Integration

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Chapter 1

Indefinite Integration

1.1 Anti-derivatives

A derivative $f'(x)$ is the result of performing differentiation on a function $f(x)$.

An anti-derivative $f(x)$ is the result of performing integration on a derivative $f'(x)$.

The result of an indeterminate integration on a derivate is a family of functions, each of which has a possibility of being the derivative's source function.

$$\int f'(x) \;\mathrm{d}x$$

$$= f(x) + c$$

I.e.

$$\int x^n \;\mathrm{d}x$$

$$= \frac{x^{n+1}}{n+1}$$

Chapter 2

Integration Techniques

2.1 Integration by Substitution

Question 1

$$\int x^2\sqrt{x^3+5}\ \mathrm{d}x \text{using } u = x^3+5$$

Solution:

$$\int x^2\sqrt{u}\ \mathrm{d}x$$

$$\frac{du}{dx} = x^3+5$$
$$\frac{du}{dx} = 3x^2$$
$$\frac{1}{3}\ \mathrm{d}u = x^2\ \mathrm{d}x$$

$$\int \sqrt{u}\frac{1}{3}\ \mathrm{d}u$$
$$\frac{1}{3}\int u\frac{1}{2}\ \mathrm{d}u$$
$$\frac{1}{3}\times(\frac{u^{\frac{3}{2}}}{\frac{3}{2}})+c$$
$$\frac{2u^{\frac{3}{2}}}{9}+c$$
$$= \frac{2}{9}\times(x^3+5)^{\frac{3}{2}}+c$$

2.2 Integration by Parts

LIATE

Logarithm

Inverse function

Algebra

Trigonometry

Exponent

Question 2

$$\int xe^x \ \mathrm{d}x$$

Solution: $\int xe^x \ \mathrm{d}x$ where $u = x$ and $\frac{dv}{dx} = e^x$ due to A 2.2 coming before E 2.2 in LIATE 2.2

$$\begin{aligned} u &= x \\ u' &= 1 \\ v &= \int \frac{dv}{dx} \ \mathrm{d}x \\ v &= \int e^x \ \mathrm{d}x \\ v &= e^x \end{aligned}$$

$$u \times v - \int v \times u' \ \mathrm{d}x \therefore$$

$$\begin{aligned} x \times e^x - \int e^x \times 1 \ \mathrm{d}x \\ xe^x - e^x + c \\ &= e^x(x-1) + c \end{aligned}$$

Chapter 3

Applications of Integration

3.1 Economics

3.2 Probability

3.2.1 Probability Density Function (P.D.F)

For a continuous random variable X, a Probability Density Function (P.D.F.) is a function $f(x)$ such that over a given interval $[a, b]$ / $a \leqslant x \leqslant b$:

- $f(x)$ must be continuous over the domain $[a, b]$
- $f(x) \geqslant 0$ for all x in $[a, b]$
- $\int_a^b f(x)\ \mathrm{d}x = 1$

Question 3

Let

$$f(x) = \begin{cases} \frac{3}{4}(2x - x^2), & 0 \leqslant x \leqslant 2 \\ 0, & \text{otherwise} \end{cases}$$

1. Show that $f(x)$ is a probability density function.
2. Find
   (a) $P(0.3 \leqslant x \leqslant 1.5)$
   (b) $P(x \leqslant 0.25)$
   (c) $P(x \geqslant 1.4)$
   (d) $P(x > 0.25)$

Solution:

1. **Cond 1:** $f(x)$ is continuous for all real numbers.

   Cond 2: $f(x) \geqslant 0$ for all real numbers / $(-\infty, \infty)$

Check:

$$
\begin{aligned}
x &= 1 \\
\frac{3}{4}(2x - x^2) \\
f(x) &= 0.75 \\
\\
x &= -1 \\
f(x) &= 0 \\
\\
x &= 3 \\
f(x) &= 0
\end{aligned}
$$

Cond 3: $\int_a^b f(x) = 1$
Check:

$$
\begin{aligned}
\int_{-\infty}^{0} f(x)\ \mathrm{d}x + \int_{0}^{2} f(x)\ \mathrm{d}x + \int_{2}^{\infty} f(x)\ \mathrm{d}x \\
\int_{-\infty}^{0} 0\ \mathrm{d}x + \int_{0}^{2} \frac{3}{4}(2x - x^2)\ \mathrm{d}x + \int_{2}^{\infty} 0\ \mathrm{d}x \\
0 + 1 + 1 \\
\therefore \int_{-\infty}^{\infty} f(x)\ \mathrm{d}x = 1
\end{aligned}
$$

2.

$$
\begin{aligned}
P\left(0.3 \leqslant x \leqslant 1.5\right) = \int_{0.3}^{1.5} \frac{3}{4}(2x + x^2)\ \mathrm{d}x \\
\frac{3}{4}[x^2 - \frac{1}{3}x^3]_{0.3}^{1.5} \\
\frac{3}{4}(\frac{9}{8} - \frac{81}{1000}) \\
P\left(0.3 \leqslant x \leqslant 1.5\right) = 0.7830
\end{aligned}
$$

3.

$$
\begin{aligned}
P\left(x \leqslant 0.25\right) = \int_{-\infty}^{0.25} f(x)\ \mathrm{d}x \\
\int_{-\infty}^{0} 0\ \mathrm{d}x + \int_{0}^{0.25} \frac{3}{4}(2x + x^2)\ \mathrm{d}x \\
\frac{3}{4}x^2 - \frac{1}{4}x^3|_{0}^{0.25} \\
P\left(x \leqslant 0.25\right) =
\end{aligned}
$$

Question 4

The continuous random variable X has a P.D.F. given by

$$f(x) = \begin{cases} 2x + k, & 3 \leqslant x \leqslant 4 \\ 0, & \text{otherwise} \end{cases}$$

1. Show that $k = -6$
2. Determine
   (a) $P(x > 3.5)$
   (b) $P(2.5 \leqslant x \leqslant 3.5)$
   (c) $P(x > 6)$
3. Find the expected value of X

Solution:

1.

$$\begin{aligned} \int_3^4 f(x)\ \mathrm{d}x &= 1 \\ \int_3^4 2x + k\ \mathrm{d}x &= 1 \\ x^2 + kx|_3^4 &= 1 \\ 16 + 4k - 9 - 3k &= 1 \\ k &= -6 \end{aligned}$$

2. (a)

$$\begin{aligned} &\int_{3.5}^{\infty} f(x)\ \mathrm{d}x \\ &\int_{3.5}^{4} 2x - 6\ \mathrm{d}x \\ &x^2 - 6x|_{3.5}^{4} \\ &= 0.7560 \end{aligned}$$

(b)

$$\begin{aligned} &\int_{2.5}^{3.5} f(x)\ \mathrm{d}x \\ &\int_{2.5}^{3.5} 2x - 6\ \mathrm{d}x \\ &x^2 - 6x|_{2.5}^{3.5} \\ &= 0.2580 \end{aligned}$$

Chapter 4

Improper Integrals

4.1 Infinite Limits

For

$$\int_a^\infty \frac{1}{x^p}\ \mathrm{d}x$$

- If $a > 0$ and $p > 1$, then the integral is Convergent.
- If $a > 0$ an $p \leqslant 1$, then the integral is Divergent.

Question 5

$$\int_a^\infty f(x)\ \mathrm{d}x$$

Solution:

$$\int_1^\infty \frac{1}{x}\ \mathrm{d}x$$

$$\int_1^t {}_{\lim_{t\to\infty}} \frac{1}{x}\ \mathrm{d}x$$

$$\ln(x)\Big|_1^t$$

$$\lim_{t\to\infty} \ln(t) - \ln(1)$$

$$\ln(\infty) \implies \infty$$

$$\infty - 0$$

$$\lim_{t\to\infty} \ln(t) - \ln(1) = \infty$$

$$\int_1^\infty \frac{1}{x}\ \mathrm{d}x = \infty$$

Since the limit of the integral is ∞, the integral is said to be **Divergent**.

Question 6

$$\int_{1}^{\infty} \frac{1}{x^2}\ \mathrm{d}x$$

Solution:

$$\int_{1}^{\infty} \frac{1}{x^2}\ \mathrm{d}x$$

$$\lim_{t\to\infty} \int_{1}^{t} \frac{1}{x^2}\ \mathrm{d}x$$

$$-\frac{1}{x}|_{1}^{t}$$

$$\lim_{t\to\infty} [-\frac{1}{t} + \frac{1}{1}]$$

$$0 - (-1)$$

$$\therefore \int_{1}^{\infty} \frac{1}{x^2}\ \mathrm{d}x = 1$$

Since the limit of the integral is infinite, the integral is said to be **Convergent**

Question 7

$$\int_{-\infty}^{\infty} \frac{1}{x^2}\ \mathrm{d}x$$

Solution:

$$\int_{-\infty}^{\infty} \frac{1}{x^2}\ \mathrm{d}x$$

$$\int_{-\infty}^{3} \frac{1}{x^2}\ \mathrm{d}x + \int_{3}^{\infty} \frac{1}{x^2}\ \mathrm{d}x$$

$$\lim_{t\to-\infty} \int_{t}^{3} \frac{1}{x^2}\ \mathrm{d}x$$

$$-\frac{1}{x}|_t^3$$

$$\lim_{t\to-\infty} [-\frac{1}{3} + \frac{1}{t}]$$

$$\lim_{t\to-\infty} [-\frac{1}{3} + \frac{1}{t^2}]$$

$$-\frac{1}{3} + \frac{1}{-\infty}$$

$$-\frac{1}{-\infty} \implies 0$$

$$\int_{-\infty}^{3} \frac{1}{x^2}\ \mathrm{d}x = -\frac{1}{3}$$

$$\lim_{t\to\infty} \int_{3}^{t} \frac{1}{x^2}\ \mathrm{d}x$$

$$\lim_{t\to\infty} [-\frac{1}{t} + \frac{1}{3}]$$

$$-\frac{1}{\infty} + \frac{1}{3}$$

$$-\frac{1}{\infty} \implies \infty$$

$$\int_{3}^{\infty} \frac{1}{x^2}\ \mathrm{d}x = \infty + \frac{1}{3}$$

$$\int_{-\infty}^{\infty} \frac{1}{x^2}\ \mathrm{d}x = -\frac{1}{3} + \infty + \frac{1}{3}$$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{x^2}\ \mathrm{d}x = \infty$$

Since the overall integral is infinite, the integral is Divergent / Since one of the sub-integrals are Divergent the overall integral is Divergent

Question 8

$$\int_0^3 \frac{1}{x-3}\ \mathrm{d}x$$

Solution:

$$\begin{aligned}
\int_0^3 \frac{1}{x-3}\ \mathrm{d}x \\
\lim_{t\to 3}\int_0^t \frac{1}{x-3}\ \mathrm{d}x \\
\lim_{t\to 3}\left(\ln|x-3|\right)_0^t \\
\lim_{t\to 3}\left(\ln|t-3| - \ln|-3|\right) \\
\ln(0) - \ln(3) \\
\ln(0) \implies -\infty \\
\int_0^3 \frac{1}{x-3}\ \mathrm{d}x = -\infty
\end{aligned}$$

Since the limit of the integral is infinite, the integral is **Divergent**