Systems of Linear Equations

Madiba Hudson-Quansah

Contents

Chapter 1

Introduction

Definition 1.0.1: Linear Equation

An equation in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where the constant b and coefficients $a_1, a_2, \ldots, a_n$ are real or complex numbers.

Definition 1.0.2: System of Linear Equations

A collection of one or more linear equations involving the same set of variables. When a system of linear equations is written in the form

$$\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 &= b_1 \\ a_4x_1 + a_6x_3 &= b_2 \end{aligned}$$

The set of variables takes on the longest subscript in the system. In this case, the variables are x_1, x_2, x_3.

Definition 1.0.3: Solution of a System of Linear Equations

The *solution* of a system of linear equations is a list of values, $(s_1, s_2, \ldots, s_n)$ that makes each equation in the system a true statement when the values are substituted for the variables, i.e. $x_1, x_2, \ldots, x_n$ and $s_1, s_2, \ldots, s_n$, where s_n is substituted for x_n

Definition 1.0.4: Solution Set

The set of all possible solutions of a system of linear equations.

Definition 1.0.5: Equivalence

Two linear systems are said to be *equivalent* if they have the same solution set.

Definition 1.0.6: Consistency

A system of linear equations is said to be *consistent* if it has at least one solution, and *inconsistent* if it has no solution.

A system of linear equations can either have:

- No solution - Equations do not intersect
- Exactly one / Unique solution - Equations intersect at a single point
- Infinitely many solutions - Equations are the same

1.1 Matrix Notation

A system of linear equations can be represented in matrix form two ways:

- Coefficient Matrix
- Augmented Matrix

1.1.1 Coefficient Matrix

Definition 1.1.1: Coefficient Matrix

Denoted by A, the coefficient matrix is a matrix that contains the coefficients of the variables in the system of linear equations with the coefficients of each equation making up each row.

Example 1.1.1

For the system of linear equations:

$$\begin{aligned} a_1x_2 + a_2x_2 + a_3x_3 &= b_1 \\ a_4x_1 + a_5x_2 + a_6x_3 &= b_2 \\ a_7x_1 + a_8x_2 + a_9x_3 &= b_3 \end{aligned}$$

The coefficient matrix is:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$$

1.1.2 Augmented Matrix

Definition 1.1.2: Augmented Matrix

Denoted by $[A|B]$, the augmented matrix is a matrix that contains the coefficients of the variables in the system of linear equations with the constant terms of each equation making up the last column.

Example 1.1.2

For the system of linear equations:

$$\begin{aligned} a_1x_2 + a_2x_2 + a_3x_3 &= b_1 \\ a_4x_1 + a_5x_2 + a_6x_3 &= b_2 \\ a_7x_1 + a_8x_2 + a_9x_3 &= b_3 \end{aligned}$$

The augmented matrix is:

$$\begin{bmatrix} a_1 & a_2 & a_3 & b_1 \\ a_4 & a_5 & a_6 & b_2 \\ a_7 & a_8 & a_9 & b_3 \end{bmatrix}$$

Definition 1.1.3: Size of a Matrix

The size of a matrix, denoted by $m \times n$, is the number of rows and columns in the matrix respectively. If $n = m$ then the matrix is said to be square, if not, it is said to be rectangular.

1.2 Solving Linear Systems

Definition 1.2.1: Pivot

Diagonal non-zero elements of a linear system

Definition 1.2.2: Forward Elimination Process

The process used to change a system into an upper triangular matrix

Definition 1.2.3: Backward Substitution Method

The process of deriving a solution from an upper triangular matrix

Definition 1.2.4: Identity Matrix

A matrix containing all zeros with pivots of 1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

One procedure used to solve linear systems is that of *simplification*. This involves replacing one linear system with a simpler equivalent system. This is done by applying the following operations to the system:

Replacement Replace one equation by the sum of itself and a multiple of another equation.

Interchange Interchange two equations.

Scaling Multiply all the terms in an equation by a non-zero constant.

Example 1.2.1

Question 1

Solve the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

Solution: Using the augmented matrix representation, we have:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

Then we times the first equation through by -5 and add it to the third equation to replace the third equation:

$$\begin{array}{r} -5x_1 + 10x_2 - 5x_3 = 0 \\ 5x_1 - 5x_3 = 10 \\ \hline 10x_2 - 10x_3 = 10 \end{array}$$

Giving us:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

We then eliminate x_2 by multiplying equation 2 by -5 and add it again to the third equation again replacing it:

$$\begin{array}{r} -10x_2 + 40x_3 = -40 \\ 10x_2 - 10x_3 = 10 \\ \hline 30x_3 = -30 \end{array}$$

Giving us:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

This new system has a triangular form, i.e.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 30x_3 &= -30 \end{aligned}$$

We then continue eliminating variables until one remains in each equation:

$$\begin{array}{r} -x_3 = 1 \\ x_1 - 2x_2 + x_3 = 0 \\ \hline x_1 - 2x_2 = 1 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

$$\begin{array}{r} 8x_3 = -8 \\ 2x_2 - 8x_3 = 8 \\ \hline 2x_2 = 0 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

$$\begin{array}{r} 2x_2 = 0 \\ x_1 - 2x_2 = 1 \\ \hline x_1 = 1 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

Giving us the system:

$$\begin{aligned} x_1 &= 1 \\ 2x_2 &= 0 \\ 30x_3 &= -30 \end{aligned}$$

Which simplifies into:

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 0 \\ x_3 &= -1 \end{aligned}$$

Definition 1.2.5: Row Equivalence

Two matrices are row equivalent if there is a sequence of elementary row operations that transforms one matrix into the other

Theorem 1.2.1

If the augmented matrices of two linear systems are row equivalent, then the two equations have the same solution set.

1.3 Identifying Existence and Uniqueness

To determine the nature of a linear system we must answer two fundamental questions:

- Is the system consistent? / Does a solution exist?
- If a solution exists, is it the only one? / Is the solution unique

Example 1.3.1

Question 2

Determine if the following system is consistent:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

Solution: Having already found the solution for this system:

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 0 \\ x_3 &= -1 \end{aligned}$$

We can determine that a solution exists, and due to the fact x_2 is uniquely determined by equation two, x_3 has only one possible value, and x_1 is also uniquely determined by equation one, we can also conclude this solution is unique.

Example 1.3.2

Question 3

Determine if the following system is consistent:

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned}$$

Solution: The augmented matrix is:

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

We interchange equations 1 and 2:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

$$\begin{array}{r} -4x_1 + 6x_2 - 4x_3 = -2 \\ 4x_1 - 8x_2 + 12x_3 = 1 \\ \hline -2x_2 + 8x_3 = -1 \end{array}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{bmatrix}$$

$$\begin{array}{r} 2x_2 - 8x_3 = 16 \\ -2x_2 + 8x_3 = -1 \\ \hline 0 = 15 \end{array}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{bmatrix}$$

Now in its triangular form, we can determine the existence and uniqueness of the solutions:

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \\ 0 &= 15 \end{aligned}$$

Since there are no coefficients for x_1, x_2, and x_3 in equation 3 equation 3 has no solution. This makes the solution set for this linear system $\{1, 8\}$. Because this set is the same as the solution set for the original linear system, $\{8, 1, 1\}$, the original system is inconsistent

1.4 Exercises

Question 4

Determine if the linear system represented by the augmented matrix below is consistent:

$$\begin{bmatrix} 1 & 5 & 2 & -6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

Solution:

$$\begin{aligned} x_1 + 5x_2 + 2x_3 &= -6 \\ 4x_2 - 7x_3 &= 2 \\ 5x_3 &= 0 \end{aligned}$$

$$\begin{aligned} x_3 &= 0 \\ x_1 + 5x_2 &= -6 \\ x_1 &= -6 - 5x_2 \end{aligned}$$

$$\begin{aligned} 4x_2 &= 2 \\ x_2 &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} x_1 &= -6 - 5\left(\frac{1}{2}\right) \\ x_1 &= -\frac{17}{2} \end{aligned}$$

Question 5

Solve the following systems:

1.

$$\begin{aligned} x_2 + 4x_3 &= -5 \\ x_1 + 3x_2 + 5x_3 &= -2 \\ 3x_1 + 7x_2 + 7x_3 &= 6 \end{aligned}$$

2.

$$\begin{aligned} x_1 - 2x_4 &= -3 \\ 2x_2 + 2x_3 &= 0 \\ x_3 + 3x_4 &= 1 \\ -2x_1 + 3x_2 + 2x_3 + x_4 &= 5 \end{aligned}$$

Solution:

1.

$$\begin{bmatrix} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 3 & 7 & 7 & 6 \\ 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \end{bmatrix}$$

$\frac{1}{3}R_1 - R_2 \longrightarrow R_2$

$$\begin{array}{r} x_1 + \frac{7}{3}x_2 + \frac{7}{3}x_3 = 2 \\ \underline{x_1 + 3x_2 + 5x_3 = -2} \\ -\frac{2}{3}x_2 - \frac{8}{3}x_3 = 4 \end{array}$$

$$\begin{bmatrix} 3 & 7 & 7 & 6 \\ 0 & -\frac{2}{3} & -\frac{8}{3} & 4 \\ 0 & 1 & 4 & -5 \end{bmatrix}$$

$3R_2$

$$\begin{bmatrix} 3 & 7 & 7 & 6 \\ 0 & -2 & -8 & 12 \\ 0 & 1 & 4 & -5 \end{bmatrix}$$

$-\frac{1}{2}R_2 - R_3 \longrightarrow R_3$

$$\begin{array}{r} x_2 + 4x_3 = -6 \\ \underline{x_2 + 4x_3 = -5} \\ 0 = -1 \end{array}$$

$$\begin{bmatrix} 3 & 7 & 7 & 6 \\ 0 & -2 & -8 & 12 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Because the system has a contradiction in row 3, $0x_1 + 0x_2 + 0x_3 = -1$, the system has no solution and is therefore inconsistent.

2.

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ -2 & 3 & 2 & 1 & 5 \end{bmatrix}$$

$R_1 \leftrightarrow R_4$

$$\begin{bmatrix} -2 & 3 & 2 & 1 & 5 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 1 & 0 & 0 & -2 & -3 \end{bmatrix}$$

$-\frac{1}{2}R_1 - R_4 \longrightarrow R_4$

$$\begin{array}{r} x_1 - \frac{3}{2}x_2 - x_3 - \frac{1}{2}x_4 = -\frac{5}{2} \\ \underline{x_1 + 0 + 0 - 2x_4 = -3} \\ -\frac{3}{2}x_2 - x_3 + \frac{3}{2}x_4 = \frac{1}{2} \end{array}$$

$$\begin{bmatrix} -2 & 3 & 2 & 1 & 5 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & -\frac{3}{2} & -1 & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

$-\frac{3}{4}R_2 - R_4 \longrightarrow R_4$

$$\begin{array}{r} 0 - \frac{3}{2}x_2 - \frac{3}{2}x_3 + 0 = 0 \\ \underline{0 - \frac{3}{2}x_2 - x_3 + \frac{3}{2}x_4 = \frac{1}{2}} \\ -\frac{1}{2}x_3 - \frac{3}{2}x_4 = -\frac{1}{2} \end{array}$$

$$\begin{bmatrix} -2 & 3 & 2 & 1 & 5 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$-\frac{1}{2}R_3 - R_4 \longrightarrow R_4$

$$\begin{array}{r} 0 + 0 - \frac{1}{2}x_3 - \frac{3}{2}x_4 = -\frac{1}{2} \\ \underline{0 + 0 - \frac{1}{2}x_3 - \frac{3}{2}x_4 = -\frac{1}{2}} \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 = 0 \end{array}$$

$$\begin{bmatrix} -2 & 3 & 2 & 1 & 5 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\frac{3}{2}R_2 - R_1 \longrightarrow R_1$

$$\begin{array}{r} 0 + 3x_2 + 3x_3 + 0 = 0 \\ \underline{-2x_1 + 3x_2 + 2x_3 + x_4 = 5} \\ 2x_1 + 0 + x_3 - x_4 = -5 \end{array}$$

$$\begin{bmatrix} 2 & 0 & 1 & -1 & -5 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 - R_1 \longrightarrow R_1$

$$\begin{array}{r} 0 + 0 + x_3 + 3x_4 = 1 \\ \underline{2x_1 + 0 + x_3 - x_4 = -5} \\ -2x_1 + 0 + 0 + 4x_4 = 6 \end{array}$$

$$\begin{bmatrix} -2 & 0 & 0 & 4 & 6 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$2R_3 - R_2 \longrightarrow R_2$

$$\begin{array}{r} 0 + 0 + 2x_3 + 6x_4 = 2 \\ \underline{0 + 2x_2 + 2x_3 + 0 = 0} \\ -2x_2 + 6x_4 = 2 \end{array}$$

$$\begin{bmatrix} -2 & 0 & 0 & 4 & 6 \\ 0 & -2 & 0 & 6 & 2 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\frac{R_1}{-2}$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & -2 & 0 & 6 & 2 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\frac{R_2}{-2}$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + -2x_4 &= -3 \\ x_2 - 3x_4 &= -1 \\ x_3 + 3x_4 &= 1 \\ 0 &= 0 \end{aligned}$$

$$\begin{aligned} x_4 &= \frac{1}{2}x_1 + \frac{3}{2} \\ x_2 &= -1 + 3x_4 \\ x_3 &= 1 - 3x_4 \\ x_1 &= -3 + 2x_4 \end{aligned}$$

Question 6

For the following matrices find the elementary row operation that transforms the first matrix into the second, and then find the reverse row operation that transforms the second matrix into the first

1.
$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & -2 & 6 \\ 0 & -5 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & -3 \\ 0 & -5 & 9 \end{bmatrix}$$

2.
$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 4 & -1 & 3 & -6 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 0 & 7 & -1 & -6 \end{bmatrix}$$

Solution:

1. Let the first matrix be M_1 and the second be M_2

$$M_1 \to M_2 = -\frac{1}{2}R_2$$
$$M_2 \to M_1 = \frac{R_2}{-\frac{1}{2}}$$

2. Let the first matrix be M_1 and the second be M_2

$$M_1 \to M_2 = -4R_1 + R_3 \to R_3$$
$$M_2 \to M_1 = R_3 - 4R_1$$

Chapter 2

Row Reduction and Echelon Forms

Definition 2.0.1: Leading Entry

The leftmost entry in a non-zero row

Definition 2.0.2: Upper triangular matrix / Echelon Form

A rectangular matrix is in *echelon form / row echelon form* if it has the following proprieties:

- All non-zero rows are above any rows of all zeros
- Each leading entry of a row is in a column to the right of the leading entry of the row above it
- All entries in a column below a leading entry are zeros

Definition 2.0.3: Reduced Row Echelon Form

A matrix is in *reduced row echelon form* if it meets all the conditions of a matrix in echelon form and:

- The leading entry in each non-zero row is 1
- Each leading 1 is the only non-zero entry in its column

Definition 2.0.4: Echelon Matrix

A matrix that is in echelon form

Definition 2.0.5: Reduced Echelon Matrix

A matrix that is in reduced row echelon form

Theorem 2.0.1 Uniqueness of a Row Reduced Echelon Form

Each matrix is row equivalent to one and only one reduced row echelon form

Therefore if matrix A is row equivalent to an echelon matrix U, U is the **echelon form of** A, and if A is row equivalent to a reduced echelon matrix R, R is the **reduced echelon form of** A.

2.1 Pivot Positions

Definition 2.1.1: Pivot Position

A *pivot position* in matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.

Definition 2.1.2: Pivot Column

A column of a matrix A that contains a pivot position.

A pivot cannot be 0 and a pivot column cannot contain any other non-zero entries. Therefore when identifying pivot positions we look for the first non-zero entry in each row, that is not in a column that already contains a pivot.

Example 2.1.1

For the reduced echelon matrix:

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivots positions are $(1,1)\,,(2,2)\,,(3,4)$ and values are $(1,2,-5)$

2.2 The Row Reduction Algorithm

The row Reduction Algorithm reduces a matrix to it's echelon form in four steps with the fifth reducing the echelon matrix to a reduced echelon matrix. These steps are:

1. Start with the leftmost non-zero column, this is the first pivot column, with its pivot position at the top of the column
2. Select a non-zero entry in the pivot column as a pivot, interchanging rows if necessary
3. Use row replacement operations to create all zeros below the pivot.
4. Cover the row containing the pivot and repeat the process for the sub matrix that remains.
5. Beginning with the rightmost pivot and working upward to the left, create zeros above each pivot. If the pivot is not 1 make it 1 via a scaling operation

Example 2.2.1

Given the matrix:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

We interchange the first and second rows to have the first item in the leftmost column be non-zero:
$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & 5 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

We then select the non-zero entry in the first column, 3, as the pivot for the first column, and use it to render zeros in the first column below the pivot $R_1 - R_3 \rightarrow R_3$

$$\begin{array}{r} 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9 \\ \underline{3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15} \\ 2x_2 - 4x_3 + 4x_4 + 2x_5 = -6 \end{array}$$

$$\begin{bmatrix} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \end{bmatrix}$$

And repeat the process for the remaining columns:

$$\frac{2}{3}R_2 - R_3 \rightarrow R_3$$

$$\begin{array}{r} 0+2x_2-4x_3+4x_4+\frac{8}{3}x_5=\frac{10}{3} \\ 0+2x_2-4x_3+4x_4+2x_5=-6 \\ \hline 0+0+0+0+\frac{2}{3}x_5=\frac{28}{3} \end{array}$$

$$\begin{bmatrix} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & 5 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{28}{3} \end{bmatrix}$$

$$6R_3-R_2 \to R_2$$

$$\begin{array}{r} 0+0+0+0+4x_5=56 \\ 0+3x_2-6x_3+6x_4+4x_5=5 \\ \hline -3x_2+6x_3-6x_4=51 \end{array}$$

$$\begin{bmatrix} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & -3 & 6 & -6 & 0 & 51 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{28}{3} \end{bmatrix}$$

$$12R_3-R_1 \to R_1$$

$$\begin{array}{r} 0+0+0+0+8x_5=612 \\ 3x_1-7x_2+8x_3-5x_4+8x_5=9 \\ \hline -3x_1+7x_2-8x_3+5x_4=603 \end{array}$$

$$\begin{bmatrix} -3 & 7 & -8 & 5 & 0 & 603 \\ 0 & -3 & 6 & -6 & 0 & 51 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{28}{3} \end{bmatrix}$$

$$-\frac{7}{3}R_2-R_1 \to R_1$$

$$\begin{array}{r} 0+7x_2-14x_3+14x_4+0=-119 \\ -3x_1+7x_2-8x_3+5x_4+0=603 \\ \hline 3x_1+0-6x_3+9x_4=-722 \end{array}$$

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -722 \\ 0 & -3 & 6 & -6 & 0 & 51 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{28}{3} \end{bmatrix}$$

$$\frac{3}{2}R_3 \to R_3$$

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -722 \\ 0 & -3 & 6 & -6 & 0 & 51 \\ 0 & 0 & 0 & 0 & 1 & 14 \end{bmatrix}$$

$$-\frac{1}{3}R_2 \to R_2$$

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -722 \\ 0 & 1 & -2 & 2 & 0 & -17 \\ 0 & 0 & 0 & 0 & 1 & 14 \end{bmatrix}$$

$$\frac{1}{3}R_1 \to R_1$$

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -\frac{722}{3} \\ 0 & 1 & -2 & 2 & 0 & -17 \\ 0 & 0 & 0 & 0 & 1 & 14 \end{bmatrix}$$

2.3 Solutions of Linear Systems

Definition 2.3.1: Free Variable

A variable that does not exist in a row of a matrix

Definition 2.3.2: Parametric Equations

Any equation that expresses the variables in a system of linear equations in terms of a free variable.

Definition 2.3.3: Basic Variable

A variable that exists in a row of a matrix

In deriving a reduced echelon matrix we can determine the consistency of a system of linear equations, and thereby describe the solution set. For example in the case below we have the reduced echelon matrix:

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

There are three variables $x_1, x_2,$ and x_3, as there are four columns in this augmented matrix. The variables x_1 and x_2 are referred to as basic variables as they have pivots in their columns, and x_3 is a free variable as it does not have a pivot in its column, thus this system is consistent has has infinitely many solutions. Describing the solution set of this equation we can say:

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases}$$

The statement "x_3 is free" means that x_3 can take on any value, and by extension the values of x_1 and x_2 are determined by the value x_3 takes on.

Example 2.3.1

Question 7

Find the general solution of the linear system whose augmented matrix has been reduced to:

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$-1R_3 - R_2 \rightarrow R_2$$

$$\begin{array}{r} 0+0+0+0-1=-7 \\ 0+0+2-8-1=-3 \\ \hline 0+0+-2+8+0=-10 \end{array}$$

$$\left[\begin{array}{llllll} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & -2 & 8 & 0 & -10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right]$$

$$-2R_3 - R_1 \longrightarrow R_1$$

$$\begin{array}{r} 0+0+0+0-2=-14 \\ 1+6+2-5-2=-4 \\ \hline -1-6-2+5+0=-10 \end{array}$$

$$\left[\begin{array}{llllll} -1 & -6 & -2 & 5 & 0 & -10 \\ 0 & 0 & -2 & 8 & 0 & -10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right]$$

$$R_2 - R_1 \longrightarrow R_1$$

$$\begin{array}{r} 0+0-2+8+0=-10 \\ -1-6-2+5+0=-10 \\ \hline 1+6+0+3+0=0 \end{array}$$

$$\left[\begin{array}{llllll} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 8 & 0 & -10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right]$$

$$-\frac{1}{2}R_2 \longrightarrow R_2$$

$$\left[\begin{array}{llllll} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right]$$

$$\begin{aligned} x_1 + 6x_2 + 3x_4 &= 0 \\ x_3 - 4x_4 - &= 5 \\ x_5 &= 7 \end{aligned}$$

$$\begin{aligned} x_1 &= -6x_2 - 3x_4 \\ x_3 &= 5 + 4x_4 \\ x_5 &= 7 \end{aligned}$$

$\therefore$

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 4x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$$

2.3.1 Parametric Description of Solution Sets

Definition 2.3.4: Parametric Description

A description of the solution set of a system of linear equations in terms of a free variable.

The descriptions given so far are all parametric, i.e., the free variables act as parameters that determine the values of the basic variables.

2.3.2 Uniqueness and Existence of Solutions

Theorem 2.3.1 Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column, i.e., if and only if an echelon form of the augmented matrix has no row of the form:

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & b \end{bmatrix}$$

where b is a non-zero number.
If a linear system is consistent then the solution set contains either:

- A unique solution, when there are no free variables.
- Infinitely many solutions in the presence of at least one free variable.

2.4 Exercises

Question 8

Find the general solutions of the systems whose augmented matrices are:

1. $$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix}$$
2. $$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}$$
3. $$\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution:

1.

$$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix}$$

$$3R_1 - R_2 \to R_2$$

$$\begin{array}{r} 3+9+12=21 \\ 3+9+7=6 \\ \hline 0+0+5=15 \end{array}$$

$$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 5 & 15 \end{bmatrix}$$

$$\frac{4}{5}R_2 - R_1 \to R_1$$

$$\begin{array}{r} 0+0+4=12 \\ 1+3+4=7 \\ \hline -1-3+0=5 \end{array}$$

$$\begin{bmatrix} -1 & -3 & 0 & 5 \\ 0 & 0 & 5 & 15 \end{bmatrix}$$

$$\frac{1}{5}R_2$$

$$\begin{bmatrix} -1 & -3 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$-R_1$$

$$\begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{aligned} x_1 + 3x_2 &= -5 \\ x_3 &= 3 \end{aligned}$$

$$\begin{cases} x_1 = -5 - 3x_2 \\ x_2 \text{ is free} \\ x_3 = 3 \end{cases}$$

2.

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ -1 & 7 & -4 & 2 & 7 \\ 0 & 0 & 1 & -2 & -3 \end{bmatrix}$$

$$R_1 + R_2 \to R_2$$

$$\begin{array}{r} 1 - 7 + 0 + 6 = 5 \\ \underline{-1 + 7 - 4 + 2 = 7} \\ 0 + 0 - 4 + 8 = 12 \end{array}$$

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & -4 & 8 & 12 \\ 0 & 0 & 1 & -2 & -3 \end{bmatrix}$$

$$\frac{1}{4}R_2 + R_3 \to R_3$$

$$\begin{array}{r} 0 + 0 - 1 + 2 = 3 \\ \underline{0 + 0 + 1 - 2 = -3} \\ 0 + 0 + 0 + 0 = 0 \end{array}$$

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & -4 & 8 & 12 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-\frac{1}{4}R_2 \to R_2$$

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - 7x_2 + 6x_4 &= 5 \\ x_3 - 2x_4 &= -3 \\ 0 &= 0 \end{aligned}$$

$$\begin{aligned} x_1 &= 5 - 7x_2 + 6x_4 \\ x_3 &= -3 + 2x_4 \end{aligned}$$

$$\begin{cases} x_1 = 5 - 7x_2 + 6x_4 \\ x_2 \text{ is free} \\ x_3 = -3 + 2x_4 \\ x_4 \text{ is free} \end{cases}$$

Question 9

Find h and k such that the system has:

1. No solutions
2. A unique solution
3. Many solutions
1. $$\begin{aligned} x_1 + hx_2 &= 2 \\ 4x_1 + 8x_2 &= k \end{aligned}$$
2. $$\begin{aligned} x_1 + 3x_2 &= 2 \\ 3x_1 + hx_2 &= k \end{aligned}$$

Solution:

1. $$\begin{aligned} x_1 + hx_2 &= 2 \\ 4x_1 + 8x_2 &= k \end{aligned}$$

   (a) $$\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix}$$

   $$-4R_1 + R_2 \longrightarrow R_2$$

   $$\begin{aligned} -4 - 4h &= 8 \\ 4 + 8 &= k \\ 0 + (-4h + 8) &= (-8 + k) \end{aligned}$$

   $$\begin{bmatrix} 1 & h & 2 \\ 0 & (-4h + 8) & (-8 + k) \end{bmatrix}$$

   $$\begin{aligned} -4h + 8 &= 0 \\ h &= 2 \end{aligned}$$

   $$\begin{aligned} -8 + k &\neq 0 \\ k &\neq 8 \end{aligned}$$

   $\therefore h = 2$ and $k \neq 8$

(b)

$$\begin{bmatrix}1 & h & 2\\ 4 & 8 & k\end{bmatrix}$$

$$-4R_1 + R_2 \longrightarrow R_2$$

$$-4 - 4h = 8$$

$$4 + 8 = k$$

$$0 + (-4h + 8) = (-8 + k)$$

$$\begin{bmatrix}1 & h & 2\\ 0 & (-4h+8) & (-8+k)\end{bmatrix}$$

$$h \neq 2$$

$\therefore h \neq 2$

(c)

$$\begin{bmatrix}1 & h & 2\\ 4 & 8 & k\end{bmatrix}$$

$$-4R_1 + R_2 \longrightarrow R_2$$

$$-4 - 4h = 8$$

$$4 + 8 = k$$

$$0 + (-4h + 8) = (-8 + k)$$

$$\begin{bmatrix}1 & h & 2\\ 0 & (-4h+8) & (-8+k)\end{bmatrix}$$

$$-4h + 8 = 0$$

$$h = 2$$

$$-8 + k = 0$$

$$k = 8$$

$\therefore h = 2$ and $k = 8$

Chapter 3

Vector Equations

Instead of matrix notation, we can represent a system of linear equations as a vector equation. For example, the system:

$$2x_1 + 4x_2 = 3$$
$$3x_1 + 2x_2 = 7$$

Can be represented as:

$$x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

3.0.1 Vectors in $\mathbb{R}^2$

Definition 3.0.1: Vector

A matrix with a single column

$\mathbb{R}^2$ refers to the set of all vectors with two real number entries this is also called the vector space. For example:

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

3.0.1.1 Axioms of a Vector Space

Commutativity $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}, \forall \mathbf{v} \in V$

Associativity $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ for all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$

Zero Vector there exists a special vector, denoted by $\mathbf{0}$ such that $\forall \mathbf{v} \in V$, $\mathbf{v} + \mathbf{0} = \mathbf{v}$

Additive Inverse For every vector $\mathbf{v} \in V$ there exists a vector $\mathbf{w} \in V$ such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$. Such additive inverse is denoted by $-\mathbf{v}$

Multiplicative Identity $1\mathbf{v} = \mathbf{v}, \forall \mathbf{v} \in V$

Multiplicative Associativity $(\alpha\beta)\,\mathbf{v} = \alpha\,(\beta\mathbf{v}), \forall \mathbf{v} \in V$ and all scalars α, β

Distributive Propety 1 $\alpha\,(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha + \mathbf{v}$, for all $\mathbf{u}$, $\mathbf{v} \in V$ and all scalars α

Distributive Propoerty 2 $(\alpha + \beta)\,\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}\ \forall \mathbf{v} \in V$ and all scalars α, β

Example 3.0.1

Question 10

Given the $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, find $4\mathbf{u}$, $(-3)\,\mathbf{v}$ and $4\mathbf{u} + (-3)\,\mathbf{v}$

Solution:

1.
$$4\mathbf{u} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$$

2.
$$-3\mathbf{v} = \begin{bmatrix} -6 \\ 15 \end{bmatrix}$$

3.
$$\begin{aligned} 4\mathbf{u} + (-3)\,\mathbf{v} &= \begin{bmatrix} 4 \\ -8 \end{bmatrix} + \begin{bmatrix} -6 \\ 15 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 7 \end{bmatrix} \end{aligned}$$

3.1 Linear Combinations

Definition 3.1.1: Linear Combination

Given vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}$ in $\mathbb{R}^n$, and given scalars $c_1, c_2, \ldots, c_p$, then the vector $\mathbf{y}$ defined by

$$y = c_1\mathbf{v}_1 + c_2\mathbf{v}_2, \ldots, c_p\mathbf{v}_p$$

Is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_p$, with weights $c_1, \ldots, c_p$

Definition 3.1.2: Mapping Function

A function in the form

$$f : \mathbb{R}^n \to R$$

That takes in a vector as input, $\mathbb{R}^n$ and returns a scalar, R.

Example 3.1.1

Question 11

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$, Determine whether $\mathbf{b}$ can be generated as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. That is determine whether weights x_1 and x_2 exist such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$$

If such an equation has a solution find it.

Solution:

$$
\begin{aligned}
x_1\begin{bmatrix}1\\-2\\-5\end{bmatrix}+x_2\begin{bmatrix}2\\5\\6\end{bmatrix}&=\begin{bmatrix}7\\4\\-3\end{bmatrix}\\
\begin{bmatrix}x_1\\-2x_1\\-5x_1\end{bmatrix}+\begin{bmatrix}2x_2\\5x_2\\6x_2\end{bmatrix}&=\begin{bmatrix}7\\4\\-3\end{bmatrix}\\
\begin{bmatrix}x_1+2x_2\\-2x_1+5x_2\\-5x_1+6x_2\end{bmatrix}&=\begin{bmatrix}7\\4\\-3\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
x_1+2x_2&=7\\
-2x_1+5x_2&=4\\
-5x_1+6x_2&=-3
\end{aligned}
$$

$$
\begin{bmatrix}1&2&7\\-2&5&4\\-5&6&-3\end{bmatrix}\sim\begin{bmatrix}1&2&7\\0&-9&-18\\-5&6&-3\end{bmatrix}\sim\begin{bmatrix}1&2&7\\0&-9&-18\\0&-16&-32\end{bmatrix}\sim
$$

$$
\begin{bmatrix}1&2&7\\0&-9&-18\\0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&3\\0&-9&-18\\0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&3\\0&1&2\\0&0&0\end{bmatrix}
$$

$$
\begin{aligned}
x_1&=3\\
x_2&=2
\end{aligned}
$$

$$
3\mathbf{a}_1+2\mathbf{a}_2=\begin{bmatrix}7\\4\\-3\end{bmatrix}
$$

This example leads to the following theorem:

Theorem 3.1.1

A vector equation

$$x_1\mathbf{a}_1+x_2\mathbf{a}_2+\ldots+x_n\mathbf{a}_n=\mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix}\mathbf{a}_1&\mathbf{a}_2&\ldots&\mathbf{a}_n&\mathbf{b}\end{bmatrix}$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1,\ldots,\mathbf{a}_n$ if an only if there exists a solution to the linear system corresponding to the augmented matrix above.

3.1.1 Span

Definition 3.1.3: Span

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned / generated by** $\mathbf{v}_1, \ldots, \mathbf{v}_p$. That is, Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p$$

with $c_1, \ldots, c_p$ scalars.

Therefore asking whether a vector $\mathbf{b}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, amounts to asking if $\mathbf{b}$ can be expressed as the vector equation:

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{b}$$

And whether the equation has a solution.

Chapter 4

The Matrix Equation $A\mathbf{x} = \mathbf{b}$

Definition 4.0.1: The Matrix Equation

If A is a $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the columns product of A and $\mathbf{x}$, denoted by $A\mathbf{x}$, is the linear combination of columns of A using the corresponding entries as weights, i.e.

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n$$

Note that the matrix equation is only defined when the number of columns in A is equal to the number rows in $\mathbf{x}$.

Example 4.0.1

$$\begin{aligned} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 & 3 & 7 \end{bmatrix} &= 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{aligned}$$

Example 4.0.2

Question 12

For $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, in $\mathbb{R}_n$, write the linear combination $3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3$ as a matrix times a vector.

Solution:

$$3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3 = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}\begin{bmatrix}3\\-5\\7\end{bmatrix}$$

$$A\mathbf{x} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}\begin{bmatrix}3\\-5\\7\end{bmatrix}$$

Theorem 4.0.1

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^n$, the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

which in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b}\end{bmatrix}$$

4.0.1 Existence of Solutions

Theorem 4.0.2

The matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

Example 4.0.3

Question 13

Let $A = \begin{bmatrix}1 & 3 & 4\\-4 & 2 & -6\\-3 & -2 & -7\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible b_1, b_2, b_3

Solution:

$$\begin{bmatrix}1 & 3 & 4 & b_1\\-4 & 2 & -6 & b_2\\-3 & -2 & -7 & b_3\end{bmatrix}$$

$$\begin{bmatrix}1 & 3 & 4 & b_1\\-4 & 2 & -6 & b_2\\-3 & -2 & -7 & b_3\end{bmatrix} \sim \begin{bmatrix}1 & 3 & 4 & b_1\\0 & -14 & -10 & -4b_1 - b_2\\-3 & -2 & -7 & b_3\end{bmatrix} \sim \begin{bmatrix}1 & 3 & 4 & b_1\\0 & -14 & -10 & -4b_1 - b_2\\0 & -7 & -5 & -3b_1 - b_3\end{bmatrix} \sim$$

$$\begin{bmatrix}1 & 3 & 4 & b_1\\0 & -14 & -10 & -4b_1 - b_2\\0 & 0 & 0 & b_1 - \frac{1}{2}b_2 + b_3\end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{b}$ is not consistent for every $\mathbf{b}$ because some choices of $\mathbf{b}$ can make $b_1 - \frac{1}{2}b_2 + b_3$ non-zero.

Theorem 4.0.3

Let A be an $m \times n$ coefficient matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

- For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
- Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
- The columns A span $\mathbb{R}^m$
- A has a pivot position in every row.

4.0.2 Computation of $A\mathbf{x}$

Definition 4.0.2: Row-Vector Rule for computing $A\mathbf{x}$

If the product $A\mathbf{x}$ is defined, then the ith entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$

Example 4.0.4

Question 14

Compute $A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Solution:

$$\begin{aligned} \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 2 \cdot x_1 + 3 \cdot x_2 + 4 \cdot x_3 \\ -1 \cdot x_1 + 5 \cdot x_2 - 3 \cdot x_3 \\ 6 \cdot x_1 - 2 \cdot x_2 + 8 \cdot x_3 \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix} \end{aligned}$$

4.0.3 Properties of Matrix-Vector Product $A\mathbf{x}$

If A is an $m \times n$ matrix, $\mathbf{u}$ are $\mathbf{v}$ are vectors in $\mathbb{R}^n$ and c is a scalar, then:

1. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
2. $A(c\mathbf{u}) = c(A\mathbf{u})$

Chapter 5

Solution Sets of Linear systems

5.1 Homogeneous Linear Systems

Definition 5.1.1: Homogeneous Linear System

A system that can be expressed in the form

$$A\mathbf{x} = \mathbf{0}$$

Where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector $\mathbb{R}^m$.

Such a system $A\mathbf{x} = \mathbf{0}$ always has at least one solution, namely $\mathbf{x} = \mathbf{0}$. This zero solution is also called the *trivial solution*. The question is then is whether there exists a *non-trivial solution* that satisfies $A\mathbf{x} = \mathbf{0}$.

Definition 5.1.2: Trivial solution

A solution of zero

Definition 5.1.3: Non-Trivial solution

A solution that is not zero

Theorem 5.1.1

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution if and only if the equation has at least one free variable.

Example 5.1.1

Question 15

Determine if the following homogeneous system has a non-trivial solution. Then describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

Solution:

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & -3 & 0 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 9 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - \frac{4}{3}x_3 = 0$$

$$x_2 = 0$$

$$0 = 0$$

This gives us x_3 as free. Due to the representation of a vector $x_3 = x_3$

In vector notation we describe the solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3\mathbf{v}$$

Where $\mathbf{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$

Example 5.1.2

Question 16

A single linear equation can be treated as a very simple system of equations. Describe all solutions of the homogeneous system

$$10x_1 - 3x_2 - 2x_3 = 0$$

Solution:

$$\begin{bmatrix} 10 & -3 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = \frac{3}{10}x_2 + \frac{1}{5}x_3$$

$$x_2 = x_2$$

$$x_3 = x_3$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{10}x_2 + \frac{1}{5}x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} \frac{3}{10} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{1}{5} \\ 0 \\ 1 \end{bmatrix}$$

5.2 Parametric Vector Form

Definition 5.2.1: Parametric Vector Form

A description of the solution set of a system of linear equations in terms of a free variable. In the form:

$$\mathbf{x} = s\mathbf{u} + t\mathbf{v}, \text{ where} s, t \in \mathbb{R}$$

5.3 Non-Homogeneous Linear Systems

Definition 5.3.1: Non-Homogeneous Linear System

A system that can be expressed in the form

$$A\mathbf{x} = b$$

Where A is a coefficient matrix, $\mathbf{x}$ is a vector in $\mathbb{R}^n$, and $\mathbf{b}$ is a vector in $\mathbb{R}^m$.

When a non-homogeneous system has many solutions, the general solution can be written in parametric vector form as one vector plus an arbitrary linear combination of other vectors that satisfy the system.

Example 5.3.1

Question 17

Describe all solutions of $A\mathbf{x} = b$, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

Solution:

Theorem 5.3.1

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Chapter 6

Linear Independence

Definition 6.0.1: Linear Independence

An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be *linearly independent* if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

Definition 6.0.2: Linear Dependence

The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be *linearly dependent* if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$$

Example 6.0.1

Question 18

Let $\mathbf{v}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}4\\5\\6\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}2\\1\\0\end{bmatrix}$

1. Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent
2. If possible find a linear dependence relationship between the vectors

Solution:

1.

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + 4x_2 + 2x_3 &= 0 \\ 2x_1 + 5x_2 + x_3 &= 0 \\ 3x_1 + 6x_2 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix}$$
$$2R_1 - R_2 \longrightarrow R_2$$
$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix}$$
$$3R_1 - R_3 \longrightarrow R_3$$
$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 6 & 6 & 0 \end{bmatrix}$$
$$2R_2 - R_3 \longrightarrow R_3$$
$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent

2.

$$\frac{4}{3}R_2 - R_1 \longrightarrow R_1$$
$$\begin{bmatrix} -1 & 0 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} -x_1 + 2x_3 &= 0 \\ 3x_2 + 3x_3 &= 0 \\ x_3 &= x_3 \end{aligned}$$

$$\begin{aligned} x_1 &= 2x_3 \\ x_2 &= -x_3 \\ x_3 &= x_3 \end{aligned}$$

$$\begin{cases} x_1 = 2x_3 \\ x_2 = -x_3 \\ x_3 \text{ is free} \end{cases}$$

6.0.1 Linear Independence of Matrix Columns

Definition 6.0.3: Linear Independence of Matrix Columns

A matrix A is said to have linearly independent columns if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Example 6.0.2

Question 19

Determine if the columns of the matrix are linearly independent

$$A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix}$$

$$5R_1 - R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 2 & -5 & 0 \end{bmatrix}$$

$$2R_2 - R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}$$

$\therefore$ The columns of the matrix A are linearly dependent as the equation $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution.

Chapter 7

Linear Transformations

The matrix equation $A\mathbf{x} = \mathbf{b}$, can be though of as the transformation of the vector $\mathbf{x}$ by A into the vector $\mathbf{b}$. For example, the equations:

$$\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \text{ and } \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Infer that multiplication by matrix A transforms the vectors in vector space $\mathbb{R}^4$ $\mathbf{x}$ and $\mathbf{u}$ into the respective $\mathbb{R}^2$ vectors

Definition 7.0.1: Transformation / Function / Mapping / T

A function T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the *domain* of T, and $\mathbb{R}^m$ is called the *codomain* of T. The notation

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

indicates that the domain of T is $\mathbb{R}^n$ and the codomain of T is $\mathbb{R}^m$.
For $\mathbf{x}$ in $\mathbb{R}^n$ the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the *image* of $\mathbf{x}$ (under the action of T). The set of all images $T(x)$ is called the *range* of T

7.1 Matrix Transformations

Definition 7.1.1: Matrix Transformation

Given an $m \times n$ matrix A, the transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by

$$T(\mathbf{x}) = A\mathbf{x}$$

is called a *matrix transformation*, denoted as

$$\mathbf{x} \mapsto A\mathbf{x}$$

Note that the domain of T is $\mathbb{R}^n$ when A has n columns, and the codomain of T is $\mathbb{R}^m$ when A has m rows.

Example 7.1.1

Question 20

Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, and define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$, by $T(\mathbf{x}) = A\mathbf{x}$, so that

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

1. Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation T
2. Find an $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\mathbf{b}$
3. Is there more than one $\mathbf{x}$, whose image under T is $\mathbf{b}$?
4. Determine if $\mathbf{c}$ is in the range of the transformation T

Solution:

1.

$$\begin{aligned} T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} &= \begin{bmatrix} (2) - 3(-1) \\ 3(2) + 5(-1) \\ -(2) + 7(-1) \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix} \end{aligned}$$

2.

$$
\begin{aligned}
T(\mathbf{x}) &= A\mathbf{x} = \mathbf{b} \\
&= \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \\
&\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \\
&3R_1 - R_2 \rightarrow R_2 \\
&\begin{bmatrix} 1 & -3 & 3 \\ 0 & -14 & 7 \\ -1 & 7 & -5 \end{bmatrix} \\
&R_2 \leftrightarrow R_3 \\
&\begin{bmatrix} 1 & -3 & 3 \\ 0 & -14 & 7 \\ -1 & 7 & -5 \end{bmatrix} \\
&-2R_2 - R_3 \rightarrow R_3 \\
&\begin{bmatrix} 1 & -3 & 3 \\ -1 & 7 & -5 \\ 2 & 0 & 3 \end{bmatrix} \\
&-1R_1 - R_2 \rightarrow R_2 \\
&\begin{bmatrix} 1 & -3 & 3 \\ 0 & -4 & 2 \\ 2 & 0 & 3 \end{bmatrix} \\
&2R_1 - R_3 \rightarrow R_3 \\
&\begin{bmatrix} 1 & -3 & 3 \\ 0 & -4 & 2 \\ 0 & -6 & 3 \end{bmatrix} \\
&\frac{3}{2}R_2 - R_3 \rightarrow R_3 \\
&\begin{bmatrix} 1 & -3 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \\
&\frac{3}{4}R_2 - R_1 \rightarrow R_1 \\
&\begin{bmatrix} -1 & 0 & \frac{-3}{2} \\ 0 & -4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \\
x_1 &= \frac{3}{2} \\
x_2 &= -\frac{1}{2}
\end{aligned}
$$

$\therefore \mathbf{x} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}$

3.

$$T(\mathbf{x}) = \mathbf{b}$$

$$\begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & \frac{-3}{2} \\ 0 & -4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

As the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, there is exactly one $\mathbf{x}$ whose image is $\mathbf{b}$

4. For $\mathbf{c}$ to be in the range of the transformation T, a $\mathbf{x}$ transformed by T must map to $\mathbf{c}$, therefore:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix}$$

$$3R_1 - R_2 \to R_2$$

$$\begin{bmatrix} 1 & -3 & 3 \\ -1 & 7 & 5 \\ 0 & -14 & 7 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & -3 & 3 \\ -1 & 7 & 5 \\ 0 & -14 & 7 \end{bmatrix}$$

$$-2R_2 - R_3 \to R_3$$

$$\begin{bmatrix} 1 & -3 & 3 \\ -1 & 7 & 5 \\ 2 & 0 & -17 \end{bmatrix}$$

$$-R_1 - R_2 \to R_2$$

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & -4 & -8 \\ 2 & 0 & -17 \end{bmatrix}$$

$$2R_1 - R_3 \to R_3$$

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & -4 & -8 \\ 0 & -6 & 23 \end{bmatrix}$$

$$\frac{3}{2}R_2 - R_3 \to R_3$$

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & -4 & -8 \\ 0 & 0 & -35 \end{bmatrix}$$

The system is inconsistent, meaning $\mathbf{c}$ is not in the range of T

7.2 Linear Transformations

Definition 7.2.1: Linear Transform

A transformation / mapping T is linear if:
- $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$, $\forall \mathbf{u}, \mathbf{v}$ in the domain of T
- $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T

Linear Transformations preserve the operations of vector addition and scalar multiplication.

Theorem 7.2.1

If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}$$

and

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

for all vectors $\mathbf{u}$, $\mathbf{v}$ in the domain of T and all scalars c, d.

Definition 7.2.2: Contraction

Given a scalar r, and a transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(\mathbf{x}) = r\mathbf{x}$. T is a contraction when $0 \leqslant r \leqslant 1$.

Definition 7.2.3: Dilation

Given a scalar r, and a transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(\mathbf{x}) = r\mathbf{x}$. T is a dilation when $r > 1$

Example 7.2.1

Question 21

Let $r = 3$, and show that T is a linear transformation.

Solution: Let $\mathbf{u}$, and $\mathbf{v}$ be vectors in $\mathbb{R}^2$, and c, and d be scalars.

$$\begin{aligned} T(c\mathbf{u} + d\mathbf{v}) &= 3(c\mathbf{u} + d\mathbf{v}) \\ &= 3c\mathbf{u} + 3d\mathbf{v} \\ &= c(3\mathbf{u}) + d(3\mathbf{v}) \\ &= cT(\mathbf{u}) + dT(\mathbf{v}) \end{aligned}$$

Thus T is a linear transformation because its satisfies the properties of linearity.

7.3 The Matrix of a Linear Transformation

Definition 7.3.1: Matrix of a Linear Transformation

Let T be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}$$

for all $\mathbf{x}$ in $\mathbb{R}^n$. The matrix A is called the *standard matrix* for the linear transformation T.

The key to finding the matrix A of a linear transformation T is determine the images of the standard unit vectors found in the columns of $n \times n$ identity matrix I_n

Example 7.3.1

Question 22

The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose T is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ such that

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \text{ and } T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

With no additional information, find a formula for the image of an arbitrary $\mathbf{x}$ in $\mathbb{R}^2$

Solution:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_1\mathbf{e}_2$$

$\therefore$

$$\begin{aligned} T(\mathbf{x}) &= x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) \\ &= x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 \end{bmatrix} \end{aligned}$$

$\therefore$

$$T(\mathbf{x}) = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Theorem 7.3.1

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Them there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}, \ \forall \mathbf{x} \in \mathbb{R}^n$$

In fact , A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$,where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$:

$$A = \left[T(\mathbf{e}_1) \ldots T(\mathbf{e}_n)\right]$$

7.4 Geometric Linear Transformations of $\mathbb{R}^2$

Chapter 8

Exercises

Question 23

1. Suppose a 3×5 coefficient matrix for a system has three pivot columns. Is the system consistent ? Why or why not?
2. Suppose the coefficient matrix of a linear system of three equations in three variables has a pivot in each column. Explain why the system has a unique solution.

Solution:

1. The system is consistent as if there are three pivot columns then there is a pivot in each row which means no row is of the form:
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
Making the value of b irrelevant to the consistency of the system. Thus the system is consistent.
2. If the coefficient matrix has a pivot in each column then there is a pivot in each row, and no row is of the form:
$$\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$
As such the system is consistent and has a unique solution.