Logic and Proofs

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Contents

Chapter 1

Propositional Logic

Definition 1.0.1

- Proof - A correct mathematical argument.
- Theorem - A proven mathematical statement.

1.1 Proposition

Definition 1.1.1: A Proposition

A declarative sentence that is either true or false, but not both. e.g.
- $1 + 1 = 2$ True
- $2 + 2 = 3$ False

Propositional variables / Statement variables are used to represent propositions, by convention one of these variables $p, q, r, s \ldots$ The truth value of a position can be denoted by T if it is a **true proposition** and F if it is a **false proposition**.

Therefore, Let p be a proposition. The *negation of* p, denoted by $\neg p$ / $\overline{p}$, is the statement

”It is not the case that p”

The proposition $\neg p$ is read ”not p”, therefore the truth value of the negation of p is the inverse of the truth value of p

Example 1.1.1

Question 1

Find the negation of the proposition
”Michael’s PC runs Linux”
and express this in simple English.

Solution: ”Michael’s PC does not run Linux”

Example 1.1.2

Question 2

Find the negation of the proposition
"Vandana's smartphone has at least 32GB of memory"
and express this in simple English.

Solution: "Vandana's has less than 32GB of memory"

p	$\neg p$
T	F
F	T

Table 1.1: The truth table for the negation of a proposition

The negation of a proposition can also be considered the result of the operation of the *negation operator* on the proposition.

1.2 Logical Operators / Connectives

1.2.1 Conjunction

Definition 1.2.1: Conjunction

Let p and q be propositions. The *conjunction* of p and q, denoted by $p \wedge q$, is the proposition "p and q". $p \wedge q$ is T when both q and p are T and is F otherwise

Table 1.2: The truth table of $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

1.2.2 Disjunction

Definition 1.2.2: Disjunction

Let p and q be propositions. The *disjunction* of p and q, dented by $p \vee q$, is the proposition, "p or q". The *disjunction* $p \vee q$ is F when both p and q are F and T otherwise.

Table 1.3: The truth table of $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The use of the **connective** *or* in a disjunction corresponds to one of the two ways the word *or* is used in English. **Inclusive or** and **Exclusive or**, e.g.

"Students who have taken calculus or computer science can take this class"

"Students who have taken calculus or computer science, but not both can take this class"

Respectively. Therefore taking the disjunction $p \lor q$ an **Exclusive or** disjunction will F when $q = T$ and $p = T$ or $q = F$ and $p = F$, and T only when $q = T$ and $p = F$ or $q = F$ and $p = T$

1.2.2.1 Exclusive or

Definition 1.2.3: Exclusive Or

Let p and q be propositions. The *exclusive or* of p and q, denoted by $p \oplus q$, is the proposition that is T when exactly one of p and q is T and is F otherwise.

Table 1.4: The truth table of $p \oplus q$

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

1.2.3 Conditional Statement / Implication

Definition 1.2.4: Conditional Statement / Implication

Let p and q be propositions. The *conditional statement* $p \rightarrow q$ is the proposition "if p, then q". $p \rightarrow q$ is F when p is T and q is F, and T otherwise. In this connective, p is called the *hypothesis / antecedent / premise* and q is called the *conclusion / consequence*.

Table 1.5: The truth table of $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

$p \rightarrow q$ is called a conditional statement because, it asserts that q is T on the condition that p holds. $p \rightarrow q$ is T when p is F no matter the value of q.

Conditional statements can be expressed in various ways, some are listed below.

"if p, then q"

"if p, q"

"p is sufficient for q"

"q if p"

"q when p"

"a necessary condition for p is q"

"q unless $\neg p$"

"p implies q"

"p only if q"

"a sufficient condition for q is p"

"q whenever p"

"q is necessary for p"

"q follows from p"

For the more confusing statements "p only if q" and "q unless $\neg p$", the explanation follows.

"p only if q" corresponds to "if p, then q", because "p only if q" says that p cannot be T when q is not T, i.e. the statement is F when is p is T but q is F. If p is F q maybe either F or T because the statement says nothing about the value of q.

"q unless $\neg p$" expresses the same conditional statement as "if p, then q", because "q unless $\neg p$" means that if if $\neg p$ is F then q must be T, That is the statement "q unless $\neg p$" is F when p is T but q is F, but T otherwise.

Example 1.2.1

Question 3

Let p be the statement "Maria learns discrete mathematics" and q be the statement "Maria will find a good job". Express the statement $p \rightarrow q$ as a statement in English.

Solution:

"Maria will find a good job, if she learns discrete mathematics"
"For Maria to get a good job, it is sufficient for her to learn discrete mathematics"
"Maria will find a good job unless she does not learn discrete mathematics"

We can form new conditional statements from a given conditional statement, lets say $p \rightarrow q$. These are

- **Converse** - $q \rightarrow p$
- **Contrapositive** - $\neg q \rightarrow \neg p$
- **Inverse** - $\neg p \rightarrow \neg q$

Definition 1.2.5: Equivalence

When two compound propositions always have the same truth value.

1.2.3.1 Contrapositive

Definition 1.2.6: Contrapositive

The *contrapositive* of the conditional statement $p \rightarrow q$ is the conditional statement $\neg q \rightarrow \neg p$

In this statement, the hypothesis ,p, and conclusion , q, are reversed and negated. This results in an identical truth table, as the contrapositive is only F when $\neg p$ is F and $\neg q$ is T.

Note:-

A **Conditional Statement** ($p \rightarrow q$) is equivalent to it's **Contrapositive** ($\neg q \rightarrow \neg p$)

Table 1.6: The truth table of $\neg q \rightarrow \neg p$

p	q	$\neg p$	$\neg q$	$\neg q \rightarrow \neg p$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

1.2.3.2 Inverse

Definition 1.2.7: Inverse

The inverse of the conditional statement $p \longrightarrow q$ is the conditional statement $\neg p \longrightarrow \neg q$

In this statement, the hypothesis ,p, and conclusion ,q, are negated. This results in a truth table differing from the original conditional statement but equivalent to the statement's **converse**.

Note:-

A conditional statement's ($p \longrightarrow q$) **Inverse** ($\neg p \longrightarrow \neg q$) is equivalent to its **Converse** ($q \longrightarrow p$)

Table 1.7: The truth table of $\neg p \longrightarrow \neg q$

p	q	$\neg p$	$\neg q$	$\neg p \longrightarrow \neg q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

1.2.3.3 Converse

Definition 1.2.8: Converse

The converse of the conditional statement $p \longrightarrow q$ is the conditional statement $q \longrightarrow p$

In this statement, the hypothesis , p and conclusion, , q, are reversed. This results in a truth table equivalent to the conditional statement's inverse.

Note:-

A conditional statement's ($p \longrightarrow q$) **Converse** ($q \longrightarrow p$) is equivalent to its **Inverse** ($\neg p \longrightarrow \neg q$)

Table 1.8: The truth table of $q \longrightarrow p$

p	q	$q \longrightarrow p$
T	T	T
T	F	T
F	T	F
F	F	T

Example 1.2.2

Question 4

What are the contrapositive, converse, and inverse of the conditional statement
"The home team wins whenever it is raining?"

Solution:

$$\begin{aligned} p &\rightarrow q \\ q &\text{ whenever } p \\ q &= \text{Home team wins} \\ p &= \text{It is raining} \\ &= \text{If it is raining, the home team wins} \end{aligned}$$

- Contrapositive - If the home team loses, then it's not raining.
- Inverse - If its not raining then the home team loses.
- Converse - If the home team wins, then it is raining.

1.2.4 Biconditionals / Bi-implications

Another way to combine proposition that expresses they have the same truth value.

Definition 1.2.9: Biconditionals / Bi-implications

Let p and q be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition "p if and only if q". The biconditional statement $p \leftrightarrow q$ is T when p and q have the same truth values, and is F otherwise.

$p \leftrightarrow q$ breaks down to $(p \rightarrow q) \wedge (q \rightarrow p)$, and can be expressed as below

"p is necessary and sufficient for q"

"if p then q, and conversely"

"p iff q"

Note:-
"iff" - If and only If

Table 1.9: The truth table of $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example 1.2.3

Let p be the statement "You can take the flight" and let q be the statement "You buy a ticket".
Then $p \leftrightarrow q$ is the statement:
"You can take the flight if and only if you buy a ticket"

1.3 Compound Propositions

Question 5

Construct the truth table of the compound proposition

$$(p \lor \neg q) \rightarrow (p \land q)$$

Solution:

p	q	$\neg q$	$p \lor \neg q$	$p \land q$	$(p \lor \neg q) \rightarrow (p \land q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

1.3.1 Precedence of Logical Operators

Table 1.10: Precedence Table

Operator	Precedence
$\neg$	1
$\land$	2
$\lor$	3
$\rightarrow$	4
$\leftrightarrow$	5

Precedence shown from 1 to 5, with 1 having the highest precedence and 5 having the lowest precedence. Operators with higher precedence are evaluated before operators with lower precedence. Precedence can be overridden by using parentheses.

1.3.2 Logic and Bit Operations

Table 1.11: Truth value to bit table

Truth value	Bit
T	1
F	0

A bit can be used to represent a truth value due to its *binary* nature. A variable representing a truth value can be called a *boolean variable*. Computer bit operations correspond to logical operations, with the operations *OR*, *AND*, and *XOR* corresponding to the connectives, $\lor$, $\land$, and $\oplus$ respectively.

Table 1.12: Bit operations table

x	y	$x \land y$	$x \lor y$	$x \oplus y$
1	1	1	1	0
1	0	0	1	1
0	1	0	1	1
0	0	0	0	0

Definition 1.3.1: Bit String

A sequence of zero or more bits. The *length* of this string is the number of bits in the string.

Example 1.3.1

101010011 is a bit string with a length of 9.

Extending bit operations to bit strings we can define *bitwise AND*, *bitwise OR*, and *bitwise XOR* of two strings of the same length. The new bit string created can be called the *AND*, *OR*, and *XOR* of the two strings respectively.

Example 1.3.2

Question 6

Find the bitwise AND, OR, and XOR of the bit strings 01 1011 0110 and 11 0001 1101.

Solution:

$$\begin{array}{c} AND \\ \\ 0110110110 \\ \underline{1100011101} \\ 0100010100 \end{array}$$

$$\begin{array}{c} OR \\ \\ 0110110110 \\ \underline{1100011101} \\ 1110111111 \end{array}$$

$$\begin{array}{c} XOR \\ \\ 0110110110 \\ \underline{1100011101} \\ 1010101011 \end{array}$$

Chapter 2

Applications of Propositional Logic