

Limits

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Chapter 1

1.1 The Limit of a Function

Reporting about the behaviour of function within the range of its dangerous values

$$f(x) = x^2 + \frac{1}{x}$$

Input variable = x

Output variable = $f(x)$

Name of function = f

”Acceptable”/Permissible input values of x - All real numbers except zero

$$\begin{aligned} x, f(x) \quad x+h, f(x+h) \\ f(x+h) - \frac{f(x)}{x+h-x} \\ f(x+h) - \frac{f(x)}{h} \end{aligned}$$

1.1.1 Proof

$$y = -16t^2 + 100t + 6$$

Points used: $(0, 6)$ $(1, 90)$ $(3, 162)$

When $t = 0$ and $y = 6$

$$\begin{aligned} y &= at^2 + bt + c \\ 6 &= a(0)^2 + b(0) + c \\ c &= 6 \end{aligned}$$

When $t = 1$ and $y = 90$

$$\begin{aligned} 90 &= a(1)^2 + b + 6 \\ 90 &= a + b + 6 \\ 84 &= a + b \\ 84 - b &= a \end{aligned}$$

When $t = 3$ and $y = 162$

$$\begin{aligned}
 162 &= a(3)^2 + b + 6 \\
 162 &= 9a + 3b + 6 \\
 162 &= 9(84 - b) + 3b + 6 \\
 162 &= 756 - 9b + 3b + 6 \\
 -594 &= -6b + 6 \\
 -600 &= -6b \\
 b &= 100
 \end{aligned}$$

$$\therefore b = 100$$

$$\begin{aligned}
 84 - 100 &= a \\
 a &= -16
 \end{aligned}$$

Therefore $a = 16$, $b = 100$ and $c = 6$

Question 1

Given $f(x) = x^2$ find the Limit of $f(x)$ at $x = 3$

Solution:

$$\begin{aligned}
 \text{As } x \rightarrow 3^-, f(x) &\rightarrow 9 \\
 \text{As } x \rightarrow 3^+, f(x) &\rightarrow 9
 \end{aligned}$$

Or

$$\begin{aligned}
 \lim_{x \rightarrow 3^-} f(x) &= 9 \\
 \lim_{x \rightarrow 3^+} f(x) &= 9
 \end{aligned}$$

The first 9 is known as the left limit of $f(x)$ and the other 9 is known as the right limit of $f(x)$

Therefore the limit of $f(x)$ at $x = 3$ is:

$$\lim_{x \rightarrow 3} f(x) = 9$$

This is because the left limit and right limit converge.

In the case where:

$$\begin{aligned}
 \lim_{x \rightarrow 1^-} f(x) &= 5 \\
 \lim_{x \rightarrow 1^+} f(x) &= 4
 \end{aligned}$$

The left and right limits do not converge so there is no limit of $f(x)$ for $x = 1$ and is written as:

$$\lim_{x \rightarrow 1} f(x) = \text{No such unique number.}$$

In the case where one limit does not exist, i.e. Increasing without bounds:

$$\begin{aligned}\lim_{x \rightarrow 1^-} f(x) &\rightarrow \infty \\ \lim_{x \rightarrow 1^+} f(x) &\rightarrow 4\end{aligned}$$

The limit does not exist because the left limit does not exist. This is written as:

$$\lim_{x \rightarrow 1} f(x) = \text{does not exist}$$

1.2 L'hôpital's Rule

Theorem 1.2.1 L'hôpital's Rule

If

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \implies \frac{0}{0} \quad \text{or} \quad \frac{\pm\infty}{\pm\infty}$$

Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}, \text{ provided } \frac{f'(x)}{g'(x)} \neq 0$$

If

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \frac{0}{0} \quad \text{or} \quad \frac{\pm\infty}{\pm\infty}$$

Then

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow a} \frac{f''(x)}{g''(x)}, \text{ provided } \frac{f''(x)}{g''(x)} \neq 0$$