Limits

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Contents

Chapter 1

1.1 The Limit of a Function

Reporting about the behaviour of function within the range of its dangerous values

$$f(x) = x^2 + \frac{1}{x}$$

Input variable = x

Output variable = $f(x)$

Name of function = f

"Acceptable"/Permissible input values of x - All real numbers except zero

$$x, f(x) \quad x+h, f(x+h)$$
$$f(x+h) - \frac{f(x)}{x+h-x}$$
$$f(x+h) - \frac{f(x)}{h}$$

1.1.1 Proof

$$y = -16t^2 + 100t + 6$$

Points used: $(0, 6)$ $(1, 90)$ $(3, 162)$

When $t = 0$ and $y = 6$

$$y = at^2 + bt + c$$
$$6 = a(0)^2 + b(0) + c$$
$$c = 6$$

When $t = 1$ and $y = 90$

$$90 = a(1)^2 + b + 6$$
$$90 = a + b + 6$$
$$84 = a + b$$
$$84 - b = a$$

When $t = 3$ and $y = 162$

$$\begin{aligned}
162 &= a(3)^2 + b + 6 \\
162 &= 9a + 3b + 6 \\
162 &= 9(84 - b) + 3b + 6 \\
162 &= 756 - 9b + 3b + 6 \\
-594 &= -6b + 6 \\
-600 &= -6b \\
b &= 100
\end{aligned}$$

$$\therefore b = 100$$

$$\begin{aligned}
84 - 100 &= a \\
a &= -16
\end{aligned}$$

Therefore $a = 16$, $b = 100$ and $c = 6$

Question 1

Given $f(x) = x^2$ find the Limit of $f(x)$ at $x = 3$

Solution:

$$\text{As x} \longrightarrow 3^-,\ f(x)\text{->}9$$
$$\text{As x} \longrightarrow 3^+,\ f(x)\text{->}9$$

Or

$$\lim_{x \to 3^-} f(x) = 9$$
$$\lim_{x \to 3^+} f(x) = 9$$

The first 9 is known as the left limit of $f(x)$ and the other 9 is known as the right limit of $f(x)$

Therefore the limit of $f(x)$ at $x = 3$ is:

$$\lim_{x \to 3} f(x) = 9$$

This is because the left limit and right limit converge.

In the case where:

$$\lim_{x \to 1^-} f(x) = 5$$
$$\lim_{x \to 1^+} f(x) = 4$$

The left and right limits do not converge so there is no limit of $f(x)$ for $x = 1$ and is written as:

$$\lim_{x \to 1} f(x) = \text{No such unique number.}$$

In the case where one limit does not exist, i.e. Increasing without bounds:

$$\lim_{x \to 1^-} f(x) \to \infty$$
$$\lim_{x \to 1^+} f(x) \to 4$$

The limit does not exist because the left limit does not exist. This is written as:

$$\lim_{x \to 1} f(x) = \text{does not exist}$$

1.2 L'hopital's Rule

Theorem 1.2.1 L'hopital's Rule

If

$$\lim_{x \to a} \frac{f(x)}{g(x)} \implies \frac{0}{0} \quad \text{or} \quad \frac{\pm\infty}{\pm\infty}$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \quad \text{, provided} \quad \frac{f'(x)}{g'(x)} \neq 0$$

If

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{0}{0} \quad \text{or} \quad \frac{\pm\infty}{\pm\infty}$$

Then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)} \quad \text{, provided} \quad \frac{f''(x)}{g''(x)} \neq 0$$