

Number Systems

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Chapter 1

Introduction

1.1 Number System

Definition 1.1.1: Number System

A set of numbers together with one or more operations that are defined on those numbers.

The most commonly used number system before the advent of computers was the decimal number system derived from the Hindu-Arabic number system. The decimal number system is a base-10 number system that uses 10 digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. With the advent of computers, other number systems have been developed to suit the needs of the computer such as:

- Binary number system - base-2, 1s and 0s
- Hexadecimal number system - base-16, 0-9 and A-F
- Octal number system - base-8, 0-7

These number systems are also referred to as weighted / positional number systems as the value of a digit depends on its position in the number. I.e. a number D

$$\begin{aligned} D &= d_{n-1}w_{n-1} + d_{n-2}w_{n-2} + \dots + d_1w_1 + d_0w_0 \\ &= \sum_{i=0}^{n-1} d_iw_i \end{aligned}$$

Consists of n digits with each at a position i . Every digit is associated with a fixed weight w_i based on its position. The weight w is derived from the base of the number system.

1.1.1 Radix / Base

Definition 1.1.2: Radix / Base

The number of digits in a number system including zero. This tells us the number of unique digits in the number system and the value of the base. The weight of a number system is derived from the radix, where:

$$w_i = \text{radix}^i$$

Therefore the left most digit, usually called the **Most Significant Digit**, has a weight of $n - 1$ and therefore the highest weight and the right most digit, the **Least Significant Digit**, has a weight of 0.

For example in the decimal number system, the radix/base is 10, $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and the weights are $w_i = 10^i$.

Therefore in the decimal number system the number 9203 is represented as:

$$\begin{aligned} D &= 9 \times 10^3 + 2 \times 10^2 + 0 \times 10^1 + 3 \times 10^0 \\ &= 9000 + 200 + 0 + 3 \\ &= 9203 \end{aligned}$$

When a number is fractional, the digits to the left of the radix point have positive position indices and the digits to the right of the radix point have negative position indices. Where a number D has n integral digits and m fractional digits, the value of the number is given by:

$$\begin{aligned} D &= d_{n-1}w_{n-1} + d_{n-2}w_{n-2} + \dots + d_1w_1 + d_0w_0 + d_{-1}w_{-1} + d_{-2}w_{-2} + \dots + d_{-m}w_{-m} \\ &= \sum_{i=-m}^{n-1} d_iw_i \end{aligned}$$

Therefore a fractional number in the decimal number system such as 57.6528 is represented as:

$$\begin{aligned} D &= 5 \times 10^1 + 7 \times 10^0 + 6 \times 10^{-1} + 5 \times 10^{-2} + 2 \times 10^{-3} + 8 \times 10^{-4} \\ &= 50 + 7 + 0.6 + 0.05 + 0.002 + 0.0008 \\ &= 57.6528 \end{aligned}$$

A number D with base r can be denoted as D_r or $(D)_r$, i.e. 57.6528_{10} or $(57.6528)_{10}$.

Chapter 2

Common Number Systems

2.1 Binary Number System

Definition 2.1.1: Binary Number System

A base-2 number system that uses two digits: 0 and 1. The weights of the binary number system are derived from the radix, where:

$$w_i = 2^i$$

A **binary digit** is called a **bit**. The left most bit is the **Most Significant Bit (MSB)** and the right most bit is the **Least Significant Bit (LSB)**.

A decimal equivalent of a binary number can be found using the same method as the decimal number system. i.e.:

$$D = \sum_{i=0}^{n-1} d_i w_i$$

Where $w_i = 2^i$. For example the binary number 1011_2 is represented as:

$$\begin{aligned} D &= 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 \\ &= 8 + 0 + 2 + 1 \\ &= 11_{10} \end{aligned}$$

And for a fractional binary number such as 110.11_2 :

$$\begin{aligned} D &= 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 + 1 \times 2^{-1} + 1 \times 2^{-2} \\ &= 4 + 2 + 0 + 0.5 + 0.25 \\ &= 6.75_{10} \end{aligned}$$

2.2 Octal Number System

Definition 2.2.1: Octal Number System

A base-8 number system that uses eight digits: 0, 1, 2, 3, 4, 5, 6, 7. The weights of the octal number system are derived from the radix, where:

$$w_i = 8^i$$

Each octal digit can represent 3 binary digits. Therefore an octal number can be converted to a binary number by converting each octal digit to a 3-bit binary number. For example the octal number 57_8 is represented as:

$$\begin{aligned} 57 &= 101\ 111 \\ &= 101111_2 \end{aligned}$$

And to decimal:

$$\begin{aligned} D &= 5 \times 8^1 + 7 \times 8^0 \\ &= 40 + 7 \\ &= 47 \end{aligned}$$

2.3 Hexadecimal Number System

Definition 2.3.1: Hexadecimal Number System

A base-16 number system that uses sixteen digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F. The weights of the hexadecimal number system are derived from the radix, where:

$$w_i = 16^i$$

All the letters are used to represent the numbers 10 to 15. Each hexadecimal digit can represent 4 binary digits. Therefore a hexadecimal number can be converted to a binary number by converting each hexadecimal digit to a 4-bit binary number. For example the hexadecimal number $A3F_{16}$ is represented as:

$$\begin{aligned} A3F &= 1010\ 0011\ 1110 \\ &= 101000111110 \end{aligned}$$

And in decimal:

$$\begin{aligned} D &= 10 \times 16^2 + 3 \times 16^1 + 15 \times 16^0 \\ &= 10 \times 256 + 3 \times 16 + 15 \\ &= 2623_{10} \end{aligned}$$

And $9E1_{16}$ in decimal:

$$\begin{aligned} D &= 9 \times 16^2 + 14 \times 16^1 + 1 \times 16^0 \\ &= 2529_{10} \end{aligned}$$

2.4 Common Notions

- The number of possible digits in a number system is the base of the number system.
- The smallest digit in a number system is 0 and the largest digit is $r - 1$ where r is the base of the number system.
- Adding 1 to the largest digit in a number system always results in a carry, i.e. a result of (10)
- For any number system with base r adding 1 to the largest n digit number results in a carry to the $n - 1$ digit, i.e. (r^n)
- The largest n -digit number in a number system with base r is $r^n - 1$

- The largest value that can be expressed in m fractional digits is $1 - r^{-m}$
- The largest value that can be expressed in n integral digits and m fractional digits is $r^n - r^{-m}$
- The total number of unique integers that can be expressed in a number system with base r is r^n and the total number of both integers and fractions is $r^n + r^{-m}$

Chapter 3

Arithmetic Operations

3.1 Binary Arithmetic

3.1.1 Addition

Definition 3.1.1: Addition

A binary operation, as in requiring two operands, that sums two numbers. The addition operation comprises of two operands the augend and the addend.

$$\begin{array}{r} 0011 \\ 0101 \\ \hline 0110 \end{array}$$

Chapter 4

Conversions

4.1 Decimal

4.1.1 Decimal to Binary

- Divide the decimal number by 2 and note the remainder.
- Repeat the process until the quotient is 0.
- The sequence of remainders in reverse order is the binary equivalent of the decimal number.

Example 4.1.1

Question 1

Convert the decimal number 57 to binary.

Solution:

$$\begin{aligned}57 \div 2 &= 28 \text{ remainder } 1 \\28 \div 2 &= 14 \text{ remainder } 0 \\14 \div 2 &= 7 \text{ remainder } 0 \\7 \div 2 &= 3 \text{ remainder } 1 \\3 \div 2 &= 1 \text{ remainder } 1 \\1 \div 2 &= 0 \text{ remainder } 1\end{aligned}$$

\therefore 57 in binary is 111001

4.1.2 Decimal to Octal

- Divide the decimal number by 8 and note the remainder.
- Repeat the process until the quotient is 0.
- The sequence of remainders in reverse order is the octal equivalent of the decimal number.

Example 4.1.2

Question 2

Convert the decimal number 57 to octal.

Solution:

$$57 \div 8 = 7 \text{ remainder } 1$$

$$7 \div 8 = 0 \text{ remainder } 7$$

\therefore 57 in octal is 71

4.1.3 Decimal to Hexadecimal

- Divide the decimal number by 16 and note the remainder.
- Repeat the process until the quotient is 0.
- The sequence of remainders in reverse order is the octal equivalent of the decimal number.

Example 4.1.3

Question 3

Convert the decimal number 57 to hexadecimal.

Solution:

$$57 \div 16 = 3 \text{ remainder } 9$$

$$3 \div 16 = 0 \text{ remainder } 3$$

\therefore 57 in hexadecimal is 39

4.1.4 Decimal Fraction

- Multiply the decimal fraction by the base of the number system.
- Repeat the process until the fractional part is 0 or the desired precision is reached.
- The sequence of remainders in forward order is the binary equivalent of the decimal number.