ETSMP - CONTINUOUS RANDOM VARIABLES

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INITIALIZATION

```
clc
clear

addpath('[0] Library');
Color = load("colors.mat");
```

PROBLEM 2.18

Show that the expected value operator has the following properties.

A: $E[a+\mathrm{bX}] = a + b[X]$

B: $E[\mathrm{aX}+\mathrm{bY}] = a \cdot E[X] + b \cdot E[Y]$

C: $\mathrm{var}(\mathrm{aX}+\mathrm{bY}) = a^2 \cdot \mathrm{var}(X) + b^2 \cdot \mathrm{var}(Y) + 2\mathrm{ab} \cdot \mathrm{covar}(X, Y)$

[A] ESTIMATED VALUE

$$E[a+b\cdot\ X]=\int a\cdot f_{X(x)}\,\mathrm{dx}+\int b\cdot x\cdot f_{X(x)}\,\mathrm{dx}$$

$$E[a+b\cdot X]=a\cdot\int f_{X(x)}\,\mathrm{dx}+b\cdot\int x\cdot f_{X(x)}\,\mathrm{dx}$$

$$E[a+b\cdot X]=a+b\cdot E[X]$$

[B] ESTIMATED VALUE

$$E[a\cdot X+b\cdot Y]=\int a\cdot\ x\cdot f_{X(x)}\mathrm{dx}+\int b\cdot y\cdot f_{Y(y)}\,\mathrm{dx}$$

$$E[a\cdot X+b\cdot Y]=a\cdot\int x\cdot f_{X(x)}\,\mathrm{dx}+b\cdot\int y\cdot f_{Y(y)}\,\mathrm{dy}$$

$$E[a\cdot X+b\cdot Y]=a\cdot E[X]+b\cdot E[Y]$$

[C] VARIANCE

$$\mathrm{VAR}[aX+bY]=E[(aX+bY)^2]-E[aX+bY]^2$$

$$\mathrm{VAR}[aX+bY]=\int(ax+by)^2\cdot f_{X,Y(x,y)}\,dx\,dy-(a\cdot E[X]+b\cdot E[Y])^2$$

$$\mathrm{VAR}[aX+bY]=\int(a^2x^2+b^2y^2+2axby)\cdot f_{X,Y}(x,y)\,dx\,dy-(a\cdot E[X]+b\cdot E[Y])^2$$

$$\mathrm{VAR}[aX+bY]=a^2\cdot\int x^2\cdot f_{X(x)}\,dx-a^2\cdot E[X]^2+b^2\cdot\int y^2\cdot f_{Y(y)}\,\mathrm{dy}-b^2\cdot E[Y]^2+2ab\cdot\int xy\cdot f_{X,Y(x,y)}\,dx\,dy-2ab\cdot E[X]\cdot E[Y]$$

$$\mathrm{VAR}[aX+bY]=a^2\cdot(E[X^2]-E[X]^2)+b^2\cdot(E[Y^2]-E[Y]^2)+2ab\cdot\mathrm{COVAR}[X,Y]$$

$$\mathrm{VAR}[aX+bY]=a^2\cdot\mathrm{VAR}[X]+b^2\cdot\mathrm{VAR}[Y]+2ab\cdot\mathrm{COVAR}[X,Y]$$

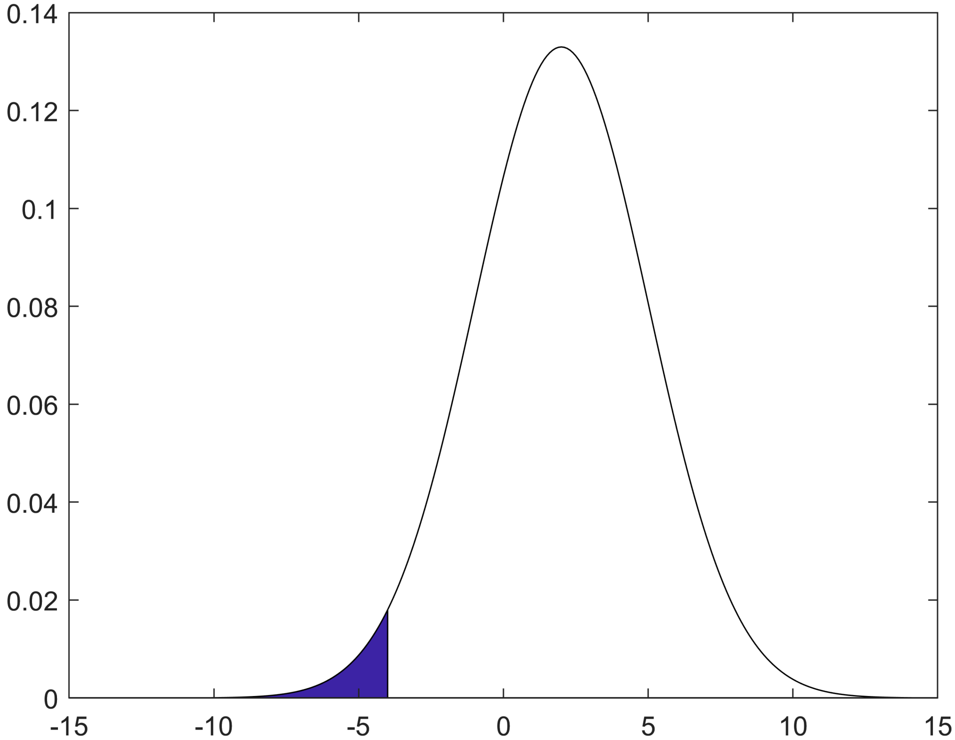
PROBLEM 2.24

X is a Gaussian random variable with $\mu_x=2$ **and** $\sigma_x^2=9$. **Find** $P(-4<X\le 5)$ **using tabulated values of** $Q(.)$

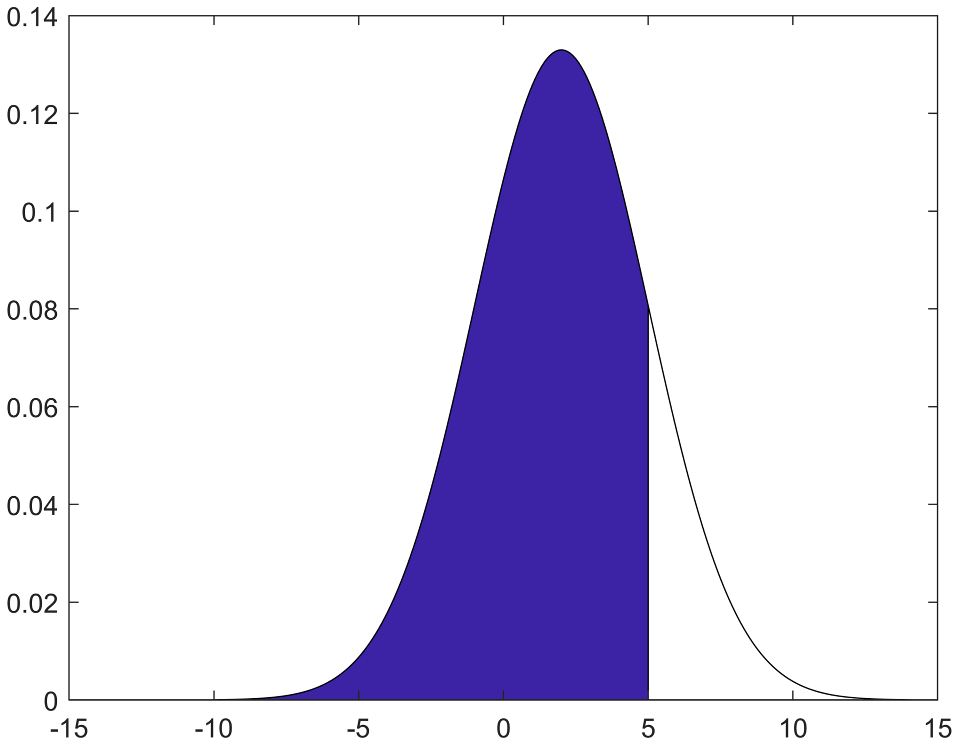
[A] PROBABILITY

```
mu = 2;
sigmasquared = 9;
sigma = sqrt(sigmasquared);
```

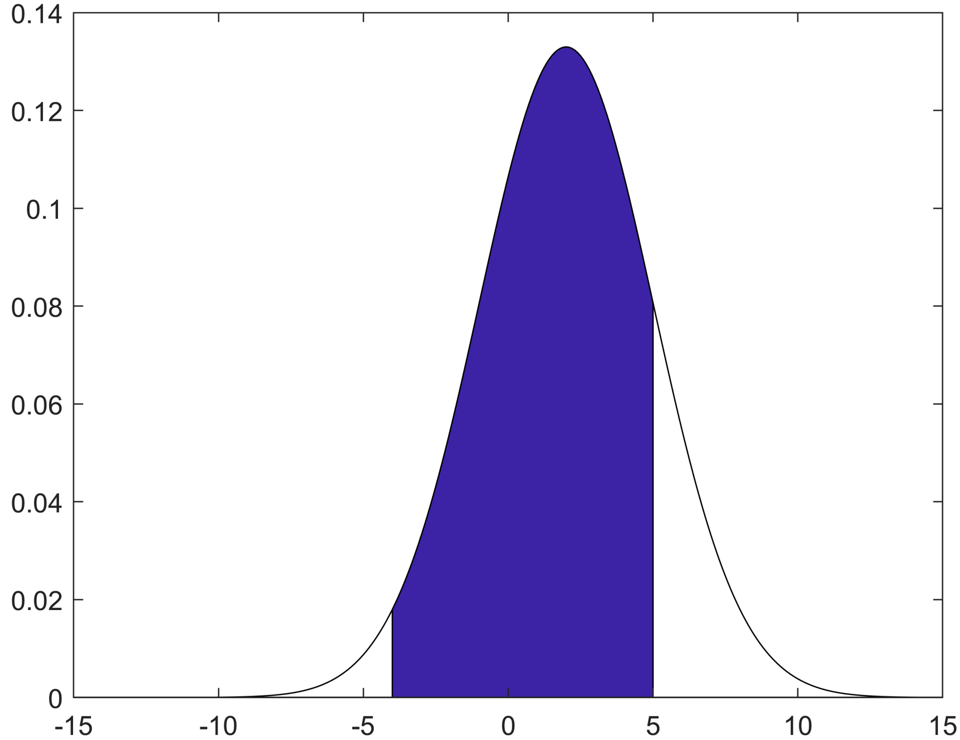
```
x = -15:.1:15;
y = normpdf(x, mu, sigma);

figure(1)
pdfarea(x, y, [], -4);
```




```
figure(2)
pdfarea(x, y, [], 5);
```




```
figure(3)
pdfarea(x, y, -4, 5);
```




```
LProbability = normcdf(-4, mu, sigma);
HProbability = normcdf(5, mu, sigma);

Probability = HProbability - LProbability;

disp("Probability of value between -4 and 5 is: " + Probability);
```

```
Probability of value between -4 and 5 is: 0.81859
```

PROBLEM 2.29

The joint pdf of random variables X and Y is

$$f_{X,Y}(x, y) = \frac{1}{2},\ 0 \leq\ x \leq\ y,\ 0 \leq y \leq 2$$

A: Find the marginal pdfs, $f_X(x)$ and $f_Y(y)$.

B: Find the conditionals pdfs $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

C: Find $E\{X|Y = 1\}$ and $E\{X|Y = 0.5\}$.

D: Are **X** and **Y** statistically independent?

E: Find ρ_{XY}.

[A] MARGINAL PDF'S

$$f_X(x) = \int_x^2 f_{X,Y}(x,y)\,dy$$

$$f_X(x) = \int_x^2 \frac{1}{2}\,dy = \left[\frac{1}{2}\cdot y\right]_x^2$$

$$f_X(x) = 1 - \frac{1}{2}\cdot x, \quad 0 \le x \le 2$$

$$f_Y(y) = \int_0^y f_{X,Y}(x,y)\,dx$$

$$f_Y(y) = \int_0^y \frac{1}{2}\,dx = \left[\frac{1}{2}\cdot x\right]_0^y$$

$$f_Y(y) = \frac{1}{2}\cdot y, \quad 0 \le y \le 2$$

[B] CONDITIONAL PDF'S

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_{X|Y}(x|y) = \frac{\frac{1}{2}}{\frac{1}{2}\cdot y} = \frac{1}{y}, \quad 0 \le x \le y, \quad 0 \le y \le 2$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$f_{Y|X}(y|x) = \frac{\frac{1}{2}}{1 - \frac{1}{2}x} = \frac{1}{2-x}, \quad 0 \le x \le 2, \quad x \le y \le 2$$

[C] ESTIMATED VALUES

$$f_{X|Y=1}(x|y=1) = \frac{1}{1} = 1, \ 0 \le x \le 1$$

$$E[X|Y=1] = \int x \cdot f_X(x)\,dx$$

$$E[X|Y=1] = \int_0^1 x \cdot 1\,dx = \left[\frac{1}{2}x^2\right]_0^1$$

$$E[X|Y=1] = \frac{1}{2} \cdot 1^2 = \frac{1}{2}$$

$$f_{X|Y=1}(x|y=1) = \frac{1}{0.5} = 2,\ \ 0 \leq x \leq 0.5$$

$$E[X|Y=0.5] = \int x \cdot f_X(x)\,dx$$

$$E[X|Y=1] = \int_0^1 x \cdot 2\,dx = 2 \cdot \left[\frac{1}{2}x^2\right]_0^{0.5}$$

$$E[X|Y=1] = 2 \cdot \left[\frac{1}{2} \cdot 0.5^2\right] = 2 \cdot \left[\frac{1}{2} \cdot \frac{1}{4}\right] = 2 \cdot \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$$

[D] STATISTICALLY INDEPENDENT

No, because

$$f_{X,Y}(x,y) \neq f_x(x) \cdot f_y(y)$$

[E] CORRELATION COEFFICIENT

RESISTOR PRODUCTION

In production of 100 ohm resistors, the resistance of each resistors will be a random variable that is Gaussian distributed with a mean of 100 ohm, and a standard deviation of sigma. The resistors are sorted in 5% and 10% resistors. Thus all resistors within 5% of 100 ohm are sorted in one package and resistors between 5% and 10% are sorted in another package. Resistors deviating more than 10% are discarded.

A: Begin by assuming that sigma is equal to 5. How many percent on average of the produced resistors are in each package, and how many is discarded. Instead of a lookup table you can use the matlab function normcdf. Use the doc in matlab to find out arguments in the function.

B: Make a matlab simulation that simulate 1000 resistors, and confirm the result obtained in question A. Use matlab randn function.

C: What should the standard deviation be if the packages of 5% resistors contains half of the resistors? Find the result with the function norminv.

D: Sample 1000 resistors with the found standard deviation. Plot the pdf and cdf with a histogram function in matlab.

E: Find the mean and standard deviation of the 1000 samples, use the mean and var functions in matlab.

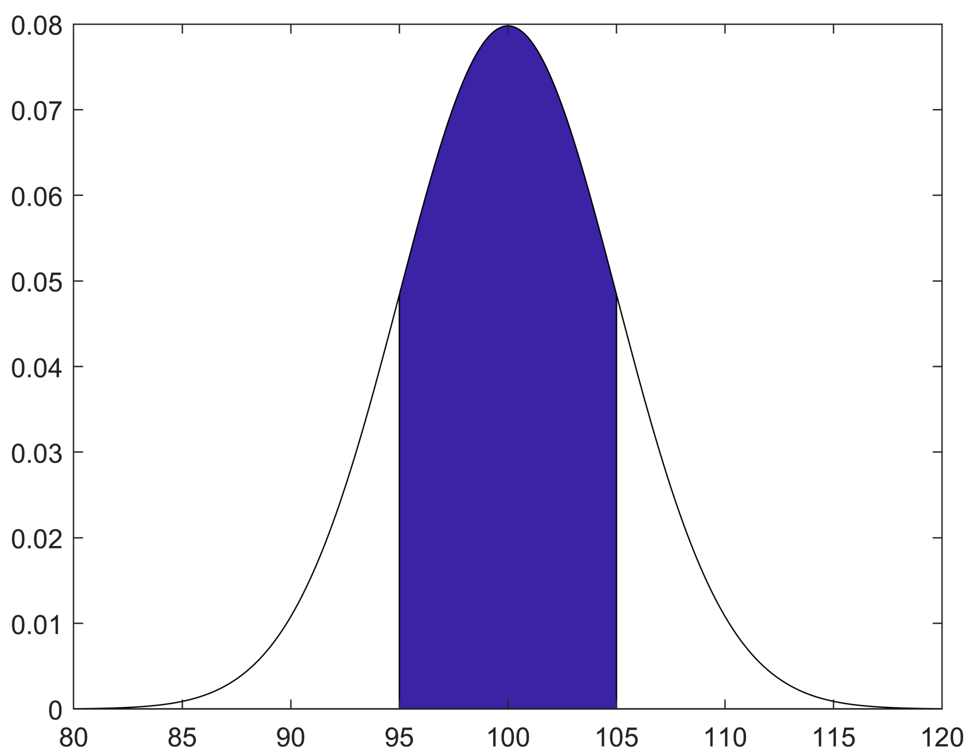
F: Why are the found mean and standard deviation not exact?

[A] CUMMULATIVE DISTRIBUTION

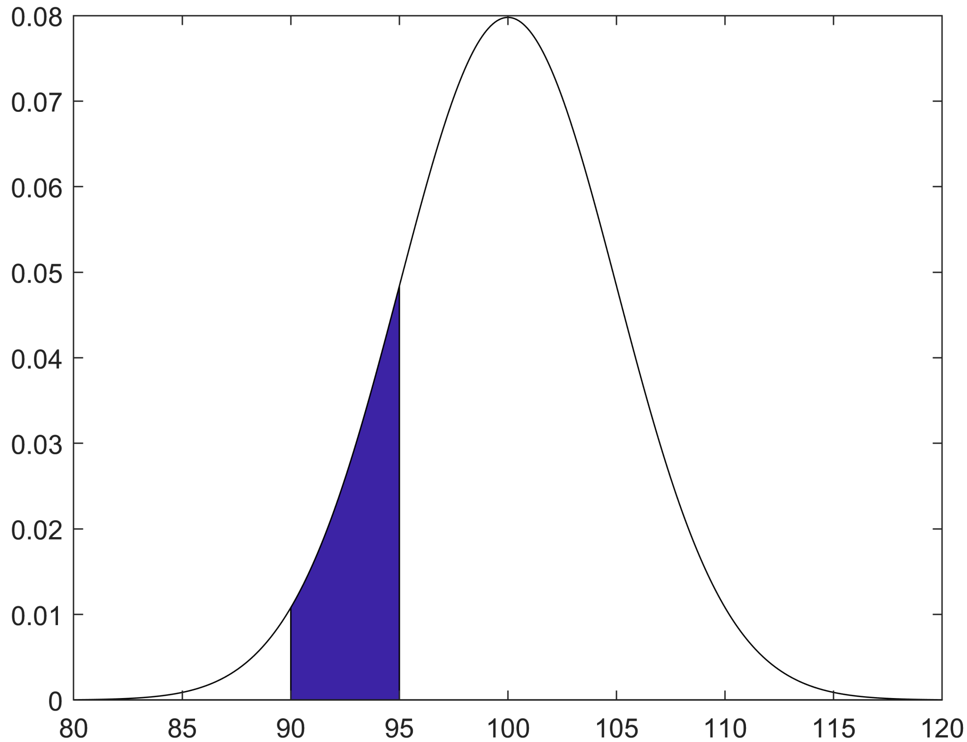
```
mu = 100;
sigma = 5;
sigma_squared = 5^2;

x = 80:0.1:120
```

```
x = 1×401
   80.0000   80.1000   80.2000   80.3000   80.4000   80.5000   80.6000   80.7000 ···
```

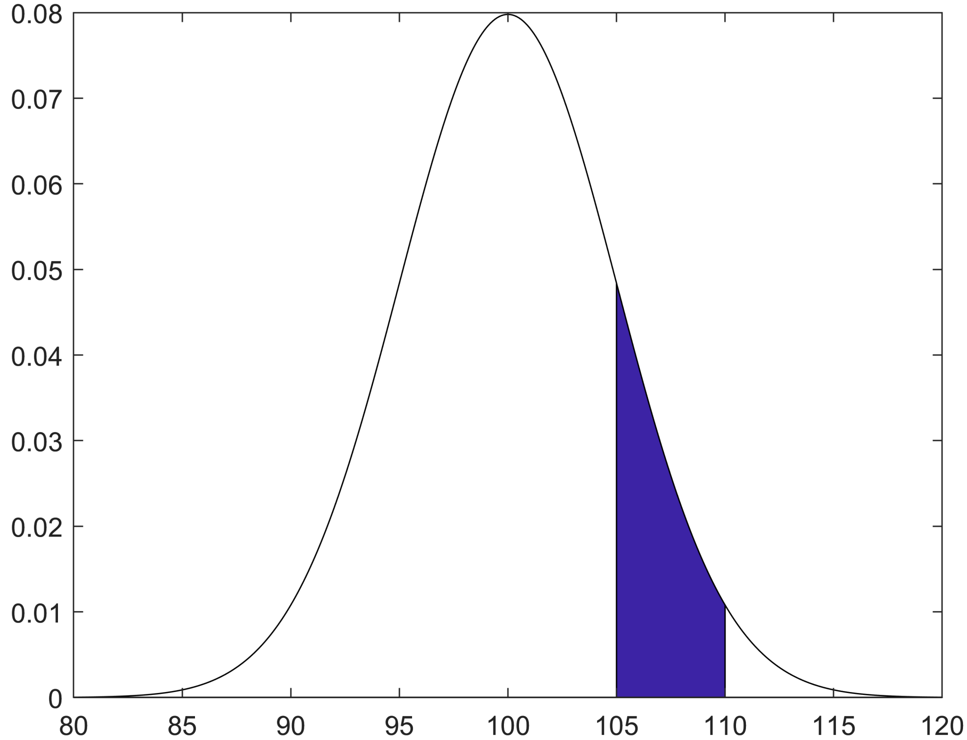
```
y = normpdf(x, mu, sigma);

probability_above_95 = normcdf(95, mu, sigma);
probability_above_90 = normcdf(90, mu, sigma);

% Probability within 5 percent
probability_5_percent = (1 - 2*probability_above_95) * 100;
pdfarea(x, y, 95, 105);
```



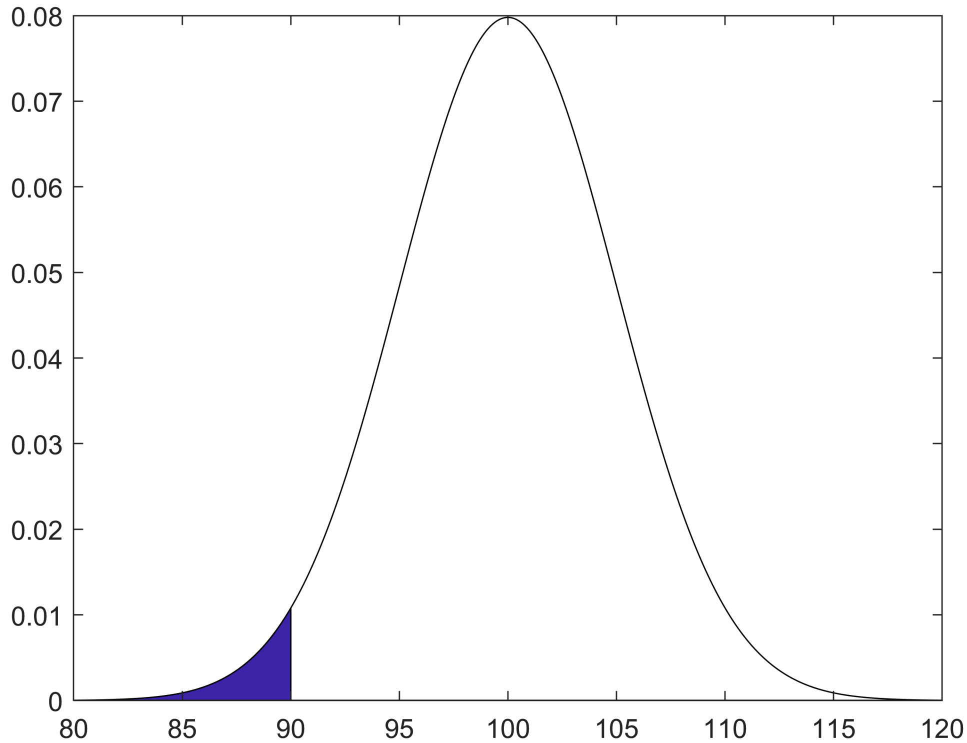
```
% Probability within 10 percent
probability_10_percent = 2 * (probability_above_95 - probability_above_90) * 100;

figure(4)
pdfarea(x, y, 90, 95);
```



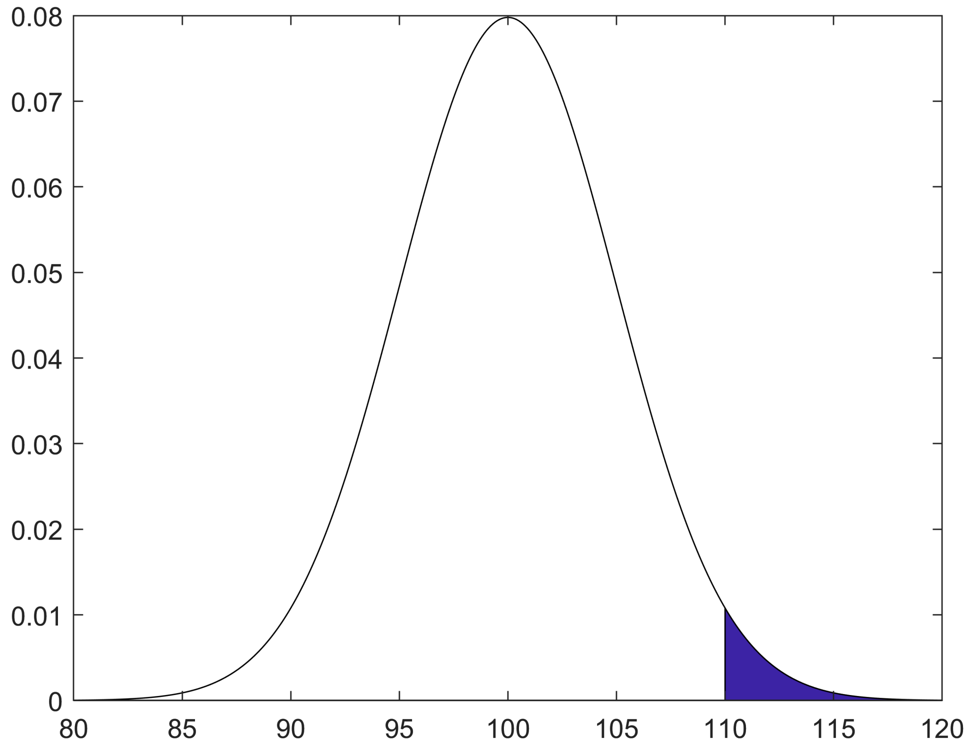

```
figure(5)
pdfarea(x, y, 105, 110);
```




```
%Probability discarded / above 10 percent
probability_discarded = 2 * probability_above_90 * 100;

figure(6)
pdfarea(x, y, [], 90)
```




```
figure(7)
pdfarea(x, y, 110)
```



[B] SIMULATION

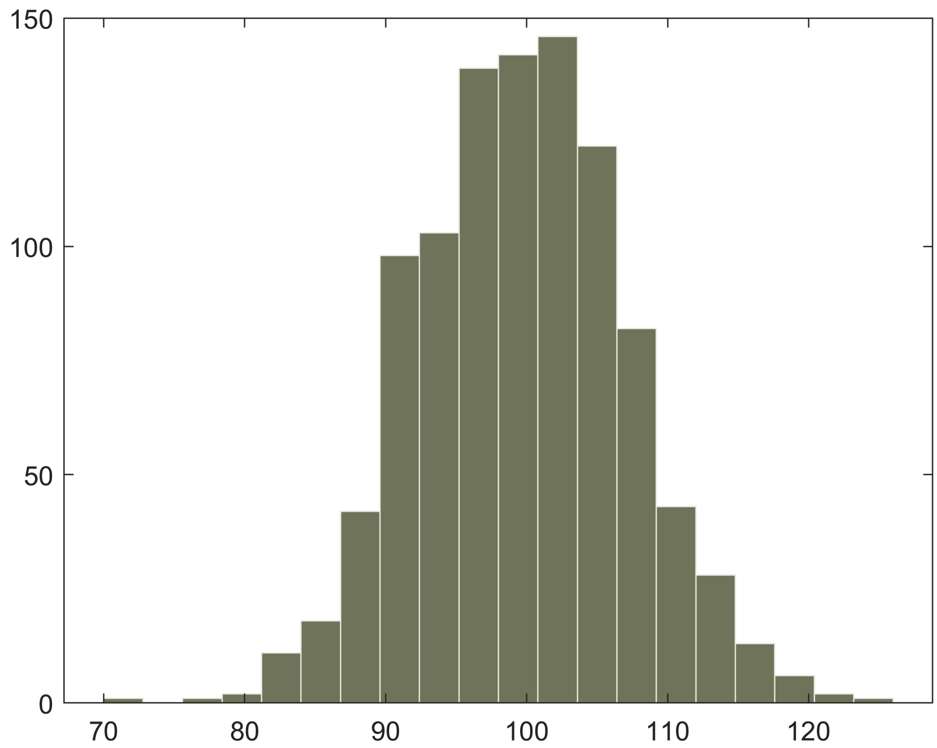
```
numberOfResistors = 1000;

numberOfDiscardedResistors = 0;
numberOf10PercentResistors = 0;
numberOf5PercentResistors = 0;

mu = 100;
sigma = 5;

resistors = sigma * randn(1, numberOfResistors) + mu;

for x = 1:numberOfResistors
    if resistors(x) > 110 || resistors(x) < 90
        numberOfDiscardedResistors = numberOfDiscardedResistors + 1;
    elseif resistors(x) > 105 || resistors(x) < 95
        numberOf10PercentResistors = numberOf10PercentResistors + 1;
    else
        numberOf5PercentResistors = numberOf5PercentResistors + 1;
    end
end


numberOf5PercentResistors = (numberOf5PercentResistors * 100) / numberOfResistors;
numberOf10PercentResistors = (numberOf10PercentResistors * 100) / numberOfResistors;
numberOfDiscardedResistors = (numberOfDiscardedResistors * 100) / numberOfResistors;
```

[C] STANDARD DEVIATION

```
sigma = 5;
mu = 100;

for x = 1:10000
    if norminv(0.25, mu, sigma) > 95
        sigma = sigma + 0.001;
    end
end
```

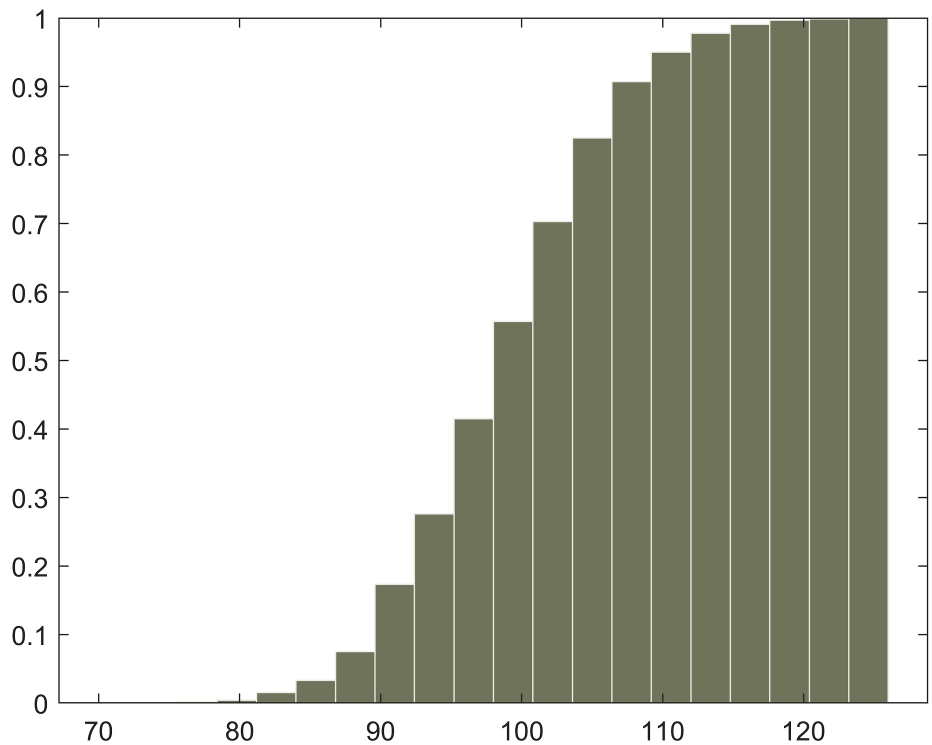
[D] PLOT

```
sigma = 7.4140;
mu = 100;
bins = 20;

numberOfResistors = 1000;
resistors = sigma * randn(1, numberOfResistors) + mu;

figure(8)
p = histogram(resistors, bins, 'FaceColor', Color.ArmyGreen, 'EdgeColor', Color.MarshMallow);
```




```
% cdfplot(resistors)

figure(9)
```

```
c = histogram(resistors, bins, 'FaceColor', Color.ArmyGreen, 'EdgeColor', Color.MarshMallow, 'N
```



[E] MEAN & STANDARD DEVIATION

```
resistorMean = mean(resistors);

resistorStdDeviation = sqrt(var(resistors));
```

[F] PRECISION

The found value for mean and standard deviation is not exact because the sample size is not big enough.