Limits and Derivatives

Madiba Hudson-Quansah

March 2023

Contents

Chapter 1

Limits

1.1 Cases

1.1.1 Case 1

Changing input values, output values remain the same $\therefore$ The change in y with respect to changes x values is 0. This function is called a constant function

$$f(x) = k, \text{ for all input } x$$

1.1.2 Case 2

Changing input values result in changing output values/ different output values.

$$f(x) = \text{ not a constant function}$$

1.1.3 Case 3

Changing input values result in a constant change in output values.

$$\Delta f(x) = \text{ constant}$$

This function is called a linear function.

1.1.4 Case 4

Changing input values result in a non constant change in output values

$$\Delta f(x) = \text{ not constant}$$

1.1.5 Questions

Question 1

Find the derivative of $f(x) = x^2$ at $x = 1$

Solution:

$$
\begin{aligned}
\frac{d}{dx}f(x)_{x \to 1^-} &= 2 \\
\frac{d}{dx}f(x)_{x \to 1^+} &= 2 \\
\frac{d}{dx}x^2 &= 2
\end{aligned}
$$

Question 2

Investigate the existence or otherwise the derivatives if the following functions:

$$f(x) = \begin{cases} x^2, & \text{if } \leqslant 2 \\ 1 + 2x & \text{if } > 2 \end{cases} \text{ at } x = 2 \tag{1.1.1}$$

$$g(x) = |x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geqslant 0 \end{cases} \text{ at } x = 0 \tag{1.1.2}$$

Solution: 1.1.1

$$
\begin{aligned}
\frac{d}{dx}f(x)_{x \to 2^-} &= 4 \\
\frac{d}{dx}f(x)_{x \to 2^+} &= \text{is not constant}
\end{aligned}
$$

Solution: 1.1.2

$$
\begin{aligned}
\frac{d}{dx}g(x)_{x \to 0^-} &= 1 \\
\frac{d}{dx}g(x)_{x \to 0^+} &= 1 \\
&= \frac{d}{dx}g(x) = 1
\end{aligned}
$$

Chapter 2

Derivatives

2.1 First Principle

- Slope or Gradient of a non-linear function
- Derivative = $\frac{dy}{dx}$ = Instantaneous rate of change of a function
- Slope of a line = $\frac{\Delta y}{\Delta x}$ = Average rate of change
- Leverage average rate of change to obtain the instantaneous rate of change = **First Principle**

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- The derivative is a function at a point and a number at another point

Definition 2.1.1: Secant

A line going though two points on a curve

2.1.1 Questions

Question 3

Find the derivative of $f(x) = x^2$ using first principle

Solution:

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$\frac{(x+h)^2 - x^2}{h}$$
$$\frac{2hx + h^2}{h}$$
$$2x + (0)$$
$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 2x$$

Question 4

Find the derivative of $f(x) = \frac{1}{x}$ using first principle

Solution:

$$\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{df}{dx} = \lim_{h\to 0} (\frac{1}{(x+h)} - \frac{1}{x}) \div h$$

$$\frac{-h}{x^2 + hx} \div h$$

$$\frac{-h}{x^2 + hx} \times \frac{1}{h}$$

$$-\frac{1}{x^2 + hx}$$

$$-\frac{1}{x^2 + 0x}$$

$$\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = -\frac{1}{x^2}$$

Question 5

Find the derivative of $f(x) = \sqrt{x}$

Solution:

$$\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{df}{dx} = \lim_{h\to 0} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$\frac{df}{dx} = \lim_{h\to 0} = \frac{(\sqrt{x+h} - \sqrt{x}) - (\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$\frac{df}{dx} = \lim_{h\to 0} = \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$\frac{df}{dx} = \lim_{h\to 0} = \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$\frac{df}{dx} = \lim_{h\to 0} = \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$\frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$\frac{1}{2\sqrt{x}}$$

$$\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \frac{1}{2\sqrt{x}}$$

2.2 Techniques of Differentiation

2.2.1 Power Rule

$$y = x^n$$

Where n is a real number

$$\frac{dy}{dx} = nx^{n-1}$$

Examples

Example 2.2.1 ($y = x^2$, n = 2)

$$\begin{aligned} \frac{dy}{dx} &= 2x^{2-1} \\ &= 2x^1 \\ &= 2x \end{aligned}$$

Example 2.2.2 ($y = \frac{1}{x^2}$)

$$y = x^{-2}, \text{ n = -2}$$
$$y' = -2x$$

Example 2.2.3 ($y = \sqrt{x}$)

$$\begin{aligned} y &= x^{\frac{1}{2}} \\ y' &= \frac{x^{-\frac{1}{2}}}{2} \\ y' &= \frac{1}{2\sqrt{x}} \end{aligned}$$

Example 2.2.4 ($y = \frac{1}{\sqrt{x}} + x^3 - 1$)

$$\begin{aligned} y' &= x^{-\frac{1}{2}} + x^3 - 1 \\ y' &= -\frac{1}{2}x^{-\frac{3}{2}} + 3x^2 \end{aligned}$$

Example 2.2.5 ($y = -x^{-8} + 3x^2$)

$$y' = -8x^{-9} + 6x$$

2.3 Chain Rule

Definition 2.3.1: Chain Function

A function that is composed of two or more functions, i.e, $f[g]$

In the case where

$$y = (2x+1)^2$$

The derivative is:

$$y' = 4(2x+1)$$

Examples

Example 2.3.1 $(y = (3x^2+2x)^5)$

$$y = (5)(6x+2)(3x^2+2x)^4$$
$$y = (30x+10)(3x^2+2x)$$

$$y = 15x^9 - 3x^{12} + 5x - 46 \tag{2.3.1}$$
$$y = 2t^6 + 7t^{-6} \tag{2.3.2}$$
$$y = 8x^3 - \frac{1}{3x^5} + x - 23 \tag{2.3.3}$$
$$y = \sqrt{x} + 9\sqrt[3]{x^4} - \frac{2}{\sqrt[5]{x^2}} \tag{2.3.4}$$
$$y = \sqrt[3]{x^2}(2x - x^2) \tag{2.3.5}$$
$$y = \frac{2t^5 + t^2 - 5}{t^2} \tag{2.3.6}$$
$$y = 2x^3 + \frac{300}{x^3} + 4 \tag{2.3.7}$$

Example 2.3.2 (2.3.1)

$$y' = 185x^8 - 36x^{11} + 5$$

Example 2.3.3 (2.3.2)

$$y' = 12t^5 = 42t^{-7}$$

Example 2.3.4 (2.3.3)

$$y = 8x^3 - \frac{1}{3x^5} + x - 23$$
$$y = 8x^3 - \frac{1}{3}x^{-5} + x - 23$$
$$y' = 24x^2 + \frac{5}{3}x^{-6} + 1$$

Example 2.3.5 (2.3.4)

$$\begin{aligned}
y &= \sqrt{x} + 9\sqrt[3]{7} - \frac{2}{\sqrt[5]{x^2}} \\
y &= x^{\frac{1}{2}} + 9(x^7)^{\frac{1}{3}} - 2(x^2)^{-\frac{1}{5}} \\
y &= x^{\frac{1}{2}} + 9x^{\frac{7}{3}} - 2x^{-\frac{2}{5}} \\
y' &= \frac{1}{2}x^{-\frac{1}{2}} + 21x^{\frac{4}{3}} + \frac{4}{5}x^{-\frac{7}{5}} \\
y' &= \frac{1}{2\sqrt{x}} + 21x^{\frac{4}{3}} + \frac{4}{5}x^{-\frac{7}{5}}
\end{aligned}$$

Example 2.3.6 (2.3.5)

$$\begin{aligned}
y &= (x^2)^{\frac{1}{3}}(2x - x^2) \\
y &= x^{\frac{2}{3}}(2x - x^2) \\
y &= 2x^{\frac{5}{3}} - x^{\frac{2}{3}} \\
y' &= \frac{10}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{\frac{5}{3}}
\end{aligned}$$

Example 2.3.7 (2.3.6)

$$\begin{aligned}
y &= 2t^3 + 1 - \frac{5}{t^2} \\
y &= 2t^3 + 1 - 5t^{-2} \\
y' &= 6t^2 + 10t^{-3}
\end{aligned}$$

Example 2.3.8 (2.3.7)

$$\begin{aligned}
y &= 2x^3 + 300x^{-3} + 4 \\
y' &= 6x^2 - 900x^{-4}
\end{aligned}$$

2.4 Product Rule

Given $y = u \times v$, then the derivative is given by:

$$\frac{dy}{dx} = u' \times v + v' \times u$$

Examples

Example 2.4.1 $(y = (x^2+1)(x^3-x))$

$$\begin{aligned} y' &= (2x)(x^3-x) + (x^2+1)(3x^2-1) \\ y' &= 2x^4 - 2x^2 + 3x^4 + 2x^2 - 1 \\ y' &= 5x^4 - 1 \end{aligned}$$

Example 2.4.2 $(y = (6x^3-x)(10-20x))$

$$\begin{aligned} y' &= (18x^2-1)(10-20x) + (-20)(6x^3-x) \\ y' &= 180x^2 - 10 - 360x^3 + 20x - 120x^3 + 20x \\ y' &= -480x^3 + 180x^2 + 40x - 10 \end{aligned}$$

Questions

Question 6

$y = (4t^2 - t)(t^3 - 8t^2 + 12)$

Solution:

$$\begin{aligned} y' &= (8t-1)(t^3-8t^2+12) + (3t^2-16t)(4t^2-t) \\ y' &= 8t^4 - 64t^3 + 96t - t^3 + 8t^2 - 12 + 12t^4 - 3t^3 - 64t^3 + 16t^2 \\ y' &= 20t^4 - 132t^3 + 24t^2 + 96t - 12 \end{aligned}$$

Question 7

$y = (1 + \sqrt{x^3})(x^{-3} - 2\sqrt[3]{x})$

Solution:

$$\begin{aligned} y &= (1 + (x^3)^{\frac{1}{2}})(x^{-3} - 2\sqrt[3]{x}) \\ y &= (1 + x^{\frac{1}{2}})(x^{-3} - 2(x^{\frac{1}{3}})) \\ y' &= (\frac{3}{2}x^{\frac{1}{2}})(x^{-3} - 2x^{\frac{1}{3}}) + (-3x^4 - \frac{2}{3}x^{-\frac{2}{3}} - \frac{2}{3}x^{\frac{5}{6}}) \\ y' &= \frac{3}{2}x^{-\frac{5}{2}} - 3x^{\frac{5}{6}} - 3x^{-4} - 3x^{-\frac{5}{2}} - \frac{2}{3}x^{-\frac{2}{3}} - \frac{2}{3}x^{\frac{5}{6}} \\ y' &= -\frac{11}{3}x^{\frac{5}{6}} - \frac{3}{2}x^{-\frac{5}{2}} - \frac{2}{3}x^{-\frac{2}{3}} - 3x^{-4} \end{aligned}$$

Question 8

$y = (4 - t^2)(1 + 5t^2)$

Solution:

$$y' = (-2t)(1 + 5t^2) + (10t)(4 - t^2)$$
$$y' = -2t - 10t^3 + 40t - 10t^3$$
$$y' = 20t^3 + 38t$$

Question 9

$y = (x - \frac{2}{x^2})(7 - 2x^3)$

Solution:

$$y = (x - 2x^{-2})(7 - 2x^3)$$
$$y' = (1 + 4x^{-3})(7 - 2x^3) + (-6x^2)(x - 2x^{-2})$$
$$y' = 7 - 2x^3 + 28x^{-3} - 8x^0 - 6x^3 + 12x^0$$
$$y' = 7 - 2x^3 + 28x^{-3} - 8 - 6x^3 + 12$$
$$y' = -8x^3 + \frac{28}{x^3} + 11$$

Question 10

$y = (3 - x)(1 - 2x + x^2)$

Solution:

$$y' = (-1)(1 - 2x + x^2) + (2x - 2)(3 - x)$$
$$y' = -1 + 2x - x^2 + 6x - 2x^2 - 6 + 2x$$
$$y' = -7 + 10x - 3x^2$$

2.5 Quotient Rule

Where $y = \frac{f(x)}{g(x)}$, the derivative is given by:

$$\frac{dy}{dx} = \frac{g(x) \times f'(x) - f(x) \times g'(x)}{(g(x))^2}$$

Or where $\frac{u}{v}$

$$y' = \frac{v \times u' - u \times v'}{v^2}$$