Optimization

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Chapter 1

Maximum and Minimum

Note:-

Local - Subsection of range Global - Whole range

1.1 Maximize

$$R(x) = 45 - \frac{x^2}{3}, \ 0 \le x \le 1$$

Find all local maximum <u>values</u> Find the global maximum value

1.2 Minimize

- Local Minimum Minimum in specified range
- Global Minimum Overall minimum

1.3 Local/Relative Maximum/Minimum (Optimum)

• Find the critical values of the function:

Stationary points, i.e. f'(?) = 0

Undefined points, i.e. $f'(?) = \emptyset$

• Assess them for potential local maximum/minimum:

Find the first derivative, input values from the left and right of the critical points and check the change in signs:

+ to -: Maximum

- to +: Minimum

Find second derivative, input the critical values and check the sign:

-: Maximum

+: Minimum

1.4 Global/Absolute Maximum/Minimum (Optima)

To find the Absolute Optima of a function whose domain is unrestricted:

$$\lim_{x\to\infty} f(x) \quad \lim_{x\to-\infty} f(x)$$

1.4.1 Conditions for finding the Absolute Optima easily

- 1. Closed Domain, i.e. $[x_1, x_2]$
- 2. Function is continuous for the duration of the closed domai

Theorem 1.4.1 Extreme Value Theorem

If a real valued function f is continuous on the closed interval [a, b], the f must attain a maximum and minimum at least once.

$$f(c) \ge f(x) \ge F(d)$$

 $\forall x \in [a, b]$

Where f(c) is the function's minimum value and F(d) is the function's maximum value.

Example 1.4.1

$$f(x) = x^3$$
 on $[-1, 10]$

- f(x) is continuous due to it being a polynomial
- The function's domain is closed due to the end values being included in the domain

By EVT(1.4.1) f(x) must attain absolute maximum and minimum at least once on the interval. Possibly at:

- 1. End points of the domain
- 2. Critical values of f(x)

$$f(x) = x^{3}$$

$$f'(x) = 3x^{2}$$

$$0 = 3x^{2}$$

$$\frac{0}{3} = x^{2}$$

$$0 = x$$

$$f(-1) = -1$$

$$f(10) = 1000$$

 \therefore Absolute Maximum is 1000

Absolute Minimum is -1

Chapter 2

Concavity

Let f be a function that is differentiable over an open interval I

- If f' is increasing over I, we say f is concave up over I, i.e. f'' > 0
- If f' is decreasing over I, we say f is concave down over I, i.e f'' < 0

2.1 Inflection

A point where a function switches concavity, i.e:

$$f''(x^-) = +ve$$
 to $f''(x^+) = -ve$
or
 $f''(x^-) = -ve$ to $f''(x^+) = +ve$

2.2 Curvature

2.2.1 Concave Up

The $\underline{\text{cave}}$ is facing up

2.2.2 Concave Down

The $\underline{\text{cave}}$ is facing down

2.3 Questions

Question 1

A closed box with a square base is to contain 252 cubic feet. The bottom costs \$ 5 per square foot, the top costs \$2 per square foot, and the sides costs \$ 3 per square foot. Find the dimensions that minimize the cost.

Solution:

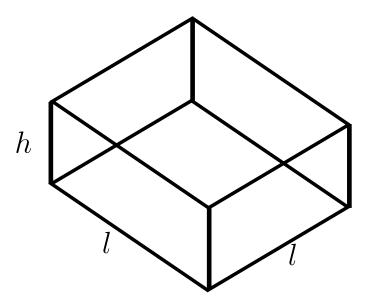


Figure 2.1: Box

$$V = L \times B \times H$$

$$V = L \times L \times H$$

$$V = 252$$

$$\therefore 252 = L^2H$$
Cost of the top (CT) = $2L^2$
Cost of the bottom (CB) = $5L^2$
Cost of one side (CS) = $3HL$
Total cost (TC) = $CT + CB + 4(CS)$

$$TC = 2L^2 + 4(3HL) + 5L^2$$

$$H = \frac{252}{L^2}$$

$$TC = 7L^2 + 12(\frac{252}{L^2})L$$

$$TC = 7L^2 + \frac{3024}{L}$$

$$TC' = 14L - \frac{3024}{L^2}$$

$$0 = 14L^3 - 3024$$

$$216 = L^3$$

$$6 = L$$

$$TC'' = 14 + \frac{6048}{L^3}$$

$$TC''(6) = 42 \therefore \text{ Minimum}$$

$$H = \frac{252}{(6)^2}$$

H = 7

: at a width of 6 ft and a height of 7 ft the total cost is minimized



A wire 16 ft long has to be formed into a rectangle. What dimensions should the rectangle have to maximize area?

Solution:

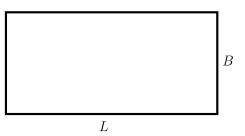


Figure 2.2: Square

$$A = L \times B$$

$$2L + 2B = 16$$

$$2(L + B) = 16$$

$$L + B = 8$$

$$B = 8 - L$$

$$A = L(8 - L)$$

$$A = 8L - L^{2}$$

$$A' = -2L + 8$$

$$0 = -2L + 8$$

$$L = 4$$

$$A'' = -2$$

$$\therefore L = 4 \text{ Maximum}$$

$$B = 8 - 4$$

B = 4

 \therefore at a length of 4 ft and a width of 4 ft the area is maximized