Systems of Linear Equations

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Chapter 1

Introduction

Definition 1.0.1: Linear Equation

An equation in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where the constant b and coefficients $a_1, a_2, \ldots, a_n$ are real or complex numbers.

Definition 1.0.2: System of Linear Equations

A collection of one or more linear equations involving the same set of variables. When a system of linear equations is written in the form

$$\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 &= b_1 \\ a_4x_1 + a_6x_3 &= b_2 \end{aligned}$$

The set of variables takes on the longest subscript in the system. In this case, the variables are x_1, x_2, x_3.

Definition 1.0.3: Solution of a System of Linear Equations

The solution of a system of linear equations is a list of values, $(s_1, s_2, \ldots, s_n)$ that makes each equation in the system a true statement when the values are substituted for the variables, i.e. $x_1, x_2, \ldots, x_n$ and $s_1, s_2, \ldots, s_n$, where s_n is substituted for x_n

Definition 1.0.4: Solution Set

The set of all possible solutions of a system of linear equations.

Definition 1.0.5: Equivalence

Two linear systems are said to be equivalent if they have the same solution set.

Definition 1.0.6: Consistency

A system of linear equations is said to be consistent if it has at least one solution, and inconsistent if it has no solution.

A system of linear equations can either have:

- No solution - Equations do not intersect
- Exactly one / Unique solution - Equations intersect at a single point
- Infinitely many solutions - Equations are the same

1.1 Matrix Notation

A system of linear equations can be represented in matrix form two ways:

- Coefficient Matrix
- Augmented Matrix

1.1.1 Coefficient Matrix

Definition 1.1.1: Coefficient Matrix

Denoted by A, the coefficient matrix is a matrix that contains the coefficients of the variables in the system of linear equations with the coefficients of each equation making up each row.

Example 1.1.1

For the system of linear equations:

$$\begin{aligned} a_1x_2 + a_2x_2 + a_3x_3 &= b_1 \\ a_4x_1 + a_5x_2 + a_6x_3 &= b_2 \\ a_7x_1 + a_8x_2 + a_9x_3 &= b_3 \end{aligned}$$

The coefficient matrix is:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$$

1.1.2 Augmented Matrix

Definition 1.1.2: Augmented Matrix

Denoted by $[A|B]$, the augmented matrix is a matrix that contains the coefficients of the variables in the system of linear equations with the constant terms of each equation making up the last column.

Example 1.1.2

For the system of linear equations:

$$\begin{aligned} a_1x_2 + a_2x_2 + a_3x_3 &= b_1 \\ a_4x_1 + a_5x_2 + a_6x_3 &= b_2 \\ a_7x_1 + a_8x_2 + a_9x_3 &= b_3 \end{aligned}$$

The augmented matrix is:

$$\begin{bmatrix} a_1 & a_2 & a_3 & b_1 \\ a_4 & a_5 & a_6 & b_2 \\ a_7 & a_8 & a_9 & b_3 \end{bmatrix}$$

Definition 1.1.3: Size of a Matrix

The size of a matrix, denoted by $m \times n$, is the number of rows and columns in the matrix respectively. If $n = m$ then the matrix is said to be square, if not, it is said to be rectangular.

1.2 Solving Linear Systems

Definition 1.2.1: Pivot

Diagonal non-zero elements of a linear system

Definition 1.2.2: Forward Elimination Process

The process used to change a system into an upper triangular matrix

Definition 1.2.3: Backward Substitution Method

The process of deriving a solution from an upper triangular matrix

Definition 1.2.4: Identity Matrix

A matrix containing all zeros with pivots of 1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

One procedure used to solve linear systems is that of simplification. This involves replacing one linear system with a simpler equivalent system. This is done by applying the following operations to the system:

Replacement Replace one equation by the sum of itself and a multiple of another equation.

Interchange Interchange two equations.

Scaling Multiply all the terms in an equation by a non-zero constant.

Example 1.2.1

Question 1

Solve the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

Solution: Using the augmented matrix representation, we have:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

Then we times the first equation through by -5 and add it to the third equation to replace the third equation:

$$\begin{array}{r} -5x_1 + 10x_2 - 5x_3 = 0 \\ 5x_1 - 5x_3 = 10 \\ \hline 10x_2 - 10x_3 = 10 \end{array}$$

Giving us:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

We then eliminate x_2 by multiplying equation 2 by -5 and add it again to the third equation again replacing it:

$$\begin{array}{r} -10x_2 + 40x_3 = -40 \\ 10x_2 - 10x_3 = 10 \\ \hline 30x_3 = -30 \end{array}$$

Giving us:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

This new system has a triangular form, i.e.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 30x_3 &= -30 \end{aligned}$$

We then continue eliminating variables until one remains in each equation:

$$\begin{array}{r} -x_3 = 1 \\ x_1 - 2x_2 + x_3 = 0 \\ \hline x_1 - 2x_2 = 1 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

$$\begin{array}{r} 8x_3 = -8 \\ 2x_2 - 8x_3 = 8 \\ \hline 2x_2 = 0 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

$$\begin{array}{r} 2x_2 = 0 \\ x_1 - 2x_2 = 1 \\ \hline x_1 = 1 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

Giving us the system:

$$\begin{aligned} x_1 &= 1 \\ 2x_2 &= 0 \\ 30x_3 &= -30 \end{aligned}$$

Which simplifies into:

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 0 \\ x_3 &= -1 \end{aligned}$$

Definition 1.2.5: Row Equivalence

Two matrices are row equivalent if there is a sequence of elementary row operations that transforms one matrix into the other

Theorem 1.2.1

If the augmented matrices of two linear systems are row equivalent, then the two equations have the same solution set.

1.3 Identifying Existence and Uniqueness

Definition 1.3.1: Free Variable

A variable that does not exist in a row of a matrix

Definition 1.3.2: Parametric Equations

Any equation that expresses the variables in a system of linear equations in terms of a free variable.

Definition 1.3.3: Basic Variable

A variable that exists in a row of a matrix

To determine the nature of a linear system we must answer two fundamental questions:

- Is the system consistent? / Does a solution exist?
- If a solution exists, is it the only one? / Is the solution unique

Example 1.3.1

Question 2

Determine if the following system is consistent:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

Solution: Having already found the solution for this system:

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 0 \\ x_3 &= -1 \end{aligned}$$

We can determine that a solution exists, and due to the fact x_2 is uniquely determined by equation two, x_3 has only one possible value, and x_1 is also uniquely determined by equation one, we can also conclude this solution is unique.

Example 1.3.2

Question 3

Determine if the following system is consistent:

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned}$$

Solution: The augmented matrix is:

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

We interchange equations 1 and 2:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

$$\begin{array}{r} -4x_1 + 6x_2 - 4x_3 = -2 \\ \underline{4x_1 - 8x_2 + 12x_3 = 1} \\ -2x_2 + 8x_3 = -1 \end{array}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{bmatrix}$$

$$\begin{array}{r} 2x_2 - 8x_3 = 16 \\ \underline{-2x_2 + 8x_3 = -1} \\ 0 = 15 \end{array}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{bmatrix}$$

Now in its triangular form, we can determine the existence and uniqueness of the solutions:

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \\ 0 &= 15 \end{aligned}$$

Since there are no coefficients for x_1, x_2, and x_3 in equation 3 equation 3 has no solution. This makes the solution set for this linear system $\{1, 8\}$. Because this set is the same as the solution set for the original linear system, $\{8, 1, 1\}$, the original system is inconsistent

1.4 Exercises

Question 4

Determine if the linear system represented by the augmented matrix below is consistent:

$$\begin{bmatrix} 1 & 5 & 2 & -6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

Solution:

$$\begin{aligned} x_1 + 5x_2 + 2x_3 &= -6 \\ 4x_2 - 7x_3 &= 2 \\ 5x_3 &= 0 \end{aligned}$$

$$\begin{aligned} x_3 &= 0 \\ x_1 + 5x_2 &= -6 \\ x_1 &= -6 - 5x_2 \end{aligned}$$

$$\begin{aligned} 4x_2 &= 2 \\ x_2 &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} x_1 &= -6 - 5\left(\frac{1}{2}\right) \\ x_1 &= -\frac{17}{2} \end{aligned}$$

Question 5

Solve the following systems:

1. $$\begin{aligned} x_2 + 4x_3 &= -5 \\ x_1 + 3x_2 + 5x_3 &= -2 \\ 3x_1 + 7x_2 + 7x_3 &= 6 \end{aligned}$$

2. $$\begin{aligned} x_1 - 2x_4 &= -3 \\ 2x_2 + 2x_3 &= 0 \\ x_3 + 3x_4 &= 1 \\ -2x_1 + 3x_2 + 2x_3 + x_4 &= 5 \end{aligned}$$

Solution:

1. $$\begin{bmatrix} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 3 & 7 & 7 & 6 \\ 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \end{bmatrix}$$

$\frac{1}{3}R_1 - R_2 \longrightarrow R_2$

$$\begin{array}{r} x_1 + \dfrac{7}{3}x_2 + \dfrac{7}{3}x_3 = 2 \\ \underline{x_1 + 3x_2 + 5x_3 = -2} \\ -\dfrac{2}{3}x_2 - \dfrac{8}{3}x_3 = 4 \end{array}$$

$$\begin{bmatrix} 3 & 7 & 7 & 6 \\ 0 & -\frac{2}{3} & -\frac{8}{3} & 4 \\ 0 & 1 & 4 & -5 \end{bmatrix}$$

$3R_2$

$$\begin{bmatrix} 3 & 7 & 7 & 6 \\ 0 & -2 & -8 & 12 \\ 0 & 1 & 4 & -5 \end{bmatrix}$$

$-\frac{1}{2}R_2 - R_3 \longrightarrow R_3$

$$\begin{array}{r} x_2 + 4x_3 = -6 \\ \underline{x_2 + 4x_3 = -5} \\ 0 = -1 \end{array}$$

$$\begin{bmatrix} 3 & 7 & 7 & 6 \\ 0 & -2 & -8 & 12 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Because the system has a contradiction in row 3, $0x_1 + 0x_2 + 0x_3 = -1$, the system has no solution and is therefore inconsistent.

2.

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ -2 & 3 & 2 & 1 & 5 \end{bmatrix}$$

$R_1 \leftrightarrow R_4$

$$\begin{bmatrix} -2 & 3 & 2 & 1 & 5 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 1 & 0 & 0 & -2 & -3 \end{bmatrix}$$

$-\frac{1}{2}R_1 - R_4 \longrightarrow R_4$

$$\begin{array}{r} x_1 - \dfrac{3}{2}x_2 - x_3 - \dfrac{1}{2}x_4 = -\dfrac{5}{2} \\ \underline{x_1 + 0 + 0 - 2x_4 = -3} \\ -\dfrac{3}{2}x_2 - x_3 + \dfrac{3}{2}x_4 = \dfrac{1}{2} \end{array}$$

$$\begin{bmatrix} -2 & 3 & 2 & 1 & 5 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & -\frac{3}{2} & -1 & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

$-\frac{3}{4}R_2 - R_4 \longrightarrow R_4$

$$\begin{array}{r} 0 - \dfrac{3}{2}x_2 - \dfrac{3}{2}x_3 + 0 = 0 \\ 0 - \dfrac{3}{2}x_2 - x_3 + \dfrac{3}{2}x_4 = \dfrac{1}{2} \\ \hline -\dfrac{1}{2}x_3 - \dfrac{3}{2}x_4 = -\dfrac{1}{2} \end{array}$$

$$\begin{bmatrix} -2 & 3 & 2 & 1 & 5 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$-\frac{1}{2}R_3 - R_4 \longrightarrow R_4$

$$\begin{array}{r} 0 + 0 - \dfrac{1}{2}x_3 - \dfrac{3}{2}x_4 = -\dfrac{1}{2} \\ 0 + 0 - \dfrac{1}{2}x_3 - \dfrac{3}{2}x_4 = -\dfrac{1}{2} \\ \hline 0x_1 + 0x_2 + 0x_3 + 0x_4 = 0 \end{array}$$

$$\begin{bmatrix} -2 & 3 & 2 & 1 & 5 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\frac{3}{2}R_2 - R_1 \longrightarrow R_1$

$$\begin{array}{r} 0 + 3x_2 + 3x_3 + 0 = 0 \\ -2x_1 + 3x_2 + 2x_3 + x_4 = 5 \\ \hline 2x_1 + 0 + x_3 - x_4 = -5 \end{array}$$

$$\begin{bmatrix} 2 & 0 & 1 & -1 & -5 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 - R_1 \longrightarrow R_1$

$$\begin{array}{r} 0 + 0 + x_3 + 3x_4 = 1 \\ 2x_1 + 0 + x_3 - x_4 = -5 \\ \hline -2x_1 + 0 + 0 + 4x_4 = 6 \end{array}$$

$$\begin{bmatrix} -2 & 0 & 0 & 4 & 6 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$2R_3 - R_2 \longrightarrow R_2$

$$\begin{array}{r} 0 + 0 + 2x_3 + 6x_4 = 2 \\ 0 + 2x_2 + 2x_3 + 0 = 0 \\ \hline -2x_2 + 6x_4 = 2 \end{array}$$

$$\begin{bmatrix} -2 & 0 & 0 & 4 & 6 \\ 0 & -2 & 0 & 6 & 2 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\frac{R_1}{-2}$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & -2 & 0 & 6 & 2 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\frac{R_2}{-2}$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + -2x_4 &= -3 \\ x_2 - 3x_4 &= -1 \\ x_3 + 3x_4 &= 1 \\ 0 &= 0 \end{aligned}$$

$$\begin{aligned} x_4 &= \frac{1}{2}x_1 + \frac{3}{2} \\ x_2 &= -1 + 3x_4 \\ x_3 &= 1 - 3x_4 \\ x_1 &= -3 + 2x_4 \end{aligned}$$

Question 6

For the following matrices find the elementary row operation that transforms the first matrix into the second, and then find the reverse row operation that transforms the second matrix into the first

1.
$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & -2 & 6 \\ 0 & -5 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & -3 \\ 0 & -5 & 9 \end{bmatrix}$$

2.
$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 4 & -1 & 3 & -6 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 0 & 7 & -1 & -6 \end{bmatrix}$$

Solution:

1. Let the first matrix be M_1 and the second be M_2

$$M_1 \longrightarrow M_2 = -\frac{1}{2}R_2$$
$$M_2 \longrightarrow M_1 = \frac{R_2}{-\frac{1}{2}}$$

2. Let the first matrix be M_1 and the second be M_2

$$M_1 \longrightarrow M_2 = -4R_1 + R_3 \longrightarrow R_3$$
$$M_2 \longrightarrow M_1 = R_3 - 4R_1$$

Chapter 2

Row Reduction and Echelon Forms

Definition 2.0.1: Leading Entry

The leftmost entry in a non-zero row

Definition 2.0.2: Upper triangular matrix / Echelon Form

A rectangular matrix is in echelon form / row echelon form if it has the following proprieties:
- All non-zero rows are above any rows of all zeros
- Each leading entry of a row is in a column to the right of the leading entry of the row above it
- All entries in a column below a leading entry are zeros

Definition 2.0.3: Reduced Row Echelon Form

A matrix is in reduced row echelon form if it meets all the conditions of a matrix in echelon form and:
- The leading entry in each non-zero row is 1
- Each leading 1 is the only non-zero entry in its column

Definition 2.0.4: Echelon Matrix

A matrix that is in echelon form

Definition 2.0.5: Reduced Echelon Matrix

A matrix that is in reduced row echelon form

Theorem 2.0.1 Uniqueness of a Row Reduced Echelon Form

Each matrix is row equivalent to one and only one reduced row echelon form

Therefore if matrix A is row equivalent to an echelon matrix U, U is the echelon form of A, and if A is row equivalent to a reduced echelon matrix R, R is the reduced echelon form of A.

2.1 Pivot Positions

Definition 2.1.1: Pivot Position

A pivot position in matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.

Definition 2.1.2: Pivot Column

A column of a matrix A that contains a pivot position.

A pivot cannot be 0 and a pivot column cannot contain any other non-zero entries. Therefore when identifying pivot positions we look for the first non-zero entry in each row, that is not in a column that already contains a pivot.

Example 2.1.1

For the reduced echelon matrix:

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivots positions are $(1, 1)$, $(2, 2)$, $(3, 4)$ and values are $(1, 2, -5)$