Chapter 1

Optimization

Question 1

Find the derivatives of the given functions.

1. $$y = \sin^{-1}(\sqrt{2})$$

2. $$y = \tan^{-1}(\pi x)$$

3. $$y = e^x \tan^{-1}(x)$$

4. $$f(x) = \ln(\sin^{-1}(x))$$

5. $$y = \tan^{-1}(\ln(x)) + \pi$$

6. $$h(x) = \frac{\sin^{-1}(x)}{1+x}$$

7. $$y = \sin(x) + x^2 \tan^{-1}(x)$$

8. $$f(t) = 4\cos^{-1}(t) - 10\tan^{-1}(t)$$

9. $$h(t) = \frac{t}{1-e^t}$$

10. $$f(x) = \frac{1+5x}{\ln(x)}$$

Solution:

1. $$y = \sin^{-1}(\sqrt{2x})$$
$$y = \sin^{-1}(2x^{\frac{1}{2}})$$
$$y' = \frac{2x^{-\frac{1}{2}}}{\sqrt{1-2x}}$$

2.

$$y = \tan^{-1}(\pi x)$$
$$y' = \frac{\pi}{1+\pi^2 x^2}$$

3.

$$y = e^x \tan^{-1}(x)$$
$$y' = e^x(\tan^{-1}(x)) + (\frac{1}{1+x^2})e^x$$
$$y' = e^x(\tan^{-1}(x)) + (\frac{1}{1+x^2})e^x$$

4.

$$f(x) = \ln(\sin^{-1}(x))$$
$$f(x) = \frac{\frac{1}{\sqrt{1+x^2}}}{\sin^{-1}(x)}$$

5.

$$y = \tan^{-1}(\ln(x)) + \pi$$
$$y' = \frac{\frac{1}{x}}{1+(\ln(x))^2}$$

6.

$$h(x) = \frac{\sin^{-1}(x)}{1+x}$$
$$h'(x) = \frac{(1+x)(\frac{1}{\sqrt{1-x^2}}) - (1)(\sin^{-1}(x))}{(1+x)^2}$$

7.

$$y = \sin(x) + x^2\tan^{-1}(x)$$
$$y' = \cos(x) + 2x\tan^{-1}(x) + \frac{x^2}{1+x^2}$$

8.

$$f(t) = 4\cos^{-1}(t) - 10\tan^{-1}(t)$$
$$f'(t) = 4(-\frac{1}{\sqrt{1-t^2}}) - 10(\frac{1}{1+t^2})$$
$$f'(t) = -\frac{4}{\sqrt{1-t^2}} - \frac{10}{1+t^2}$$

9.

$$h(t) = \frac{t}{1-e^t}$$
$$h'(t) = \frac{(1)(1-e^t) - (t)(-e^t)}{(1-e^t)^2}$$
$$h' = \frac{1-e^t-te^t}{(1-e^t)^2}$$

10.

$$f(x) = \frac{1+5x}{\ln(x)}$$
$$f'(x) = \frac{(\ln(x))(5) - (\frac{1}{x})(1+5x)}{(\ln(x))^2}$$
$$f'(x) = \frac{5\ln(x) - \frac{1+5x}{x}}{(\ln(x))^2}$$

Question 2

In 2009, the population of Hungary was approximated by $P = 9.906(0.997)^t$, where P is in millions and t is in years since 2009. Assume the trend continues.

- What does this model predict for the population of Hungary in the year 2020?
- How fast (in people/year) does this model predict Hungary's population will be changing in 2020?

Solution:

- $$P = 9.906(0.997)^t$$
  $$t = 2020 - 2009$$
  $$t = 11$$
  $$P = 9.906(0.997)^{11}$$
  $$P = 9.584 \text{ million}$$

- $$P' = (0.997^t \times (1) \times \ln(0.997)) \times 9.906$$
  $$P' = 9.906\ln(0.997) \times 0.997^{11}$$
  $$P' = -0.0288 \text{ million}$$

Question 3

Find the equation of the tangent line to the graph of $y = \ln(x)\log_2(x)$ at $x = 2$

Solution:

$$y = \ln(x)\log_2(x)$$
$$y' = \frac{1}{x}\log_2(x) + \ln(x)\frac{1}{x\ln(2)}$$

$$y = \ln(2)\log_2(2)$$

$$y' = \frac{\log_2(x)}{x} + \frac{\ln(x)}{x\ln(2)}$$
$$y' = \frac{1}{2} + \frac{1}{2}$$
$$y' = 1$$

$$\ln(2)\log_2(2) = 2 + c$$
$$\ln(2)\log_2(2) - 2 = c$$
$$y = x + (\ln(2)\log_2(2) - 2)$$

Question 4

Find the tangent line to $f(x) = 7^x + 4e^x$ at $x = 0$

Solution:

$$
\begin{aligned}
f(x) &= 7^x + 4e^x \\
f'(x) &= 7^x \ln(7) + 4e^x \\
f'(0) &= 7^0 \ln(7) + 4e^0 \\
f'(0) &= 1 \ln(7) + 4 \\
f'(0) &= \ln(7) + 4 \\
\\
y &= 5 \\
\\
5 &= 0(\ln(7) + 4) + c \\
5 &= c
\end{aligned}
$$

$$y = (\ln(7) + 4)x + 5$$

Question 5

The cost of producing a quantity, q, of a product is given by $C(q) = 1000 + 30e^{0.05q}$ dollars. Find the cost and the marginal cost when $q = 50$. Interpret these answers in economic terms.

Solution:

$$
\begin{aligned}
C(q) &= 1000 + 30e^{0.05q} \\
C(50) &= 1000 + 30e^{0.05(50)} \\
C(50) &= 1365.47 \\
\\
C'(q) &= \frac{3}{2}e^{0.05q}
\end{aligned}
$$

$$C'(50) = \$18.27$$

$\therefore$ When the business produces 50 units of a good it's cost incurred is \$1365.47 and the rate at which the cost increases is \$18.27

Question 6

At a time t hours after it was administered, the concentration of a drug in the body is $f(t) = 27e^{-0.14t}$ mg/ml.

- What is the concentration 4 hours after it was administered?
- At what rate is the concentration changing at that time?

Solution:

-

$$
\begin{aligned}
f(t) &= 27e^{-0.14t} \\
f(4) &= 27e^{-0.14(4)}
\end{aligned}
$$

$$f(4) = 15.42 \ \frac{mg}{ml}$$

-

$$f'(t) = -3.78e^{-0.14t}$$
$$f'(4) = -2.16 \ \frac{mg}{h}$$

Question 7

For the cost function $C = 1000 + 300\ln(q)$ (in dollars), find the cost and marginal cost at a production level of 500. Interpret your answers in economic terms.

Solution:

$$C = 1000 + 300\ln(q)$$
$$C = 1000 + 300\ln(500)$$
$$C = 2864.38$$

$$C' = \frac{300}{q}$$
$$C' = \$0.6$$

$\therefore$ At a production level of 500 the business is incurring a cost of \$2864.38 and this cost is increasing at a rate of \$0.6

Question 8

Carbon-14 is a radioactive isotope used to date objects. If A_0 represents the initial amount of carbon-14 in the object, then the quantity remaining at time, t, in years, is $A(t) = A_0e^{-0.000121t}$. A tree, originally 185 micrograms of carbon-14, is now 500 years old. At what rate is the carbon-14 decaying now?

Solution:

$$A(t) = A_0e^{-0.000121t}$$
$$A'(t) = -0.000121A_0e^{-0.000121t}$$
$$A'(500) = -0.000121(185)e^{-0.000121(500)}$$
$$A'(500) = -0.0211 \ \frac{\mu g}{y}$$

Question 9

In 2009, the population of Mexico was 111 million and growing 1.13% annually, while the population of the US was 307 million and growing 0.975% annually. If we measure growth rates in people/year, which population was growing faster in 2009?

Solution:

$$M = 111 + 1.13x$$
$$U = 307 + 0.975x$$

$$M' = 1.13$$
$$U' = 0.975$$

$\therefore$ Mexico's population is growing faster than the US's population

Question 10

With t in years since January 1, 2010, the population P of Slim Chance is predicted by $P = 35000(0.98)^t$. At what rate will the population be changing on January 1, 2023?

Solution:

$$t = 2023 - 2010$$
$$t = 13$$

$$P = 35000(0.98)^{13}$$
$$P' = 35000 \times 0.98^{13} \times \ln(0.98)$$
$$P' = 543.772 \ \frac{people}{year}$$

Question 11

Two nonnegative numbers are such that the first plus the square of the second is 10. Find the numbers if their sum is as large as possible.

Solution:

$$a \geqslant 0$$
$$b \geqslant 0$$

$$a + b^2 = 10$$
$$a = 10 - b^2 b = \sqrt{10 - a}$$

$$S = a + b$$
$$S = 10 - b^2 + \sqrt{10 - a}$$
$$S = 10 - b^2 + \sqrt{10 - (10 - b^2)}$$
$$S = 10 - b^2 + \sqrt{10 - 10 + b^2}$$
$$S = 10 - b^2 + \sqrt{b^2}$$
$$S = 10 - b^2 + b$$
$$S' = -2b + 1$$
$$0 = -2b + 1$$
$$2b = 1$$
$$b = \frac{1}{2}$$
$$S'' = -2 \therefore b = \frac{1}{2} \quad \text{Max}$$

$$a = 10 - (\frac{1}{2})^2$$
$$a = \frac{39}{4}$$

$$\therefore a = \frac{39}{4} \quad b = \frac{1}{2}$$

Chapter 2

Differential Equations

Question 12

A company earns revenue (income) and also makes payroll payments. Assume that revenue is earned continuously, that payroll payments are made continuously, and that the only factors of affecting net worth are revenue and payroll. The company's revenue is earned at a continuous annual rate of 5% times the net worth. At the same time, the company's payroll obligations are paid out at a constant rate of 200 million dollars a year. Use this information to write a differential equation to model the net worth of the company, W, in millions of dollars, as a function of time, t, in years.

Solution: Let R be the company's revenue

Let W be the company's net worth

Let $\frac{dR}{dt}$ be the change in revenue

Let $\frac{dP}{dt}$ be the change in payroll paid

$$\frac{dW}{dt} = 0.05W - 200$$

$$dW = (0.05W - 200)dt$$

$$\frac{dW}{0.05W - 100} = dt$$

Question 13

A patient having major surgery is given the antibiotic Vancomycin intravenously at a rate of 85 mg per hour. The rate at which the drug is excreted from the body is proportional to the quantity present, with proportionality constant 0.1 if time is in hours. Write a differential equation for the quantity, Q in mg of Vancomycin in the body after t hours.

Solution: Let $\frac{dQ}{dt}$ be the rate at which the antibiotic is administered

Let $\frac{dE}{dt}$ be the rate at which the antibiotic is excreted

$$\frac{dQ}{dt} = 85 - \frac{dE}{dt}$$
$$\frac{dE}{dt} \propto Q$$
$$\frac{dE}{dt} = KQ$$
$$\frac{dE}{dt} = 0.1Q$$
$$\frac{dQ}{dt} = 85 - 0.1Q$$
$$\frac{dQ}{85 - 0.1Q} = dt$$

Question 14

Money in a bank account earns interest at a continuous annual rate of 5% times the current balance. Write a differential equation for the balance, B, in the account as a function of time, t, in years.

Solution: Let $\frac{dB}{dt}$ be the change in balance

$$\frac{dB}{dt} = 0.05B$$
$$\frac{dB}{B} = 0.05dt$$

Question 15

A bank account that initially contains \$25,000 earns interest at a continuous rate of 4% per year. Withdrawals are made out of the account at a constant rate of \$2,000 per year. Write a differential equation for the balance, B, in the account as a function of the number of years, t.

Solution: Let B_0 be the principal

Let $\frac{dB}{dt}$ be the change in balance

Let $\frac{dW}{dt}$ be the rate of withdrawal

$$B_0 = 25000$$
$$\frac{dW}{dt} = 2000$$
$$\frac{dB}{dt} = 0.04B - 2000$$
$$dB = (0.04B - 2000)dt$$
$$\frac{dB}{0.04B - 2000} = dt$$

Question 16

Morphine is administered to a patient intravenously at a rate of 2.5 mg per hour. About 34.7% of the morphine is metabolized and leaves the body each hour. Write a differential equation for the amount of morphine, M, in milligrams, in the body as a function of time, t, in hours.

Solution: Let $\frac{dM}{dt}$ be the rate at which the morphine is administered

Let $\frac{dMet}{dt}$ be the rate at which the morphine is metabolized

$$\frac{dM}{dt} = 2.5$$
$$\frac{dMet}{dt} = 0.347M$$
$$\frac{dM}{dt} = 2.5 - 0.347M$$
$$\frac{dM}{2.5 - 0.347M} = dt$$

Question 17

A cup of coffee contains about 100 mg of caffeine. Caffeine is metabolized and leaves the body at a continuous rate of about 17% every hour. Write a differential equation for the amount, A, of caffeine in the body as a function of the number of hours, t since the coffee was consumed.

Solution: Let A_0 be the initial amount of caffeine ingested

Let $\frac{dMet}{dt}$ be the rate the caffeine is metabolized

Let $\frac{dA}{dt}$ be the rate at which the amount of caffeine in the body is changing

$$A_0 = 100$$
$$\frac{dMet}{dt} = 0.17A$$
$$\frac{dA}{dt} = -0.17A$$
$$\frac{dA}{dt} = -0.17A$$
$$\frac{dA}{A} = -0.17dt$$

Question 18

A person deposits money into an account at a continuous rate of \$6000 a year, and the account earns interest at a continuous rate of 7% per year. Write a differential equation for the balance in the account, B, in dollars, as a function of years, t.

Solution: Let $\frac{dB}{dt}$ be the change in balance

$$\frac{dB}{dt} = 6000 + 0.07B$$
$$\frac{dB}{6000 + 0.07B} = dt$$

Chapter 3

L'hopital's Rule

Question 19

$$\lim_{x\to 0}(\frac{1}{x} - \frac{1}{\sin(x)})$$

Solution:

$$\lim_{x\to 0}(\frac{1}{x} - \frac{1}{\sin(x)})$$
$$\frac{1}{x} - \frac{1}{\sin(x)}$$
$$\frac{\sin(x) - x}{x\sin(x)}$$
$$\lim_{x\to 0}(\frac{\sin(x) - x}{x\sin(x)})$$
$$\text{Using L'hopital's Rule}$$
$$\lim_{x\to 0}\frac{\frac{dy}{dx}(\sin(x) - x)}{\frac{dy}{dx}(x\sin(x))} \implies \frac{0}{0}$$
$$\lim_{x\to 0}\frac{\frac{dy}{dx}(\cos(x) - 1)}{\frac{dy}{dx}(x\cos(x) + \sin(x))} \implies \frac{0}{0}$$
$$\lim_{x\to 0}\frac{-\sin(x)}{-x\sin(x) + \cos(x) + \cos(x)} = \frac{0}{0+2}$$
$$\frac{0}{2}$$
$$= 0$$

$$\therefore \lim_{x\to 0}(\frac{1}{x} - \frac{1}{\sin(x)}) = 0$$

Question 20

$$\lim_{x\to\infty}\frac{e^x}{x^2}$$

Solution:

$$\lim_{x \to \infty} \frac{e^x}{x^2}$$

$$\lim_{x \to \infty} \frac{e^x}{x^2} \implies \frac{e^\infty}{x^\infty} = \frac{\infty}{\infty}$$

$$\lim_{x \to \infty} \frac{e^x}{2x} \implies \frac{e^\infty}{2(\infty)} = \frac{\infty}{\infty}$$

$$\lim_{x \to \infty} \frac{e^x}{2} \implies \frac{e^\infty}{2} = \frac{\infty}{2}$$

$$\therefore \lim_{x \to \infty} \frac{e^x}{x^2} \text{ does not exist}$$