Orthogonality and Least Squares

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Contents

Chapter 1

Inner Product, Length and Orthogonality

1.1 Inner Product

Definition 1.1.1: Inner / Dot Product

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, then we regard $\mathbf{u}$ and $\mathbf{v}$ as $n \times 1$ matrices. The transpose of $\mathbf{u}^T$ is a $1 \times n$ matrix, and the matrix product $\mathbf{u}^T\mathbf{v}$ is a 1×1 matrix, a scalar. This scalar is called the *inner / dot product* of $\mathbf{u}$ and $\mathbf{v}$ which can also be referred to as:

$$\mathbf{u} \cdot \mathbf{v}$$

Which breaks down into:

$$\mathbf{u}^T \times \mathbf{v}$$

When $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, is then defined as:

$$\begin{bmatrix} u_1 & u_2 & \dots u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

Example 1.1.1

Question 1

Compute $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$ for $\mathbf{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$

Solution:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^T \times \mathbf{v} = \begin{bmatrix}2 & -5 & -1\end{bmatrix}\begin{bmatrix}3\\2\\-3\end{bmatrix} \\ &= 3(2) + (-5)(2) + (-1)(-3) \\ &= -1\end{aligned}$$

$$\begin{aligned}\mathbf{v}\cdot\mathbf{u} = \mathbf{v}^T \times \mathbf{u} = \begin{bmatrix}3 & 2 & -3\end{bmatrix}\begin{bmatrix}2\\-5\\-1\end{bmatrix} \\ &= 2(3) + 2(-5) + (-3)(-1) \\ &= -1\end{aligned}$$

Theorem 1.1.1 Axioms of Inner / Dot products

Let $\mathbf{u}$ and $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

1. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$
2. $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$
3. $(c\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(c\mathbf{v})$
4. $\mathbf{u}\cdot\mathbf{u} \geqslant 0$, and $\mathbf{u}\cdot\mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

1.2 Length of a Vector

Definition 1.2.1: Length of a Vector

If $\mathbf{v}$ is in $\mathbb{R}^n$, with entries $v_1, \ldots, v_n$, then the square root of $\mathbf{v}\cdot\mathbf{v}$ is defined because $\mathbf{v}\cdot\mathbf{v}$ is non-negative. Therefore the *length / norm* of $\mathbf{v}$ is the non-negative scalar $||\mathbf{v}||$, defined:

$$||\mathbf{v}|| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2} \text{ and } ||\mathbf{v}||^2 = \mathbf{v}\cdot\mathbf{v}$$

And similarly for any scalar c, the length of $c\mathbf{v}$ is $|c|$ times the length of $\mathbf{v}$, i.e:

$$||c\mathbf{v}|| = |c| \times ||\mathbf{v}||$$

Definition 1.2.2: Unit Vector

A vector whose length is 1. If we divide a non zero vector by it's length, i.e. multiply by $\frac{1}{||\mathbf{v}||}$, we obtain a unit vector $\mathbf{u}$. This process of creating a unit vector $\mathbf{u}$ from $\mathbf{v}$ can be called *normalizing* $\mathbf{v}$, and the resulting $\mathbf{u}$ is in the same direction as $\mathbf{v}$

1.3 Distance in $\mathbb{R}^n$

Definition 1.3.1: Distance between two vectors

For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the *distance between* $\mathbf{u}$ and $\mathbf{v}$, expressed as $\text{dist}(\mathbf{u}, \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$:

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

Then defined:

$$\begin{aligned} \text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| &= \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})} \\ &= \sqrt{(u_1 - v_1)^2 + \ldots + (u_n - v_n)^2} \end{aligned}$$

In $\mathbb{R}^2$ and $\mathbb{R}^3$, this is basically the same as the Euclidean distance between two points.

Example 1.3.1

Question 2

Compute the distance between the vectors $\mathbf{u} = (7, 1)$ and $\mathbf{v} = (3, 2)$

Solution:

$$\begin{aligned} \text{dist}(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\| \\ &= \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})} \\ &= \sqrt{\begin{bmatrix} 4 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \end{bmatrix}} \\ &= \sqrt{4^2 + (-1)^2} \\ &= \sqrt{17} \end{aligned}$$

1.4 Orthogonal Vectors

Consider $\mathbb{R}^2$ and $\mathbb{R}^3$ and two lines through the origin determined by vectors $\mathbf{u}$ and $\mathbf{v}$. These lines are geometrically perpendicular if and only if the distance from $\mathbf{u}$ to $\mathbf{v}$ is the same as the distance from $\mathbf{u}$ to $-\mathbf{v}$. This is equivalent to saying the squares of the distances are the same. Therefore:

$$\begin{aligned} [\text{dist}(\mathbf{u}, -\mathbf{v})]^2 = \|\mathbf{u} - (-\mathbf{v})\|^2 = \|\mathbf{u} + \mathbf{v}\|^2 \\ &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot (\mathbf{u} + \mathbf{v}) + \mathbf{v} \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}^2\| + \|\mathbf{v}\|^2 + 2\mathbf{u} \cdot \mathbf{v} \end{aligned}$$

And then $\text{dist}(\mathbf{u}, \mathbf{v})$:

$$[\text{dist}(\mathbf{u}, \mathbf{v})] = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}$$

This shows that the two squared distances are only equal if and only if $2\mathbf{u} \cdot \mathbf{v} = -2\mathbf{u} \cdot \mathbf{v}$, which happens if and only if $\mathbf{u} \cdot \mathbf{v} = 0$

Definition 1.4.1: Orthogonality

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are orthogonal, to each other, if $\mathbf{u} \cdot \mathbf{v} = 0$

This then confirms that the zero vector $\mathbf{0}$ is orthogonal to every vector in $\mathbb{R}^n$, since $\mathbf{0}^T\mathbf{v} = 0$ for every $\mathbf{v}$.

Theorem 1.4.1 The Pythagorean Theorem

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in $\mathbb{R}^n$, then:

$$||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$$

1.5 Exercises

Question 3

Let $\mathbf{a} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. Compute $\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}$ and $\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a}$

Solution:

$$\begin{aligned}
\mathbf{a} \cdot \mathbf{b} &= (-2)(-3) + 1 \\
&= 7 \\
\mathbf{a} \cdot \mathbf{a} &= (-2)^2 + 1 \\
&= 5 \\
\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} &= \frac{7}{5}
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a} &= \frac{7}{5}\begin{bmatrix} -2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -2.8 \\ \frac{7}{5} \end{bmatrix}
\end{aligned}$$

Question 4

Let $\mathbf{c} = \begin{bmatrix} \frac{4}{3} \\ -1 \\ \frac{2}{3} \end{bmatrix}$ and $\mathbf{d} = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}$.

1. Find a unit vector $\mathbf{u}$ in the direction of $\mathbf{c}$
2. Show that $\mathbf{d}$ is orthogonal to $\mathbf{c}$.
3. Use the results of parts (1) and (2) to explain why d must be orthogonal to the unit vector $\mathbf{u}$

Solution:

1.

$$
\begin{aligned}
\|\mathbf{c}\| &= \sqrt{\mathbf{c}\cdot\mathbf{c}} \\
&= \sqrt{\left(\frac{4}{3}\right)^2 + (-1)^2 + \left(\frac{2}{3}\right)^2} \\
&= \frac{\sqrt{29}}{3} \\
\mathbf{u} &= \frac{1}{\frac{\sqrt{29}}{3}} \begin{bmatrix} \frac{4}{3} \\ -1 \\ \frac{2}{3} \end{bmatrix} \\
&= \frac{3\sqrt{29}}{29} \begin{bmatrix} \frac{4}{3} \\ -1 \\ \frac{2}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{4\sqrt{29}}{29} \\ \frac{3\sqrt{29}}{29} \\ \frac{2\sqrt{29}}{29} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\|\mathbf{u}\| &= \sqrt{\mathbf{u}\cdot\mathbf{u}} \\
&= \sqrt{\left(\frac{4\sqrt{29}}{29}\right)^2 + \left(\frac{3\sqrt{29}}{29}\right)^2 + \left(\frac{2\sqrt{29}}{29}\right)^2} \\
&= 1
\end{aligned}
$$

2. If $\mathbf{d}$ is orthogonal to $\mathbf{c}$ then $\mathbf{d}\cdot\mathbf{c} = 0$

$$
\begin{aligned}
\mathbf{d}\cdot\mathbf{c} &= \mathbf{d}^T \times \mathbf{c} \\
&= \begin{bmatrix} 5 & 6 & -1 \end{bmatrix} \begin{bmatrix} \frac{4}{3} \\ -1 \\ \frac{2}{3} \end{bmatrix} \\
&= 5\left(\frac{4}{3}\right) + 6\,(-1) - 1\left(\frac{2}{3}\right) \\
&= \frac{20}{3} - 6 - \frac{2}{3} \\
&= 0
\end{aligned}
$$

$\therefore$ $\mathbf{c}$ and $\mathbf{d}$ are orthogonal to each other.

3. $\mathbf{d}$ is orthogonal to the unit vector $\mathbf{u}$ because $\mathbf{d}$ is orthogonal to $\mathbf{c}$ of which $\mathbf{u}$ is a scalar multiple of. I.e $\mathbf{u}$ is in the form $k\mathbf{c}$ for some k and:

$$\mathbf{d}\cdot\mathbf{u} = \mathbf{d}\cdot(k\mathbf{c}) = k\,(\mathbf{d}\cdot c) = k\,(0) = 0$$

Chapter 2

Orthogonal Sets

Definition 2.0.1: Orthogonal Set

If $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal set of non-zero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

Definition 2.0.2: Orthogonal Basis

An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

Theorem 2.0.1

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y}$ in W, the weights in the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p$$

are given by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \qquad (j = 1, \ldots, p)$$

Example 2.0.1

Question 5

The set $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -\frac{1}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix}$$

is an orthogonal basis for $\mathbb{R}^3$. Express the vector $\mathbf{y} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as a linear combination of the vectors in S

Solution:

$$\begin{aligned}\mathbf{y} &= \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{y}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}\mathbf{u}_3 \\ &= \frac{11}{11}\mathbf{u}_1 + -\frac{12}{6}\mathbf{u}_2 + -\frac{33}{\frac{33}{2}}\mathbf{u}_3 \\ &= \mathbf{u}_1 - 2\mathbf{u}_2 - 2\mathbf{u}_3\end{aligned}$$

Chapter 3

Orthogonal Projections

Definition 3.0.1: Orthogonal Projection

Let W be a subspace of $\mathbb{R}^n$, and let $\mathbf{y}$ be in $\mathbb{R}^n$. The *orthogonal projection* of $\mathbf{y}$ onto W, denoted $\text{proj}_W\mathbf{y}$, is the closest point in W to $\mathbf{y}$. This point is obtained by adding the orthogonal projection of $\mathbf{y}$ onto the orthogonal complement of W to the orthogonal projection of $\mathbf{y}$ onto W.

Example 3.0.1

Question 6

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_5\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^5$, and let

$$\mathbf{y} = c_1\mathbf{u}_1 + \ldots + c_5\mathbf{u}_5$$

Consider the subspace $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, and write $\mathbf{y}$ as the sum of a vector $\mathbf{z}_1$ in W and a vector $\mathbf{z}_2$ in $W^\perp$

Solution:

$$\begin{aligned}\mathbf{z}_1 &= \mathbf{y} - \mathbf{z}_2\\ \mathbf{y} &= \mathbf{z}_2 + \mathbf{z}_1\end{aligned}$$

$$\begin{aligned}\mathbf{z}_1 &= c_1\mathbf{u}_1 + c_2\mathbf{u}_2\\ \mathbf{z}_2 &= c_3\mathbf{u}_3 + c_4\mathbf{u}_4 + c_5\mathbf{u}_5\end{aligned}$$

$$\mathbf{y} = \mathbf{z}_1 + \mathbf{z}_2 = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4 + c_5\mathbf{u}_5$$

Theorem 3.0.1 The Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form:

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^\perp$. In fact, if $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$, is any orthogonal basis of W, then:

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \ldots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p$$

and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$

Example 3.0.2

Question 7

Let $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Observe that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W

Solution: The orthogonal projection of $\mathbf{y}$ onto W is:

$$\begin{aligned}
\hat{\mathbf{y}} &= \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 \\
&= \frac{9}{30}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{5} \\ \frac{3}{2} \\ -\frac{3}{10} \end{bmatrix} + \begin{bmatrix} -1 \\ \frac{3}{6} \\ \frac{3}{6} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{2}{5} \\ 2 \\ \frac{1}{5} \end{bmatrix}
\end{aligned}$$

So:

$$\begin{aligned}
\mathbf{z} &= \mathbf{y} - \hat{\mathbf{y}} \\
&= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -\frac{2}{5} \\ 2 \\ \frac{1}{5} \end{bmatrix} \\
&= \begin{bmatrix} \frac{7}{5} \\ 0 \\ \frac{14}{5} \end{bmatrix}
\end{aligned}$$

$\mathbf{z}$ is orthogonal to W due to 3, so $\mathbf{y}$ can be expressed as:

$$\mathbf{y} = \begin{bmatrix} -\dfrac{2}{5} \\ 2 \\ \dfrac{1}{5} \end{bmatrix} + \begin{bmatrix} \dfrac{7}{5} \\ 0 \\ \dfrac{14}{5} \end{bmatrix}$$

3.1 Properties of Orthogonal Projections

If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal basis for W and if $\mathbf{y}$ happens to be in W, then the formula for $\text{proj}_W \mathbf{y}$ is exactly the same as the representation of $\mathbf{y}$ in terms of the basis. In this case, $\text{proj}_W \mathbf{y} = \mathbf{y}$.

Theorem 3.1.1

If $\mathbf{y}$ is in $W = \text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$, then $\text{proj}_W \mathbf{y} = \mathbf{y}$

This leads to the next theorem:

Theorem 3.1.2 The Best Approximation Theorem

Let W be a subspace of $\mathbb{R}^n$, let $\mathbf{y}$ be any vector in $\mathbb{R}^n$, and let $\hat{\mathbf{y}}$ be an orthogonal projection of $\mathbf{y}$ onto W. Then $\hat{\mathbf{y}}$ is the closest point in W to $\mathbf{y}$, in the sense that

$$||\mathbf{y} - \hat{\mathbf{y}}|| < ||\mathbf{y} - \mathbf{v}||$$

For all $\mathbf{v}$ in W distinct from $\hat{\mathbf{y}}$

Definition 3.1.1: Orthonormality

A set of vectors is orthonormal if each of them are orthogonal to each other and have a length of 1.

Theorem 3.1.3 Orthonormal Basis

If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an Orthonormal basis for a subspace W of $\mathbb{R}^n$, then

$$\text{proj}_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\,\mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\,\mathbf{u}_2 + \ldots + \left(\mathbf{y} \cdot \mathbf{u}_p\right)\mathbf{u}_p$$

If $U = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_p\end{bmatrix}$, then

$$\text{proj}_W \mathbf{y} = UU^T \mathbf{y} \forall \mathbf{y} \in \mathbb{R}^n$$

3.1.1 Exercises

Question 8

The distance from a point $\mathbf{y}$ in $\mathbb{R}^n$ to a subspace W is defined as the distance from $\mathbf{y}$ to the nearest point in W. Find the distance from $\mathbf{y}$ to $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, where

$$\mathbf{y} = \begin{bmatrix}-1\\-5\\10\end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix}5\\-2\\1\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}1\\2\\-1\end{bmatrix}$$

Solution:

$$\begin{aligned}
\hat{\mathbf{y}} &= \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2} \\
&= \frac{15}{30}\begin{bmatrix}5\\-2\\1\end{bmatrix} + -\frac{21}{6}\begin{bmatrix}1\\2\\-1\end{bmatrix} \\
&= \begin{bmatrix}\frac{5}{2}\\-1\\\frac{15}{30}\end{bmatrix} + \begin{bmatrix}-\frac{21}{6}\\-7\\\frac{21}{6}\end{bmatrix} \\
&= \begin{bmatrix}-1\\-8\\4\end{bmatrix} \\
\mathbf{z} &= \mathbf{y} - \hat{\mathbf{y}} \\
&= \begin{bmatrix}-1\\-5\\10\end{bmatrix} - \begin{bmatrix}-1\\-8\\4\end{bmatrix} \\
&= \begin{bmatrix}0\\3\\6\end{bmatrix} \\
\|\mathbf{z}\| &= \sqrt{3^2+6^2} \\
&= \sqrt{45}
\end{aligned}$$

Chapter 4

The Gram-Schmidt Process

The Gram-Schmidt process is a simple algorithm for producing an orthogonal or orthonormal basis for any non-zero subspace of $\mathbb{R}^n$.

Example 4.0.1

Question 9

Let $\mathbf{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$. Then $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is clearly linearly independent and thus a basis for a subspace W of $\mathbb{R}^4$. Construct and orthogonal basis for W.

Solution:

Step 1 Let $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{Span}\,\{\mathbf{x}_1\} = \text{Span}\,\{\mathbf{v}_1\}$

Step 2 Let $\mathbf{v}_2$ be the vector produced by subtracting from $\mathbf{x}_2$ is projection onto the subspace W_1. That is, let

$$\begin{aligned}\mathbf{v}_2 &= \mathbf{x}_2 - \text{proj}_{W_1}\mathbf{x}_2\\ &= \mathbf{x}_2 - \frac{\mathbf{x}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1\\ \text{Since } \mathbf{v}_1 = \mathbf{x}_1 & \\ &= \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{3}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}-\frac{3}{4}\\ \frac{1}{4}\\ \frac{1}{4}\\ \frac{1}{4}\end{bmatrix}\end{aligned}$$

$\mathbf{v}_2$ is the component of $\mathbf{x}_2$ orthogonal to $\mathbf{x}_1$, and $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for the subspace W_2 spanned by $\mathbf{x}_1$ and $\mathbf{x}_2$

Step 2' (Optional) If possible scale $\mathbf{v}_2$ to simplify future calculations. Since $\mathbf{v}_2$ has fractional entries, it is convenient to scale it by a factor of 4 and replace $\{\mathbf{v}_1, \mathbf{v}_2\}$ by the orthogonal basis

$$\mathbf{v}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix},\ \mathbf{v}_2' = \begin{bmatrix}-3\\1\\1\\1\end{bmatrix}$$

Step 3 Let $\mathbf{v}_3$ be the vector produced by subtracting from $\mathbf{x}_3$ its projection onto the subspace W_2, using the orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2'\}$ to compute this projection to W_2:

$$\begin{aligned}
\mathrm{proj}_{W_2}\mathbf{x}_3 &= \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{x}_3 \cdot \mathbf{v}_2'}{\mathbf{v}_2' \cdot \mathbf{v}_2'}\mathbf{v}_2' \\
\mathbf{x}_3 \cdot \mathbf{v}_1 &= 0 + 0 + 1 + 1 \\
&= 2 \\
\mathbf{v}_1 \cdot \mathbf{v}_1 &= 1 + 1 + 1 + 1 \\
&= 4 \\
\mathbf{x}_3 \cdot \mathbf{v}_2' &= 0 + 0 + 1 + 1 \\
&= 2 \\
\mathbf{v}_2' \cdot \mathbf{v}_2' &= 9 + 1 + 1 + 1 \\
&= 12 \\
&= \frac{2}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{2}{12}\begin{bmatrix}-3\\1\\1\\1\end{bmatrix} \\
&= \begin{bmatrix}\frac{2}{4}\\\frac{2}{4}\\\frac{2}{4}\\\frac{2}{4}\end{bmatrix} + \begin{bmatrix}-\frac{1}{2}\\\frac{1}{6}\\\frac{1}{6}\\\frac{1}{6}\end{bmatrix} \\
&= \begin{bmatrix}0\\\frac{2}{3}\\\frac{2}{3}\\\frac{2}{3}\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_3 &= \mathbf{x}_3 - \mathrm{proj}_{W_2}\mathbf{x}_3 \\
&= \begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \begin{bmatrix}0\\\frac{2}{3}\\\frac{2}{3}\\\frac{2}{3}\end{bmatrix} \\
&= \begin{bmatrix}0\\-\frac{2}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}
\end{aligned}$$

Therefore the orthogonal basis for W is $\{\mathbf{v}_1, \mathbf{v}_2', \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \mathbf{v}_2' = \begin{bmatrix}-3\\1\\1\\1\end{bmatrix} \mathbf{v}_3 = \begin{bmatrix}0\\-\frac{2}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}$$

Theorem 4.0.1 The Gram-Schmidt Process

Given a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ for a non-zero subspace W of $\mathbb{R}^n$, define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \ldots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}
\end{aligned}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W. In addition

$$\text{Span}\,\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \text{Span}\,\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \quad \text{for } 1 \leqslant k \leqslant p$$

4.1 Orthonormal Bases

An orthonormal basis is constructed easily from an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, by normalizing all the $\mathbf{v}_k$.

Example 4.1.1

Given the constructed basis

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

An orthonormal basis is

$$\begin{aligned}
\mathbf{u}_1 &= \frac{1}{||\mathbf{v}_1||} = \frac{1}{\sqrt{45}}\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix} \\
\mathbf{u}_2 &= \frac{1}{||\mathbf{v}_2||}\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}$$

4.1.1 Exercises

Question 10

Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$. Find an orthonormal basis for the column space of A.

Solution:

$$\begin{bmatrix}1 & -1 & 4\\ 1 & 4 & -2\\ 1 & 4 & 2\\ 1 & -1 & 0\end{bmatrix}$$

$$R_1 - R_2 \rightarrow R_2$$

$$\begin{bmatrix}1 & -1 & 4 & 0\\ 0 & -5 & 6 & 0\\ 1 & 4 & 2 & 0\\ 1 & -1 & 0 & 0\end{bmatrix}$$

$$R_1 - R_3 \rightarrow R_3$$

$$\begin{bmatrix}1 & -1 & 4 & 0\\ 0 & -5 & 6 & 0\\ 0 & -5 & 2 & 0\\ 1 & -1 & 0 & 0\end{bmatrix}$$

$$R_1 - R_4 \rightarrow R_4$$

$$\begin{bmatrix}1 & -1 & 4 & 0\\ 0 & -5 & 6 & 0\\ 0 & -5 & 2 & 0\\ 0 & 0 & 4 & 0\end{bmatrix}$$

$$R_2 - R_3 \rightarrow R_3$$

$$\begin{bmatrix}1 & -1 & 4 & 0\\ 0 & -5 & 6 & 0\\ 0 & 0 & 4 & 0\\ 0 & 0 & 4 & 0\end{bmatrix}$$

$$R_3 - R_4 \longrightarrow R_4$$

$$\begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & -5 & 6 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{5}R_2 - R_1 \longrightarrow R_1$$

$$\begin{bmatrix} -1 & 0 & \frac{-14}{5} & 0 \\ 0 & -5 & 6 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{3}{2}R_3 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix} -1 & 0 & \frac{-14}{5} & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{-7}{10}R_3 - R_1 \longrightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{5}R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{4}R_3 \longrightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore Col $A = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} \right\}$

Let $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{Span}\{\mathbf{v}1\}$

$$\begin{aligned}
\mathbf{v}_2 &= \mathbf{x}_2 - \text{proj}_{W_1}\mathbf{x}_2 \\
&= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
&= \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} - \frac{6}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{5}{2} \\ \frac{5}{2} \\ \frac{5}{2} \\ -\frac{5}{2} \end{bmatrix} \\
\mathbf{v}_2' &= \begin{bmatrix} -5 \\ 5 \\ 5 \\ -5 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{Now let } W_2 &= \text{Span}\{\mathbf{v}_1, \mathbf{v}_2'\} \\
\mathbf{v}_3 &= \mathbf{x}_3 - \text{proj}_{W_2}\mathbf{x}_3 \\
&= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2'}{\mathbf{v}_2' \cdot \mathbf{v}_2'}\mathbf{v}_2' \\
&= \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \frac{4}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left(-\frac{20}{100}\right)\begin{bmatrix} -5 \\ 5 \\ 5 \\ -5 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}
\end{aligned}$$

Therefore the orthogonal basis for the column space of a is $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 5 \\ 5 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}\right\}$. The orthonormal basis is

$$\mathbf{u}_1 = \frac{1}{||\mathbf{v}_1||} = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{1}{||\mathbf{v}_2||} = \frac{1}{\sqrt{100}}\begin{bmatrix} -5 \\ 5 \\ 5 \\ -5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\mathbf{u}_3 = \frac{1}{||\mathbf{v}_3||} = \frac{1}{\sqrt{16}}\begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Therefore the orthonormal basis for the column space of A is $\left\{\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}\right\}$

Chapter 5

Exercises

Question 11

Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. Then express $\mathbf{x}$ as a linear combination of the $\mathbf{u}s$

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \text{ and } \mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$$

Solution: For the basis be orthogonal $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, $\mathbf{u}_2 \cdot \mathbf{u}_3 = 0$, and $\mathbf{u}_1 \cdot \mathbf{u}_3 = 0$.

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{u}_2 &= \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \\ &= 3(2) + (-3)(2) + 0(-1) \\ &= 6 - 6 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{u}_2 \cdot \mathbf{u}_3 &= \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \\ &= 2(1) + 2(1) + (-1)(4) \\ &= 2 + 2 - 4 \\ &= 0 \\ \mathbf{u}_3 \cdot \mathbf{u}_1 &= \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \\ &= 1(3) + 1(-3) + 4(0) \\ &= 3 - 3 \\ &= 0 \end{aligned}$$

Therefore $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. To express $\mathbf{x}$ as a linear combination of the $\mathbf{u}s$:

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$$

$$\begin{bmatrix} 3 & 2 & 1 & 5 \\ -3 & 2 & 1 & -3 \\ 0 & -1 & 4 & 1 \end{bmatrix}$$

$$-1R_1 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 3 & 2 & 1 & 5 \\ 0 & -4 & -2 & -2 \\ 0 & -1 & 4 & 1 \end{bmatrix}$$

$$\frac{1}{4}R_2 - R_3 \longrightarrow R_3$$

$$\begin{bmatrix} 3 & 2 & 1 & 5 \\ 0 & -4 & -2 & -2 \\ 0 & 0 & \frac{-9}{2} & \frac{-3}{2} \end{bmatrix}$$

$$\frac{-1}{2}R_2 - R_1 \longrightarrow R_1$$

$$\begin{bmatrix} -3 & 0 & 0 & -4 \\ 0 & -4 & -2 & -2 \\ 0 & 0 & \frac{-9}{2} & \frac{-3}{2} \end{bmatrix}$$

$$\frac{4}{9}R_3 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix} -3 & 0 & 0 & -4 \\ 0 & 4 & 0 & \frac{4}{3} \\ 0 & 0 & \frac{-9}{2} & \frac{-3}{2} \end{bmatrix}$$

$$\frac{-1}{3}R_1 \longrightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 4 & 0 & \frac{4}{3} \\ 0 & 0 & \frac{-9}{2} & \frac{-3}{2} \end{bmatrix}$$

$$\frac{1}{4}R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{-9}{2} & \frac{-3}{2} \end{bmatrix}$$

$$\frac{-2}{9}R_3 \longrightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

$$c_1 = \frac{4}{3}$$

$$c_2 = \frac{1}{3}$$

$$c_3 = \frac{1}{3}$$

$$\mathbf{x} = \frac{4}{3}\begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

Question 12

Let W be the subspace spanned by the $\mathbf{u}s$, and write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

$$\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

Solution: We can express $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W using the orthogonal decomposition theorem, finding the orthogonal projection of $\mathbf{y}$ onto W, and subtracting that from $\mathbf{y}$ to find the vector orthogonal to W.

$$\begin{aligned}
\hat{\mathbf{y}} &= \frac{0}{14}\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \frac{28}{42}\begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} \frac{10}{3} \\ \frac{2}{3} \\ \frac{8}{3} \end{bmatrix} \\
\mathbf{z} &= \mathbf{y} - \hat{\mathbf{y}} \\
&= \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} \frac{10}{3} \\ \frac{2}{3} \\ \frac{8}{3} \end{bmatrix} \\
\mathbf{z} &= \begin{bmatrix} -\frac{7}{3} \\ \frac{7}{3} \\ \frac{7}{3} \end{bmatrix} \\
\mathbf{y} &= \mathbf{z} + \hat{\mathbf{y}} \\
&= \begin{bmatrix} -\frac{7}{3} \\ \frac{7}{3} \\ \frac{7}{3} \end{bmatrix} + \begin{bmatrix} \frac{10}{3} \\ \frac{2}{3} \\ \frac{8}{3} \end{bmatrix}
\end{aligned}$$