Limits and Continuity

Madiba Hudson-Quansah

CONTENTS

Chapter 1

Preliminaries

1.1 The Real Number system

> **Definition 1.1.1: Rational Number**
>
> An integer that can be expressed in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Rational numbers have decimal expansions that either terminate or repeat.

> **Definition 1.1.2: Irrational Number**
>
> An integer that cannot be expressed in the form $\frac{p}{q}$. Irrational numbers do not have decimal expansions that terminate or repeat.

> **Theorem 1.1.1**
>
> If a and b are real numbers and $a < b$ then:
> - For any real number c, $a + c < b + c$
> - For real numbers c and d if $c < d$, then $a + c < b + d$
> - For any real number $c > 0$, $a \times c < b \times c$
> - For any real number $c < 0$, $a \times c > b \times c$

> **Theorem 1.1.2** Triangle of Inequality
>
> For any real numbers a and b:
> - $|a \times b| = |a| \times |b|$
> - $|a + b| \neq |a| + |b|$
> - $|a + b| \leqslant |a| + |b|$

> **Example 1.1.1**
>
> > **Question 1**
> >
> > Solve the inequality
> > $$|x - 2| < 5$$

Solution:

$$|x - 2| < 5$$
$$x - 2 < 5 \quad x - 2 > -5$$
$$x < 7 \quad x > -3$$
$$\therefore -3 > x > 7$$

1.2 Lines and Functions

Definition 1.2.1: Slope of a Line

For $x_1 \neq x_2$, the slope of a straight line through the points (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

When $x_1 = x_2$ and $y_1 \neq y_2$, the line through (x_1, y_1) and (x_2, y_2) is vertical and its slope is undefined.

The slope of a line can also be described as the change in y, (Δy) divided by the change in x (Δx), or *Rise* over *Run* where rise it the change in height and run is the change in width in this case.

The equation of a line can be written in point slope form like:

$$y = m(x - x_0) + y_0$$

Where x_0 and y_0 are the starting coordinates of the line

Two (non-vertical) lines are parallel if they have the same slope, therefore any two vertical lines are parallel.
Two (non-vertical) lines of slope m_1 and m_2 are perpendicular if the product of their slopes is -1 this indicates that they are negative reciprocals of each other.

Definition 1.2.2: Function

For any two subsets A and B of the real line, a function f is a rule that assigns exactly one element y in set B to each element x in set A, where $y = f(x)$.

The set A is referred to as the *domain* of f, and the set of B is called the *range* of f, i.e. $\{y \mid y = f(x), \text{ for some } x \in A\}$

Definition 1.2.3: Polynomial

A function that can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers (coefficients) with $a_n \neq 0$ and $n \geqslant 0$ is an integer (degree)

> **Definition 1.2.4: Rational function**
>
> Any function that can be written in the form
>
> $$f(x) = \frac{p(x)}{q(x)}$$
>
> Where p and q are polynomials.

> **Theorem 1.2.1** Factor Theorem
>
> For any polynomial function, f, $f(a) = 0$ if and only if $(x - a)$ is a factor of $f(x)$

1.3 Trigonometric Functions

> **Definition 1.3.1: Periodic Function**
>
> A function f is periodic of period T if
> $$f(x + T) = f(x)$$
> For all x such that x and $x + T$ are in the domain of of f. The smallest such number $T > 0$ is called the *fundamental period.*

$$\sin(x + \frac{\pi}{2}) = \cos(x)$$

> **Definition 1.3.2**
>
> **Tangent**
> $$\tan(x) = \frac{\sin(x)}{\cos(x)}$$
>
> **Cotangent**
> $$\cot(x) = \frac{\cos(x)}{\sin(x)} \quad / \quad \cot(x) = \frac{1}{\tan(x)}$$
>
> **Secant**
> $$\sec(x) = \frac{1}{\cos(x)}$$
>
> **Cosecant**
> $$\csc(x) = \frac{1}{\sin(x)}$$

> **Definition 1.3.3: Proprieties of a periodic function**
>
> For a periodic function in the form
> $$y = Af(cx)$$
> A is the amplitude of the values generated, and for any positive integer c, the period of the function f is $\frac{T}{c}$, where T is the period of the function f.
>
> The frequency of a period function can be found:
>
> $$f = \frac{c}{T}$$

> **Theorem 1.3.1** Trigonometric Identities
>
> For any real numbers α and β, the following identities hold:
> $$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$$
> $$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
> $$\sin^2(\alpha) = \frac{1}{2}\left(1 - 2\cos(2\alpha)\right)$$
> $$\cos^2(\alpha) = \frac{1}{2}\left(1 + \cos(2\alpha)\right)$$

1.4 Transformations of Functions

> **Definition 1.4.1: Function Combinations**
>
> Suppose that f and g are functions with domains D_1 and D_2, respectively. The functions $f + g$, $f - g$ and $f \times g$ are defined by:
> $$\left(f + g\right)(x) = f(x) + g(x)$$
> $$\left(f - g\right)(x) = f(x) - g(x)$$
> $$\left(f \times g\right)(x) = f(x) \times g(x)$$
>
> $\forall x$ where $x \in D_1 \cap D_2$. The function $\frac{f}{g}$ is defined by:
> $$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$
>
> $\forall x$ where $x \in D_1 \cap D_2$ such that $g(x) \neq 0$

> **Definition 1.4.2: Composition**
>
> The composition of functions f and g is written $f \circ g$ and is defined by
> $$\left(f \circ g\right)(x) = f\left(g(x)\right)$$
>
> $\forall x$ where x is in the domain of g and $g(x)$ is in the domain of f

Chapter 2

The Concept of Limit