Graphs

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Contents

Chapter 1

Graphs and Graph Models

1.1 Introduction

Definition 1.1.1: Graph

A graph $G = (V, E)$ consist of V, a non empty set of *vertices / nodes* and E a set of *edges*. Each edge has either one or two vertices associated with it called its *endpoints*. An edge is said to *connect* its endpoints.

The set of vertices V of a graph may be infinite, in this case the graph is called an *infinite graph*, conversely if the set of vertices is finite, the graph is called a *finite graph*.

The set of edges E contains ordered pairs or sets of elements in the set of vertices V indicating a connection between the two nodes or a node to itself.

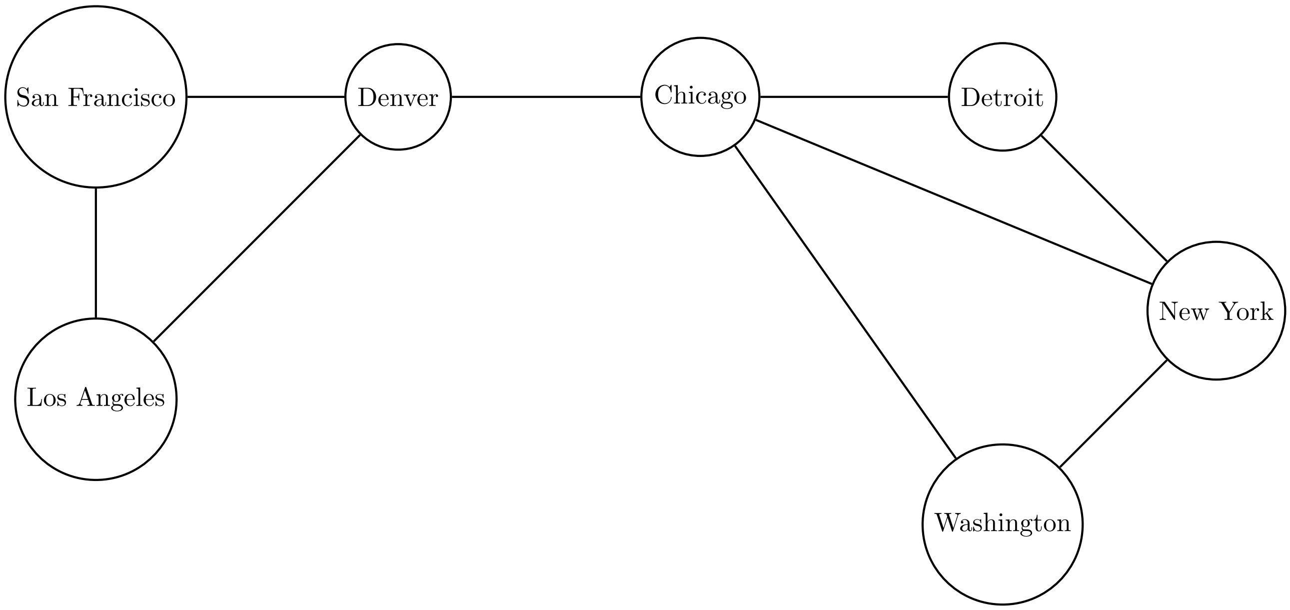
Definition 1.1.2: Vertex

A *vertex* is a point in a graph.

Definition 1.1.3: Edge

An *edge* is a line connecting two vertices in a graph.

Below is a graph representing a network of data centres and communication links between computers, where locations are represented by points and the links are represented by lines connecting the points.



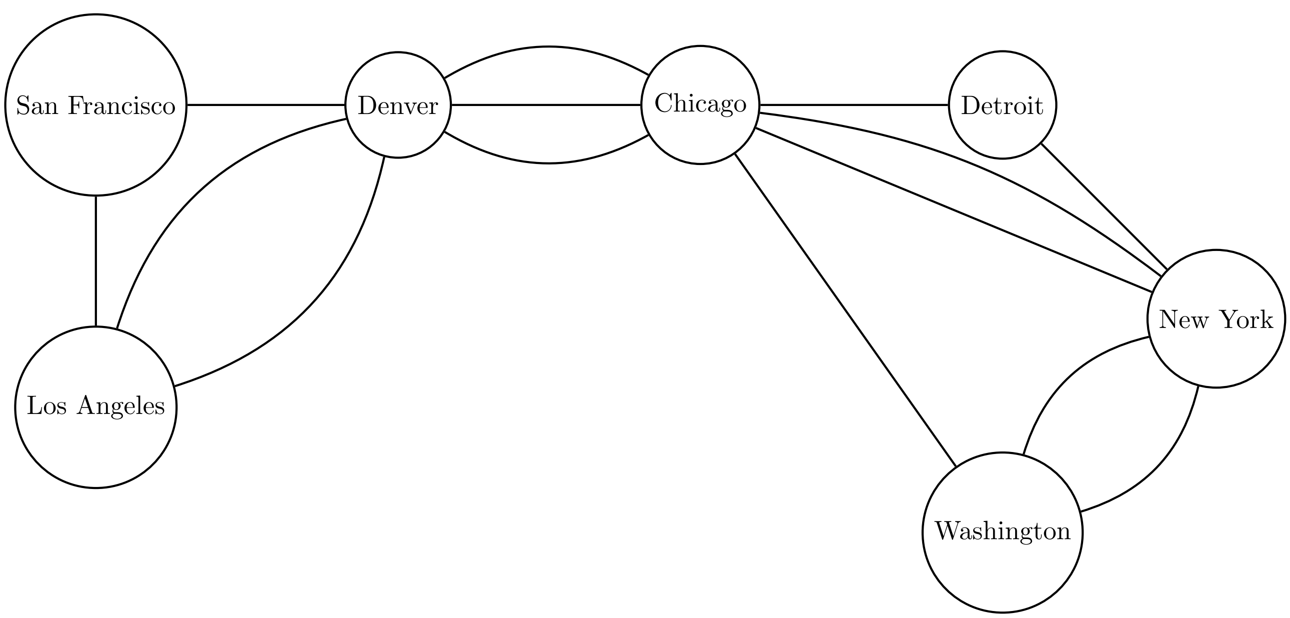
This is an example of a simple graph.

Definition 1.1.4: Simple Graph

A graph is said to be *simple* if it has no loops or multiple edges. A *loop* is an edge that connects a vertex to itself. A *multiple edge* is two or more edges that connect the same pair of vertices.

1.1.1 Multigraph

This graph could be re-drawn to model multiple links between the same pair of locations, as shown in Figure 1.1.1.



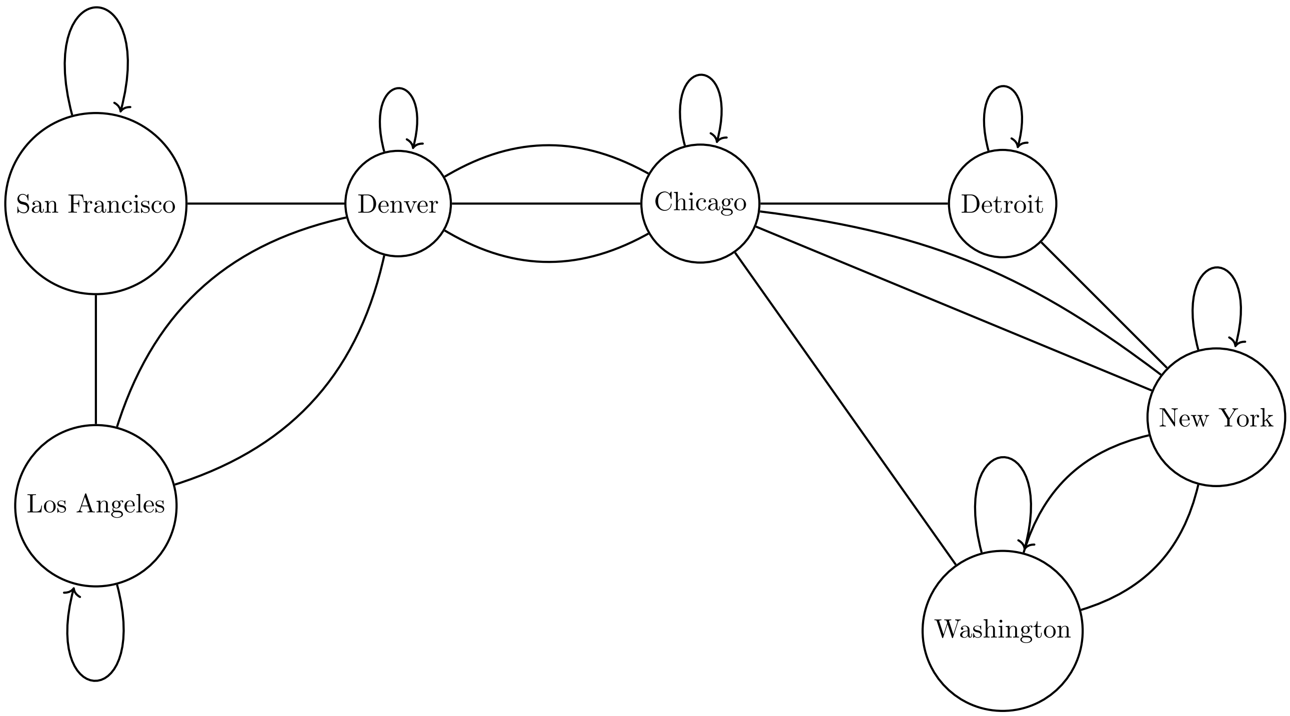
This is an example of a multigraph.

Definition 1.1.5: Multigraph

A graph that has multiple edges connecting the same vertices. When there are m distinct edges connecting the same unordered pair of vertices $\{u, v\}$, we say that $\{u, v\}$ is an edge of *multiplicity* m. i.e. j

1.1.2 Loops

Sometimes vertices may be connected to themselves, as shown in Figure 1.1.2.



Edges connecting vertices to themselves are called *loops*.

Definition 1.1.6: Loop

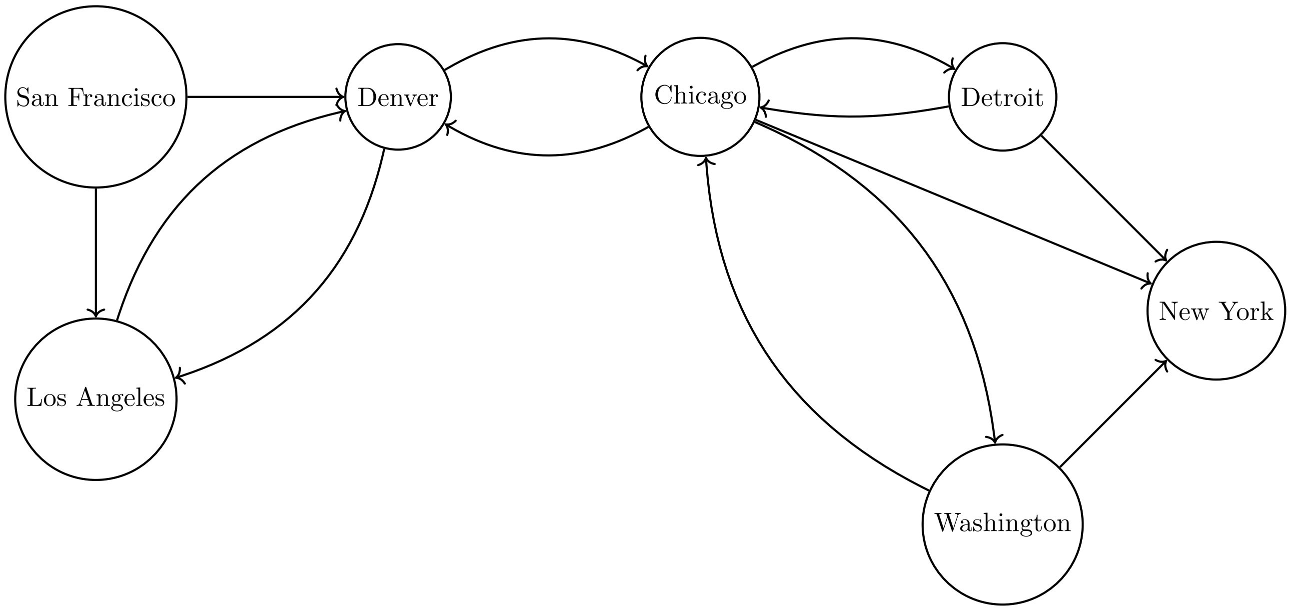
An edge that connects a vertex to itself.

Definition 1.1.7: Psuedograph

A graph that allows loops and multiple edges.

1.1.3 Directed Graphs

So far the examples given have been *undirected graphs*, with undirected edges. It is also possible to assign directions to the edges, as shown in Figure 1.1.3.



Such a graph is called a *directed graph* or *digraph*.

Definition 1.1.8: Directed Graph

A graph (V, E) that consists of a non-empty set of vertices V and a set of directed edges / arcs E. Each directed edge is associated with an ordered pair of vertices. The arc associated with the ordered pair (u, v) is said to *start* at u and *end* at v.

1.1.4 Simple Directed Graph

Definition 1.1.9: Simple Directed Graph

A directed graph with no loops or multiple edges.

1.1.5 Directed Multigraph

Definition 1.1.10: Directed Multigraph

A directed graph with multiple edges connecting the same pair of vertices. When there are m directed edges, each associated to an ordered pair of vertices (u, v), then (u, v) is an edge of multiplicity m.

1.1.6 Mixed Graph

Definition 1.1.11: Mixed graph

A graph with both direct and undirected edges, that may have multiple edges and loops.

1.1.7 Graph Terminology

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Psuedograph	Undirected	Yes	Yes
Simple Directed Graph	Directed	No	No
Directed Multigraph	Directed	Yes	Yes
Mixed Graph	Directed and Undirected	Yes	Yes

Table 1.1: Graph Terminology

Chapter 2

Graph Terminology and Special Graphs

2.1 Basic Terminology

Definition 2.1.1: Adjacent / Neighbours

Two vertices u and v in an undirected graph G, that are endpoints of an edge e. Such an edge e is called *incident with* the nodes u and v and e is said to *connect* u and u

Definition 2.1.2: Neighbourhood

The set of all neighbours of a vertex v of $G = (V, E)$, denoted by:

$$N(v)$$

If A is a subset of V, we denote by $N(A)$ the set of all nodes in G that are adjacent to at least one node in A. i.e. $N(A) = \bigcup_{v \in A} N(v)$

Definition 2.1.3: Degree of a Vertex / Node

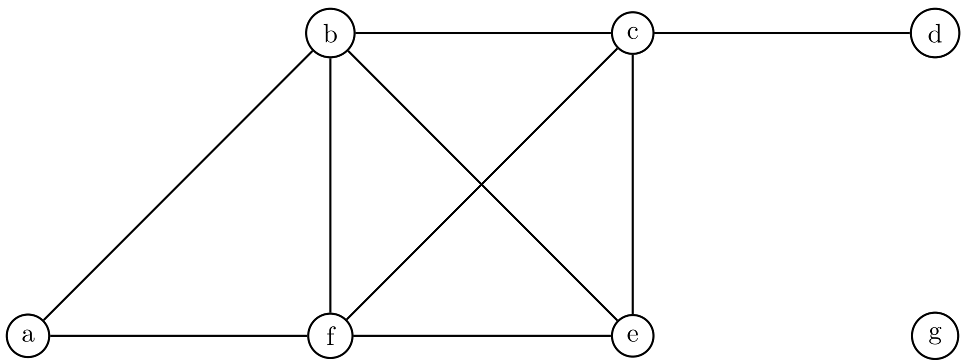
The number of edges incident with a particular node, except that a loop at a node is counted twice to the degree of that node, denoted by:

$$\deg(v)$$

Example 2.1.1

Question 1

What are the degrees and neighbourhoods of the vertices of the graph below



Solution:

Degrees:

deg(a) - 2
deg(b) - 4
deg(c) - 4
deg(d) - 1
deg(e) - 3
deg(f) - 4
deg(g) - 0

Neighbourhoods:

$$N(a) = \{b, f\}$$
$$N(b) = \{c, f, e, a\}$$
$$N(c) = \{b, d, f, e\}$$
$$N(d) = \{c\}$$
$$N(e) = \{b, f, c\}$$
$$N\left(f\right) = \{b, c, e, a\}$$
$$N\left(g\right) = \emptyset$$

Theorem 2.1.1 The Handshaking Theorem

Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} \deg(v)$$

Theorem 2.1.2

An undirected graph has an even number of nodes of odd degree.