Trigonometry and Derivatives

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Chapter 1

1.1 Sine

$$\begin{aligned} f(x) &= \sin(x) \\ -1 \leqslant \sin(x) &\leqslant 1 \\ \sin(0) &= 0 \end{aligned}$$

For all integer multiples π, sin() attains 0

$$\sin(k\pi) = 0, \text{ Where k is an integer}$$

The graph of sine is periodic with a period of 2π, meaning it repeats itself every interval of 2π

1.1.1 Derivative of $\sin(x)$

Definition 1.1.1: $\sin(x)$

if $y = \sin(x)$

$$y' = \cos(x)$$

Proof:

$$y' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{\sin(x+h) - \sin(x)}{h}$$

$$\therefore \frac{\sin(x+h) - \sin(x)}{h}$$

$$\frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$\frac{\sin(\cos(h) - 1) + \cos(x)\sin(h)}{h}$$

$$y' = \lim_{h \to 0} \frac{\sin(\cos(h) - 1) + \cos(x)\sin(h)}{h}$$

$$y' = \lim_{h \to 0} \frac{\sin(\cos(h) - 1)}{h} + \frac{\cos(x)\sin(h)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(\cos(h) - 1)}{h} + \lim_{h \to 0} \frac{\cos(x)\sin(h)}{h}$$

$$\because \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = \lim_{x \to a}(f(x) + g(x))$$

$$= \sin(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h}$$

$$\because \lim_{x \to a}(kf(x)) = k \lim_{x \to a} f(x)$$

$$\lim_{h \to 0^-} \frac{\cos(h) - 1}{h} = 0$$

$$\lim_{h \to 0^+} \frac{\cos(h) - 1}{h} = 0$$

$$\therefore \lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0$$

$$= \sin(x) \times 0 + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h}$$

$$= \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h}$$

$$\lim_{h \to 0^-} \frac{\sin(h)}{h} = 1$$

$$\lim_{h \to 0^+} \frac{\sin(h)}{h} = 1$$

$$\therefore \lim_{h \to 0} \frac{\sin(h)}{h} = 1$$

$$\therefore \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h} = \cos(x) \times 1$$

$$= \cos(x)$$

1.2 Cosine

$$f(x) = \cos(x)$$
$$-1 \leqslant \cos(x) \leqslant 1$$
$$\cos(0) = 1$$

1.2.1 Derivative of $\cos(x)$

Definition 1.2.1: $\cos(x)$

if $y = \cos(x)$

$$y' = -\sin(x)$$

1.3 Tangent

$$f(x) = \tan(x)$$
$$-1 \leqslant \tan(x) \leqslant 1$$
$$\tan(0) = 0$$

1.3.1 Vertical Asymptote

Note:-

A vertical line $(x, 0)$ where the values of a function rise or fall infinitely

The line $x = a$ is a vertical asymptote of $f(x)$ if

$$\lim_{x \to a} f(x) = \pm\infty$$

The zero points of $\cos(x)$ create a vertical asymptote in relation to $\tan(x)$

1.3.2 Derivative of $\tan(x)$

Definition 1.3.1: $\tan(x)$

if $y = \tan(x)$

$$y' = \sec^2(x)$$

Proof:

$$y = \tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$y' = \frac{\cos(x)(\cos(x)) - \sin(x)(-\sin(x))}{(\cos(x))^2}$$

$$\begin{aligned}
\because y' &= \frac{v \times u' - u \times v'}{v^2} \\
y' &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\
y' &= \frac{\cos^2(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)} \\
y' &= 1 + (\frac{\sin(x)}{\cos(x)})^2 \\
y' &= 1 + \tan^2(x) \\
&= \sec^2(x)
\end{aligned}$$

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1.4 Secant

$$\begin{aligned}
f(x) &= \sec(x) \\
1 \leqslant \sec(x) &\leqslant 1 \\
\sec(0) &= 1
\end{aligned}$$

1.4.1 Derivative of $\sec(x)$

Definition 1.4.1: $\sec(x)$

if $y = \sec(x)$

$$y' = \sec(x)\tan(x)$$

Chapter 2

2.1 Inverse Trigonometric Functions

In the case where

$$f[g(x)] = x$$

and

$$g[f(x)] = x$$

When can say the function f is the inverse of function g, due to the function g being able to extract the origninal input of the function f.

Therefore the derivative of the inverse function $y = \sin^{-1}(x)$ is as follows:

$$\begin{aligned}
\sin(y) &= x \\
\frac{d}{dx}\sin(y) &= \frac{d}{dx}x \\
\cos(y)\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\cos(y)}
\end{aligned}$$

$$\begin{aligned}
\text{Using the identity } \cos^2(y) + \sin^2(y) &= 1 \\
\cos(x) &= \sqrt{1 - \sin^2(x)} \\
\frac{dy}{dx} &= \frac{1}{\sqrt{1 - \sin^2(y)}} \\
\text{And since } x &= \sin(y) \\
\sin^2(y) &= x^2
\end{aligned}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

Example 2.1.1 (Find the derivative of $\cos^{-1}(x)$)

$$
\begin{aligned}
y &= \cos^{-1}(x) \\
\cos(y) &= x \\
-\sin(y)\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= -\frac{1}{\sin(y)} \\
\text{Using the identity } \cos^2(y) + \sin^2(y) &= 1 \\
\sin(y) &= \sqrt{1-\cos^2(y)} \\
\frac{dy}{dx} &= -\frac{1}{\sqrt{1-\cos^2(y)}} \\
\text{And since } x &= \cos(y) \\
\cos^2(y) &= x^2 \\
&= \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}
\end{aligned}
$$

Example 2.1.2 (Find the derivative of $\tan^{-1}(x)$)

$$
\begin{aligned}
\tan(y) &= x \\
\sec^2(y)\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\sec^2(y)} \\
\text{From the identity } 1 + \tan^2(y) &= \sec^2(y) \\
\frac{dy}{dx} &= \frac{1}{1+\tan^2(y)} \\
\text{And since } x &= \tan(y) \\
\tan^2(y) &= x^2 \\
&= \frac{dy}{dx} = \frac{1}{1+x^2}
\end{aligned}
$$

2.1.1 Questions

Question 1

$$y = \sin^{-1}(5x + 9)$$

Solution:

$$\begin{aligned}
\sin(y) &= 5x + 9 \\
\cos(y)\frac{dy}{dx} &= 5 \\
\frac{dy}{dx} &= \frac{5}{\cos(y)} \\
\text{Using the identity } \cos^2(y) + \sin^2(y) &= 1 \\
\cos(y) &= \sqrt{1 - \sin^2(y)} \\
\frac{dy}{dx} &= \frac{5}{\sqrt{1 - \sin^2(y)}} \\
\text{And since } x &= \sin(y) \\
\frac{dy}{dx} &= \frac{5}{\sqrt{1 - (5x + 9)^2}}
\end{aligned}$$

Question 2

$$y = \sin^{-1}(x)$$

Solution:

$$\begin{aligned}
\sin(y) &= x \\
\cos(y)\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\cos(y)} \\
\text{Using the identity } \cos^2(y) + \sin^2(y) &= 1 \\
\cos(y) &= \sqrt{1 - \sin^2(y)} \\
\frac{dy}{dx} &= \frac{1}{\sqrt{1 - \sin^2(y)}} \\
\text{And since } x &= \sin(y) \\
\frac{dy}{dx} &= \frac{1}{\sqrt{1 - x^2}}
\end{aligned}$$

Question 3

$$y = \cos^{-1}(x)$$

Solution:

$$\cos(y) = x$$
$$-\sin(y)\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = -\frac{1}{\sin(y)}$$
Using the identity $\cos^2(y) + \sin^2(y) = 1$
$$\sin(y) = \sqrt{1 - \cos^2(y)}$$
$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - \cos^2(y)}}$$
And since $x = \cos(y)$
$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$$

Chapter 3

3.1 Relationships

$$\cos(x - \frac{\pi}{2}) = \sin(x)$$
$$\sin(x + \frac{\pi}{2}) = \cos(x)$$

3.2 Identities

3.2.1 Reciprocal Identities

$$\sin(\theta) = \frac{1}{\csc(\theta)} \quad \text{or} \quad \csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\cos(\theta) = \frac{1}{\sec(\theta)} \quad \text{or} \quad \sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\tan(\theta) = \frac{1}{\cot(\theta)} \quad \text{or} \quad \cot(\theta) = \frac{1}{\tan(\theta)}$$

3.2.2 Pythagorean Identities

$$\sin^2(\theta) + \cos^2(\theta) = 1$$
$$1 + \tan^2(\theta) = \sec^2(\theta)$$
$$\csc^2(\theta) = 1 + \cot^2(\theta)$$

3.2.3 Ratio Identities

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$
$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

3.2.4 Sum and Difference of Angles

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin(\alpha)\times\cos(\beta)+\cos(\alpha)\times\cos(\beta)\\
\sin(\alpha-\beta) &= \sin(\alpha)\times\cos(\beta)-\cos(\alpha)\times\sin(\beta)\\
\cos(\alpha+\beta) &= \cos(\alpha)\times\cos(\beta)-\sin(\alpha)\times\sin(\beta)\\
\cos(\alpha-\beta) &= \cos(\alpha)\times\cos(\beta)+\sin(\alpha)\times\sin(\beta)\\
\tan(\alpha+\beta) &= \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\times\tan(\beta)}\\
\tan(\alpha-\beta) &= \frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\times\tan(\beta)}
\end{aligned}$$

3.2.5 Double Angles

$$\begin{aligned}
\sin(2\theta) &= 2\sin(\theta)\cos(\theta)\\
\cos(2\theta) &= \cos^2(\theta)-\sin^2(\theta)\\
&= 2\cos^2(\theta)-1\\
&= 1-2\sin^2(\theta)\\
\tan(2\theta) &= (2\tan(\theta))/(1-\tan^2(\theta))
\end{aligned}$$