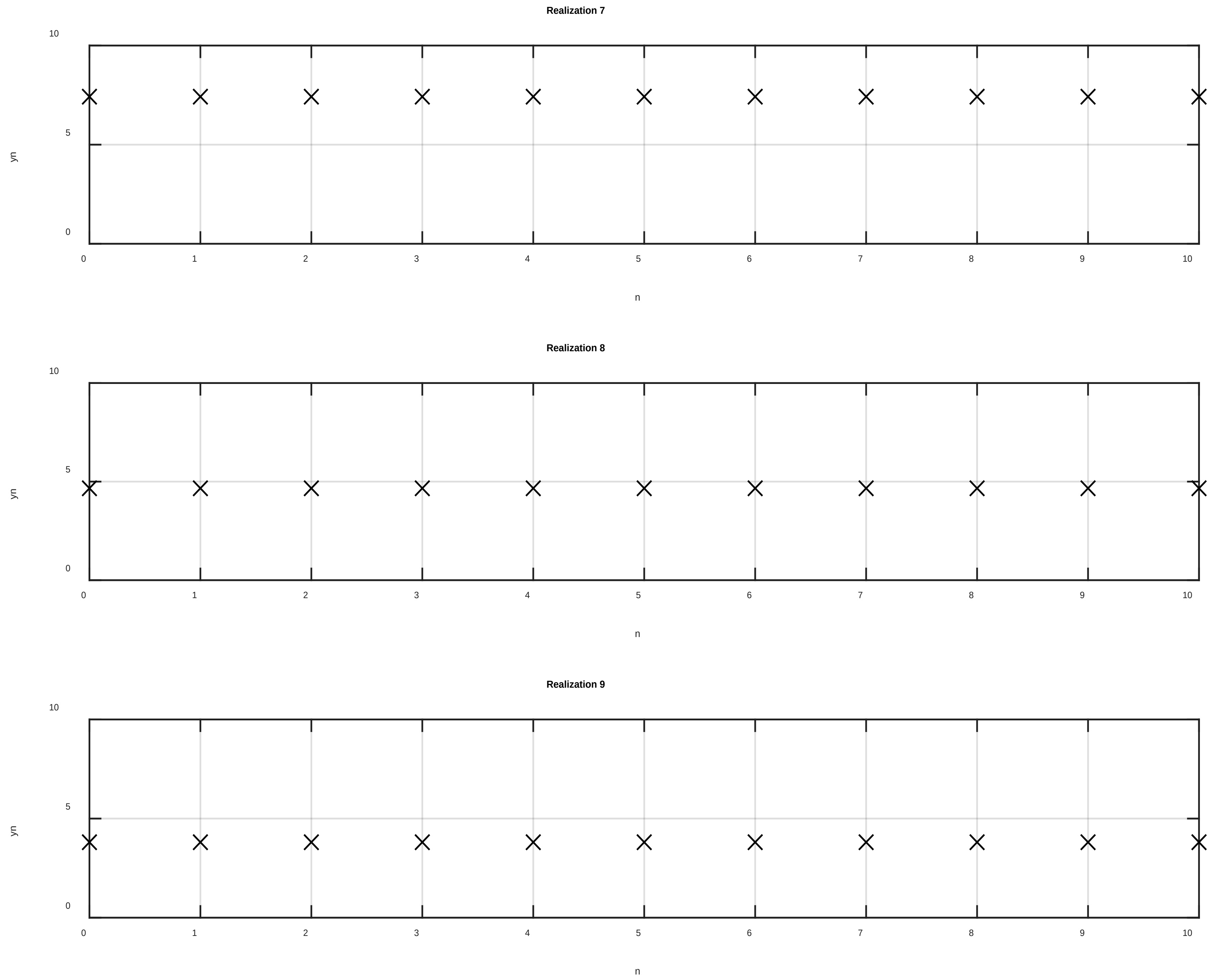
$Y(t) = W;$

$W \sim \mathcal{N}(5,2)$

3 realizations
11 samples
(n=0,..,10)

WWS ✓

Ergodic ÷



Realizations / Samples - Example

Continous stochastic process:

$$\mathrm{X}(t) = \mathrm{A}(1 - e^{-k \cdot t});$$

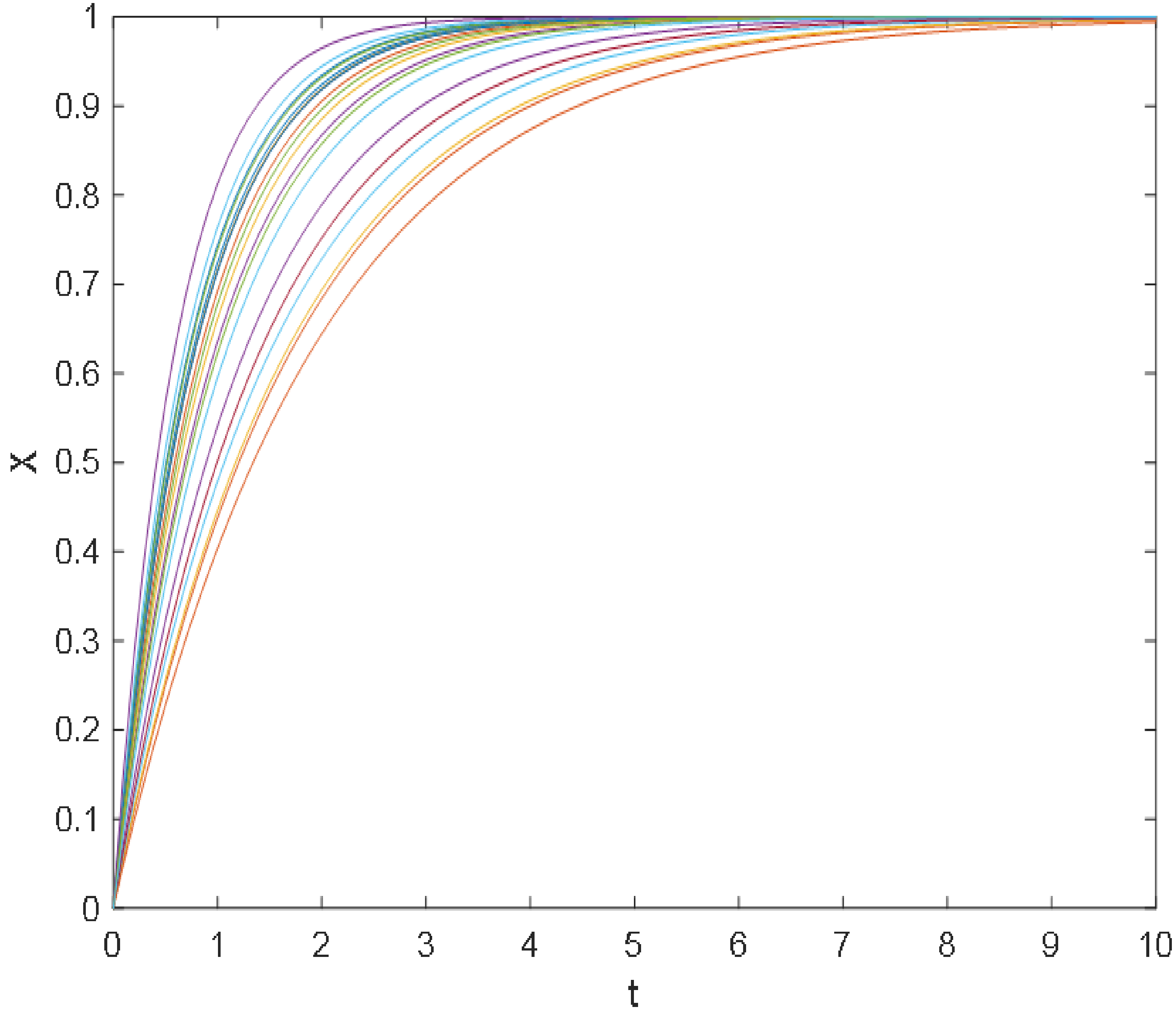
$$A = 1;\ k \sim \mathcal{N}(1{,}0.4)$$

20 realizations

$0 \leq \mathrm{t} \leq 10$

WWS ÷

Ergodic ÷



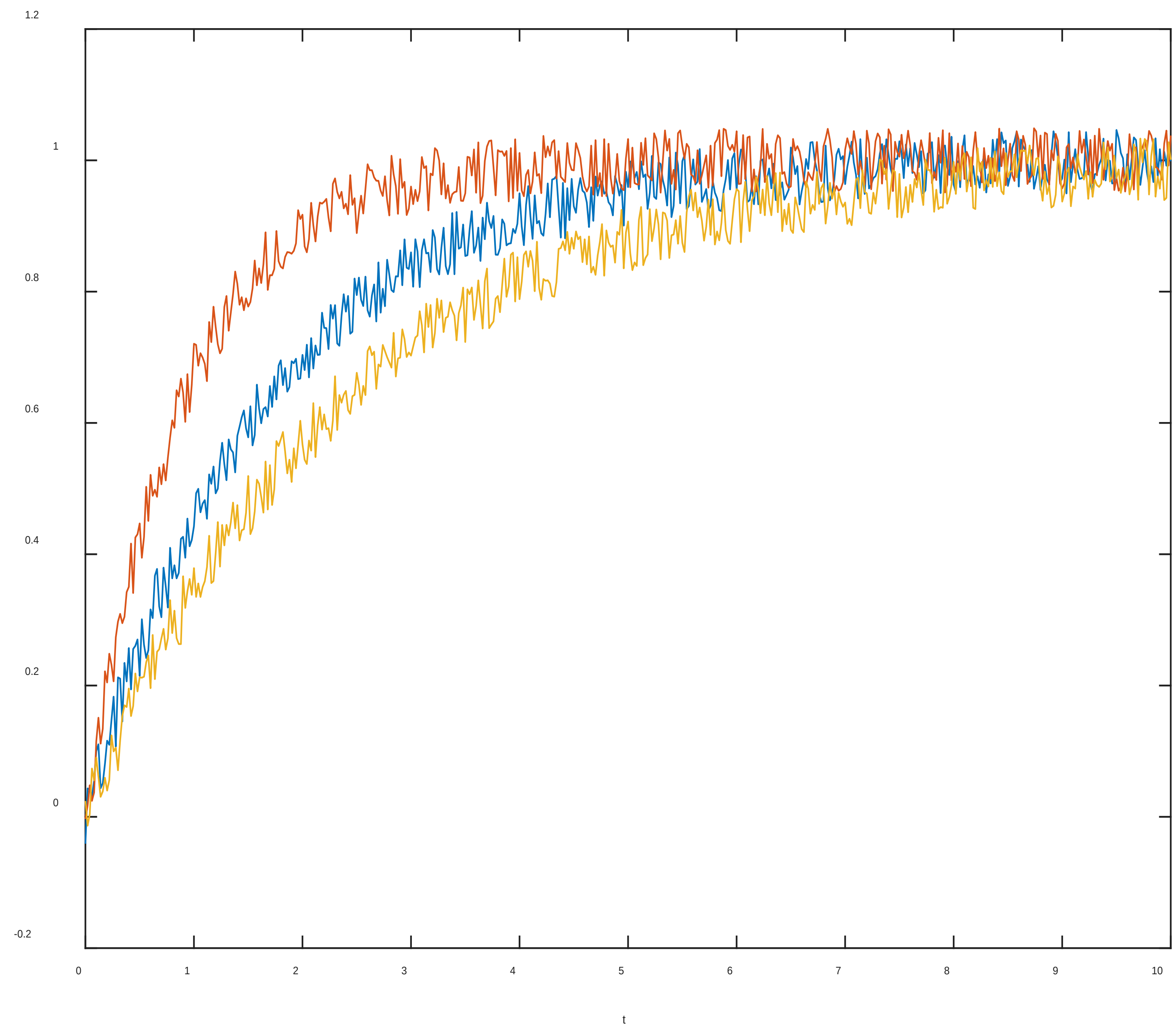
Realizations / Samples - Example

Continous stochastic process:

$$X(t) = A(1 - e^{-k \cdot t}) + w(t);$$

$$A = 1;\ k \sim \mathcal{N}(1{,}0.4);$$
$$w(t) \sim \mathcal{U}[-0.1{,}0.1]$$

3 realizations
$0 \leq t \leq 10$



WWS ÷

Ergodic ÷

Realizations / Samples - Example

Continous stochastic process:

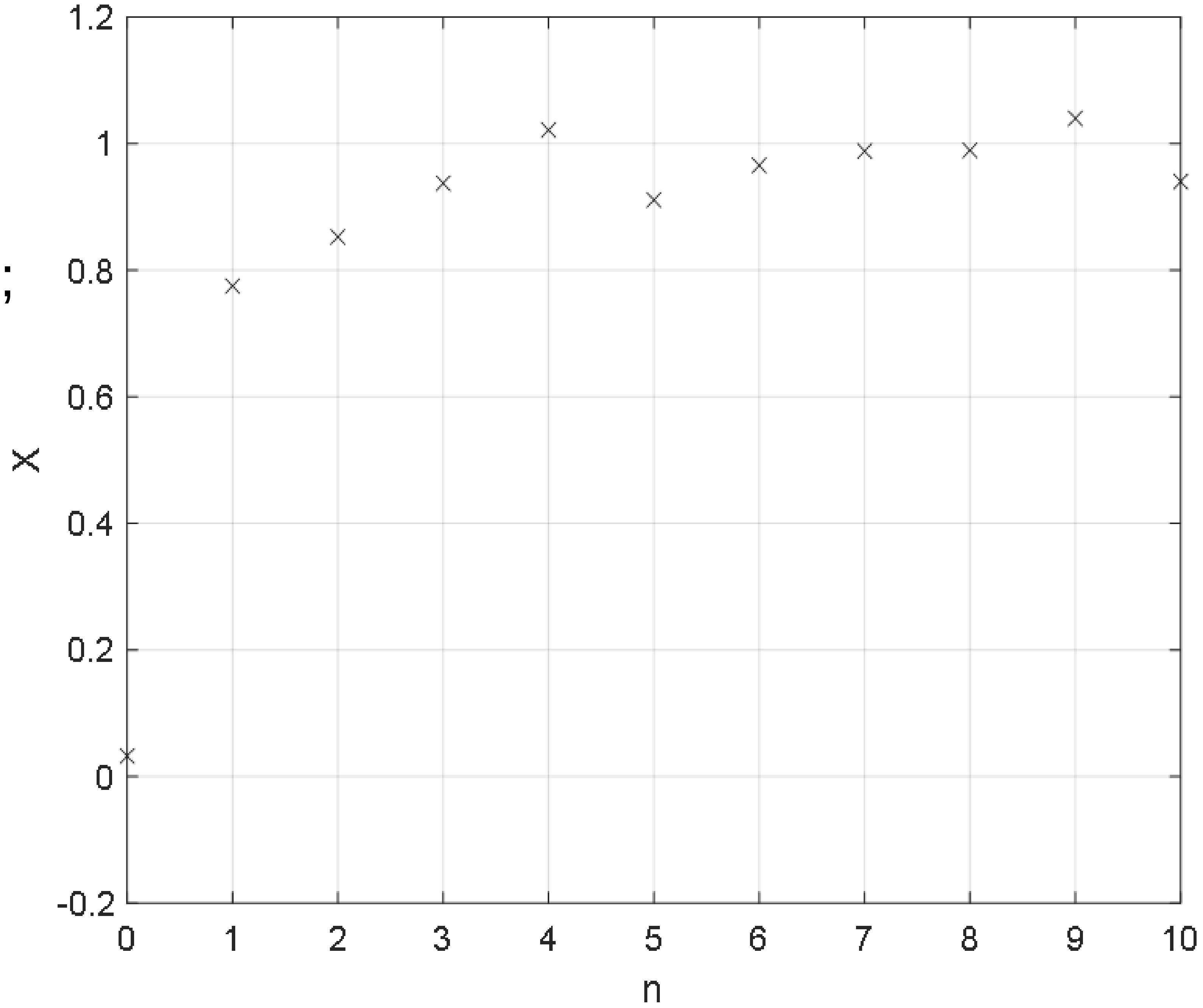
$X(t) = A(1 - e^{-k \cdot t}) + w(t);$

$A = 1;\ k \sim \mathcal{N}(1{,}0.4);$
$w(t) \sim \mathcal{U}[-0.1{,}0.1]$

1 realization
11 samples
(n=0,..,10)

WWS ÷

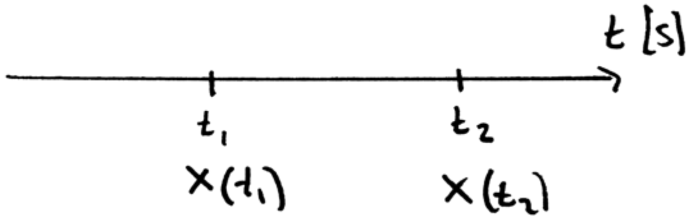
Ergodic ÷



Comparing realizations

Correlations

- We compare the process at two different times.



Correlation of a realization with itself

- <u>Autocorrelation</u>: $R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)^*]$
  - Says something about how much the signal $X(t_1)$ resembles itself at time t_2
  - Must depent on how rapidly the signal changes over time
  - Larger if $|t_1 - t_2|$ is small

Correlation of two realizations

- <u>Crosscorrelation</u>: $R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)^*]$
  - Can be used to look for places where the signal $X(t)$ is similar to the signal $Y(t)$

Autocorrelation

Tells about the connection at two different times

- In general:

Complex conjugated

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)^*]$$
$$= \iint_{-\infty}^{\infty} x(t_1)\, x(t_2)^* f_{X(t_1),X(t_2)}\big(x(t_1), x(t_2)\big) dx(t_1) dx(t_2)$$

- For a stationary process (WSS):

$$R_{XX}(t_1, t_2) = R_{XX}(t_1 + T, t_2 + T) = E[X(t_1 + T)X(t_2 + T)^*]$$

Independent of time (t_1)
Depends only on $\tau = t_2 - t_1$

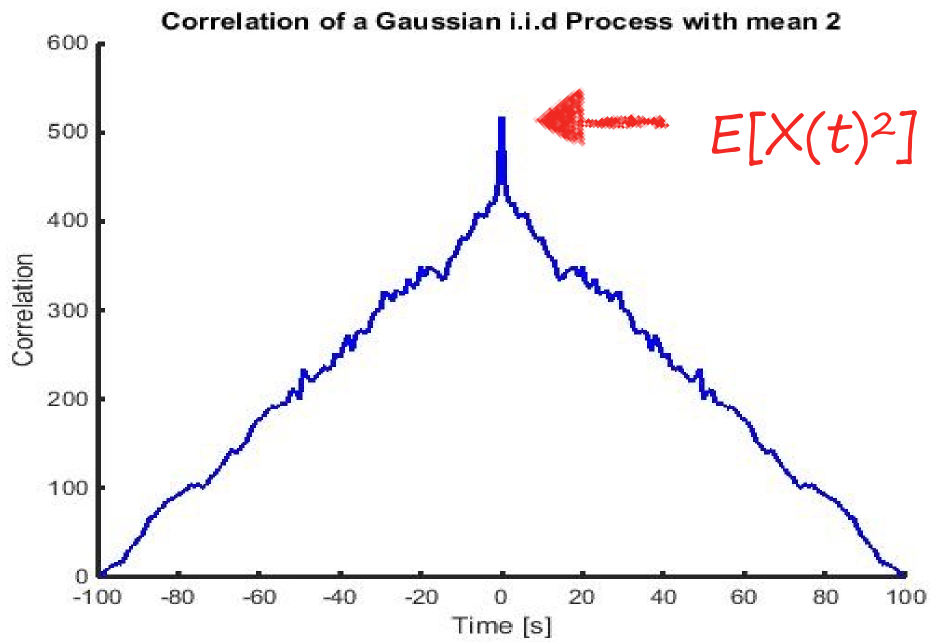
- We rewrite to: $R_{XX}(\tau) = E[X(t)X(t+\tau)^*]$

$\tau = t_2 - t_1$ is the lag!

Autocorrelation

- For Real WSS: $R_{XX}(\tau) = E[X(t)X(t+\tau)]$

- Properties of the autocorrelation function $R_{XX}(\tau)$:
  - An even function of τ $(R_{XX}(\tau) = R_{XX}(-\tau))$
  - Bounded by: $|R_{XX}(\tau)| \leq R_{XX}(0) = E\left[X^2\right]$ (max. in $\tau = 0$)
  - If X(t) changes fast, then $R_{XX}(\tau)$ decreases fast from $\tau = 0$
  - If X(t) changes slowly, then $R_{XX}(\tau)$ decreases slowly from $\tau = 0$
  - If X(t) is periodic, then $R_{XX}(\tau)$ is also periodic



Temporal Autocorrelation

Temporal only looks at one realization of the stochastic process.

Convolution

- Temporal autocorrelation:

$$\mathcal{R}_{XX}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x\,(t)\cdot x\,(t+\tau)dt$$

- If the process is <u>ergodic</u> the temporal autocorrelation is equal to the ensemble autocorrelation:

$$R_{XX}(\tau) = \mathcal{R}_{XX}(\tau)$$

Ensemble

Temporal

Estimate Autocorrelation

We only have measurements of one realization of X(t)

Autocorrelation function:

- In practise, with respect to the lag:

temporal

$$\mathcal{R}_{XX}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x\,(t)\cdot x\,(t+\tau)dt$$

N+1 measurements

$$x(0), x(\Delta t), x(2\Delta t), \dots, x(N\Delta t)$$

- The estimated autocorrelation function:

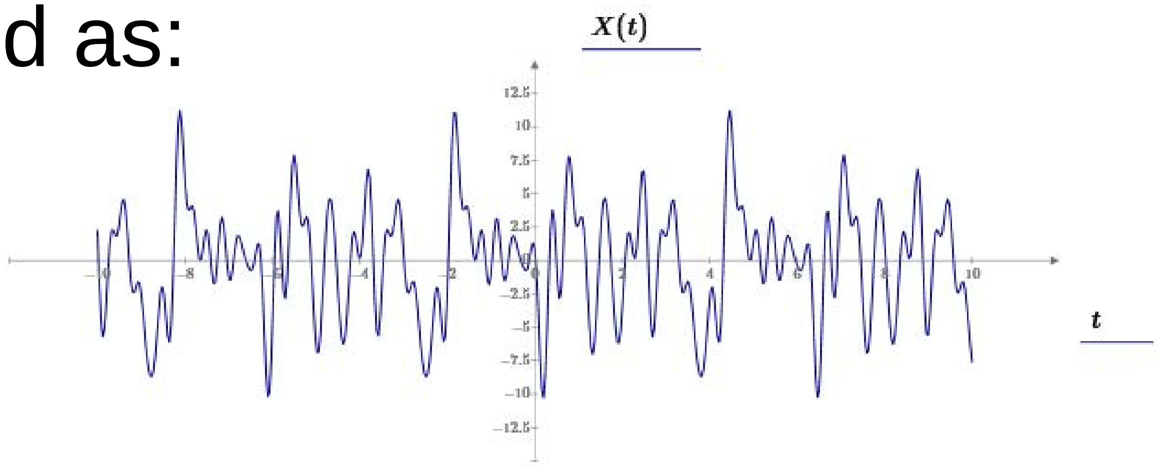
hat = estimation

$$\hat{R}_{XX}(n\Delta t) = \frac{1}{\underset{\text{Number of terms } (T/\Delta t)}{N-n+1}} \sum_{k=0}^{N-n} \underset{t}{x(k\Delta t)} \cdot \underset{t+\tau}{x((k+n)\Delta t)}$$

Autocorrelation Functions – Example

- Let a stochastic process be defined as:

$$X(t) = \sum_{i=1}^{n} (A_i cos\omega_i t + B_i sin\omega_i t)$$



- where A_i , $B_i \sim \mathcal{N}(0, \sigma^2)$ and i.i.d., and $\omega_i = i \cdot \omega_0$

- Find the autocorrelation:

$$E[X(t)X(t+\tau)] = E\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(A_i cos\omega_i t + B_i sin\omega_i t) \cdot \right.$$

$$\left. \cdot\, (A_j cos\omega_j (t+\tau) + B_j sin\omega_j (t+\tau))\ \right]$$

Autocorrelation Functions – Example (cont'd)

$$E[X(t)X(t+\tau)] = E\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(A_i cos\omega_i t + B_i sin\omega_i t) \cdot \right.$$
$$\left. \cdot \left(A_j cos\omega_j (t+\tau) + B_j sin\omega_j (t+\tau)\right) \right]$$

- Since A and B are i.i.d. (and $E[A_i] = E[B_i] = 0$):

$$i \neq j : E[A_i A_j] = 0, E[A_i B_j] = 0, E[B_i A_j] = 0, E[B_i B_j] = 0$$

- We get: $$E[X(t)X(t+\tau)] = \sum_{i=1}^{n}(E\left[A_i^2\right] \cdot cos\,\omega_i t \cdot cos\,\omega_i (t+\tau)$$
$$+ E\left[B_i^2\right] \cdot sin\,\omega_i t \cdot sin\,\omega_i (t+\tau))$$

Autocorrelation Functions – Example (cont'd)

- We can rewrite to:

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

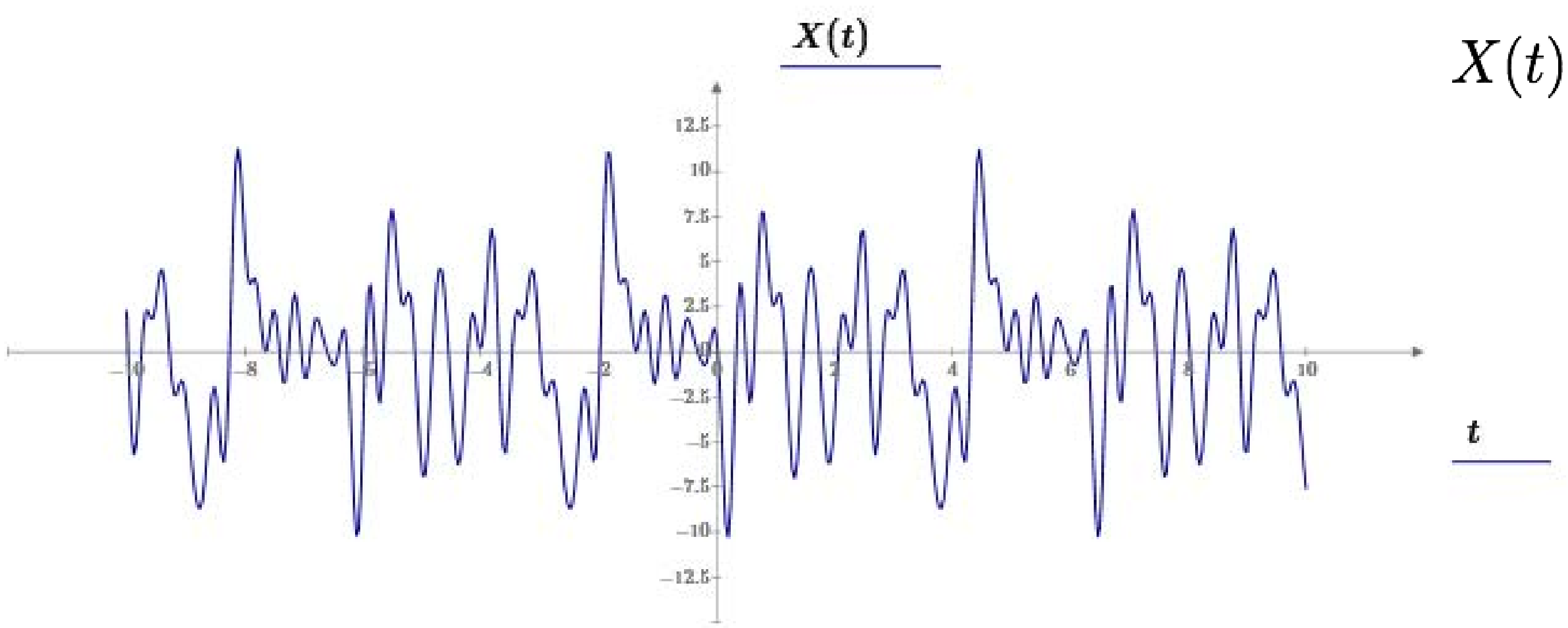
$$= \sum_{i=1}^{n} \left(E\left[A_i^2\right] \cdot cos\, \omega_i t \cdot cos\, \omega_i (t+\tau) + E\left[B_i^2\right] \cdot sin\, \omega_i t \cdot sin\, \omega_i (t+\tau)\right)$$

$$= \sigma^2 \sum_{i=1}^{n} cos\, \omega_i \tau$$

(since $E\left[A_i^2\right] = E\left[B_i^2\right] = \sigma^2$ and $\cos(\theta_1 - \theta_2) = cos\theta_1 \cdot cos\theta_2 + sin\theta_1 \cdot sin\theta_2$)

- We have: $R_{XX}(0) = n\sigma^2$

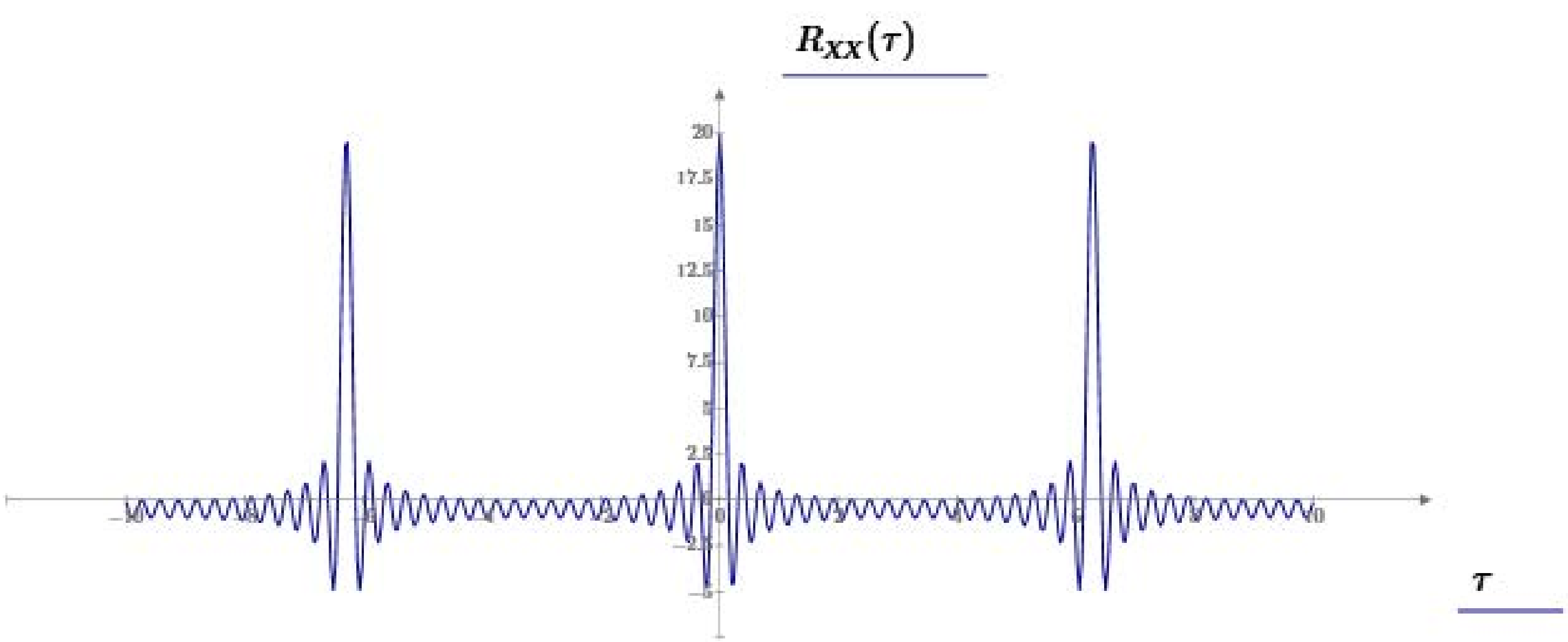
Autocorrelation Functions – Example (cont'd)



$$X(t) = \sum_{i=1}^{n} A_i \cos \omega_i t + B_i \sin \omega_i t$$

$$A_i, B_i \sim \mathcal{N}(0, \sigma^2)$$

$$\omega_i = i \cdot \omega_0$$
$$\omega_0 = 1$$



$$\sigma = 1, n = 20$$

$$R_{XX}(0) = n\sigma^2 = 20$$

Tells about how much we can predict the future

Autocovariances

- Autocovariance function:

$$C_{XX}(t_1, t_2) = E\left[\left(X(t_1) - \mu_X(t_1)\right)\left(X(t_2) - \mu_X(t_2)\right)^*\right]$$
$$= R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$

Especially: $C_{XX}(t, t\ ) = E\left[(X(t) - \mu_X(t))^2\right] = E\left[X(t)^2\right] - E[X(t)]^2 = \sigma_X^2(t)$

- Autocorrelation coefficient:

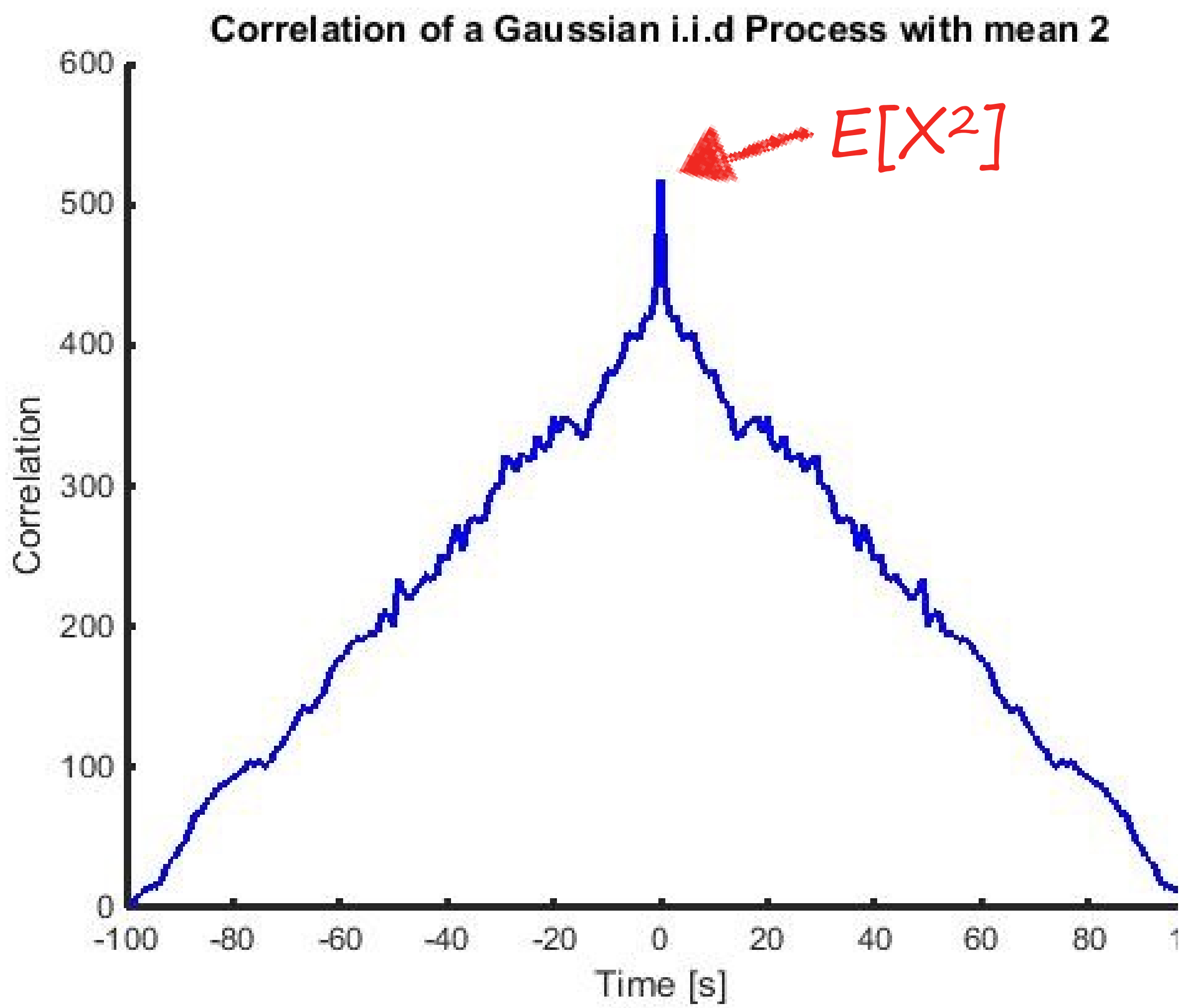
$$r_{XX}(t_1, t_2) = \frac{C_{XX}(t_1, t_2)}{\sqrt{C_{XX}(t_1, t_1)C_{XX}(t_2, t_2)}}; \qquad 0 \leq r_{XX}(t_1, t_2) \leq 1$$

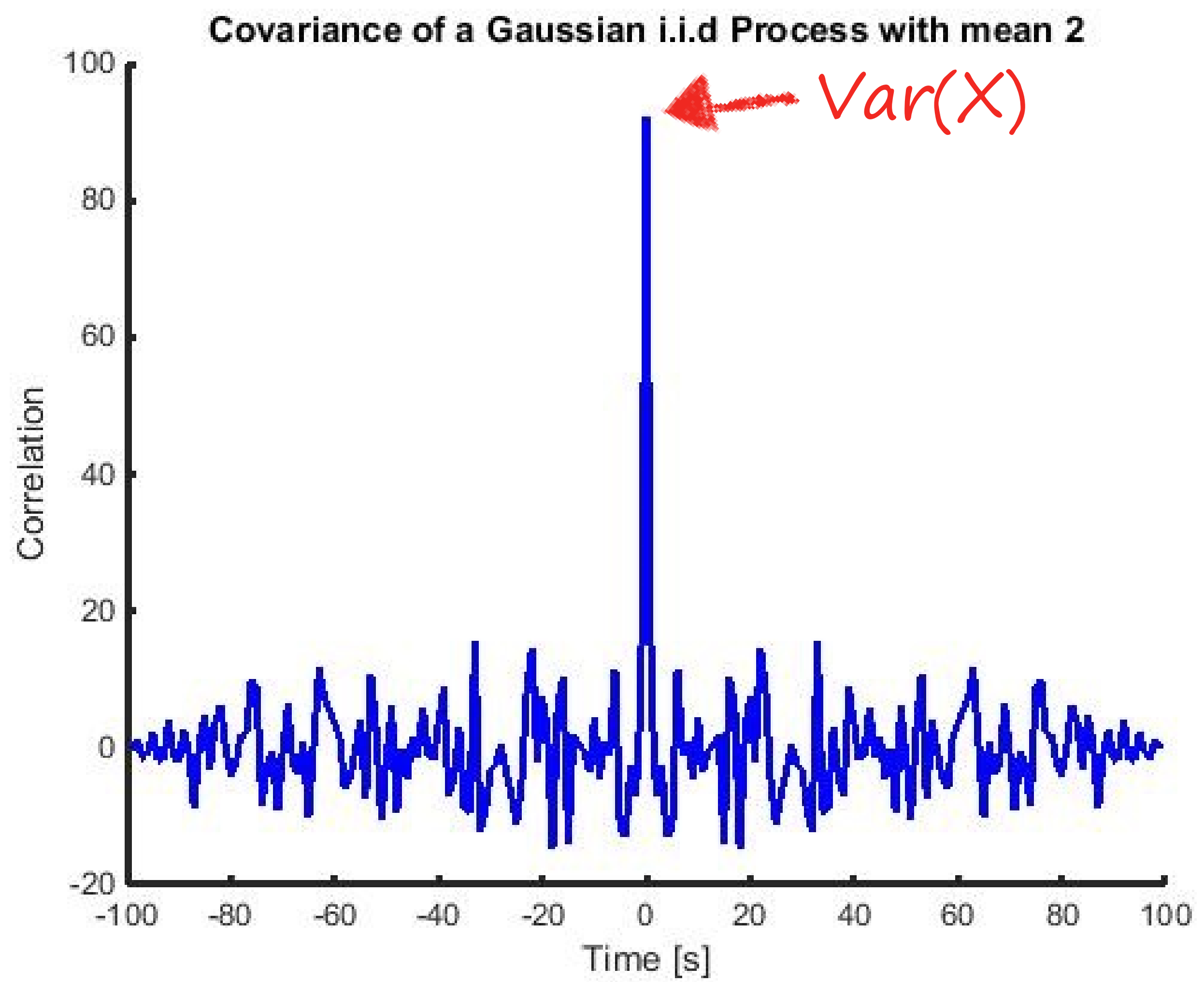
Especially: $r_{XX}(t, t) = 1$ ($X(\mathrm{t})$ is totally correlated to itself!)

Autocovariances

For i.i.d. Gaussian (stationary) noise

- Autocorrelation and autocovariance





Two Stochastic Processes

- If we have two stochastic processes $X(t)$ and $Y(t)$
  - We can compare them by looking at the cross-correlation and cross-covariance:

Cross-correlation $$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)^*]$$

Cross-covariance $$C_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)^*] - E[X(t_1)]E[Y(t_2)]$$

Ensemble Cross-correlation

Ensemble means that it applied for the ensemble of the two processes

- In general:

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)^*]$$

$$= \iint_{-\infty}^{\infty} x(t_1)\, y(t_2)^* f_{X(t_1),Y(t_2)}\big(x(t_1), y(t_2)\big) dx(t_1) dy(t_2)$$

- For two WSS stationary processes:

$$R_{XY}(t_1, t_2) = R_{XY}(t_1 + T, t_2 + T) = E[X(t_1 + T)Y(t_2 + T)^*]$$

- We write: $R_{XY}(\tau) = E[X(t) \cdot Y(t + \tau)^*]$

Cross-Correlation Functions

- For Real WSS processes $X(t)$ and $Y(t)$:

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$

- Properties of the cross-correlation function $R_{XY}(\tau)$:
  - $R_{XY}(\tau) = R_{YX}(-\tau))$
  - $|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)R_{YY}(0)} = \sqrt{E[X^2]E[Y^2]}$
  - $|R_{XY}(\tau)| \leq \frac{1}{2}(R_{XX}(0) + R_{YY}\ (0))$
  - If $X(t)$ and $\mathrm{Y}(t)$ are orthogonal, then $R_{XY}(\tau) = 0$
  - If $X(t)$ and $\mathrm{Y}(t)$ are independant, then $R_{XY}(\tau) = \mu_X \cdot \mu_Y$

Temporal Cross-correlation

Temporal only looks at one realization of the two stochastic processes.

- The temporal cross-correlation between X and Y:

Convolution

$$\mathcal{R}_{XY}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x\,(t) \cdot y\,(t+\tau) dt$$

- If the two processes are <u>ergodic</u> the temporal cross-correlation is equal to the ensemble cross-correlation:

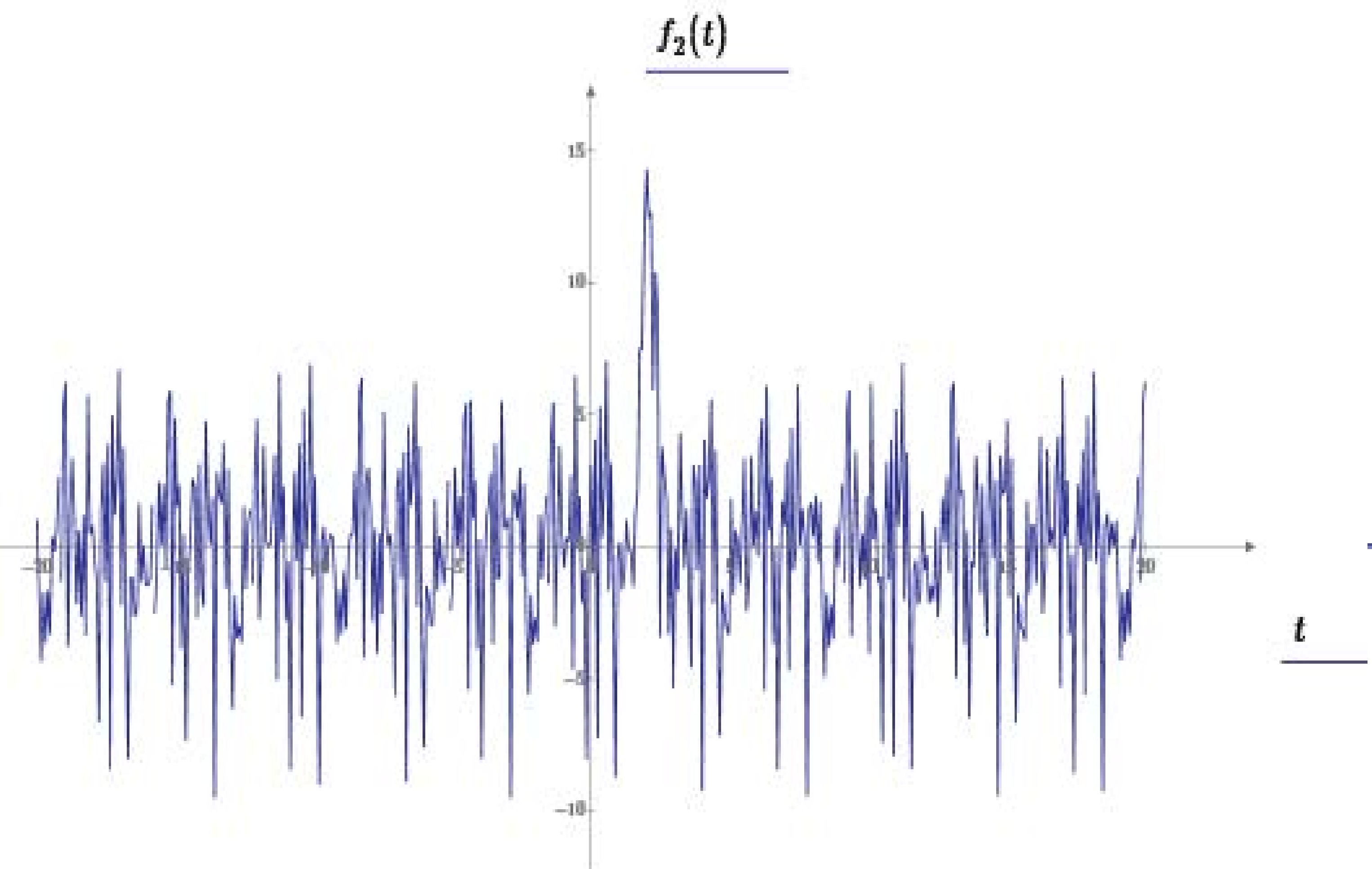
$$R_{XY}(\tau) = \mathcal{R}_{XY}(\tau)$$
$$R_{YX}(\tau) = \mathcal{R}_{YX}(\tau)$$

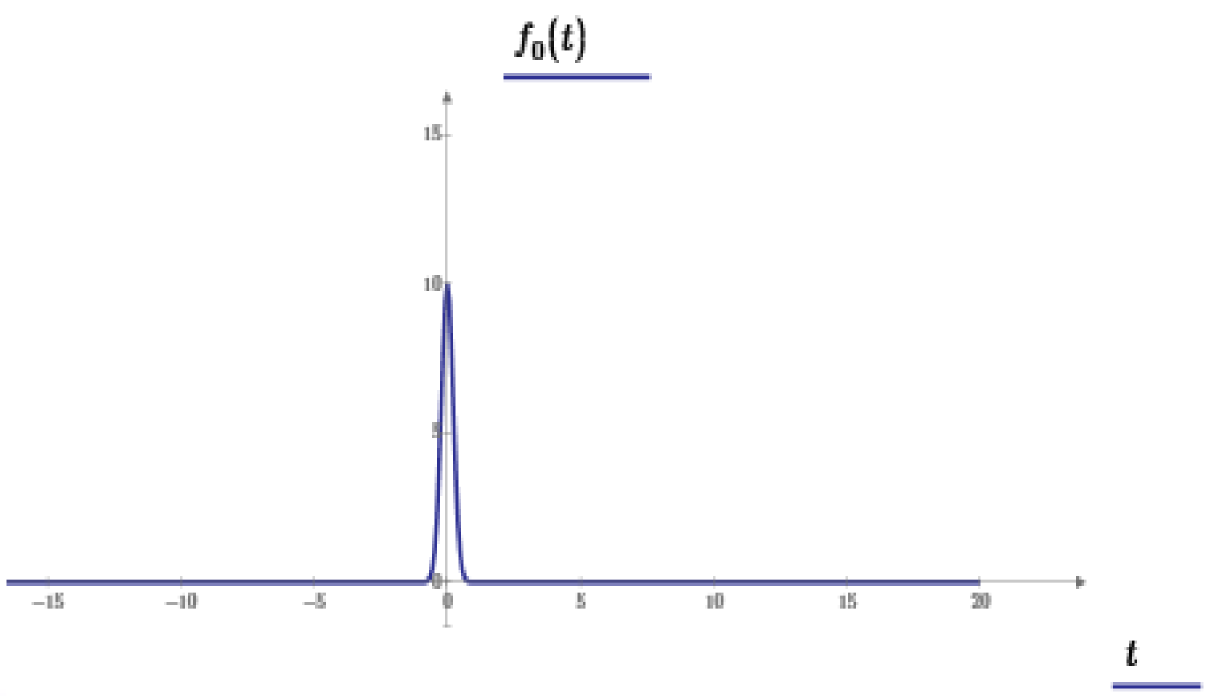
Ensemble

Temporal

Cross-correlation – Uncalibrated noisy signal

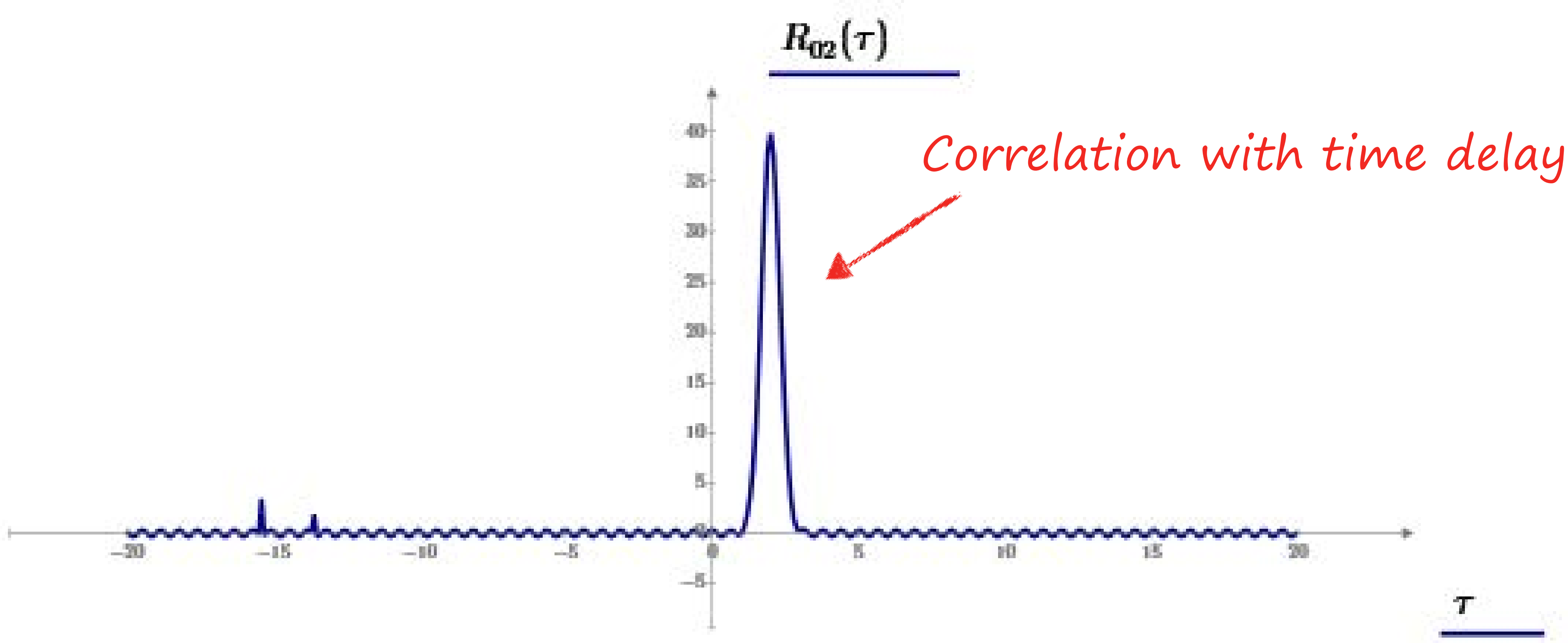
- Comparing two signals:
  - An uncalibrated and noisy signal: $f_2(t)$
  - Reference signal: $f_0(t) = 10 \cdot e^{-10t^2}$





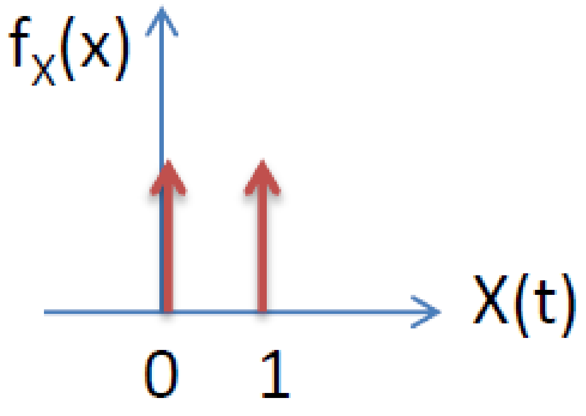
Cross-correlation – Uncalibrated noisy signal

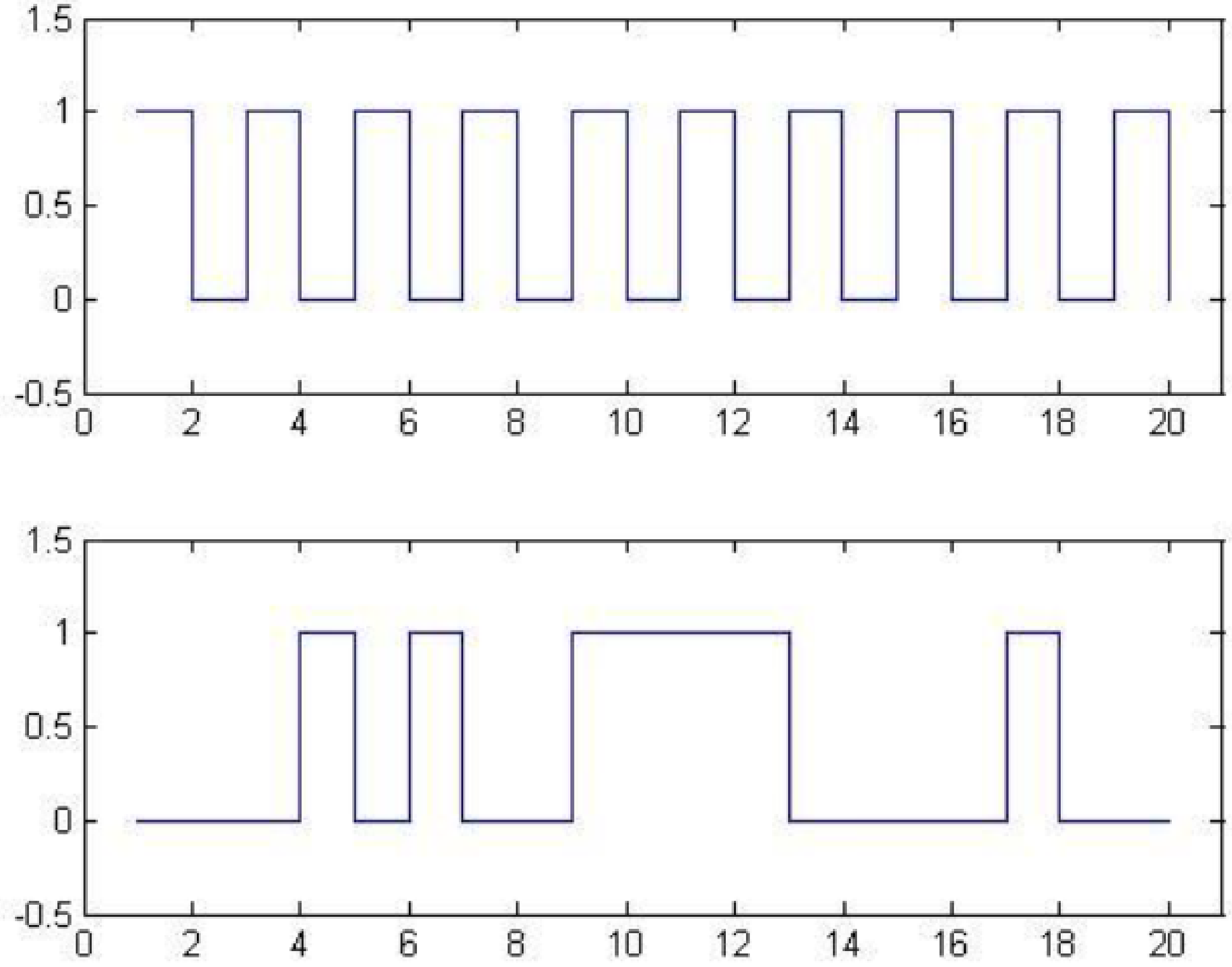
- Comparing two signals:
  - An uncalibrated and noisy signal $f_2(t)$
  - Reference signal $f_0(t) = 10 \cdot e^{-10t^2}$
- Cross-correlation: $R_{02}(\tau) = \int_{-\infty}^{\infty} f_0(t) \cdot f_2(t+\tau)dt$



Deterministic vs. Stochastic

The probability mass function:





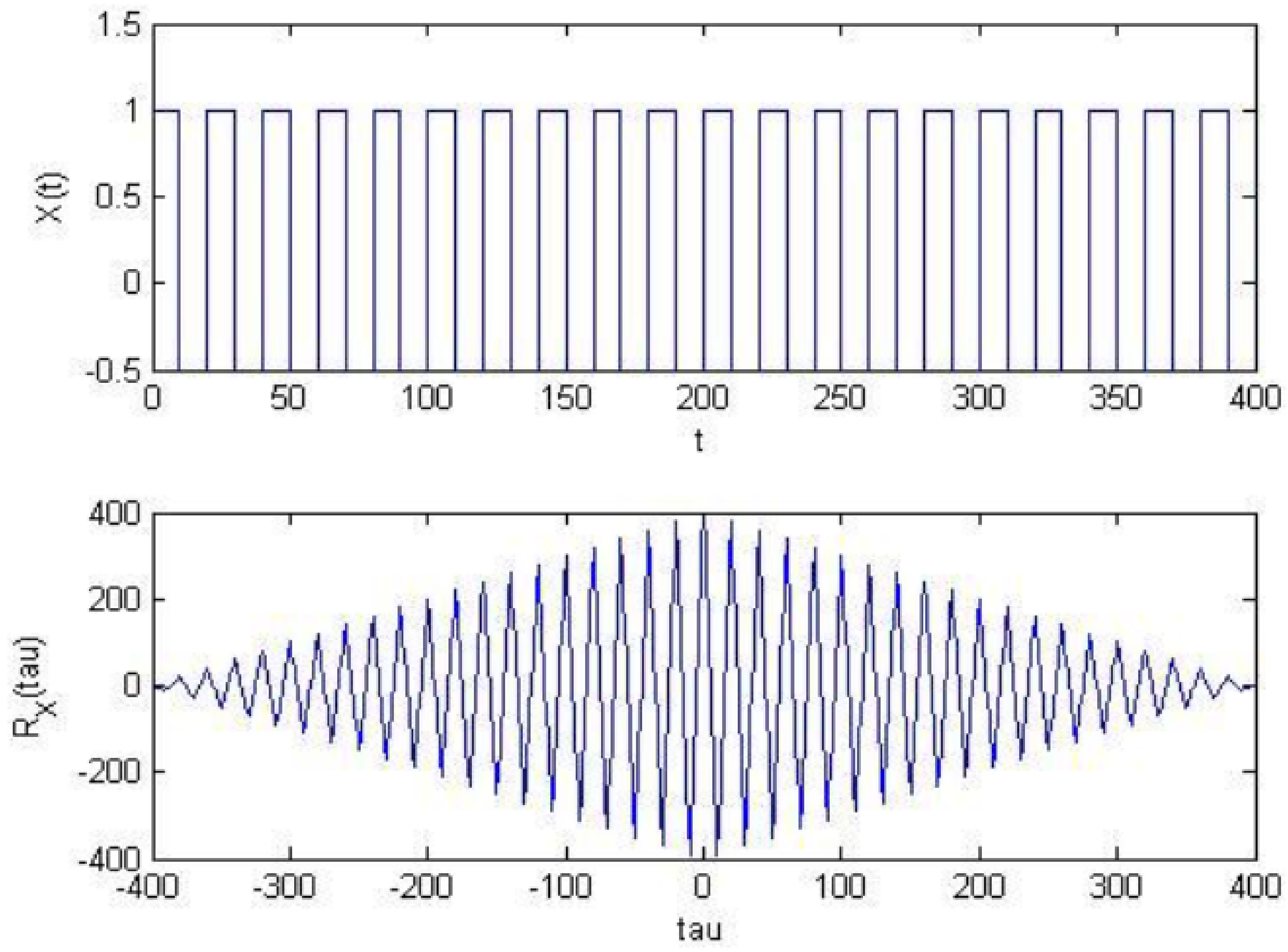
The two random processes have the same pmf.

Deterministic

Periodic signal

↓

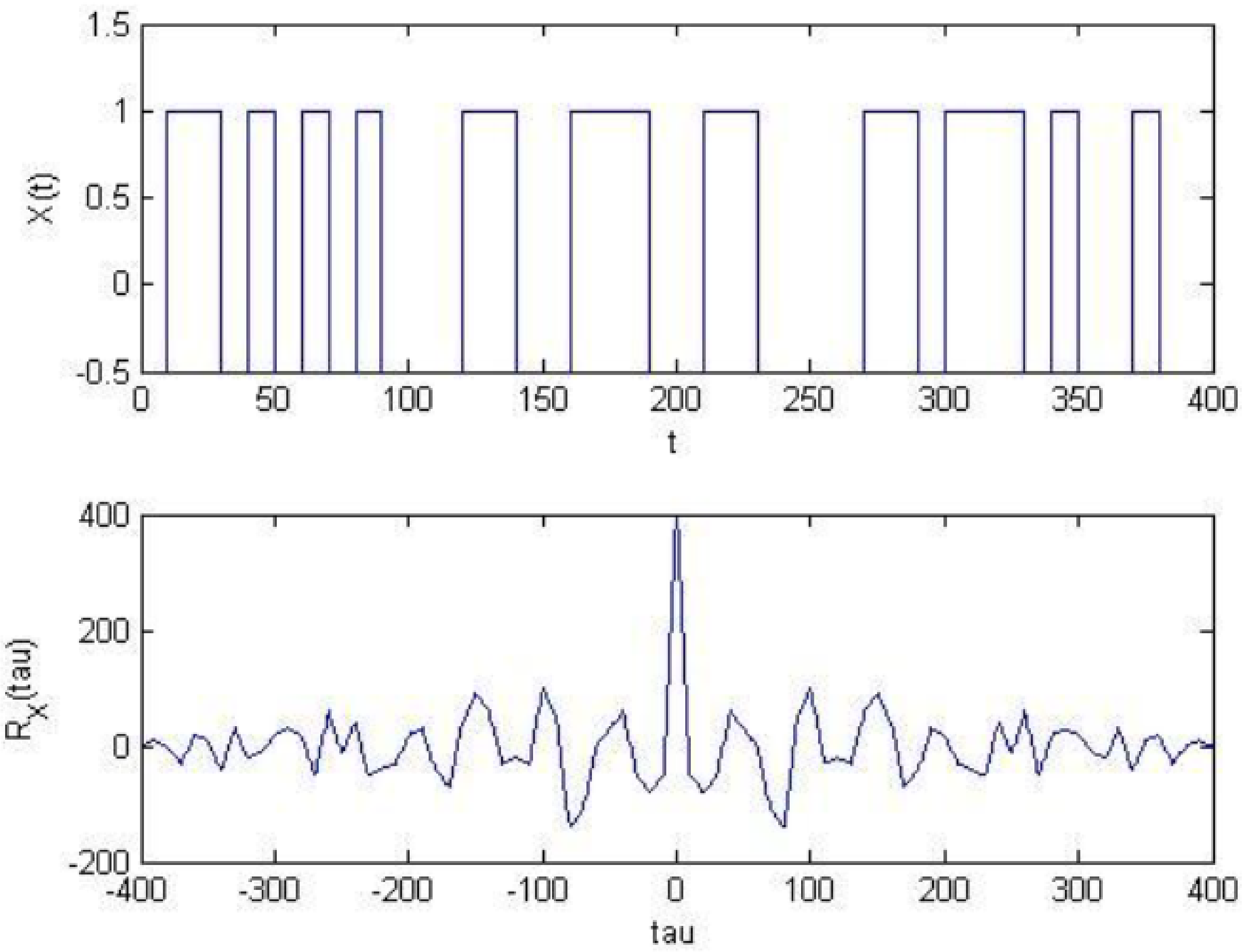
R_{XX} periodic




```
Rx = conv(x,fliplr(x));
```

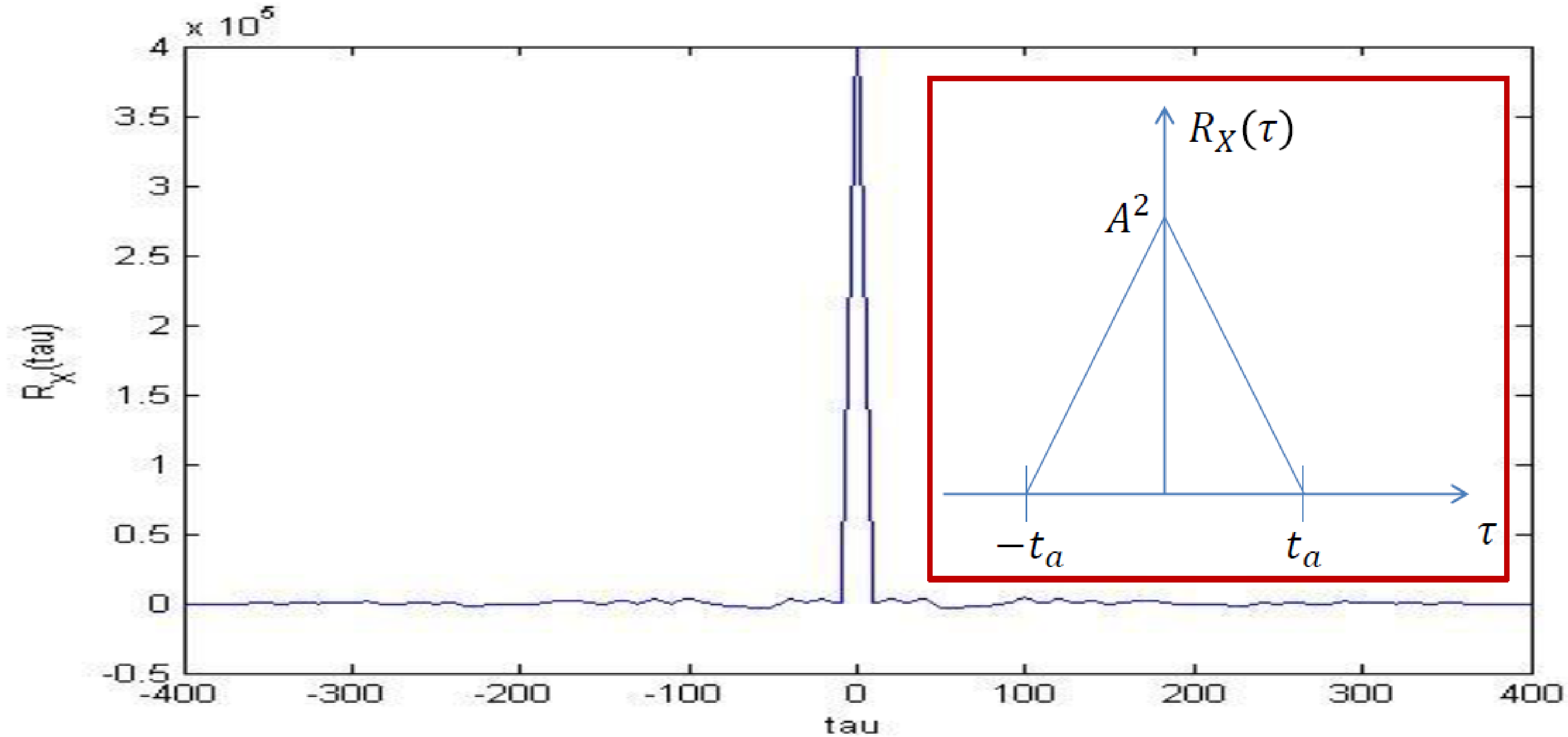
Stochastic

Also called Non-deterministic

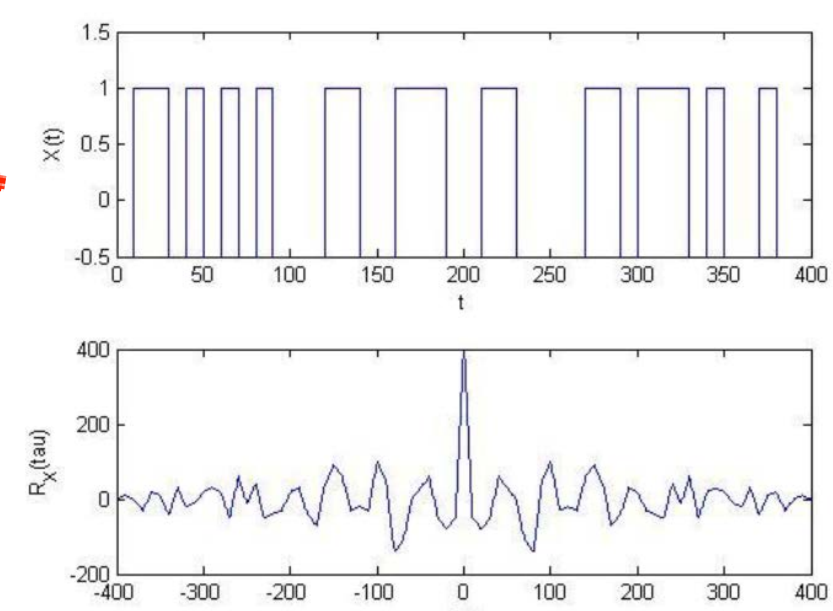



```
Rx = conv(x,fliplr(x));
```

Autocorrelation for Stochastic



Autocorrelation function averaged over 1000 simulations.



Power Spectral Density (psd)

- Frequency domain:
  - Deterministic signals f(t) → Fourier-transformation $\mathcal{F}(\mathrm{f(t)})$
  - Random signals X(t) → ÷Fourier-transformation

- For Real WSS:
- Properties of the autocorrelation function $R_{XX}(\tau)$:
  - If X(t) changes fast, then $R_{XX}(\tau)$ decreases fast from $\tau = 0$
  - If X(t) changes slowly, then $R_{XX}(\tau)$ decreases slowly from $\tau = 0$
  - If X(t) is periodic, then $R_{XX}(\tau)$ is also periodic

→ $R_{XX}(\tau)$ contain information about the frequency content in X(t)

Power Spectral Density (psd)

- Deterministic signals $x(t)$:

Time-average

- Average power: $P_X = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)^2 dt$
- $x(t)$ periodic T_0: $\langle R_{XX}(\tau)\rangle_{T_0} = \frac{1}{T_0} \int_0^{T_0} x(t)x(t+\tau)dt$
- Power Spectral Density Function (psd):

$$S_{XX}(f) = \mathcal{F}(\langle R_{XX}(\tau)\rangle_{T_0}) \;\Rightarrow\; P_X = \int_{-\infty}^{\infty} S_{XX}(f)df$$

Fourier-transform

Average power in $x(t)$

Power Spectral Density (psd)

- WSS random signals X(t):
- Power Spectral Density Function (psd):

Fourier-transform

$$S_{XX}(f) = \mathcal{F}(R_{XX}(\tau)) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\cdot 2\pi f\cdot \tau} d\tau$$

Invers Fourier-transform

$$\Rightarrow \quad R_{XX}(\tau) = \mathcal{F}^{-1}(S_{XX}(f)) = \int_{-\infty}^{\infty} S_{XX}(f) e^{j\cdot 2\pi f\cdot \tau} df$$

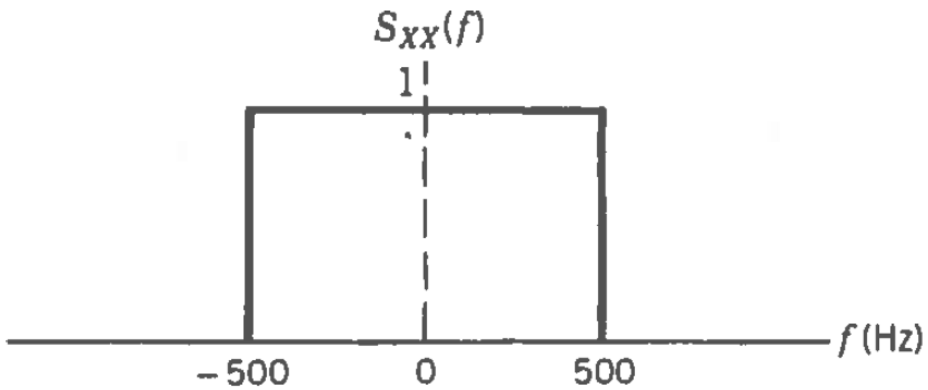


Figure 3.19a Psd of a lowpass random process $X(t)$.

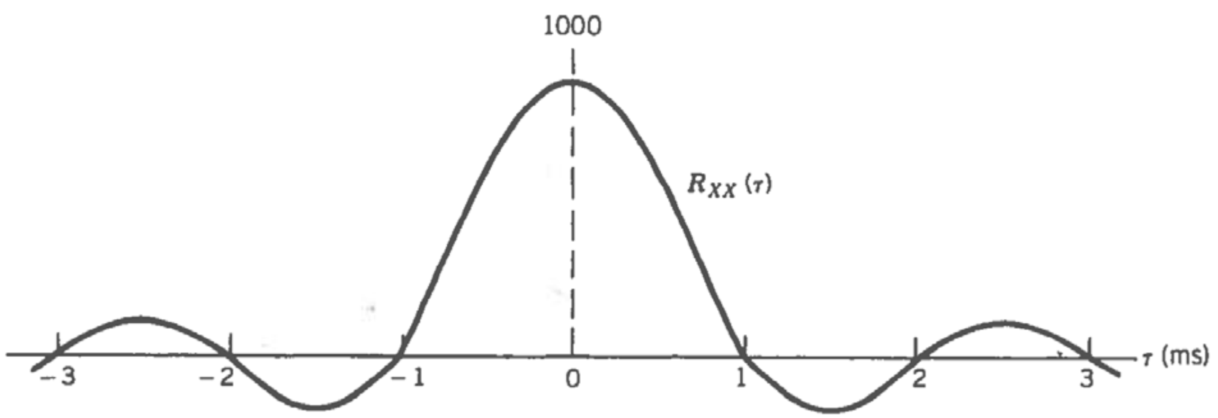


Figure 3.19b Autocorrelation function of $X(t)$.

Figures from ”Random Signals”

Power Spectral Density (psd)

- Properties of psd $S_{XX}(f)$ (spectrum of $X(t)$):
  - $S_{XX}(f) \in \mathbb{R}$
  - $S_{XX}(f) \geq 0$
  - If $X(t) \in \mathbb{R}$: $R_{XX}(-\tau) = R_{XX}(\tau)$ and $S_{XX}(-f) = S_{XX}(f) \rightarrow$ even functions
  - If $X(t)$ periodic components: $S_{XX}(f)$ will have impulses (δ−functions)
  - $[S_{XX}(f)] = \frac{W}{Hz} \rightarrow$ Distribution of power with frequency (power spectral density of the stationary random process X(t))
  - $P_X = E\left[X(t)^2\right] = R_{XX}(0) = \int_{-\infty}^{\infty} S_{XX}(f)df$

    i.e. if $X(t) = V(t)$ (voltage signal)

    $\rightarrow P_X =$ power in 1Ω-resistor

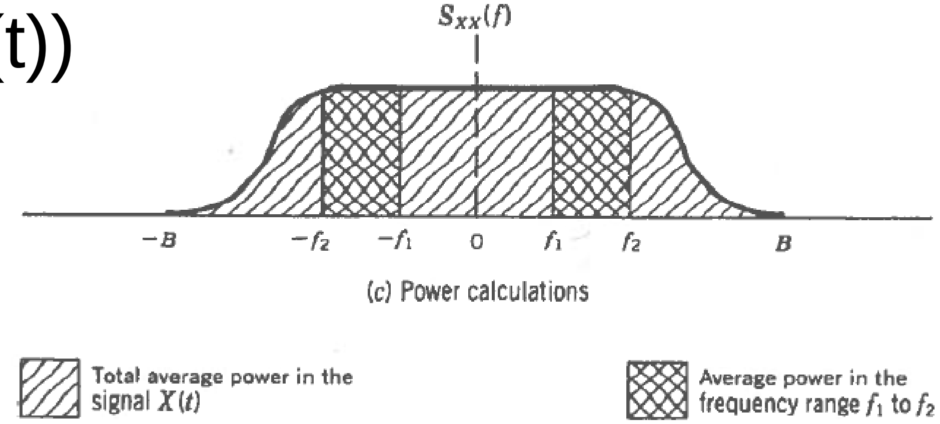


Figure from ”Random Signals”

  - $P_X[f_1, f_2] = 2\int_{f_1}^{f_2} S_{XX}(f)df \rightarrow$ Power in the frequency-interval $[f_1, f_2]$

Power Spectral Density – Random Binary Signal

Figures from ”Random Signals”

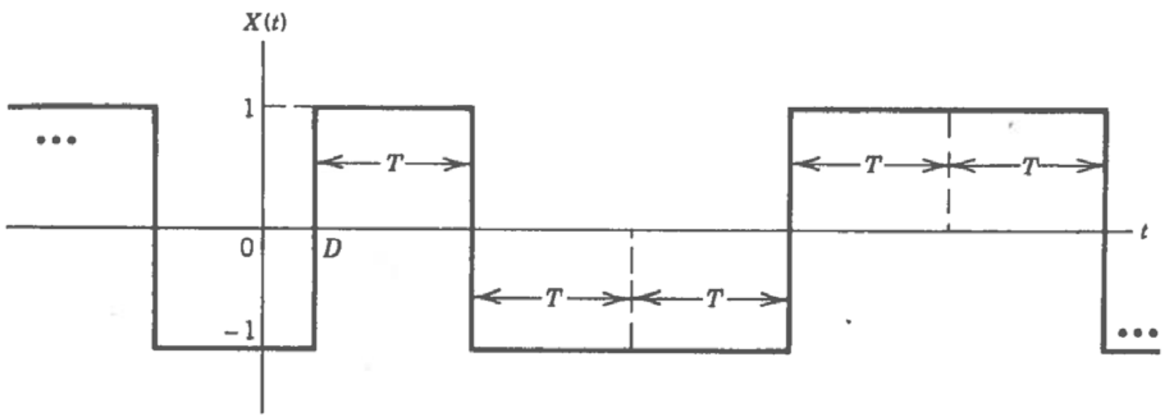


Figure 3.7 Random binary waveform.

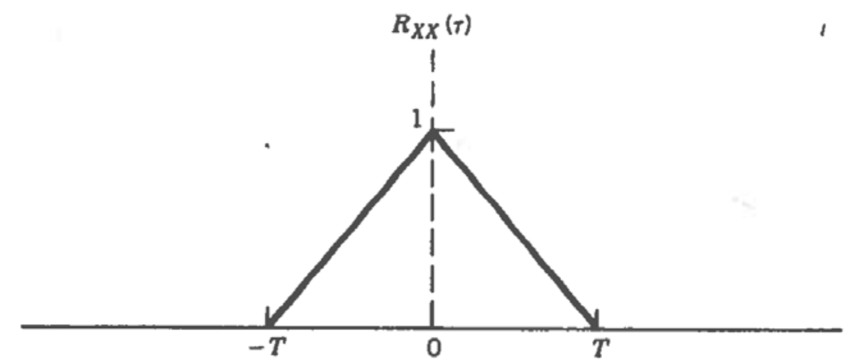


Figure 3.18a Autocorrelation function of the random binary waveform.

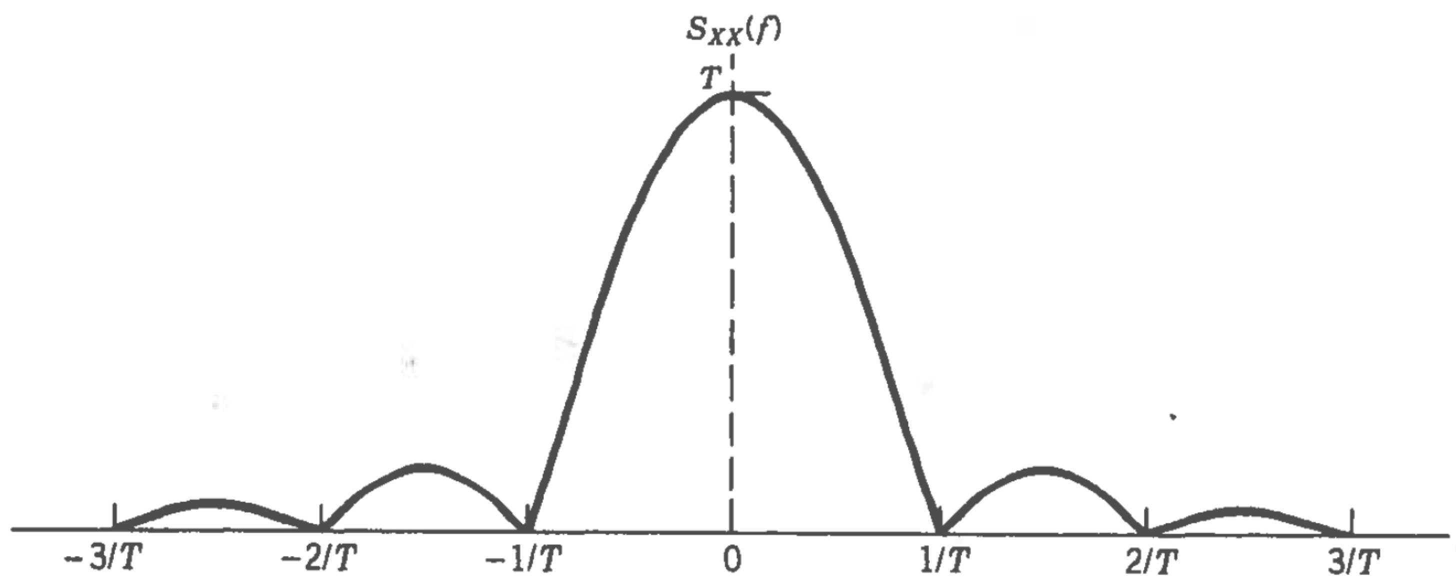


Figure 3.18b Power spectral density function of the random binary waveform.

Words and Concepts to Know

Cross-correlation

Power Spectral Density

Deterministic

Cross-covariance

psd

Temporal Autocovariance

Autocorrelation Coefficient

Temporal cross-correlation

Non-deterministic