Eigenvalues and Eigenvectors

Madiba Hudson-Quansah

Contents

Chapter 1

Introduction

Definition 1.0.1: Eigenvector

An eigenvector of a of a $n \times n$ matrix A is a non-zero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a non-trivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.

Example 1.0.1

Question 1

Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are $\mathbf{u}$ and $\mathbf{v}$ eigenvectors of A?

Solution:

$$A\mathbf{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\mathbf{u}$$
$$A\mathbf{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Question 2

Show that 7 is an eigenvalue of A and find the corresponding eigenvector.

Solution: To show this we need to prove that $A\mathbf{x} = \lambda\mathbf{x}$ where $\lambda = 7$ has non-trivial solutions.

$$
\begin{aligned}
A\mathbf{x} &= \lambda\mathbf{x} \\
A\mathbf{x} &= 7\mathbf{x} \\
A\mathbf{x} - 7\mathbf{x} &= \mathbf{0} \\
(A - 7I)\,\mathbf{x} &= \mathbf{0} \\
\left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\mathbf{x} &= \mathbf{0} \\
\left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}\right)\mathbf{x} &= \mathbf{0} \\
\begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}\mathbf{x} &= \mathbf{0} \\
&\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \\
&\frac{-5}{6}R_1 - R_2 \longrightarrow R_2 \\
&\begin{bmatrix} -6 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&\frac{-1}{6}R_1 \longrightarrow R_1 \\
&\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&x_1 - x_2 = 0 \\
&x_1 = x_2 \\
&x_2 = x_2 \\
\mathbf{x} = &\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

This system has non-trivial solutions as the columns are multiples of themselves and such linearly dependent. Therefore 7 is a eigenvalue of A, with the corresponding eigenvectors in the form $x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where $x_2 \neq 0$

This brings us to the next conclusion:

A scalar λ is an eigenvalue of a matrix A if and only if

$$(A - \lambda I)\,\mathbf{x} = \mathbf{0} \tag{1.1}$$

Has a non-trivial solution, where the corresponding eigenvectors is in the form of the parametric vector equation of the solution set of this non-homogeneous system.

The set of all solutions of 1.1 is just the null space of the matrix $A - \lambda I$. This solution set is a subspace of $\mathbb{R}^n$ and is called the *eigenspace* of A corresponding to λ

Definition 1.0.2: Eigenspace

The eigenspace of a matrix A corresponding to an eigenvalue λ is the set of all eigenvectors of A corresponding to λ, together with the zero vector.

Example 1.0.2

Question 3

Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis of for the eigenspace of A corresponding to $\lambda = 2$.

Solution:

$$(A - 2I)$$

$$\left(\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right)$$

$$\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

$$R_1 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 2 & -1 & 6 \end{bmatrix}$$

$$R_1 - R_3 \longrightarrow R_3$$

$$\begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{2} R_1 \longrightarrow R_1$$

$$\begin{bmatrix} 1 & \frac{-1}{2} & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = \frac{1}{2} x_2 - 3x_3 x_2 = x_2$$

$$x_3 = x_3$$

$$\mathbf{x} = \begin{bmatrix} \frac{1}{2} x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

1.0.1 Exercises

Question 4

Is $\lambda = 2$ an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$? Why or why not?

Solution:

$$\left(\begin{bmatrix}3 & 2\\3 & 8\end{bmatrix} - \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\right)$$

$$\begin{bmatrix}1 & 2\\3 & 6\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 0\\3 & 6 & 0\end{bmatrix}$$

$$3R_1 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix}1 & 2 & 0\\0 & 0 & 0\end{bmatrix}$$

$$x_1 = -2x_2$$

$$x_2 = x_2$$

$$\mathbf{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-2x_2\\x_2\end{bmatrix} = x_2\begin{bmatrix}-2\\1\end{bmatrix}$$

This columns of the matrix are linearly dependent therefore 2 is an eigenvalue of the matrix. And the eigenspace is the set of all vectors in the form $x_2\begin{bmatrix}-2\\1\end{bmatrix}$ where $x_2 \neq 0$, i.e:

$$\left\{x_2\begin{bmatrix}-2\\1\end{bmatrix} : x_2 \in \mathbb{R} \wedge x_2 \neq 0\right\}$$

Question 5

Is $\begin{bmatrix}4\\-3\\1\end{bmatrix}$ and eigenvector of in $\begin{bmatrix}3 & 7 & 9\\-4 & -5 & 1\\2 & 4 & 4\end{bmatrix}$? If so find the corresponding eigenvalue.

Solution:

$$\begin{bmatrix}3 & 7 & 9\\-4 & -5 & 1\\2 & 4 & 4\end{bmatrix}\begin{bmatrix}4\\-3\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$0\begin{bmatrix}-4\\-5\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\therefore \lambda = 0$$

$\therefore \begin{bmatrix}4\\-3\\1\end{bmatrix}$ is an eigenvector of the matrix, with 0 as its eigenvalue.

Question 6

Is $\lambda = 4$ an eigenvalue of $\begin{bmatrix}3 & 0 & -1\\2 & 3 & 1\\-3 & 4 & 5\end{bmatrix}$? If so, find one corresponding eigenvector.

Solution:

$$(A - 4I) = \mathbf{0}$$

$$\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & -1 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 4 & 1 & 0 \end{bmatrix}$$

$$-2R_1 - R_2 \to R_2$$

$$\begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -3 & 4 & 1 & 0 \end{bmatrix}$$

$$3R_1 - R_3 \to R_3$$

$$\begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -4 & -4 & 0 \end{bmatrix}$$

$$-4R_2 - R_3 \to R_3$$

$$\begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-1R_1 \to R_1$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + x_3 = 0$$
$$x_2 + x_3 = 0$$
$$x_3 = 0$$

$$x_1 = -x_3$$
$$x_2 = -x_3$$
$$x_3 = x_3$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore$ Since the columns of $(A - 4I)$ are linearly dependent, 4 is an eigenvalue of the matrix A

One eigenvector is found when $x_3 = 1$, $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

Question 7

Find a basis for the eigenspace of $A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$ with eigenvalues $\lambda = 1, 5$

Solution:

$$\begin{aligned}
&(A - 1I) \\
&\begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&\begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix} \\
&\begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \\
&\frac{1}{2}R_1 - R_2 \longrightarrow R_2 \\
&\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&\frac{1}{4}R_1 \longrightarrow R_1 \\
&\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&x_1 = 0 \\
&x_2 = x_2
\end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore \text{ the basis of the eigenspace of } A \text{ with } \lambda = 1 \text{ is } \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$(A - 5I)$$

$$\begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 2 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{2} R_1 \to R_1$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 2x_2 = 0$$

$$x_2 = x_2$$

$$x_1 = 2x_2$$

$$x_2 = x_2$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\therefore \text{ the basis of the eigenspace of } A \text{ with } \lambda = 5 \text{ is } \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

Chapter 2

The Characteristic Equation

Theorem 2.0.1

Let A be an $n \times n$ matrix. Then A is only invertible if and only if:

- The number 0 is not an eigenvalue of A
- The determinant of A is not zero

Therefore the updated properties of determinants are:

Theorem 2.0.2

Let A and B be $n \times n$ matrices

1. A is invertible if and only if $\det A \neq 0$
2. $\det AB = (\det A)(\det B)$
3. $\det A^T = \det A$
4. If A is triangular, then $\det A$ is the product of the entries on the main diagonal of A
5. A row replacement operation on A does not change the determinant of A. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same factor.

Useful information about the eigenvalues of a square matrix A is found in a special scalar equation called the characteristic equation of A.

Question 8

Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

Solution: We must find all scalars λ such that the matrix equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

Has the non-trivial solution. By the invertible matrix theorem, this is the same as finding all the scalars λ where the matrix $A - \lambda I$ is non-invertible, i.e. $\det(A - \lambda I) = 0$. Therefore

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\ &= \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \det(A - \lambda I) = 0 \therefore \\ \det\left(\begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}\right) = 0 \\ (2-\lambda)(-6-\lambda) - 9 = 0 \\ \lambda^2 + 4\lambda - 21 = 0 \\ (\lambda - 7)(\lambda - 3) = 0 \\ \lambda = 7 \\ \lambda = 3 \end{aligned}$$

Definition 2.0.1: Characteristic Equation

A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation

$$\det(A - \lambda I) = 0$$

2.1 Characteristic Polynomial

The characteristic polynomial of a matrix A is a polynomial of degree n in the variable λ, where n is the the size of the matrix A. The characteristic polynomial of A is defined as:

$$\lambda^n - (\text{trace}A)\,\lambda^{n-1} + (\text{trace}A)\,\lambda^{n-2} + \ldots + (-1)^n \det A$$

2.2 Similarity

Definition 2.2.1: Similarity

If A and B are $n \times n$ matrices, then A is similar to B if there is an invertible matrix P such that $P^{-1}AP = B$. If A is similar to B, then B is also similar to A, therefore A and B are similar.

Theorem 2.2.1

If $n \times n$ matrices of A and B are similar, then they have the same characteristic polynomial and hence have the same eigenvalues, with the same multiplicities. Therefore

$$\det(A - \lambda I) = \det(B - \lambda I)$$

Proof: If $B = P^{-1}AP$, Then

$$\begin{aligned}
B - \lambda I &= P^{-1}AP - \lambda I \\
B - \lambda I &= P^{-1}AP - \lambda P^{-1}P \\
&= P^{-1}\left(AP - \lambda P\right) \\
&= P^{-1}\left(AP - \lambda PI\right) \\
&= P^{-1}\left(A - \lambda I\right)P
\end{aligned}$$

The determinants of the two matrices are equal, then:

$$\begin{aligned}
\det\left(B - \lambda I\right) &= \det\left(P^{-1}\left(A - \lambda I\right)P\right) \\
&= \det P^{-1} \det\left(A - \lambda I\right) \det P \\
&= \det P^{-1} \det P \det\left(A - \lambda I\right) \\
&= 1 \det\left(A - \lambda I\right) \\
\det\left(B - \lambda I\right) &= \det\left(A - \lambda I\right)
\end{aligned}$$

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2.3 Exercises

Question 9

Find the characteristic polynomial and the eigenvalues of the matrices

1.
$$\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$$

Solution:

1.

$$\begin{aligned}
\det\left(A - \lambda I\right) &= 0 \\
\det\begin{bmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{bmatrix} &= 0 \\
\left(2 - \lambda\right)^2 - 49 &= 0 \\
\lambda^2 - 4\lambda + 4 - 49 &= 0 \\
\left(\lambda - 9\right)\left(\lambda + 5\right) &= 0 \\
\lambda &= 9 \\
\lambda &= -4
\end{aligned}$$

Chapter 3

Diagonalization

In many cases the eigenvalue-eigenvector information contained in a matrix A can be displayed in the factorization $A = PDP^{-1}$, where D is a diagonal matrix. This makes it easy compute A^k for large values of k.

Example 3.0.1

If $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$, then:

$$D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \times \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}$$

$$D^3 = \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5^3 & 0 \\ 0 & 3^3 \end{bmatrix}$$

Therefore generally

$$D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}$$

Theorem 3.0.1 Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
$A = PDP^{-1}$, with the diagonal matrix D, if an only if the columns of P are n linearly independent eigenvectors of A. In this case the diagonal entries of D are the eigenvalues of A that correspond, respectively to the eigenvectors in P

Example 3.0.2

Question 10

Diagonalize the following matrix if possible:

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Solution: To do this we must complete the following steps:

1. Find the eigenvalues of A
2. Find three linearly independent eigenvectors of A
3. Construct P from the vectors found in step 2
4. Construct D from the eigenvalues found in step 1

 Therefore

1. $$\det(A - \lambda I) = 0$$

   Eigenvalues: $1, -2, -2$
2. $\lambda = -2$

$$A + 2I = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}$$

$$-1R_1 - R_2 \to R_2$$

$$\begin{bmatrix} 3 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix}$$

$$R_1 - R_3 \to R_3$$

$$\begin{bmatrix} 3 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{3}R_1 \to R_1$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -x_2 - x_3$$
$$x_2 = x_2$$
$$x_3 = x_2$$

$$\mathbf{x} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 1$

$$A - I = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} -3 & -6 & -3 & 0 \\ 0 & 3 & 3 & 0 \\ 3 & 3 & 0 & 0 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix}$$

$$-1R_1 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix} -3 & -6 & -3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix}$$

$$R_2 - R_3 \longrightarrow R_3$$

$$\begin{bmatrix} -3 & -6 & -3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-2R_2 - R_1 \longrightarrow R_1$$

$$\begin{bmatrix} 3 & 0 & -3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{3}R_1 \longrightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{3}R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = x_3$$

$$x_2 = -x_3$$

$$x_3 = x_3$$

$$\mathbf{x} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Our linearly dependent eigenvectors are therefore

$$\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\}$$

3. Therefore our P:

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

4. We start the matrix with the columns from the eigenspace from the repeated eigenvalue -2, therefore we must list the entries in D in the same order.

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In checking our answers we check if the sides of the equation below are equal

$$AP = PD$$

Theorem 3.0.2

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable

Example 3.0.3

Question 11

Determine if the following matrix is diagonalizable

$$A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$$

Solution:

$$\det(A - \lambda I) = 0$$

$$\begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} 5-\lambda & -8 & 1 \\ 0 & -\lambda & 7 \\ 0 & 0 & -2-\lambda \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= \sum_{j=1}^{n} a_{ij} C_{ij} \\ &= 0 + 0 + (-1)^{3+3} \left(-2 - \lambda \begin{vmatrix} 5-\lambda & -8 \\ 0 & -\lambda \end{vmatrix} \right) \\ &= \lambda^3 - 3\lambda^2 - 10\lambda \\ \lambda &= 5 \\ \lambda &= -3 \end{aligned}$$

3.1 Exercises

Question 12

Let $A = PDP^{-1}$ and compute A^4

$$P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution:

$$\begin{aligned} A &= PDP^{-1} \\ A^2 &= PDP^{-1} \times PDP^{-1} \\ P \times P^{-1} &= I \\ A^2 &= PD^2IP^{-1} \\ A^4 &= PD^2P^{-1} \times PD^2P^{-1} \\ &= PD^4P^{-1} \end{aligned}$$

Question 13

The matrix A is factored in the form PDP^{-1}, Use the Diagonalization Theorem to find the eigenvalues and the basis of each eigenspace

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & -\frac{3}{4} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

Solution: By the Diagonalization theory, the eigenvalues of A are the diagonal entries of D, and the corresponding basis of the eigenspace of each eigenvalue is the respective column of P. Therefore the eigenvalues are $5, 1, 1$ and the basis for the eigenspace $\lambda = 5$:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

And the basis for the eigenspace $\lambda = 1$:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\}$$

Question 14

Diagonalize the matrices below

1. $$\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$$

Solution:

1.

$$\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$$

The eigenvalues are on the diagonal as this is a triangular matrix

$$\lambda = 1$$

$$A - I$$

$$\begin{bmatrix} 0 & 0 \\ 6 & -2 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 6 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{6}R_1 \longrightarrow R_1$$

$$\begin{bmatrix} 1 & \frac{-1}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = \frac{1}{3}x_2$$

$$x_2 = x_2$$

$$\mathbf{x} = \begin{bmatrix} \frac{1}{3}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

$$\lambda = -1$$

$$A + I$$

$$\begin{bmatrix} 2 & 0 \\ 6 & 0 \end{bmatrix}$$

$$3R_1 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{2}R_1 \longrightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = 0$$

$$x_2 = x_2$$

$$\mathbf{x} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{3} & 0 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Chapter 4

Eigenvectors and Linear Transformations

4.1 The Matrix of a Linear Transformation

Let V be an n-dimensional matrix, and W be an m-dimensional vector space, and let T be any linear transformation from V to W. To associate a matrix with T, choose ordered bases $\mathcal{B}$ and $\mathcal{C}$ for V and W, respectively.

Given any $\mathbf{x}$ in V the coordinates vector $[\mathbf{x}]_{\mathcal{B}}$ is in $\mathbb{R}^n$ and the coordinate vector of its image, $[T(\mathbf{x})]_{\mathcal{C}}$, is in $\mathbb{R}^m$. The connection between the coordinate vectors $[\mathbf{x}]_{\mathcal{B}}$ and $[T(\mathbf{x})]_{\mathcal{C}}$ can be found. Let $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be the basis $\mathcal{B}$ for V. If $\mathbf{x} = r_1\mathbf{b}_1 + \ldots + r_n\mathbf{b}_n$, then:

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

And therefore:

$$T(\mathbf{x}) = T(r_1\mathbf{b}_1 + \ldots + r_n\mathbf{b}_n) = r_1 T(\mathbf{b}_1) + \ldots + r_n T(\mathbf{b}_n)$$

Chapter 5

Discrete Dynamical Systems

5.1 Introduction

Eigenvalues and vectors can be used to understand the long term behaviour / evolution of a dynamical system, described by the difference equation:

$$\mathbf{x}_{k+1} = A\mathbf{x}_k$$

This equation can be used to model population movement, Markov chains, and other systems. The matrix A is called the stage matrix of the system.

Assuming A is diagonalizable, with n linearly independent eigenvectors, $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and corresponding eigenvalues, $\lambda_1, \ldots, \lambda_n$, we assume that the eigenvalues are arranged so that $|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant \lambda_n$. Since as $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is set of linearly independent vectors in $\mathbb{R}^n$, it is a basis for $\mathbb{R}^n$ as any vector in $\mathbb{R}^n$ can be written as a linear combination of the vectors in $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Therefore since $\mathbf{x}_0$ is in $\mathbb{R}^n$:

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n$$

This is called the *eigenvector decomposition* of $\mathbf{x}_0$ and determines what happens to the sequence $\{\mathbf{x}_k\}$. The sequence is:

$$\begin{aligned}\mathbf{x}_1 = A\mathbf{x}_0 &= c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \ldots + c_nA\mathbf{v}_n \\ &= c_1\lambda_1\mathbf{v}_1 + c_1\lambda_2\mathbf{v}_2 + \ldots + c_n\lambda_n\mathbf{v}_n\end{aligned}$$

And generally:

$$\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + \ldots + c_n\lambda_n^k\mathbf{v}_n, \quad (k = 0, 1, 2, \ldots)$$

5.2 Exercises

Question 15

In a certain town, 30 percent of the married women get divorced each year and 20 percent of the single women get married each year. There are currently 8000 married women and 200 single women.
Assuming the total population of women remains constant, how many married women and how many single women will there be after on year? Two years? Can you predict what would happen in the long run, say in a hundred years?

Solution: First we express the first year as an equation:

$$\begin{aligned} m_k &= 0.7m_k + 0.2s_k \\ s_k &= 0.3m_k + 0.8s_k \end{aligned}$$

$$\begin{aligned} \mathbf{x}_k &= A\mathbf{x}_{k-1} \\ \begin{bmatrix} m_k \\ s_k \end{bmatrix} &= \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \end{aligned}$$

$x_0 = \begin{bmatrix} 8000 \\ 2000 \end{bmatrix}$. We then need to see if A is diagonalizable, and if so, find the eigenvalues and eigenvectors.

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} &- \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\ &\begin{bmatrix} 0.7-\lambda & 0.2 \\ 0.3 & 0.8-\lambda \end{bmatrix} \\ (0.7-\lambda)(0.8-\lambda) - (0.2 \times 0.3) &= 0 \\ \lambda^2 - 1.5\lambda + 0.5 &= 0 \\ (\lambda - 1)(2\lambda - 1) \\ \lambda &= 1, 0.5 \end{aligned}$$

$$\begin{aligned} \lambda &= 1 \\ \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} &- \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &\begin{bmatrix} -0.3 & 0.2 \\ 0.3 & -0.2 \end{bmatrix} \\ -1R_1 - R_2 &\to R_2 \\ &\begin{bmatrix} \frac{-3}{10} & \frac{1}{5} \\ 0 & 0 \end{bmatrix} \\ \frac{-10}{3}R_1 &\to R_1 \\ \begin{bmatrix} 1 & \frac{-2}{3} \\ 0 & 0 \end{bmatrix} x_1 &= \frac{2}{3}x_2 \\ x_2 &= x_2 \\ \mathbf{x} &= x_2 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \\ &\left\{ \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \right\} \end{aligned}$$

$$\lambda = 0.5$$

$$\begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} - \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$\begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.3 \end{bmatrix}$$

$$\frac{3}{2}R_1 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ 0 & 0 \end{bmatrix}$$

$$5R_1 \longrightarrow R_1$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$x_1 = -x_2$$

$$x_2 = x_2$$

$$\mathbf{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

$$P = \begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Since the columns of P are linearly independent and in $\mathbb{R}^2$, they form a basis for $\mathbb{R}^2$. As $\mathbf{x}_0$ exists in $\mathbb{R}^2$, therefore:

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

Where $\mathbf{v}_1 = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Next we solve for c_1 and c_2 in the linear combination:

$$\begin{aligned} \mathbf{x}_0 &= c_1\lambda_1^0\mathbf{v}_1 + c_2\lambda_2^0\mathbf{v}_2 \\ &= c_1 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

Giving us the equation

$$\begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 8000 \\ 200 \end{bmatrix}$$

$$\frac{3}{2}R_1 - R_2 \to R_2$$

$$\begin{bmatrix} \frac{2}{3} & -1 & 8000 \\ 0 & \frac{-5}{2} & \frac{11800}{1} \end{bmatrix}$$

$$R_1 - R_2 \to R_2$$

$$\begin{bmatrix} \frac{2}{3} & -1 & 8000 \\ 0 & \frac{5}{2} & \frac{-11800}{1} \end{bmatrix}$$

$$R_1 - R_2 \to R_2$$

$$\begin{bmatrix} \frac{2}{3} & -1 & 8000 \\ 0 & \frac{-5}{2} & \frac{11800}{1} \end{bmatrix}$$

$$R_1 - R_2 \to R_2$$

$$\begin{bmatrix} \frac{2}{3} & -1 & 8000 \\ 0 & \frac{5}{2} & \frac{-11800}{1} \end{bmatrix}$$

$$R_1 - R_2 \to R_2$$

$$\begin{bmatrix} \frac{2}{3} & -1 & 8000 \\ 0 & \frac{-5}{2} & \frac{11800}{1} \end{bmatrix}$$

$$R_1 - R_2 \to R_2$$

$$\begin{bmatrix} \frac{2}{3} & -1 & 8000 \\ 0 & \frac{5}{2} & \frac{-11800}{1} \end{bmatrix}$$

$$R_1 - R_2 \to R_2$$

$$\begin{bmatrix} \frac{2}{3} & -1 & 8000 \\ 0 & \frac{-5}{2} & \frac{11800}{1} \end{bmatrix}$$

$$R_1 - R_2 \to R_2$$

$$\begin{bmatrix} \frac{2}{3} & -1 & 8000 \\ 0 & \frac{5}{2} & \frac{-11800}{1} \end{bmatrix}$$

$$R_1 - R_2 \to R_2$$

$$\begin{bmatrix} \frac{2}{3} & -1 & 8000 \\ 0 & \frac{-5}{2} & \frac{11800}{1} \end{bmatrix}$$

$$R_1 - R_2 \to R_2$$

$$\begin{bmatrix} \frac{2}{3} & -1 & 8000 \\ 0 & \frac{5}{2} & \frac{-11800}{1} \end{bmatrix}$$

$$R_1 - R_2 \to R_2$$

$$\begin{bmatrix} \frac{2}{3} & -1 & 8000 \\ 0 & \frac{-5}{2} & \frac{11800}{1} \end{bmatrix}$$

$$R_1 - R_2 \to R_2$$

$$\begin{bmatrix} \frac{2}{3} & -1 & 8000 \\ 0 & \frac{5}{2} & \frac{-11800}{1} \end{bmatrix}$$

$$R_1 - R_2 \to R_2$$

$$\begin{bmatrix} \frac{2}{3} & -1 & 8000 \\ 0 & \frac{-5}{2} & \frac{11800}{1} \end{bmatrix}$$

$$R_1 - R_2 \to R_2$$

$$\begin{bmatrix} \frac{2}{3} & -1 & 8000 \\ 0 & \frac{5}{2} & \frac{-11800}{1} \end{bmatrix}$$

$$R_1 - R_2 \to R_2$$

$$\begin{bmatrix} \frac{2}{3} & -1 & 8000 \\ 0 & \frac{-5}{2} & \frac{11800}{1} \end{bmatrix}$$

$$R_1 - R_2 \to R_2$$

$$\begin{bmatrix} \frac{2}{3} & -1 & 8000 \\ 0 & \frac{5}{2} & \frac{-11800}{1} \end{bmatrix}$$

$$R_1 - R_2 \rightarrow R_2$$
$$\begin{bmatrix} \frac{2}{3} & -1 & 8000 \\ 0 & \frac{-5}{2} & \frac{11800}{1} \end{bmatrix}$$
$$R_1 - R_2 \rightarrow R_2$$
$$\begin{bmatrix} \frac{2}{3} & -1 & 8000 \\ 0 & \frac{5}{2} & \frac{-11800}{1} \end{bmatrix}$$
$$R_1 - R_2 \rightarrow R_2$$
$$\begin{bmatrix} \frac{2}{3} & -1 & 8000 \\ 0 & \frac{-5}{2} & \frac{11800}{1} \end{bmatrix}$$
$$R_1 - R_2 \rightarrow R_2$$
$$\begin{bmatrix} \frac{2}{3} & -1 & 8000 \\ 0 & \frac{5}{2} & \frac{-11800}{1} \end{bmatrix}$$
$$R_1 - R_2 \rightarrow R_2$$
$$\begin{bmatrix} \frac{2}{3} & -1 & 8000 \\ 0 & \frac{-5}{2} & \frac{11800}{1} \end{bmatrix}$$
$$\frac{2}{5}R_2 - R_1 \rightarrow R_1$$
$$\begin{bmatrix} \frac{-2}{3} & 0 & \frac{-3280}{1} \\ 0 & \frac{-5}{2} & \frac{11800}{1} \end{bmatrix}$$
$$\frac{-3}{2}R_1 \rightarrow R_1$$
$$\begin{bmatrix} 1 & 0 & \frac{4920}{1} \\ 0 & \frac{-5}{2} & \frac{11800}{1} \end{bmatrix}$$
$$\frac{-2}{5}R_2 \rightarrow R_2$$
$$\begin{bmatrix} 1 & 0 & \frac{4920}{1} \\ 0 & 1 & \frac{-4720}{1} \end{bmatrix}$$
$$R_2 - R_1 \rightarrow R_1$$
$$\begin{bmatrix} -1 & 0 & \frac{-4920}{1} \\ 0 & 1 & \frac{-4720}{1} \end{bmatrix}$$
$$x_2 - 1R_1 \rightarrow R_1$$
$$\begin{bmatrix} 1 & 0 & \frac{4920}{1} \\ 0 & 1 & \frac{-4720}{1} \end{bmatrix}$$
$$c_1 = 4920$$
$$c_2 = -4720$$

$$x_k = 4920\,(1)^k\,\mathbf{v}_1 + -4720\,(0.5)^k\,\mathbf{v}_2$$

Question 16

Construct a stage matrix model for an animal speices that has two life stages, juvenile and adule. Suppsoe

Solution:

1.

$$
\begin{aligned}
j_k &= 1.6a_k \\
a_k &= 0.3j_k + 0.8a_k
\end{aligned}
$$

$$
A = \begin{bmatrix} 0 & 1.6 \\ 0.3 & 0.8 \end{bmatrix}
$$

2.

$$\det(A - \lambda I) = 0$$

$$\begin{bmatrix} 0 & 1.6 \\ 0.3 & 0.8 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} -\lambda & 1.6 \\ 0.3 & 0.8 - \lambda \end{bmatrix}$$

$$(-\lambda)(0.8 - \lambda) - 0.48 = 0$$

$$\lambda^2 - 0.8\lambda - 0.48 = 0$$

$$\lambda = 1.2, -0.4$$

$$\lambda = 1.2$$

$$\begin{bmatrix} -1.2 & 1.6 \\ 0.3 & -0.4 \end{bmatrix}$$

$$\frac{-1}{4} R_1 - R_2 \to R_2$$

$$\begin{bmatrix} \frac{-6}{5} & \frac{8}{5} \\ 0 & 0 \end{bmatrix}$$

$$\frac{-5}{6} R_1 \to R_1$$

$$\begin{bmatrix} 1 & \frac{-4}{3} \\ 0 & 0 \end{bmatrix}$$

$$x = x_2 \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix} \right\}$$

$$\lambda = -0.4$$

$$\begin{bmatrix} 0 & 1.6 \\ 0.3 & 0.8 \end{bmatrix} - \begin{bmatrix} -0.4 & 0 \\ 0 & -0.4 \end{bmatrix}$$

$$\begin{bmatrix} 0.4 & 1.6 \\ 0.3 & 1.2 \end{bmatrix}$$

$$\frac{3}{4} R_1 - R_2 \to R_2$$

$$\begin{bmatrix} \frac{2}{5} & \frac{8}{5} \\ 0 & 0 \end{bmatrix}$$

$$\frac{5}{2} R_1 \to R_1$$

$$\begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{x} = x_2 \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\}$$

$$P = \begin{bmatrix} \frac{4}{3} & -4 \\ 1 & 1 \end{bmatrix}$$
$$D = \begin{bmatrix} 1.2 & 0 \\ 0 & -0.4 \end{bmatrix}$$

Therefore:

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

Where $\mathbf{v}_1 = \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

And in the long run:

$$\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2$$

Therefore:

$$\mathbf{x}_k = c_1\left(1.2\right)^k\mathbf{v}_1 + c_2\left(-0.4\right)^k\mathbf{v}_2$$

For large values of k:

$$\mathbf{x}_k \sim c_1\left(1.2\right)^k\mathbf{v}_1$$

Therefore $\mathbf{x}_{k+1}$:

$$\mathbf{x}_{k+1} \sim 1.2\mathbf{x}_k$$

Therefore the number of juveniles will increase by 20 percent each year, and the number of adults will increase by 20 percent each year.