

# Homework 1

Madiba Hudson-Quansah

### Question 1

Find the general solution of the system whose augmented matrix is given below

$$\begin{bmatrix} 1 & 0 & -5 & 0 & -8 & 3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Solution:**

$$\begin{bmatrix} 1 & 0 & -5 & 0 & -8 & 3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-\frac{8}{1}R_3 - R_1 \rightarrow R_1$$

$$\begin{bmatrix} -1 & 0 & 5 & 0 & 0 & -3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-R_1 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & -5 & 0 & 0 & 3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 5x_2 = 3$$

$$x_2 + 4x_3 - x_4 = 6$$

$$x_3 = x_3$$

$$x_5 = 0$$

$$x_4 = x_4$$

$$x_1 = 3 + 5x_2$$

$$x_2 = 6 - 4x_3 + x_4$$

$$x_3 = x_3$$

$$x_4 = x_4$$

$$x_5 = 0$$

$$\begin{cases} x_1 = 3 + 5x_2 \\ x_2 = 6 - 4x_3 + x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \\ x_5 = 0 \end{cases}$$

### Question 2

Choose  $h$  and  $k$  such that the following system has:

1. No solution
2. A unique solution
3. Many solutions

Give separate answers for each part.

$$\begin{aligned}x_1 - 3x_2 &= 1 \\ 2x_1 + hx_2 &= k\end{aligned}$$

**Solution:**

1.

$$\begin{bmatrix} 1 & -3 & 1 \\ 2 & h & k \end{bmatrix}$$
$$2R_1 - R_2 \rightarrow R_2$$
$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & (-6-h) & (2-k) \end{bmatrix}$$

$$\begin{aligned}-6 - h &= 0 \\ h &= -6\end{aligned}$$

$$\begin{aligned}2 - k &\neq 0 \\ k &\neq 2\end{aligned}$$

$\therefore$  the system has no solution when  $h = -6$  and  $k \neq 2$ . So for example  $h = -6$  and  $k = 0$

2.

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & (-6-h) & (2-k) \end{bmatrix}$$

$$\begin{aligned}-6 - h &\neq 0 \\ h &\neq -6\end{aligned}$$

$$\begin{aligned}2 - k &\neq 0 \\ k &\neq 2\end{aligned}$$

$\therefore$  the system has a unique solution when  $h \neq -6$  and  $k \neq 2$ . So for example  $h = 0$  and  $k = 0$

3.

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & (-6-h) & (2-k) \end{bmatrix}$$

$$\begin{aligned}-6 - h &= 0 \\ h &= -6\end{aligned}$$

$$\begin{aligned}2 - k &= 0 \\ k &= 2\end{aligned}$$

$\therefore$  the system has many solutions when  $h = -6$  and  $k = 2$ .

### Question 3

A system of linear equations with fewer equations than unknowns is sometimes called an underdetermined system. Give an example of an inconsistent underdetermined system of two equations in three unknowns.

**Solution:**

$$2x_1 + 3x_2 + 4x_3 = 1$$

$$2x_1 + 3x_2 + 4x_3 = 10$$

#### Question 4

Determine if  $\mathbf{b}$  is a linear combination of  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ .

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$$

**Solution:** To determine if  $\mathbf{b}$  is a linear combination of the given vectors I must prove that there exists weights  $x_1, x_2, x_3$  such that

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + x_3 \mathbf{a}_3 = \mathbf{b}$$

Therefore:

$$x_1 - 2x_2 - 6x_3 = 11$$

$$3x_2 + 7x_3 = -5$$

$$x_1 - 2x_2 + 5x_3 = 9$$

$$\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & -2 & -6 & 11 \\ 1 & -2 & 5 & 9 \\ 0 & 3 & 7 & -5 \end{bmatrix}$$

$$R_1 - R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 0 & -11 & 2 \\ 0 & 3 & 7 & -5 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & -11 & 2 \end{bmatrix}$$

$$\frac{-2}{3}R_2 - R_1 \rightarrow R_1$$

$$\begin{bmatrix} -1 & 0 & \frac{4}{3} & \frac{-23}{3} \\ 0 & 3 & 7 & -5 \\ 0 & 0 & -11 & 2 \end{bmatrix}$$

$$\frac{-7}{11}R_3 - R_2 \rightarrow R_2$$

$$\begin{bmatrix} -1 & 0 & \frac{4}{3} & \frac{-23}{3} \\ 0 & -3 & 0 & \frac{\frac{41}{11}}{\frac{11}{11}} \\ 0 & 0 & -11 & 2 \end{bmatrix}$$

$$\frac{-4}{33}R_3 - R_1 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{\frac{245}{33}}{\frac{33}{33}} \\ 0 & -3 & 0 & \frac{\frac{41}{11}}{\frac{11}{11}} \\ 0 & 0 & -11 & 2 \end{bmatrix}$$

$$-\frac{1}{3}R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{245}{33} \\ 0 & 1 & 0 & -\frac{\frac{41}{11}}{\frac{33}{33}} \\ 0 & 0 & -11 & 2 \end{bmatrix}$$

$$-\frac{1}{11}R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{245}{33} \\ 0 & 1 & 0 & -\frac{\frac{41}{11}}{\frac{33}{33}} \\ 0 & 0 & 1 & -\frac{2}{11} \end{bmatrix}$$

∴ the vector  $\mathbf{b}$  is a linear combination of the vectors  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  for weights:

$$\begin{aligned}x_1 &= \frac{245}{33} \\x_2 &= -\frac{41}{33} \\x_3 &= -\frac{2}{11}\end{aligned}$$

### Question 5

Show that the equation  $A\mathbf{x} = \mathbf{b}$  does not have a solution for all possible  $\mathbf{b}$  and describe the set of all  $\mathbf{b}$  for which the equation does have a solution.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

**Solution:**

The equation  $A\mathbf{x} = \mathbf{b}$  is as follows:

$$\begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

resulting in the system:

$$\begin{aligned}x_1 - 2x_2 - x_3 &= b_1 \\ -2x_1 + 2x_2 &= b_2 \\ 4x_1 - x_2 + 3x_3 &= b_3\end{aligned}$$

Finding the solutions to the system:

$$\begin{aligned}&\begin{bmatrix} 1 & -2 & -1 & b_1 \\ -2 & 2 & 0 & b_2 \\ 4 & -1 & 3 & b_3 \end{bmatrix} \\&\quad -2R_1 - R_2 \rightarrow R_2 \\&\begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & 2 & 2 & -2b_1 - b_2 \\ 4 & -1 & 3 & b_3 \end{bmatrix} \\&\quad 4R_1 - R_3 \rightarrow R_3 \\&\begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & 2 & 2 & -2b_1 - b_2 \\ 0 & -7 & -7 & 4b_1 - b_3 \end{bmatrix} \\&\quad -\frac{7}{2}R_2 - R_3 \rightarrow R_3 \\&\begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & 2 & 2 & -2b_1 - b_2 \\ 0 & 0 & 0 & 3b_1 + \frac{7}{2}b_2 + b_3 \end{bmatrix}\end{aligned}$$

From the echelon form I can see that this system cannot be consistent for all values of  $\mathbf{b}$  because not all values of  $\mathbf{b}$  will result in the expression  $3b_1 + \frac{7}{2}b_2 + b_3$  evaluating to 0.

$$3b_1 + \frac{7}{2}b_2 + b_3 = 0$$

The set of  $\mathbf{b}$  for which the equation does have a solution is the set of all  $\mathbf{b}$  that satisfy the above equation.