Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Madiba Hudson-Quansah

Contents

Chapter 1

Sets

Definition 1.0.1: Set

An unordered collection of objects, called *elements* or *members* of the set. A set contains elements and, we can denote this as $a \in A$ where a is an element of the set A, or $a \notin A$, where a is not an element of the set A.

There are several ways to describe a set:

Roster notation $\{1, 2, 3, 4, 5\}$

Set-Builder notation Where all the elements of a set are described by a property they satisfy.i.e. The set O of all odd positive numbers less than 10 can be expressed as $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$ or specifying the domain of discourse, $O = \{x \in \mathbb{Z}^+ \mid x \text{ is odd and } x < 10\}$, or the set of all positive rational numbers $\mathbb{Q}^+$ can be expressed as $\mathbb{Q}^+ = \left\{x \in \mathbb{R} \mid x = \frac{p}{q}, \text{for some positive integers } q \text{ and } p\right\}$

Definition 1.0.2: Equality of Sets

Two sets A and B are equal if and only if they have the same elements. Therefore, $\forall x\,(x \in A \leftrightarrow x \in B)$, We write $A = B$ if this is the case.

Definition 1.0.3: Empty / Null Set

A set with no elements, denoted by $\emptyset$ or $\{\}$. Can be expressed as $\{x \mid F\}$

Definition 1.0.4: Singleton Set

A set with exactly one element, denoted by $\{a\}$. The set $\{\emptyset\}$ is a singleton set as it is a set with one element, the empty set.

1.1 Set Definitions

1.1.1 Natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

1.1.2 Integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

1.1.3 Positive Integers

$$\mathbb{Z}^+ = \{1, 2, 3, 4, 5, \ldots\}$$

1.1.4 Rational numbers

$$\mathbb{Q} = \left\{\frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0\right\}$$

1.1.5 Irrational Numbers

$$\mathbb{I} = \{x \mid x \text{ is a number that cannot be expressed as a fraction}\}$$

1.1.6 Real numbers

$$\mathbb{R} = \{x \mid x \text{ is a point on the number line}\}$$

Or

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$$

1.1.7 Positive Real numbers

$$\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$$

1.1.8 Complex numbers

$$\mathbb{C} = \left\{a + bi \mid a, b \in \mathbb{R} \text{ and } i^2 = -1\right\}$$

1.2 Venn Diagrams

Definition 1.2.1: Universal Set

The set of all objects under consideration, denoted by U. Can be expressed as $\{x \mid T\}$

Sets can be graphically represented using Venn diagrams. A Venn diagram is a collection of simple closed curves, especially circles, that represent sets. In Venn diagrams the universal set U which contains all the objects under consideration is represented by a rectangle, and the sets are represented by circles within the rectangle, with points inside the circles representing elements of the sets.

1.3 Subsets

Definition 1.3.1: Subset

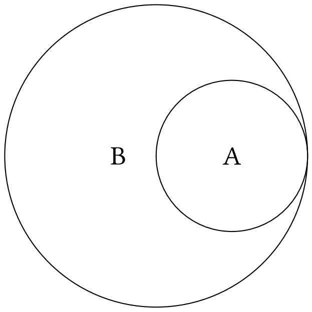
A set A is a *subset* of a set B if and only if every element of A is also an element of B. Denoted by $A \subseteq B$.

We see that $A \subseteq B$ if and only if

$$\forall x\,(x \in A \to x \in B)$$

Is true. I.e. If $x \in A$, then $x \in B$. To disprove this we need to show that $\exists x\,(x \in A \land x \notin B)$

Shown graphically:



Example 1.3.1

The set of integers with squares less than 100 is not a subset of the set of nonnegative integers because -1 is in the former set [as $(-1)^2 < 10$], but not the later set. The set of people who have taken discrete mathematics at your school is not a subset of the set of all computer science majors at your school if there is at least one student who has taken discrete mathematics who is not a computer science major.

Theorem 1.3.1

For every set S

1. $\emptyset \subseteq S$
2. $S \subseteq S$

1. ***Proof:*** We will prove that $\emptyset \subseteq S$, using a vacuous proof
   Let S be a set.
   To show $\emptyset \subseteq S$ we must show that $\forall x\,(x \in \emptyset \longrightarrow x \in S)$ is T.
   Because $\emptyset$ contains no elements $x \in \emptyset$ is always F
   This follows that the implication $x \in \emptyset \longrightarrow x \in S$ is always T
   Hence $\emptyset \subseteq S$

   😊

2. ***Proof:*** We will prove that $S \subseteq S$, using a direct proof
   Let S be a set
   To show $S \subseteq S$ we must show that $\forall x\,(x \in S \longrightarrow x \in S)$ is T
   Assume $x \in S$
   Because $x \in S$ is always T, the implication $x \in S \longrightarrow x \in S$ is always T
   $\therefore \forall x\,(x \in S \longrightarrow x \in S)$ is T
   Hence $S \subseteq S$

   😊

Definition 1.3.2: Proper subset

A set A is *proper subset* of a set B if and only if every element of A is also an element of B and $A \neq B$. Denoted by $A \subset B$. I.e.

$$\exists x\,(x \notin A \wedge x \in B) \wedge \forall x\,(x \in A \longrightarrow x \in B)$$

Is T.

Definition 1.3.3: Further Equality

Two sets A and B are equal if $A \subseteq B \wedge B \subseteq A$ is T. I.e. $A = \{\emptyset, \{a\}, \{a\}, \{b\}, \{a, b\}\}$ and $B = \{x \mid x \text{ is a subset of the set } \{a, b\}\}$ are equal.

1.4 Cardinality

Definition 1.4.1: Cardinality

The number of distinct elements n in a set A. Denoted by $|A| = n$. Where n is a non-negative integer, we say that A is a finite set.

Definition 1.4.2: Infinite set

A set A is infinite if it is not finite. I.e. $|A| = \infty$

1.5 Power Set

Definition 1.5.1: Power Set

A set containing all the subsets of a given set A. Denoted by $\mathcal{P}(A)$. If a set has n distinct elements, then the cardinality of the power set is 2^n.

Example 1.5.1

Question 1

What is the power set of the set $\{0, 1, 2\}$

Solution:

$$\mathcal{P}(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, , \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$$

Example 1.5.2

Question 2

What is the power set of $\emptyset$

Solution:

$$\mathcal{P}(\emptyset) = \{\emptyset\}$$

Question 3

What is the power set of $\{\emptyset\}$

Solution:

$$\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$$

1.6 N-Tuples

Definition 1.6.1: Ordered N-Tuple

N-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, ..., and a_n as its nth element.

Two n-tuples are equal if an only if each corresponding pair of their elements is equal, i.e. $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ are equal if and only if $a_i = b_i$, for $i = 1, 2, \ldots, n$.

Ordered 2-tuples are called *ordered pairs*. The ordered pairs, (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

1.7 Cartesian Products

Definition 1.7.1: Cartesian Product

Let A and B be sets. The *Cartesian Product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. I.e.

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

The number of items in the Cartesian product of two sets is the product of the cardinality of each set.

Example 1.7.1

Question 4

What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$

Solution:

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

Question 5

Show that the Cartesian product $B \times A$ is not equal to the Cartesian product $A \times B$.

Solution:

$$B \times A = \{(a, 1)(a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$$

$\therefore A \times B \neq B \times A$

Definition 1.7.2: Cartesian Product of more than two sets

The Cartesian product of the sets $A_1, A_2, \ldots, A_n$,denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$. I.e.

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$$

Example 1.7.2

Question 6

What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, $C = \{0, 1, 2\}$.

Solution:

$$A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}$$

We use the notation A^2 to denote $A \times A$, the Cartesian product of A and itself. Therefore

$$A^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for } i = 1, 2, \ldots, n\}$$

Example 1.7.3

Suppose $A = \{1, 2\}$.
It follows $A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$,
and $A^3 = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$

Example 1.7.4

Question 7

What are the ordered pairs in the less than or equal to relation, which contains, (a, b) if $a \leqslant b$, on the set $\{0, 1, 2, 3\}$

Solution: Let R be the relation on the set $\{0, 1, 2, 3\}$, if $a \leqslant b$.

$$R = \{(0,0), (1,1), (2,2), (3,3), (0,1), (0,2), (0,3), (1,2), (1,3), (2,3)\}$$

1.8 Set Notation with Quantifiers

We can restrict the domain of a quantifier to a set, I.e. Where S is a set $\forall x \in S\,(P(x))$, denotes the universal quantification of $P(x)$ for all elements in the set S. Which is shorthand for $\forall x\,(x \in S \longrightarrow P(x))$

Example 1.8.1

$\forall x \in \mathbb{R}\left(x^2 \geqslant 0\right)$ means "the square of any real number is greater than or equal to 0".
$\exists x \in \mathbb{Z}\left(x^2 = 1\right)$ means "there exists an integer whose square is 1"

1.9 Truth Sets and Quantifiers

Definition 1.9.1: Truth Set

For a predicate P the truth set of P is the set of all elements in the domain of discourse that make P true. I.e. let S be a set. The truth set of $P(x)$ is

$$\{x \in S \mid P(x)\}$$

Example 1.9.1

Question 8

What are the truth set of the predicates $P(x)$, $Q(x)$, and $R(x)$, where the domain is the set of integers, and $P(x)\colon |x| = 1$, $Q(x)\colon x^2 = 2$, and $R(x)\colon |x| = x$

Solution:
The truth set of P is $\{x \in \mathbb{Z} \mid |x| = 1\}$
The truth set of Q is $\left\{x \in Z \mid x^2 = 2\right\}$
The truth set of R is $\{x \in \mathbb{Z} \mid |x| = x\}$

Note:-

$\forall x P(x)$ is T over the domain U if and only if the truth set of P is U.
$\exists x P(x)$ is T over the domain U if and only if the truth set of P is not empty.

1.10 Set Operations

1.10.1 Union

Definition 1.10.1: Union

Let A and B be sets. The *union* of A and B, denoted by $A \cup B$, is the set of all elements that are either in A or in B or in both. I.e.

$$A \cup B = \{x \mid x \in A \lor x \in B\}$$

1.10.2 Intersection

Definition 1.10.2: Intersection

Let A and B be sets. The *intersection* of A and B, denoted by $A \cap B$, is the set of all elements that are in both A and B. I.e.

$$A \cap B = \{x \mid x \in A \land x \in B\}$$

1.10.3 Complement

Definition 1.10.3: Complement

Let A be a set. The *complement* of the set A (with respect to U), denoted by $\overline{A}$ is the set $U - A$. I.e.

$$\overline{A} = \{x \in U \mid x \notin A\}$$

1.10.4 Difference

Definition 1.10.4: Difference

Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set of all elements that are in A but not in B. I.e.

$$A - B = \{x \mid x \in A \land x \notin B\}$$

Or

$$A - B = A \cap \overline{B}$$

1.10.5 Symmetric Difference

Definition 1.10.5: Symmetric Difference

Let A and B be sets. The *symmetric difference* of A and B, denoted by $A \oplus B$, is the set of all elements that are in exactly one of A and B. I.e.

$$A \oplus B = (A - B) \cup (B - A)$$

Example 1.10.1

Question 9

$$
\begin{aligned}
U &= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \\
A &= \{1, 2, 3, 4, 5\} \\
B &= (4, 5, 6, 7, 8)
\end{aligned}
$$

What is $A \oplus B$

Solution:

$$A \oplus B = \{1, 2, 3, 6, 7, 8\}$$

1.10.6 The Cardinality of the Union of Two Sets

The cardinality of the union of two sets A and B is given by

$$|A \cup B| = |A| + |B| - |A \cap B|$$

1.11 Set Identities

1.11.1 Identity Laws

$$
\begin{aligned}
A \cap U &= A \\
A \cup \emptyset &= A
\end{aligned}
$$

1.11.2 Domination Laws

$$
\begin{aligned}
A \cup U &= U \\
A \cap \emptyset &= \emptyset
\end{aligned}
$$

1.11.3 Idempotent Laws

$$
\begin{aligned}
A \cup A &= A \\
A \cap A &= A
\end{aligned}
$$

1.11.4 Complementation Law

$$\overline{\left(\overline{A}\right)} = A$$

1.11.5 Commutative Laws

$$
\begin{aligned}
A \cup B &= B \cup A \\
A \cap B &= B \cap A
\end{aligned}
$$

1.11.6 Associative Laws

$$
\begin{aligned}
A \cup (B \cup C) &= (A \cup B) \cup C \\
A \cap (B \cap C) &= (A \cap B) \cap C
\end{aligned}
$$

1.11.7 Distributive Laws

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

1.11.8 De Morgan's Laws

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

1.11.9 Absorption Laws

$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$

1.11.10 Complement Laws

$$A \cup \overline{A} = U$$
$$A \cap \overline{A} = \emptyset$$

1.11.11 Proving Set Identities

There are different ways to prove set identities, these include:

- Proving each set is a subset of the other
- Using set builder notation and propositional logic
- Using Membership tables

Definition 1.11.1: Membership Table

A table that shows the truth value of a predicate for all possible combinations of truth values of its variables.

Example 1.11.1

Question 10

Prove that
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Using propositional logic:

Proof: We prove this identity by showing that each set is a subset of the other. I.e.

$$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B} \land \overline{A} \cap \overline{B} \subseteq \overline{A \cap B}$$

$$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B} \text{ means } \forall x \left(x \in \overline{A \cap B} \rightarrow x \in \overline{A} \cup \overline{B} \right)$$
$$\overline{A} \cup \overline{B} \subseteq \overline{A \cap B} \text{ means } \forall x \left(x \in \overline{A} \cup \overline{B} \rightarrow x \in \overline{A \cap B} \right)$$

Assume that $x \in \overline{A \cap B}$

$$\begin{aligned}
x &\in \overline{A \cap B} && \text{Assumption} \\
x &\notin A \cap B && \text{Definition of Complement} \\
\neg\,(x &\in A \cap B) && \text{Definition of } \notin \\
\neg\,(x \in A &\land x \in B) && \text{Definition of intersection} \\
\neg\,(x \in A) &\lor \neg\,(x \in B) && \text{By First De Moragn's Law for propositional logic} \\
x \notin A &\lor x \notin B && \text{Definition of Complement} \\
x &\in \overline{A} \cup \overline{B} && \text{Definition of union}
\end{aligned}$$

Then we assume $x \in \overline{A} \cup \overline{B}$

$$\begin{aligned}
x &\in \overline{A} \cup \overline{B} && \text{Assumption} \\
x \notin A &\lor x \notin B && \text{Definition of union} \\
\neg\,(x \in A) &\lor \neg\,(x \in B) && \text{Definition of Complement} \\
\neg\,(x \in A &\land x \in B) && \text{By Second De Morgan's Law for propositional logic} \\
x \notin A &\land x \notin B && \text{Definition of Complement} \\
x &\notin A \cap B && \text{Definition of intersection} \\
x &\in \overline{A \cap B} && \text{Definition of Complement}
\end{aligned}$$

😊

Using set builder notation

$$\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{Definition of Complement} \\
&= \{x \mid \neg\,(x \in (A \cap B))\} && \text{Definition of } \notin \\
&= \{x \mid \neg\,(x \in A \land x \in B)\} && \text{Definition of Intersection} \\
&= \{x \mid \neg\,(x \in A) \lor \neg\,(x \in B)\} && \text{By First De Morgan's Law for propositional logic} \\
&= \{x \mid x \notin A \lor x \notin B\} && \text{Definition of Complement} \\
&= \{x | x \in \overline{A} \cup \overline{B}\} && \text{Definition of union} \\
&= \overline{A} \cup \overline{B}
\end{aligned}$$

1.12 Generalized Unions and Intersections

Definition 1.12.1: Generalized Union

The union of a collection of sets that contains those elements that are members of at least one set in the collection. Denoted by

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$$

Where $A_1 \cup A_2 \cup \ldots A_n$ is the union of sets $A_1, A_2, \ldots, A_n$

Definition 1.12.2: Generalized Intersection

The intersection of a collection of sets that contains those elements that are members of all the sets in the collection. Denoted by

$$A_1 \cap A_2 \cap \ldots A_n = \bigcap_{i=1}^{n} A_i$$

Where $A_1 \cap A_2 \cap \ldots \cap A_n$ is the intersection of sets $A_1, A_2, \ldots, A_n$

Example 1.12.1

For $i = 1, 2, \ldots,$ let $A_i = \{i, i+1, i+2, \ldots\}$. Then.

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\}$$

and

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2\} = A_n$$

We can extend this notation to other families of sets I.e.

$$A_1 \cup A_2 \cup \ldots \cup A_n \cup \ldots = \bigcup_{i=1}^{\infty} A_i$$

Denotes the union of the sets $A_1, A_2, \ldots, A_n, \ldots$, and the intersection of these sets is denoted by

$$A_1 \cap A_2 \cap \ldots \cap A_n \cap \ldots = \bigcap_{i=1}^{\infty} A_i$$

Generally when I is set, the notations $\bigcap_{i \in I} A_i$ and $\bigcup_{i \in I} A_i$ are used to denote the intersection and union of the sets A_i for $i \in I$, respectively, where

$$\bigcap_{i \in I} A_i = \{x \mid \forall i \in I \, (x \in A_i)\}$$

and

$$\bigcup_{i \in I} A_i = \{x \mid \exists i \in I \, (x \in A_i)\}$$

Example 1.12.2

Suppose $A_i = \{1, 2, 3, \ldots, i\}$ for $i = \{1, 2, 3, \ldots\}$ Then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \{1, 2, 3, \ldots, i\} = \mathbb{Z}^+$$

and

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} \{1, 2, 3, \ldots, i\} = \{1\}$$

1.13 Computer Representation Of Sets

Assume that the universal set U is finite. First, specify an arbitrary ordering of the elements of U, e.g. $a_1, a_2, a_3, \ldots, a_n$. Represent subset A of U with the bit string of length n, where the ith bit in this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A.

Example 1.13.1

Question 11

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the ordering of the elements of U has the elements in increasing order, i.e. $a_i = i$. What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of all integers not exceeding 5 in U.

Solution: Odd integers - 1010101010
Even integers - 0101010101
Less than or equal to 5 - 1111100000

To find the bit strings that represent the union, intersection, and complement of two sets, we can use the bitwise OR, bitwise AND, and bitwise NOT operations, respectively.

Example 1.13.2

Question 12

The bit strings for the sets $\{1, 2, 3, 4, 5\}$ and $\{1, 3, 5, 7, 9\}$ are 1111100000 and 1010101010 respectively. Find the union and intersection of these sets.

Solution:

$$\begin{array}{r r} & 11\,1110\,0000 \\ \vee & \underline{10\,1010\,1010} \\ & 11\,1110\,1010 \end{array}$$

$$\begin{array}{r r} & 11\,1110\,0000 \\ \wedge & \underline{10\,1010\,1010} \\ & 10\,1010\,0000 \end{array}$$

1.14 Exercises

Question 13

List the members of these sets

1. $\{x \mid x \text{ is the square of an integer and } x < 100\}$
2. $\left\{x \mid x \text{ is an integer such that } x^2 = 2\right\}$

Solution:

1. $\{1, 4, 9, 16, 25, 36, 49, 64, 81\}$
2. $\emptyset$

Question 14

Use set builder notation to describe the following sets

1. $\{-3, -2, -1, 0, 1, 2, 3\}$
2. $\left\{m, n, o, p\right\}$

Solution:

1. $\{x \mid -3 \leqslant x \leqslant 3\}$
2. $\{x \mid x$ is a letter in the word monopoly excluding ”l” and ”y” $\}$

Question 15

Suppose that $A = \{2, 4, 6\}$, $B = \{2, 6\}$, $C = \{4, 6\}$ and $D = \{4, 6, 8\}$. Determine which of these sets are subsets of which other sets.

Solution:

$$B \subseteq A$$
$$C \subseteq A$$
$$C \subseteq D$$

Question 16

Suppose that A, B, C, are sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$

Solution:

$$A \subseteq B \text{ means } \forall x\,(x \in A \longrightarrow x \in B)$$
$$B \subseteq C \text{ means } \forall x\,(x \in B \longrightarrow x \in C)$$
$$A \subseteq C \text{ means } \forall x\,(x \in A \longrightarrow x \in C)$$

$$\begin{array}{rl} & \forall x\,(x \in A \longrightarrow x \in B) \\ & \forall x\,(x \in B \longrightarrow x \in C) \\ \hline \therefore & \forall x\,(x \in A \longrightarrow x \in C) \end{array}$$

	Steps	Reasons
1	$\forall x\,(x \in A \longrightarrow x \in B)$	Premise 1
2	$\forall x\,(x \in B \longrightarrow x \in C)$	Premise 2
3	$x \in A \longrightarrow x \in B$	Universal Instantiation of 1
4	$x \in B \longrightarrow x \in C$	Universal Instantiation of 2
5	$x \in A \longrightarrow x \in C$	By Hypothetical Syllogism of 3 and 4
6	$\forall x\,(x \in A \longrightarrow x \in C)$	Universal generalization of 5

Question 17

Find the power set of each of these sets, where a and b are distinct elements.

1. $\{a\}$
2. $\{a, b\}$
3. $\{\emptyset, \{\emptyset\}\}$

Solution:

1. $\mathcal{P}\,(\{a\}) = \{\emptyset, \{a\}\}$
2. $\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
3. $\mathcal{P}(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}\ \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

Question 18

List the members of these sets

1. $\left\{x \mid x \text{ is a real number such that } x^2 = 1\right\}$
2. $\{x|x \text{ is a positive integer less than } 12\}$
3. $\{x|x \text{ is the square of an integer and } x < 100\}$
4. $\left\{x \mid x \text{ is an integer such that } x^2 = 2\right\}$

Solution:

1. $\{-1, 1\}$
2. $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$
3. $\{0, 1, 4, 9, 16, 25, 36, 49, 64, 81\}$
4. $\emptyset$

Question 19

Use set builder notation to show that:

1. $A \cup U = U$
2. $A \cap \emptyset = \emptyset$
3. $A \cup \overline{A} = U$
4. $A \cap \overline{A} = \emptyset$

Solution:

1.

$$\begin{aligned}
A \cup U &= \{x \mid x \in A \cup U\} && \text{Set builder notation} \\
&= \{x \mid x \in A \vee x \in U\} && \text{Definition of Union} \\
&= \{x \mid x \in A \vee T\} && \text{Definition of Universal Set} \\
&= \{x \mid T\} && \text{By First Domination law for propositional logic} \\
&= U && \text{Definition of Universal Set}
\end{aligned}$$

2.

$$\begin{aligned}
A \cap \emptyset &= \{x \mid x \in A \cap \emptyset\} && \text{Set builder notation} \\
&= \{x \mid x \in A \wedge x \in \emptyset\} && \text{Definition of Intersection} \\
&= \{x \mid x \in A \wedge F\} && \text{Definition of Empty set} \\
&= \{x \mid F\} && \text{By Second Domination Law for propositional logic} \\
&= \emptyset && \text{Definition of Empty set}
\end{aligned}$$

3.

$$\begin{aligned}
A \cup \overline{A} &= \{x \mid x \in A \cap \overline{A}\} && \text{Set builder notation} \\
&= \{x \mid x \in A \vee x \in \overline{A}\} && \text{Definition of Union} \\
&= \{x \mid x \in A \vee x \notin A\} && \text{Definition of Complement} \\
&= \{x \mid x \in A \vee \neg(x \in A)\} && \text{Definition of Complement} \\
&= \{x \mid T\} && \text{By First Negation Law for propositional logic} \\
&= U && \text{Definition of Universal set}
\end{aligned}$$

4.

$$
\begin{aligned}
A \cap \overline{A} &= \{x \mid x \in A \cap \overline{A}\} && \text{Set builder notation} \\
&= \{x \mid x \in A \land x \in \overline{A}\} && \text{Definition of intersection} \\
&= \{x \mid x \in A \land (x \notin A)\} && \text{Definition of Complement} \\
&= \{x|x \in A \land \neg(x \in A)\} && \text{Definition of Complement} \\
&= \{x \mid F\} && \text{By Second Negation law of propositional logic} \\
&= \emptyset && \text{Definition of } \emptyset
\end{aligned}
$$

Question 20

Let A and B be sets. Show that

1. $(A \cap B) \subseteq A$
2. $A \subseteq (A \cup B)$
3. $A - B \subseteq A$
4. $A \cap (B - A) = \emptyset$
5. $A \cup (B - A) = A \cup B$

Solution:

1.

$$(A \cap B) \subseteq A \text{ means } \forall x\,(x \in (A \cap B) \longrightarrow x \in A)$$

Assume $x \in (A \cap B)$

	Steps	Reasons
1	$x \in A \cap B$	Assumption
2	$x \in A \land x \in B$	Definition of intersection
3	$x \in A$	Simplification of 2

$\therefore\ x \in (A \cap B) \longrightarrow x \in A$
Conclusion: $(A \cap B) \subseteq A$

2. ***Proof:***

$$A \subseteq (A \cup B) \text{ means } \forall x\,(x \in A \longrightarrow x \in A \cup B)$$

	Steps	Reasons
1	$x \in A$	Premise
2	$x \in A \land x \in B$	Addition
3	$x \in A \cup B$	Definition of Union

$\therefore\ x \in A \longrightarrow x \in A \cup B$
Conclusion: $A \subseteq A \cup B$ 😊

3. ***Proof:*** $A - B \subseteq A$ means $\forall x\,(x \in A - B \longrightarrow x \in A)$
Assume $x \in A - B$

$\therefore\ x \in A - B \longrightarrow x \in A$
Hence $A - B \subseteq A$ 😊

	Steps	Reasons
1	$x \in A - B$	Assumption
2	$x \in A \wedge x \notin B$	Definition of Difference
3	$x \in A$	By Simplification on 2

4.

$$\begin{aligned}
A \cap (B - A) &= A \cap \left(B \cap \overline{A}\right) && \text{Definition of Difference}\\
&= B \cap \left(A \cap \overline{A}\right) && \text{By Second Associative Law}\\
&= B \cap \emptyset && \text{By Second Complement Law}\\
&= \emptyset && \text{By Second Domination Law}
\end{aligned}$$

Hence $A \cap (B - A) = \emptyset$

5.

$$\begin{aligned}
A \cup (B - A) &= A \cup \left(B \cap \overline{A}\right) && \text{Definition of Difference}\\
&= (A \cup B) \cap \left(A \cup \overline{A}\right) && \text{By First Distributive Law}\\
&= (A \cup B) \cap U && \text{By First Complement Law}\\
&= (A \cap U) \cup (B \cap U) && \text{By Second Distributive Law}\\
&= A \cup B && \text{By First Identity Law}
\end{aligned}$$

Hence $A \cup (B - A) = A \cup B$

Question 21

Show that if A is a subset of a universal set U, then

1. $A \oplus A = \emptyset$
2. $A \oplus \emptyset = A$
3. $A \oplus U = \overline{A}$
4. $A \oplus \overline{A} = U$

Solution:

1.

$$\begin{aligned}
A \oplus A &= \{x \mid A \oplus A\} && \text{Set Builder Notation}\\
&= \{x \mid x \in (A - A) \cup (A - A)\} && \text{Definition of Symmetric Difference}\\
&= \{x \mid (x \in A - A) \vee (x \in A - A)\} && \text{Definition of Union}\\
&= \{x \mid x \in A - A\} && \text{By Idempotent Law for propositional logic}\\
&= \{x \mid (x \in A) \wedge (x \notin A)\} && \text{Definition of Difference}\\
&= \{x \mid (x \in A) \wedge \neg (x \in A)\} && \text{Definition of Complement}\\
&= \{x \mid F\} && \text{By First Negation Law for propositional logic}\\
&= \emptyset
\end{aligned}$$

Hence $A \oplus A = \emptyset$

2.

$$
\begin{aligned}
A \oplus \emptyset &= \{x \mid A \oplus \emptyset\} && \text{Set Builder Notation}\\
&= \{x \mid x \in (A - \emptyset) \cup (\emptyset - A)\} && \text{Definition of Symmetric Difference}\\
&= \{x \mid x \in (A - \emptyset) \vee x \in (\emptyset - A)\} && \text{Definition of Union}\\
&= \{x \mid (x \in A \wedge x \notin \emptyset) \vee (x \in \emptyset \wedge x \notin A)\} && \text{Definition of Difference}\\
&= \{x \mid (x \in A \wedge \neg(x \in \emptyset)) \vee (x \in \emptyset \wedge \neg(x \in A))\} && \text{Definition of Complement}\\
&= \{x \mid (x \in A \wedge \neg(F)) \vee (F \wedge \neg(x \in A))\} && \text{Definition of } \in\\
&= \{x \mid (x \in A \wedge T) \vee (F)\} && \text{By Second Domination Law for propositional logic}\\
&= \{x \mid x \in A \vee F\} && \text{By First Identity Law for propositional logic}\\
&= \{x \mid x \in A\} && \text{By Second Identity Law for propositional logic}\\
&= A && \text{Definition of set } A
\end{aligned}
$$

By set identities:

$$
\begin{aligned}
A \oplus \emptyset &= (A - \emptyset) \cup (\emptyset - A) && \text{Definition of Symmetric Difference}\\
&= \left(A \cap \overline{\emptyset}\right) \cup \left(\emptyset \cap \overline{A}\right) && \text{Definition of Difference}\\
&= (A \cap U) \cup \left(\emptyset \cap \overline{A}\right) && \text{Complementation of an Empty set}\\
&= (A) \cup \left(\emptyset \cap \overline{A}\right) && \text{By First Identity Law}\\
&= A \cup \left(\overline{A} \cap \emptyset\right) && \text{By Second Commutative Law}\\
&= A \cup \emptyset && \text{By Second Domination Law}\\
&= A && \text{By Second Identity Law}
\end{aligned}
$$

Hence $A \oplus \emptyset = A$

3.

$$
\begin{aligned}
A \oplus U &= (A - U) \cup (U - A) && \text{Definition of Symmetric Difference}\\
&= \left(A \cap \overline{U}\right) \cup \left(U \cap \overline{A}\right) && \text{Definition of Difference}\\
&= (A \cap \emptyset) \cup \left(U \cap \overline{A}\right) && \text{Complementation of the Universal set}\\
&= \emptyset \cup \left(U \cap \overline{A}\right) && \text{By Second Domination Law}\\
&= \emptyset \cup \left(\overline{A} \cap U\right) && \text{By Second Commutative Law}\\
&= \emptyset \cup \overline{A} && \text{By First Identity Law}\\
&= \overline{A} && \text{By Second Identity Law}
\end{aligned}
$$

Hence $A \oplus U = \overline{A}$

4.

$$
\begin{aligned}
A \oplus \overline{A} &= \left(A - \overline{A}\right) \cup \left(\overline{A} - A\right) && \text{Definition of Symmetric Difference} \\
&= \left(A \cap \overline{\overline{A}}\right) \cup \left(\overline{A} \cap \overline{A}\right) && \text{Definition of Difference} \\
&= (A \cap A) \cup \left(\overline{A} \cap \overline{A}\right) && \text{By Complementation Law} \\
&= A \cup \overline{A} && \text{By Second Idempotent Law} \\
&= U && \text{By First Complement Law}
\end{aligned}
$$

Hence $A \oplus \overline{A} = U$

Question 22

Find two sets A and B such that $A \in B$ and $A \subseteq B$

Solution:

$$
\begin{aligned}
\text{Let } A &= \emptyset \\
B &= \{\emptyset\}
\end{aligned}
$$

Question 23

Find the power set of each of these sets, where a and b are distinct elements.

1. $\{a, b\}$
2. $\{\emptyset, \{\emptyset\}\}$

Solution:

1.

$$\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

2.

$$\mathcal{P}(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$$

Question 24

Find the truth set of each of these predicates where the domain is the set of integers

1. $Q(x)$: $x^2 = 2$
2. $R(x)$: $x < x^2$

Solution:

1. $\emptyset$
2. $\{x \in \mathbb{Z} \mid x \neq 0 \wedge x \neq 1\}$

Question 25

Let A and B be sets. Show that

1. $A \subseteq (A \cup B)$
2. $A - B \subseteq A$
3. $A \cap (B - A) = \emptyset$

Solution:

1. ***Proof:***

$$A \subseteq (A \cup B) \text{ means } \forall x\,(x \in A \longrightarrow x \in A \cup B)$$

Assume $x \in A$

	Steps	Reasons
1	$x \in A$	Assumption
2	$x \in A \vee x \in B$	Addition on 1
3	$x \in A \cup B$	Definition of Union

$\therefore x \in A \longrightarrow x \in A \cup B$
Hence $A \subseteq (A \cup B)$ ☺

2. ***Proof:***

$$A - B \subseteq A \text{ means } \forall x\,(x \in A - B \longrightarrow x \in A)$$

Assume $x \in A - B$

	Steps	Reasons
1	$x \in A - B$	Assumption
2	$x \in A \wedge x \notin B$	Definition of Difference
3	$x \in A$	Simplification of 2

$\therefore x \in A - B \longrightarrow x \in A$
Hence $A - B \subseteq A$ ☺

3.

$$\begin{aligned} A \cap (B - A) &= A \cap \left(B \cap \overline{A}\right) && \text{Definition of Difference} \\ &= B \cap \left(A \cap \overline{A}\right) && \text{By Second Distributive Law} \\ &= B \cap \emptyset && \text{By Second Complement Law} \\ &= \emptyset && \text{By Second Identity Law} \end{aligned}$$

Hence $A \cap (B - A) = \emptyset$

Chapter 2

Functions

Chapter 3

Sequences and Summations

3.1 Sequences

Definition 3.1.1: Sequence

A sequence is a function from a subset of the set of integers (usually the set $\mathbb{N}$ or the set $\mathbb{Z}$) to a set S. Denoted a_n, where a is the image of the integer n under the function. We call a_n a term of a sequence.

We use $\{a_n\}$ to describe a sequence, where a_n represents an individual term of the sequence $\{a_n\}$. Sequences are described by listing the terms of the sequence in order of increasing subscripts. I.e. $\{a_n\}$, where:

$$a_n = \frac{1}{n}$$

The list of terms of the sequence $\{a_n\}$ beginning with a_1:

$$a_1, a_2, a_3, a_4, a_5, \ldots$$

starts with

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

Definition 3.1.2: Geometric Progression

A sequence in the form

$$a, ar, ar^2, ar^3, ar^4, \ldots, ar^n, \ldots$$

Where the initial term a and the common ratio r are real numbers.

Note:-

A geometric progression is a discrete analogue of the exponential function $f(x) = ar^x$

Example 3.1.1

The sequences $\{b_n\}$ with $b_n = (-1)^n$, $\{c_n\}$, with $c_n = 2 \times 5^n$, and $\{d_n\}$, with $d_n = 6 \times \left(\frac{1}{3}\right)^n$ are geometric progressions, with initial terms of 1, 2, and 6, and common ratios of -1, 5, and $\frac{1}{3}$ respectively. Starting at $n = 0$, the list of terms $b_0, b_1, b_2, b_3, b_4, \ldots$:

$$1, -1, 1, -1, 1, \ldots$$

And so on for the other sequences.

Definition 3.1.3: Arithmetic Progression

A sequence in the form

$$a, a+d, a+2d, \ldots, a+nd, \ldots$$

where the initial term a and the common difference d are real numbers.

Note:-

An arithmetic sequence is a discrete analogue of the linear function $f(x) = a + dx$

Example 3.1.2

The sequences $\{s_n\}$, with $s_n = -1 + 4n$ and $\{t_n\}$, with $t_n = 7 - 3n$ are arithmetic progressions, with initial terms of -1 and 7 and common differences of 4 and -3 respectively. The list of terms $s_0, s_1, s_2, s_3, s_4, \ldots$:

$$-1, 3, 7, 11, 15, \ldots$$

3.1.1 Recurrence Relations

Definition 3.1.4: Recurrence Relation

A recurrence relation for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, a_2, \ldots, a_{n-1}$ $\forall n \in \mathbb{Z}$ where$n \geqslant n_0$, where n_0 is a nonnegative integer.

A sequence is called a solution of a recurrence relation if its term satisfy the recurrence relation. A recurrence relation is said to recursively define a sequence.

For the previous sequences to be defined we specified explicit formulas for their terms. However there are many other ways to define a sequence. One way is to specify a sequence based on one or more initial terms and a rule for determining subsequent terms from those that precede them. This is called a recurrence relation.

Example 3.1.3

Question 26

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots$ and suppose $a_0 = 2$. What are a_1, a_2 and a_3

Solution:

$$a_n = a_{n-1} + 3$$

$$\begin{aligned} a_1 &= a_0 + 3 \\ &= 2 + 3 \\ &= 5 \end{aligned}$$

$$\begin{aligned} a_2 &= a_1 + 3 \\ &= 8 \end{aligned}$$

$$\begin{aligned} a_3 &= a_2 + 3 \\ &= 11 \end{aligned}$$

Example 3.1.4

Question 27

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$ and suppose $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3

Solution:

$$a_n = a_{n-1} - a_{n-2}$$

$$\begin{aligned} a_2 &= a_1 - a_0 \\ &= 5 - 3 \\ &= 2 \end{aligned}$$

$$\begin{aligned} a_3 &= a_2 - a_1 \\ &= 2 - 5 \\ &= -3 \end{aligned}$$

The initial conditions for a recursively defined sequence specify the terms that precede the first term where the recurrence relation takes effect. The Fibonacci Sequence is a famous example of a recursively defined sequence and has many applications in computer science and mathematics.

Definition 3.1.5: Fibonacci Sequence

The Fibonacci sequence $f_0, f_1, f_2, \ldots$ is defined by the intimal conditions, $f_0 = 0$, $f_1 = 1$, and the recurrence relation

$$f_n = f_{n-1} + f_{n-2} \text{ for } n = 2, 3, 4, \ldots$$

Chapter 4

Exercises

4.1 Sets

Question 28

Determine whether each of these statements is true or false.

1. $0 \in \emptyset$
2. $\emptyset \in \{0\}$
3. $\{0\} \subset \emptyset$
4. $\emptyset \subset \{0\}$
5. $\{0\} \in \{0\}$
6. $\{0\} \subset \{0\}$
7. $\{\emptyset\} \subseteq \{\emptyset\}$

Solution:

1. False, as the empty set contains no elements.
2. False, as the empty set is not an element of the set $\{0\}$.
3. False, as the set $\{0\}$ cannot be a subset of $\emptyset$ as the definition of a subset $\forall x\,(x \in \{0\} \longrightarrow x \in \emptyset)$ fails for 0 as 0 is not in the empty set.
4. True, as $\emptyset$ is a subset of any set and $\emptyset \neq \{0\}$
5. False, as the element $\{0\}$ is not found in the set $\{0\}$
6. False, as although any set is a subset of itself, for a set to be a proper subset of another set, the sets cannot be equal but in this case $\{0\} = \{0\}$
7. True, as any set is a subset of itself.

Question 29

Let A and B be sets. Show that

1. $(A \cap B) \subseteq A$
2. $A \subseteq (A \cup B)$
3. $A - B \subseteq A$

Solution:

1. $(A \cap B) \subseteq A$ means $\forall x\,(x \in (A \cap B) \rightarrow x \in A)$.
   Assume $x \in (A \cap B)$ is T.

	Steps	Reasons
1	$x \in (A \cap B)$	Premise
2	$x \in A \wedge x \in B$	By Definition of Intersection
3	$x \in A$	By Simplification on 2

   $\therefore$ $x \in (A \cap B) \rightarrow x \in A$ is T.
   Hence $(A \cap B) \subseteq A$

2. $A \subseteq (A \cup B)$ mean $\forall x\,(x \in A \rightarrow x \in (A \cup B))$.
   Assume $x \in A$ is T.

	Steps	Reasons
1	$x \in A$	Premise
2	$x \in A \vee x \in B$	By Addition on 1.
3	$x \in A \cup B$	By Definition of Union

   $\therefore$ $x \in A \rightarrow x \in (A \cup B)$
   Hence $A \subseteq (A \cup B)$

3. $A - B \subseteq A$ means $\forall x\,(x \in A - B \rightarrow x \in A)$
   Assume $x \in A - B$ is T.

	Steps	Reasons
1	$x \in A - B$	Premise
2	$x \in A \cap \overline{B}$	By Definition of Difference
3	$x \in A \wedge x \in \overline{B}$	By Definition of Intersection
4	$x \in A$	By Simplification on 3

   $\therefore$ $x \in A - B \rightarrow x \in A$
   Hence $A - B \subseteq A$

Question 30

Let A and B be sets. Using set builder notation show that:

1. $A \cap \emptyset = \emptyset$
2. $A \cup \overline{A} = U$
3. $A \cap (B - A) = \emptyset$

Solution:

1.

$$
\begin{aligned}
A \cap \emptyset &= \{x \mid x \in A \cap \emptyset\} && \text{Set Builder Notation}\\
&= \{x \mid x \in A \land x \in \emptyset\} && \text{By Definition of Intersection}\\
&= \{x \mid x \in A \land F\} && \text{By Definition of } \emptyset\\
&= \{x \mid F\} && \text{By Second Domination Law for propositional logic}\\
&= \emptyset && \text{By Definition of } \emptyset
\end{aligned}
$$

2.

$$
\begin{aligned}
A \cup \overline{A} &= \{x \mid x \in A \cup \overline{A}\} && \text{Set Builder Notation}\\
&= \{x \mid x \in A \lor x \in \overline{A}\} && \text{By Definition of Union}\\
&= \{x \mid x \in A \lor x \notin A\} && \text{By Definition of Complement}\\
&= \{x \mid x \in A \lor \neg(x \in A)\} && \text{By Definition of Complement}\\
&= \{x \mid T\} && \text{By First Negation Law for propositional logic}\\
&= U && \text{By Definition of Universal set}
\end{aligned}
$$

3.

$$
\begin{aligned}
A \cap (B - A) &= \{x \mid x \in A \cap (B - A)\} && \text{Set Builder Notation}\\
&= \{x \mid x \in A \land x \in (B - A)\} && \text{By Definition of Intersection}\\
&= \{x \mid x \in A \land x \in B \cap \overline{A}\} && \text{By Definition of Difference}\\
&= \{x \mid x \in A \land x \in B \land x \in \overline{A}\} && \text{By Definition of Intersection}\\
&= \{x \mid x \in A \land x \in B \land x \notin A\} && \text{By Definition of Complement}\\
&= \{x \mid x \in A \land x \notin A \land x \in B\} && \text{By Second Commutative Law for propositional logic}\\
&= \{x \mid x \in A \land \neg(x \in A) \land x \in B\} && \text{By Definition of } \notin\\
&= \{x \mid F \land x \in B\} && \text{By Second Negation Law for propositional logic}\\
&= \{x \mid F\} && \text{By Second Domination Law for propositional logic}\\
&= \emptyset && \text{By Definition of Empty set}
\end{aligned}
$$