

Continuous Random Variables

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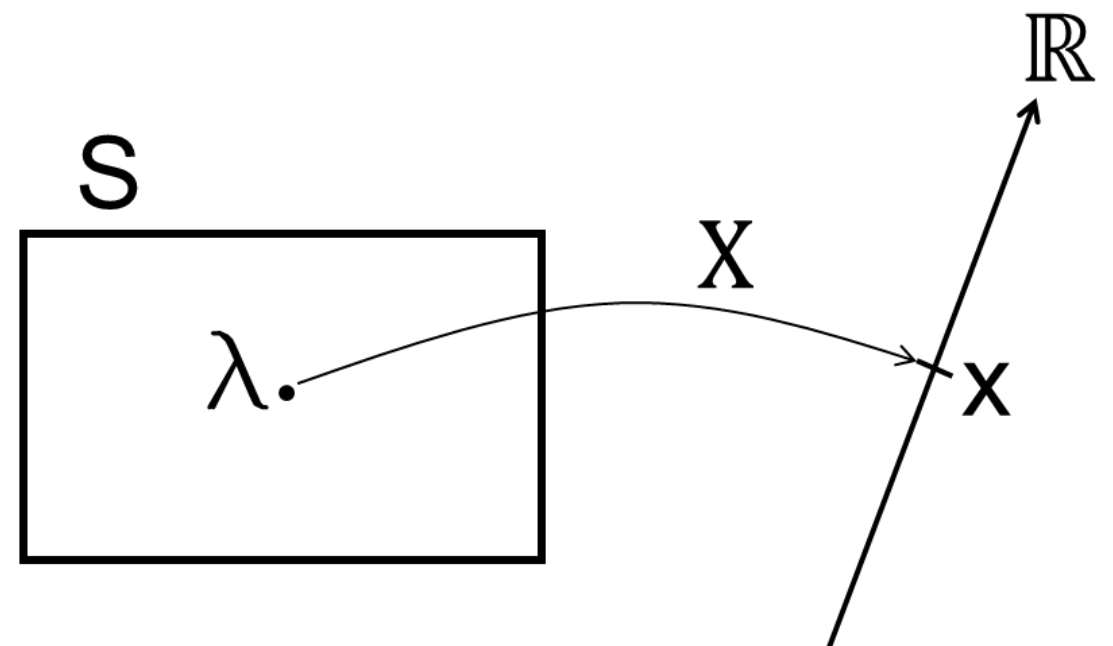
Agenda for Today

- Repetition from last time
 - Discrete Random Variables
- Continuous Random Variables

Also just called a random variables

Stochastic Random Variables

- A random variable tells something important about a stochastic experiment.
- Can be discrete or continuous



Examples:

- The numbers on a dice (discrete):
 - Sample space for variable X is : $\{1, 2, 3, 4, 5, 6\}$
 - Sample space for variable Y “Even (1)/Uneven (-1)”: $\{1, -1\}$
- The height of students at IHA (continuous):
 - Sample space for variable H is all real numbers: $[100; 250]$ cm.

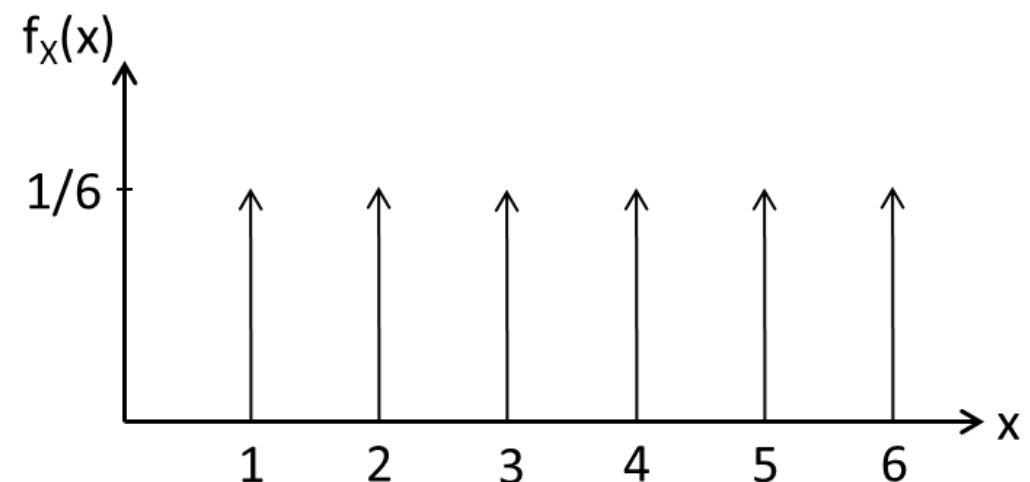
Probability Mass Function (PMF)

- Sample space for X .
- X is a discrete stochastic variable.

$$f_X(x) = \begin{cases} \Pr(X = x_i) & \text{for } X = x_i \\ 0 & \text{otherwise} \end{cases} \quad 0 \leq f_X(x) \leq 1$$

- We have that: $\sum_{i=1}^n f_X(x_i) = \sum_{i=1}^n \Pr(X = x_i) = 1$

Example: Laplace Dice
(perfect dice)



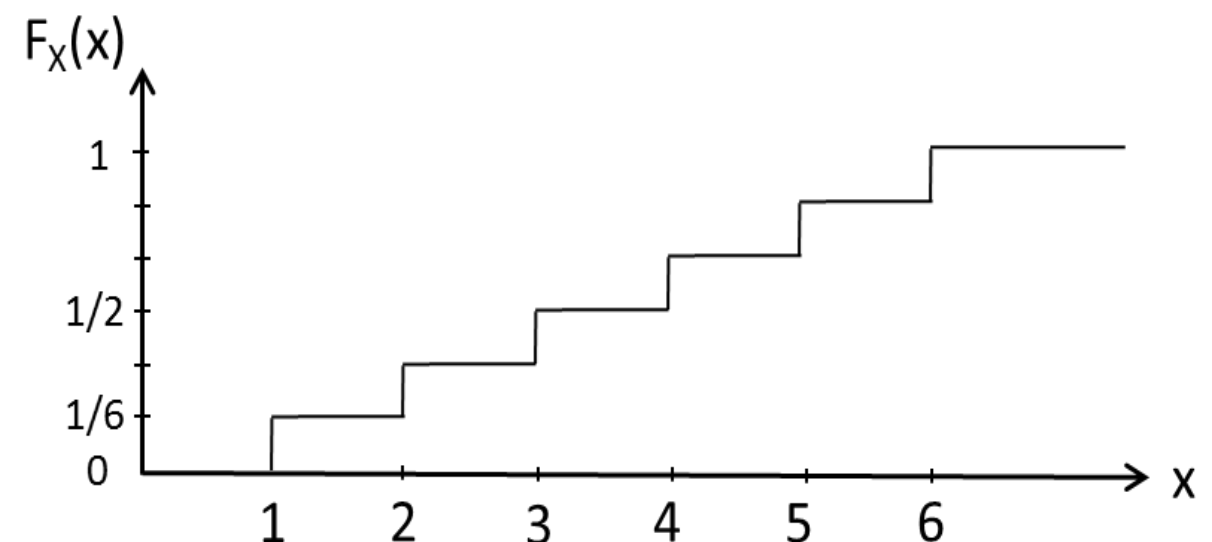
Cumulative Distribution Function (CDF)

- Sample space for X .
- X is a discrete stochastic variable.
- $F_X(x)$ is a non-decreasing step-function.

$$F_X(x) = \Pr(X \leq x) \quad 0 \leq F_X(x) \leq 1$$

- We have that: $\lim_{x \rightarrow -\infty} F_X(x) = 0$ and $\lim_{x \rightarrow \infty} F_X(x) = 1$

Example: Laplace Dice
(perfect dice)



Mean, Variance and Standard deviation

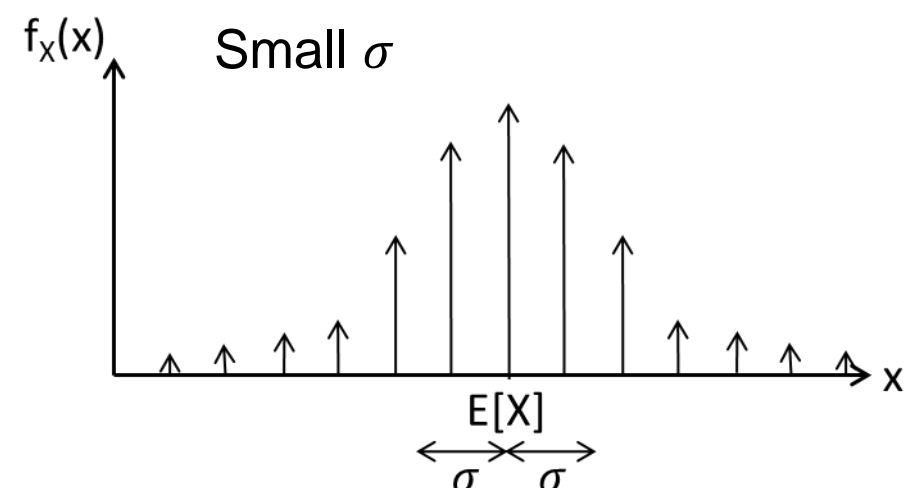
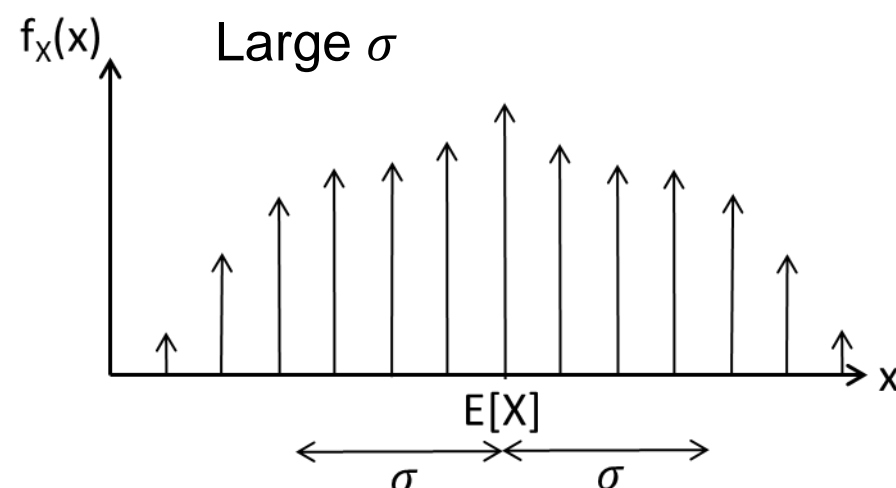
- The mean or the expectation of a discrete random variable X

$$\bar{X} = E[X] = \sum_{i=1}^n x_i f_X(x_i)$$

Variance and standard deviation tells of the spreading of the data

- The variance σ^2 or the standard deviation σ of a random variable X

$$Var(X) = \sigma_X^2 = E[X^2] - E[X]^2$$



The Binomial Distribution

- n repeated trials – each with two possible outcomes

Also called a Bernoulli trial

- **Success** — probability p
- **Failure** — probability 1-p

- Probability mass function (pmf):

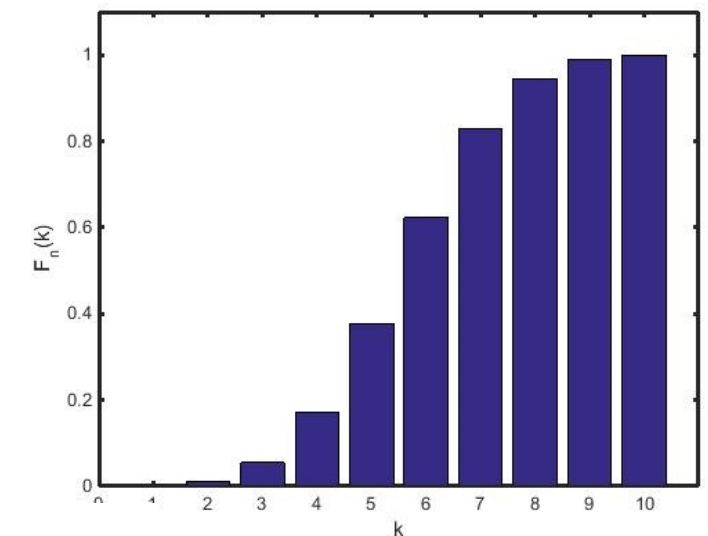
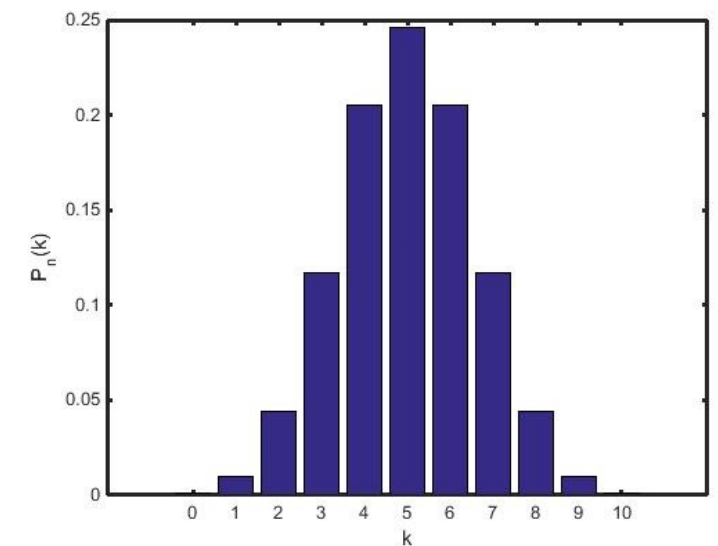
$$f(k|n, p) = \frac{n!}{k! (n - k)!} p^k (1 - p)^{n-k}$$

- Cumulative distribution function (cdf):

$$F(k|n, p) = \sum_{i=0}^k f(i|n, p)$$

- Mean and variance:

$$E[k] = n \cdot p$$
$$Var(X) = n \cdot p \cdot (1 - p)$$



Two Simultaneous Discrete Random Variables

Joint (Simultaneous) pmfs:

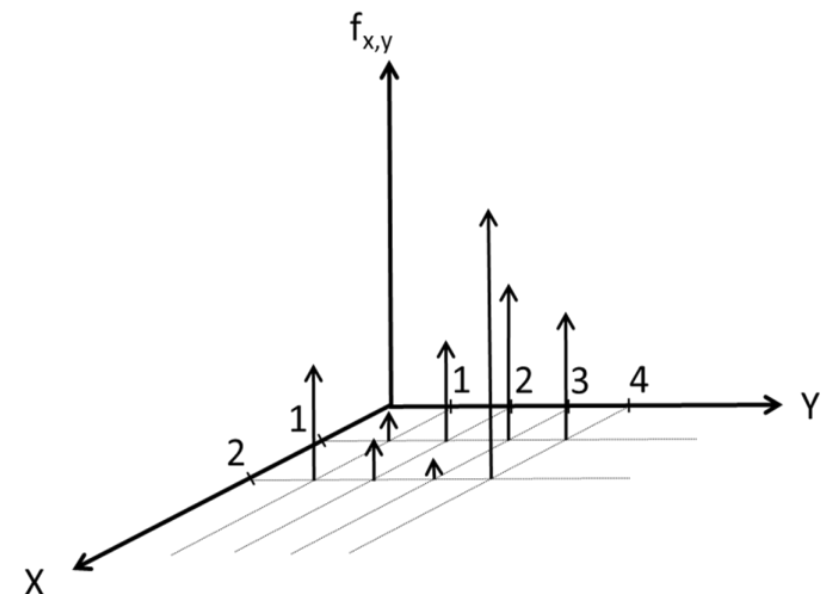
$$f_{X,Y}(x,y) = \begin{cases} \Pr((X = x_i) \cap (Y = y_j)) & \text{for } X = x_i \wedge Y = y_j \\ 0 & \text{otherwise} \end{cases}$$

Marginal pmfs:

$$f_X(x) = \sum_y f_{X,Y}(x,y) \quad f_Y(y) = \sum_x f_{X,Y}(x,y)$$

Conditional pmfs / Bayes Rule:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \Pr(X = x|Y = y)$$



Correlation Coefficient

Correlation tells of the coupling between variables

- The correlation coefficient, is an indicator on how much two random variables X and Y are correlated.

$$\rho = E \left[\frac{X - \bar{X}}{\sigma_X} \cdot \frac{Y - \bar{Y}}{\sigma_Y} \right] = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y}$$

- We have that: $-1 \leq \rho \leq 1$

Independence

Independence: $f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$

- Bayes Rule: $f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$

gives that if X and Y are independent, then:

$$f_{X|Y}(x|y) = f_X(x)$$

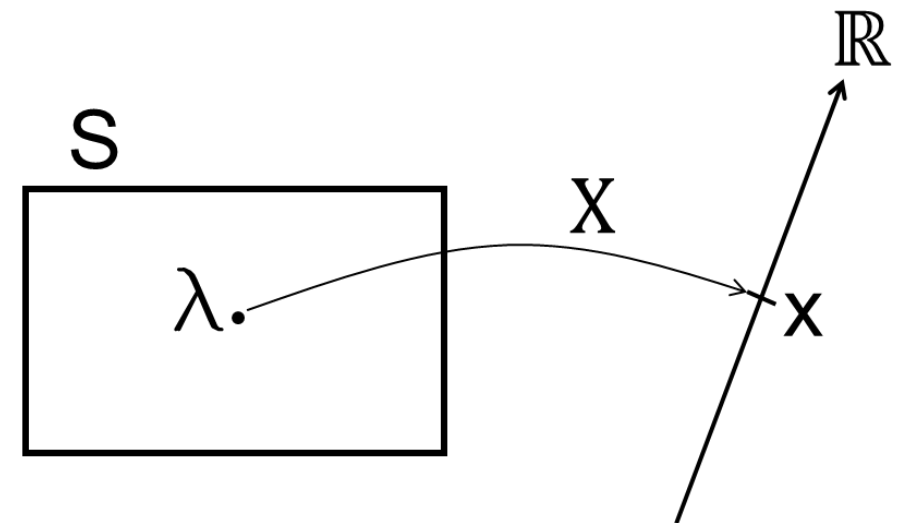
- Also:

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y) \Rightarrow E[XY] = E[X]E[Y] \Rightarrow \rho = 0$$

but the opposite is not always true!

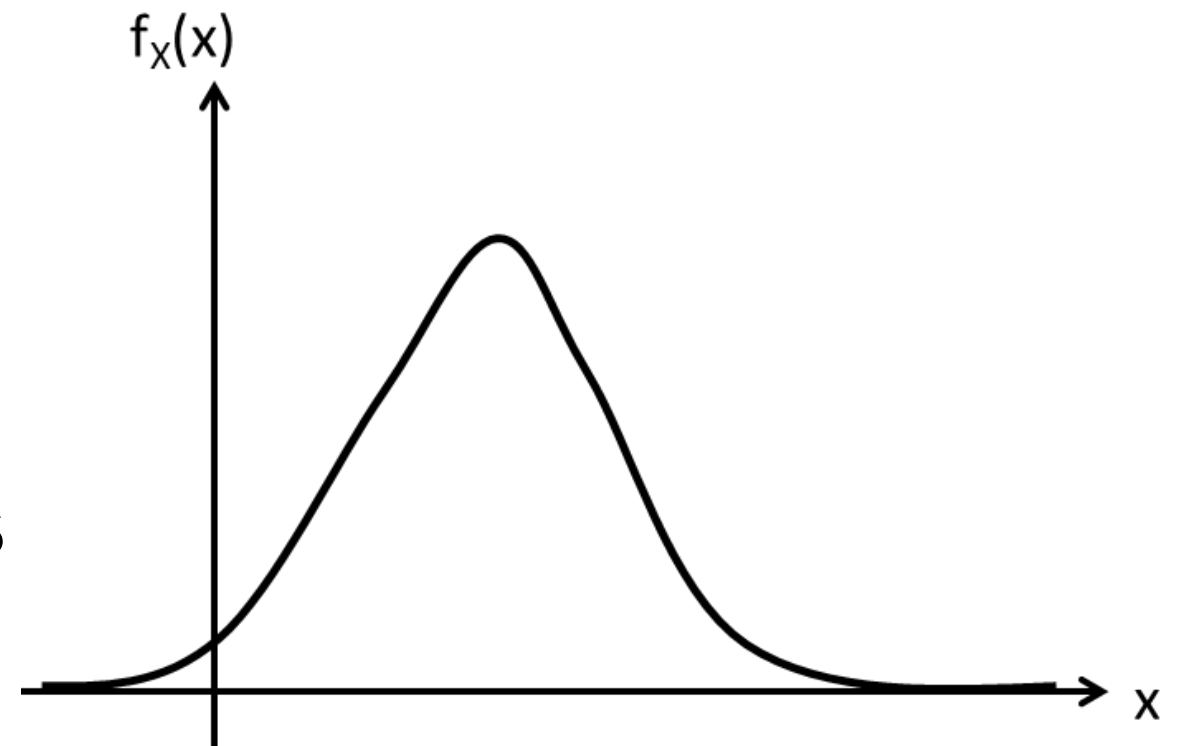
Continuous Random Variables

- We define a stochastic variable X
- X is continuous on \mathbb{R}



- Ex. The exact value R of a resistor
- X is defined by a density function $f_X(x)$
- The probability of one exact value of the variable is always zero:

$$Pr(X = x) = 0$$



Continuous Random Variables — PDF

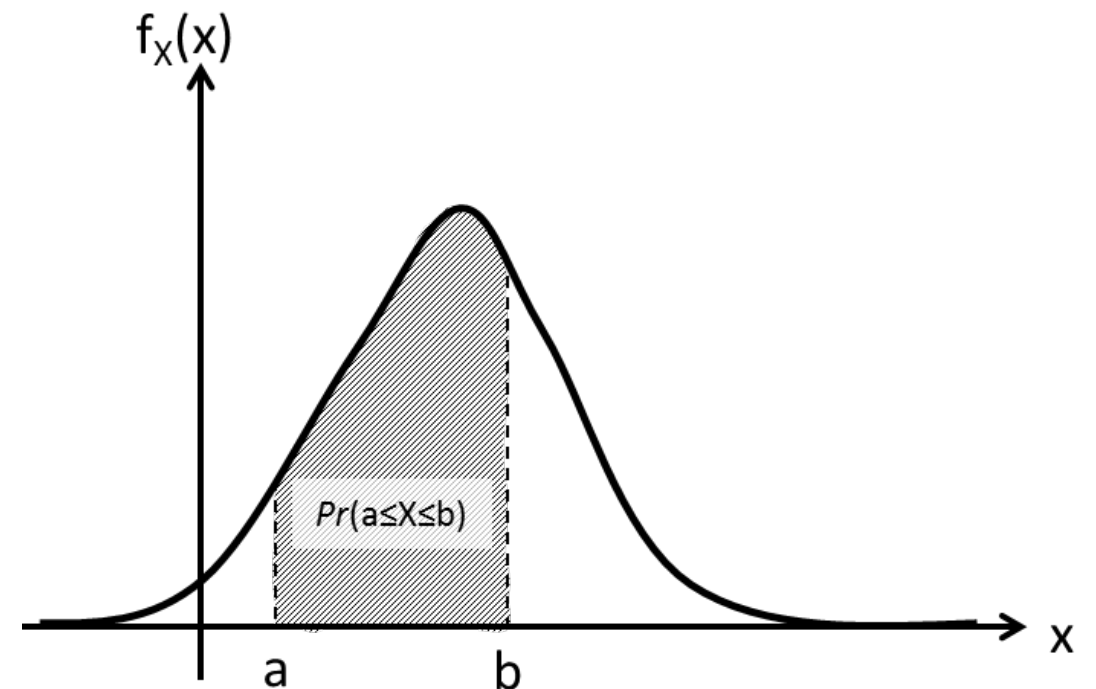
- We define a probability density function (**pdf**): $f_X(x)$

$$Pr(a \leq X \leq b) = \int_a^b f_X(x) dx$$

Properties:

$$f_X(x) \geq 0$$

$$\int_{-\infty}^{\infty} f_X(x) dx = 1$$



Total probability is 1.

Notice: $f_X(x) > 1$ is possible

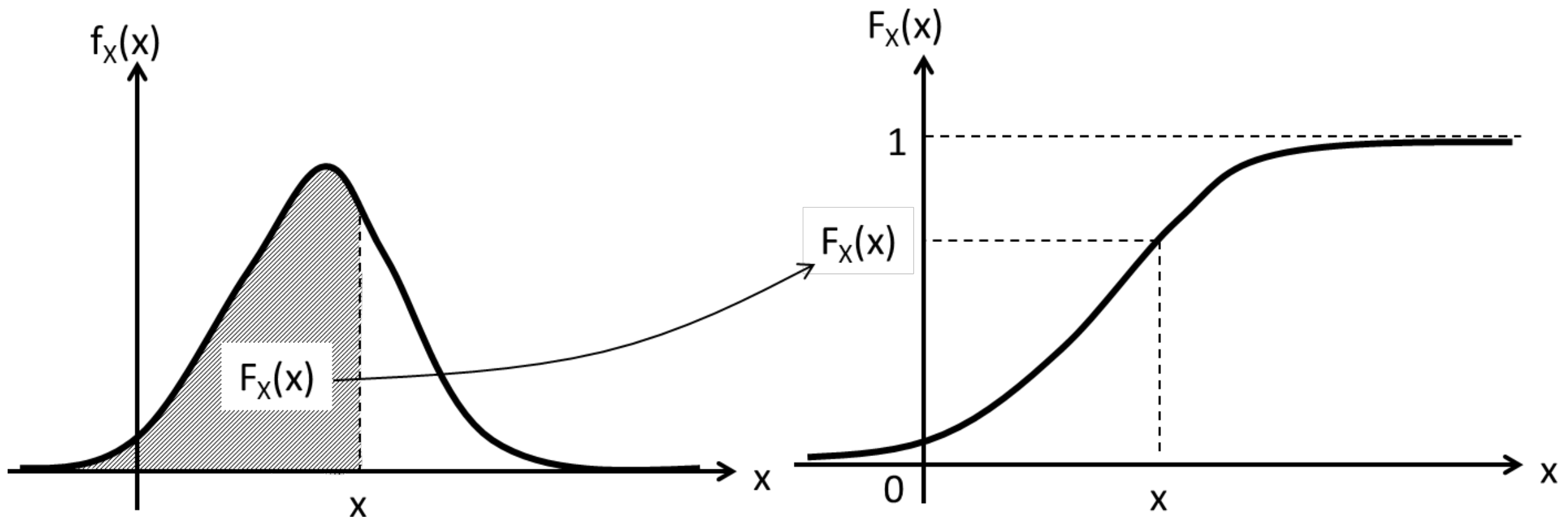
$$Pr(X = x) = 0$$

$$Pr(a < X < b) = Pr(a \leq X < b) = Pr(a < X \leq b) = Pr(a \leq X \leq b)$$

Cumulative Distribution Function (CDF)

- We define a cumulative distribution function (**cdf**): $F_X(x)$
Accumulates the probabilities from minus infinite to x .

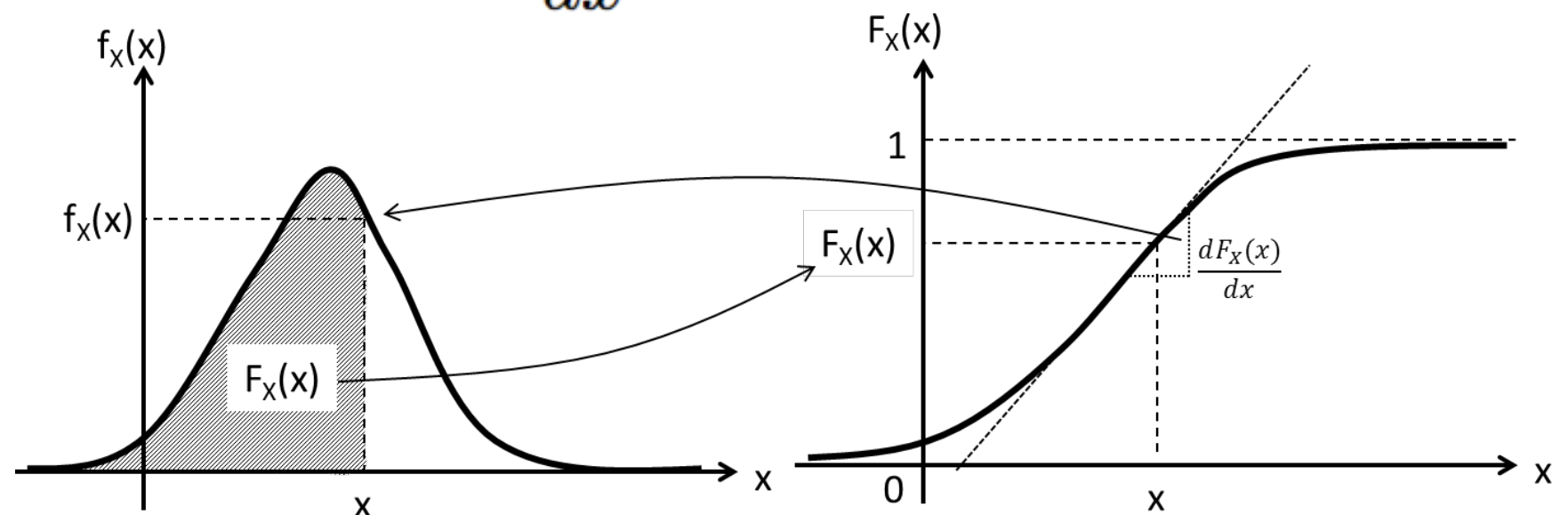
$$F_X(x) = \int_{-\infty}^x f_X(u) du = \Pr(X \leq x)$$



The cdf and pdf contains the same information.

Cumulative Distribution Function (CDF)

- From pdf to cdf:
$$F_X(x) = \int_{-\infty}^x f_X(u) du = \Pr(X \leq x)$$
- From cdf to pdf:
$$f_X(x) = \frac{dF_X(x)}{dx}$$



Properties:

- $0 \leq F_X(x) \leq 1$
- $F_X(x)$ is always non-decreasing and continuous
- $\Pr(a \leq X \leq b) = \int_a^b f_X(x) dx = F_X(b) - F_X(a)$
- $\Pr(X > x) = 1 - \Pr(X \leq x) = 1 - F_X(x)$

Definition of Expectation

- We define the expectation of $g(X)$ with respect to a pdf $f_X(x)$ as the integral:

$$E[g(X)] = \overline{g(X)} = \int_{-\infty}^{\infty} g(x) \cdot f_X(x) dx$$

Example:

- DC voltage with a noise-signal.

Mean Value

- The mean value is the expectation of X :

$$E[X] = \bar{X} = \mu_X = \int_{-\infty}^{\infty} x \cdot f_X(x) dx$$

Example:

- The value of 5% 1k Ω resistors.

Expectation

- Linear function: $g(X) = aX + b$

$$E[aX + b] = \int_{-\infty}^{\infty} (ax + b) \cdot f_X(x) dx = a \cdot E[X] + b$$

- Square function: $g(X) = X^2$

$$E[g(X)] = E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f_X(x) dx$$

$$\neq \left(\int_{-\infty}^{\infty} x \cdot f_X(x) dx \right)^2 = E[X]^2$$

Definition of Variance

- We define the variance of $g(X)$ with respect to a pdf $f_X(x)$ as the integral:

$$\begin{aligned} \text{Var}(g(X)) &= \int_{-\infty}^{\infty} (g(x) - \overline{g(x)})^2 \cdot f_X(x) dx \\ &= E[g(X)^2] - E[g(X)]^2 \end{aligned}$$

- The variance of a continuous random variable X :

$$\text{Var}(X) = \int_{-\infty}^{\infty} (x - \bar{x})^2 \cdot f_X(x) dx = E[X^2] - E[X]^2$$

Variance

- Linear function: $g(X) = aX + b$

$$\text{Var}[aX + b] = E[(aX + b)^2] - E[aX + b]^2$$

$$= \int_{-\infty}^{\infty} (ax + b)^2 \cdot f_X(x) dx - (a \cdot E[X] + b)^2$$

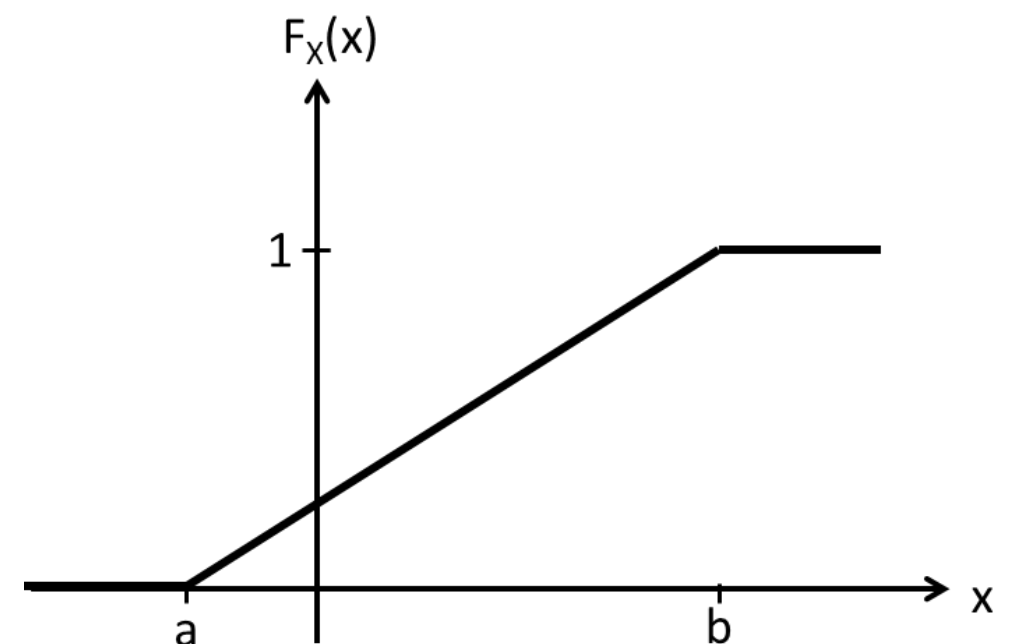
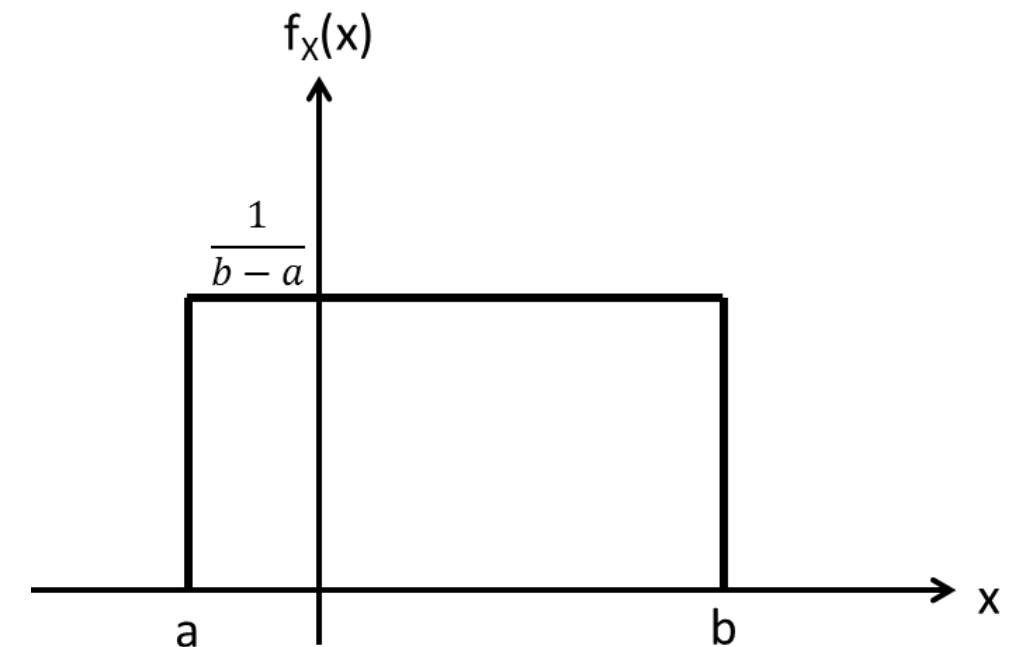
$$= (a^2 E[X^2] + b^2 + 2abE[X]) - (a^2 E[X]^2 + b^2 + 2abE[X])$$

$$= a^2 (E[X^2] - E[X]^2)$$

$$= a^2 \cdot \text{Var}(X)$$

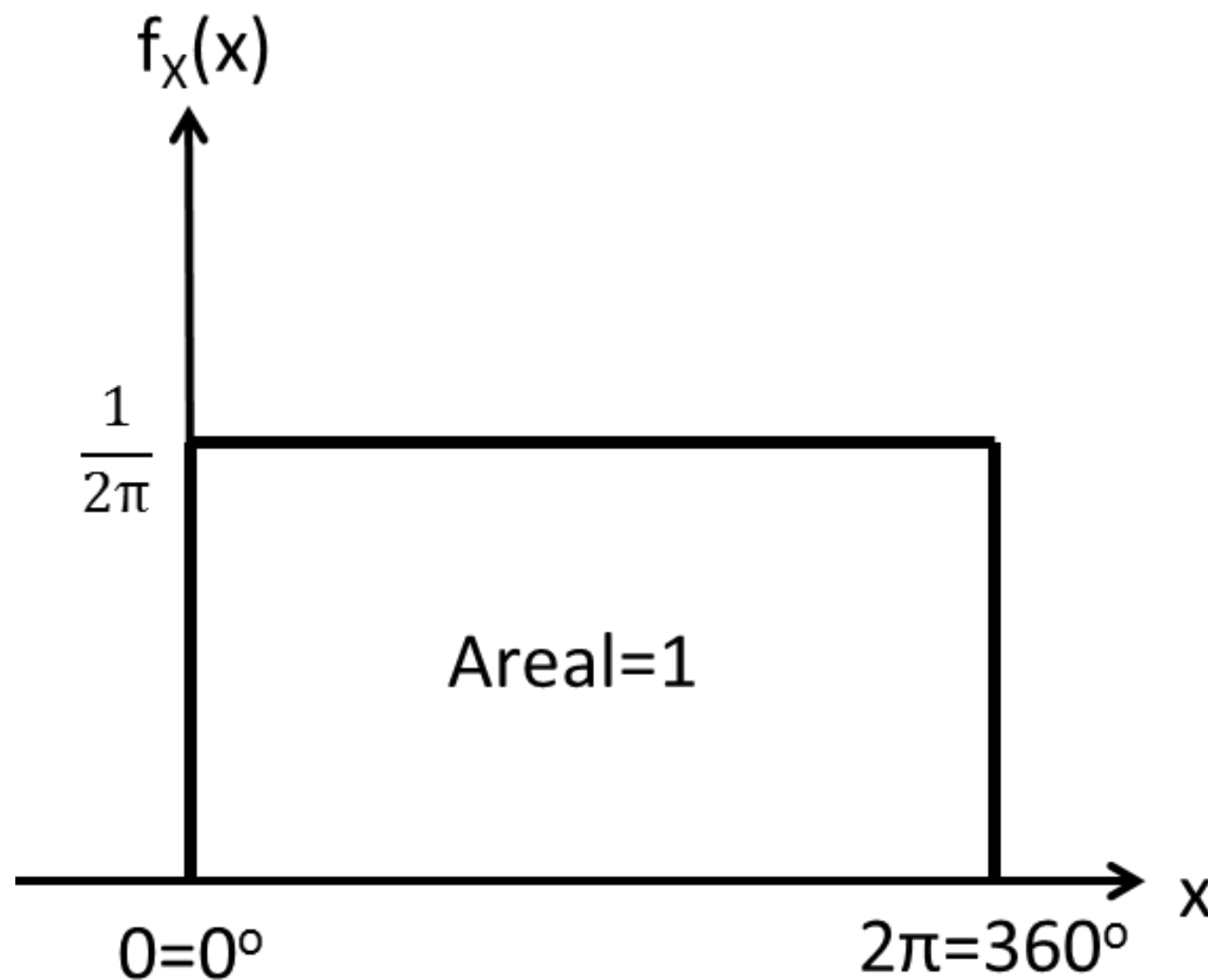
Uniform Distribution (continuous)

- $\mathcal{U}(a,b)$
- Mean value: $\mu = \frac{a+b}{2}$
- Variance: $\sigma^2 = \frac{1}{12}(b-a)^2$
- pdf: $f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$
- cdf: $F_X(x) = \begin{cases} 0 & \text{for } x \leq a \\ \frac{x-a}{b-a} & \text{for } a \leq x \leq b \\ 1 & \text{for } x \geq b \end{cases}$

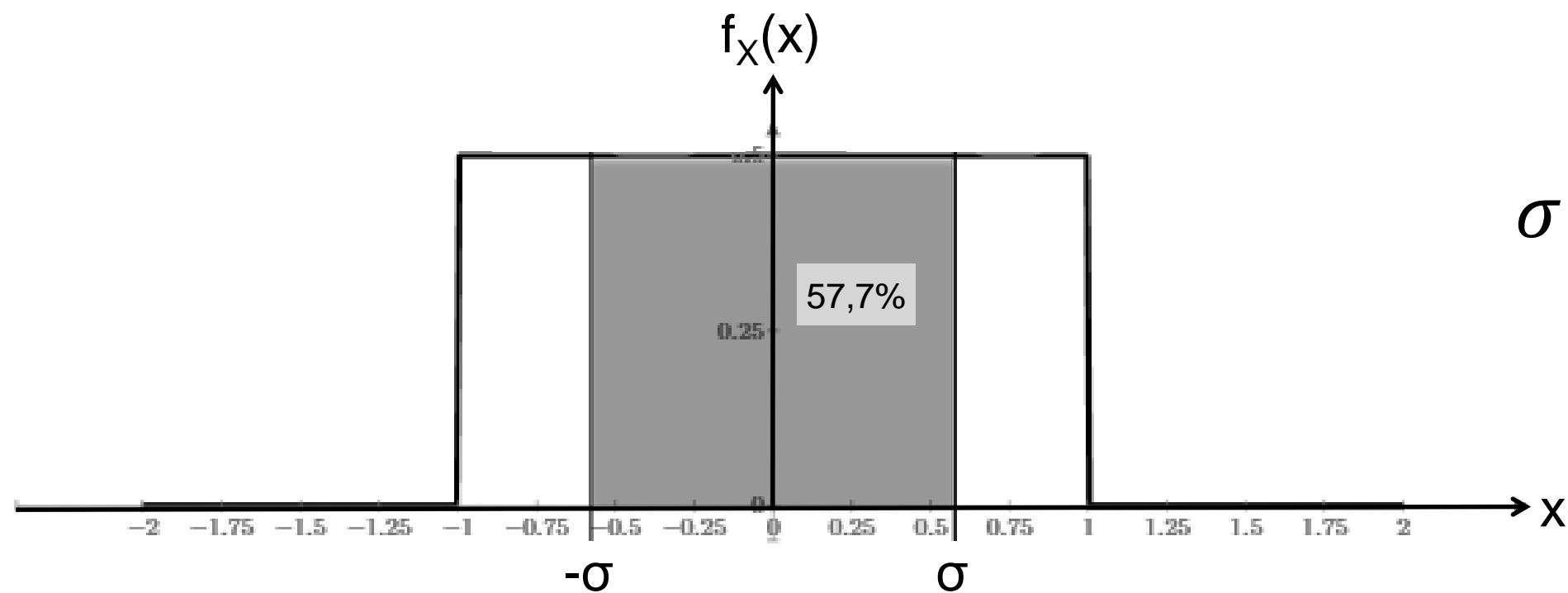


Uniform Distribution — Example

- A phase noise is uniformly distributed.



Uniform Distribution: Standard deviation



$$\sigma = \frac{b - a}{\sqrt{12}}$$

$$\Pr(|X - \mu| \leq \sigma) = 57,7\%$$

$$\Pr(|X - \mu| \leq 2\sigma) = 100\%$$

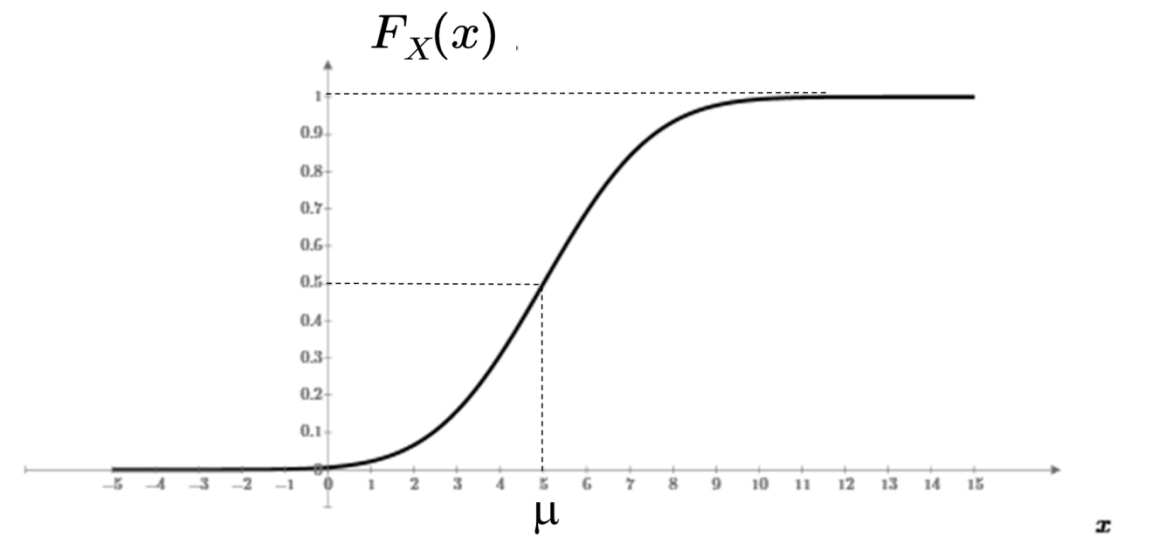
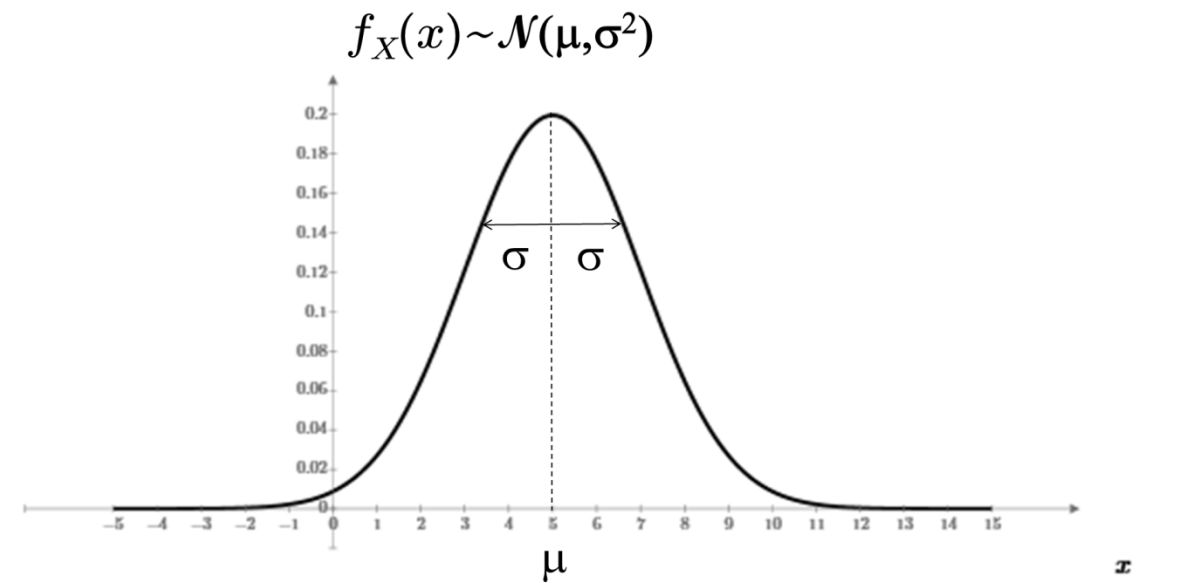
Gaussian Distribution = Normal Distribution

- $\mathcal{N}(\mu, \sigma^2)$
- Mean value: μ
- Variance: σ^2

- pdf: $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

- cdf: $F_X(x) = \frac{1}{2} \left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right) \right]$

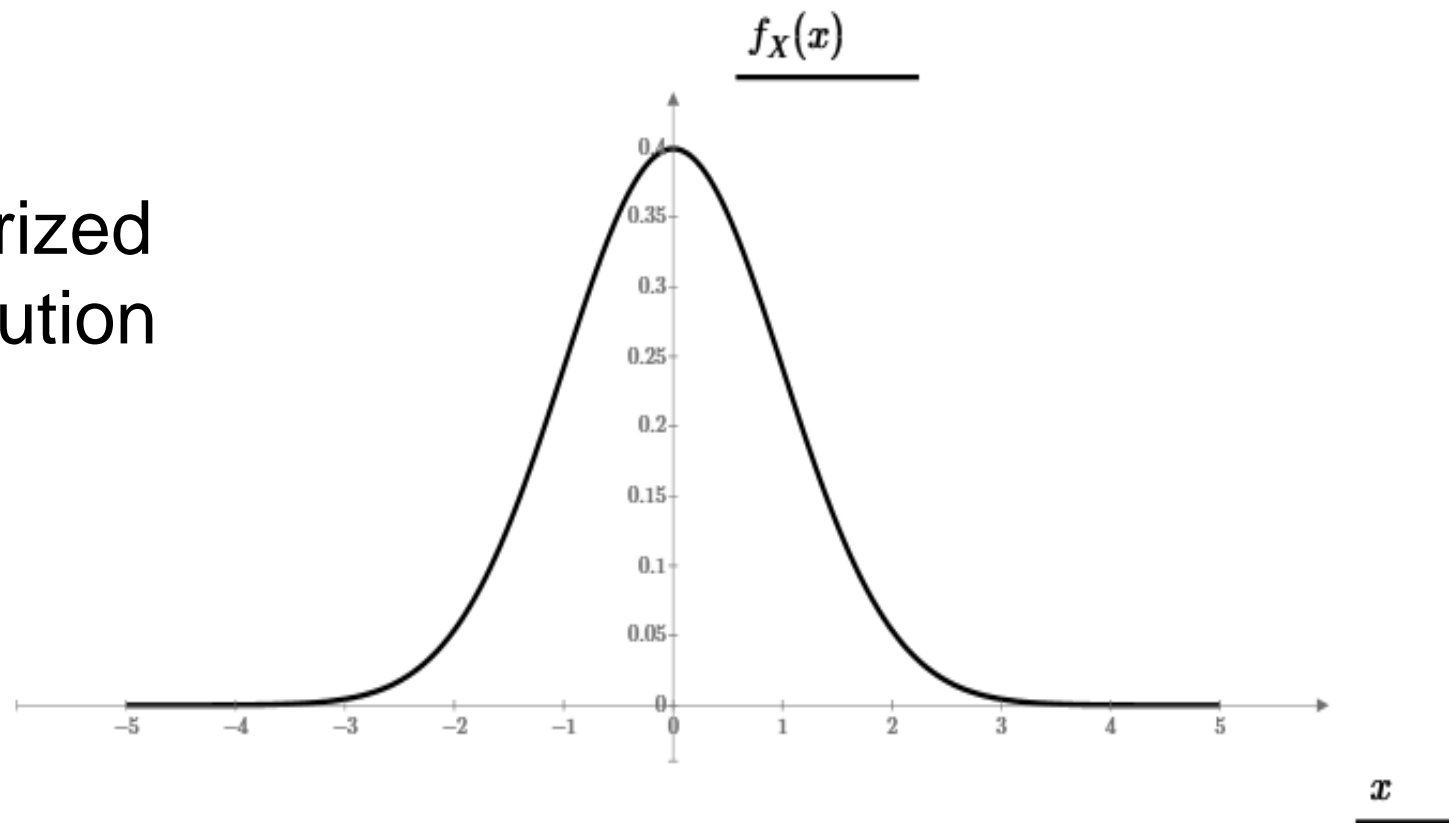
No closed expression for the cdf
erf = error-function: $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$



Gaussian Distribution = Normal Distribution

$\mathcal{N}(0,1)$

→ the standardized
normal distribution



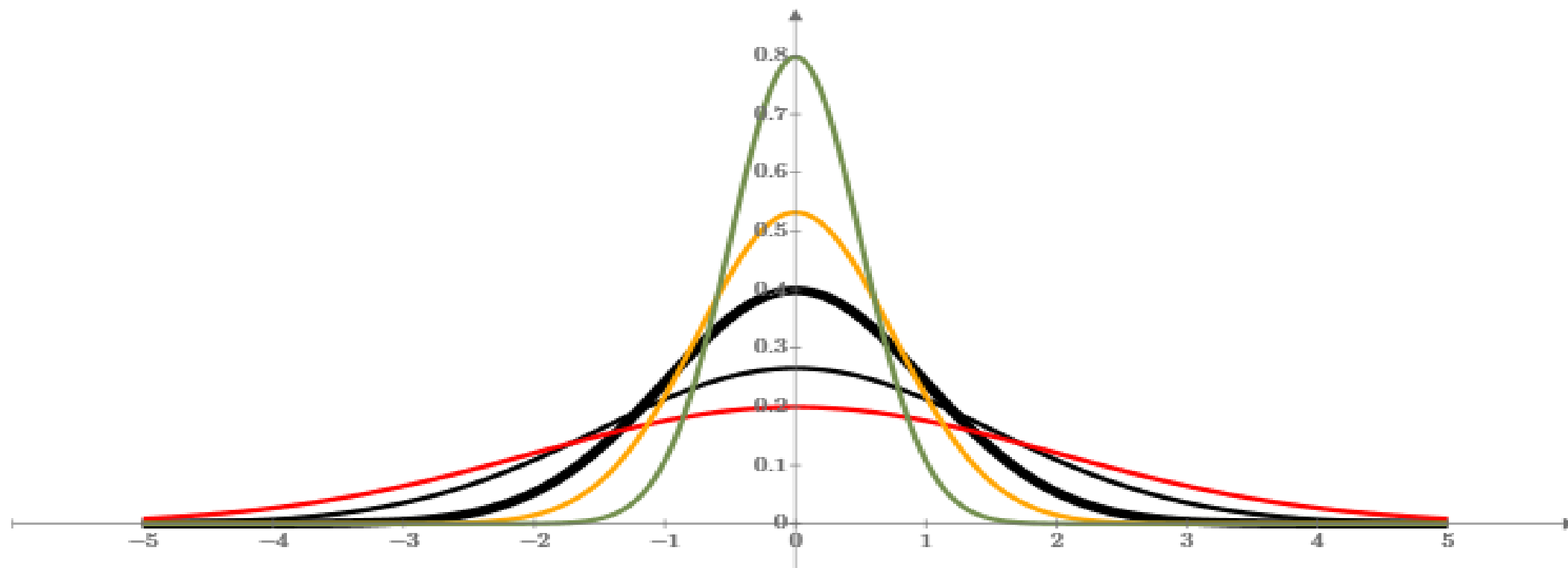
- A lot of things in nature are Gaussian distributed
 - Fx. Examination marks
- Central Limit Theorem → Gaussian distribution

Gaussian Distribution = Normal Distribution

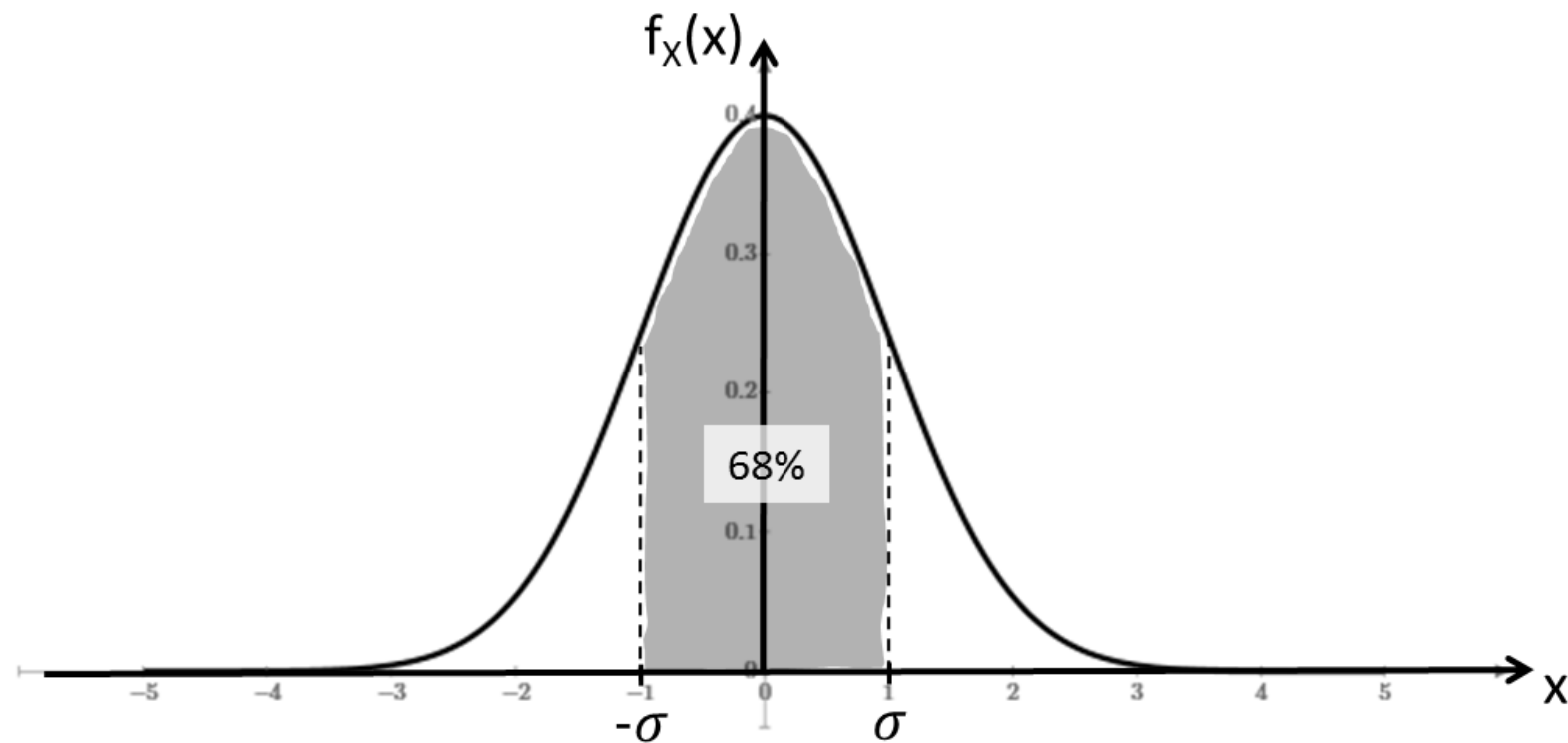
- Maximum probability density at the mean value μ
- The standard deviation (variance) σ determines the form (width and height)

$$f_X(x, \sigma) \sim \mathcal{N}(0, \sigma^2)$$

$\frac{f_X(x, 1)}{f_X(x, 0.75)}$	$\frac{f_X(x, 1.5)}{f_X(x, 0.5)}$	$f_X(x, 2)$
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Normal Distribution: Standard Deviation



$$\Pr(|X - \mu| \leq \sigma) = 68,3\%$$

$$\Pr(|X - \mu| \leq 2\sigma) = 95,4\%$$

$$\Pr(|X - \mu| \leq 3\sigma) = 99,7\%$$

Gaussian Distribution = Normal Distribution

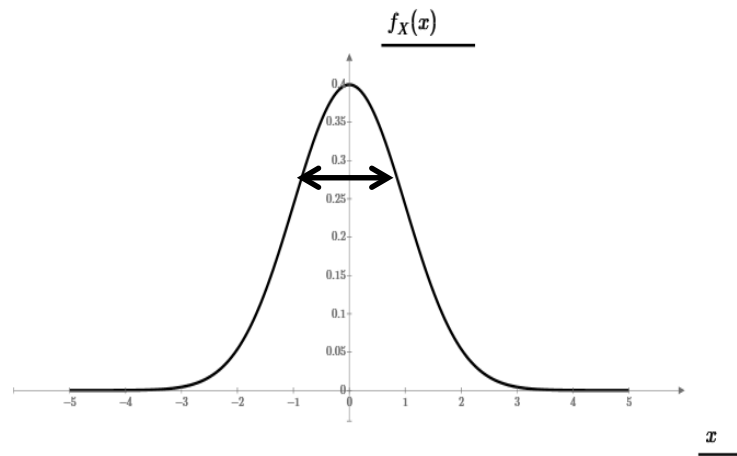
- Beregninger med normalfordelinger: Tabelopslag og Matlab:
- $X \sim \mathcal{N}(\mu, \sigma^2) \rightarrow Z = \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$ (Standard Normal Distribution)
- $F_X(x) = \Pr(X \leq x) = \Pr\left(Z \leq \frac{x - \mu}{\sigma}\right) = F_Z(z)$ hvor $z = \frac{x - \mu}{\sigma}$
$$= \begin{cases} \Phi(z) & \text{Tabel 1 ("Statistik og Sandsynlighedsregning")} \\ 1 - Q(z) & \text{App. D ("Random Signals")} \end{cases}$$
- $\Phi(z) = \Pr(Z \leq z)$
- $\Phi(-z) = 1 - \Phi(z)$
- $Q(z) = \Pr(Z \geq z) = 1 - \Pr(Z \leq z) = 1 - \Phi(z)$
- $Q(-z) = 1 - Q(z)$
- Matlab:
 - $\Pr(X \leq x) = F_X(x) = \text{normcdf}(x, \mu, \sigma)$
 - $\Pr(Z \leq z) = F_Z(z) = \text{normcdf}(z, 0, 1) = \text{normcdf}(z)$

↙ Obs: Standard variation

Summary of Expectations

- Mean value: $E[X] = \bar{X} = \mu_X = \int_{-\infty}^{\infty} x \cdot f_X(x) dx \quad (\sum_{i=1}^n x_i f_X(x_i))$
- Mean square: $E[X^2] = \overline{X^2} = \int_{-\infty}^{\infty} x^2 \cdot f_X(x) dx \quad (\sum_{i=1}^n x_i^2 f_X(x_i))$
- Variance: $Var(X) = \sigma_X^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 \cdot f_X(x) dx = E[X^2] - E[X]^2$

- Standard deviation: $\sigma_X = \sqrt{Var(X)}$



- A function: $E[g(X)] = \overline{g(X)} = \int_{-\infty}^{\infty} g(x) \cdot f_X(x) dx \quad (\sum_{i=1}^n g(x_i) f_X(x_i))$
 $Var(g(X)) = \int_{-\infty}^{\infty} (g(x) - \overline{g(x)})^2 \cdot f_X(x) dx = E[g(X)^2] - E[g(X)]^2$
- Linear function: $E[aX + b] = a \cdot E[X] + b$
 $Var[aX + b] = a^2(E[X^2] - E[X]^2) = a^2 \cdot Var(X)$

Two Stochastic Variables X,Y

- The simultaneous (joint) density function
- The marginal probability density function
- Bayes rule
- Discrete → Continuous stochastic random variable

$$\sum \rightarrow \int$$

Continuous Random Variables

- We have a simultaneous (joint) pdf: $f_{X,Y}(x, y)$

- We have the probability:

$$Pr((a \leq X \leq b) \cap (c \leq Y \leq d)) = \int_c^d \int_a^b f_{X,Y}(x, y) dx dy$$

- We have for the pdf: $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1$

$$0 \leq f_{X,Y}(x, y)$$

The Marginal PDF

- For a two dimensional pdf $f_{X,Y}(x, y)$, we can find the marginals

Marginals:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx$$

Relationship between pdf and cdf

- For a two dimensional pdf $f_{X,Y}(x, y)$, the cdf and the pdf correspond to each other

$$\text{cdf} \quad F_{X,Y}(x, y) = \int_{-\infty}^y \int_{-\infty}^x f_{X,Y}(x, y) dx dy = Pr(X \leq x \wedge Y \leq y)$$

$$\text{pdf} \quad f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y}$$

The Conditional PDF

- For a two dimensional pdf $f_{X,Y}(x,y)$, we can find the conditional pdf with Bayes rule:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Independence:

- X and Y are independent if:

$$f_{X|Y}(x|y) = f_X(x)$$

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

Correlation

Correlation tells of the (biased) coupling between variables

- Correlation:

$$\text{corr}(X, Y) = E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot y \cdot f_{X,Y}(x, y) dx dy$$

➤ If X and Y are independent: $E[XY] = E[X] \cdot E[Y]$

➤ If $X = Y$: $\text{corr}(X, X) = E[X^2]$

Covariance

Covariance is without bias from the mean

- Covariance:

$$\begin{aligned} \text{cov}(X, Y) &= E[(X - \bar{X})(Y - \bar{Y})] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \bar{x}) \cdot (y - \bar{y}) \cdot f_{X,Y}(x, y) dx dy \\ &= E[XY] - E[X] \cdot E[Y] = \text{corr}(X, Y) - E[X] \cdot E[Y] \end{aligned}$$

- If X and Y are independent: $\text{corr}(X, Y) = 0$

*OBS: The
opposite not
always true*

- If $X = Y$: $\text{cov}(X, X) = E[X^2] - E[X]^2 = \text{Var}(X)$

Correlation Coefficient

Correlation Coefficient is the normalized Covariance

- The correlation coefficient, is an indicator on how much two random variables X and Y are correlated.

$$\rho = E \left[\frac{X - \bar{X}}{\sigma_X} \cdot \frac{Y - \bar{Y}}{\sigma_Y} \right] = \frac{E[XY] - E[X]E[Y]}{\sigma_X \cdot \sigma_Y} = \frac{\text{cov}(X, Y)}{\sigma_X \cdot \sigma_Y}$$

- We have that: $-1 \leq \rho \leq 1$
- If X and Y are independent: $\rho = 0$

Dependence

- We have independence between X and Y if and only if:

$$f_{X,Y} = f_X(x) f_Y(y)$$

Example of independent random variables:

- A persons height and the current exact distance from the earth to the moon.

Example of dependent random variables:

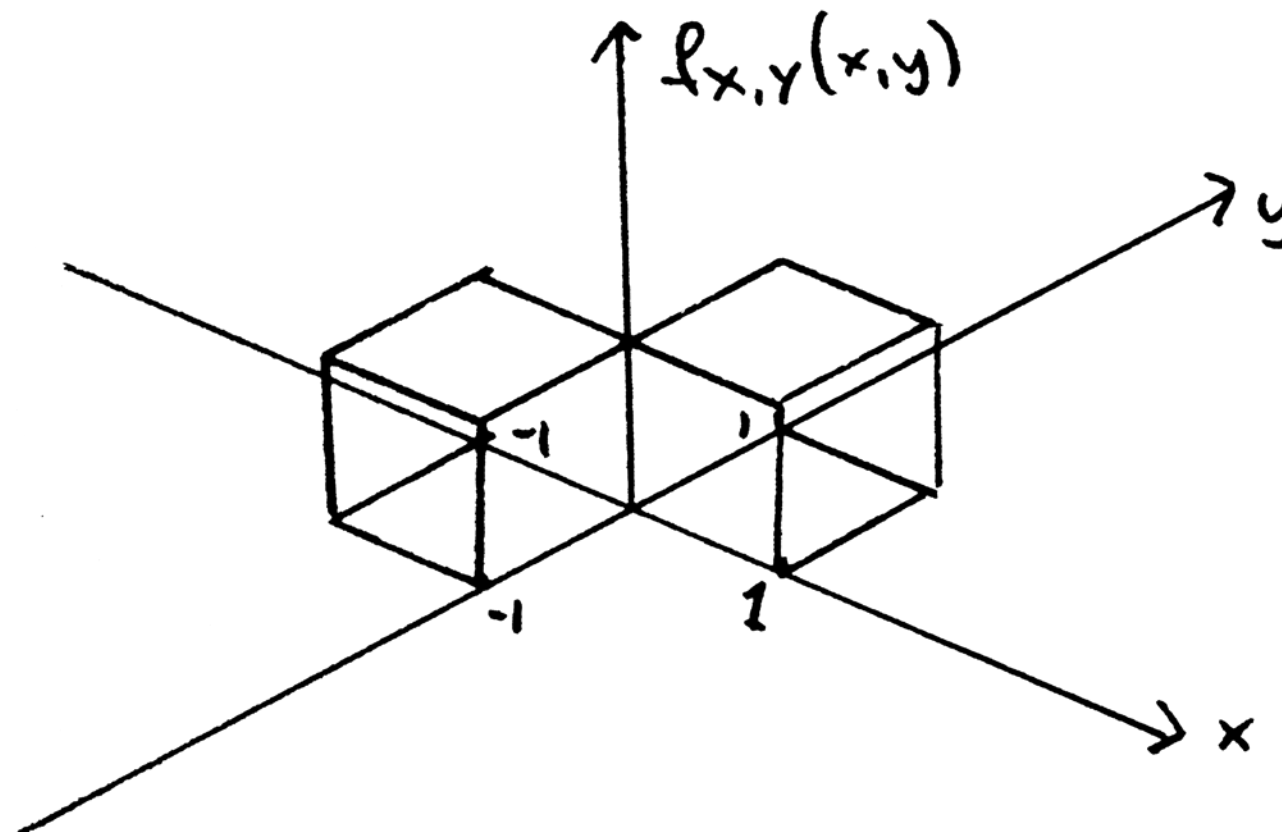
- The time of day and the amount of bicycles parked the at the engineering college.
- The energy of a mobile signal and the length in meters to a basestation.

Dependence - Example

- We want to find out whether two random variables are independent:

Simultaneous pdf for X and Y:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{for } -1 \leq x < 0 \text{ and } -1 \leq y < 0 \\ \frac{1}{2} & \text{for } 0 \leq x < 1 \text{ and } 0 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$



Dependence - Example

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{for } -1 \leq x < 0 \text{ and } -1 \leq y < 0 \\ \frac{1}{2} & \text{for } 0 \leq x < 1 \text{ and } 0 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find marginals:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy \\ &= \begin{cases} \int_{-1}^0 \frac{1}{2} dy & \text{for } -1 \leq x < 0 \\ \int_0^1 \frac{1}{2} dy & \text{for } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{2} & \text{for } -1 \leq x < 0 \\ \frac{1}{2} & \text{for } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx \\ &= \begin{cases} \int_{-1}^0 \frac{1}{2} dx & \text{for } -1 \leq y < 0 \\ \int_0^1 \frac{1}{2} dx & \text{for } 0 \leq y < 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{2} & \text{for } -1 \leq y < 0 \\ \frac{1}{2} & \text{for } 0 \leq y < 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Dependence - Example

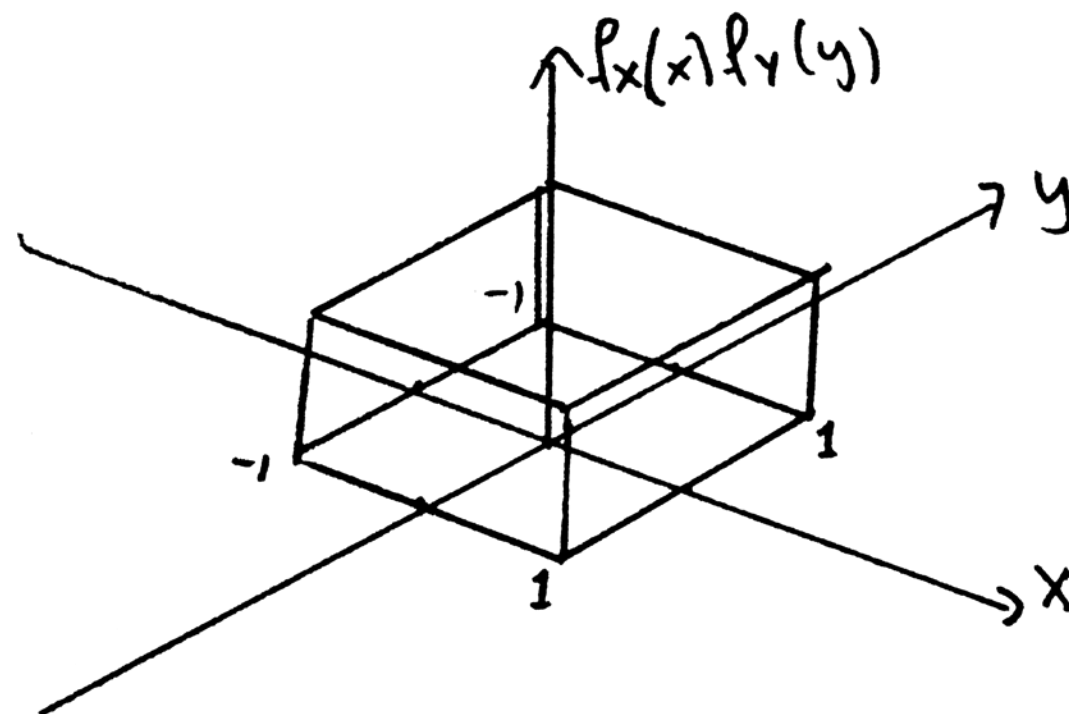
- Independence if and only if: $f_{X,Y} = f_X(x)f_Y(y)$

Multiply marginals:

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{for } -1 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{2} & \text{for } -1 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$

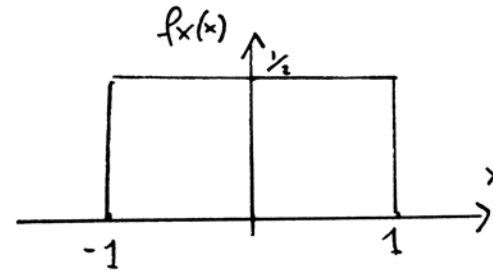
$$f_X(x)f_Y(y) = \begin{cases} \frac{1}{4} & \text{for } -1 \leq x < 1 \text{ and } -1 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$



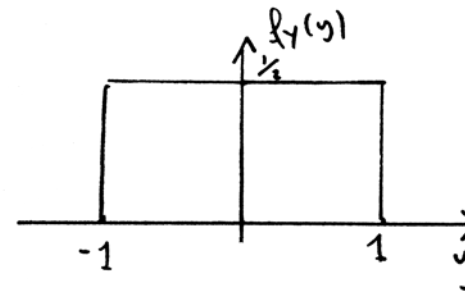
Dependence - Example

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{for } -1 \leq x < 0 \text{ and } -1 \leq y < 0 \\ \frac{1}{2} & \text{for } 0 \leq x < 1 \text{ and } 0 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$

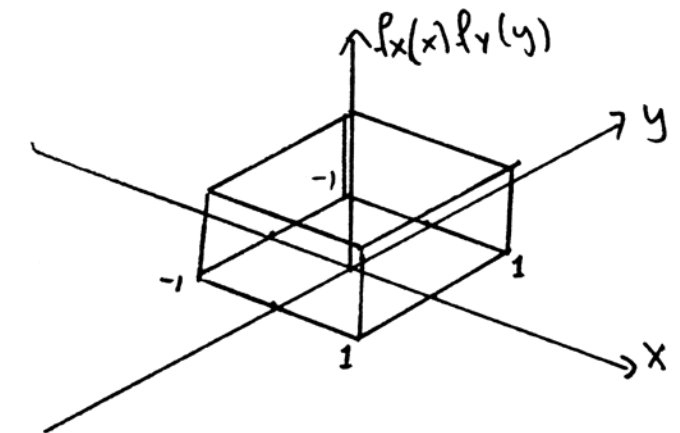
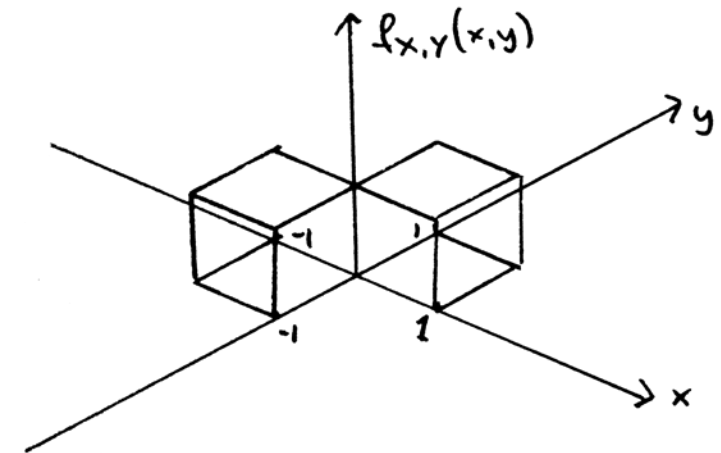
$$f_X(x) = \begin{cases} \frac{1}{2} & \text{for } -1 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$f_Y(y) = \begin{cases} \frac{1}{2} & \text{for } -1 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$f_X(x)f_Y(y) = \begin{cases} \frac{1}{4} & \text{for } -1 \leq x < 1 \text{ and } -1 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$



$f_{X,Y}(x,y) \neq f_X(x) \cdot f_Y(y) \Rightarrow X \text{ and } Y \text{ er ikke uafhængige}$

Correlation calculation

*Assignment:
Verify the results by doing
the detailed calculations*

$$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot y \cdot f_{X,Y}(x, y) dx dy = \frac{1}{4}$$

$$E[X] = E[Y] = \int_{-\infty}^{\infty} x \cdot f_X(x) dx = \int_{-\infty}^{\infty} y \cdot f_Y(y) dy = 0$$

$$\sigma_X^2 = \sigma_Y^2 = E[X^2] - E[X]^2 = E[Y^2] - E[Y]^2 = \frac{1}{3}$$

$$\text{corr}(X, Y) = E[XY] = \frac{1}{4}$$

$$\text{cov}(X, Y) = E[XY] - E[X]E[Y] = \frac{1}{4} - 0 \cdot 0 = \frac{1}{4}$$

$$\rho = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{1/4}{1/3} = \frac{3}{4} = 0,75$$

Very important!

i.i.d.: Independent and Identically distributed

- We define that for series of random variables that is taken from the same distribution (identically distributed), and are sampled independent of each other, that they are i.i.d.

i.i.d. = Independent and Identically distributed

- i.i.d. is a very important characteristic in stochastic variable processing and statistics

Example:

- Quantisation noise.

Words and Concepts to Know

Probability density function

i.i.d.

Correlation

Marginal probability density function

Continuous random variable

Uniform distribution

Gaussian distribution

pdf

Independent and Identically Distributed

Normal distribution

Correlation coefficient

Simultaneous density function

Covariance

Joint density function