

# Basic Notions

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# Chapter 1

## Vector Spaces

### 1.1 Axioms of a Vector Space

#### Definition 1.1.1: Vector Spaces

A *vector space*  $V$  is a collection of objects, called *vectors*, denoted by lower-case bold letters  $\mathbf{v}$ , with two operations, addition and scalar multiplication.

These two operations must satisfy the following properties:

**Commutativity**  $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ ,  $\forall \mathbf{v} \in V$

**Associativity**  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$  for all  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$

**Zero Vector** there exists a special vector, denoted by  $\mathbf{0}$  such that  $\forall \mathbf{v} \in V, \mathbf{v} + \mathbf{0} = \mathbf{v}$

**Additive Inverse** For every vector  $\mathbf{v} \in V$  there exists a vector  $\mathbf{w} \in V$  such that  $\mathbf{v} + \mathbf{w} = \mathbf{0}$ . Such additive inverse is denoted by  $-\mathbf{v}$

**Multiplicative Identity**  $1\mathbf{v} = \mathbf{v}$ ,  $\forall \mathbf{v} \in V$

**Multiplicative Associativity**  $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$ ,  $\forall \mathbf{v} \in V$  and all scalars  $\alpha, \beta$

**Distributive Property 1**  $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ , for all  $\mathbf{u}, \mathbf{v} \in V$  and all scalars  $\alpha$

**Distributive Property 2**  $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$   $\forall \mathbf{v} \in V$  and all scalars  $\alpha, \beta$

If the scalars are real numbers, then the vector space  $V$  is called a *real vector space*, and if the scalars are complex numbers,  $V$  is a *complex vector space*.

#### Note:-

Every complex vector space is also a real vector space, but not vice versa.

The space  $\mathbb{R}^n$  consists of all columns of size  $n$ , whose entries are real numbers.

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Addition and scalar multiplication are defined as follows:

$$\alpha \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix}$$

The space  $\mathbb{C}^n$ , consists of all columns of size  $n$ , whose entries are complex numbers. Addition and scalar multiplication are defined exactly as in  $\mathbb{R}^n$ , but the scalars can now also be complex numbers, as  $\mathbb{C}^n$  is a complex vector space.

**Note:-**

Where a result is true for both complex ( $\mathbb{C}^n$ ) and real ( $\mathbb{R}^n$ ) vector spaces,  $\mathbb{F}^n$ , will be used to denote the vector space.

In the space  $M_{m \times n} / M_{m,n}$  of  $m \times n$  matrices, addition and scalar multiplication are also defined entrywise. If this vector space is over real entries, then it is a real vector space, and if it is over complex entries, then it is a complex vector space. These spaces are denoted by  $M_{m,n}^{\mathbb{R}}$  and  $M_{m,n}^{\mathbb{C}}$  respectively.

### 1.1.0.0.1 Matrix Notation

#### Definition 1.1.2: $m \times n$ Matrix

An  $m \times n$  matrix is a regular array with  $m$  rows and  $n$  columns, in which elements of such an array are called entries of the matrix.

It is common practice to denote matrix entries by indexed letters, with the first index denoting the number of the row and the second index denoted the number of the column where the entry is located. For example a  $m \times n$  matrix is written as:

$$A = (a_{j,k})_{j=1,k=1}^{m,n} = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix}$$

#### Definition 1.1.3: Transposition

Given a matrix  $A$ , its *transpose*, denoted by  $A^T$ , is defined by transforming the rows of  $A$  into columns. For example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Therefore formally, the transpose of a matrix  $A_{j,k}$  is defined as:

$$A_{j,k}^T = A_{k,j}$$

Transposition allows for column vectors to be written horizontally, i.e.,  $\mathbf{x} \in \mathbb{F}$ ,  $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$

## 1.2 Exercises

### Question 1

Let  $\mathbf{x} = (1, 2, 3)^T$ ,  $\mathbf{y} = (y_1, y_2, y_3)^T$ , and  $\mathbf{z} = (4, 2, 1)^T$ . Compute:

1.  $2\mathbf{x}$
2.  $3\mathbf{y}$
3.  $\mathbf{x} + 2\mathbf{y} - 3\mathbf{z}$

**Solution:**

1.

$$\begin{aligned} 2\mathbf{x} &= 2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \end{aligned}$$

2.

$$\begin{aligned} 3\mathbf{y} &= 3 \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\ &= \begin{pmatrix} 3y_1 \\ 3y_2 \\ 3y_3 \end{pmatrix} \end{aligned}$$

3.

$$\begin{aligned} \mathbf{x} + 2\mathbf{y} - 3\mathbf{z} &= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2y_1 \\ 2y_2 \\ 2y_3 \end{pmatrix} - \begin{pmatrix} 12 \\ 6 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 + 2y_1 - 12 \\ 2 + 2y_2 - 6 \\ 3 + 2y_3 - 3 \end{pmatrix} \\ &= \begin{pmatrix} -11 + 2y_1 \\ -3 + 2y_2 \\ 2y_3 \end{pmatrix} \end{aligned}$$

### Question 2

Which of the following sets (with natural addition and multiplication by a scalar) are vector spaces. Justify your answer.

1. The set of all continuous functions on the interval  $[0, 1]$
2. The set of all non-negative functions on the interval  $[0, 1]$
3. The set of all polynomials of degree exactly  $n$
4. The set of all symmetric  $n \times n$  matrices, i.e. the set of matrices  $A = \{a_{j,k}\}_{j,k=1}^n$  such that  $A^T = A$

**Solution:**

1.

## Chapter 2

# Linear Combinations and Bases

### 2.1 Introduction

#### Definition 2.1.1: Linear Combination

Let  $V$  be a vector space, and let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p \in V$  be a collection of vectors. A *linear combination* of vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$  is a sum of form:

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_p \mathbf{v}_p = \sum_{k=1}^p \alpha_k \mathbf{v}_k$$

#### Definition 2.1.2: Basis

A system of vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in V$ , is called a *basis*, for the vector space  $V$ , if any vector  $v \in V$  admits a *unique* representation as a linear combination

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n = \sum_{k=1}^n \alpha_k \mathbf{v}_k$$

Where the coefficients  $\alpha_1, \alpha_2, \dots, \alpha_n$  are *coordinates* of the vector  $\mathbf{v}$ , with respect to the basis.

A *basis* can also be described as the situation in which for the collection of vectors,  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ , for any possible choice of the right side  $\mathbf{v}$ , the equation  $x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \dots + x_n \mathbf{v}_n = \mathbf{v}$ , with unknowns  $x_k$ , has a unique solution. This means that the basis of a vector space  $V$  can be used to generate every possible vector found in a vector space  $V$ .

#### Example 2.1.1

In the space  $V$ , where  $V$  is  $\mathbb{F}^n$ , we have the vectors:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Where the vectors  $\mathbf{e}_k$  has all entries 0 except the entry number  $k$ , which is 1.

The system of vectors  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_n$  is a basis in  $\mathbb{F}^n$  and can be represented as the linear combination:

$$\mathbf{v} = \sum_{k=1}^n x_k \mathbf{e}_k$$

We can represent any vector in the vector space  $\mathbb{F}^n$  as this linear combination, making the system  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n \in \mathbb{F}^n$  the *standard basis* in  $\mathbb{F}^n$

### Example 2.1.2

In the space  $\mathbb{P}_n$  of the polynomials of degree at most  $n$ . The system of vectors (polynomials),  $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n \in \mathbb{P}_n$  defined by:

$$\mathbf{e}_0 := 1, \mathbf{e}_1 := t, \mathbf{e}_2 := t^2, \mathbf{e}_3 := t^3, \dots, \mathbf{e}_n := t^n$$

Any polynomial  $p, p(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$  admits a unique representation

$$p = a_0 \mathbf{e}_0 + a_1 \mathbf{e}_1 + \dots + a_n \mathbf{e}_n$$

So the system  $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n \in \mathbb{P}_n$  is a basis in  $\mathbb{P}_n$ , and is the standard basis in  $\mathbb{P}_n$

If a vector space  $V$  has a basis  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ , then any vector  $\mathbf{v} \in V$  can be uniquely defined by its coefficients in the decomposition  $\mathbf{v} = \sum_{k=1}^n \alpha_k \mathbf{v}_k$ . We can then stack the coefficients  $\alpha_k$  in a column, rendering it a column vector giving us the usual vector operations, i.e. if  $\mathbf{v} = \sum_{k=1}^n \alpha_k \mathbf{v}_k$  and  $\mathbf{w} = \sum_{k=1}^n \beta_k \mathbf{v}_k$

$$\mathbf{v} + \mathbf{w} = \sum_{k=1}^n \alpha_k \mathbf{v}_k + \sum_{k=1}^n \beta_k \mathbf{v}_k = \sum_{k=1}^n (\alpha_k + \beta_k) \mathbf{v}_k$$

This means to get the column of coordinates of the sum, we just need to add the column of coordinates of the vectors being summed,  $\alpha_k + \beta_k$ , and similarly to get the column of coordinates of a product,  $\alpha \mathbf{v}$ , we just need to multiply the column of coordinates of  $\mathbf{v}$  by  $\alpha$

## 2.2 Generating and Linearly Independent Systems

The definition of a basis posits that any vector can be uniquely expressed as a linear combination. Breaking this statement down we get two central ideas:

- There exists a linear combination representation of any vector
- The resultant linear combination is unique for that specific vector

### 2.2.1 Existence / Generating System

Looking at the first idea of existence we get the notion of *Generating systems*

#### Definition 2.2.1: Generating / Spanning / Complete System

A system of vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p \in V$  is termed as a *generating system* in  $V$  if any vector  $\mathbf{v} \in V$  can be represented as a linear combination

$$\mathbf{v} = \sum_{k=1}^p \alpha_k \mathbf{v}_k$$

This definition although similar to that of a basis differs in the fact that we do not assume that this representation is unique to the vector  $\mathbf{v}$ .

### 2.2.2 Uniqueness / Linearly Independent System

Now only looking at uniqueness, we consider the zero vector  $\mathbf{0}$ , which is always representable as a linear combination.

**Definition 2.2.2: Triviality**

A linear combination  $\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_p \mathbf{v}_p$  is termed as *trivial* if  $\alpha_k = 0 \forall k$ .  
As a result any trivial linear combination is equal to  $\mathbf{0}$ .

**Definition 2.2.3: Linearly Independent**

A system of vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p \in V$  is termed as *linearly independent* if and only if the trivial linear Combination

$$\sum_{k=1}^p \alpha_k \mathbf{v}_k \text{ with } \alpha_k = 0 \forall k$$

of vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  is equal to  $\mathbf{0}$ .

In other words the system of equations  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  is linearly independent iff all the coordinates are equal to 0, i.e.,  $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

In the same vein if a system is not linearly independent, it is termed as *linearly dependent*.

**Definition 2.2.4: Linearly Dependent**

A system of vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$  is *linearly dependent* if  $\mathbf{0}$  can be represented as a non-trivial linear combination  $\sum_{k=1}^p \alpha_k \mathbf{v}_k$ , where non-trivial means  $\exists k, \alpha_k \neq 0$ , and thus,  $\sum_{k=1}^p \alpha_k \neq 0$ .

Therefore a system of vectors is *linearly dependent* iff there exists scalars  $\alpha_1, \alpha_2, \dots, \alpha_p, \sum_{k=1}^p \alpha_k \neq 0$  such that

$$\mathbf{0} = \sum_{k=1}^p \alpha_k \mathbf{v}_k$$