Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

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Contents

Chapter 1

Sets

Definition 1.0.1: Set

An unordered collection of objects, called *elements* or *members* of the set. A set contains elements and, we can denote this as $a \in A$ where a is an element of the set A, or $a \notin A$, where a is not an element of the set A.

There are several ways to describe a set:

Roster notation $\{1, 2, 3, 4, 5\}$

Set-Builder notation Where all the elements of a set are described by a property they satisfy.i.e. The set O of all odd positive numbers less than 10 can be expressed as $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$ or specifying the domain of discourse, $O = \{x \in \mathbb{Z}^+ \mid x \text{ is odd and } x < 10\}$, or the set of all positive rational numbers $\mathbb{Q}^+$ can be expressed as $\mathbb{Q}^+ = \left\{x \in \mathbb{R} \mid x = \frac{p}{q}, \text{ for some positive integers } q \text{ and } p\right\}$

Definition 1.0.2: Equality of Sets

Two sets A and B are equal if and only if they have the same elements. Therefore, $\forall x\,(x \in A \leftrightarrow x \in B)$, We write $A = B$ if this is the case.

Definition 1.0.3: Empty / Null Set

A set with no elements, denoted by $\emptyset$ or $\{\}$. Can be expressed as $\{x \mid F\}$

Definition 1.0.4: Singleton Set

A set with exactly one element, denoted by $\{a\}$. The set $\{\emptyset\}$ is a singleton set as it is a set with one element, the empty set.

1.0.1 Set Definitions

1.0.1.1 Natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

1.0.1.2 Integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

1.0.1.3 Positive Integers

$$\mathbb{Z}^+ = \{1, 2, 3, 4, 5, \ldots\}$$

1.0.1.4 Rational numbers

$$\mathbb{Q} = \left\{\frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0\right\}$$

1.0.1.5 Irrational Numbers

$$\mathbb{I} = \{x \mid x \text{ is a number that cannot be expressed as a fraction}\}$$

1.0.1.6 Real numbers

$$\mathbb{R} = \{x \mid x \text{ is a point on the number line}\}$$

Or

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$$

1.0.1.7 Positive Real numbers

$$\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$$

1.0.1.8 Complex numbers

$$\mathbb{C} = \left\{a + bi \mid a, b \in \mathbb{R} \text{ and } i^2 = -1\right\}$$

1.0.2 Venn Diagrams

Definition 1.0.5: Universal Set

The set of all objects under consideration, denoted by U. Can be expressed as $\{x \mid T\}$

Sets can be graphically represented using Venn diagrams. A Venn diagram is a collection of simple closed curves, especially circles, that represent sets. In Venn diagrams the universal set U which contains all the objects under consideration is represented by a rectangle, and the sets are represented by circles within the rectangle, with points inside the circles representing elements of the sets.

1.0.3 Subsets

Definition 1.0.6: Subset

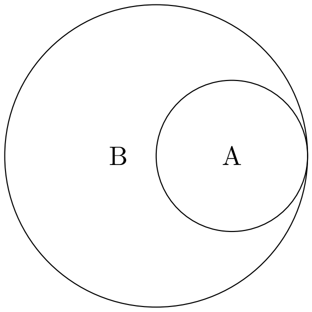
A set A is a *subset* of a set B if and only if every element of A is also an element of B. Denoted by $A \subseteq B$.

We see that $A \subseteq B$ if and only if

$$\forall x\, (x \in A \rightarrow x \in B)$$

Is true. I.e. If $x \in A$, then $x \in B$. To disprove this we need to show that $\exists x\, (x \in A \wedge x \notin B)$

Shown graphically:



Example 1.0.1

The set of integers with squares less than 100 is not a subset of the set of nonnegative integers because -1 is in the former set [as $(-1)^2 < 10$], but not the later set. The set of people who have taken discrete mathematics at your school is not a subset of the set of all computer science majors at your school if there is at least one student who has taken discrete mathematics who is not a computer science major.

Theorem 1.0.1

For every set S

1. $\emptyset \subseteq S$
2. $S \subseteq S$

1. ***Proof:*** We will prove that $\emptyset \subseteq S$, using a vacuous proof
   Let S be a set.
   To show $\emptyset \subseteq S$ we must show that $\forall x\,(x \in \emptyset \rightarrow x \in S)$ is T.
   Because $\emptyset$ contains no elements $x \in \emptyset$ is always F
   This follows that the implication $x \in \emptyset \rightarrow x \in S$ is always T
   Hence $\emptyset \subseteq S$

   😊

2. ***Proof:*** We will prove that $S \subseteq S$, using a direct proof
   Let S be a set
   To show $S \subseteq S$ we must show that $\forall x\,(x \in S \rightarrow x \in S)$ is T
   Assume $x \in S$
   Because $x \in S$ is always T, the implication $x \in S \rightarrow x \in S$ is always T
   $\therefore\ \forall x\,(x \in S \rightarrow x \in S)$ is T
   Hence $S \subseteq S$

   😊

Definition 1.0.7: Proper subset

A set A is *proper subset* of a set B if and only if every element of A is also an element of B and $A \neq B$. Denoted by $A \subset B$. I.e.

$$\exists x\,(x \notin A \wedge x \in B) \wedge \forall x\,(x \in A \rightarrow x \in B)$$

Is T.

Definition 1.0.8: Further Equality

Two sets A and B are equal if $A \subseteq B \wedge B \subseteq A$ is T. I.e. $A = \{\emptyset, \{a\}, \{a\}, \{b\}, \{a, b\}\}$ and $B = \{x \mid x \text{ is a subset of the set } \{a, b\}\}$ are equal.

1.0.4 Cardinality

Definition 1.0.9: Cardinality

The number of distinct elements n in a set A. Denoted by $|A| = n$. Where n is a non-negative integer, we say that A is a finite set.

Definition 1.0.10: Infinite set

A set A is infinite if it is not finite. I.e. $|A| = \infty$

1.0.5 Power Set

Definition 1.0.11: Power Set

A set containing all the subsets of a given set A. Denoted by $\mathcal{P}(A)$. If a set has n distinct elements, then the cardinality of the power set is 2^n.

Example 1.0.2

Question 1

What is the power set of the set $\{0, 1, 2\}$

Solution:

$$\mathcal{P}(\{0,1,2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, , \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$$

Example 1.0.3

Question 2

What is the power set of $\emptyset$

Solution:

$$\mathcal{P}(\emptyset) = \{\emptyset\}$$

Question 3

What is the power set of $\{\emptyset\}$

Solution:

$$\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$$

1.0.6 N-Tuples

Definition 1.0.12: Ordered N-Tuple

N-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, $\ldots$, and a_n as its nth element.

Two n-tuples are equal if an only if each corresponding pair of their elements is equal, i.e. $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ are equal if and only if $a_i = b_i$, for $i = 1, 2, \ldots, n$.

Ordered 2-tuples are called *ordered pairs*. The ordered pairs, (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

1.0.7 Cartesian Products

Definition 1.0.13: Cartesian Product

Let A and B be sets. The *Cartesian Product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. I.e.

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

The number of items in the Cartesian product of two sets is the product of the cardinality of each set.

Example 1.0.4

Question 4

What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$

Solution:

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

Question 5

Show that the Cartesian product $B \times A$ is not equal to the Cartesian product $A \times B$.

Solution:

$$B \times A = \{(a, 1)(a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$$

$\therefore A \times B \neq B \times A$

Definition 1.0.14: Cartesian Product of more than two sets

The Cartesian product of the sets $A_1, A_2, \ldots, A_n$,denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$. I.e.

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$$

Example 1.0.5

Question 6

What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, $C = \{0, 1, 2\}$.

Solution:

$$A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}$$

We use the notation A^2 to denote $A \times A$, the Cartesian product of A and itself. Therefore

$$A^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for } i = 1, 2, \ldots, n\}$$

Example 1.0.6

Suppose $A = \{1, 2\}$.

It follows $A^2 = \{(1,1),(1,2),(2,1),(2,2)\}$,
and $A^3 = \{(1,1,1),(1,1,2),(1,2,1),(1,2,2),(2,1,1),(2,1,2),(2,2,1),(2,2,2)\}$

Example 1.0.7

Question 7

What are the ordered pairs in the less than or equal to relation, which contains, (a,b) if $a \leqslant b$, on the set $\{0,1,2,3\}$

Solution: Let R be the relation on the set $\{0,1,2,3\}$, if $a \leqslant b$.

$$R = \{(0,0),(1,1),(2,2),(3,3),(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)\}$$

1.0.8 Set Notation with Quantifiers

We can restrict the domain of a quantifier to a set, I.e. Where S is a set $\forall x \in S\,(P(x))$, denotes the universal quantification of $P(x)$ for all elements in the set S. Which is shorthand for $\forall x\,(x \in S \rightarrow P(x))$

Example 1.0.8
$\forall x \in \mathbb{R}\left(x^2 \geqslant 0\right)$ means "the square of any real number is greater than or equal to 0".
$\exists x \in \mathbb{Z}\left(x^2 = 1\right)$ means "there exists an integer whose square is 1"

1.0.9 Truth Sets and Quantifiers

Definition 1.0.15: Truth Set

For a predicate P the truth set of P is the set of all elements in the domain of discourse that make P true. I.e. let S be a set. The truth set of $P(x)$ is

$$\{x \in S \mid P(x)\}$$

Example 1.0.9

Question 8

What are the truth set of the predicates $P(x)$, $Q(x)$, and $R(x)$, where the domain is the set of integers, and $P(x)\colon\ |x| = 1$, $Q(x)\colon\ x^2 = 2$, and $R(x)\colon\ |x| = x$

Solution:
The truth set of P is $\{x \in \mathbb{Z} \mid |x| = 1\}$
The truth set of Q is $\left\{x \in Z \mid x^2 = 2\right\}$
The truth set of R is $\{x \in \mathbb{Z} \mid |x| = x\}$

Note:-
$\forall x P(x)$ is T over the domain U if and only if the truth set of P is U.
$\exists x P(x)$ is T over the domain U if and only if the truth set of P is not empty.

1.1 Exercises

Question 9

List the members of these sets

1. $\{x \mid x \text{ is the square of an integer and } x < 100\}$
2. $\left\{x \mid x \text{ is an integer such that } x^2 = 2\right\}$

Solution:

1. $\{1, 4, 9, 16, 25, 36, 49, 64, 81\}$
2. $\emptyset$

Question 10

Use set builder notation to describe the following sets

1. $\{-3, -2, -1, 0, 1, 2, 3\}$
2. $\left\{m, n, o, p\right\}$

Solution:

1. $\{x \mid -3 \leqslant x \leqslant 3\}$
2. $\{x \mid x$ is a letter in the word monopoly excluding "l" and "y" $\}$

Question 11

Suppose that $A = \{2, 4, 6\}$, $B = \{2, 6\}$, $C = \{4, 6\}$ and $D = \{4, 6, 8\}$. Determine which of these sets are subsets of which other sets.

Solution:

$$B \subseteq A$$
$$C \subseteq A$$
$$C \subseteq D$$

Question 12

Suppose that A, B, C, are sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$

Solution:

$$A \subseteq B \text{ means } \forall x\,(x \in A \longrightarrow x \in B)$$
$$B \subseteq C \text{ means } \forall x\,(x \in B \longrightarrow x \in C)$$
$$A \subseteq C \text{ means } \forall x\,(x \in A \longrightarrow x \in C)$$

$$\begin{array}{rl} & \forall x\,(x \in A \longrightarrow x \in B) \\ & \forall x\,(x \in B \longrightarrow x \in C) \\ \hline \therefore & \forall x\,(x \in A \longrightarrow x \in C) \end{array}$$

	Steps	Reasons
1	$\forall x\,(x \in A \longrightarrow x \in B)$	Premise 1
2	$\forall x\,(x \in B \longrightarrow x \in C)$	Premise 2
3	$x \in A \longrightarrow x \in B$	Universal Instantiation of 1
4	$x \in B \longrightarrow x \in C$	Universal Instantiation of 2
5	$x \in A \longrightarrow x \in C$	By Hypothetical Syllogism of 3 and 4
6	$\forall x\,(x \in A \longrightarrow x \in C)$	Universal generalization of 5

Question 13

Find the power set of each of these sets, where a and b are distinct elements.

1. $\{a\}$
2. $\{a, b\}$
3. $\{\emptyset, \{\emptyset\}\}$

Solution:

1. $\mathcal{P}\,(\{a\}) = \{\emptyset, \{a\}\}$
2. $\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
3. $\mathcal{P}(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}\, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

Chapter 2

Set Operations

2.1 Set Operations

2.1.1 Union

Definition 2.1.1: Union

Let A and B be sets. The *union* of A and B, denoted by $A \cup B$, is the set of all elements that are either in A or in B or in both. I.e.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

2.1.2 Intersection

Definition 2.1.2: Intersection

Let A and B be sets. The *intersection* of A and B, denoted by $A \cap B$, is the set of all elements that are in both A and B. I.e.

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

2.1.3 Complement

Definition 2.1.3: Complement

Let A be a set. The *complement* of the set A (with respect to U), denoted by $\overline{A}$ is the set $U - A$. I.e.

$$\overline{A} = \{x \in U \mid x \notin A\}$$

2.1.4 Difference

Definition 2.1.4: Difference

Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set of all elements that are in A but not in B. I.e.

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

Or

$$A - B = A \cap \overline{B}$$

2.1.5 Symmetric Difference

Definition 2.1.5: Symmetric Difference

Let A and B be sets. The *symmetric difference* of A and B, denoted by $A \oplus B$, is the set of all elements that are in exactly one of A and B. I.e.

$$A \oplus B = (A - B) \cup (B - A)$$

Example 2.1.1

Question 14

$$\begin{aligned} U &= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \\ A &= \{1, 2, 3, 4, 5\} \\ B &= (4, 5, 6, 7, 8) \end{aligned}$$

What is $A \oplus B$

Solution:

$$A \oplus B = \{1, 2, 3, 6, 7, 8\}$$

2.1.6 The Cardinality of the Union of Two Sets

The cardinality of the union of two sets A and B is given by

$$|A \cup B| = |A| + |B| - |A \cap B|$$

2.2 Set Identities

2.2.1 Identity Laws

$$\begin{aligned} A \cap U &= A \\ A \cup \emptyset &= A \end{aligned}$$

2.2.2 Domination Laws

$$\begin{aligned} A \cup U &= U \\ A \cap \emptyset &= \emptyset \end{aligned}$$

2.2.3 Idempotent Laws

$$\begin{aligned} A \cup A &= A \\ A \cap A &= A \end{aligned}$$

2.2.4 Complementation Law

$$\overline{\left(\overline{A}\right)} = A$$

2.2.5 Commutative Laws

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

2.2.6 Associative Laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

2.2.7 Distributive Laws

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

2.2.8 De Morgan's Laws

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

2.2.9 Absorption Laws

$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$

2.2.10 Complement Laws

$$A \cup \overline{A} = U$$
$$A \cap \overline{A} = \emptyset$$

2.2.11 Proving Set Identities

There are different ways to prove set identities, these include:

- Proving each set is a subset of the other
- Using set builder notation and propositional logic
- Using Membership tables

Definition 2.2.1: Membership Table

A table that shows the truth value of a predicate for all possible combinations of truth values of its variables.

Example 2.2.1

Question 15

Prove that

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Using propositional logic:

Proof: We prove this identity by showing that each set is a subset of the other. I.e.

$$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B} \wedge \overline{A} \cap \overline{B} \subseteq \overline{A \cap B}$$

$$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B} \text{ means } \forall x \left(x \in \overline{A \cap B} \rightarrow x \in \overline{A} \cup \overline{B} \right)$$
$$\overline{A} \cup \overline{B} \subseteq \overline{A \cap B} \text{ means } \forall x \left(x \in \overline{A} \cup \overline{B} \rightarrow x \in \overline{A \cap B} \right)$$

Assume that $x \in \overline{A \cap B}$

$x \in \overline{A \cap B}$	Assumption
$x \notin A \cap B$	Definition of Complement
$\neg (x \in A \cap B)$	Definition of $\notin$
$\neg (x \in A \wedge x \in B)$	Definition of intersection
$\neg (x \in A) \vee \neg (x \in B)$	By First De Moragn's Law for propositional logic
$x \notin A \vee x \notin B$	Definition of Complement
$x \in \overline{A} \cup \overline{B}$	Definition of union

Then we assume $x \in \overline{A} \cup \overline{B}$

$x \in \overline{A} \cup \overline{B}$	Assumption
$x \notin A \vee x \notin B$	Definition of union
$\neg (x \in A) \vee \neg (x \in B)$	Definition of Complement
$\neg (x \in A \wedge x \in B)$	By Second De Morgan's Law for propositional logic
$x \notin A \wedge x \notin B$	Definition of Complement
$x \notin A \cap B$	Definition of intersection
$x \in \overline{A \cap B}$	Definition of Complement

☺

Using set builder notation

$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$	Definition of Complement
$= \{x \mid \neg (x \in (A \cap B))\}$	Definition of $\notin$
$= \{x \mid \neg (x \in A \wedge x \in B)\}$	Definition of Intersection
$= \{x \mid \neg (x \in A) \vee \neg (x \in B)\}$	By First De Morgan's Law for propositional logic
$= \{x \mid x \notin A \vee x \notin B\}$	Definition of Complement
$= \{x \| x \in \overline{A} \cup \overline{B}\}$	Definition of union
$= \overline{A} \cup \overline{B}$	

2.3 Generalized Unions and Intersections

Definition 2.3.1: Generalized Union

The union of a collection of sets that contains those elements that are members of at least one set in the collection. Denoted by

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$$

Where $A_1 \cup A_2 \cup \ldots A_n$ is the union of sets $A_1, A_2, \ldots, A_n$

Definition 2.3.2: Generalized Intersection

The intersection of a collection of sets that contains those elements that are members of all the sets in the collection. Denoted by

$$A_1 \cap A_2 \cap \ldots A_n = \bigcap_{i=1}^{n} A_i$$

Where $A_1 \cap A_2 \cap \ldots \cap A_n$ is the intersection of sets $A_1, A_2, \ldots, A_n$

Example 2.3.1

For $i = 1, 2, \ldots,$ let $A_i = \{i, i+1, i+2, \ldots\}$. Then.

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\}$$

and

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2\} = A_n$$

We can extend this notation to other families of sets I.e.

$$A_1 \cup A_2 \cup \ldots \cup A_n \cup \ldots = \bigcup_{i=1}^{\infty} A_i$$

Denotes the union of the sets $A_1, A_2, \ldots, A_n, \ldots$, and the intersection of these sets is denoted by

$$A_1 \cap A_2 \cap \ldots \cap A_n \cap \ldots = \bigcap_{i=1}^{\infty} A_i$$

Generally when I is set, the notations $\bigcap_{i \in I} A_i$ and $\bigcup_{i \in I} A_i$ are used to denote the intersection and union of the sets A_i for $i \in I$, respectively, where

$$\bigcap_{i \in I} A_i = \{x \mid \forall i \in I\, (x \in A_i)\}$$

and

$$\bigcup_{i \in I} A_i = \{x \mid \exists i \in I\, (x \in A_i)\}$$

Example 2.3.2

Suppose $A_i = \{1, 2, 3, \ldots, i\}$ for $i = \{1, 2, 3, \ldots\}$ Then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \{1, 2, 3, \ldots, i\} = \mathbb{Z}^+$$

and

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} \{1, 2, 3, \ldots, i\} = \{1\}$$

2.4 Exercises

Question 16

List the members of these sets

1. $\left\{x \mid x \text{ is a real number such that } x^2 = 1\right\}$
2. $\{x|x \text{ is a positive integer less than } 12\}$
3. $\{x|x \text{ is the square of an integer and } x < 100\}$
4. $\left\{x \mid x \text{ is an integer such that } x^2 = 2\right\}$

Solution:

1. $\{-1, 1\}$
2.
3.
4. $\emptyset$

Question 17

Use set builder notation to show that:

1. $A \cup U = U$
2. $A \cap \emptyset = \emptyset$
3. $A \cup \overline{A} = U$
4. $A \cap \overline{A} = \emptyset$

Solution:

1.

$$\begin{aligned}
A \cup U &= \{x \mid x \in A \cup U\} && \text{Set builder notation}\\
&= \{x \mid x \in A \lor x \in U\} && \text{Definition of Union}\\
&= \{x \mid x \in A \lor T\} && \text{Definition of Universal Set}\\
&= \{x \mid T\} && \text{By Domination law for propositional logic}\\
&= U && \text{Definition of Universal Set}
\end{aligned}$$

2.
3.

4.

$$
\begin{aligned}
A \cap \overline{A} &= \{x \mid x \in A \cap \overline{A}\} && \text{Set builder notation} \\
&= \{x \mid x \in A \wedge x \in \overline{A}\} && \text{Definition of intersection} \\
&= \{x \mid x \in A \wedge (x \notin A)\} && \text{Definition of Complement} \\
&= \{x | x \in A \wedge \neg (x \in A)\} && \text{Definition of } \emptyset \\
&= \{x \mid F\} && \text{By Identity law of propositional logic} \\
&= \emptyset && \text{Definition of } \emptyset
\end{aligned}
$$

Question 18

Let A and B be sets. Show that

1. $(A \cap B) \subseteq A$
2. $A \subseteq (A \cup B)$
3. $A \subseteq (A \cup B)$
4. $A - B \subseteq A$
5. $A \cap (B - A) = \emptyset$
6. $A \cup (B - A) = A \cup B$

Solution:

1.

$$(A \cap B) \subseteq A \text{ means } \forall x\, (x \in (A \cap B) \rightarrow x \in A)$$

Assume $x \in (A \cap B)$

	Steps	Reasons
1	$x \in A \cap B$	Assumption
2	$x \in A \wedge x \in B$	Definition of intersection
3	$x \in A$	Simplification of 2

$\therefore\ x \in (A \cap B) \rightarrow x \in A$
Conclusion: $(A \cap B) \subseteq A$

2. ***Proof:***

$$A \subseteq (A \cup B) \text{ means } \forall x\, (x \in A \rightarrow x \in A \cup B)$$

	Steps	Reasons
1	$x \in A$	Premise
2	$x \in A \wedge x \in B$	Addition
3	$x \in A \cup B$	Definition of Union

$\therefore\ x \in A \rightarrow x \in A \cup B$
Conclusion: $A \subseteq A \cup B$ ☺

3. ***Proof:*** ☺

4. ***Proof:*** ☺

5. ***Proof:*** ☺

6. ***Proof:*** ☺

Question 19

Show that if A is a subset of a universal set U, then

1. $A \oplus A = \emptyset$
2. $A \oplus \emptyset = A$
3. $A \oplus U = \overline{A}$
4. $A \oplus \overline{A} = U$

Solution:

1.

$$\begin{aligned}
A \oplus A &= (A - A) \cup (A - A) && \text{Definition of Symmetric Difference} \\
&= \{x \mid (x \in A \land x \notin A) \cup (x \in A \land x \notin A)\} && \text{Definition of Difference} \\
&= \{x \mid \left(x \in A \land x \in \overline{A}\right) \cup \left(x \in A \land x \in \overline{A}\right)\} && \text{Definition of Complement} \\
&= \{x \mid \left(x \in A \cap \overline{A}\right) \cup \left(x \in A \cap \overline{A}\right)\} && \text{Definition of Intersection} \\
&= \{x \mid \emptyset \cup \emptyset\} && \text{By Second Complement Law} \\
&= \{x \mid \emptyset\} && \text{By First Idempotent Law} \\
&= \emptyset && \text{By Definition of } \emptyset
\end{aligned}$$