Question 1

Find the derivatives of the given functions.

1.
$$y = \sin^{-1}(\sqrt{2})$$

2.
$$y = \tan^{-1}(\pi x)$$

3.
$$y = e^x \tan^{-1}(x)$$

4.
$$f(x) = \ln(\sin^{-1}(x))$$

5.
$$y = \tan^{-1}(\ln(x)) + \pi$$

6.
$$h(x) = \frac{\sin^{-1}(x)}{1+x}$$

7.
$$y = \sin(x) + x^2 \tan^{-1}(x)$$

8.
$$f(t) = 4\cos^{-1}(t) - 10\tan^{-1}(t)$$

9.
$$h(t) = \frac{t}{1-e^t}$$

10.
$$f(x) = \frac{1+5x}{\ln(x)}$$

Solution:

1.
$$y = \sin^{-1}(\sqrt{2x})$$
$$y = \sin^{-1}(2x^{\frac{1}{2}})$$
$$y' = \frac{2x^{-\frac{1}{2}}}{\sqrt{1-2x}}$$

2.
$$y = \tan^{-1}(\pi x)$$
$$y' = \frac{\pi}{1+\pi^2 x^2}$$

3.
$$y = e^x \tan^{-1}(x)$$
$$y' = e^x(\tan^{-1}(x)) + (\frac{1}{1+x^2})e^x$$
$$y' = e^x(\tan^{-1}(x)) + (\frac{1}{1+x^2})e^x$$

4.

$$f(x) = \ln(\sin^{-1}(x))$$

$$f(x) = \frac{\frac{1}{\sqrt{1+x^2}}}{\sin^{-1}(x)}$$

5.

$$y = \tan^{-1}(\ln(x)) + \pi$$

$$y' = \frac{\frac{1}{x}}{1 + (\ln(x))^2}$$

6.

$$h(x) = \frac{\sin^{-1}(x)}{1+x}$$

$$h'(x) = \frac{(1+x)(\frac{1}{\sqrt{1-x^2}}) - (1)(\sin^{-1}(x))}{(1+x)^2}$$

7.

$$y = \sin(x) + x^2 \tan^{-1}(x)$$

$$y' = \cos(x) + 2x \tan^{-1}(x) + \frac{x^2}{1+x^2}$$

8.

$$f(t) = 4\cos^{-1}(t) - 10\tan^{-1}(t)$$

$$f'(t) = 4(-\frac{1}{\sqrt{1-t^2}}) - 10(\frac{1}{1+t^2})$$

$$f'(t) = -\frac{4}{\sqrt{1-t^2}} - \frac{10}{1+t^2}$$

9.

$$h(t) = \frac{t}{1-e^t}$$

$$h'(t) = \frac{(1)(1-e^t) - (t)(-e^t)}{(1-e^t)^2}$$

$$h' = \frac{1 - e^t - te^t}{(1-e^t)^2}$$

10.

$$f(x) = \frac{1+5x}{\ln(x)}$$

$$f'(x) = \frac{(\ln(x))(5) - (\frac{1}{x})(1+5x)}{(\ln(x))^2}$$

$$f'(x) = \frac{5\ln(x) - \frac{1+5x}{x}}{(\ln(x))^2}$$

Question 2

In 2009, the population of Hungary was approximated by $P = 9.906(0.997)^t$, where P is in millions and t is in years since 2009. Assume the trend continues.

- What does this model predict for the population of Hungary in the year 2020?
- How fast (in people/year) does this model predict Hungary's population will be changing in 2020?

Solution:

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$$\begin{aligned} P &= 9.906(0.997)^t \\ t &= 2020 - 2009 \\ t &= 11 \\ P &= 9.906(0.997)^{11} \\ P &= 9.584 \ \text{ million} \end{aligned}$$

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$$\begin{aligned} P' &= (0.997^t \times (1) \times \ln(0.997)) \times 9.906 \\ P' &= 9.906 \ln(0.997) \times 0.997^{11} \\ P' &= -0.0288 \ \text{ million} \end{aligned}$$

Question 3

Find the equation of the tangent line to the graph of $y = \ln(x)\log_2(x)$ at $x = 2$

Solution:

$$\begin{aligned} y &= \ln(x)\log_2(x) \\ y' &= \frac{1}{x}\log_2(x) + \ln(x)\frac{1}{x\ln(2)} \end{aligned}$$

$$y = \ln(2)\log_2(2)$$

$$\begin{aligned} y' &= \frac{\log_2(x)}{x} + \frac{\ln(x)}{x\ln(2)} \\ y' &= \frac{1}{2} + \frac{1}{2} \\ y' &= 1 \end{aligned}$$

$$\begin{aligned} \ln(2)\log_2(2) &= 2 + c \\ \ln(2)\log_2(2) - 2 &= c \\ y &= x + (\ln(2)\log_2(2) - 2) \end{aligned}$$

Question 4

Find the tangent line to $f(x) = 7^x + 4e^x$ at $x = 0$

Solution:

$$f(x) = 7^x + 4e^x$$
$$f'(x) = 7^x \ln(7) + 4e^x$$
$$f'(0) = 7^0 \ln(7) + 4e^0$$
$$f'(0) = 1 \ln(7) + 4$$
$$f'(0) = \ln(7) + 4$$

$$y = 5$$

$$5 = 0(\ln(7) + 4) + c$$
$$5 = c$$
$$y = (\ln(7) + 4)x + 5$$

Question 5

The cost of producing a quantity, q, of a product is given by $C(q) = 1000 + 30e^{0.05q}$ dollars. Find the cost and the marginal cost when $q = 50$. Interpret these answers in economic terms.

Solution:

$$C(q) = 1000 + 30e^{0.05q}$$
$$C(50) = 1000 + 30e^{0.05(50)}$$
$$C(50) = 1365.47$$

$$C'(q) = \frac{3}{2}e^{0.05q}$$
$$C'(50) = \$18.27$$

$\therefore$ When the business produces 50 units of a good it's cost incurred is \$1365.47 and the rate at which the cost increases is \$18.27

Question 6

At a time t hours after it was administered, the concentration of a drug in the body is $f(t) = 27e^{-0.14t}$ mg/ml.

- What is the concentration 4 hours after it was administered?
- At what rate is the concentration changing at that time?

Solution:

- $$f(t) = 27e^{-0.14t}$$
  $$f(4) = 27e^{-0.14(4)}$$
  $$f(4) = 15.42 \ \frac{mg}{ml}$$

- $$f'(t) = -3.78e^{-0.14t}$$
  $$f'(4) = -2.16 \ \frac{mg}{h}$$

Question 7

For the cost function $C = 1000 + 300\ln(q)$ (in dollars), find the cost and marginal cost at a production level of 500. Interpret your answers in economic terms.

Solution:

$$C = 1000 + 300\ln(q)$$
$$C = 1000 + 300\ln(500)$$
$$C = 2864.38$$

$$C' = \frac{300}{q}$$
$$C' = \$0.6$$

$\therefore$ At a production level of 500 the business is incurring a cost of \$2864.38 and this cost is increasing at a rate of \$0.6

Question 8

Carbon-14 is a radioactive isotope used to date objects. If A_0 represents the initial amount of carbon-14 in the object, then the quantity remaining at time, t, in years, is $A(t) = A_0e^{-0.000121t}$. A tree, originally 185 micrograms of carbon-14, is now 500 years old. At what rate is the carbon-14 decaying now?

Solution:

$$A(t) = A_0e^{-0.000121t}$$
$$A'(t) = -0.000121A_0e^{-0.000121t}$$
$$A'(500) = -0.000121(185)e^{-0.000121(500)}$$
$$A'(500) = -0.0211 \ \frac{\mu g}{y}$$

Question 9

In 2009, the population of Mexico was 111 million and growing 1.13% annually, while the population of the US was 307 million and growing 0.975% annually. If we measure growth rates in people/year, which population was growing faster in 2009?

Solution:

$$M = 111 + 1.13x$$
$$U = 307 + 0.975x$$

$$M' = 1.13$$
$$U' = 0.975$$

$\therefore$ Mexico's population is growing faster than the US's population

Question 10

With t in years since January 1, 2010, the population P of Slim Chance is predicted by $P = 35000(0.98)^t$. At what rate will the population be changing on January 1, 2023?

Solution:

$$t = 2023 - 2010$$
$$t = 13$$

$$P = 35000(0.98)^{13}$$
$$P' = 35000 \times 0.98^{13} \times \ln(0.98)$$
$$P' = 543.772 \ \frac{people}{year}$$