

# Vector Spaces

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## CHAPTER 2

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# Chapter 1

## Vector Spaces and Subspaces

### 1.1 Introduction

#### Definition 1.1.1: Vector Space

A *vector space* is a non empty set  $V$  of objects, called vectors, on which are defined two operations, addition and multiplication by scalars, e.g. real numbers, subject to the following axioms which must hold for all vectors  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  in  $V$  and for all scalars  $c$  and  $d$ .

1. The sum of  $\mathbf{u}$  and  $\mathbf{v}$ , denoted by  $\mathbf{u} + \mathbf{v}$ , is in  $V$ .
2.  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3.  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is a zero vector  $\mathbf{0}$  in  $V$  such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$
5. For each  $\mathbf{u}$  in  $V$ , there is a vector  $-\mathbf{u}$  in  $V$  such that  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
6. The scalar multiple of  $\mathbf{u}$  by  $c$ , denoted by  $c\mathbf{u}$ , is in  $V$ .
7.  $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8.  $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9.  $c(d\mathbf{u}) = (cd)\mathbf{u}$
10.  $1\mathbf{u} = \mathbf{u}$

Using these axioms one can show that the zero vector in axiom 4 is unique, and the vector  $-\mathbf{u}$ , called the **negative** of  $\mathbf{u}$  in axiom 5 is unique for each  $\mathbf{u}$  in  $V$ , outlined in:

#### Theorem 1.1.1

$$0\mathbf{u} = \mathbf{0} \quad (1.1)$$

$$c\mathbf{0} = \mathbf{0} \quad (1.2)$$

$$-\mathbf{u} = (-1)\mathbf{u} \quad (1.3)$$

### 1.2 Subspaces

In many problems, a vector space consists of an appropriate set of vectors from a larger vector space. In this case only, three of the ten axioms need to be checked to determine if the subset is a vector space, the rest are satisfied automatically.

### Definition 1.2.1: Subspace

A subset  $H$  of the vector space  $V$ , where:

1. The zero vector of  $V$  is in  $H$ .
2.  $H$  is closed under vector addition. That is for each  $\mathbf{u}$  and  $\mathbf{v}$  in  $H$ , the sum of  $\mathbf{u} + \mathbf{v}$  is in  $H$ .
3.  $H$  is closed under scalar multiplication. That is for each  $\mathbf{u}$  in  $H$  and each scalar  $c$ , the scalar multiple  $c\mathbf{u}$  is in  $H$ .

These properties guarantee that a subspace  $H$  of  $V$  is also a vector space, under the defined vector space operations. This means that every subspace is a vector space and conversely every vector space is a subspace (of itself and possibly of a larger vector space).

### Example 1.2.1

#### Question 1

The vector space  $\mathbb{R}^2$  is not a vector space of  $\mathbb{R}^3$  because  $\mathbb{R}^2$  is not even a subset of  $\mathbb{R}^3$ . The set

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real} \right\}$$

is a subset of  $\mathbb{R}^3$  that "looks" and "acts" like  $\mathbb{R}^2$  even though it is logically distinct from  $\mathbb{R}^2$ . Show that  $H$  is subset of  $\mathbb{R}^3$

**Solution:**

- The zero vector is in  $H$
- $H$  is closed under vector addition and scalar multiplication as these operations on vectors in  $H$  always produce vectors whose third entry is zero and thus belong to  $H$ .

Thus  $H$  is as subspace of  $\mathbb{R}^3$

### 1.2.1 Subspace Spanned by a Set

One way of describing a subspace is as a linear combination of vectors that span the subspace.

### Example 1.2.2

#### Question 2

Given  $\mathbf{v}_1$  and  $\mathbf{v}_2$  in a vector space  $V$ , let  $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ . Show that  $H$  is a subspace of  $V$

**Solution:**

- The zero vector is in  $H$  as:

$$\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2$$

- To show that  $H$  is closed under vector addition and scalar multiplication, take two arbitrary vectors in  $H$ , say

$$\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 \text{ and } \mathbf{w} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$$

By axioms 2, 3, and 8 for the vector space  $V$ :

$$\begin{aligned} \mathbf{u} + \mathbf{w} &= (s_1\mathbf{v}_1 + s_2\mathbf{v}_2) + (t_1\mathbf{v}_1 + t_2\mathbf{v}_2) \\ &= (s_1 + t_1)\mathbf{v}_1 + (s_2 + t_2)\mathbf{v}_2 \end{aligned}$$

The result is still in  $H$  as it can still be spanned from  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ , with weights  $(s_1 + t_1)$  and  $(s_2 + t_2)$

Furthermore:

$$\begin{aligned}c\mathbf{u} &= c(s_1\mathbf{v}_1 + s_2\mathbf{v}_2) \\ &= (cs_1)\mathbf{v}_1 + (cs_2)\mathbf{v}_2\end{aligned}$$

therefore  $H$  is also closed under scalar multiplication.

### Theorem 1.2.1

If  $\mathbf{v}_1, \dots, \mathbf{v}_p$  are in vector space  $V$ , then  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is a subspace of  $V$ .

We can call  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  **the subspace spanned by**  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ . Therefore given any subspace  $H$  of  $V$ , a **spanning set** for  $H$  is a set  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  in  $H$  such that  $H = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ .

### Example 1.2.3

#### Question 3

Let  $H$  be the set of all vectors of the form  $(a - 3b, b - a, a, b)$ , where  $a$  and  $b$  are arbitrary scalars. That is let  $H = \{(a - 3b, b - a, a, b) : a \text{ and } b \text{ in } \mathbb{R}\}$ . Show that  $H$  is a subspace of  $\mathbb{R}^4$

**Solution:**

$$\begin{aligned}H &= \begin{bmatrix} a - 3b \\ b - a \\ a \\ b \end{bmatrix} \\ &= a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \\ &= a\mathbf{v}_1 + b\mathbf{v}_2\end{aligned}$$

Thus  $H$  is a subspace of  $\mathbb{R}^4$  by theorem 1.2.1

### Example 1.2.4

#### Question 4

For what value(s) of  $h$  will  $\mathbf{y}$  be in the subspace of  $\mathbb{R}^3$  spanned by  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ , if

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

**Solution:** The subspace of  $\mathbb{R}^3$   $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ .  $\mathbf{y}$  will be in the subspace if the span of  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  contains  $\mathbf{y}$ , that is if  $\mathbf{y}$  can be written

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{y}$$

And thus the matrix equation:

$$A\mathbf{x} = \mathbf{y}$$

Where  $A = \begin{bmatrix} 1 & 5 & -3 \\ -1 & -4 & 1 \\ -2 & -7 & 0 \end{bmatrix}$  and  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix}$$

$$-R_1 - R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ -2 & -7 & 0 & h \end{bmatrix}$$

$$-2R_2 + R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & -3 & 6 & 8-h \end{bmatrix}$$

$$-3R_2 - R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & -5+h \end{bmatrix}$$

$\therefore$  The system  $A\mathbf{x} = \mathbf{y}$  is only consistent if  $h = 5$ , and thus  $\mathbf{y}$  is in the subspace spanned by  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  if and only if  $h = 5$

### 1.3 Exercises

#### Question 5

Show that the set  $H$  of all points in  $\mathbb{R}^2$  of the form  $(3s, 2 + 5s)$  is not a vector space, by showing that it is not closed under scalar multiplication. (Find a specific vector  $\mathbf{u}$  in  $H$  and a scalar  $c$  such that  $c\mathbf{u}$  is not in  $H$ )

**Solution:** Let  $\mathbf{u} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$  and  $c = 2$ . Then:

$$\begin{aligned} 2\mathbf{u} &= 2 \begin{pmatrix} 3 \\ 7 \end{pmatrix} \\ &= \begin{bmatrix} 6 \\ 14 \end{bmatrix} \end{aligned}$$

This implies there is some  $s$  such that  $\begin{bmatrix} 3s \\ 2 + 5s \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$ , but for this to be true  $s$  would need to be equal to 2 and 2.4 which is impossible. Therefore  $H$  is not closed under scalar multiplication and thus is not a vector space.

#### Question 6

Let  $W = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ , where  $\mathbf{v}_1, \dots, \mathbf{v}_p$  are in a vector space  $V$ . Show that  $\mathbf{v}_k$  is in  $W$  for  $1 \leq k \leq p$ .

**Solution:** If  $W = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ , then the contents of  $W$  for example  $\mathbf{v}_1$  can be written as linear combination of the spanned vectors, that is:

$$\mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + \dots + 0\mathbf{v}_p$$

Therefore if  $1 \leq k \leq p$ , then  $\mathbf{v}_k$  is in  $W$  because:

$$\mathbf{v}_k = 0\mathbf{v}_1 + \dots + 0\mathbf{v}_{k-1} + 1\mathbf{v}_k + 0\mathbf{v}_{k+1} + \dots + 0\mathbf{v}_p$$

### Question 7

An  $n \times n$  matrix  $A$  is said to be *symmetric* if  $A = A^T$ . Let  $S$  be the set of all  $3 \times 3$  symmetric matrices. Show that  $S$  is a subspace of  $M_{3 \times 3}$ , the vector space of all  $3 \times 3$  matrices.

**Solution:** To prove that  $S$  is a subspace of  $M_{3 \times 3}$ , I must show:

**The Zero vector Is in  $S$**  Since the zero vector is symmetric  $S$  contains the zero vector as:

$$\mathbf{0} = \mathbf{0}^T$$

**$S$  is closed under vector addition** Let  $A$  and  $B$  be in  $S$ , hence  $A = A^T$  and  $B = B^T$

$$\begin{aligned}(A + B)^T &= A^T + B^T \\ &= A + B\end{aligned}$$

Thus  $A + B$  is symmetric and is in  $S$

**$S$  is closed under scalar multiplication** Let  $A$  be in  $S$  and  $c$  be a scalar

$$\begin{aligned}(cA)^T &= c(A)^T \\ &= cA\end{aligned}$$

Thus  $cA$  is symmetric and is in  $S$

$\therefore S$  is a subspace of  $M_{3 \times 3}$

### Question 8

Let  $V$  be the first quadrant in the  $xy$ -plane; that is, let

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0 \text{ and } y \geq 0 \right\}$$

1. If  $\mathbf{u}$  and  $\mathbf{v}$  are in  $V$ , is  $\mathbf{u} + \mathbf{v}$  in  $V$ ? Why?
2. Find a specific vector  $\mathbf{u}$  in  $V$  and specific scalar  $c$  such that  $c\mathbf{u}$  is not in  $V$ .

**Solution:**

1. If  $\mathbf{u}$  and  $\mathbf{v}$  are in  $V$ , then indeed  $\mathbf{u} + \mathbf{v}$  are in  $V$ , because the sum of these two vectors will always have positive  $x$  and  $y$  components and will therefore always be in the first quadrant of the  $xy$ -plane.

2. For  $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$  and  $c = -2$

$$\begin{aligned}-2\mathbf{u} &= -2 \left( \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right) \\ &= \begin{bmatrix} -6 \\ -8 \end{bmatrix}\end{aligned}$$

### Question 9

Determine if the given sets are subspaces of  $\mathbb{P}_n$  for an appropriate value of  $n$ . Justify your answers.

1. All polynomials in the form  $\mathbf{p}(t) = at^2$ , where  $a \in \mathbb{R}$ .
2. All polynomials in the form  $\mathbf{p}(t) = a + t^2$ , where  $a \in \mathbb{R}$ .
3. All polynomials of degree at most 3, with integers as coefficients.
4. All polynomials in  $\mathbb{P}_n$  such that  $\mathbf{p}(0) = 0$ .

**Solution:**

1. Yes this is a subspace of  $\mathbb{P}_n$  as:

**Contains the zero vector** When  $a = 0$ ,  $\mathbf{p}(t) = 0t^2 = 0$ .

**Closed under vector addition** Let  $\mathbf{w}$  and  $\mathbf{q}$  be polynomials in the appropriate form

$$\begin{aligned}\mathbf{w} + \mathbf{q} &= (wt^2) + (qt^2) \\ &= (w + q)t^2\end{aligned}$$

$$\begin{aligned}\text{Let } w + q &= a, \text{ then} \\ &= at^2\end{aligned}$$

**Closed under scalar multiplication** Let  $\mathbf{w}$  be a polynomial in the appropriate form and  $c$  be a scalar.

$$\begin{aligned}c\mathbf{w} &= c(wt^2) \\ &= (cw)t^2\end{aligned}$$

$$\begin{aligned}\text{Let } cw &= a, \text{ then} \\ &= at^2\end{aligned}$$



## Chapter 2

# Null Space, Column Space, and Linear Transformations

### 2.1 The Null Space of a Matrix

#### Definition 2.1.1: Null Space

The *null space* of an  $m \times n$  matrix  $A$ , denoted by  $\text{Nul } A$ , is the set of all solutions of the homogeneous equation  $A\mathbf{x} = \mathbf{0}$ . In set notation:

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

#### Example 2.1.1

##### Question 10

Let  $A$  be the matrix  $\begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ , and let  $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ . Determine if  $\mathbf{u}$  belongs to the null space of  $A$ .

**Solution:** This is basically asking us to verify if  $\mathbf{u}$  satisfies the equation  $A\mathbf{u} = \mathbf{0}$

$$\begin{aligned} \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 5 - 9 + 4 \\ -25 + 27 - 2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

$\therefore \mathbf{u}$  is in the null space of  $A$ .

#### Theorem 2.1.1

The null space of an  $m \times n$  matrix  $A$  is a subspace of  $\mathbb{R}^n$ , equivalently, the set of all solutions to a system  $A\mathbf{x} = \mathbf{0}$  of  $m$  homogeneous linear equations in  $n$  unknowns is a subspace of  $\mathbb{R}^n$ .

### 2.1.1 An Explicit Description of the Null Space of a Matrix

There is no obvious relation between the vectors in  $\text{Nul } A$  and the entries  $A$ . We say that  $\text{Nul } A$  is defined implicitly, as it is defined by a condition that must be checked. However solving the equation  $A\mathbf{x} = \mathbf{0}$  amounts to producing an explicit description of  $\text{Nul } A$ .

#### Example 2.1.2

##### Question 11

Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

**Solution:** The first step is to find the general solution of  $A\mathbf{x} = \mathbf{0}$  in terms of free variables. Therefore:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 2x_2 - x_4 + 3x_5 = 0$$

$$x_3 + 2x_4 - 2x_5 = 0$$

$$x_2 = x_2$$

$$x_4 = x_4$$

$$x_5 = x_5$$

$$x_1 = 2x_2 + x_4 - 3x_5$$

$$x_2 = x_2$$

$$x_3 = -2x_4 + 2x_5$$

$$x_4 = x_4$$

$$x_5 = x_5$$

$$\mathbf{x} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = x_2 \mathbf{u} + x_4 \mathbf{v} + x_5 \mathbf{w}$$

Every linear combination of  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  is an element of  $\text{Nul } A$  and vice versa. Thus  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$  is a spanning set for  $\text{Nul } A$ .

Two points are made apparent by the previous example:

1. The spanning set produced by the general solution of  $A\mathbf{x} = \mathbf{0}$  is automatically linearly independent because the free variables are weights on the spanning vectors.
2. When  $\text{Nul } A$  contains non-zero vectors, the number of vectors in the spanning set for  $\text{Nul } A$  equals the number of free variables in the equation  $A\mathbf{x} = \mathbf{0}$

## 2.2 The Column Space of a Matrix

### Definition 2.2.1: The Column Space of a Matrix

The column space of an  $m \times n$  matrix  $A$ , denoted by  $\text{Col } A$ , is the set of all linear combinations of the columns of  $A$ . If  $A = [\mathbf{a}_1 \ \dots \ \mathbf{a}_n]$ , then

$$\text{Col } A = \text{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$$

Since  $\text{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$  is a subspace, by theorem 1.2.1, the next theorem follows from the definition of  $\text{Col } A$  and the fact that the columns of  $A$  are in  $\mathbb{R}^m$ .

### Theorem 2.2.1

The column space of an  $m \times n$  matrix  $A$  is a subspace of  $\mathbb{R}^m$ .

### Example 2.2.1

#### Question 12

Find a matrix  $A$  such that  $W = \text{Col } A$ .

$$W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

#### Solution:

We first write  $W$  as a set of linear combinations:

$$W = \left\{ a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Then we create a matrix  $A$  with these columns:

$$\begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$$

### Theorem 2.2.2

The column space of an  $m \times n$  matrix  $A$  is all of  $\mathbb{R}^m$  if and only if the equation  $A\mathbf{x} = \mathbf{b}$  has a solution for each  $\mathbf{b}$  in  $\mathbb{R}^m$ .