

Limits and Continuity

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Chapter 1

Preliminaries

1.1 The Real Number system

Definition 1.1.1: Rational Number

An integer that can be expressed in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Rational numbers have decimal expansions that either terminate or repeat.

Definition 1.1.2: Irrational Number

An integer that cannot be expressed in the form $\frac{p}{q}$. Irrational numbers do not have decimal expansions that terminate or repeat.

Theorem 1.1.1

If a and b are real numbers and $a < b$ then:

- For any real number c , $a + c < b + c$
- For real numbers c and d if $c < d$, then $a + c < b + d$
- For any real number $c > 0$, $a \times c < b \times c$
- For any real number $c < 0$, $a \times c > b \times c$

Theorem 1.1.2 Triangle of Inequality

For any real numbers a and b :

- $|a \times b| = |a| \times |b|$
- $|a + b| \neq |a| + |b|$
- $|a + b| \leq |a| + |b|$

Example 1.1.1

Question 1

Solve the inequality

$$|x - 2| < 5$$

Solution:

$$\begin{aligned}|x - 2| &< 5 \\ x - 2 &< 5 \quad x - 2 > -5 \\ x &< 7 \quad x > -3 \\ \therefore -3 &< x < 7\end{aligned}$$

1.2 Lines and Functions

Definition 1.2.1: Slope of a Line

For $x_1 \neq x_2$, the slope of a straight line through the points (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

When $x_1 = x_2$ and $y_1 \neq y_2$, the line through (x_1, y_1) and (x_2, y_2) is vertical and its slope is undefined.

The slope of a line can also be described as the change in y , (Δy) divided by the change in x (Δx), or *Rise* over *Run* where rise is the change in height and run is the change in width in this case.

The equation of a line can be written in point slope form like:

$$y = m(x - x_0) + y_0$$

Where x_0 and y_0 are the starting coordinates of the line

Two (non-vertical) lines are parallel if they have the same slope, therefore any two vertical lines are parallel.

Two (non-vertical) lines of slope m_1 and m_2 are perpendicular if the product of their slopes is -1 this indicates that they are negative reciprocals of each other.

Definition 1.2.2: Function

For any two subsets A and B of the real line, a function f is a rule that assigns exactly one element y in set B to each element x in set A , where $y = f(x)$.

The set A is referred to as the *domain* of f , and the set of B is called the *range* of f , i.e. $\{y \mid y = f(x), \text{ for some } x \in A\}$

Definition 1.2.3: Polynomial

A function that can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where a_0, a_1, \dots, a_n are real numbers (coefficients) with $a_n \neq 0$ and $n \geq 0$ is an integer (degree)

Definition 1.2.4: Rational function

Any function that can be written in the form

$$f(x) = \frac{p(x)}{q(x)}$$

Where p and q are polynomials.

Theorem 1.2.1 Factor Theorem

For any polynomial function, f , $f(a) = 0$ if and only if $(x - a)$ is a factor of $f(x)$

1.3 Trigonometric Functions

Definition 1.3.1: Periodic Function

A function f is periodic of period T if

$$f(x + T) = f(x)$$

For all x such that x and $x + T$ are in the domain of f . The smallest such number $T > 0$ is called the *fundamental period*.

$$\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$$

Definition 1.3.2

Tangent

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

Cotangent

$$\cot(x) = \frac{\cos(x)}{\sin(x)} \quad / \quad \cot(x) = \frac{1}{\tan(x)}$$

Secant

$$\sec(x) = \frac{1}{\cos(x)}$$

Cosecant

$$\csc(x) = \frac{1}{\sin(x)}$$

Definition 1.3.3: Proprieties of a periodic function

For a periodic function in the form

$$y = Af(cx)$$

A is the amplitude of the values generated, and for any positive integer c , the period of the function f is $\frac{T}{c}$, where T is the period of the function f .

The frequency of a period function can be found:

$$f = \frac{c}{T}$$

Theorem 1.3.1 Trigonometric Identities

For any real numbers α and β , the following identities hold:

$$\sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \sin(\beta) \cos(\alpha)$$

$$\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)$$

$$\sin^2(\alpha) = \frac{1}{2} (1 - \cos(2\alpha))$$

$$\cos^2(\alpha) = \frac{1}{2} (1 + \cos(2\alpha))$$

1.4 Transformations of Functions

Definition 1.4.1: Function Combinations

Suppose that f and g are functions with domains D_1 and D_2 , respectively. The functions $f + g$, $f - g$ and $f \times g$ are defined by:

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(f \times g)(x) = f(x) \times g(x)$$

$\forall x$ where $x \in D_1 \cap D_2$. The function $\frac{f}{g}$ is defined by:

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

$\forall x$ where $x \in D_1 \cap D_2$ such that $g(x) \neq 0$

Definition 1.4.2: Composition

The composition of functions f and g is written $f \circ g$ and is defined by

$$(f \circ g)(x) = f(g(x))$$

$\forall x$ where x is in the domain of g and $g(x)$ is in the domain of f

Chapter 2

The Concept of Limit