Eigenvalues and Eigenvectors

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Chapter 1

Introduction

Definition 1.0.1: Eigenvector

An eigenvector of a of a $n \times n$ matrix A is a non-zero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a non-trivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.

Example 1.0.1

Question 1

Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are $\mathbf{u}$ and $\mathbf{v}$ eigenvectors of A?

Solution:

$$
\begin{aligned}
A\mathbf{u} &= \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\mathbf{u} \\
A\mathbf{v} &= \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \begin{bmatrix} 3 \\ -2 \end{bmatrix}
\end{aligned}
$$

Question 2

Show that 7 is an eigenvalue of A and find the corresponding eigenvector.

Solution: To show this we need to prove that $A\mathbf{x} = \lambda\mathbf{x}$ where $\lambda = 7$ has non-trivial solutions.

$$\begin{aligned}
A\mathbf{x} &= \lambda\mathbf{x} \\
A\mathbf{x} &= 7\mathbf{x} \\
A\mathbf{x} - 7\mathbf{x} &= \mathbf{0} \\
(A - 7I)\,\mathbf{x} &= \mathbf{0} \\
\left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\mathbf{x} &= \mathbf{0} \\
\left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}\right)\mathbf{x} &= \mathbf{0} \\
\begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}\mathbf{x} &= \mathbf{0} \\
\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix}& \\
\frac{-5}{6}R_1 - R_2 &\rightarrow R_2 \\
\begin{bmatrix} -6 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}& \\
\frac{-1}{6}R_1 &\rightarrow R_1 \\
\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}& \\
x_1 - x_2 &= 0 \\
x_1 &= x_2 \\
x_2 &= x_2 \\
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}$$

This system has non-trivial solutions as the columns are multiples of themselves and such linearly dependent. Therefore 7 is a eigenvalue of A, with the corresponding eigenvectors in the form $x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where $x_2 \neq 0$

This brings us to the next conclusion:

A scalar λ is an eigenvalue of a matrix A if and only if

$$(A - \lambda I)\,\mathbf{x} = \mathbf{0} \tag{1.1}$$

Has a non-trivial solution, where the corresponding eigenvectors is in the form of the parametric vector equation of the solution set of this non-homogeneous system.

The set of all solutions of 1.1 is just the null space of the matrix $A - \lambda I$. This solution set is a subspace of $\mathbb{R}^n$ and is called the *eigenspace* of A corresponding to λ

Definition 1.0.2: Eigenspace

The eigenspace of a matrix A corresponding to an eigenvalue λ is the set of all eigenvectors of A corresponding to λ, together with the zero vector.

Example 1.0.2

Question 3

Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis of for the eigenspace of A corresponding to $\lambda = 2$.

Solution:

$$(A - 2I)$$

$$\left(\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right)$$

$$] \, [$$

1.0.1 Exercises

Question 4

Is $\lambda = 2$ an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$? Why or why not?

Solution:

$$\left(\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right)$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

$$3R_1 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -2x_2$$

$$x_2 = x_2$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

This columns of the matrix are linearly dependent therefore 2 is an eigenvalue of the matrix. And the eigenspace is the set of all vectors in the form $x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ where $x_2 \neq 0$, i.e:

$$\left\{ x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} : x_2 \in \mathbb{R} \wedge x_2 \neq 0 \right\}$$

Question 5

Is $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ and eigenvector of in $\begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}$? If so find the corresponding eigenvalue.

Solution:

$$\begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0 \begin{bmatrix} -4 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \lambda = 0$$

$\therefore \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ is an eigenvector of the matrix, with 0 as its eigenvalue.

Question 6

Is $\lambda = 4$ an eigenvalue of $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$? If so, find one corresponding eigenvector.

Solution:

$$(A - 4I) = \mathbf{0}$$

$$\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & -1 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 4 & 1 & 0 \end{bmatrix}$$

$$-2R_1 - R_2 \rightarrow R_2$$

$$\begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -3 & 4 & 1 & 0 \end{bmatrix}$$

$$3R_1 - R_3 \rightarrow R_3$$

$$\begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -4 & -4 & 0 \end{bmatrix}$$

$$-4R_2 - R_3 \rightarrow R_3$$

$$\begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-1R_1 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 + x_3 &= 0 \\ x_3 &= 0 \end{aligned}$$

$$\begin{aligned} x_1 &= -x_3 \\ x_2 &= -x_3 \\ x_3 &= x_3 \end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore$ Since the columns of $(A - 4I)$ are linearly dependent, 4 is an eigenvalue of the matrix A

One eigenvector is found when $x_3 = 1$, $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

Question 7

Find a basis for the eigenspace of $A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$ with eigenvalues $\lambda = 1, 5$

Solution:

$$(A - 1I)$$

$$\begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{2}R_1 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{4}R_1 \longrightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = 0$$

$$x_2 = x_2$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore \text{ the basis of the eigenspace of } A \text{ with } \lambda = 1 \text{ is } \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$(A - 5I)$$

$$\begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 2 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{2}R_1 \to R_1$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 2x_2 = 0$$

$$x_2 = x_2$$

$$x_1 = 2x_2$$

$$x_2 = x_2$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\therefore \text{ the basis of the eigenspace of } A \text{ with } \lambda = 5 \text{ is } \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

Chapter 2

The Characteristic Equation

Theorem 2.0.1

Let A be an $n \times n$ matrix. Then A is only invertible if and only if:

- The number 0 is not an eigenvalue of A
- The determinant of A is not zero

Therefore the updated properties of determinants are:

Theorem 2.0.2

A

1. A is invertible if and only if $\det A \neq 0$
2. $\det AB = (\det A)(\det B)$
3. $\det A^T = \det A$
4. If A is triangular, then $\det A$ is the product of the entries on the main diagonal of A
5. A row replacement operation on A does not change the determinant of A. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same factor.

Useful information about the eigenvalues of a square matrix A is found in a special scalar equation called the characteristic equation of A.

Question 8

Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

Solution: We must find all scalars λ such that the matrix equation

$$(A - \lambda I)\,\mathbf{x} = \mathbf{0}$$

Has the non-trivial solution. By the invertible matrix theorem, this is the same as finding all the scalars λ where the matrix $A - \lambda I$ is non-invertible, i.e. $\det(A - \lambda I) = 0$. Therefore

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\ &= \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \det(A - \lambda I) = 0 \therefore \\ \det\left(\begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}\right) = 0 \\ (2-\lambda)(-6-\lambda) - 9 = 0 \\ \lambda^2 + 4\lambda - 21 = 0 \\ (\lambda - 7)(\lambda - 3) = 0 \\ \lambda = 7 \\ \lambda = 3 \end{aligned}$$

Definition 2.0.1: Characteristic Equation

A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation

$$\det(A - \lambda I) = 0$$

2.1 Characteristic Polynomial

The characteristic polynomial of a matrix A is a polynomial of degree n in the variable λ, where n is the the size of the matrix A. The characteristic polynomial of A is defined as:

$$\lambda^n - (\text{trace}A)\,\lambda^{n-1} + (\text{trace}A)\,\lambda^{n-2} + \ldots + (-1)^n \det A$$

2.2 Similarity

Definition 2.2.1: Similarity

If A and B are $n \times n$ matrices, then A is similar to B if there is an invertible matrix P such that $P^{-1}AP = B$. If A is similar to B, then B is also similar to A, therefore A and B are similar.

Theorem 2.2.1

If $n \times n$ matrices of A and B are similar, then they have the same characteristic polynomial and hence have the same eigenvalues, with the same multiplicities. Therefore

$$\det(A - \lambda I) = \det(B - \lambda I)$$

Proof: If $B = P^{-1}AP$, Then

$$\begin{aligned} B - \lambda I &= P^{-1}AP - \lambda I \\ B - \lambda I &= P^{-1}AP - \lambda P^{-1}P \\ &= P^{-1}(AP - \lambda P) \\ &= P^{-1}(AP - \lambda PI) \\ &= P^{-1}(A - \lambda I)P \end{aligned}$$

The determinants of the two matrices are equal, then:

$$\begin{aligned} \det(B - \lambda I) &= \det\left(P^{-1}(A - \lambda I)P\right) \\ &= \det P^{-1} \det(A - \lambda I) \det P \\ &= \det P^{-1} \det P \det(A - \lambda I) \\ &= 1 \det(A - \lambda I) \\ \det(B - \lambda I) &= \det(A - \lambda I) \end{aligned}$$

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2.3 Exercises

Question 9

Find the characteristic polynomial and the eigenvalues of the matrices

1. $$\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$$

Solution:

1. $$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det\begin{bmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{bmatrix} &= 0 \\ (2-\lambda)^2 - 49 &= 0 \\ \lambda^2 - 4\lambda + 4 - 49 &= 0 \\ (\lambda - 9)(\lambda + 5) &= 0 \\ \lambda &= 9 \\ \lambda &= -4 \end{aligned}$$

Chapter 3

Diagonalization

In many cases the eigenvalue-eigenvector information contained in a matrix A can be displayed in the factorization $A = PDP^{-1}$, where D is a diagonal matrix. This makes it easy compute A^k for large values of k.

Example 3.0.1

If $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$, then:

$$D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \times \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}$$

$$D^3 = \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5^3 & 0 \\ 0 & 3^3 \end{bmatrix}$$

Therefore generally

$$D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}$$

Theorem 3.0.1 Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
$A = PDP^{-1}$, with the diagonal matrix D, if an only if the columns of P are n linearly independent eigenvectors of A. In this case the diagonal entries of D are the eigenvalues of A that correspond, respectively to the eigenvectors in P

Example 3.0.2

Question 10

Diagonalize the following matrix if possible:

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Solution: To do this we must complete the following steps:

1. Find the eigenvalues of A
2. Find three linearly independent eigenvectors of A
3. Construct P from the vectors found in step 2
4. Construct D from the eigenvalues found in step 1

   Therefore

1. $$\det(A - \lambda I) = 0$$

   Eigenvalues: $1, -2, -2$
2. $\lambda = -2$

$$A + 2I = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}$$

$$-1R_1 - R_2 \to R_2$$

$$\begin{bmatrix} 3 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix}$$

$$R_1 - R_3 \to R_3$$

$$\begin{bmatrix} 3 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{3}R_1 \to R_1$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -x_2 - x_3$$

$$x_2 = x_2$$

$$x_3 = x_2$$

$$\mathbf{x} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 1$

$$A - I = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} -3 & -6 & -3 & 0 \\ 0 & 3 & 3 & 0 \\ 3 & 3 & 0 & 0 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix}$$

$$-1R_1 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix} -3 & -6 & -3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix}$$

$$R_2 - R_3 \longrightarrow R_3$$

$$\begin{bmatrix} -3 & -6 & -3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-2R_2 - R_1 \longrightarrow R_1$$

$$\begin{bmatrix} 3 & 0 & -3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{3}R_1 \longrightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{3}R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = x_3$$

$$x_2 = -x_3$$

$$x_3 = x_3$$

$$\mathbf{x} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Our linearly dependent eigenvectors are therefore

$$\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\}$$

3. Therefore our P:

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

4. We start the matrix with the columns from the eigenspace from the repeated eigenvalue -2, therefore we must list the entries in D in the same order.

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In checking our answers we check if the sides of the equation below are equal

$$AP = PD$$

Theorem 3.0.2

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable

Example 3.0.3

Question 11

Determine if the following matrix is diagonalizable

$$A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$$

Solution:

3.1 Exercises

Question 12

Let $A = PDP^{-1}$ and compute A^4

$$P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution:

$$\begin{aligned} A &= PDP^{-1} \\ A^2 &= PDP^{-1} \times PDP^{-1} \\ P \times P^{-1} &= I \\ A^2 &= PD^2IP^{-1} \\ A^4 &= PD^2P^{-1} \times PD^2P^{-1} \\ &= PD^4P^{-1} \end{aligned}$$

Question 13

The matrix A is factored in the form PDP^{-1}, Use the Diagonalization Theorem to find the eigenvalues and the basis of each eigenspace

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & -\frac{3}{4} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

Solution:

Question 14

Diagonalize the matrices below

1. $$\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$$

Solution:

1.

$$\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$$

The eigenvalues are on the diagonal as this is a triangular matrix

$$\lambda = 1$$

$$A - I$$

$$\begin{bmatrix} 0 & 0 \\ 6 & -2 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 6 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{6}R_1 \rightarrow R_1$$

$$\begin{bmatrix} 1 & \frac{-1}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = \frac{1}{3}x_2$$

$$x_2 = x_2$$

$$\mathbf{x} = \begin{bmatrix} \frac{1}{3}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

$$\lambda = -1$$

$$A + I$$

$$\begin{bmatrix} 2 & 0 \\ 6 & 0 \end{bmatrix}$$

$$3R_1 - R_2 \rightarrow R_2$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{2}R_1 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = 0$$

$$x_2 = x_2$$

$$\mathbf{x} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{3} & 0 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$