Homework 3

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Question 1

Is $\lambda = -2$ an eigenvalue of

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix}$$

If so, find one corresponding eigenvector.

Solution:

If $\lambda = -2$, then $A\mathbf{x} = -2\mathbf{x}$

$$\begin{aligned} A + 2I &= \\ &= \begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 & -1 \\ 1 & -1 & 0 \\ 4 & -13 & 3 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 3 & 0 & -1 \\ 1 & -1 & 0 \\ 4 & -13 & 3 \end{bmatrix}$$

$$\frac{1}{3}R_1 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & \frac{-1}{3} \\ 4 & -13 & 3 \end{bmatrix}$$

$$\frac{4}{3}R_1 - R_3 \longrightarrow R_3$$

$$\begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & \frac{-1}{3} \\ 0 & 13 & \frac{-13}{3} \end{bmatrix}$$

$$13R_2 - R_3 \longrightarrow R_3$$

$$\begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & \frac{-1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{3}R_1 \longrightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & \frac{-1}{3} \\ 0 & 1 & \frac{-1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - \frac{1}{3}x_3 = 0$$

$$x_2 - \frac{1}{3}x_3 = 0$$

$$x_3 = x_3$$

$$x_1 = \frac{1}{3}x_3$$

$$x_2 = \frac{1}{3}x_3$$

$$x_3 = x_3$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x_3 \\ \frac{1}{3}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

Therefore $\lambda = -2$ is an eigenvector of the matrix and an eigenvector is $\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$ when $x_3 = 1$

Question 2

Diagonalize the following matrix if possible:

$$\begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}$$

Solution: The matrix is diagonalizable if and only if the matrix has 2 linearly independent eigenvectors. To prove this I must first find the eigenvalues of the matrix

$$
\begin{aligned}
\det(A - \lambda I) &= 0 \\
A - \lambda I &= \begin{bmatrix} -2-\lambda & 12 \\ -1 & 5-\lambda \end{bmatrix} \\
(-2-\lambda)(5-\lambda) - (-12) &= 0 \\
-10 + 2\lambda - 5\lambda + \lambda^2 + 12 = 0 \\
\lambda^2 - 3\lambda + 2 &= 0 \\
(\lambda - 2)(\lambda - 1) &= 0 \\
\lambda &= 2 \\
\lambda &= 1
\end{aligned}
$$

Next I find the corresponding eigenvectors of the eigenvalues $\lambda = 2, 1$

$$
\begin{aligned}
\lambda &= 2 \\
A - 2I &= \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \\
\frac{1}{4}R_1 - R_2 &\longrightarrow R_2 \\
&\begin{bmatrix} -4 & 12 \\ 0 & 0 \end{bmatrix} \\
\frac{-1}{4}R_1 &\longrightarrow R_1 \\
&\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \\
x_1 &= 3x_2 \\
x_2 &= x_2 \\
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \end{bmatrix} &= x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\lambda &= 1 \\
A - I &= \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \\
\frac{1}{3}R_1 - R_2 &\longrightarrow R_2 \\
&\begin{bmatrix} -3 & 12 \\ 0 & 0 \end{bmatrix} \\
\frac{-1}{3}R_1 &\longrightarrow R_1 \\
&\begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \\
x_1 &= 4x_2 \\
x_2 &= x_2 \\
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= x_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix}
\end{aligned}
$$

Therefore:

$$P = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$$

And

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AP = PD$$

$$\begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 4 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & 1 \end{bmatrix}$$