Vector Spaces

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Contents

Chapter 1

Vector Spaces and Subspaces

1.1 Introduction

Definition 1.1.1: Vector Space

A *vector space* is a non empty set V of objects, called vectors, on which are defined two operations, addition and multiplication by scalars, e.g. real numbers, subject to the following axioms which must hold for all vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$, is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is a zero vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

Using these axioms one can show that the zero vector in axiom 4 is unique, and the vector $-\mathbf{u}$, called the **negative** of $\mathbf{u}$ in axiom 5 is unique for each $\mathbf{u}$ in V, outlined in:

Theorem 1.1.1

$$0\mathbf{u} = \mathbf{0} \tag{1.1}$$

$$c\mathbf{0} = \mathbf{0} \tag{1.2}$$

$$-\mathbf{u} = (-1)\mathbf{u} \tag{1.3}$$

1.2 Subspaces

In many problems, a vector space consists of an appropriate set of vectors from a larger vector space. In this case only, three of the ten axioms need to be checked to determine if the subset is a vector space, the rest are satisfied automatically.

Definition 1.2.1: Subspace

A subset H of the vector space V, where:

1. The zero vector of V is in H.
2. H is closed under vector addition. That is for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum of $\mathbf{u} + \mathbf{v}$ is in H.
3. H is closed under scalar multiplication. That is for each $\mathbf{u}$ in H and each scalar c, the scalar multiple $c\mathbf{u}$ is in H.

These properties guarantee that a subspace H of V is also a vector space, under the defined vector space operations. This means that every subspace is a vector space and conversely every vector space is a subspace (of itself and possibly of a larger vector space).

Example 1.2.1

Question 1

The vector space $\mathbb{R}^2$ is not a vector space of $\mathbb{R}^3$ because $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$. The set

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real} \right\}$$

is a subset of $\mathbb{R}^3$ that ”looks” and ”acts” like $\mathbb{R}^2$ even though it is logically distinct from $\mathbb{R}^2$. Show that H is subset of $\mathbb{R}^3$

Solution:

- The zero vector is in H
- H is closed under vector addition and scalar multiplication as these operations on vectors in H always produce vectors whose third entry is zero and thus belong to H.

Thus H is as subspace of $\mathbb{R}^3$

1.2.1 Subspace Spanned by a Set

One way of describing a subspace is as a linear combination of vectors that span the subspace.

Example 1.2.2

Question 2

Given $\mathbf{v}_1$ and $\mathbf{v}_2$ in a vector space V, let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Show that H is a subspace of V

Solution:

- The zero vector is in H as:
$$\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2$$
- To show that H is closed under vector addition and scalar multiplication, take two arbitrary vectors in H, say
$$\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 \text{ and } \mathbf{w} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$$
By axioms 2, 3, and 8 for the vector space V:
$$\begin{aligned} \mathbf{u} + \mathbf{w} &= (s_1\mathbf{v}_1 + s_2\mathbf{v}_2) + (t_1\mathbf{v}_1 + t_2\mathbf{v}_2) \\ &= (s_1 + t_1)\mathbf{v}_1 + (s_2 + t_2)\mathbf{v}_2 \end{aligned}$$
The result is still in H as it can still be spanned from $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, with weights $(s_1 + t_1)$ and $(s_2 + t_2)$

Furthermore:

$$\begin{aligned} c\mathbf{u} &= c\left(s_1\mathbf{v}_1 + s_2\mathbf{v}_2\right) \\ &= \left(cs_1\right)\mathbf{v}_1 + \left(cs_2\right)\mathbf{v}_2 \end{aligned}$$

therefore H is also closed under scalar multiplication.

Theorem 1.2.1

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in vector space V, then $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

We can call $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ **the subspace spanned** by $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Therefore given any subspace H of V, a **spanning set** for H is a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in H such that $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Example 1.2.3

Question 3

Let H be the set of all vectors of the form $(a - 3b, b - a, a, b)$, where a and b are arbitrary scalars. That is let $H = \{(a - 3b, b - a, a, b) : a \text{ and } b \text{ in } \mathbb{R}\}$. Show that H is a subspace of $\mathbb{R}^4$

Solution:

$$\begin{aligned} H &= \begin{bmatrix} a - 3b \\ b - a \\ a \\ b \end{bmatrix} \\ &= a\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \\ &= a\mathbf{v}_1 + b\mathbf{v}_2 \end{aligned}$$

Thus H is a subspace of $\mathbb{R}^4$ by theorem 1.2.1

Example 1.2.4

Question 4

For what value(s) of h will $\mathbf{y}$ be in the subspace of $\mathbb{R}^3$ spanned by $\mathbf{v}_1, \mathbf{v}_2, pmboldv_3$, if

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

Solution: The subspace of $\mathbb{R}^3$ $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. $\mathbf{y}$ will be in the subspace if the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ contains $\mathbf{y}$, that is if $\mathbf{y}$ can be written

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{y}$$

And thus the matrix equation:

$$A\mathbf{x} = \mathbf{y}$$

Where $A = \begin{bmatrix} 1 & 5 & -3 \\ -1 & -4 & 1 \\ -2 & -7 & 0 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix}$$
$$-R_1 - R_2 \rightarrow R_2$$
$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ -2 & -7 & 0 & h \end{bmatrix}$$
$$-2R_2 + R_3 \rightarrow R_3$$
$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & -3 & 6 & 8-h \end{bmatrix}$$
$$-3R_2 - R_3 \rightarrow R_3$$
$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & -5+h \end{bmatrix}$$

$\therefore$ The system $A\mathbf{x} = \mathbf{y}$ is only consistent if $h = 5$, and thus $\mathbf{y}$ is in the subspace spanned by $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if $h = 5$

1.3 Exercises

Question 5

Show that the set H of all points in $\mathbb{R}^2$ of the form $(3s, 2+5s)$ is not a vector space, by showing that it is not closed under scalar multiplication. (Find a specific vector $\mathbf{u}$ in H and a scalar c such that $c\mathbf{u}$ is not in H)

Solution: Let $\mathbf{u} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ and $c = 2$. Then:

$$\begin{aligned} 2\mathbf{u} &= 2\left(\begin{bmatrix} 3 \\ 7 \end{bmatrix}\right) \\ &= \begin{bmatrix} 6 \\ 14 \end{bmatrix} \end{aligned}$$

This implies there is some s such that $\begin{bmatrix} 3s \\ 2+5s \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$, but for this to be true s would need to be equal to 2 and 2.4 which is impossible. Therefore H is not closed under scalar multiplication and thus is not a vector space.

Question 6

Let $W = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$, where $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in a vector space V. Show that $\mathbf{v}_k$ is in W for $1 \leqslant k \leqslant p$.

Solution: If $W = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$, then the contents of W for example $\mathbf{v}_1$ can be written as linear combination of the spanned vectors, that is:

$$\mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_p$$

Therefore if $1 \leqslant k \leqslant p$, then $\mathbf{v}_k$ is in W because:

$$\mathbf{v}_k = 0\mathbf{v}_1 + \ldots + 0\mathbf{v}_{k-1} + 1\mathbf{v}_k + 0\mathbf{v}_{k+1} + \ldots + 0\mathbf{v}_p$$

Question 7

An $n \times n$ matrix A is said to be *symmetric* if $A = A^T$. Let S be the set of all 3×3 symmetric matrices. Show that S is a subspace of $M_{3\times3}$, the vector space of all 3×3 matrices.

Solution: To prove that S is a subspace of $M_{3\times3}$, I must show:

The Zero vector Is in S Since the zero vector is symmetric S contains the zero vector as:

$$\mathbf{0} = \mathbf{0}^T$$

S **is closed under vector addition** Let A and B be in S, hence $A = A^T$ and $B = B^T$

$$\begin{aligned}(A + B)^T &= A^T + B^T \\ &= A + B\end{aligned}$$

Thus $A + B$ is symmetric and is in S

S **is closed under scalar multiplication** Let A be in S and c be a scalar

$$\begin{aligned}(cA)^T &= c\,(A)^T \\ &= cA\end{aligned}$$

Thus cA is symmetric and is in S

$\therefore$ S is a subspace of $M_{3\times3}$

Question 8

Let V be the first quadrant in the xy-plane; that is, let

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geqslant 0 \text{ and } y \geqslant 0 \right\}$$

1. If $\mathbf{u}$ and $\mathbf{v}$ are in V, is $\mathbf{u} + \mathbf{v}$ in V? Why?
2. Find a specific vector $\mathbf{u}$ in V and specific scalar c such that $c\mathbf{u}$ is not in V.

Solution:

1. If $\mathbf{u}$ and $\mathbf{v}$ are V, then indeed $\mathbf{u} + \mathbf{v}$ are in V, because the sum of these two vectors will always have positive x and y components and will therefore always be in the first quadrant of the xy-plane.

2. For $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $c = -2$

$$\begin{aligned}-2\mathbf{u} &= -2\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) \\ &= \begin{bmatrix} -6 \\ -8 \end{bmatrix}\end{aligned}$$

Question 9

Determine if the given sets are subspaces of $\mathbb{P}_n$ for an appropriate value of n. Justify your answers.

1. All polynomials in the form $\mathbf{p}(t) = at^2$, where $a \in \mathbb{R}$.
2. All polynomials in the form $\mathbf{p}(t) = a + t^2$, where $a \in \mathbb{R}$
3. All polynomials of degree at most 3, with integers as coefficients.
4. All polynomials in $\mathbb{P}_n$ such that $\mathbf{p}(0) = 0$

Solution:

1. Yes this is a subspace of $\mathbb{P}_n$ as:

   Contains the zero vector When $a = 0$, $\mathbf{p}(t) = 0t^2 = 0$.

   Closed under vector additon Let $\mathbf{w}$ and $\mathbf{q}$ be polynomials in the appropriate form

$$\begin{aligned}\mathbf{w} + \mathbf{q} &= \left(wt^2\right) + \left(qt^2\right) \\ &= \left(w + q\right) t^2 \\ \text{Let } w + q = a, \text{ then} \\ &= at^2\end{aligned}$$

   Closed under scalar multiplication Let $\mathbf{w}$ be a polynomial in the appropriate form and c be a scalar.

$$\begin{aligned}c\mathbf{w} &= c\left(wt^2\right) \\ &= (cw)\, t^2 \\ \text{Let } cw = a, \text{ then} \\ &= at^2\end{aligned}$$

Chapter 2

Null Space, Column Space, and Linear Transformations

2.1 The Null Space of a Matrix

Definition 2.1.1: Null Space

The *null space* of an $m \times n$ matrix A, denoted by Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation:

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

Example 2.1.1

Question 10

Let A be the matrix $\begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$, and let $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$. Determine if $\mathbf{u}$ belongs to the null space of A.

Solution: This is basically asking us to verify if $\mathbf{u}$ satisfies the equation $A\mathbf{u} = \mathbf{0}$

$$\begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 - 9 + 4 \\ -25 + 27 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ $\mathbf{u}$ is in the null space of A.

Theorem 2.1.1

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$, equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

2.1.1 An Explicit Description of the Null Space of a Matrix

There is no obvious relation between the vectors in Nul A and the entries A. We say that Nul A is defined implicitly, as it is defined by a condition that must be checked. However solving the equation $A\mathbf{x} = \mathbf{0}$ amounts to producing an explicit description of Nul A.

Example 2.1.2

Question 11

Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Solution: The first step is to find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of free variables. Therefore:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - 2x_2 - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ x_2 &= x_2 \\ x_4 &= x_4 \\ x_5 &= x_5 \end{aligned}$$

$$\begin{aligned} x_1 &= 2x_2 + x_4 - 3x_5 \\ x_2 &= x_2 \\ x_3 &= -2x_4 + 2x_5 \\ x_4 &= x_4 \\ x_5 &= x_5 \end{aligned}$$

$$\begin{aligned} \mathbf{x} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \\ = x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w} \end{aligned}$$

Every linear combination of $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$ is an element of Nul A and vice versa. Thus $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a spanning set for Nul A.

Two points are made apparent by the previous example:

1. The spanning set produced by the general solution of $A\mathbf{x} = \mathbf{0}$ is automatically linearly independent because the free variables are weights on the spanning vectors.
2. When Nul A contains non-zero vectors, the number of vectors in the spanning set for Nul A equals the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$

2.2 The Column Space of a Matrix

Definition 2.2.1: The Column Space of a Matrix

The column space of an $m \times n$ matrix A, denoted by Col A, is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$, then

$$\text{Col } A = \text{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$$

Since Span $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is a subspace, by theorem 1.2.1, the next theorem follows from the definition of Col A and the fact that the columns of A are in $\mathbb{R}^m$.

Theorem 2.2.1

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Example 2.2.1

Question 12

Find a matrix A such that $W = \text{Col } A$.

$$W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

Solution:

We first write W as a set of linear combinations:

$$W = \left\{ a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Then we create a matrix A with these columns:

$$\begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$$

Theorem 2.2.2

The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^m$