

Question 1

Find the derivatives of the given functions.

1.

$$y = \sin^{-1}(\sqrt{2x})$$

2.

$$y = \tan^{-1}(\pi x)$$

3.

$$y = e^x \tan^{-1}(x)$$

4.

$$f(x) = \ln(\sin^{-1}(x))$$

5.

$$y = \tan^{-1}(\ln(x)) + \pi$$

6.

$$h(x) = \frac{\sin^{-1}(x)}{1+x}$$

7.

$$y = \sin(x) + x^2 \tan^{-1}(x)$$

8.

$$f(t) = 4 \cos^{-1}(t) - 10 \tan^{-1}(t)$$

9.

$$h(t) = \frac{t}{1-e^t}$$

10.

$$f(x) = \frac{1+5x}{\ln(x)}$$

Solution:

1.

$$y = \sin^{-1}(\sqrt{2x})$$

$$y = \sin^{-1}(2x^{\frac{1}{2}})$$

$$y' = \frac{2x^{-\frac{1}{2}}}{\sqrt{1-2x}}$$

2.

$$y = \tan^{-1}(\pi x)$$

$$y' = \frac{\pi}{1+\pi^2 x^2}$$

3.

$$y = e^x \tan^{-1}(x)$$

$$y' = e^x(\tan^{-1}(x)) + \left(\frac{1}{1+x^2}\right)e^x$$

$$y' = e^x(\tan^{-1}(x)) + \left(\frac{1}{1+x^2}\right)e^x$$

4.

$$f(x) = \ln(\sin^{-1}(x))$$

$$f'(x) = \frac{\frac{1}{\sqrt{1-x^2}}}{\sin^{-1}(x)}$$

5.

$$y = \tan^{-1}(\ln(x)) + \pi$$

$$y' = \frac{\frac{1}{x}}{1 + (\ln(x))^2}$$

6.

$$h(x) = \frac{\sin^{-1}(x)}{1+x}$$

$$h'(x) = \frac{(1+x)(\frac{1}{\sqrt{1-x^2}}) - (1)(\sin^{-1}(x))}{(1+x)^2}$$

7.

$$y = \sin(x) + x^2 \tan^{-1}(x)$$

$$y' = \cos(x) + 2x \tan^{-1}(x) + \frac{x^2}{1+x^2}$$

8.

$$f(t) = 4 \cos^{-1}(t) - 10 \tan^{-1}(t)$$

$$f'(t) = 4(-\frac{1}{\sqrt{1-t^2}}) - 10(\frac{1}{1+t^2})$$

$$f'(t) = -\frac{4}{\sqrt{1-t^2}} - \frac{10}{1+t^2}$$

9.

$$h(t) = \frac{t}{1-e^t}$$

$$h'(t) = \frac{(1)(1-e^t) - (t)(-e^t)}{(1-e^t)^2}$$

10.

$$f(x) = \frac{1+5x}{\ln(x)}$$

$$f'(x) = \frac{(\ln(x))(5) - (\frac{1}{x})(1+5x)}{(\ln(x))^2}$$

$$f'(x) = \frac{5 \ln(x) - \frac{1+5x}{x}}{(\ln(x))^2}$$

Question 2

In 2009, the population of Hungary was approximated by $P = 9.906(0.997)^t$, where P is in millions and t is in years since 2009. Assume the trend continues.

- What does this model predict for the population of Hungary in the year 2020?
- How fast (in people/year) does this model predict Hungary's population will be changing in 2020?

Solution:

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$$P = 9.906(0.997)^t$$

$$t = 2020 - 2009$$

$$t = 11$$

$$P = 9.906(0.997)^{11}$$

$$P = 9.584$$

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$$P' = (0.997^t \times (1) \times \ln(0.997)) \times 9.906$$

$$P' = 9.906 \ln(0.997) \times 0.997^{11}$$

$$P' = -0.0288$$

Question 3

Find the equation of the tangent line to the graph of $y = \ln(x) \log_2(x)$ at $x = 2$

Solution:

$$y = \ln(x) \log_2(x)$$

$$y' = \frac{1}{x} \log_2(x) + \ln(x) \frac{1}{x \ln(2)}$$

$$y = \ln(2) \log_2(2)$$

$$y' = \frac{\log_2(x)}{x} + \frac{\ln(x)}{x \ln(2)}$$

$$y' = \frac{1}{2} + \frac{1}{2}$$

$$y' = 1$$

$$\ln(2) \log_2(2) = 2 + c$$

$$\ln(2) \log_2(2) - 2 = c$$

$$y = x + (\ln(2) \log_2(2) - 2)$$

Question 4

Find the tangent line to $f(x) = 7^x + 4e^x$ at $x = 0$

Solution:

$$\begin{aligned}f(x) &= 7^x + 4e^x \\f'(x) &= 7^x \ln(7) + 4e^x \\f'(0) &= 7^0 \ln(7) + 4e^0 \\f'(0) &= 1 \ln(7) + 4 \\f'(0) &= \ln(7) + 4 \\y &= 5 \\5 &= 0(\ln(7) + 4) + c \\5 &= c \\y &= (\ln(7) + 4)x + 5\end{aligned}$$

Question 5

The cost of producing a quantity, q , of a product is given by $C(q) = 1000 + 30e^{0.05q}$ dollars. Find the cost and the marginal cost when $q = 50$. Interpret these answers in economic terms.

Solution:

$$\begin{aligned}C(q) &= 1000 + 30e^{0.05q} \\C(50) &= 1000 + 30e^{0.05(50)} \\C(50) &= 1365.47 \\C'(q) &= \frac{3}{2}e^{0.05q} \\C'(50) &= 18.27\end{aligned}$$

\therefore When the business produces 50 units of a good its cost incurred is \$1365.47 and the rate at which the cost increases is \$18.27

Question 6

At a time t hours after it was administered, the concentration of a drug in the body is $f(t) = 27e^{-0.14t}$ mg/ml.

- What is the concentration 4 hours after it was administered?
- At what rate is the concentration changing at that time?

Solution:

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$$\begin{aligned}f(t) &= 27e^{-0.14t} \\f(4) &= 27e^{-0.14(4)} \\f(4) &= 15.42 \frac{mg}{ml}\end{aligned}$$

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$$\begin{aligned}f'(t) &= -3.78e^{-0.14t} \\f'(4) &= -2.16 \frac{mg}{h}\end{aligned}$$

Question 7

For the cost function $C = 1000 + 300 \ln(q)$ (in dollars), find the cost and marginal cost at a production level of 500. Interpret your answers in economic terms.

Solution:

$$\begin{aligned}C &= 1000 + 300 \ln(q) \\C &= 1000 + 300 \ln(500) \\C &= 2864.38\end{aligned}$$

$$\begin{aligned}C' &= \frac{300}{q} \\C' &= 0.6\end{aligned}$$

\therefore At a production level of 500 the business is incurring a cost of \$2864.38 and this cost is increasing at a rate of \$0.6

Question 8

Carbon-14 is a radioactive isotope used to date objects. If A_0 represents the initial amount of carbon-14 in the object, then the quantity remaining at time, t , in years, is $A(t) = A_0 e^{-0.000121t}$. A tree, originally 185 micrograms of carbon-14, is now 500 years old. At what rate is the carbon-14 decaying now?

Solution:

$$\begin{aligned}A(t) &= A_0 e^{-0.000121t} \\A'(t) &= -0.000121 A_0 e^{-0.000121t} \\A'(500) &= -0.000121(185)e^{-0.000121(500)} \\A'(500) &= -0.0211 \frac{\mu g}{y}\end{aligned}$$

Question 9

In 2009, the population of Mexico was 111 million and growing 1.13% annually, while the population of the US was 307 million and growing 0.975% annually. If we measure growth rates in people/year, which population was growing faster in 2009?

Solution:

$$\begin{aligned}M &= 111 + 1.13x \\U &= 307 + 0.975x \\M' &= 1.13 \\U' &= 0.975\end{aligned}$$

\therefore Mexico's population is growing faster than the US's population

Question 10

With t in years since January 1, 2010, the population P of Slim Chance is predicted by $P = 35000(0.98)^t$. At what rate will the population be changing on January 1, 2023?

Solution:

$$t = 2023 - 2010$$

$$t = 13$$

$$P = 35000(0.98)^{13}$$

$$P' = 35000 \times 0.98^{13} \times \ln(0.98)$$

$$P' = 543.772 \frac{\text{people}}{\text{year}}$$