Preliminaries

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Chapter 1

$\mathbb{R}^n$

$\mathbb{R}^n$ denotes the set of real numbers / scalars. If n is a positive integer then $\mathbb{R}^n$ is defined to be the set of all sequences $\mathbf{x}$ of n real numbers

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

Multivariable calculus studies functions that act on these sets, functions in the form of

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

or more accurately

$$f : A \to \mathbb{R}^m$$

Where A is a subset of $\mathbb{R}^n$.

1.1 Vector Arithmetic

Every vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ can be decomposed as a sum along the coordinate directions

$$\begin{aligned}\mathbf{x} &= (x_1, 0, \ldots, 0) + (0, x_2, 0, \ldots, 0) + (0, \ldots, x_n) \\ &= x_1 (1, 0, \ldots, 0) + x_2 (0, 1, 0, \ldots, 0) + x_n (0, \ldots, 1)\end{aligned}$$

The vectors $\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, \ldots, 0)$, $\mathbf{e}_n = (0, \ldots, 1)$, with a 1 in a single component corresponding to the value of n and zeros everywhere else, are called the **standard basis vectors**. I.e.:

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_n\mathbf{e}_n$$

Where the scalar coefficients x_i are the coordinates of $\mathbf{x}$.

1.2 Linear Transformations

Definition 1.2.1: Linear Transformation

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if:

- $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$
- $T(c\mathbf{x}) = cT(\mathbf{x})$

$\forall c \in \mathbb{R}$ and $\forall \mathbf{x} \in \mathbb{R}^n \wedge \forall \mathbf{y} \in \mathbb{R}^n$

1.3 The Matrix of a linear Transformation

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Thus:

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_n\mathbf{e}_n) \\ &= x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) + \ldots + x_n T(\mathbf{e}_n) \\ &= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n \end{aligned}$$

Where $\mathbf{a}_j = T(\mathbf{e}_j)$ for $j = 1, 2, \ldots, n$. A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is completely determined by the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$

Definition 1.3.1: Dot Product

Given $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, the dot product, denoted by $\mathbf{x} \cdot \mathbf{y}$, is defined by:

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \ldots + x_ny_n$$

We have shown the every real valued linear transformation, $T : \mathbb{R}^n \longrightarrow \mathbb{R}$ has the form

$$T(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$$

Where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ in $\mathbb{R}^n$. Generalizing for the case $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, the objects in $\mathbf{a}$ are now vectors in $\mathbb{R}^m$, i.e. $\mathbf{a}_j = T(\mathbf{e}_j)$, $\mathbf{a}$ then becomes the matrix A, thus we have the form:

$$T(\mathbf{x}) = A\mathbf{x}$$

Example 1.3.1

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the counter-clockwise rotation by $\frac{\pi}{3}$ about the origin. Then T rotates the vector $\mathbf{e}_1 = (1, 0)$ to the vector on the unit circle that makes an angle of $\frac{\pi}{3}$ with the positive x_1-axis. That is $T(\mathbf{e}_1) = \left(\cos(\frac{\pi}{3}), \sin(\frac{\pi}{3})\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Similarly, $T(\mathbf{e}_2) = \left(\cos(\frac{5\pi}{6}), \sin(\frac{5\pi}{6})\right) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Hence the matrix of T with respect to the standard bases is:

$$A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

Example 1.3.2

If $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is the projection of the $x_1x_2x_3$ space onto the x_1x_2-plane, then $T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0)$, $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1)$, and $T(\mathbf{e}_3) = T(0, 0, 0) = (0, 0)$, therefore:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

1.4 The geometry of the dot product

Definition 1.4.1: Norm / Magnitude

Denoted by $\|\mathbf{x}\|$ is defined as:

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

Proposition 1.1 |

$\mathbf{x} \in \mathbb{R}^n$, then $\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$

Proof. Both sides equal $x_1^2 + x_2^2 + \ldots + x_n^2$ 😄

Examining these notions in $\mathbb{R}^2$, If $\mathbf{x} = (x_1, x_2)$, then $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$. By the Pythagorean theorem, this is the length of the hypotenuse of a right triangle with legs $|x_1|$ and $|x_2|$. If we think of $\mathbf{x}$ as an arrow originating from the origin, then $\|\mathbf{x}\|$ is the length of the arrow, if instead we think of $\mathbf{x}$ as point $\|\mathbf{x}\|$ is the distance from the point to the origin.

Given two points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$, the distance between them is the length of the arrow that connects them $\vec{\mathbf{yx}} = \mathbf{x} - \mathbf{y}$, Hence:

$$\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$$