Graphs

Madiba Hudson-Quansah

Contents

Chapter 1

Graphs and Graph Models

1.1 Introduction

Definition 1.1.1: Graph

A graph $G = (V, E)$ consist of V, a non empty set of *vertices / nodes* and E a set of *edges*. Each edge has either one or two vertices associated with it called its *endpoints*. An edge is said to *connect* its endpoints.

The set of vertices V of a graph may be infinite, in this case the graph is called an *infinite graph*, conversely if the set of vertices is finite, the graph is called a *finite graph*.

The set of edges E contains ordered pairs or sets of elements in the set of vertices V indicating a connection between the two nodes or a node to itself.

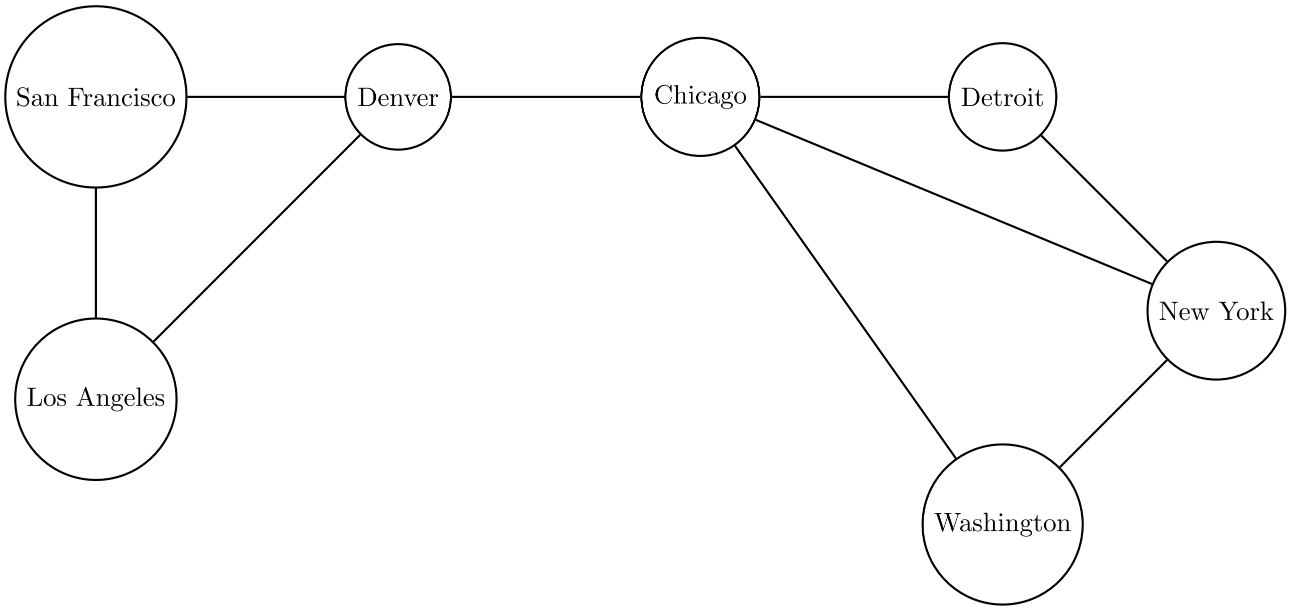
Definition 1.1.2: Vertex

A *vertex* is a point in a graph.

Definition 1.1.3: Edge

An *edge* is a line connecting two vertices in a graph.

Below is a graph representing a network of data centres and communication links between computers, where locations are represented by points and the links are represented by lines connecting the points.



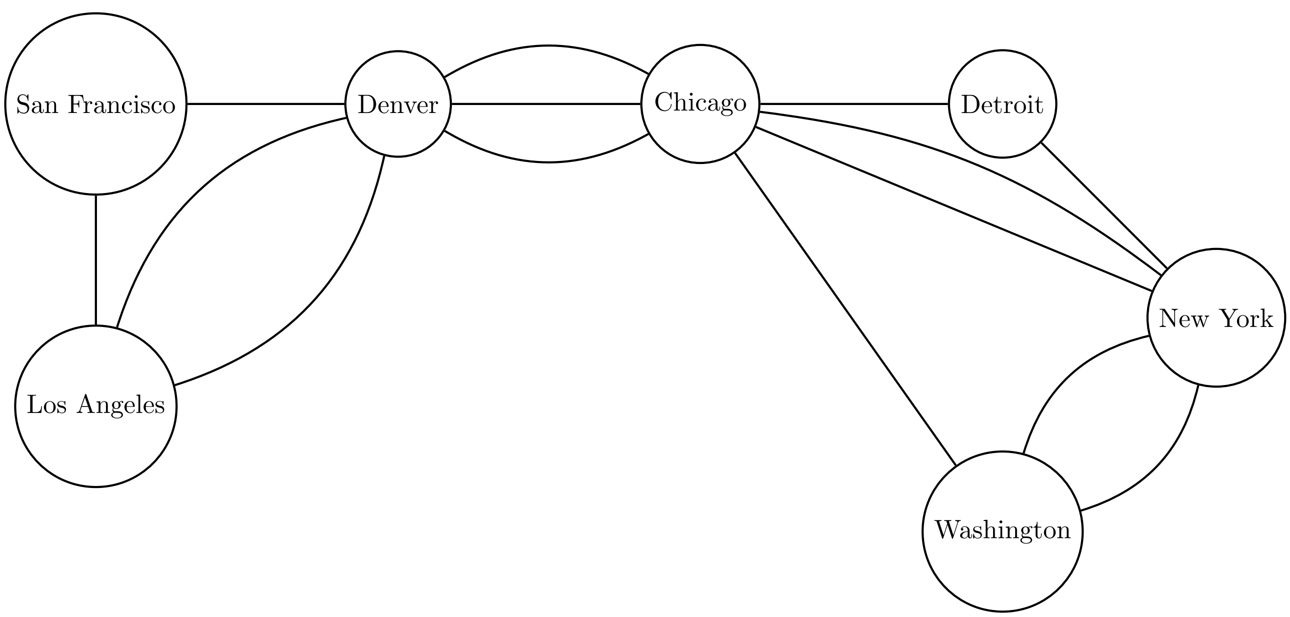
This is an example of a simple graph.

Definition 1.1.4: Simple Graph

A graph is said to be *simple* if it has no loops or multiple edges. A *loop* is an edge that connects a vertex to itself. A *multiple edge* is two or more edges that connect the same pair of vertices.

1.1.1 Multigraph

This graph could be re-drawn to model multiple links between the same pair of locations, as shown in Figure 1.1.1.



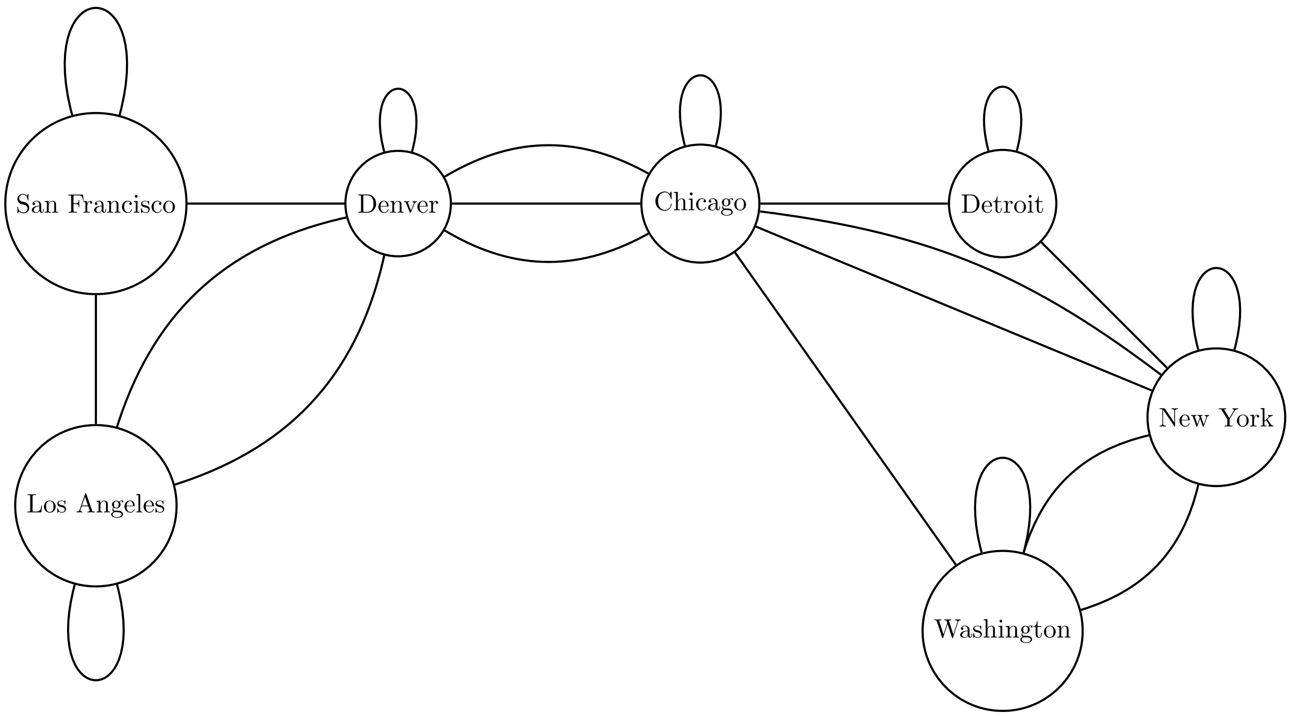
This is an example of a multigraph.

Definition 1.1.5: Multigraph

A graph that has multiple edges connecting the same vertices. When there are m distinct edges connecting the same unordered pair of vertices $\{u, v\}$, we say that $\{u, v\}$ is an edge of *multiplicity* m. i.e. j

1.1.2 Loops

Sometimes vertices may be connected to themselves, as shown in Figure 1.1.2.



Edges connecting vertices to themselves are called *loops*.

Definition 1.1.6: Loop

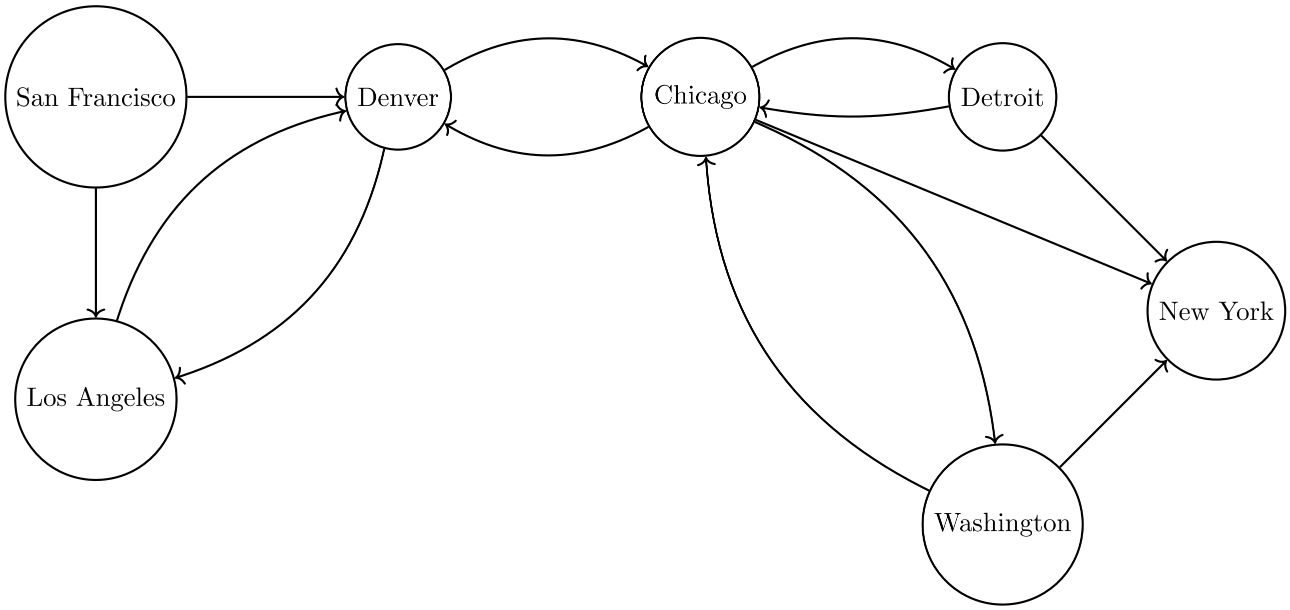
An edge that connects a vertex to itself.

Definition 1.1.7: Psuedograph

A graph that allows loops and multiple edges.

1.1.3 Directed Graphs

So far the examples given have been *undirected graphs*, with undirected edges. It is also possible to assign directions to the edges, as shown in Figure 1.1.3.



Such a graph is called a *directed graph* or *digraph*.

Definition 1.1.8: Directed Graph

A graph (V, E) that consists of a non-empty set of vertices V and a set of directed edges / arcs E. Each directed edge is associated with an ordered pair of vertices. The arc associated with the ordered pair (u, v) is said to *start* at u and *end* at v.

1.1.4 Simple Directed Graph

Definition 1.1.9: Simple Directed Graph

A directed graph with no loops or multiple edges.

1.1.5 Directed Multigraph

Definition 1.1.10: Directed Multigraph

A directed graph with multiple edges connecting the same pair of vertices. When there are m directed edges, each associated to an ordered pair of vertices (u, v), then (u, v) is an edge of multiplicity m.

1.1.6 Mixed Graph

Definition 1.1.11: Mixed graph

A graph with both direct and undirected edges, that may have multiple edges and loops.

1.1.7 Graph Terminology

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Psuedograph	Undirected	Yes	Yes
Simple Directed Graph	Directed	No	No
Directed Multigraph	Directed	Yes	Yes
Mixed Graph	Directed and Undirected	Yes	Yes

Table 1.1: Graph Terminology

Chapter 2

Graph Terminology and Special Graphs

2.1 Basic Terminology

Definition 2.1.1: Adjacent / Neighbours

Two vertices u and v in an undirected graph G, that are endpoints of an edge e. Such an edge e is called *incident with* the nodes u and v and e is said to *connect* u and u

Definition 2.1.2: Neighbourhood

The set of all neighbours of a vertex v of $G = (V, E)$, denoted by:

$$N(v)$$

If A is a subset of V, we denote by $N(A)$ the set of all nodes in G that are adjacent to at least one node in A. i.e. $N(A) = \bigcup_{v \in A} N(v)$

Definition 2.1.3: Degree of a Vertex / Node

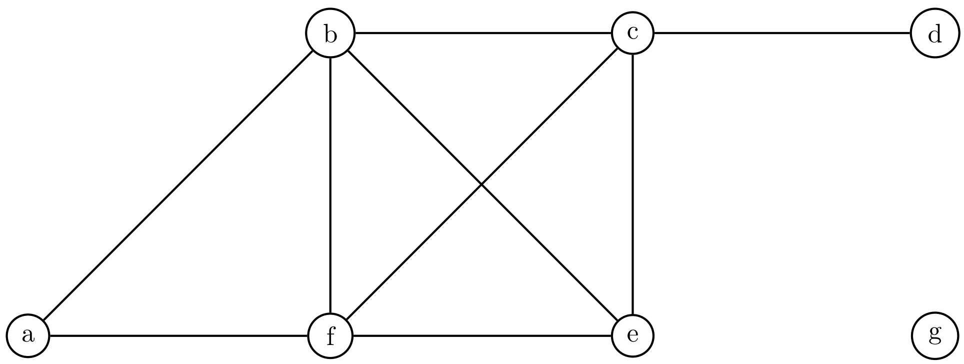
The number of edges incident with a particular node, except that a loop at a node is counted twice to the degree of that node, denoted by:

$$\deg(v)$$

Example 2.1.1

Question 1

What are the degrees and neighbourhoods of the vertices of the graph below



Solution:

Degrees:

deg(a) - 2
deg(b) - 4
deg(c) - 4
deg(d) - 1
deg(e) - 3
deg(f) - 4
deg(g) - 0

Neighbourhoods:

$$N(a) = \{b, f\}$$
$$N(b) = \{c, f, e, a\}$$
$$N(c) = \{b, d, f, e\}$$
$$N(d) = \{c\}$$
$$N(e) = \{b, f, c\}$$
$$N(f) = \{b, c, e, a\}$$
$$N(g) = \emptyset$$

Theorem 2.1.1 The Handshaking Theorem

Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} \deg(v)$$

Theorem 2.1.2

An undirected graph has an even number of nodes of odd degree.

Definition 2.1.4: Initial and End / Terminal Vertex

When an edge (u, v), of a directed graph G, u is said to be adjacent to v and v is said to be adjacent from u. u is called the *initial vertex* of (u, v) and v is called the *terminal / end vertex* of (u, v)

Definition 2.1.5: In-degree and Out degree

In a directed graph G, the *in-degree* of a vertex v, denoted by $\deg^{-}(v)$, is the number of edges with v as their terminal vertex and the *out-degree* of v, denoted by $\deg^{+}(v)$, is the number of edges with v as their initial vertex.

Theorem 2.1.3

Let $G = (V, E)$ a graph with directed edges. Then

$$\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = | E |$$

Where $| E |$ is the number of edges in the graph.

2.2 Special Graphs

2.2.1 Complete Graphs

Definition 2.2.1: Complete Graph

A complete graph K_n, on n vertices, is a simple graph that contains exactly one edge between each pair of distinct vertices. A simple graph where there it at least one pair of distinct vertices that are not connected by a edge is called an *incomplete graph*.

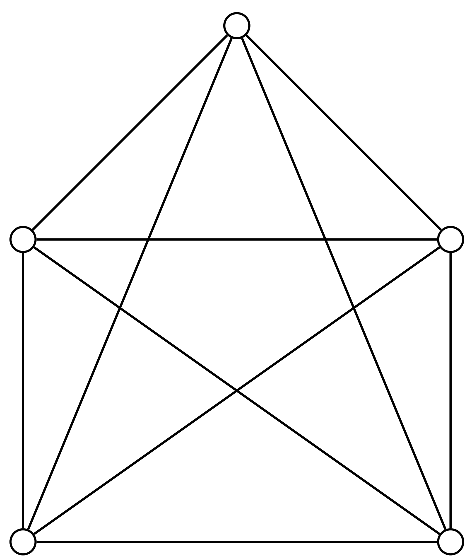


Figure 2.1: K_5

2.2.2 Cycle Graphs

Definition 2.2.2: Cycle Graphs

A cycle C_n, where $n \geqslant 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1},\}$ and $\{v_n, v_1\}$.

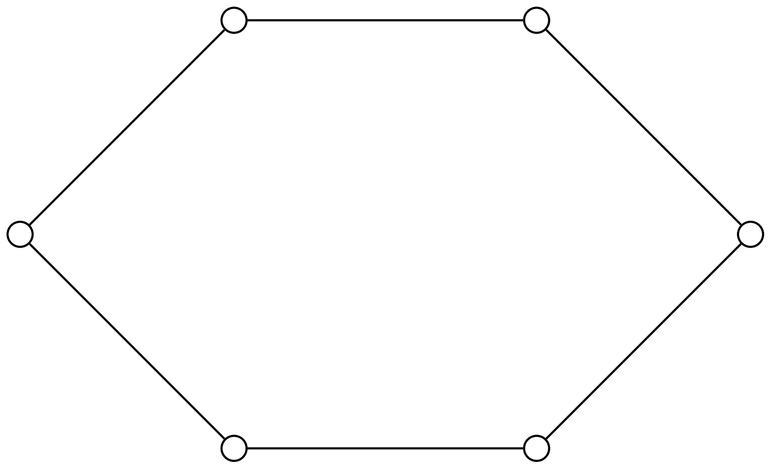


Figure 2.2: C_6

2.2.3 Wheel Graphs

Definition 2.2.3: Wheel Graph

A wheel W_n is obtained when an additional vertex is added to a cycle C_n, for $n \geqslant 3$, and connected to each of the n vertices of C_n, by new edges.

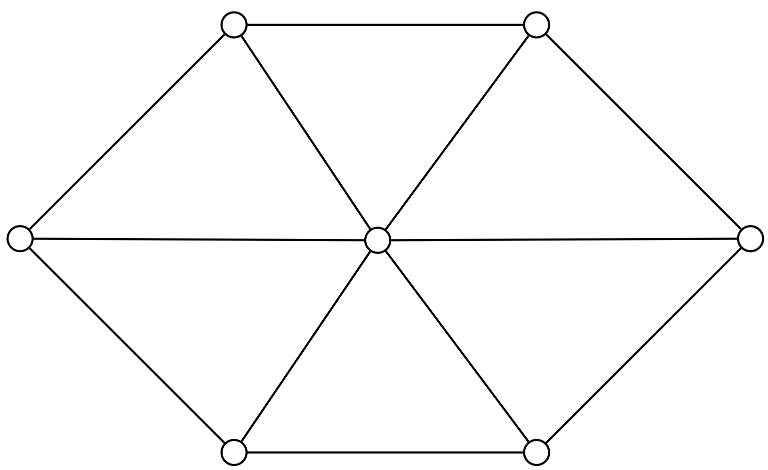


Figure 2.3: W_6

2.3 Bipartite Graphs

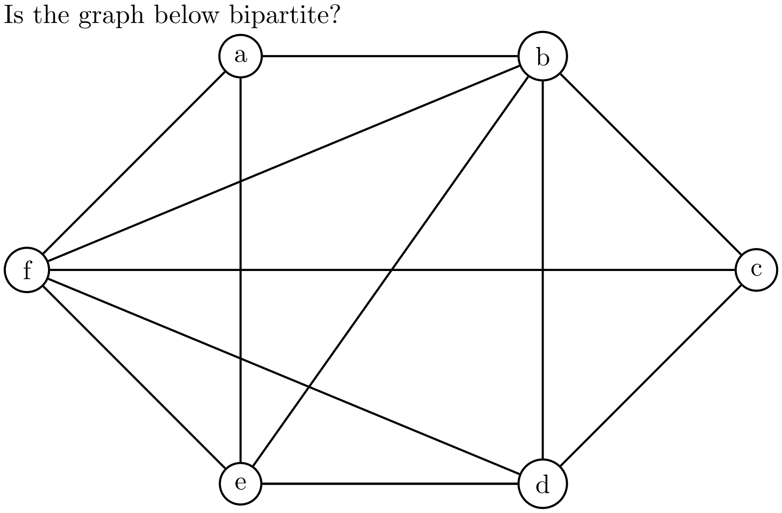
Definition 2.3.1: Bipartite Graphs

A simple graph G is called *bipartite* if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in G connects a vertex in V_1 and a vertex in V_2, so that no edge in G connects either two vertices in V_1 or two vertices in V_2. When this condition holds we call pair (V_1, V_2) a bipartition of the vertex set V of G.

Example 2.3.1

Question 2

Is the graph below bipartite?



Solution: This graph is not bipartite as there is no vertex set partition which would result in each of elements in set V_1 connecting to at least one element in set V_2.

Theorem 2.3.1

A simple graph is bipartite if and only if it is possible to assign one of two different colours to each vertex of the graph so that no two adjacent vertices have the same colour.

Theorem 2.3.2

A simple graph is bipartite if and only if it is not possible to start a node and return to this node by traversing an odd number of distinct edges.

2.3.1 Complete Bipartite Graphs

Definition 2.3.2: Complete Bipartite Graph

A complete bipartite graph $K_{m,n}$ is a bipartite graph with bipartition (m, n) such that each vertex in m is adjacent to every vertex in n.

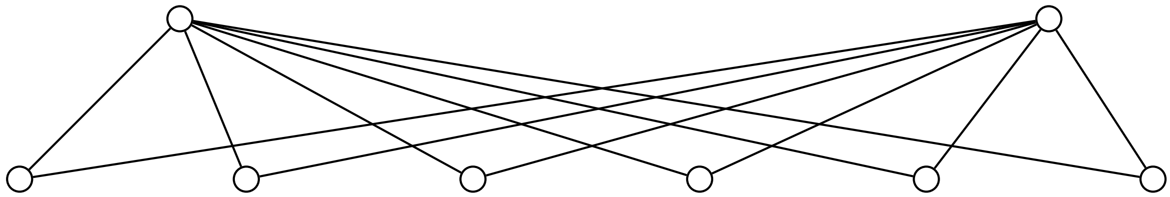


Figure 2.4: $K_{2,6}$

2.4 New Graphs From Old

2.4.1 Subgraphs

Definition 2.4.1: Subgraph

A *subgraph* of graph $G = (V, E)$, is a graph $H = (W, F)$ where $W \subseteq V$ and $F \subseteq E$. A subgraph H is *proper* if $H \neq G$

Definition 2.4.2: Induced Subgraph

Let $G = (V, E)$ be a simple graph. The subgraph induced by a subset of W of the node set V is the graph (W, F), where the edge set F contains an edge E if and only if the endpoints of this edge are in the node set W.

Given a graph $G = (V, E)$, and an edge $e \in E$, we can produce a subgraph of G by removing e from the set of edges E. The resulting subgraph denoted by $G - e$, has the same node set V as G but with its edge set $E - e$. Hence:

$$G - e = (V, E - \{e\})$$

Similarly if E' is a subset of E, a subgraph of G can be produced by finding the difference of these two sets. Hence

$$G - E' = (V, E - E')$$

We can also add an edge e to a graph to produce a larger graph when this edge connects to two nodes already in G, denoted by $G + e$, where e connects two non-incident nodes in G. Hence:

$$G + e = (V, E \cup \{e\})$$

Note that $G + e$ is not a subgraph of G.

Definition 2.4.3: Graph Union

The union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, is the simple graph with node set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$, and is denoted:

$$G_1 \cup G_2$$

2.5 Exercises

Question 3

What do the in-degree and out-degree of a vertex in a web graph represent?

Solution: The in-degree of a node in a web graph represent the number of pages that link to the node, while the out-degree represents the links from the node to other pages.

Question 4

What do the in-degree and out-degree of in a directed graph modelling a round-robin tournament represent?

Solution: In in-degree of a node in this case represents the number of wins that node has had, while the out-degree represents the number of losses.

Question 5

Show that in a simple graph with at least two vertices, there must be two vertices with the same degree.

Solution: Using the handshaking theorem, we know that the sum of the degrees of all the vertices in a graph is even. If all the vertices in the graph have distinct degrees, then the sum of the degrees of all the vertices in the graph is odd. This is a contradiction, hence there must be two vertices with the same degree.

Question 6

Draw these graphs:

1. $K_{1,8}$
2. W_7
3. C_7
4. Q_4

Solution:

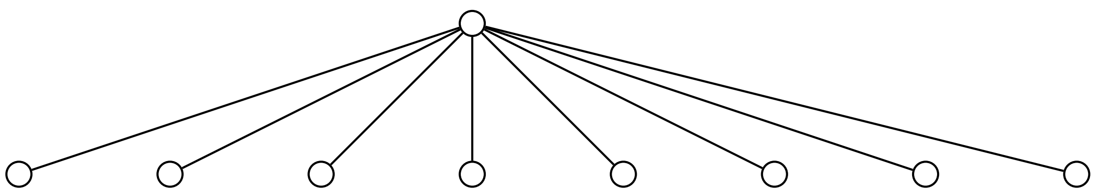


Figure 2.5: $K_{1,8}$

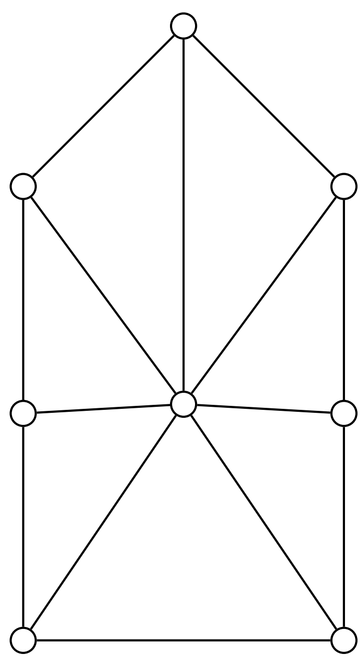


Figure 2.6: W_7

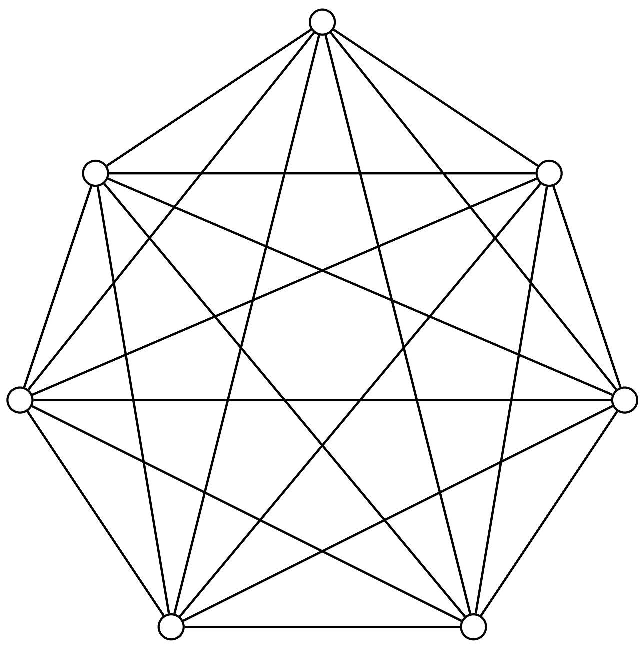


Figure 2.7: C_7

Chapter 3

Representing Graphs and Graph Isomorphism

3.1 Adjacency List

Definition 3.1.1: Adjacency List

A list of all the neighbours of each vertex in a graph, showing all the neighbours of each node.

3.1.1 Simple graph

Node	Adjacent nodes
a	b, c, e
b	a
c	a, d, e
d	c, e
e	d, a, c

3.1.2 Directed Graph

Initial Node	Terminal nodes
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

3.2 Adjacency Matrix

Definition 3.2.1: Adjacency Matrix

A matrix that represents a graph, either by representing the adjacency of nodes in the graph or the incidences of nodes and edges in the graph.

3.2.1 Adjacency of Nodes

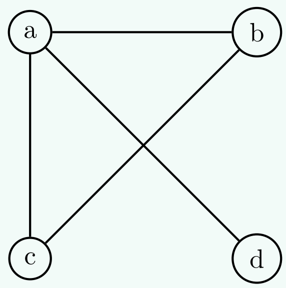
3.2.1.1 Simple Graphs

Let $G = (V, E)$, be a simple graph where $|V| = n$. Suppose that the nodes of G are listed as $v_1, v_2, \ldots, v_n$. The corresponding adjacency matrix A (or A_G) of G, with respect to this listing of nodes is the $n \times n$ zero matrix with 1 as its (i, j) entry when v_i and v_j are adjacent, and 0 as its (i, j) entry when they are not adjacent, i.e.:

$$A = [a_{ij}]$$

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases}$$

Example 3.2.1



Representing the graph above in an adjacency matrix with nodes listed as a, b, c, d:

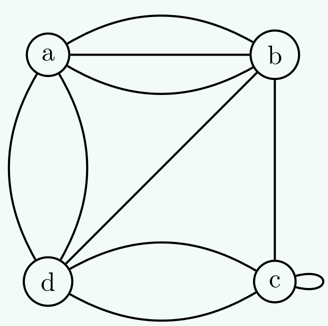
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The adjacency matrix of a simple graph is symmetric, i.e. $a_{ij} = a_{ji}$, because both these entries are 1 when v_i or v_j are adjacent. Furthermore, because a simple graph has no loops each entry a_{ii}, $i = 1, 2, 3, \ldots, n$ is 0.

3.2.1.2 Undirected Graphs

Adjacency matrices can also be used to represent undirected graphs with loops and multiple edges, with loops at the node v_i represented by a 1 at the (i, i)th position of the matrix, and multiple edges between nodes v_i and v_j or multiple loops at the same node being represented the number of edges associated with $\{v_i, v_j\}$

Example 3.2.2



Representing the graph above with an adjacency matrix with nodes listed as a, b, c, d:

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

3.2.1.3 Directed Graphs

The matrix for a directed graph $G = (V, E)$ has a 1 in its (i, j)th position if there is an edge from v_i to v_j, where $v_1, v_2, \ldots, v_n$ is an arbitrary listing of the nodes of the graph. Therefore:

$$A = \left[aij\right]$$

$$a_{ij} = \begin{cases} 1 & \text{if } \left(v_i, v_j\right) \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases}$$

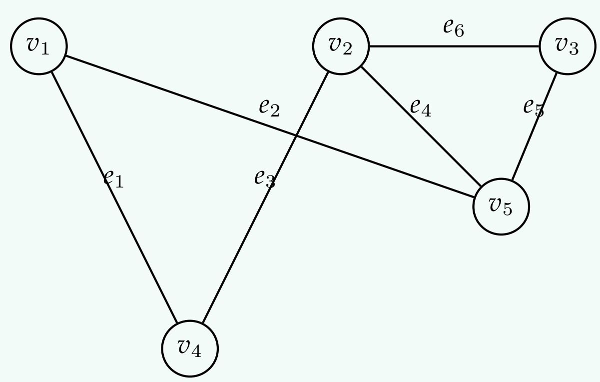
3.2.2 Incidence of Nodes and Edges / Incidence Matrix

Definition 3.2.2: Incidence Matrix

Let $G = (V, E)$, be an undirected graph, with an arbitrary listing of nodes $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_m$. The incidence matrix with respect to this listing of nodes and edges is the $n \times m$ matrix:

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with node } v_i \\ 0 & \text{otherwise} \end{cases}$$

Example 3.2.3



Representing the graph above with an incidence matrix with the already listed order of nodes and edges:

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Incidence matrices can also be used to represent multiple edges and loops, with multiple edges represented in the matrix using columns with identical entries as these edges are incident with the same pair of nodes, and loops represented as a column with exactly one entry equal to 1, corresponding to the node that is incident with the loop.

3.3 Isomorphism of Graphs

Definition 3.3.1: Isomorphic Graphs

Two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there exists a one-to-one and onto function f from V_1 to v_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. This function f is called an *isomorphism*, and two graphs that are not isomorphic are called *nonisomorphic*.

Example 3.3.1

Question 7

Show that the graphs $G = (V, E)$ and $H = (W, F)$ below are isomorphic:

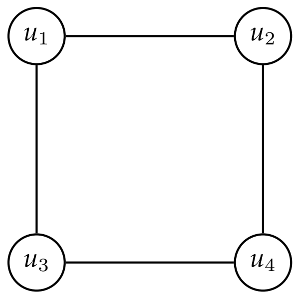


Figure 3.1: G

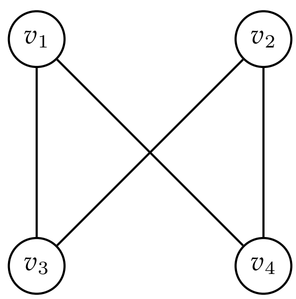


Figure 3.2: H

Solution:

The function f with $f(u_1) = v_1, f(u_2) = v_4, f(u_3) = v_3, f(u_4) = v_2$ is a one-to-one correspondence between V and W. With the correspondence preserving adjacency as all the adjacent pairs in graph G, $\{u_1, u_2\}$, $\{u_2, u_4\}$, $\{u_3, u_4\}$, $\{u_3, u_4\}$, are preserved in the graph H as $\{v_1, v_4\}$, $\{v_1, v_3\}$, $\{v_3, v_2\}$, $\{v_2, v_4\}$

3.3.1 Determining Isomorphism of Two Simple Graphs

It is difficult to determine the isomorphism of two simple graphs as there are $n!$ possible one-to-one correspondences between the nodes of the two graphs with n nodes. However sometimes it is not hard to show that two graphs are nonisomorphic. We can do this by finding one property only one of the two graphs has, but that is preserved by isomorphism, this is called a graph *invariant*.

Definition 3.3.2: Graph Invariant

A property of one graph that is preserved by isomorphism, and that is not shared by another graph. For example the number of nodes in a graph is a graph invariant.

Two graphs cannot be isomorphic if they do not have the following properties:

- The same number of nodes as there must be a one-to-one correspondence between the nodes of the two graphs.
- The same number of edges as the one-to-one correspondence between node preserves the one-to-one correspondence between edges.
- The same degrees for each node correspondence between each graph.

Chapter 4

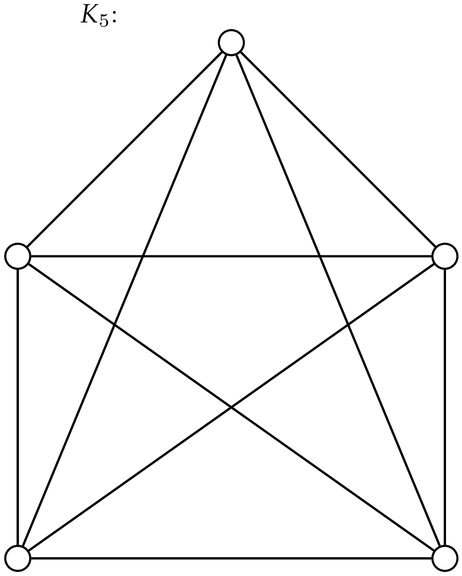
Exercises

Question 8

Draw K_5 and $\overline{K_5}$

Solution:

K_5:



$\overline{K_5}$: