

Instantaneous rate of change

In the case where f is a function of x $f'(x)$ measures the instantaneous rate of change of f with respect to x .

Example

The term widget is an economic term for a generic unit of manufacturing output. Suppose a company produces widgets and knows that the market supports a price of \$10 per widget. Let $P(n)$ give the profit, in dollars, earned by manufacturing and selling n widgets. The company likely cannot make a (positive) profit making just one widget; the start-up costs will likely exceed \$10. Mathematically, we would write this as $P(1) < 0$.

What do $P(1000) = 500$ and $P'(1000) = 0.25$ mean? Approximate $P(1100)$.

The equation $P(1000) = 500$ means that selling 1,000 widgets returns a profit of \$500. We interpret $P'(1000) = 0.25$ as meaning that the profit is increasing at rate of \$0.25 per widget (the units are “dollars per widget.”). Since we have no other information to use, our best approximation for $P(1100)$ is:

$$\begin{aligned} P(1100) &\approx P(1000) + P'(1000) \times 100 \\ &= P(1000) + P'(1000) \times 100 \\ &= 500 + 0.25 \times 100 \\ &= 525 \end{aligned}$$

We approximate that selling 1,100 widgets returns a profit of \$525.

The Slope of the Tangent Line

We can measure the instantaneous rate of change at a given x value c of a non-linear function by computing $f'(c)$. We can determine the behaviour of the function f by observing the slopes of its tangent lines.

Increasing Functions

$f(x)$ is increasing whenever $x_1 < x_2$ and $f(x_1) < f(x_2)$, i.e as you go up the x axis the y or function values increase.

$f(x)$ is increasing if the slope on any point on it's graph is positive throughout the function's entire domain.

Decreasing Functions

$f(x)$ is decreasing whenever $x_1 < x_2$ and $f(x_1) > f(x_2)$, .i.e as you go up the x axis the y or function values decrease

$f(x)$ is increasing if the slope on any point on it's graph is negative throughout the function's entire domain.

Critical Points

- Points where the gradient is equal 0, i.e. $f'(x) = 0$
- Points where the gradient does not exist, i.e. $f'(x) = \emptyset$

Examples

$$t\sqrt[3]{t^2 - 4}$$

$$g(t) = t\sqrt[3]{t^2 - 4}$$

$$g(t) = t(t^2 - 4)^{\frac{1}{3}}$$

$$g'(t) = (1)(t^2 - 4)^{\frac{1}{3}} + \left(\frac{1}{3}\right)(2t)(t^2 - 4)^{-\frac{2}{3}}$$

$$g'(t) = (t^2 - 4)^{\frac{1}{3}} + \frac{2}{3}t^2(t^2 - 4)^{-\frac{2}{3}}$$

$$g'(t) = (t^2 - 4)^{\frac{1}{3}} + \frac{2t^2}{3(t^2 - 4)^{\frac{2}{3}}}$$

$$g'(t) = \frac{(t^2 - 4)^{\frac{1}{3}}}{1} + \frac{2t^2}{3(t^2 - 4)^{\frac{2}{3}}}$$

$$g'(t) = \frac{3(t^2 - 4) + t^2}{3(t^2 - 4)^{\frac{2}{3}}}$$

$$0 = \frac{3(t^2 - 4) + t^2}{3(t^2 - 4)^{\frac{2}{3}}}$$

$$0 = 3t^2 - 12 + t^2$$

$$0 = 4t^2 - 12$$

$$12 = 4t^2$$

$$\frac{12}{4} = t^2$$

$$\pm\sqrt{\frac{12}{4}} = t$$

$$3(t^2 - 4)^{\frac{2}{3}} = 0$$

$$(t^2 - 4)^{\frac{2}{3}} = 0$$

$$t = \pm 2$$

Interval	Test Value	Slope _{$g'(x)$}
$x < -2$	-3	+
$-2 < x < -\sqrt{\frac{12}{5}}$	-1.7	+
$-\sqrt{\frac{12}{5}} < x < \sqrt{\frac{12}{5}}$	0	-
$\sqrt{\frac{12}{5}} < x < 2$	2	+
$x > 2$	7	+

$$\therefore \text{When } g'(x) = 0, x = -\sqrt{\frac{12}{5}}, x = \sqrt{\frac{12}{5}}$$

$$\therefore \text{Increasing } (-\infty, -2), (2, \infty), (-2, -\sqrt{\frac{12}{5}}), (\sqrt{\frac{12}{5}}, 2)$$

$$\text{Decreasing } (-\sqrt{\frac{12}{5}}, \sqrt{\frac{12}{5}})$$