Homework 2

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Question 1

Suppose an economy has four sectors, Agriculture (A), Energy (E), Manufacturing (M), and Transportation (T). Sector A sells 10 % of it's output to E and 25 % to M and retains the rest. Sector E sells 30 % of its output to A, 35 % to M and 25 % to T and retains the rest. Sector M sells 30 % of its output to A, 15 % to E, and 40 % to T and retains the rest. Sector T sells 20 % of its output to A, 10 % to E, and 30 % to M and retains the rest.

1. Construct the exchange table for this economy.
2. Find the set of equilibrium prices for the economy.

Solution:

1.

Agriculture	Energy	Manufacturing	Transportation	Sold By
0.65	0.1	0.25	0	Agriculture
0.3	0.1	0.35	0.25	Energy
0.3	0.15	0.15	0.4	Manufacturing
0.2	0.1	0.3	0.4	Transportation

Question 2

Ashesi University owns two farms. When it is operated for a month, Farm #1 produces 50 kg of kale, 56 kg of onions and 34 kg of tomatoes. A month's operation at Farm #2 on the other hand yields 35 kg of kale, 14 kg of onions and 123 kg of tomatoes. The vectors $\mathbf{v}_1 = \begin{bmatrix} 50 \\ 56 \\ 34 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 35 \\ 14 \\ 123 \end{bmatrix}$, then represent the output per month of Famr #1 and Farm #2 respectively.

1. What physical interpretation can be given to the vector $3\mathbf{v}_2$
2. Suppose the university operates Farm #1 for x_1 months and Farm #2 for x_2 months. Write a vector equation whose solution gives the number of months each farm should operate in order to produce in total 830 kg of kale, 728 kg of onions, 1358 kg of tomatoes. Explain. Do not solve the equation

Solution:

1. This represents the output of Farm #2 if it were to be operated for 3 months.
2. $$x_1 \begin{bmatrix} 50 \\ 56 \\ 34 \end{bmatrix} + x_2 \begin{bmatrix} 35 \\ 14 \\ 123 \end{bmatrix} = \begin{bmatrix} 830 \\ 728 \\ 1358 \end{bmatrix}$$
   $$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \begin{bmatrix} 830 \\ 728 \\ 1358 \end{bmatrix}$$
   This equation represents the total output of the two farms if they were to be operated for x_1 and x_2 months respectively. This is a linear combination of the output of the two farms, weighted by the number of months each farm is operated.

Question 3

Intersections in England are often constructed as one-way "roundabouts" such as the one shown in the figure. Assume that traffic must travel in the directions shown.

1. Find the general solution of the network flow.
2. Find the smallest possible value for x_6.

$$\begin{aligned}
A &: x_1 = 100 + x_2 \\
B &: 50 + x_2 = x_3 \\
C &: x_3 = x_4 + 120 \\
D &: x_4 + 150 = x_5 \\
E &: x_5 = x_6 + 80 \\
F &: x_6 + 100 = x_1
\end{aligned}$$

$$\begin{aligned}
x_1 - x_2 &= 100 \\
x_2 - x_3 &= -50 \\
x_3 - x_4 &= 120 \\
x_4 - x_5 &= -150 \\
x_5 - x_6 &= 80 \\
x_6 - x_1 &= -100
\end{aligned}$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ -1 & 0 & 0 & 0 & 0 & 1 & -100 \end{bmatrix}$$

$$R_4 \leftrightarrow R_6$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ -1 & 0 & 0 & 0 & 0 & 1 & -100 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \end{bmatrix}$$

$$-R_1 - R_4 \rightarrow R_4$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \end{bmatrix}$$

$$R_2 - R_4 \rightarrow R_4$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & -1 & 0 & 0 & 1 & -50 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \end{bmatrix}$$

$$-R_3 - R_4 \rightarrow R_4$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \end{bmatrix}$$

$$R_4 - R_6 \to R_6$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \end{bmatrix}$$

$$R_5 - R_6 \to R_6$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-R_2 - R_1 \to R_1$$

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & -50 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-R_3 - R_2 \to R_2$$

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & -50 \\ 0 & -1 & 0 & 1 & 0 & 0 & -70 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3 - R_1 \to R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 170 \\ 0 & -1 & 0 & 1 & 0 & 0 & -70 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-R_4 - R_3 \longrightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 170 \\ 0 & -1 & 0 & 1 & 0 & 0 & -70 \\ 0 & 0 & -1 & 0 & 0 & 1 & -50 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_4 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 170 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & -50 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-R_4 - R_1 \longrightarrow R_1$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 & -100 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & -50 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-R_1 \longrightarrow R_1$$
$$-R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 100 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 50 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - x_6 = 100$$
$$x_2 - x_6 = 0$$
$$x_3 - x_6 = 50$$
$$x_4 - x_6 = -70$$
$$x_5 - x_6 = 80$$
$$x_6 = x_6$$

$$\begin{cases} x_1 = x_6 + 100 \\ x_2 = x_6 \\ x_3 = x_6 + 50 \\ x_4 = x_6 - 70 \\ x_5 = x_6 + 80 \\ x_6 \text{ is free} \end{cases}$$

The smallest value of x_6 is 0 as x_6 is the same as the outflow of x_2 which cannot be negative.

Question 4

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a linear transformation with $T(x_1, x_2) = (2x_1 - x_2, -3x_1 + x_2, 2x_1 - 3x_2)$

1. Find the standard matrix of T
2. Find $\mathbf{x}$ such that $T(\mathbf{x}) = (0, -1, 4)$

3. Is T onto? Is it one-to-one? Explain.

Solution:

1.

$$
\begin{gathered}
T(\mathbf{x}) = A\mathbf{x} \\
I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \\
T(\mathbf{x}) = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) \\
\therefore\ T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} \quad T(\mathbf{e}_2) = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} \\
T(\mathbf{x}) = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\mathbf{x} \\
\text{Hence } A = \begin{bmatrix} 2 & -1 \\ -3 & 1 \\ 2 & -3 \end{bmatrix}
\end{gathered}
$$

2.

$$
\begin{gathered}
T(\mathbf{x}) = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} \\
\begin{bmatrix} 2 & -1 \\ -3 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}
\end{gathered}
$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -3 & 1 & -1 \\ 2 & -3 & 4 \end{bmatrix}$$

$$\frac{-3}{2}R_1 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 2 & -3 & 4 \end{bmatrix}$$

$$R_1 - R_3 \longrightarrow R_3$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 2 & -4 \end{bmatrix}$$

$$4R_2 - R_3 \longrightarrow R_3$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 8 \end{bmatrix}$$

$$-2R_2 - R_1 \longrightarrow R_1$$

$$\begin{bmatrix} -2 & 0 & -2 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 8 \end{bmatrix}$$

$$-\frac{1}{2}R_1 \longrightarrow R_1$$

$$2R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 8 \end{bmatrix}$$

$$x_1 = 1$$

$$x_2 = 2$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

3. T is onto for $(0, -1, 4)$ as $(0, -1, 4)$ produces an image in the $\mathbb{R}^3$ vector space, and T is also one-to-one as it produces only one, a unique, image.