

### Instantaneous rate of change

In the case where  $f$  is a function of  $x$   $f'(x)$  measures the instantaneous rate of change of  $f$  with respect to  $x$ .

#### Example

The term widget is an economic term for a generic unit of manufacturing output. Suppose a company produces widgets and knows that the market supports a price of \$10 per widget. Let  $P(n)$  give the profit, in dollars, earned by manufacturing and selling  $n$  widgets. The company likely cannot make a (positive) profit making just one widget; the start-up costs will likely exceed \$10. Mathematically, we would write this as  $P(1) < 0$ .

What do  $P(1000) = 500$  and  $P'(1000) = 0.25$  mean? Approximate  $P(1100)$ .

The equation  $P(1000) = 500$  means that selling 1,000 widgets returns a profit of \$500. We interpret  $P'(1000) = 0.25$  as meaning that the profit is increasing at rate of \$0.25 per widget (the units are “dollars per widget.”). Since we have no other information to use, our best approximation for  $P(1100)$  is:

$$\begin{aligned} P(1100) &\approx P(1000) + P'(1000) \times 100 \\ &= P(1000) + P'(1000) \times 100 \\ &= 500 + 0.25 \times 100 \\ &= 525 \end{aligned}$$

We approximate that selling 1,100 widgets returns a profit of \$525.

### The Slope of the Tangent Line

We can measure the instantaneous rate of change at a given  $x$  value  $c$  of a non-linear function by computing  $f'(c)$ . We can determine the behaviour of the function  $f$  by observing the slopes of its tangent lines.

### Increasing Functions

$f(x)$  is increasing whenever  $x_1 < x_2$  and  $f(x_1) < f(x_2)$ , i.e as you go up the  $x$  axis the  $y$  or function values increase.

$f(x)$  is increasing if the slope on any point on it's graph is positive throughout the function's entire domain.

### Decreasing Functions

$f(x)$  is decreasing whenever  $x_1 < x_2$  and  $f(x_1) > f(x_2)$ , .i.e as you go up the  $x$  axis the  $y$  or function values decrease

$f(x)$  is increasing if the slope on any point on it's graph is negative throughout the function's entire domain.

### Critical Points

- Points where the gradient is equal 0, i.e.  $f'(x) = 0$
- Points where the gradient does not exist, i.e.  $f'(x) = \emptyset$

### Examples

$$t\sqrt[3]{t^2 - 4}$$

$$g(t) = t\sqrt[3]{t^2 - 4}$$

$$g(t) = t(t^2 - 4)^{\frac{1}{3}}$$

$$g'(t) = (1)(t^2 - 4)^{\frac{1}{3}} + \left(\frac{1}{3}\right)(2t)(t^2 - 4)^{-\frac{2}{3}}$$

$$g'(t) = (t^2 - 4)^{\frac{1}{3}} + \frac{2}{3}t^2(t^2 - 4)^{-\frac{2}{3}}$$

$$g'(t) = (t^2 - 4)^{\frac{1}{3}} + \frac{2t^2}{3(t^2 - 4)^{\frac{2}{3}}}$$

$$g'(t) = \frac{(t^2 - 4)^{\frac{1}{3}}}{1} + \frac{2t^2}{3(t^2 - 4)^{\frac{2}{3}}}$$

$$g'(t) = \frac{3(t^2 - 4) + t^2}{3(t^2 - 4)^{\frac{2}{3}}}$$

$$0 = \frac{3(t^2 - 4) + t^2}{3(t^2 - 4)^{\frac{2}{3}}}$$

$$0 = 3t^2 - 12 + t^2$$

$$0 = 4t^2 - 12$$

$$12 = 4t^2$$

$$\frac{12}{4} = t^2$$

$$\pm\sqrt{\frac{12}{4}} = t$$

$$3(t^2 - 4)^{\frac{2}{3}} = 0$$

$$(t^2 - 4)^{\frac{2}{3}} = 0$$

$$t = \pm 2$$

| Interval   | Test Value | Slope <sub><math>g'(x)</math></sub> |
|--|------------|-------------------------------------|
| $x < -2$   | -3         | +                                   |
| $-2 < x < -\sqrt{\frac{12}{5}}$                  | -1.7       | +                                   |
| $-\sqrt{\frac{12}{5}} < x < \sqrt{\frac{12}{5}}$ | 0          | -                                   |
| $\sqrt{\frac{12}{5}} < x < 2$                    | 2          | +                                   |
| $x > 2$  | 7          | +                                   |

$$\therefore \text{When } g'(x) = 0, x = -\sqrt{\frac{12}{5}}, x = \sqrt{\frac{12}{5}}$$

$$\therefore \text{Increasing } (-\infty, -2), (2, \infty), (-2, -\sqrt{\frac{12}{5}}), (\sqrt{\frac{12}{5}}, 2)$$

$$\text{Decreasing } (-\sqrt{\frac{12}{5}}, \sqrt{\frac{12}{5}})$$