

Preliminaries

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Chapter 1

\mathbb{R}^n

\mathbb{R}^n denotes the set of real numbers / scalars. If n is a positive integer then \mathbb{R}^n is defined to be the set of all sequences \mathbf{x} of n real numbers

$$\mathbf{x} = (x_1, x_2, \dots, x_n)$$

Multivariable calculus studies functions that act on these sets, functions in the form of

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

or more accurately

$$f : A \rightarrow \mathbb{R}^m$$

Where A is a subset of \mathbb{R}^n .

1.1 Vector Arithmetic

Every vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$ in \mathbb{R}^n can be decomposed as a sum along the coordinate directions

$$\begin{aligned}\mathbf{x} &= (x_1, 0, \dots, 0) + (0, x_2, 0, \dots, 0) + (0, \dots, x_n) \\ &= x_1 (1, 0, \dots, 0) + x_2 (0, 1, 0, \dots, 0) + x_n (0, \dots, 1)\end{aligned}$$

The vectors $\mathbf{e}_1 = (1, 0, \dots, 0)$, $\mathbf{e}_2 = (0, 1, \dots, 0)$, $\mathbf{e}_n = (0, \dots, 1)$, with a 1 in a single component corresponding to the value of n and zeros everywhere else, are called the **standard basis vectors**. I.e.:

$$\mathbf{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \dots + x_n \mathbf{e}_n$$

Where the scalar coefficients x_i are the coordinates of \mathbf{x} .

1.2 Linear Transformations

Definition 1.2.1: Linear Transformation

A function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation if:

- $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$
- $T(c\mathbf{x}) = cT(\mathbf{x})$

$\forall c \in \mathbb{R}$ and $\forall \mathbf{x} \in \mathbb{R}^n \wedge \forall \mathbf{y} \in \mathbb{R}^n$

1.3 The Matrix of a linear Transformation

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Thus:

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \dots + x_n\mathbf{e}_n) \\ &= x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + \dots + x_nT(\mathbf{e}_n) \\ &= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n \end{aligned}$$

Where $\mathbf{a}_j = T(\mathbf{e}_j)$ for $j = 1, 2, \dots, n$. A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is completely determined by the vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$

Definition 1.3.1: Dot Product

Given $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ in \mathbb{R}^n , the dot product, denoted by $\mathbf{x} \cdot \mathbf{y}$, is defined by:

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \dots + x_ny_n$$

We have shown the every real valued linear transformation, $T : \mathbb{R}^n \rightarrow \mathbb{R}$ has the form

$$T(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$$

Where $\mathbf{a} = (a_1, a_2, \dots, a_n)$ in \mathbb{R}^n . Generalizing for the case $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, the objects in \mathbf{a} are now vectors in \mathbb{R}^m , i.e. $\mathbf{a}_j = T(\mathbf{e}_j)$, \mathbf{a} then becomes the matrix A , thus we have the form:

$$T(\mathbf{x}) = A\mathbf{x}$$

Example 1.3.1

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the counter-clockwise rotation by $\frac{\pi}{3}$ about the origin. Then T rotates the vector $\mathbf{e}_1 = (1, 0)$ to the vector on the unit circle that makes an angle of $\frac{\pi}{3}$ with the positive x_1 -axis. That is $T(\mathbf{e}_1) = (\cos(\frac{\pi}{3}), \sin(\frac{\pi}{3})) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Similarly, $T(\mathbf{e}_2) = (\cos(\frac{5\pi}{6}), \sin(\frac{5\pi}{6})) = (-\frac{\sqrt{3}}{2}, \frac{1}{2})$. Hence the matrix of T with respect to the standard bases is:

$$A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

Example 1.3.2

If $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is the projection of the $x_1x_2x_3$ space onto the x_1x_2 -plane, then $T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0)$, $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1)$, and $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0)$, therefore:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

1.4 The geometry of the dot product

Definition 1.4.1: Norm / Magnitude

Denoted by $\|\mathbf{x}\|$ is defined as:

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

Proposition 1.1

$\mathbf{x} \in \mathbb{R}^n$, then $\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$

Proof. Both sides equal $x_1^2 + x_2^2 + \dots + x_n^2$



Examining these notions in \mathbb{R}^2 , If $\mathbf{x} = (x_1, x_2)$, then $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$. By the Pythagorean theorem, this is the length of the hypotenuse of a right triangle with legs $|x_1|$ and $|x_2|$. If we think of \mathbf{x} as an arrow originating from the origin, then $\|\mathbf{x}\|$ is the length of the arrow, if instead we think of \mathbf{x} as point $\|\mathbf{x}\|$ is the distance from the point to the origin.

Given two points \mathbf{x} and \mathbf{y} in \mathbb{R}^2 , the distance between them is the length of the arrow that connects them $\vec{\mathbf{y}\mathbf{x}} = \mathbf{x} - \mathbf{y}$, Hence:

$$\text{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$$

Next let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ be non-zero elements of \mathbb{R}^2 , represented as arrows emitting from the origin. If the lines are perpendicular then the slopes of these lines through the origin that contain them are defined and are negative reciprocals, i.e. let m be the gradient of a line, the gradient of the line perpendicular to this line is $-\frac{1}{m}$. As the slope is the ratio of vertical to horizontal displacement, then

$$\begin{aligned} \frac{x_2}{x_1} &= -\frac{y_2}{y_1} \\ 0 &= x_1 y_1 + x_2 y_2 \\ \text{Or} \\ 0 &= \mathbf{x} \cdot \mathbf{y} \end{aligned}$$

Thus:

In \mathbb{R}^2 \mathbf{x} and \mathbf{y} are perpendicular if and only if $\mathbf{x} \cdot \mathbf{y} = 0$

For vectors in a plane in general, let θ be the angle between two vectors \mathbf{x} and \mathbf{y} . Assuming none of the vectors are scalar products of each other, i.e. $\theta \neq 0$ we can study the relationship by recognizing the triangle whose vertices are $\mathbf{0}$, \mathbf{x} and \mathbf{y} . Two of the sides of this triangle have lengths $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$ and the length of the third is $\text{dist}(\mathbf{x}, \mathbf{y})$. Using the cosine rule:

$$\begin{aligned} \text{dist}(\mathbf{x}, \mathbf{y})^2 &= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\|\|\mathbf{y}\|\cos(\theta) \\ (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) &= \mathbf{x} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} - 2\|\mathbf{x}\|\|\mathbf{y}\|\cos(\theta) \\ \mathbf{x} \cdot \mathbf{x} - 2\mathbf{y} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} &= \mathbf{x} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} - 2\|\mathbf{x}\|\|\mathbf{y}\|\cos(\theta) \\ -2\mathbf{x} \cdot \mathbf{y} &= -2\|\mathbf{x}\|\|\mathbf{y}\|\cos(\theta) \\ \mathbf{x} \cdot \mathbf{y} &= \|\mathbf{x}\|\|\mathbf{y}\|\cos(\theta) \end{aligned}$$

When our assumption does not hold, i.e. the vectors are scalar multiples of each other then $\theta = 0$ or π . For instance when $\mathbf{y} = c\mathbf{x}$, then $\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot c(\mathbf{x}) = c\|\mathbf{x}\|^2$. Meanwhile $\|\mathbf{y}\| = |c|\|\mathbf{x}\|$ so $\|\mathbf{x}\|\|\mathbf{y}\|\cos(\theta) = |c|\|\mathbf{x}\|^2\cos(\theta)$.

We can use these notions about \mathbb{R}^2 to create some intuition about \mathbb{R}^n where $n \geq 3$. For instance, if \mathbf{v} is a nonzero vector in \mathbb{R}^n , we think of the set of all scalar multiple $c\mathbf{v}$ as a "line". If a second vector \mathbf{w} is not a scalar multiple of \mathbf{v} and c and d are scalars, by the parallelogram law of addition the combination $c\mathbf{v} + d\mathbf{w}$ lies in a "plane" determined by \mathbf{v} and \mathbf{w} , and that c and d range over all possible scalars, $c\mathbf{v} + d\mathbf{w}$ should sweep out this plane. As a result the set of all vectors $c\mathbf{v} + d\mathbf{w}$, where $c, d \in \mathbb{R}$ is called the plane spanned by \mathbf{v} and \mathbf{w} , denoted by \mathcal{P} .

Relating this plane \mathcal{P} to \mathbb{R}^2 , we begin by choosing two vector \mathbf{u}_1 and \mathbf{u}_2 in \mathcal{P} such that $\mathbf{u}_1 \cdot \mathbf{u}_1 = \mathbf{u}_2 \cdot \mathbf{u}_2 = 1$ and $\mathbf{u}_2 \cdot \mathbf{u}_1 = 0$. These vectors become the standard basis vectors \mathbf{e}_1 and \mathbf{e}_2 , which of course satisfy the same relations, i.e. orthogonal bases. These two vectors can be used to establish a two-dimensional coordinate space internal to \mathcal{P} , where every element in \mathcal{P} can be written in the form $\mathbf{x} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2$ for some scalars a_1 and a_2 . We can then think of $\sqrt{a_1^2 + a_2^2}$ as the

distance from the origin, i.e. the length of \mathbf{x} . If $\mathbf{y} = b_1\mathbf{u}_1 + b_2\mathbf{u}_2$ is also in \mathcal{P} , then:

$$\begin{aligned}\mathbf{x} \cdot \mathbf{y} &= (a_1\mathbf{u}_1 + a_2\mathbf{u}_2) \cdot (b_1\mathbf{u}_1 + b_2\mathbf{u}_2) \\ &= a_1b_1\mathbf{u}_1 \cdot \mathbf{u}_1 + (a_1b_2 + a_2b_1)\mathbf{u}_1\mathbf{u}_2 + a_2b_2\mathbf{u}_2\mathbf{u}_2 \\ &= a_1b_1 + a_2b_2\end{aligned}$$

Thus the dot product in \mathbb{R}^n agrees with the general expectation in our internal coordinate system in \mathcal{P} . Continuing in this way we can build up \mathcal{P} as a copy of \mathbb{R}^2 porting over the notions familiar in this plane of geometry such as angle and distance.

Definition 1.4.2: Unit vector

A vector \mathbf{x} in \mathbb{R}^n such that:

$$\|\mathbf{x}\| = 1$$

The standard basis vectors $\mathbf{e}_i = (0, 0, \dots, 0, 1, 0, \dots, 0)$ are unit vectors.

Corollary 1.4.1

If \mathbf{x} is a vector in \mathbb{R}^n , then $\mathbf{u} = \frac{1}{\|\mathbf{x}\|}\mathbf{x}$ is a unit vector in the same direction as \mathbf{x}

Theorem 1.4.1

If \mathbf{x} and \mathbf{y} are vectors \mathbb{R}^n then:

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos(\theta)$$

Definition 1.4.3: Orthogonal Vectors

Two vectors \mathbf{x} and \mathbf{y} are orthogonal if $\mathbf{x} \cdot \mathbf{y} = 0$

1.5 Determinants

The determinant is a function that assigns a real number to any $n \times n$ matrix. The determinant is not defined for $m \times n$ matrices. The determinant of a 2×2 matrix is defined as:

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

I.e, the product of the entries on the main diagonal minus the product of the entries on the secondary diagonal.

One often thinks of the determinant as a function of the rows of the matrix. If $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, let $[\mathbf{x} \ \mathbf{y}]$, denote a matrix whose rows are \mathbf{x} and \mathbf{y} :

$$\det [\mathbf{x} \ \mathbf{y}] = \det \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = x_1y_2 - x_2y_1$$