Limits and Continuity

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Contents

Chapter 1

Preliminaries

1.1 The Real Number system

Definition 1.1.1: Rational Number

An integer that can be expressed in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Rational numbers have decimal expansions that either terminate or repeat.

Definition 1.1.2: Irrational Number

An integer that cannot be expressed in the form $\frac{p}{q}$. Irrational numbers do not have decimal expansions that terminate or repeat.

Theorem 1.1.1

If a and b are real numbers and $a < b$ then:

- For any real number c, $a + c < b + c$
- For real numbers c and d if $c < d$, then $a + c < b + d$
- For any real number $c > 0$, $a \times c < b \times c$
- For any real number $c < 0$, $a \times c > b \times c$

Theorem 1.1.2 Triangle of Inequality

For any real numbers a and b:

- $|a \times b| = |a| \times |b|$
- $|a + b| \neq |a| + |b|$
- $|a + b| \leqslant |a| + |b|$

Example 1.1.1

Question 1

Solve the inequality

$$|x - 2| < 5$$

Solution:

$$|x-2| < 5$$
$$x-2<5 \quad x-2>-5$$
$$x<7 \quad x>-3$$
$$\therefore -3>x>7$$

1.2 Lines and Functions

Definition 1.2.1: Slope of a Line

For $x_1 \neq x_2$, the slope of a straight line through the points (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

When $x_1 = x_2$ and $y_1 \neq y_2$, the line through (x_1, y_1) and (x_2, y_2) is vertical and its slope is undefined.

The slope of a line can also be described as the change in y, (Δy) divided by the change in x (Δx), or *Rise* over *Run* where rise it the change in height and run is the change in width in this case.

The equation of a line can be written in point slope form like:

$$y = m(x - x_0) + y_0$$

Where x_0 and y_0 are the starting coordinates of the line

Two (non-vertical) lines are parallel if they have the same slope, therefore any two vertical lines are parallel.
Two (non-vertical) lines of slope m_1 and m_2 are perpendicular if the product of their slopes is -1 this indicates that they are negative reciprocals of each other.

Definition 1.2.2: Function

For any two subsets A and B of the real line, a function f is a rule that assigns exactly one element y in set B to each element x in set A, where $y = f(x)$.

The set A is referred to as the *domain* of f, and the set of B is called the *range* of f, i.e. $\{y \mid y = f(x), \text{ for some } x \in A\}$

Definition 1.2.3: Polynomial

A function that can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers (coefficients) with $a_n \neq 0$ and $n \geqslant 0$ is an integer (degree)

Definition 1.2.4: Rational function

Any function that can be written in the form

$$f(x) = \frac{p(x)}{q(x)}$$

Where p and q are polynomials.

Theorem 1.2.1 Factor Theorem

For any polynomial function, f, $f(a) = 0$ if and only if $(x - a)$ is a factor of $f(x)$

1.3 Trigonometric Functions

Definition 1.3.1: Periodic Function

A function f is periodic of period T if

$$f(x + T) = f(x)$$

For all x such that x and $x+T$ are in the domain of of f. The smallest such number $T > 0$ is called the *fundamental period.*

$$\sin(x + \frac{\pi}{2}) = \cos(x)$$

Definition 1.3.2

Tangent

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

Cotangent

$$\cot(x) = \frac{\cos(x)}{\sin(x)} \quad / \quad \cot(x) = \frac{1}{\tan(x)}$$

Secant

$$\sec(x) = \frac{1}{\cos(x)}$$

Cosecant

$$\csc(x) = \frac{1}{\sin(x)}$$

Definition 1.3.3: Proprieties of a periodic function

For a periodic function in the form

$$y = Af(cx)$$

A is the amplitude of the values generated, and for any positive integer c, the period of the function f is $\frac{T}{c}$, where T is the period of the function f.

The frequency of a period function can be found:

$$f = \frac{c}{T}$$

Theorem 1.3.1 Trigonometric Identities

For any real numbers α and β, the following identities hold:

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha)\\
\cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)\\
\sin^2(\alpha) &= \frac{1}{2}(1-2\cos(2\alpha))\\
\cos^2(\alpha) &= \frac{1}{2}(1+\cos(2\alpha))
\end{aligned}$$

1.4 Transformations of Functions

Definition 1.4.1: Function Combinations

Suppose that f and g are functions with domains D_1 and D_2, respectively. The functions $f+g$, $f-g$ and $f\times g$ are defined by:

$$\begin{aligned}
(f+g)(x) &= f(x)+g(x)\\
(f-g)(x) &= f(x)-g(x)\\
(f\times g)(x) &= f(x)\times g(x)
\end{aligned}$$

$\forall x$ where $x\in D_1\cap D_2$. The function $\frac{f}{g}$ is defined by:

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$$

$\forall x$ where $x\in D_1\cap D_2$ such that $g(x)\neq 0$

Definition 1.4.2: Composition

The composition of functions f and g is written $f\circ g$ and is defined by

$$(f\circ g)(x)=f(g(x))$$

$\forall x$ where x is in the domain of g and $g(x)$ is in the domain of f

Chapter 2

The Concept of Limit