Matrix Algebra

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Chapter 1

Matrix Operations

If A is a $n \times m$ matrix then the scalar entry in the ith row and the jth column of A is denoted by a_{ij}, and is called the (i, j)-entry. Each column of A is a list of m real numbers in the $\mathbb{R}^m$ vector space. Therefore the columns of A can be represented as vectors in $\mathbb{R}^m$:

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots \mathbf{a}_n \end{bmatrix}$$

Definition 1.0.1: Diagonals

The diagonal entries of a matrix A of dimension $n \times m$, are the entries a_{ij}, where $i = j$. This is called the **main diagonal** of the matrix A. A **diagonal matrix** is a square matrix $n \times n$ whose non-diagonal entries are all zero.

1.1 Sums and Scalar Multiples

Definition 1.1.1: Equality of Matrices

Two matrices A and B, are equal if:

- The are of the same size i.e, $m \times x$
- The corresponding entries are equal i.e, $A_{ij} = B_{ij}$

Theorem 1.1.1 Axioms of Matrix Addition

Let A, B and C be matrices of the same size, and let r and s be scalars. Then the following axioms hold:
Communtativity $A + B = B + A$
Associativity $(A + B) + C = A + (B + C)$
Additive Identity $A + 0 = A$
Distruibutivity 1 $r(A + B) = rA + rB$
Distruibutivity 2 $(r + s)A = rA + sA$
Compatibility with Scalar Multiplication $r(sA) = (rs)A$

1.2 Matrix Multiplication

When a matrix B multiples a vector $\mathbf{x}$, it transforms $\mathbf{x}$ into the vector $B\mathbf{x}$. If this vector is then multiples by another matrix A, the result is the vector $A(B\mathbf{x})$. Thus $A(B\mathbf{x})$ is produced by a composition of mappings / linear transformations. This can be also expressed as:

$$A(B\mathbf{x}) = (AB)\mathbf{x}$$

Because, if A is $m \times n$, B is $n \times p$ and $\mathbf{x}$ is in $\mathbb{R}^p$, can denote the columns of B, by $\mathbf{b}_1, \ldots, \mathbf{b}_p$ and the entries of $\mathbf{x}$ by, $x_1, \ldots, x_p$. Then

$$B\mathbf{x} = x_1\mathbf{b}_1 + \ldots + x_p\mathbf{b}_p$$

By the linearity of matrix multiplication, we have:

$$\begin{aligned} A(B\mathbf{x}) &= A(x_1\mathbf{b}_1) + \ldots + A(x_p\mathbf{b}_p) \\ &= x_1(A\mathbf{b}_1) + \ldots + x_p(A\mathbf{b}_p) \end{aligned}$$

The vector $A(B\mathbf{x})$ is then a linear combination of the vectors $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$, using the entries of $\mathbf{x}$ as weights. This can be expressed in matrix notation as:

$$A(B\mathbf{x}) = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \ldots & A\mathbf{b}_p \end{bmatrix} \mathbf{x}$$

Theorem 1.2.1

If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$.That is:

$$A(B\mathbf{x}) = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \ldots & A\mathbf{b}_p \end{bmatrix} \mathbf{x}$$

Example 1.2.1

Question 1

Compute AB where $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$, and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$

Solution:

$$\begin{aligned} A\mathbf{b}_1 &= \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 8+3 \\ 4+-5 \end{bmatrix} \\ &= \begin{bmatrix} 11 \\ -1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A\mathbf{b}_2 &= \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 6-6 \\ 3+10 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 13 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A\mathbf{b}_3 &= \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 21 \\ -9 \end{bmatrix} \end{aligned}$$

$$AB = \begin{bmatrix} 11 & 0 & 12 \\ -1 & 13 & -9 \end{bmatrix}$$

Theorem 1.2.2 Row-Column Rule

If the product AB is defined, them the entry in row i and column j of AB is the sum of the products of corresponding entries of the row i of A and column j of B. If $(AB)_{ij}$ denotes the (i, j)-entry in AB, and if A is an $m \times n$, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

Example 1.2.2

Use the row–column rule to compute two of the entries in AB for the matrices:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$$

An inspection of the numbers involved will make it clear how the two methods for calculating AB produce the same matrix.

The dimensions of the resultant matrix is 2×3, therefore the entries of AB are:

$$\begin{aligned} AB &= \begin{bmatrix} 2(4) + 3(1) & 2(3) + 3(-2) & 2(6) + 3(3) \\ 1(4) - 5(1) & 1(3) - 5(-2) & 1(6) - 5(3) \end{bmatrix} \\ &= \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & 9 \end{bmatrix} \end{aligned}$$

Example 1.2.3

Question 2

Find the entries in the second row of AB where,

$$A = \begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix}, B = \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} -1 & 3 & -4 \end{bmatrix} \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}$$
$$\begin{bmatrix} -4 + 21 - 12 & 6 + 3 - 8 \end{bmatrix}$$
$$\begin{bmatrix} 5 & 1 \end{bmatrix}$$

Theorem 1.2.3 Axioms of Matrix Multiplication

Let A be an $m \times n$ matrix and let B and C have sizes for which the indicated sums and products are defined:
Associativity $A(BC) = (AB)C$
Left Distruibutivity $A(B + C) = AB + AC$
Right Distruibutivity $(B + C)A = BA + CA$
Scalar Associativity $r(AB) = (rA)B = A(rB), \forall r, r \in \mathbb{F}$
Mutliplicative Identitiy $I_m A = A = AI_n$

Example 1.2.4

Question 3

Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$. Show that these matrices do not commute, I.e, verify $AB \neq BA$

Solution:

$$\begin{aligned} AB &= \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 2 \\ 29 & -2 \end{bmatrix} \end{aligned}$$

$$\therefore AB \neq BA$$

1.2.1 Powers of a Matrix

Definition 1.2.1: Powers of a Matrix

If A is an $n \times n$ matrix and if k is a positive integer, then A^k denotes the product of k copies of A:

$$A^k = A_1 \dots A_k$$

Where $A_1 = A_2 \wedge A_2 = A_3 \wedge \dots \wedge A_{k-1} = A_k$

If A is non-zero and if $\mathbf{x}$ is in $\mathbb{R}^n$, then $A^k\mathbf{x}$ is the result of left-multiplying $\mathbf{x}$ by A repeatedly k times.

If $k = 0$, then $A^0\mathbf{x}$ is $\mathbf{x}$. Thus A^0 is interpreted as the Identity matrix.

1.3 The Transpose of a Matrix

Definition 1.3.1: The Transpose of a Matrix

Given a matrix A, its *transpose*, denoted by A^T, is defined by transforming the rows of A into columns. For example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Therefore formally, the transpose of a matrix $A_{m,n}$ is defined as:

$$A^T_{m,n} = A_{n,m}$$

Therefore, let A and B denote matrices whose sizes are appropriate for the following sums and products:

1. $\left(A^T\right)^T = A$
2. $(A + B)^T = A^T + B^T$

3. $\forall r \in \mathbb{F}, (rA)^T = rA^T$
4. $(AB)^T = B^T A^T$

Usually $(AB)^T$ is not equal $A^T B^T$, even when A and B have dimensions such that $A^T B^T$ is defined. The generalization of axiom 4 to products more than two factors is as follows:

Theorem 1.3.1

The transpose of a product of matrices equals the product of their transpose in the reverse order.

Chapter 2

The Inverse Of A Matrix

2.1 Invertibility