Mathematical Induction

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Contents

Chapter 1

Mathematical Induction

1.1 Introduction

Definition 1.1.1: Mathematical Induction

Mathematical induction can be used to prove statements that assert $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function. A proof by mathematical induction has two parts, a **basis step**, where we show that $P(1)$ is true and an **inductive step**, where we show that for all positive integers k if $P(k)$ is true then $P(k+1)$ is also true.

Mathematical Induction can be expressed as the rule of inference $P(1) \wedge (P(k) \to P(k+1)) \to \forall n P(n)$, where the domain is the set of positive integers.

Example 1.1.1

Question 1

Show that

$$P(n) : 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Solution: We first start with the basis step, $P(1)$, which is

$$\begin{aligned} P(1) : 1 &= \frac{1(1+1)}{2} \\ 1 &= 1 \end{aligned}$$

$\therefore P(1)$ is true.
Next is the inductive step, where we need to show that $P(k) \to P(k+1)$
We assume

$$P(k) : 1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$$

and prove

$$P(k+1) : 1 + 2 + 3 + \ldots + (k+1) = \frac{(k+1)(k+2)}{2}$$

To prove this we need to relate these two predicates. And because $P(k+1)$ can be expressed as $P(k) + (k+1)$ we can write

$$P(k+1) : 1 + 2 + 3 + \ldots + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

So we simplify

$$\frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2}$$
$$= \frac{(k+1)(k+2)}{2}$$

Example 1.1.2

Question 2

Show that

$$P(n) : 1 + 3 + 5 + \ldots + (2n-1) = n^2 \ \forall n \in \mathbb{Z}$$

Solution: The basis step is $P(1)$

$$P(n) : 1 = 1^2$$

Assume

$$P(k) : 1 + 3 + 5 + \ldots + (2k-1) = k^2$$

and show that

$$P(k+1) : 1 + 3 + 5 + \ldots + (2(k+1)-1) = (k+1)^2$$

$P(k+1)$ can be expressed as $P(k) + (2k+1)$

$$P(k+1) : 1 + 3 + 5 \ldots + (2k-1) + (2k+1)$$
$$= k^2 + 2k + 1$$
$$= (k+1)^2$$

$\therefore P(k+1)$ is true

Conclusion: Having completed the basis and induction steps we can conclude that $1 + 3 + 5 + \ldots + (2n-1) = n^2$

1.2 Why Mathematical Induction is Valid

Definition 1.2.1: Well Ordering Property of Positive Integers

This axiom states that every non-empty subset of positive integers has a least element

The validity of mathematical induction as a proof technique comes from the well ordering property, as an axiom of positive integers. So suppose we know that $P(1)$ is true for all positive integers n and that the proposition $P(k) \longrightarrow P(k+1)$ is true for all positive integers k. To show that $P(n)$ must be true for all positive integers n we can use a proof by contradiction.

We first assume that there is at least positive integer for which $P(n)$ is false, i.e. $\exists x \in \mathbb{Z}^+ \neg P(x)$

Then the set of positive integers for which $P(n)$ is false, let this be S, is non-empty. Thus by the well ordering property, S has a least element, which we will denote as m.

We know that m cannot be 1, because $P(1)$ is true, and since m is a positive integer it must be greater than one ergo $m-1$ is also a positive integer.

Again due to well ordering property we know the least value in S is m therefore $m-1$ is not in set S, as a result $P(m-1)$ is true.

Because the statements $P(m-1) \longrightarrow P(m)$ is also true, it must be the case that $P(m)$ is true, which is a contradiction of the statement that $P(m)$ must be false.

Hence $P(n)$ must be true for every positive integer.

1.3 Examples of Proofs by Mathematical Induction

1.3.1 Proving Summation Formulae

Question 3

Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathematical induction.

Solution: The sums of the first n positive odd integers for $n = 1, 2, 3, 4, 5$ are

$$\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ 1+3+5 &= 9 \\ 1+3+5+7 &= 16 \\ 1+3+5+7+9 &= 25 \end{aligned}$$

From these values it is reasonable to conjecture that the sum of the first n odd integers is n^2, i.e.

$$1+3+5+\ldots+(2n-1) = n^2$$

Let $P(n)$ be $1+3+5+\ldots+(2n-1) = n^2$

Basis Step:

$$P(1) : 1 = 1^2$$

Inductive Step:

To complete this step we need to prove that $\forall k \in \mathbb{Z}^+ \ (P(k) \longrightarrow P(k+1))$

Assume $\exists k \in Z^+ \ P(k) = T$, then

$$P(k) : 1+3+5+\ldots+(2k-1) = k^2$$

$$P(k+1): 1+3+5+\ldots+(2k+1) = (k+1)^2$$
$$P(k+1): 1+3+5+\ldots+(2k-1)+(2k+1) = (k+1)^2$$

$$\begin{aligned} P(k+1) &: P(k)+(2k+1) \\ 1+3+5+\ldots+(2k+1) &= k^2+2k+1 \\ &= (k+1)(k+1) \\ &= (k+1)^2 \end{aligned}$$

$\therefore P(k+1)$ is T
Hence we can conclude that $P(n)$ is true for all positive integers n.

Question 4

Use mathematical induction to show that

$$1+2+2^2+\ldots+2^n = 2^{n+1}-1$$

For all non-negative integers n

Solution:

Let $P(n)$ be $1+2+2^2+\ldots+2^n = 2^{n+1}-1$

Basis Step:

$$\begin{aligned} P(0): 2^0 &= 2^{0+1}-1 \\ 1 &= 2-1 \\ 1 &= 1 \end{aligned}$$

Induction Step

To complete this step I must prove $P(k) \longrightarrow P(k+1)$ where k is any non-negative integer.

Assume $P(k)$ is true for some non-negative integer k, then

$$P(k): 1+2+2^2+\ldots+2^k = 2^{k+1}-1$$

Then $P(k+1)$ is

$$P(k+1): 1+2+2^2+\ldots+2^k+2^{k+1} = 2^{k+2}-1$$

$P(k+1)$ can also be expressed as $P(k+1) = P(k) + 2^{k+1}$

$$
\begin{aligned}
P(k+1): 1+2+2^2+\ldots+2^{k+1} &= 2^{k+1}-1+2^{k+1} \\
&= 2^k \times 2^1 + 2^k \times 2^1 - 1 \\
&= 2^k\left(2^1+2^1\right)-1 \\
&= 2^k\left(2^2\right)-1 \\
&= 2^{k+2}-1
\end{aligned}
$$

$\therefore P(k+1)$ is T
Hence we can conclude that $P(n)$ is true for all non-negative integers n.

1.3.1.0.1 Sums of Geometric Progressions

Question 5

Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression with initial term a and a common ratio r

$$
\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \ldots + ar^n = \frac{ar^{n+1}-a}{r-1} \quad \text{where } r \neq 1
$$

where n is a non-negative integer.

Solution: Let $P(n)$ be "the sum of the first $n+1$ terms of a geometric progression in this formula is correct".

Basis Step

$$
\begin{aligned}
P(0): \sum_{j=0}^{0} ar^j &= \frac{ar^{0+1}-a}{r-1} \\
a &= \frac{ar-a}{r-1} \\
a &= \frac{a(r-1)}{r-1} \\
a &= a
\end{aligned}
$$

Induction Step

To complete this step I must prove $P(k) - P(k+1)$ where k is any non-negative integer.

Assume $P(k)$ for some non-negative integer k, then

$$
P(k): \sum_{j=0}^{k} ar^j = a + ar + ar^2 + \ldots + ar^k = \frac{ar^{k+1}-a}{r-1}
$$

Then $P(k+1)$ is

$$
P(k+1): \sum_{j=0}^{k+1} ar^j = a + ar + ar^2 + \ldots + ar^{k+1} = \frac{ar^{k+2}-a}{r-1}
$$

And can be expressed as $P(k+1) = P(k) + ar^{k+1}$, therefore:

$$\begin{aligned} P(k+1) : \sum_{j=0}^{k+1} ar^j = a + ar + ar^2 + \ldots + ar^k + ar^{k+1} &= \frac{ar^{k+1} - a}{r-1} + ar^{k+1} \\ &= \frac{ar^{k+1} - a}{r-1} + \frac{ar^{k+1}}{1} \\ &= \frac{ar^{k+1} - a + \left(ar^{k+1}\right)(r-1)}{r-1} \\ &= \frac{ar^{k+1} + ar^{k+2} - ar^{k+1} - a}{r-1} \\ &= \frac{ar^{k+2} - a}{r-1} \end{aligned}$$

$\therefore P(k+1)$ is T

Hence we can conclude that $P(n)$ is true for all non-negative integers n

1.4 Exercises

Question 6

Prove that

$$3 + 3 \times 5 + 3 \times 5^2 + \ldots + 3 \times 5^n = \frac{3\left(5^{n+1} - 1\right)}{4}$$

whenever n is non-negative integer

Solution:

Let $P(n)$ be $3 + 3 \times 5 + 3 \times 5^2 + \ldots + 3 \times 5^n = \frac{3\left(5^{n+1}-1\right)}{4}$

Basis Step

$$\begin{aligned} P(0) : 3 &= \frac{3\left(5^1 - 1\right)}{4} \\ 3 &= \frac{12}{4} \\ 3 &= 3 \end{aligned}$$

Induction Step

To complete this step I must prove $P(k) \longrightarrow P(k+1)$ for any non-negative integer k

Assume $P(k)$ is T for some non-negative integer k, then

$$P(k) : 3 + 3 \times 5 + 3 \times 5^2 + \ldots + 3 \times 5^k = \frac{3\left(5^{k+1} - 1\right)}{4}$$

Then $P(k+1)$ is

$$P(k+1): 3 + 3\times 5 + 3\times 5^2 + \ldots + 3\times 5^{k+1} = \frac{3\left(5^{k+2}-1\right)}{4}$$

And can be expressed as $P(k+1) = P(k) + 3\times 5^{k+1}$, therefore:

$$\begin{aligned}
P(k+1): 3 + 3\times 5 + 3\times 5^2 + \ldots + 3\times 5^{k} + 3\times 5^{k+1} &= \frac{3\left(5^{k+1}-1\right)}{4} + \left(3\times 5^{k+1}\right)\\
&= \frac{3\left(5^{k+1}-1\right) + 4\left(3\times 5^{k+1}\right)}{4}\\
&= \frac{3\times 5^{k+1} - 3 + 12\times 5^{k+1}}{4}\\
&= \frac{3\left(5^{k+1} + 4\times 5^{k+1} - 1\right)}{4}\\
\text{Let } x = 5^{k+1} &\\
&= \frac{3\left(x + 4x - 1\right)}{4}\\
&= \frac{3\left(5x - 1\right)}{4}\\
&= \frac{3\left(5^1\times 5^{k+1}\right) - 1}{4}\\
&= \frac{3\left(5^{k+2}-1\right)}{4}
\end{aligned}$$

$\therefore P(k+1)$ is T
Hence we can conclude that $P(n)$ is true for all non-negative integers n

Question 7

Prove that

$$1\times 1! + 2\times 2! + \ldots + n\times n! = (n+1)! - 1$$

Whenever n is a positive integer.

Solution:

Let $P(n)$ be $1\times 1! + 2\times 2! + \ldots + n\times n! = (n+1)! - 1$

Basis Step

$$\begin{aligned}
P(1): 1\times 1! &= (1+1)! - 11 &= 2 - 1\\
1 &= 1
\end{aligned}$$

Induction Step

To conclude this step I need to prove $P(k) \rightarrow P(k+1)$ for any positive integer k

Assume $P(k)$ is T for some positive integer k, then

$$P(k) : 1 \times 1! + 2 \times 2! + \ldots + k \times k! = (k+1)! - 1$$

Then $P(k+1)$ is:

$$P(k+1) : 1 \times 1! + 2 \times 2! + \ldots + (k+1) \times (k+1)! = (k+2)! - 1$$

And can be expressed as $P(k+1) : P(k) + (k+1) \times (k+1)!$, therefore

$$\begin{aligned} P(k+1) : 1 \times 1! + 2 \times 2! + \ldots + k \times k! + (k+1) \times (k+1)! &= ((k+1)! - 1) + ((k+1) \times (k+1)!) \\ &= (k+1)! + (k+1) \times (k+1)! - 1 \\ &= (k+1)!\,(k+2) - 1 \\ &= (k+2)! - 1 \end{aligned}$$

Question 8

1. Find a formula for
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n}$$
by examining the values of this expression for small values of n
2. Prove the formula you conjectured in part 1.

Solution:

1.

$$\begin{aligned} S_n &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} \\ S_1 &= \frac{1}{2} \\ S_2 &= \frac{3}{4} \\ S_3 &= \frac{7}{8} \\ S_4 &= \frac{15}{16} \\ S_n &= \frac{2^n - 1}{2^n} \end{aligned}$$

2. ***Proof:*** Let $P(n)$ be $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$

Basis Step

$$\begin{aligned} P(1) : \frac{1}{2} &= \frac{2^1 - 1}{2^1} \\ \frac{1}{2} &= \frac{1}{2} \end{aligned}$$

Induction Step

To complete this step I need to prove $P(k) \longrightarrow P(k+1)$ for any positive integer k.

Assume $P(k)$ is T for some positive integer k. Then:

$$P(k) : \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^k} = \frac{2^k - 1}{2^k}$$

Then $P(k+1)$ is:

$$P(k+1) : \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}}$$

And can be expressed as $P(k+1) : P(k) + \frac{1}{2^{k+1}}$, therefore

$$\begin{aligned} P(k+1) : \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^{k+1}} &= \frac{2^k - 1}{2^k} + \frac{1}{2^{k+1}} \\ &= \frac{\left(2^k - 1\right)\left(2^{k+1}\right) + 2^k}{\left(2^k\right)\left(2^{k+1}\right)} \end{aligned}$$

😆

Question 9

Show that 3 divides $n^3 + 2n$, whenever $n \in \mathbb{Z}^+$

Proof: This has the same meaning as:

”$n^3 + 2n$ is divisible by 3”

”$n^3 + 2n$ is a multiple of 3”

Let $P(n)$ be $n^3 + 2n$ is a multiple of 3

Basis Step

$$P(1) : 1^3 + 2 \times 1 \text{ is a multiple of } 3$$

True because 3 is a multiple of three.

Induction Step

Assume $P(k)$ is T then

”$k^3 + 2k$ is multiple of 3 ” - $\exists m \in \mathbb{Z}, k^3 + 2k = 3m$

And show $P(k+1)$ is

”$(k+1)^3 + 2(k+1)$ is a multiple of 3”

We have

$$
\begin{aligned}
(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\
&= \left(k^3 + 2k\right) + 3k^2 + 3k + 3 \\
&= 3m + 3k^2 + 3k + 3 \\
&= 3\left(m + k^2 + k + 1\right) \\
\text{Let } z = m + k^2 + k + 1 \\
&= 3z
\end{aligned}
$$

Since z is made up of positive integers z is a positive integer
$\therefore P(k+1)$
Hence we can conclude that $P(n)$ is true for all positive integers.

😄

Chapter 2

Exercises

Question 10

Use mathematical induction to prove that 43 divides $6^{n+1} + 7^{2n-1}$ for every positive integer n

Proof: The given statements can be written as "$6^{n+1} + 7^{2n-1}$ is a multiple of 43".
Let $P(n)$ be $6^{n+1} + 7^{2n-1}$ is a multiple of 43.

Basis Step

$$\begin{aligned} P(1) &: 6^{1+1} + 7^{2\times 1-1} \text{ is a multiple of } 43 \\ &: 43 \text{ is a multiple of } 43 \end{aligned}$$

$P(1)$ is true as 43 is a multiple of 43.

Induction Step

Assume $P(k)$ is T, then:

"$6^{k+1} + 7^{2k-1}$" is a multiple of 43", means $\exists m \in \mathbb{Z}, 6^{k+1} + 7^{2k-1} = 43m$

I must now show that $P(k+1)$: "$6^{k+2} + 7^{2(k+1)-1}$" is a multiple of 43.

$$
\begin{aligned}
6^{k+2} + 7^{2(k+1)-1} &= 6^{k+2} + 7^{2k+1} \\
&= 6^{k+1+1} + 7^{2k+2-1} \\
&= 6^{k+1} \times 6^{1} + 7^{2k-1} \times 7^{2} \\
&= 6^{k+1} \times 6^{1} + 7^{2k-1} \times 49 \\
&= 6^{k+1} \times 6^{1} + 7^{2k-1} \times 6 + 7^{2k-1} \times 43 \\
&= 6\left(6^{k+1} + 7^{2k-1}\right) + 7^{2k-1} \times 43 \\
&= 6\,(43m) + 7^{2k-1} \times 43 \\
&= 43\left(6m + 7^{2k-1}\right) \\
\text{Let } u = 6m + 7^{2k-1} & \\
&= 43u
\end{aligned}
$$

Since u is made up of integers u is an integer
$\therefore\ P(k+1)$ is T.
Hence I can conclude $P(n)$ is true for all positive integers

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Question 11

Prove by induction that $\sum_{j=0}^{n} \left(-\frac{1}{2}\right)^{j} = \frac{2^{n+1}+(-1)^{n}}{3\times 2^{n}}$, whenever n is a non-negative integer

Proof: Let $P(n)$ be:

$$1 + \left(-\frac{1}{2}\right)^{1} + \left(-\frac{1}{2}\right)^{2} + \ldots + \left(-\frac{1}{2}\right)^{n} = \frac{2^{n+1} + (-1)^{n}}{3 \times 2^{n}}$$

Basis Step

$$
\begin{aligned}
P(0) : 1 &= \frac{2+1}{3\times 1} \\
1 &= 1
\end{aligned}
$$

$\therefore P(0)$ is T

Induction Step

To complete this step I must prove $P(k) \longrightarrow P(k+1)$ for every non non-negative integer k

Assume $P(k)$ is true for some post integer k, then:

$$P(k) : 1 + \left(-\frac{1}{2}\right)^{1} + \left(-\frac{1}{2}\right)^{2} + \ldots + \left(-\frac{1}{2}\right)^{k} = \frac{2^{k+1} + (-1)^{k}}{3 \times 2^{k}}$$

Then $P(k+1)$ is:

$$P(k+1) : 1 + \left(-\frac{1}{2}\right)^{1} + \left(-\frac{1}{2}\right)^{2} + \ldots + \left(-\frac{1}{2}\right)^{k+1} = \frac{2^{k+2} + (-1)^{k+1}}{3 \times 2^{k+1}}$$

And can be expressed as $P\left(k+1\right): P\left(k\right) + \left(-\frac{1}{2}\right)^{k+1}$, therefore:

$$\begin{aligned}
P\left(k+1\right): 1 + \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)^{2} + \ldots + \left(-\frac{1}{2}\right)^{k} + \left(-\frac{1}{2}\right)^{k+1} &= \frac{2^{k+1} + (-1)^{k}}{3 \times 2^{k}} + \left(-\frac{1}{2}\right)^{k+1} \\
&= \frac{2^{2k+2} + (-1)^{k} \times 2^{k+1} - 3 \times 2^{k} \times 1^{k+1}}{3 \times 2^{2k+1}} \\
&= \frac{2^{k}\left(2^{k+2} + (-1)^{k} \times 2 - 3\right)}{2^{k} \times 3 \times 2^{k+1}} \\
&= \frac{2^{k+2} + (-1)^{k} \times 2 - 3}{3 \times 2^{k+1}}
\end{aligned}$$

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