Counting

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Contents

Chapter 1

Basics of Counting

1.1 Basic Counting Principles

1.1.1 Product Rule

Definition 1.1.1: Product Rule

This rule applies when a procedure is made up of separate tasks. Suppose that a procedure can be broken down into two tasks. If there are n_1 ways to do task 1 and for each of these ways of doing task 1, there are n_2 ways to do task 2, then there are $n_1 n_2$ ways to do the procedure.

If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets in the product of the number of elements of each set.
Therefore it follows that the product rule then becomes

$$| A_1 \times A_2 \times \ldots \times A_m | = | A_1 | \cdot | A_2 | \cdot \ldots \cdot | A_m |$$

Example 1.1.1

Question 1

A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees.

Solution: This procedure is made up of two tasks, assigning an office to Sanchez, then assigning an office to Patel. The first task can be done in 12 ways, and the second can be done in 11 since one office would be occupied. this comes to 12×11 ways

1.1.2 Sum Rule

Definition 1.1.2: Sum Rule

If a task can be done in either one of n_1 ways or in one of n_2 ways, where none of the set n_1 ways is the same of any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task

The sum rule can be phrased in terms of sets

$$| A \cup B | = | A | + | B | + \ldots + | A_m | \quad \text{as long as } A \text{ and } B \text{ are disjoint sets}$$

Or

$$|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| \text{ when } A_i \cap A_j = \emptyset \text{ for all } i, j$$

Example 1.1.2

Question 2

The mathematics depart must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 member of the mathematics faculty and 83 math majors and no one is both a faculty member and a student.

Solution:

$$37 + 83 = 120$$

Example 1.1.3

Question 3

How many bit strings are there of length 6 or less, not including the empty strings

Solution: First we add all the bit strings of lengths 6, 5, 4, 3, 2, 1.
To find the number of bit string of each length we use the product rule, i.e.

$$\sum_{i=1}^{6} 2^n = 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6$$
$$= 126$$

1.1.3 Subtraction Rule

Definition 1.1.3: Subtraction Rule

If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to both n_1 and n_2.

Example 1.1.4

Question 4

How many bit strings of length 8 either start with a 1 bit or end with the two bits 00

Solution: We can construct a bit string of length 8 that starts with 1 in 2^7 ways, and we can construct a bit string of length 8 that ends with the two bits 00 in 2^6 ways. We can do both in 2^5 ways
Using the subtraction rule the number of ways to construct a bit string of length 8 that either starts with a 1 or ends with the two bits 00 is:

$$2^7 + 2^6 - 2^5 = 160$$

1.2 Combining the sum and product rule

Example 1.2.1

Question 5

Count all passwords of length 6,7,or 8.
A character in a password can either be an upper-case letter or a digit
A password must contain at least 1 digit

Solution: Passwords of length 6 with either upper-case letter or digit - $(26+10)^6$
Minus number of passwords that are only made up letters - $(26+10)^6$
Times the number of digit orders - 10×6

$$(26+10)^6 - 26^6$$

Passwords of length 6 with either upper-case letter or digit - $(26+10)^7$
Minus number of passwords that are only made up letters - 26^7

$$(26+10)^7 - 26^7$$

Passwords of length 6 with either upper-case letter or digit - $(26+10)^8$
Minus number of passwords that are only made up letters - 26^8

$$(26+10)^8 - 26^8$$

$$(26+10)^6 - 26^6 + (26+10)^7 - 26^7 + (26+10)^8 - 26^8$$

Question 6

How many bit strings of length 8 start with a 1 or end with a 00

Solution:

$$2^7 + 2^6 - 2^5$$

Number of bit strings that start with 1 + Number of bit strings that end with 00 - Number of the intersection of both

1.3 The Pigeon-hole Principle

Definition 1.3.1: Pigeon-hole Principle

If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects

Corollary 1.3.1 A function f from a set with $k+1$ or more elements to a set with k elements is not one-to-one

1.3.1 Generalized Pigeon-hole Principle

Definition 1.3.2: Generalized Pigeon-hole Principle

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil \frac{N}{k} \rceil$ objects. i.e.

$$k\left(\left\lceil\frac{N}{k}-1\right\rceil\right) < k\left(\left(\frac{N}{k}+1\right)-1\right) = N$$

Example 1.3.1

Question 7

Among 100 people how many must be born in the same month

Solution:

$$\left\lceil\frac{100}{12}\right\rceil = 9$$

1.4 Permutations and Combinations

1.4.1 Permutations

Definition 1.4.1: Permutation

An arrangement of r objects from a set of n objects is called a permutation of n objects taken r at a time, where the order of the objects is important. The number of permutations of n objects taken r at a time is denoted by $P(n,r)$ and is given by

$$P(n,r) = \frac{n!}{(n-r)!}$$

Example 1.4.1

Question 8

How many permutations of the letters ABCDEFGH contain the string ABC

Solution: We treat the string ABC as a single object, then we have 6 objects to permute.

$$\begin{aligned} P(6,6) &= 6! \\ &= 720 \end{aligned}$$

1.4.2 Combinations

Definition 1.4.2: Combination

An arrangement of r objects from a set of n objects is called a combination of n objects taken r at a time, where the order of the objects is not important. The number of combinations of n objects taken r at a time is denoted by $C(n,r)$ and is given by

$$C(n,r) = \frac{n!}{r!\,(n-r)!}$$

1.4.2.0.1 Binomial Theorem

Theorem 1.4.1 Binomial Theorem

Let x and y be variables, and let n be a non-negative integer then

$$(x+y)^n = \sum_{k=0}^{n} C\left(n,k\right) x^{n-k} y^k$$

Example 1.4.2

Question 9

How many 2-combinations of the set $\{a,b,c,d\}$ are there.

Solution:

$$\begin{aligned} C\left(4,2\right) &= \frac{4!}{2!\,(4-2)!} \\ &= 6 \end{aligned}$$

Example 1.4.3

Question 10

Suppose that there are 9 faculty members in the mathematics department. How many ways are there to select a committee and 11 in the computer science department. How many ways are there to develop a discrete mathematics course at a school if the committee is to consist of three faculty member from the mathematics department and four from the computer science?

Solution: Using the product rule, we can split the problem into two tasks, selecting the committee from the math department and selecting the committee from the computer science department, giving us

$$\begin{aligned} C\left(9,3\right) \times C\left(11,4\right) &= \frac{9!}{(9-3)! \times 3!} \times \frac{11!}{(11-4)! \times 4!} \\ &= 27,720 \end{aligned}$$

1.5 Exercises

Question 11

There are four major auto routes from Boston to Detroit and six from Detroit to Los Angeles. How many major auto routes are there from Boston to Los Angeles via Detroit.

Solution:

Using the product rule we can split the trips into two steps, Boston to Detroit (4) and Detroit to Los Angeles (6), giving us

$$4 \times 6 = 24$$

Question 12

1. How many different three-letter initials can people have
2. How many different three letter initials with none of the letters repeated can people have

Solution:

1. $26^3 = 17576$
2. $26 \times 25 \times 24 = 15600$

Question 13

1. How many bit strings of length ten both begin and end with a 1
2. How many bit strings of length n, where n is a positive integer, start and end with 1s

Solution:

1. 2^8
2. 2^{n-2} where $n \geqslant 2$ and when $n = 1$, there is one bit string

Question 14

How many stings are there of four letters that have the letter x in them

Solution:

$$26^4 - 25^4 = 66351$$

Question 15

Suppose that a password for a computer system must have at least 8 but no more 12, characters where each character in the password is a lower-case English letter, an upper-case English letter, a digit, or one of the six special characters, *, ¿, ¡, !, + and =

1. How many different passwords are available for this computer system
2. How many of these passwords contain at least one occurrence of at least one of the six special characters.
3. Using your answer to part 1., determine how long it takes a hacker to try every possible password assuming that it takes one nanosecond for a hacker to check each possible password

Solution:

1. The possible number of ways to choose one character is $26 + 26 + 10 + 6 = 68$

$$\begin{aligned} 8 &: 68^8 \\ 9 &: 68^9 \\ 10 &: 68^{10} \\ 11 &: 68^{10} \\ 12 &: 68^{12} \\ &\sum_{12}^{i=8} 68^i \end{aligned}$$

2.

$$\sum_{12}^{i=8} 68^i - \sum_{12}^{i=8} 62^i$$

Question 16

Show that if there are 30 students in a class, then at least two have last names that begin with the same letter

Solution: $N = 30$ objects to be placed into $k = 26$ boxes
According to the generalized pigeon hole principle:

$$\left\lceil \frac{30}{26} \right\rceil = 2$$

$\therefore$ at least 2

Question 17

Show that there are at least siz people in California (population: 37 million) with the same three initials who were born on the same day of the year (but not necessarily in the same year). Assume that everyone has three initials

Solution: $N = 37$ million to be place into k boxes
$k = 26^3 \times 366$
According to the generalized pigeon hole principle:

$$\left\lceil \frac{37,000,000}{26^3 \times 366} \right\rceil = 6$$

$\therefore$ at least 6

Question 18

Let $S = \{1, 2, 3, 4, 5\}$
1. List all the 3-permutations of S
2. List all the 3-combinations of S

Solution:

1.

$$\begin{aligned} P(5,3) &= \frac{5!}{3!} \\ &= 60 \end{aligned}$$

2.

$$\begin{aligned}C(5,3) &= \frac{5!}{2! \times 3!} \\ &= 10\end{aligned}$$

$$\begin{aligned}&\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,5\}, \\ &\{1,3,4\},\{3,4,5\},\{2,4,5\}, \\ &\{1,4,5\},\{2,3,5\},\{2,3,4\}\end{aligned}$$

Question 19

A club has 25 members

1. How many ways are there to choose four members of the club to serve on an executive committee
2. How many ways are there to choose a president, vice-president, secretary, and treasurer of the club where no person can hold more than one office

Solution:

1.

$$\begin{aligned}C(25,4) &= \frac{25!}{4! \times (25-4)!} \\ &= 12,650\end{aligned}$$

2.

$$25 \times 24 \times 23 \times 22 = 303,600$$

Question 20

Seven women and nine men are on the faculty in the mathematics department of a school

1. How many ways are there to select a committee of five members of the department if at least one woman must be on the committee
2. How many ways are there to select a committee of five members of the department if at least one woman and at least one man must be on the committee.

Solution:

1. First we find the number of all the possible ways a five member committee can be selected, i.e. $C(16,5)$, then we find the number of ways a committee can be selected consisting of all men, i.e. $C(9,5)$

$$C(16,5) - C(9,5) = 4242$$

2. First we find the number of all the possible ways a five member committee can be selected, i.e. $C(16,5)$, then we find the number of ways a committee can be selected consisting of all men and all women, $C(9,5)$ and $C(7,5)$, respectively

$$C(16,5) - (C(9,5) + C(7,5)) = 4221$$

Question 21

How many subsets of a set with 10 elements

1. Have fewer than 4 elements

2. Have more than 7 elements

Solution:

1.

$$C(10,4) + C(10,3) + C(10,2) + C(10,1) + C(10,0) = 386$$

Question 22

Find the expansion of $(x+y)^6$

Solution:

$$\begin{aligned}(x+y)^6 &= \sum_{i=0}^{6} C(6,i) \times x^{n-i} \times y^i \\ &= C(6,0)\,x^6 + C(6,1)\,x^5y^1 + C(6,2)\,x^4y^2 + C(6,3)\,x^3y^3 + C(6,4)\,x^2y^4 + C(6,5)\,x^1y^5 + C(6,6)\,y^6 \\ &= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6\end{aligned}$$

Chapter 2

Exercises

Question 23

Suppose you have 30 books (15 novels, 10 history books, and 5 math books). Assume that all 30 books are different.

1. In how many ways can you put the 30 books in a row on a shelf?
2. In how many ways can you get a bunch of four books to give to a friend?
3. In how many ways can you get a bunch of three history books and seven novels to give to a friend?
4. In how many ways can you put the 30 books in a row on a shelf if the novels are on the left, the math books are in the middle, and the history books are on the right?

Solution:

1. As each of the books are distinct the order matters, therefore using permutation:

$$\begin{aligned} P(30, 30) &= \frac{30!}{0!} \\ &= 30! \end{aligned}$$

2. As the order and type of book doesn't matter in this case, using combination:

$$\begin{aligned} C(30, 4) &= \frac{30!}{(30-4)!4!} \\ &= 27405 \end{aligned}$$

3. As we want 3 out of the 10 history books and 7 out of the 15 novels where the order doesn't matter using combinations and product rule:

$$\begin{aligned} C(10, 3) \times C(15, 7) &= \frac{10!}{(10-3)!3!} \times \frac{15!}{(15-7)!7!} \\ &= 120 \times 6435 \\ &= 772200 \end{aligned}$$

4. Breaking this task into 3 separate steps as this deals with the ordering of distinct items the order matters in each step, with the first choosing novels, the second choosing math books, and the third choosing history

books, using permutation and the product rule:

$$\begin{aligned} P(15,15) \times P(10,10) \times P(5,5) &= 15! \times 10! \times 5! \\ &= 15! \times 10! \times 5! \end{aligned}$$

Question 24

What is the minimum number of students, each of whom comes from one of the 50 states of the USA, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

Solution: Using the generalized Pigeon-hole principle we have our k as the number of states, 50 and we must find N such that $\lceil \frac{N}{50} \rceil$:

$$\begin{aligned} \left\lceil \frac{N}{50} \right\rceil &= 100 \\ 99 < \frac{N}{50} &\leqslant 100 \\ \\ \frac{N}{50} &> 99 \\ N &> 4950 \\ \frac{N}{50} \leqslant 100 N &\leqslant 5000 \\ \therefore 4950 < N &\leqslant 5000 \end{aligned}$$

Hence the minimum N is 4951

Question 25

1. Find the expansion of $(x+y)^6$
2. Find the coefficient of x^6y^9 in the expansion of $(3x-2y)^{15}$

Solution:

1.

$$\begin{aligned} (x+y)^6 &= C(6,0)\,x^6 + C(6,1)\,x^5y^1 + C(6,2)\,x^4y^2 + C(6,3)\,x^3y^3 + C(6,4)\,x^2y^4 + C(6,5)\,x^1y^5 + C(6,6)\,x^0y^6 \\ &= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6 \end{aligned}$$

2. Let a be the coefficient of x^6y^9

$$\begin{aligned} ax^6y^9 &= C(15,9)\,(3x)^6\,(-2y)^9 \\ &= 5005 \times 729x^6 \times -512y^9 \\ &= -5005 \times 729 \times 512x^6y^9 \\ a &= -5005 \times 729 \times 512 \end{aligned}$$