Matrix Algebra

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Contents

Chapter 1

Matrix Operations

If A is a $n \times m$ matrix then the scalar entry in the ith row and the jth column of A is denoted by a_{ij}, and is called the (i, j)-entry. Each column of A is a list of m real numbers in the $\mathbb{R}^m$ vector space. Therefore the columns of A can be represented as vectors in $\mathbb{R}^m$:

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots \mathbf{a}_n \end{bmatrix}$$

Definition 1.0.1: Diagonals

The diagonal entries of a matrix A of dimension $n \times m$, are the entries a_{ij}, where $i = j$. This is called the **main diagonal** of the matrix A. A **diagonal matrix** is a square matrix $n \times n$ whose non-diagonal entries are all zero.

1.1 Sums and Scalar Multiples

Definition 1.1.1: Equality of Matrices

Two matrices A and B, are equal if:

- The are of the same size i.e, $m \times x$
- The corresponding entries are equal i.e, $A_{ij} = B_{ij}$

Theorem 1.1.1 Axioms of Matrix Addition

Let A, B and C be matrices of the same size, and let r and s be scalars. Then the following axioms hold:
Communtativity $A + B = B + A$
Associativity $(A + B) + C = A + (B + C)$
Additive Identity $A + 0 = A$
Distruibutivity 1 $r(A + B) = rA + rB$
Distruibutivity 2 $(r + s)A = rA + sA$
Compatibility with Scalar Multiplication $r(sA) = (rs)A$

1.2 Matrix Multiplication

When a matrix B multiplies a vector $\mathbf{x}$, it transforms $\mathbf{x}$ into the vector $B\mathbf{x}$. If this vector is then multiples by another matrix A, the result is the vector $A(B\mathbf{x})$. Thus $A(B\mathbf{x})$ is produced by a composition of mappings / linear transformations. This can be also expressed as:

$$A(B\mathbf{x}) = (AB)\mathbf{x}$$

Because, if A is $m \times n$, B is $n \times p$ and $\mathbf{x}$ is in $\mathbb{R}^p$, can denote the columns of B, by $\mathbf{b}_1, \ldots, \mathbf{b}_p$ and the entries of $\mathbf{x}$ by, $x_1, \ldots, x_p$. Then

$$B\mathbf{x} = x_1\mathbf{b}_1 + \ldots + x_p\mathbf{b}_p$$

By the linearity of matrix multiplication, we have:

$$\begin{aligned} A\left(B\mathbf{x}\right) &= A\left(x_1\mathbf{b}_1\right) + \ldots + A\left(x_p\mathbf{b}_p\right) \\ &= x_1\left(A\mathbf{b}_1\right) + \ldots + x_p\left(A\mathbf{b}_p\right) \end{aligned}$$

The vector $A\left(B\mathbf{x}\right)$ is then a linear combination of the vectors $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$, using the entries of $\mathbf{x}$ as weights. This can be expressed in matrix notation as:

$$A\left(B\mathbf{x}\right) = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \ldots & A\mathbf{b}_p \end{bmatrix} \mathbf{x}$$

Theorem 1.2.1

If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$.That is:

$$A\left(B\mathbf{x}\right) = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \ldots & A\mathbf{b}_p \end{bmatrix} \mathbf{x}$$

Example 1.2.1

Question 1

Compute AB where $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$, and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$

Solution:

$$\begin{aligned} A\mathbf{b}_1 &= \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 8+3 \\ 4+-5 \end{bmatrix} \\ &= \begin{bmatrix} 11 \\ -1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A\mathbf{b}_2 &= \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 6-6 \\ 3+10 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 13 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A\mathbf{b}_3 &= \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 21 \\ -9 \end{bmatrix} \end{aligned}$$

$$AB = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

Theorem 1.2.2 Row-Column Rule

If the product AB is defined, them the entry in row i and column j of AB is the sum of the products of corresponding entries of the row i of A and column j of B. If $(AB)_{ij}$ denotes the (i, j)-entry in AB, and if A is an $m \times n$, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

Example 1.2.2

Use the row–column rule to compute two of the entries in AB for the matrices:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$$

An inspection of the numbers involved will make it clear how the two methods for calculating AB produce the same matrix.

The dimensions of the resultant matrix is 2×3, therefore the entries of AB are:

$$\begin{aligned} AB &= \begin{bmatrix} 2(4) + 3(1) & 2(3) + 3(-2) & 2(6) + 3(3) \\ 1(4) - 5(1) & 1(3) - 5(-2) & 1(6) - 5(3) \end{bmatrix} \\ &= \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & 9 \end{bmatrix} \end{aligned}$$

Example 1.2.3

Question 2

Find the entries in the second row of AB where,

$$A = \begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix}, B = \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} -1 & 3 & -4 \end{bmatrix} \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -4 + 21 - 12 & 6 + 3 - 8 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 \end{bmatrix}$$

Theorem 1.2.3 Axioms of Matrix Multiplication

Let A be an $m \times n$ matrix and let B and C have sizes for which the indicated sums and products are defined:

Associativity $A(BC) = (AB)C$

Left Distruibutivity $A(B + C) = AB + AC$

Right Distruibutivity $(B + C)A = BA + CA$

Scalar Associativity $r(AB) = (rA)B = A(rB), \forall r, r \in \mathbb{F}$

Mutliplicative Identitiy $I_m A = A = AI_n$

Example 1.2.4

Question 3

Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$. Show that these matrices do not commute, I.e, verify $AB \neq BA$

Solution:

$$\begin{aligned} AB &= \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 2 \\ 29 & -2 \end{bmatrix} \end{aligned}$$

$$\therefore AB \neq BA$$

1.2.1 Powers of a Matrix

Definition 1.2.1: Powers of a Matrix

If A is an $n \times n$ matrix and if k is a positive integer, then A^k denotes the product of k copies of A:

$$A^k = A_1 \ldots A_k$$

Where $A_1 = A_2 \wedge A_2 = A_3 \wedge \ldots \wedge A_{k-1} = A_k$

If A is non-zero and if $\mathbf{x}$ is in $\mathbb{R}^n$, then $A^k\mathbf{x}$ is the result of left-multiplying $\mathbf{x}$ by A repeatedly k times.

If $k = 0$, then $A^0\mathbf{x}$ is $\mathbf{x}$. Thus A^0 is interpreted as the Identity matrix.

1.3 The Transpose of a Matrix

Definition 1.3.1: The Transpose of a Matrix

Given a matrix A, its *transpose*, denoted by A^T, is defined by transforming the rows of A into columns. For example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Therefore formally, the transpose of a matrix $A_{m,n}$ is defined as:

$$A^T_{m,n} = A_{n,m}$$

Therefore, let A and B denote matrices whose sizes are appropriate for the following sums and products:

1. $\left(A^T\right)^T = A$
2. $(A + B)^T = A^T + B^T$

3. $\forall r \in \mathbb{F}, (rA)^T = rA^T$
4. $(AB)^T = B^T A^T$

Usually $(AB)^T$ is not equal $A^T B^T$, even when A and B have dimensions such that $A^T B^T$ is defined. The generalization of axiom 4 to products more than two factors is as follows:

> **Theorem 1.3.1**
> The transpose of a product of matrices equals the product of their transpose in the reverse order.

Chapter 2

The Inverse Of A Matrix

2.1 Invertibility

Definition 2.1.1: Invertibility

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible. Where $ad - bc$ is known as the *determinant* and denoted by

$$\det A = ad - bc$$

Theorem 2.1.1

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution:

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Theorem 2.1.2

1. If A is an invertible matrix, then A^{-1} is invertible and
$$\left(A^{-1}\right)^{-1} = A$$
2. If A and B are $n \times n$ invertible matrices, then so AB, and the inverse of AB is the product of the inverses of A and B in the reverse order:
$$(AB)^{-1} = B^{-1}A^{-1}$$
3. If A is an invertible matrix, then so is A^T and the inverse of A^T is the transpose of A^{-1}:
$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

2.2 Elementary Matrices

Definition 2.2.1: Elementary Matrix

A matrix obtained by performing a single elementary row operation on an identity matrix.

Example 2.2.1

Question 4

Let

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}, A \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Compute E_1A, E_2A, E_3A, and describe how these products can be obtained by elementary row operations on A.
Solution:

$$\begin{aligned} E_1A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \\ &= \begin{bmatrix} a & b & c \\ d & e & f \\ -4a+g & -4b+h & -4c+1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} E_2A &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \\ &= \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix} \end{aligned}$$

$$\begin{aligned} E_3A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \\ &= \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix} \end{aligned}$$

- E_1A could be obtained by the elementary row operation $-4R_1 + R_3 \longrightarrow R_3$
- E_2A could be obtained by the elementary row operation $R_1 \leftrightarrow R_2$
- E_3A could be obtained by the elementary row operation $5R_3 \longrightarrow R_3$

Corollary 2.2.1

If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be expressed as EA, where E is the $m \times m$ matrix created by performing the same row operation on I_m

Since row operations are reversible, all elementary matrices are invertible. Therefore there exists an elementary matrix F such that

$$FE = I$$

And since E and F correspond to reverse operations $EF = I$, also.

Example 2.2.2

Question 5

Find the inverse of $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

Solution: To transform this matrix into I_3 we must get rid of the -4 entry in the third row. This can be done by the row operation $4R_1 + R_3 \longrightarrow R_3$, which corresponds to the elementary matrix:

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

Checking our answer:

$$\begin{aligned} E_1E_1^{-1} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

This is indeed the identity matrix I_m

Theorem 2.2.1

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}

2.2.1 Finding A^{-1}

To find the inverse of a matrix A, we can augment A with the $n \times n$ identity matrix I_n and then row reduce. If A is row equivalent to I_n then $\begin{bmatrix} A & I \end{bmatrix}$ is row equivalent to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$. Otherwise, A does not have an inverse.

Example 2.2.3

Question 6

Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

Solution:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} 4 & -3 & 8 & 0 & 0 & 1 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{4}R_1 - R_2 \rightarrow R_2$$

$$\begin{bmatrix} 4 & -3 & 8 & 0 & 0 & 1 \\ 0 & \frac{-3}{4} & -1 & 0 & -1 & \frac{1}{4} \\ 0 & 1 & 2 & 1 & 0 & 0 \end{bmatrix}$$

$$\frac{-4}{3}R_2 - R_3 \rightarrow R_3$$

$$\begin{bmatrix} 4 & -3 & 8 & 0 & 0 & 1 \\ 0 & \frac{-3}{4} & -1 & 0 & -1 & \frac{1}{4} \\ 0 & 0 & \frac{-2}{3} & -1 & \frac{4}{3} & \frac{-1}{3} \end{bmatrix}$$

$$4R_2 - R_1 \rightarrow R_1$$

$$\begin{bmatrix} -4 & 0 & -12 & 0 & -4 & 0 \\ 0 & \frac{-3}{4} & -1 & 0 & -1 & \frac{1}{4} \\ 0 & 0 & \frac{-2}{3} & -1 & \frac{4}{3} & \frac{-1}{3} \end{bmatrix}$$

$$\frac{3}{2}R_3 - R_2 \rightarrow R_2$$

$$\begin{bmatrix} -4 & 0 & -12 & 0 & -4 & 0 \\ 0 & \frac{3}{4} & 0 & \frac{-3}{2} & 3 & \frac{-3}{4} \\ 0 & 0 & \frac{-2}{3} & -1 & \frac{4}{3} & \frac{-1}{3} \end{bmatrix}$$

$$18R_3 - R_1 \rightarrow R_1$$

$$\begin{bmatrix} 4 & 0 & 0 & -18 & 28 & \frac{-6}{1} \\ 0 & \frac{3}{4} & 0 & \frac{-3}{2} & 3 & \frac{-3}{4} \\ 0 & 0 & \frac{-2}{3} & -1 & \frac{4}{3} & \frac{-1}{3} \end{bmatrix}$$

$$-R_1 \rightarrow R_1$$

$$\begin{bmatrix} -4 & 0 & 0 & 18 & -28 & \frac{6}{1} \\ 0 & \frac{3}{4} & 0 & \frac{-3}{2} & 3 & \frac{-3}{4} \\ 0 & 0 & \frac{-2}{3} & -1 & \frac{4}{3} & \frac{-1}{3} \end{bmatrix}$$

$$\frac{-1}{4}R_1 \rightarrow R_1$$

$$\frac{4}{3}R_2 \rightarrow R_2$$

$$\frac{-3}{2}R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{-9}{2} & 7 & \frac{-3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$$

Since $A \sim I$, A is invertible and

$$A^{-1} = \begin{bmatrix} \frac{-9}{2} & 7 & \frac{-3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$$

Checking our answer:

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} \frac{-9}{2} & 7 & \frac{-3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

2.3 Characteristics of Invertible Matrices

Theorem 2.3.1 The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

1. A is an invertible matrix.
2. A is row equivalent to the $n \times n$ identity matrix.
3. A has n pivot positions.
4. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
5. The columns of A form a linearly independent set.
6. The linear transformation $\mathbf{x} \longrightarrow A\mathbf{x}$ is one-to-one.
7. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
8. The columns of A span $\mathbb{R}^n$.
9. The linear transformation $\mathbf{x} \longrightarrow A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
10. There is an $n \times n$ matrix C such that $CA = I$
11. There is an $n \times n$ matrix D such that $AD = I$.
12. A^T is an invertible matrix.

Due to the definition of solutions statement 7 could also be written as ”The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$ in $\mathbb{R}^n$”. This further implies the statement 2 and hence implies all other statements. This next fact follows directly from the invertible matrix theorem:

Theorem 2.3.2

Let A and B be square matrices. If $AB = I$, then A and B are both invertible, with $B = A^{-1}$ and $A = B^{-1}$

The Invertible Matrix theorem divides the set of $n \times n$ matrices into two disjoint subsets: the invertible (nonsingular) matrices, and the noninvertible (singular) matrices, with each statement in the theorem characterizing a property of invertible matrices and the negation of each statement characterizing a property of noninvertible matrices.

Chapter 3

Determinants

3.1 Introduction

To extend the concept of the determinant to $n \times n$ matrices we must use this recursive definition:

Definition 3.1.1: The Determinant of a $n \times n$ matrix

For $n \geqslant 2$, the determinant of an $n \times n$ matrix $A = \left[a_{ij}\right]$ is the sum of terms of the form $\pm a_{1j}$ $\det A_{1j}$, with plus and minus signs alternating, where the entries of $a_{11}, a_{12}, \ldots, a_{1n}$ are form the first row of A, i.e.:

$$\begin{aligned}\det A &= a_{11}\det A_{11} - a_{12}\det A_{12} + \ldots + (-1)^{1+n} a_{1n}\det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j}\, a_{1j} \det A_{1j}\end{aligned}$$

Where A_{1j} refers to the matrix obtained by crossing out the first row and the jth column of A, which if A is a 3×3 matrix would result in a 2×2 one allowing us to find the determinant of A_{1j} using 2.1

Example 3.1.1

Question 7

Compute the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Solution:

$$\begin{aligned}\det A &= \sum_{j=1}^{n} (-1)^{1+j}\, a_{1j} \det A_{1j} \\ &= a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} \\ &= 1 \begin{vmatrix} 4 & 1 \\ 2 & 0 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} \\ &= 1\,(0-2) - 5\,(0) + 0\,(-4) \\ &= -2\end{aligned}$$

The definition of $\det A$ can also be written in the form of a *cofactor expansion*, Given $A = \left[a_{ij}\right]$, the (i, j) -cofactor of A is the number C_{ij} defined by:

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Allowing us to express $\det A$ as:

$$\begin{aligned}\det A &= \sum_{j=1}^{n} a_{1j} C_{1j} \\ &= a_{11}C_{11} + \ldots + a_{1n}C_{1n}\end{aligned}$$

This is termed as the *cofactor expansion of the determinant along the first row* of A.

Theorem 3.1.1 Cofactor Expansion

The determinant of any $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row is:

$$\begin{aligned}\det A &= \sum_{j=1}^{n} a_{ij} C_{ij} \\ &= a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}\end{aligned}$$

And the expansion down the jth column is:

$$\begin{aligned}\det A &= \sum_{i=1}^{n} a_{ij} C_{ij} \\ &= a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}\end{aligned}$$

Example 3.1.2

Question 8

Use a cofactor expansion across the third row to compute the determinant of A, where

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Solution:

$$\begin{aligned}\det A &= \sum_{j=1}^{n} a_{3j} C_{3j} \\ &= 0 \begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0 \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} \\ &= 0 + 2(-1) + 0 \\ &= -2\end{aligned}$$

In the case where we are computing the determinant of a matrix with great dimension, we take the cofactor across the row or column with the most zeros.

Example 3.1.3

Question 9

Compute $\det A$, where

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

Solution: We take the cofactor expansion down the first column of A.

$$\begin{aligned}
\det A &= \sum_{i=1}^{n} a_{i1}C_{i3} \\
&= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + a_{41}C_{41} + a_{51}C_{51} \\
&= 3\begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} + 0C_{21} + 0C_{31} + 0C_{41} + 0C_{51} \\
&\text{We disregard the zero terms} \\
&= 3\begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} \\
&\text{Next we perform a cofactor expansion down the 1st column of our determinant} \\
&= 3\left(\sum_{i=1}^{n} a_{i1}C_{i1}\right) \\
&= 3\left(a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + a_{41}C_{41}\right) \\
&= 3\left(2\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} - 0C_{21} + 0C_{31} - 0C_{41}\right) \\
&= 3 \times 2\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} \\
&= 3 \times 2\left(\sum_{j=1}^{n} a_{3j}C_{3j}\right) \\
&= 3 \times 2\left(a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}\right) \\
&= 3 \times 2\left(0C_{31} + 2\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0C_{33}\right) \\
&= 3 \times 2 \times 2\,(-1) \\
&= -12
\end{aligned}$$

Theorem 3.1.2
If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

3.1.1 Exercises

Question 10

Compute

$$\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$$

Solution:

$$\begin{aligned}
\det A &= \sum_{j=1}^{n} a_{4j}C_{4j} \\
&= a_{41}C_{41} + a_{42}C_{42} + a_{43}C_{43} + a_{44}C_{44} \\
&= 0C_{41} - 5\begin{vmatrix} 5 & 2 & 2 \\ 0 & 0 & -4 \\ -5 & 0 & 3 \end{vmatrix} + 0C_{43} + 6\begin{vmatrix} 5 & -7 & 2 \\ 0 & 3 & 0 \\ -5 & -8 & 0 \end{vmatrix} \\
&= 5\left(\sum_{j=1}^{n} a_{2j}C_{2j}\right) + 6\left(\sum_{j=1}^{n} a_{2j}C_{2j}\right) \\
&= 5\left(0C_{21} - 0C_{22} - 4\begin{vmatrix} 5 & 2 \\ -5 & 0 \end{vmatrix}\right) + 6\left(\sum_{j=1}^{n} a_{2j}C_{2j}\right) \\
&= 5\,(0 + 40) + 6\left(0C_{21} - 3\begin{vmatrix} 5 & 2 \\ -5 & 0 \end{vmatrix} + 0C_{23}\right) \\
&= 200 + 6\,(-3 \times 10) \\
&= 200 - 180 \\
&= 20
\end{aligned}$$

3.2 Properties of Determinants

Theorem 3.2.1 Row Operations
Let A be a square matrix, Then:
1. If a multiple of one row A is added to another row to produce a matrix B, then $\det B = \det A$
2. If two rows of A are interchanged to produce B, then $\det B = -\det A$
3. If one row of A is multiple by k to produce B, then $\det B = k \cdot \det A$

Example 3.2.1

Question 11

Compute $\det A$, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

Solution: We can reduce the matrix A to reduced row echelon form then use the fact that the determinant of a triangular matrix is the product of main diagonal entries.

$$\begin{aligned} \det A &= \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} \\ &= 1 \times 3 \times -5 \\ &= -15 \end{aligned}$$

Theorem 3.2.2 Invertible Matrix Theorem

A square matrix A is invertible if and only if $\det A \neq 0$

This theorem adds the statement "$\det A \neq 0$" to the Invertible Matrix theorem 2.3. This then implies a matrix A with $\det A = 0$ has its columns as linearly dependent, also implying the rows of A are also linearly dependent.

Example 3.2.2

Question 12

Compute $\det A$, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$$

$$-2R_1 + R_3 \to R_3$$

$$\begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{bmatrix}$$

$\det A = 0$ because rows 2 and 3 are the same making the system linearly dependent

3.2.1 Column Operations

Similar to row operations, we can also perform column operations on matrices. The following theorem shows that column operations have the same effect on the determinant as row operations.

Theorem 3.2.3

If A is an $n \times n$ matrix, then $\det A^T = \det A$

Because of this theorem, each statement in 3.2 is true when the word row is replaced with column.

3.2.2 Determinants and Matrix Products

Theorem 3.2.4 Multiplicative Property

If A and B are $n \times n$ matrices then $\det AB = (\det A)(\det B)$

Example 3.2.3

For $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$,

$$\begin{aligned} AB &= \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix} \\ \det AB &= 25 \times 13 - 20 \times 14 \\ &= 45 \\ \det A &= 6 \times 2 - 1 \times 3 \\ &= 9 \\ \det B &= 4 \times 2 - 3 \times 1 \\ &= 5 \\ (\det A)(\det B) &= 9 \times 5 \\ &= 45 \end{aligned}$$

Chapter 4

Cramer's Rule, Volume, and Linear Transformations

4.1 Cramer's Rule

Cramer's rule is often used in many theoretical calculations, for example in studying how the solution of $A\mathbf{x} = \mathbf{b}$ is affected by changes in the entries of $\mathbf{b}$. However, Cramer's rule is not practical for solving systems of equations larger than 3×3.

For any $n \times n$ matrix A and any $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(b)$ be the matrix obtained from A by replacing column i by the vector $\mathbf{b}$.

$$A_i(\mathbf{b}) = \begin{bmatrix} a_1 & \dots & a_{i-1} & \mathbf{b} & \dots & a_n \end{bmatrix}$$

Definition 4.1.1: Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = b$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, 2, \dots, n$$

Example 4.1.1

Question 13

Use Cramer's rule to solve the system

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8 \end{aligned}$$

Solution:

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$\begin{aligned}
\det A &= (3 \times 4) - (-2 \times -5) \\
&= 2 \\
x_1 &= \frac{\begin{vmatrix} 6 & -2 \\ 8 & 4 \end{vmatrix}}{2} \\
&= \frac{40}{2} \\
&= 20
\end{aligned}$$

$$\begin{aligned}
x_2 &= \frac{\begin{vmatrix} 3 & 6 \\ -5 & 8 \end{vmatrix}}{2} \\
&= \frac{54}{2} \\
&= 27
\end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} 20 \\ 27 \end{bmatrix}$$

4.1.1 Determinants as Area or Volume

Theorem 4.1.1

If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$.
If A is a 3×3 matrix, the volume of the parallelogram determined by the columns of A is $|\det A|$

Theorem 4.1.2

Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be non-zero vectors. Then for any scalar c, the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2$ equals the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2 + c\mathbf{a}_1$

Example 4.1.2

Question 14

Calculate the area of the parallelogram determined by the points $(-2, -2), (0, 3), (4, -1),$ and $(6, 4)$

Solution:

First we must translate the parallelogram to one having the origin as a vertex. We can do this by subtracting any one of the points from all the other points. We choose to subtract $(-2, -2)$ from all the other points. This gives us the points $(0, 0), (2, 5), (6, 1), (8, 6)$, this parallelogram still has the same area as the original one.

Then we can form the matrix A with the columns as the vectors formed by the points. We can choose any two of the vectors to form the parallelogram, we choose $(2, 5)$ and $(6, 1)$:

$$\begin{aligned}
\text{Area} &= \left\| \begin{matrix} 2 & 6 \\ 5 & 1 \end{matrix} \right\| \\
&= |(2 - 30)| \\
&= |-28| \\
&= 28
\end{aligned}$$

4.2 Exercises

Question 15

Use Cramer's rule to compute the solutions of the system

$$
\begin{aligned}
x_1 + 3x_2 + x_3 &= 4 \\
-x_1 + 2x_3 &= 2 \\
3x_1 + x_2 &= 2
\end{aligned}
$$

Solution:

$$A = \begin{bmatrix} 1 & 3 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{aligned}
\det A &= \sum_{j=1}^{n} a_{2j} C_{2j} \\
&= (-1)^{1+2}(-1)\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} + (-1)^{3+2}(2)\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} \\
&= 1(-1) + -2(1-9) \\
&= -1 + -2(-8) \\
&= 15
\end{aligned}$$

$$\begin{aligned}
x_1 &= \frac{\begin{vmatrix} 4 & 3 & 1 \\ 2 & 0 & 2 \\ 2 & 1 & 0 \end{vmatrix}}{15} \\
x_1 &= \frac{6}{15} \\
x_1 &= \frac{2}{5}
\end{aligned}$$

$$\begin{aligned}
x_2 &= \frac{\begin{vmatrix} 1 & 4 & 1 \\ -1 & 2 & 2 \\ 3 & 2 & 0 \end{vmatrix}}{15} \\
x_2 &= \frac{12}{15} \\
x_2 &= \frac{4}{5}
\end{aligned}$$

$$\begin{aligned}
x_3 &= \frac{\begin{vmatrix} 1 & 3 & 4 \\ -1 & 0 & 2 \\ 3 & 1 & 2 \end{vmatrix}}{15} \\
x_3 &= \frac{18}{15} \\
x_3 &= \frac{6}{5}
\end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} \frac{2}{5} \\ \frac{4}{5} \\ \frac{6}{5} \end{bmatrix}$$

Question 16

Find the area of the parallelogram whose vertices are listed:

$$(0,-2), (5,-2), (-3,1), \text{ and } (2,1)$$

Solution: Minus $(0, -2)$, giving $(0,0), (5,0), (-3,3), (2,3)$

$$\begin{aligned} A &= \begin{bmatrix} 5 & -3 \\ 0 & 3 \end{bmatrix} \\ \text{Area} &= |\det A| \\ &= |15| \\ &= 15 \end{aligned}$$

Question 17

Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1,0,3), (1,2,4), (5,1,0)$

Solution:

$$\begin{aligned} A &= \begin{bmatrix} 1 & 1 & 5 \\ 0 & 2 & 1 \\ 5 & 4 & 0 \end{bmatrix} \\ \det A &= \left| \sum_{j=1}^{n} a_{ij} C_{ij} \right| \\ &= \left| (-1)^{3+1} (5) \begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} + (-1)^{2+3} (4) \begin{vmatrix} 1 & 5 \\ 0 & 1 \end{vmatrix} \right| \\ &= |5(1-10) + -4(1)| \\ &= |-45 - 4| \\ &= 49 \end{aligned}$$

Chapter 5

Exercises

Question 18

Compute the product AB using:

- The definition where Ab_1, Ab_2 are computed separately.
- The row-column rule.

$$A = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix}, B \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$$

Solution:

1.

$$\begin{aligned} Ab_1 &= \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} -3-4 \\ 15-8 \\ 6+6 \end{bmatrix} \\ &= \begin{bmatrix} -7 \\ 7 \\ 12 \end{bmatrix} \\ Ab_2 &= \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ -6 \\ -7 \end{bmatrix} \end{aligned}$$

$$AB = \begin{bmatrix} -7 & 4 \\ 7 & -6 \\ 12 & -7 \end{bmatrix}$$

2.

$$
\begin{aligned}
AB &= \begin{bmatrix} -1\times 3 + 2\times -2 & -1\times -2 + 2\times 1 \\ 5\times 3 + 4\times -2 & 5\times -2 + 4\times 1 \\ 2\times 3 + -3\times -2 & -2\times 2 + -3\times 1 \end{bmatrix} \\
&= \begin{bmatrix} -7 & 4 \\ 7 & -6 \\ 12 & -7 \end{bmatrix}
\end{aligned}
$$

Question 19

Suppose the last column of AB is entirely zero but B itself has no column of zeros. What can you say about the columns of A?

Solution: If the last column of AB is entirely zero, then the last column of A must be a linear combination of the columns of B. Therefore the columns of A are linearly dependent.

Question 20

Find the inverses of the following matrices:

1. $$\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$$

2. $$\begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}$$

Solution:

1.

$$
\begin{aligned}
\det(A) &= 32 - 30 \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
A^{-1} &= \frac{1}{2}\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}
\end{aligned}
$$

2.

$$
\begin{aligned}
\det(A) &= -24 + 28 \\
&= 4 \\
A^{-1} &= \frac{1}{4}\begin{bmatrix} -8 & 4 \\ -7 & 3 \end{bmatrix} \\
&= \begin{bmatrix} -4 & 1 \\ -\frac{7}{4} & \frac{3}{4} \end{bmatrix}
\end{aligned}
$$

Question 21

Use the inverse found in 6 1 to solve the system:

$$\begin{aligned} 8x_1 + 6x_2 &= 2 \\ 5x_1 + 4x_2 &= -1 \end{aligned}$$

Solution:

$$\begin{aligned} \mathbf{b} &= \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 7 \\ -9 \end{bmatrix} \end{aligned}$$

Question 22

Find the inverse of the following matrix if it exists:

$$\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$$

$$4R_1 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ -2 & 6 & -4 \end{bmatrix}$$

$$-2R_1 - R_3 \longrightarrow R_3$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix}$$

$$2R_2 - R_3 \longrightarrow R_3$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 1 \times -1 \times 0 \\ &= 0 \end{aligned}$$

$\therefore$ the matrix does not have an inverse

Question 23

Suppose the system below is consistent for all possible values of f and g. What can you say about the coefficients c and d? Justify your answer.

$$\begin{aligned} x_1 + 3x_2 &= f \\ cx_1 + dx_2 &= g \end{aligned}$$

Solution:

$$\begin{bmatrix} 1 & 3 & f \\ c & d & g \end{bmatrix}$$

$$cR_1 - R_2 \to R_2$$

$$\begin{bmatrix} 1 & 3 & f \\ 0 & 3c - d & cf - g \end{bmatrix}$$

Question 24

Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is in Span$\{u, v\}$ for all h and k.

Solution:

$$x_1\mathbf{u} + x_2\mathbf{v} = \begin{bmatrix} h \\ k \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 2 & 2 & h \\ -1 & 1 & k \end{bmatrix}$$

$$-\frac{1}{2}R_1 - R_2 \to R_2$$

Question 25

A steam plant burns two types of coal: anthracite (A) and bituminous (B). For each ton of A burned, the plant produces 27.6 million Btu of heat, 3100 grams (g) of sulfur dioxide, and 250 g of particulate matter (solid-particle pollutants). For each ton of B burned, the plant produces 30.2 million Btu, 6400 g of sulfur dioxide, and 360 g of particulate matter.

1. How much heat does the steam plant produce when it burns x_1 tons of A and x_2 tons of B?
2. Suppose the output of the steam plant is described by a vector that lists the amounts of heat, sulfur dioxide, and particulate matter. Express this output as a linear combination of two vectors, assuming that the plant burns x_1 tons of A and x_2 tons of B.
3. Over a certain time period, the steam plant produced 162 million Btu of heat, 23,610 g of sulfur dioxide, and 1623 g of particulate matter. Determine how many tons of each type of coal the steam plant must have burned. Include a vector equation as part of your solution.

Solution:

$$\begin{aligned} 27.6x_1 + 30.2x_2 &= \text{Heat} \\ 3100x_1 + 6400x_2 &= \text{Sulfur Dioxide} \\ 250x_1 + 360x_2 &= \text{Particulate Matter} \end{aligned}$$

1.

$$27.6x_1 + 30.2x_2$$

2.

$$\begin{aligned} 27.6x_1 + 30.2x_2 &= H \\ 3100x_1 + 6400x_2 &= SO_2 \\ 250x_1 + 360x_2 &= P \end{aligned}$$

$$\mathbf{u}x_1 + \mathbf{v}x_2 = \begin{bmatrix} H \\ SO_2 \\ P \end{bmatrix}$$

Where $\mathbf{u} = \begin{bmatrix} 27.6 \\ 3100 \\ 250 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 30.2 \\ 6400 \\ 360 \end{bmatrix}$

3.

$$
\begin{aligned}
27.6x_1 + 30.2x_2 &= 162\\
3100x_1 + 6400x_2 &= 23610\\
250x_1 + 360x_2 &= 1623
\end{aligned}
$$

$$\begin{bmatrix} 27.6 & 30.2 & 162 \\ 3100 & 6400 & 23610 \\ 250 & 360 & 1623 \end{bmatrix}$$

$$\frac{7750}{69}R_1 - R_2 \to R_2$$

$$\begin{bmatrix} \frac{138}{5} & \frac{151}{5} & 162 \\ 0 & \frac{-207550}{69} & \frac{-124530}{23} \\ 250 & 360 & 1623 \end{bmatrix}$$

$$\frac{625}{69}R_1 - R_3 \to R_3$$

$$\begin{bmatrix} \frac{138}{5} & \frac{151}{5} & 162 \\ 0 & \frac{-207550}{69} & \frac{-124530}{23} \\ 0 & \frac{-5965}{69} & \frac{-3579}{23} \end{bmatrix}$$

$$\frac{112}{3897}R_2 - R_3 \to R_3$$

$$\begin{bmatrix} \frac{138}{5} & \frac{151}{5} & 162 \\ 0 & \frac{-207550}{69} & \frac{-124530}{23} \\ 0 & 0 & \frac{0}{1} \end{bmatrix}$$

$$0R_1 - R_3 \to R_3$$

$$\begin{bmatrix} \frac{138}{5} & \frac{151}{5} & 162 \\ 0 & \frac{-207550}{69} & \frac{-124530}{23} \\ 0 & 0 & \frac{0}{1} \end{bmatrix}$$

$$0R_2 - R_3 \to R_3$$

$$\begin{bmatrix} \frac{138}{5} & \frac{151}{5} & 162 \\ 0 & \frac{-207550}{69} & \frac{-124530}{23} \\ 0 & 0 & \frac{0}{1} \end{bmatrix}$$

$$\frac{-98}{9761}R_2 - R_1 \to R_1$$

$$\begin{bmatrix} \frac{-138}{5} & 0 & \frac{-2691}{25} \\ 0 & \frac{-207550}{69} & \frac{-124530}{23} \\ 0 & 0 & \frac{0}{1} \end{bmatrix}$$

$$0R_2 - R_1 \to R_1$$

$$\begin{bmatrix} \frac{138}{5} & 0 & \frac{2691}{25} \\ 0 & \frac{-207550}{69} & \frac{-124530}{23} \\ 0 & 0 & \frac{0}{1} \end{bmatrix}$$

$$0R_2 - R_1 \to R_1$$

$$\frac{5}{138}R_1 \to R_1$$

$$\frac{-1}{3008}R_2 \to R_2$$

$$\begin{bmatrix} 1 & 0 & \frac{39}{10} \\ 0 & 1 & \frac{9}{5} \\ 0 & 0 & \frac{0}{1} \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} \frac{39}{10} \\ \frac{9}{5} \end{bmatrix}$$

Question 26

Describe and compare the solution sets of $x_1 - 3x_2 + 5x_3 = 0$ and $x_1 - 3x_2 + 5x_3 = 4$.

Solution:

$$\begin{bmatrix} 1 & -3 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - 3x_2 + 5x_3 &= 0 \\ x_2 &= x_2 \\ x_3 &= x_3 \\ x_1 &= 3x_2 - 5x_3 \end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} x_3$$