Vector Spaces

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Contents

Chapter 1

Vector Spaces and Subspaces

1.1 Introduction

Definition 1.1.1: Vector Space

A *vector space* is a non empty set V of objects, called vectors, on which are defined two operations, addition and multiplication by scalars, e.g. real numbers, subject to the following axioms which must hold for all vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$, is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is a zero vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

1.2 Subspaces

Definition 1.2.1: Subspace

A subset H of the vector space V, where:

1. The zero vector of V is in H.
2. H is closed under vector addition. That is for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum of $\mathbf{u} + \mathbf{v}$ is in H.
3. H is closed under scalar multiplication. That is for each $\mathbf{u}$ in H and each scalar c, the scalar multiple $c\mathbf{u}$ is in H.

1.2.1 Subspace Spanned by a Set

Theorem 1.2.1

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in vector space V, then $\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Chapter 2

Null Space, Column Space, and Linear Transformations

2.1 The Null Space of a Matrix

Definition 2.1.1: Null Space

The *null space* of an $m \times n$ matrix A, denoted by Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation:

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

Example 2.1.1

Question 1

Let A be the matrix $\begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$, and let $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$. Determine if $\mathbf{u}$ belongs to the null space of A.

Solution: This is basically asking us to verify if $\mathbf{u}$ satisfies the equation $A\mathbf{u} = \mathbf{0}$

$$\begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 - 9 + 4 \\ -25 + 27 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ $\mathbf{u}$ is in the null space of A.

Theorem 2.1.1

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$, equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

2.1.1 An Explicit Description of the Null Space of a Matrix