

Orthogonality and Least Squares

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Chapter 1

Inner Product, Length and Orthogonality

1.1 Inner Product

Definition 1.1.1: Inner / Dot Product

If \mathbf{u} and \mathbf{v} are vectors in \mathbb{R}^n , then we regard \mathbf{u} and \mathbf{v} as $n \times 1$ matrices. The transpose of \mathbf{u}^T is a $1 \times n$ matrix, and the matrix product $\mathbf{u}^T \mathbf{v}$ is a 1×1 matrix, a scalar. This scalar is called the *inner / dot product* of \mathbf{u} and \mathbf{v} which can also be referred to as:

$$\mathbf{u} \cdot \mathbf{v}$$

Which breaks down into:

$$\mathbf{u}^T \times \mathbf{v}$$

When $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, is then defined as:

$$\begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

Example 1.1.1

Question 1

Compute $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$ for $\mathbf{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$

Solution:

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= \mathbf{u}^T \times \mathbf{v} = \begin{bmatrix} 2 & -5 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix} \\ &= 3(2) + (-5)(2) + (-1)(-3) \\ &= -1\end{aligned}$$

$$\begin{aligned}\mathbf{v} \cdot \mathbf{u} &= \mathbf{v}^T \times \mathbf{u} = \begin{bmatrix} 3 & 2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} \\ &= 2(3) + 2(-5) + (-3)(-1) \\ &= -1\end{aligned}$$

Theorem 1.1.1 Axioms of Inner / Dot products

Let \mathbf{u} and \mathbf{v} , and \mathbf{w} be vectors in \mathbb{R}^n , and let c be a scalar. Then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
3. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$
4. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

1.2 Length of a Vector

Definition 1.2.1: Length of a Vector

If \mathbf{v} is in \mathbb{R}^n , with entries v_1, \dots, v_n , then the square root of $\mathbf{v} \cdot \mathbf{v}$ is defined because $\mathbf{v} \cdot \mathbf{v}$ is non-negative. Therefore the *length / norm* of \mathbf{v} is the non-negative scalar $\|\mathbf{v}\|$, defined:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2} \text{ and } \|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$$

And similarly for any scalar c , the length of $c\mathbf{v}$ is $|c|$ times the length of \mathbf{v} , i.e:

$$\|c\mathbf{v}\| = |c| \times \|\mathbf{v}\|$$

Definition 1.2.2: Unit Vector

A vector whose length is 1. If we divide a non zero vector by its length, i.e. multiply by $\frac{1}{\|\mathbf{v}\|}$, we obtain a unit vector \mathbf{u} . This process of creating a unit vector \mathbf{u} from \mathbf{v} can be called *normalizing* \mathbf{v} , and the resulting \mathbf{u} is in the same direction as \mathbf{v}

1.3 Distance in \mathbb{R}^n

Definition 1.3.1: Distance between two vectors

For \mathbf{u} and \mathbf{v} in \mathbb{R}^n , the *distance between \mathbf{u} and \mathbf{v}* , expressed as $\text{dist}(\mathbf{u}, \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$:

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

Then defined:

$$\begin{aligned}\text{dist}(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\| = \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})} \\ &= \sqrt{(u_1 - v_1)^2 + \dots + (u_n - v_n)^2}\end{aligned}$$

In \mathbb{R}^2 and \mathbb{R}^3 , this is basically the same as the Euclidean distance between two points.

Example 1.3.1

Question 2

Compute the distance between the vectors $\mathbf{u} = (7, 1)$ and $\mathbf{v} = (3, 2)$

Solution:

$$\begin{aligned}\text{dist}(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\| \\ &= \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})} \\ &= \sqrt{\begin{bmatrix} 4 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \end{bmatrix}} \\ &= \sqrt{4^2 + (-1)^2} \\ &= \sqrt{17}\end{aligned}$$

1.4 Orthogonal Vectors

Consider \mathbb{R}^2 and \mathbb{R}^3 and two lines through the origin determined by vectors \mathbf{u} and \mathbf{v} . These lines are geometrically perpendicular if and only if the distance from \mathbf{u} to \mathbf{v} is the same as the distance from \mathbf{u} to $-\mathbf{v}$. This is equivalent to saying the squares of the distances are the same. Therefore:

$$\begin{aligned}[\text{dist}(\mathbf{u}, -\mathbf{v})]^2 &= \|\mathbf{u} - (-\mathbf{v})\|^2 = \|\mathbf{u} + \mathbf{v}\|^2 \\ &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot (\mathbf{u} + \mathbf{v}) + \mathbf{v} \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u} \cdot \mathbf{v}\end{aligned}$$

And then $\text{dist}(\mathbf{u}, \mathbf{v})$:

$$[\text{dist}(\mathbf{u}, \mathbf{v})]^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}$$

This shows that the two squared distances are only equal if and only if $2\mathbf{u} \cdot \mathbf{v} = -2\mathbf{u} \cdot \mathbf{v}$, which happens if and only if $\mathbf{u} \cdot \mathbf{v} = 0$

Definition 1.4.1: Orthogonality

Two vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n are orthogonal, to each other, if $\mathbf{u} \cdot \mathbf{v} = 0$

This then confirms that the zero vector $\mathbf{0}$ is orthogonal to every vector in \mathbb{R}^n , since $\mathbf{0}^T \mathbf{v} = 0$ for every \mathbf{v} .

Theorem 1.4.1 The Pythagorean Theorem

If \mathbf{u} and \mathbf{v} are orthogonal vectors in \mathbb{R}^n , then:

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

1.5 Exercises

Question 3

Let $\mathbf{a} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. Compute $\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}$ and $\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a}$

Solution:

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (-2)(-3) + 1 \\ &= 7 \end{aligned}$$

$$\begin{aligned} \mathbf{a} \cdot \mathbf{a} &= (-2)^2 + 1 \\ &= 5 \end{aligned}$$

$$\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} = \frac{7}{5}$$

$$\begin{aligned} \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a} &= \frac{7}{5} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -2.8 \\ \frac{7}{5} \end{bmatrix} \end{aligned}$$

Question 4

Let $\mathbf{c} = \begin{bmatrix} \frac{4}{3} \\ -1 \\ \frac{2}{3} \end{bmatrix}$ and $\mathbf{d} = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}$.

1. Find a unit vector \mathbf{u} in the direction of \mathbf{c}
2. Show that \mathbf{d} is orthogonal to \mathbf{c} .
3. Use the results of parts (1) and (2) to explain why \mathbf{d} must be orthogonal to the unit vector \mathbf{u}

Solution:

1.

$$\begin{aligned}
 \|\mathbf{c}\| &= \sqrt{\mathbf{c} \cdot \mathbf{c}} \\
 &= \sqrt{\left(\frac{4}{3}\right)^2 + (-1)^2 + \left(\frac{2}{3}\right)^2} \\
 &= \frac{\sqrt{29}}{3} \\
 \mathbf{u} &= \frac{1}{\frac{\sqrt{29}}{3}} \begin{bmatrix} \frac{4}{3} \\ -1 \\ \frac{2}{3} \end{bmatrix} \\
 &= \frac{3\sqrt{29}}{29} \begin{bmatrix} \frac{4}{3} \\ -1 \\ \frac{2}{3} \end{bmatrix} \\
 &= \begin{bmatrix} \frac{4\sqrt{29}}{29} \\ \frac{3\sqrt{29}}{29} \\ \frac{2\sqrt{29}}{29} \end{bmatrix} \\
 \|\mathbf{u}\| &= \sqrt{\mathbf{u} \cdot \mathbf{u}} \\
 &= \sqrt{\left(\frac{4\sqrt{29}}{29}\right)^2 + \left(\frac{3\sqrt{29}}{29}\right)^2 + \left(\frac{2\sqrt{29}}{29}\right)^2} \\
 &= 1
 \end{aligned}$$

2. If \mathbf{d} is orthogonal to \mathbf{c} then $\mathbf{d} \cdot \mathbf{c} = 0$

$$\begin{aligned}
 \mathbf{d} \cdot \mathbf{c} &= \mathbf{d}^T \times \mathbf{c} \\
 &= \begin{bmatrix} 5 & 6 & -1 \end{bmatrix} \begin{bmatrix} \frac{4}{3} \\ -1 \\ \frac{2}{3} \end{bmatrix} \\
 &= 5\left(\frac{4}{3}\right) + 6(-1) - 1\left(\frac{2}{3}\right) \\
 &= \frac{20}{3} - 6 - \frac{2}{3} \\
 &= 0
 \end{aligned}$$

$\therefore \mathbf{c}$ and \mathbf{d} are orthogonal to each other.

3. \mathbf{d} is orthogonal to the unit vector \mathbf{u} because \mathbf{d} is orthogonal to \mathbf{c} of which \mathbf{u} is a scalar multiple of. I.e \mathbf{u} is in the form $k\mathbf{c}$ for some k and:

$$\mathbf{d} \cdot \mathbf{u} = \mathbf{d} \cdot (k\mathbf{c}) = k(\mathbf{d} \cdot \mathbf{c}) = k(0) = 0$$

Chapter 2

Orthogonal Sets

Definition 2.0.1: Orthogonal Set

If $S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthogonal set of non-zero vectors in \mathbb{R}^n , then S is linearly independent and hence is a basis for the subspace spanned by S .

Definition 2.0.2: Orthogonal Basis

An orthogonal basis for a subspace W of \mathbb{R}^n is a basis for W that is also an orthogonal set.

Theorem 2.0.1

Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of \mathbb{R}^n . For each \mathbf{y} in W , the weights in the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + \dots + c_p\mathbf{u}_p$$

are given by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \quad (j = 1, \dots, p)$$

Example 2.0.1

Question 5

The set $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -\frac{1}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix}$$

is an orthogonal basis for \mathbb{R}^3 . Express the vector $\mathbf{y} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as a linear combination of the vectors in S

Solution:

$$\begin{aligned} \mathbf{y} &= \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 + \frac{\mathbf{y} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3} \mathbf{u}_3 \\ &= \frac{11}{11} \mathbf{u}_1 + -\frac{12}{6} \mathbf{u}_2 + -\frac{\frac{33}{2}}{\frac{33}{2}} \mathbf{u}_3 \\ &= \mathbf{u}_1 - 2\mathbf{u}_2 - 2\mathbf{u}_3 \end{aligned}$$

Chapter 3

Orthogonal Projections

Definition 3.0.1: Orthogonal Projection

Let W be a subspace of \mathbb{R}^n , and let \mathbf{y} be in \mathbb{R}^n . The *orthogonal projection* of \mathbf{y} onto W , denoted $\text{proj}_W \mathbf{y}$, is the closest point in W to \mathbf{y} . This point is obtained by adding the orthogonal projection of \mathbf{y} onto the orthogonal complement of W to the orthogonal projection of \mathbf{y} onto W .

Example 3.0.1

Question 6

Let $\{\mathbf{u}_1, \dots, \mathbf{u}_5\}$ be an orthogonal basis for a subspace W of \mathbb{R}^5 , and let

$$\mathbf{y} = c_1 \mathbf{u}_1 + \dots + c_5 \mathbf{u}_5$$

Consider the subspace $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, and write \mathbf{y} as the sum of a vector \mathbf{z}_1 in W and a vector \mathbf{z}_2 in W^\perp .

Solution:

$$\mathbf{z}_1 = \mathbf{y} - \mathbf{z}_2$$

$$\mathbf{y} = \mathbf{z}_2 + \mathbf{z}_1$$

$$\mathbf{z}_1 = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2$$

$$\mathbf{z}_2 = c_3 \mathbf{u}_3 + c_4 \mathbf{u}_4 + c_5 \mathbf{u}_5$$

$$\mathbf{y} = \mathbf{z}_1 + \mathbf{z}_2 = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + c_3 \mathbf{u}_3 + c_4 \mathbf{u}_4 + c_5 \mathbf{u}_5$$

Theorem 3.0.1 The Orthogonal Decomposition Theorem

Let W be a subspace of \mathbb{R}^n . Then each \mathbf{y} in \mathbb{R}^n can be written uniquely in the form:

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}}$ is in W and \mathbf{z} is in W^\perp . In fact, if $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is any orthogonal basis of W , then:

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \dots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p} \mathbf{u}_p$$

and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$

Example 3.0.2

Question 7

Let $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Observe that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Write \mathbf{y} as the sum of a vector in W and a vector orthogonal to W

Solution: The orthogonal projection of \mathbf{y} onto W is:

$$\begin{aligned}\hat{\mathbf{y}} &= \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 \\ &= \frac{9}{30} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{5} \\ \frac{3}{2} \\ -\frac{1}{10} \end{bmatrix} + \begin{bmatrix} -1 \\ \frac{1}{2} \\ \frac{1}{6} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{2}{5} \\ 2 \\ \frac{1}{5} \end{bmatrix}\end{aligned}$$

So:

$$\begin{aligned}\mathbf{z} &= \mathbf{y} - \hat{\mathbf{y}} \\ &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -\frac{2}{5} \\ 2 \\ \frac{1}{5} \end{bmatrix} \\ &= \begin{bmatrix} \frac{7}{5} \\ 0 \\ \frac{14}{5} \end{bmatrix}\end{aligned}$$

\mathbf{z} is orthogonal to W due to 3, so \mathbf{y} can be expressed as:

$$\mathbf{y} = \begin{bmatrix} -\frac{2}{5} \\ 2 \\ \frac{1}{5} \end{bmatrix} + \begin{bmatrix} \frac{7}{5} \\ 0 \\ \frac{14}{5} \end{bmatrix}$$

3.1 Properties of Orthogonal Projections

If $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthogonal basis for W and if \mathbf{y} happens to be in W , then the formula for $\text{proj}_W \mathbf{y}$ is exactly the same as the representation of \mathbf{y} in terms of the basis. In this case, $\text{proj}_W \mathbf{y} = \mathbf{y}$.

Theorem 3.1.1

If \mathbf{y} is in $W = \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$, then $\text{proj}_W \mathbf{y} = \mathbf{y}$

This leads to the next theorem:

Theorem 3.1.2 The Best Approximation Theorem

Let W be a subspace of \mathbb{R}^n , let \mathbf{y} be any vector in \mathbb{R}^n , and let $\hat{\mathbf{y}}$ be an orthogonal projection of \mathbf{y} onto W . Then $\hat{\mathbf{y}}$ is the closest point in W to \mathbf{y} , in the sense that

$$\|\mathbf{y} - \hat{\mathbf{y}}\| < \|\mathbf{y} - \mathbf{v}\|$$

For all \mathbf{v} in W distinct from $\hat{\mathbf{y}}$

Definition 3.1.1: Orthonormality

A set of vectors is orthonormal if each of them are orthogonal to each other and have a length of 1.

Theorem 3.1.3 Orthonormal Basis

If $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an Orthonormal basis for a subspace W of \mathbb{R}^n , then

$$\text{proj}_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1) \mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2) \mathbf{u}_2 + \dots + (\mathbf{y} \cdot \mathbf{u}_p) \mathbf{u}_p$$

If $U = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \dots \quad \mathbf{u}_p]$, then

$$\text{proj}_W \mathbf{y} = UU^T \mathbf{y} \quad \forall \mathbf{y} \in \mathbb{R}^n$$

3.1.1 Exercises**Question 8**

The distance from a point \mathbf{y} in \mathbb{R}^n to a subspace W is defined as the distance from \mathbf{y} to the nearest point in W . Find the distance from \mathbf{y} to $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, where

$$\mathbf{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Solution:

$$\begin{aligned}\hat{\mathbf{y}} &= \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 \\&= \frac{15}{30} \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + -\frac{21}{6} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \\&= \begin{bmatrix} \frac{5}{2} \\ -1 \\ \frac{15}{30} \end{bmatrix} + \begin{bmatrix} -\frac{21}{6} \\ -7 \\ \frac{21}{6} \end{bmatrix} \\&= \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{z} &= \mathbf{y} - \hat{\mathbf{y}} \\&= \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix} - \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix} \\&= \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\|\mathbf{z}\| &= \sqrt{3^2 + 6^2} \\&= \sqrt{45}\end{aligned}$$

Chapter 4

The Gram-Schmidt Process

The Gram-Schmidt process is a simple algorithm for producing an orthogonal or orthonormal basis for any non-zero subspace of \mathbb{R}^n .

Example 4.0.1

Question 9

Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. Then $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is clearly linearly independent and thus a basis for a subspace W of \mathbb{R}^4 . Construct an orthogonal basis for W .

Solution:

Step 1 Let $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{Span}\{\mathbf{x}_1\} = \text{Span}\{\mathbf{v}_1\}$

Step 2 Let \mathbf{v}_2 be the vector produced by subtracting from \mathbf{x}_2 its projection onto the subspace W_1 . That is, let

$$\begin{aligned}\mathbf{v}_2 &= \mathbf{x}_2 - \text{proj}_{W_1} \mathbf{x}_2 \\ &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1\end{aligned}$$

Since $\mathbf{v}_1 = \mathbf{x}_1$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

\mathbf{v}_2 is the component of \mathbf{x}_2 orthogonal to \mathbf{x}_1 , and $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for the subspace W_2 spanned by \mathbf{x}_1 and \mathbf{x}_2

Step 2' (Optional) If possible scale \mathbf{v}_2 to simplify future calculations. Since \mathbf{v}_2 has fractional entries, it is convenient to scale it by a factor of 4 and replace $\{\mathbf{v}_1, \mathbf{v}_2\}$ by the orthogonal basis

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2' = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Step 3 Let \mathbf{v}_3 be the vector produced by subtracting from \mathbf{x}_3 its projection onto the subspace W_2 , using the

orthogonal basis $\{\mathbf{v}_1, \mathbf{v}'_2\}$ to compute this projection to W_2 :

$$\text{proj}_{W_2} \mathbf{x}_3 = \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 + \frac{\mathbf{x}_3 \cdot \mathbf{v}'_2}{\mathbf{v}'_2 \cdot \mathbf{v}'_2} \mathbf{v}'_2$$

$$\begin{aligned} \mathbf{x}_3 \cdot \mathbf{v}_1 &= 0 + 0 + 1 + 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_1 &= 1 + 1 + 1 + 1 \\ &= 4 \end{aligned}$$

$$\begin{aligned} \mathbf{x}_3 \cdot \mathbf{v}'_2 &= 0 + 0 + 1 + 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \mathbf{v}'_2 \cdot \mathbf{v}'_2 &= 9 + 1 + 1 + 1 \\ &= 12 \end{aligned}$$

$$= \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{2}{12} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{4} \\ \frac{2}{4} \\ \frac{2}{4} \\ \frac{2}{4} \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \text{proj}_{W_2} \mathbf{x}_3$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

Therefore the orthogonal basis for W is $\{\mathbf{v}_1, \mathbf{v}'_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}'_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

Theorem 4.0.1 The Gram-Schmidt Process

Given a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ for a non-zero subspace W of \mathbb{R}^n , define

$$\begin{aligned}\mathbf{v}_1 &= \mathbf{x}_1 \\ \mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 \\ \mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 \\ &\vdots \\ \mathbf{v}_p &= \mathbf{x}_p - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 - \dots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}} \mathbf{v}_{p-1}\end{aligned}$$

Then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an orthogonal basis for W . In addition

$$\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\} \quad \text{for } 1 \leq k \leq p$$

4.1 Orthonormal Bases

An orthonormal basis is constructed easily from an orthogonal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$, by normalizing all the \mathbf{v}_k .

Example 4.1.1

Given the constructed basis

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

An orthonormal basis is

$$\begin{aligned}\mathbf{u}_1 &= \frac{1}{\|\mathbf{v}_1\|} \mathbf{v}_1 = \frac{1}{\sqrt{45}} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix} \\ \mathbf{u}_2 &= \frac{1}{\|\mathbf{v}_2\|} \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\end{aligned}$$

4.1.1 Exercises

Question 10

Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$. Find an orthonormal basis for the column space of A .

Solution:

$$\begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

$$R_1 - R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & -5 & 6 & 0 \\ 1 & 4 & 2 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$R_1 - R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & -5 & 6 & 0 \\ 0 & -5 & 2 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$R_1 - R_4 \rightarrow R_4$$

$$\begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & -5 & 6 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$$R_2 - R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & -5 & 6 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$$R_3 - R_4 \rightarrow R_4$$

$$\begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & -5 & 6 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{5}R_2 - R_1 \rightarrow R_1$$

$$\begin{bmatrix} -1 & 0 & \frac{-14}{5} & 0 \\ 0 & -5 & 6 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{3}{2}R_3 - R_2 \rightarrow R_2$$

$$\begin{bmatrix} -1 & 0 & \frac{-14}{5} & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{-7}{10}R_3 - R_1 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{5}R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{4}R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Therefore Col } A = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} \right\}$$

Let $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{Span}\{\mathbf{v}_1\}$

$$\begin{aligned}
 \mathbf{v}_2 &= \mathbf{x}_2 - \text{proj}_{W_1} \mathbf{x}_2 \\
 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 \\
 &= \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} - \frac{6}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
 &= \begin{bmatrix} -\frac{5}{2} \\ \frac{5}{2} \\ \frac{5}{2} \\ -\frac{5}{2} \end{bmatrix} \\
 \mathbf{v}'_2 &= \begin{bmatrix} -5 \\ 5 \\ 5 \\ -5 \end{bmatrix}
 \end{aligned}$$

Now let $W_2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}'_2\}$

$$\begin{aligned}
 \mathbf{v}_3 &= \mathbf{x}_3 - \text{proj}_{W_2} \mathbf{x}_3 \\
 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}'_2}{\mathbf{v}'_2 \cdot \mathbf{v}'_2} \mathbf{v}'_2 \\
 &= \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \frac{4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left(-\frac{20}{100}\right) \begin{bmatrix} -5 \\ 5 \\ 5 \\ -5 \end{bmatrix} \\
 &= \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}
 \end{aligned}$$

Therefore the orthogonal basis for the column space of a is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 5 \\ 5 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix} \right\}$. The orthonormal basis is

$$\begin{aligned}
 \mathbf{u}_1 &= \frac{1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \\
 \mathbf{u}_2 &= \frac{1}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{100}} \begin{bmatrix} -5 \\ 5 \\ 5 \\ -5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \\
 \mathbf{u}_3 &= \frac{1}{\|\mathbf{v}_3\|} = \frac{1}{\sqrt{16}} \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}
 \end{aligned}$$

Therefore the orthonormal basis for the column space of A is $\left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \right\}$

Chapter 5

Exercises

Question 11

Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for \mathbb{R}^3 . Then express \mathbf{x} as a linear combination of the \mathbf{u} s

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \text{ and } \mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$$

Solution: For the basis to be orthogonal $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, $\mathbf{u}_2 \cdot \mathbf{u}_3 = 0$, and $\mathbf{u}_1 \cdot \mathbf{u}_3 = 0$.

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{u}_2 &= \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \\ &= 3(2) + (-3)(2) + 0(-1) \\ &= 6 - 6 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{u}_2 \cdot \mathbf{u}_3 &= \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \\ &= 2(1) + 2(1) + (-1)(4) \\ &= 2 + 2 - 4 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{u}_3 \cdot \mathbf{u}_1 &= \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \\ &= 1(3) + 1(-3) + 4(0) \\ &= 3 - 3 \\ &= 0 \end{aligned}$$

Therefore $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for \mathbb{R}^3 . To express \mathbf{x} as a linear combination of the \mathbf{u} s:

$$\mathbf{x} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + c_3 \mathbf{u}_3$$

$$\begin{bmatrix} 3 & 2 & 1 & 5 \\ -3 & 2 & 1 & -3 \\ 0 & -1 & 4 & 1 \end{bmatrix}$$

$$-1R_1 - R_2 \rightarrow R_2$$

$$\begin{bmatrix} 3 & 2 & 1 & 5 \\ 0 & -4 & -2 & -2 \\ 0 & -1 & 4 & 1 \end{bmatrix}$$

$$\frac{1}{4}R_2 - R_3 \rightarrow R_3$$

$$\begin{bmatrix} 3 & 2 & 1 & 5 \\ 0 & -4 & -2 & -2 \\ 0 & 0 & \frac{-9}{2} & \frac{-3}{2} \end{bmatrix}$$

$$\frac{-1}{2}R_2 - R_1 \rightarrow R_1$$

$$\begin{bmatrix} -3 & 0 & 0 & -4 \\ 0 & -4 & -2 & -2 \\ 0 & 0 & \frac{-9}{2} & \frac{-3}{2} \end{bmatrix}$$

$$\frac{4}{9}R_3 - R_2 \rightarrow R_2$$

$$\begin{bmatrix} -3 & 0 & 0 & -4 \\ 0 & 4 & 0 & \frac{4}{3} \\ 0 & 0 & \frac{-9}{2} & \frac{-3}{2} \end{bmatrix}$$

$$\frac{-1}{3}R_1 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 4 & 0 & \frac{4}{3} \\ 0 & 0 & \frac{-9}{2} & \frac{-3}{2} \end{bmatrix}$$

$$\frac{1}{4}R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{-9}{2} & \frac{-3}{2} \end{bmatrix}$$

$$\frac{-2}{9}R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

$$c_1 = \frac{4}{3}$$

$$c_2 = \frac{1}{3}$$

$$c_3 = \frac{1}{3}$$

$$\mathbf{x} = \frac{4}{3} \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

Question 12

Let W be the subspace spanned by the \mathbf{u} s, and write \mathbf{y} as the sum of a vector in W and a vector orthogonal to W .

$$\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

Solution: We can express \mathbf{y} as the sum of a vector in W and a vector orthogonal to W using the orthogonal decomposition theorem, finding the orthogonal projection of \mathbf{y} onto W , and subtracting that from \mathbf{y} to find the vector orthogonal to W .

$$\hat{\mathbf{y}} = \frac{0}{14} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \frac{28}{42} \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{10}{3} \\ \frac{10}{3} \\ \frac{10}{3} \end{bmatrix}$$

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$$

$$= \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} \frac{10}{3} \\ \frac{10}{3} \\ \frac{10}{3} \end{bmatrix}$$

$$\mathbf{z} = \begin{bmatrix} -\frac{7}{3} \\ -\frac{7}{3} \\ \frac{5}{3} \end{bmatrix}$$

$$\mathbf{y} = \mathbf{z} + \hat{\mathbf{y}}$$

$$= \begin{bmatrix} -\frac{7}{3} \\ -\frac{7}{3} \\ \frac{5}{3} \end{bmatrix} + \begin{bmatrix} \frac{10}{3} \\ \frac{10}{3} \\ \frac{10}{3} \end{bmatrix}$$