Matrix Algebra

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Contents

Chapter 1

Matrix Operations

If A is a $n \times m$ matrix then the scalar entry in the ith row and the jth column of A is denoted by a_{ij}, and is called the (i, j)-entry. Each column of A is a list of m real numbers in the $\mathbb{R}^m$ vector space. Therefore the columns of A can be represented as vectors in $\mathbb{R}^m$:

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots \mathbf{a}_n \end{bmatrix}$$

Definition 1.0.1: Diagonals

The diagonal entries of a matrix A of dimension $n \times m$, are the entries a_{ij}, where $i = j$. This is called the **main diagonal** of the matrix A. A **diagonal matrix** is a square matrix $n \times n$ whose non-diagonal entries are all zero.

1.1 Sums and Scalar Multiples

Definition 1.1.1: Equality of Matrices

Two matrices A and B, are equal if:

- The are of the same size i.e, $m \times x$
- The corresponding entries are equal i.e, $A_{ij} = B_{ij}$

Theorem 1.1.1 Axioms of Matrix Addition

Let A, B and C be matrices of the same size, and let r and s be scalars. Then the following axioms hold:
Communtativity $A + B = B + A$
Associativity $(A + B) + C = A + (B + C)$
Additive Identity $A + 0 = A$
Distruibutivity 1 $r(A + B) = rA + rB$
Distruibutivity 2 $(r + s)A = rA + sA$
Compatibility with Scalar Multiplication $r(sA) = (rs)A$

1.2 Matrix Multiplication

When a matrix B multiples a vector $\mathbf{x}$, it transforms $\mathbf{x}$ into the vector $B\mathbf{x}$. If this vector is then multiples by another matrix A, the result is the vector $A(B\mathbf{x})$. Thus $A(B\mathbf{x})$ is produced by a composition of mappings / linear transformations. This can be also expressed as:

$$A(B\mathbf{x}) = (AB)\mathbf{x}$$

Because, if A is $m \times n$, B is $n \times p$ and $\mathbf{x}$ is in $\mathbb{R}^p$, can denote the columns of B, by $\mathbf{b}_1, \ldots, \mathbf{b}_p$ and the entries of $\mathbf{x}$ by, $x_1, \ldots, x_p$. Then

$$B\mathbf{x} = x_1\mathbf{b}_1 + \ldots + x_p\mathbf{b}_p$$

By the linearity of matrix multiplication, we have:

$$\begin{aligned} A(B\mathbf{x}) &= A(x_1\mathbf{b}_1) + \ldots + A(x_p\mathbf{b}_p) \\ &= x_1(A\mathbf{b}_1) + \ldots + x_p(A\mathbf{b}_p) \end{aligned}$$

The vector $A(B\mathbf{x})$ is then a linear combination of the vectors $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$, using the entries of $\mathbf{x}$ as weights. This can be expressed in matrix notation as:

$$A(B\mathbf{x}) = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \ldots & A\mathbf{b}_p \end{bmatrix} \mathbf{x}$$

Theorem 1.2.1

If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$.That is:

$$A(B\mathbf{x}) = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \ldots & A\mathbf{b}_p \end{bmatrix} \mathbf{x}$$

Example 1.2.1

Question 1

Compute AB where $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$, and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$

Solution:

$$\begin{aligned} A\mathbf{b}_1 &= \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 8+3 \\ 4+-5 \end{bmatrix} \\ &= \begin{bmatrix} 11 \\ -1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A\mathbf{b}_2 &= \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 6-6 \\ 3+10 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 13 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A\mathbf{b}_3 &= \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 21 \\ -9 \end{bmatrix} \end{aligned}$$

$$AB = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

Theorem 1.2.2 Row-Column Rule

If the product AB is defined, them the entry in row i and column j of AB is the sum of the products of corresponding entries of the row i of A and column j of B. If $(AB)_{ij}$ denotes the (i, j)-entry in AB, and if A is an $m \times n$, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

Example 1.2.2

Use the row−column rule to compute two of the entries in AB for the matrices:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$$

An inspection of the numbers involved will make it clear how the two methods for calculating AB produce the same matrix.

The dimensions of the resultant matrix is 2×3, therefore the entries of AB are:

$$\begin{aligned} AB &= \begin{bmatrix} 2(4) + 3(1) & 2(3) + 3(-2) & 2(6) + 3(3) \\ 1(4) - 5(1) & 1(3) - 5(-2) & 1(6) - 5(3) \end{bmatrix} \\ &= \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & 9 \end{bmatrix} \end{aligned}$$

Example 1.2.3

Question 2

Find the entries in the second row of AB where,

$$A = \begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix}, B = \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} -1 & 3 & -4 \end{bmatrix} \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -4 + 21 - 12 & 6 + 3 - 8 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 \end{bmatrix}$$

Theorem 1.2.3 Axioms of Matrix Multiplication

Let A be an $m \times n$ matrix and let B and C have sizes for which the indicated sums and products are defined:

Associativity $A(BC) = (AB)C$

Left Distruibutivity $A(B + C) = AB + AC$

Right Distruibutivity $(B + C)A = BA + CA$

Scalar Associativity $r(AB) = (rA)B = A(rB), \forall r, r \in \mathbb{F}$

Mutliplicative Identitiy $I_m A = A = A I_n$

Example 1.2.4

Question 3

Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$. Show that these matrices do not commute, I.e, verify $AB \neq BA$

Solution:

$$\begin{aligned} AB &= \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 2 \\ 29 & -2 \end{bmatrix} \end{aligned}$$

$$\therefore AB \neq BA$$

1.2.1 Powers of a Matrix

Definition 1.2.1: Powers of a Matrix

If A is an $n \times n$ matrix and if k is a positive integer, then A^k denotes the product of k copies of A:

$$A^k = A_1 \dots A_k$$

Where $A_1 = A_2 \wedge A_2 = A_3 \wedge \dots \wedge A_{k-1} = A_k$

If A is non-zero and if $\mathbf{x}$ is in $\mathbb{R}^n$, then $A^k\mathbf{x}$ is the result of left-multiplying $\mathbf{x}$ by A repeatedly k times.

If $k = 0$, then $A^0\mathbf{x}$ is $\mathbf{x}$. Thus A^0 is interpreted as the Identity matrix.

1.3 The Transpose of a Matrix

Definition 1.3.1: The Transpose of a Matrix

Given a matrix A, its *transpose*, denoted by A^T, is defined by transforming the rows of A into columns. For example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Therefore formally, the transpose of a matrix $A_{m,n}$ is defined as:

$$A^T_{m,n} = A_{n,m}$$

Therefore, let A and B denote matrices whose sizes are appropriate for the following sums and products:

1. $\left(A^T\right)^T = A$
2. $(A + B)^T = A^T + B^T$

3. $\forall r \in \mathbb{F}, (rA)^T = rA^T$
4. $(AB)^T = B^T A^T$

Usually $(AB)^T$ is not equal $A^T B^T$, even when A and B have dimensions such that $A^T B^T$ is defined. The generalization of axiom 4 to products more than two factors is as follows:

> **Theorem 1.3.1**
> The transpose of a product of matrices equals the product of their transpose in the reverse order.

Chapter 2

The Inverse Of A Matrix

2.1 Invertibility

Definition 2.1.1: Invertibility

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible. Where $ad - bc$ is known as the *determinant* and denoted by

$$\det A = ad - bc$$

Theorem 2.1.1

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution:

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Theorem 2.1.2

1. If A is an invertible matrix, then A^{-1} is invertible and

$$\left(A^{-1}\right)^{-1} = A$$

2. If A and B are $n \times n$ invertible matrices, then so AB, and the inverse of AB is the product of the inverses of A and B in the reverse order:

$$(AB)^{-1} = B^{-1}A^{-1}$$

3. If A is an invertible matrix, then so is A^T and the inverse of A^T is the transpose of A^{-1}:

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

2.2 Elementary Matrices

Definition 2.2.1: Elementary Matrix

A matrix obtained by performing a single elementary row operation on an identity matrix.

Example 2.2.1

Question 4

Let

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}, A \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Compute E_1A, E_2A, E_3A, and describe how these products can be obtained by elementary row operations on A.
Solution:

$$\begin{aligned} E_1A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \\ &= \begin{bmatrix} a & b & c \\ d & e & f \\ -4a+g & -4b+h & -4c+1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} E_2A &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \\ &= \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix} \end{aligned}$$

$$\begin{aligned} E_3A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \\ &= \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix} \end{aligned}$$

- E_1A could be obtained by the elementary row operation $-4R_1 + R_3 \longrightarrow R_3$
- E_2A could be obtained by the elementary row operation $R_1 \leftrightarrow R_2$
- E_3A could be obtained by the elementary row operation $5R_3 \longrightarrow R_3$

Corollary 2.2.1

If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be expressed as EA, where E is the $m \times m$ matrix created by performing the same row operation on I_m

Since row operations are reversible, all elementary matrices are invertible. Therefore there exists an elementary matrix F such that

$$FE = I$$

And since E and F correspond to reverse operations $EF = I$, also.

Example 2.2.2

Question 5

Find the inverse of $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

Solution: To transform this matrix into I_3 we must get rid of the -4 entry in the third row. This can be done by the row operation $4R_1 + R_3 \longrightarrow R_3$, which corresponds to the elementary matrix:

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

Checking our answer:

$$\begin{aligned} E_1 E_1^{-1} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

This is indeed the identity matrix I_m

Theorem 2.2.1

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}

2.2.1 Finding A^{-1}

To find the inverse of a matrix A, we can augment A with the $n \times n$ identity matrix I_n and then row reduce. If A is row equivalent to I_n then $\begin{bmatrix} A & I \end{bmatrix}$ is row equivalent to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$. Otherwise, A does not have an inverse.

Example 2.2.3

Question 6

Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

Solution:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} 4 & -3 & 8 & 0 & 0 & 1 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{4}R_1 - R_2 \to R_2$$

$$\begin{bmatrix} 4 & -3 & 8 & 0 & 0 & 1 \\ 0 & \frac{-3}{4} & -1 & 0 & -1 & \frac{1}{4} \\ 0 & 1 & 2 & 1 & 0 & 0 \end{bmatrix}$$

$$\frac{-4}{3}R_2 - R_3 \to R_3$$

$$\begin{bmatrix} 4 & -3 & 8 & 0 & 0 & 1 \\ 0 & \frac{-3}{4} & -1 & 0 & -1 & \frac{1}{4} \\ 0 & 0 & \frac{-2}{3} & -1 & \frac{4}{3} & \frac{-1}{3} \end{bmatrix}$$

$$4R_2 - R_1 \to R_1$$

$$\begin{bmatrix} -4 & 0 & -12 & 0 & -4 & 0 \\ 0 & \frac{-3}{4} & -1 & 0 & -1 & \frac{1}{4} \\ 0 & 0 & \frac{-2}{3} & -1 & \frac{4}{3} & \frac{-1}{3} \end{bmatrix}$$

$$\frac{3}{2}R_3 - R_2 \to R_2$$

$$\begin{bmatrix} -4 & 0 & -12 & 0 & -4 & 0 \\ 0 & \frac{3}{4} & 0 & \frac{-3}{2} & 3 & \frac{-3}{4} \\ 0 & 0 & \frac{-2}{3} & -1 & \frac{4}{3} & \frac{-1}{3} \end{bmatrix}$$

$$18R_3 - R_1 \to R_1$$

$$\begin{bmatrix} 4 & 0 & 0 & -18 & 28 & \frac{-6}{1} \\ 0 & \frac{3}{4} & 0 & \frac{-3}{2} & 3 & \frac{-3}{4} \\ 0 & 0 & \frac{-2}{3} & -1 & \frac{4}{3} & \frac{-1}{3} \end{bmatrix}$$

$$-R_1 \to R_1$$

$$\begin{bmatrix} -4 & 0 & 0 & 18 & -28 & \frac{6}{1} \\ 0 & \frac{3}{4} & 0 & \frac{-3}{2} & 3 & \frac{-3}{4} \\ 0 & 0 & \frac{-2}{3} & -1 & \frac{4}{3} & \frac{-1}{3} \end{bmatrix}$$

$$\frac{-1}{4}R_1 \to R_1$$

$$\frac{4}{3}R_2 \to R_2$$

$$\frac{-3}{2}R_3 \to R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{-9}{2} & 7 & \frac{-3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$$

Since $A \sim I$, A is invertible and

$$A^{-1} = \begin{bmatrix} \frac{-9}{2} & 7 & \frac{-3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$$

Checking our answer:

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} \frac{-9}{2} & 7 & \frac{-3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Chapter 3

Determinants

3.1 Introduction

To extend the concept of the determinant to $n \times n$ matrices we must use this recursive definition:

Definition 3.1.1: The Determinant of a $n \times n$ matrix

For $n \geqslant 2$, the determinant of an $n \times n$ matrix $A = \left[a_{ij}\right]$ is the sum of terms of the form $\pm a_{1j}$ $\det A_{1j}$, with plus and minus signs alternating, where the entries of $a_{11}, a_{12}, \ldots, a_{1n}$ are form the first row of A, i.e.:

$$\begin{aligned}\det A &= a_{11}\det A_{11} - a_{12}\det A_{12} + \ldots + (-1)^{1+n} a_{1n}\det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j}\, a_{1j} \det A_{1j}\end{aligned}$$

Where A_{1j} refers to the matrix obtained by crossing out the first row and the jth column of A, which if A is a 3×3 matrix would result in a 2×2 one allowing us to find the determinant of A_{1j} using 2.1

Example 3.1.1

Question 7

Compute the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Solution:

$$\begin{aligned}\det A &= \sum_{j=1}^{n} (-1)^{1+j}\, a_{1j} \det A_{1j} \\ &= a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} \\ &= 1 \begin{vmatrix} 4 & 1 \\ 2 & 0 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} \\ &= 1\,(0-2) - 5\,(0) + 0\,(-4) \\ &= -2\end{aligned}$$

The definition of $\det A$ can also be written in the form of a *cofactor expansion*, Given $A = \left[a_{ij}\right]$, the $\left(i, j\right)$ -cofactor of A is the number C_{ij} defined by:

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Allowing us to express $\det A$ as:

$$\begin{aligned} \det A &= \sum_{j=1}^{n} a_{1j} C_{1j} \\ &= a_{11} C_{11} + \ldots + a_{1n} C_{1n} \end{aligned}$$

This is termed as the *cofactor expansion of the determinant along the first row* of A.

Theorem 3.1.1 Cofactor Expansion

The determinant of any $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row is:

$$\begin{aligned} \det A &= \sum_{j=1}^{n} a_{ij} C_{ij} \\ &= a_{i1} C_{i1} + a_{i2} C_{i2} + \ldots + a_{in} C_{in} \end{aligned}$$

And the expansion down the jth column is:

$$\begin{aligned} \det A &= \sum_{i=1}^{n} a_{ij} C_{ij} \\ &= a_{1j} C_{1j} + a_{2j} C_{2j} + \ldots + a_{nj} C_{nj} \end{aligned}$$

Example 3.1.2

Question 8

Use a cofactor expansion across the third row to compute the determinant of A, where

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Solution:

$$\begin{aligned} \det A &= \sum_{j=1}^{n} a_{3j} C_{3j} \\ &= 0 \begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0 \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} \\ &= 0 + 2\left(-1\right) + 0 \\ &= -2 \end{aligned}$$

In the case where we are computing the determinant of a matrix with great dimension, we take the cofactor across the row or column with the most zeros.

Example 3.1.3

Question 9

Compute $\det A$, where

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

Solution: We take the cofactor expansion down the first column of A.

$$\begin{aligned}
\det A &= \sum_{i=1}^{n} a_{i1}C_{i3} \\
&= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + a_{41}C_{41} + a_{51}C_{51} \\
&= 3\begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} + 0C_{21} + 0C_{31} + 0C_{41} + 0C_{51} \\
&\text{We disregard the zero terms} \\
&= 3\begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} \\
&\text{Next we perform a cofactor expansion down the 1st column of our determinant} \\
&= 3\left(\sum_{i=1}^{n} a_{i1}C_{i1}\right) \\
&= 3\left(a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + a_{41}C_{41}\right) \\
&= 3\left(2\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} - 0C_{21} + 0C_{31} - 0C_{41}\right) \\
&= 3 \times 2\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} \\
&= 3 \times 2\left(\sum_{j=1}^{n} a_{3j}C_{3j}\right) \\
&= 3 \times 2\left(a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}\right) \\
&= 3 \times 2\left(0C_{31} + 2\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0C_{33}\right) \\
&= 3 \times 2 \times 2\,(-1) \\
&= -12
\end{aligned}$$

Theorem 3.1.2
If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

3.1.1 Exercises

Question 10

Compute

$$\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$$

Solution:

$$\begin{aligned}
\det A &= \sum_{j=1}^{n} a_{4j} C_{4j} \\
&= a_{41}C_{41} + a_{42}C_{42} + a_{43}C_{43} + a_{44}C_{44} \\
&= 0C_{41} - 5 \begin{vmatrix} 5 & 2 & 2 \\ 0 & 0 & -4 \\ -5 & 0 & 3 \end{vmatrix} + 0C_{43} + 6 \begin{vmatrix} 5 & -7 & 2 \\ 0 & 3 & 0 \\ -5 & -8 & 0 \end{vmatrix} \\
&= 5 \left(\sum_{j=1}^{n} a_{2j} C_{2j} \right) + 6 \left(\sum_{j=1}^{n} a_{2j} C_{2j} \right) \\
&= 5 \left(0C_{21} - 0C_{22} - 4 \begin{vmatrix} 5 & 2 \\ -5 & 0 \end{vmatrix} \right) + 6 \left(\sum_{j=1}^{n} a_{2j} C_{2j} \right) \\
&= 5\,(0 + 40) + 6 \left(0C_{21} - 3 \begin{vmatrix} 5 & 2 \\ -5 & 0 \end{vmatrix} + 0C_{23} \right) \\
&= 200 + 6\,(-3 \times 10) \\
&= 200 - 180 \\
&= 20
\end{aligned}$$

3.2 Properties of Determinants

Theorem 3.2.1 Row Operations
Let A be a square matrix, Then:

1. If a multiple of one row A is added to another row to produce a matrix B, then $\det B = \det A$
2. If two rows of A are interchanged to produce B, then $\det B = -\det A$
3. If one row of A is multiple by k to produce B, then $\det B = k \cdot \det A$

Example 3.2.1

Question 11

Compute $\det A$, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

Solution: We can reduce the matrix A to reduced row echelon form then use the fact that the determinant of a triangular matrix is the product of main diagonal entries.

$$\begin{aligned}
\det A &= \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix} \\
&= \begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} \\
&= 1 \times 3 \times -5 \\
&= -15
\end{aligned}$$

Chapter 4

Exercises

Question 12

Compute the product AB using:

- The definition where Ab_1, Ab_2 are computed separately.
- The row-column rule.

$$A = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix}, B \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$$

Solution:

1.

$$\begin{aligned} Ab_1 &= \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} -3-4 \\ 15-8 \\ 6+6 \end{bmatrix} \\ &= \begin{bmatrix} -7 \\ 7 \\ 12 \end{bmatrix} \\ Ab_2 &= \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ -6 \\ -7 \end{bmatrix} \end{aligned}$$

$$AB = \begin{bmatrix} -7 & 4 \\ 7 & -6 \\ 12 & -7 \end{bmatrix}$$

2.

$$
\begin{aligned}
AB &= \begin{bmatrix} -1\times 3 + 2\times -2 & -1\times -2 + 2\times 1 \\ 5\times 3 + 4\times -2 & 5\times -2 + 4\times 1 \\ 2\times 3 + -3\times -2 & -2\times 2 + -3\times 1 \end{bmatrix} \\
&= \begin{bmatrix} -7 & 4 \\ 7 & -6 \\ 12 & -7 \end{bmatrix}
\end{aligned}
$$

Question 13

Suppose the last column of AB is entirely zero but B itself has no column of zeros. What can you say about the columns of A?

Solution: If the last column of AB is entirely zero, then the last column of A must be a linear combination of the columns of B. Therefore the columns of A are linearly dependent.

Question 14

Find the inverses of the following matrices:

1.
$$\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$$

2.
$$\begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}$$

Solution:

1.

$$
\begin{aligned}
\det(A) &= 32 - 30 \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
A^{-1} &= \frac{1}{2}\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}
\end{aligned}
$$

2.

$$
\begin{aligned}
\det(A) &= -24 + 28 \\
&= 4 \\
A^{-1} &= \frac{1}{4}\begin{bmatrix} -8 & 4 \\ -7 & 3 \end{bmatrix} \\
&= \begin{bmatrix} -4 & 1 \\ -\frac{7}{4} & \frac{3}{4} \end{bmatrix}
\end{aligned}
$$

Question 15

Use the inverse found in 6 1 to solve the system:

$$\begin{aligned} 8x_1 + 6x_2 &= 2 \\ 5x_1 + 4x_2 &= -1 \end{aligned}$$

Solution:

$$\begin{aligned} \mathbf{b} &= \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 7 \\ -9 \end{bmatrix} \end{aligned}$$

Question 16

Find the inverse of the following matrix if it exists:

$$\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$$

$$4R_1 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ -2 & 6 & -4 \end{bmatrix}$$

$$-2R_1 - R_3 \longrightarrow R_3$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix}$$

$$2R_2 - R_3 \longrightarrow R_3$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 1 \times -1 \times 0 \\ &= 0 \end{aligned}$$

$\therefore$ the matrix does not have an inverse

Question 17

Suppose the system below is consistent for all possible values of f and g. What can you say about the coefficients c and d? Justify your answer.

$$\begin{aligned} x_1 + 3x_2 &= f \\ cx_1 + dx_2 &= g \end{aligned}$$

Solution:

$$\begin{bmatrix} 1 & 3 & f \\ c & d & g \end{bmatrix}$$

$$cR_1 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 1 & 3 & f \\ 0 & 3c-d & cf-g \end{bmatrix}$$

Question 18

Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is in Span$\{u, v\}$ for all h and k.

Solution:

$$x_1\mathbf{u} + x_2\mathbf{v} = \begin{bmatrix} h \\ k \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 2 & 2 & h \\ -1 & 1 & k \end{bmatrix}$$

$$-\frac{1}{2}R_1 - R_2 \longrightarrow R_2$$

Question 19

A steam plant burns two types of coal: anthracite (A) and bituminous (B). For each ton of A burned, the plant produces 27.6 million Btu of heat, 3100 grams (g) of sulfur dioxide, and 250 g of particulate matter (solid-particle pollutants). For each ton of B burned, the plant produces 30.2 million Btu, 6400 g of sulfur dioxide, and 360 g of particulate matter.

1. How much heat does the steam plant produce when it burns x_1 tons of A and x_2 tons of B?
2. Suppose the output of the steam plant is described by a vector that lists the amounts of heat, sulfur dioxide, and particulate matter. Express this output as a linear combination of two vectors, assuming that the plant burns x_1 tons of A and x_2 tons of B.
3. Over a certain time period, the steam plant produced 162 million Btu of heat, 23,610 g of sulfur dioxide, and 1623 g of particulate matter. Determine how many tons of each type of coal the steam plant must have burned. Include a vector equation as part of your solution.

Solution:

$$\begin{aligned} 27.6x_1 + 30.2x_2 &= \text{Heat} \\ 3100x_1 + 6400x_2 &= \text{Sulfur Dioxide} \\ 250x_1 + 360x_2 &= \text{Particulate Matter} \end{aligned}$$

1.

$$27.6x_1 + 30.2x_2$$

2.

$$\begin{aligned} 27.6x_1 + 30.2x_2 &= H \\ 3100x_1 + 6400x_2 &= SO_2 \\ 250x_1 + 360x_2 &= P \end{aligned}$$

$$\mathbf{u}x_1 + \mathbf{v}x_2 = \begin{bmatrix} H \\ SO_2 \\ P \end{bmatrix}$$

Where $\mathbf{u} = \begin{bmatrix} 27.6 \\ 3100 \\ 250 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 30.2 \\ 6400 \\ 360 \end{bmatrix}$

3.

$$27.6x_1 + 30.2x_2 = 162$$
$$3100x_1 + 6400x_2 = 23610$$
$$250x_1 + 360x_2 = 1623$$

$$\begin{bmatrix} 27.6 & 30.2 & 162 \\ 3100 & 6400 & 23610 \\ 250 & 360 & 1623 \end{bmatrix}$$
$$\frac{7750}{69}R_1 - R_2 \to R_2$$
$$\begin{bmatrix} \frac{138}{5} & \frac{151}{5} & 162 \\ 0 & \frac{-207550}{69} & \frac{-124530}{23} \\ 250 & 360 & 1623 \end{bmatrix}$$
$$\frac{625}{69}R_1 - R_3 \to R_3$$
$$\begin{bmatrix} \frac{138}{5} & \frac{151}{5} & 162 \\ 0 & \frac{-207550}{69} & \frac{-124530}{23} \\ 0 & \frac{-5965}{69} & \frac{-3579}{23} \end{bmatrix}$$
$$\frac{112}{3897}R_2 - R_3 \to R_3$$
$$\begin{bmatrix} \frac{138}{5} & \frac{151}{5} & 162 \\ 0 & \frac{-207550}{69} & \frac{-124530}{23} \\ 0 & 0 & \frac{0}{1} \end{bmatrix}$$
$$0R_1 - R_3 \to R_3$$
$$\begin{bmatrix} \frac{138}{5} & \frac{151}{5} & 162 \\ 0 & \frac{-207550}{69} & \frac{-124530}{23} \\ 0 & 0 & \frac{0}{1} \end{bmatrix}$$
$$0R_2 - R_3 \to R_3$$
$$\begin{bmatrix} \frac{138}{5} & \frac{151}{5} & 162 \\ 0 & \frac{-207550}{69} & \frac{-124530}{23} \\ 0 & 0 & \frac{0}{1} \end{bmatrix}$$
$$\frac{-98}{9761}R_2 - R_1 \to R_1$$
$$\begin{bmatrix} \frac{-138}{5} & 0 & \frac{-2691}{25} \\ 0 & \frac{-207550}{69} & \frac{-124530}{23} \\ 0 & 0 & \frac{0}{1} \end{bmatrix}$$
$$0R_2 - R_1 \to R_1$$
$$\begin{bmatrix} \frac{138}{5} & 0 & \frac{2691}{25} \\ 0 & \frac{-207550}{69} & \frac{-124530}{23} \\ 0 & 0 & \frac{0}{1} \end{bmatrix}$$
$$0R_2 - R_1 \to R_1$$
$$\frac{5}{138}R_1 \to R_1$$
$$\frac{-1}{3008}R_2 \to R_2$$
$$\begin{bmatrix} 1 & 0 & \frac{39}{10} \\ 0 & 1 & \frac{9}{5} \\ 0 & 0 & \frac{0}{1} \end{bmatrix}$$
$$\mathbf{x} = \begin{bmatrix} \frac{39}{10} \\ \frac{9}{5} \end{bmatrix}$$

Question 20

Describe and compare the solution sets of $x_1 - 3x_2 + 5x_3 = 0$ and $x_1 - 3x_2 + 5x_3 = 4$.

Solution:

$$\begin{bmatrix} 1 & -3 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - 3x_2 + 5x_3 &= 0 \\ x_2 &= x_2 \\ x_3 &= x_3 \\ x_1 &= 3x_2 - 5x_3 \end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} x_3$$