Vector Valued functions of one variable

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Contents

Chapter 1

Paths and Curves

Describing curves in $\mathbb{R}^n$ space, formally does not involve a defining equation which is intuitive in $\mathbb{R}^2$ and $\mathbb{R}^3$ but rather by how it it swept out like the trace of a skywriter. Thus the characteristics of a curve are not only given by its length, but also by the tracer's velocity and acceleration.

1.1 Parametrizations

Definition 1.1.1: Path

Let I be an interval of real numbers, typically $I = [a, b], (a, b)$ or $\mathbb{R}$.
A continuous function $\alpha : I \longrightarrow \mathbb{R}^n$, is called a **path**. As t varies over I, $\alpha(t)$ traces out a **curve**, C, i.e.

$$C = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \alpha(t) \ \exists t \in I\}$$

This is also called the **image** of α, we then say that α **parametrizes** the curve C. We often refer to the input variable t as the **time** and $\alpha(t)$ as the **position** of a moving object at time t.

A path is a vector valued function of one variable, i.e. t. For each t in I, $\alpha(t)$ is a point in $\mathbb{R}^n$, so can be written $\alpha(t) = (x_1(t), x_2(t), \ldots x_n(t))$, where each of the n coordinates is a real number that depends on t.

Example 1.1.1 (Circles in $\mathbb{R}^2$)
I.e. $x^2 + y^2 = a^2$, where a is the radius of the circle. This can be rewritten:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

To parametrize a circle we use the identity $\cos^2 t + \sin^2 t = 1$, and let $\frac{x}{a} = \cos(t)$ and $\frac{y}{a} = \sin(t)$, or $x = a\cos(t)$ and $y = a\sin(t)$, Thus we have the parametrization:

$$\alpha(t) = (a\cos(t), a\sin(t)), \ a > 0, \ t \in \mathbb{R}$$

Example 1.1.2 (Graphs of $y = f(x)$)
Consider the curve described by $y = \sin(x)$ in $\mathbb{R}^2$, where $0 \leqslant x \leqslant \pi$, it consists of points of the form $(x, \sin(x))$, where x varies over the interval $[0, \pi]$. x can be used as a parameter, thus we can write the parametrization as:

$$\alpha(x) = (x, \sin(x))$$

1.2 Velocity, Acceleration, Speed and Arc Length

Definition 1.2.1: Derivative of a path

Given a path $\alpha : I \longrightarrow \mathbb{R}^n$, the **derivative** of α is defined by

$$\begin{aligned}\alpha'(t) &= \lim_{h\to 0}\frac{\alpha(t+h)-\alpha(t)}{h}\\ &= \lim_{h\to -0}\left(\frac{1}{h}\left((x_1(t+h), x_2(t+h), \ldots, x_n(t+h)) - (x_1(t), x_2(t), \ldots, x_n(t))\right)\right)\\ &= \lim_{h\to 0}\left(\frac{x_1(t+h)-x_1(t)}{h}, \frac{x_2(t+h)-x_2(t)}{h}, \ldots, \frac{x_n(t+h)-x_n(t)}{h}\right)\\ &= \left(x_1'(t), x_2'(t), \ldots x_n'(t)\right)\end{aligned}$$

Provided the limit exists. The derivative is also called the **velocity** ($\mathbf{v}(t)$) of α. And thus α'' is the **acceleration** ($\mathbf{a}(t)$) of α.

For some time t_0, and for any time t, let $s(t)$ be the distance travelled by the path from $\alpha(t_0)$ to $\alpha(t)$, $s(t)$ is called the **arclength** function and the magnitude of it's derivative is the speed.

Definition 1.2.2: Speed

If $\alpha : I \longrightarrow \mathbb{R}^n$ is a differentiable path, then its speed, denoted bu $v(t)$, is defined:

$$v(t) = \|\mathbf{v}(t)\|$$

Where $v(t)$ is a scalar quantity.

Definition 1.2.3: Arclength

The arclength from $t = a$ to $t = b$ is defined to be

$$s(t) = \int_a^b v(t)\ dt = \int_a^b \|\mathbf{v}(t)\|\ dt$$