

# Boolean Algebra

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February 2024

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# Chapter 1

## Boolean Functions

### 1.1 Introduction

Boolean Algebra provides the operations and rules for working with the set  $\{0, 1\}$ . The three operations that will be discussed are the:

- Boolean sum (**OR**) -  $0 + 1 = 1$
- Boolean product (**AND**) -  $0 \cdot 1 = 0$
- Complementation (**NOT**) -  $\overline{0} = 1$

#### 1.1.1 Boolean Product (AND)

##### Definition 1.1.1: Boolean Product

The Boolean product of two variables  $x$  and  $y$  is denoted by  $x \cdot y$  and is defined by the following values:

$$1 \cdot 1 = 1$$

$$1 \cdot 0 = 0$$

$$0 \cdot 1 = 0$$

$$0 \cdot 0 = 0$$

#### 1.1.2 Boolean Sum (OR)

##### Definition 1.1.2: Boolean Sum

The Boolean sum of two variables  $x$  and  $y$  is denoted by  $x + y$  and is defined by the following values:

$$1 + 1 = 1$$

$$1 + 0 = 1$$

$$0 + 1 = 1$$

$$0 + 0 = 0$$

### 1.1.3 Complementation (NOT)

#### Definition 1.1.3: Complementation

The complement of a variable  $x$  is denoted by  $\bar{x}$  and is defined by the following values:

$$\bar{1} = 0$$

$$\bar{0} = 1$$

#### Example 1.1.1

##### Question 1

Find the value of  $1 \cdot 0 + \overline{(0 + 1)}$

**Solution:**

$$\begin{aligned} 1 \cdot 0 + \overline{(0 + 1)} &= 1 \cdot 0 + \bar{1} \\ &= 0 + \bar{1} \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

#### Example 1.1.2

##### Question 2

Translate  $1 \cdot 0 + \overline{(0 + 1)} = 0$ , into a logical equivalence.

**Solution:**

$$T \wedge F \vee \neg (F \vee T) \equiv F$$

## 1.2 Boolean Expressions and Functions

Let  $B = \{0, 1\}$ , then  $B^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in B \text{ for } 1 \leq i \leq n\}$  is the set of all possible  $n$ -tuples of 0's and 1's. The variable  $x$  is a *Boolean variable*.

#### Definition 1.2.1: Boolean variable

A variable that can take on the values 0 or 1.

#### Definition 1.2.2: Boolean Function

A function  $f : B^n \rightarrow B$  is called a *Boolean function* of degree  $n$ . I.e. takes  $n$  inputs and returns a single output.

##### Example 1.2.1

The function  $F(x, y) = x$  from the set of ordered pairs of Boolean variables to the set  $\{0, 1\}$ , has a degree of 2.

### 1.2.1 Complement of a Boolean function

#### Definition 1.2.3: Complement of a Boolean function

The complement of a Boolean function  $F$  is denoted by  $\overline{F}$  and is defined by:

$$\overline{F}(x_1, x_2, \dots, x_n) = \overline{F(x_1, x_2, \dots, x_n)}$$

## 1.3 Boolean Identities

### 1.3.1 Law of Double Complement

$$\overline{\overline{x}} = x$$

### 1.3.2 Idempotent Laws

$$x + x = x$$

$$x \cdot x = x$$

### 1.3.3 Identity Laws

$$x + 0 = x$$

$$x \cdot 1 = x$$

### 1.3.4 Domination Laws

$$x + 1 = 1$$

$$x \cdot 0 = 0$$

### 1.3.5 Commutative Laws

$$x + y = y + x$$

$$xy = yx$$

### 1.3.6 Associative Laws

$$x + (y + z) = (x + y) + z$$

$$x(yz) = (xy)z$$

### 1.3.7 Distributive Laws

$$x + yz = (x + y)(x + z)$$

$$x(y + z) = xy + xz$$

### 1.3.8 De Morgan's Laws

$$\overline{(xy)} = \overline{x} + \overline{y}$$

$$\overline{(x + y)} = \overline{x} \cdot \overline{y}$$

### 1.3.9 Absorption Laws

$$x + xy = x$$
$$x(x + y) = x$$

### 1.3.10 Unit Property

$$x + \bar{x} = 1$$

### 1.3.11 Zero Property

$$x\bar{x} = 0$$

## 1.4 Duality

### Definition 1.4.1: Dual

The dual of a Boolean expression is obtained by replacing the **AND** operation with **OR** and the **OR** operation with **AND**, and interchanging 1s and 0s.

### Example 1.4.1

#### Question 3

Find the duals of  $x(y + 0)$  and  $\bar{x} \cdot 1 + (\bar{y} + z)$

**Solution:**

$$x(y + 0) = x + (y \cdot 1)$$

$$\bar{x} \cdot 1 + (\bar{y} + z) = (\bar{x} + 0) \cdot (\bar{y}z)$$

The dual of a boolean function  $F$  is the function representing the dual of the expression representing  $F$ , denoted by  $F^d$

### Definition 1.4.2: Duality Principle

An identity between functions represented by boolean expressions remains valid when the duals of both sides of the expression are taken.

### Example 1.4.2

#### Question 4

Construct an identity from the absorption law  $x(x + y) = x$  by taking duals

**Solution:**

$$x(x + y) = x$$

Let  $F(x, y) = x(x + y)$  and  $G(x) = x$

$$F(x, y) = G(x)$$

$$F^d(x, y) = G^d(x)$$

$$F^d(x, y) = x + xy$$

$$G^d(x) = x$$

$$x + xy = x$$

## 1.5 Exercises

### Question 5

Show that these identities hold

1.  $x \oplus y = (x + y)(xy)$
2.  $x \oplus y = (x\bar{y}) + (\bar{x}y)$

**Solution:**

$x$	$y$	$\bar{x}$	$\bar{y}$	$x + y$	$\overline{(xy)}$	$x\bar{y}$	$\bar{x}y$	$x \oplus y$	$(x + y)\overline{(xy)}$	$(x\bar{y}) + (\bar{x}y)$
1	1	0	0	1	0	0	0	0	0	0
1	0	0	1	1	1	1	0	1	1	1
0	0	1	1	0	1	0	1	0	0	0
0	1	1	0	1	1	0	0	1	1	1

### Question 6

Show that you obtain De Morgan's Laws for propositions when you transform De Morgan's Laws for boolean algebra into logical equivalences

**Solution:**

$$\overline{(xy)} = \bar{x} + \bar{y}$$

Let  $x$  be  $p$  and  $y$  be  $q$

$$\neg(p \wedge q) = \neg p \vee \neg q$$

$$\overline{(x + y)} = \bar{x} \cdot \bar{y}$$

$$\neg(p \vee q) = \neg p \wedge \neg q$$

### Question 7

Show that in a boolean algebra, the Idempotent laws  $x \vee x = x$  and  $x \wedge x = x$  hold for every element  $x$

***Solution:***

$$x = x \vee 0$$

$$= x \vee (x \wedge \neg x)$$

$$= (x \vee x) \wedge (x \vee \neg x)$$

$$= (x \vee x) \wedge 1$$

$$= x \vee x$$

By First Identity Law

By Second Complement Law

By First Distributive Law

By First Complement Law

By Second Identity Law



## Chapter 2

# Representing Boolean Functions

### 2.1 Sum of Products Expansion

#### Definition 2.1.1: Literal

A variable or its complement.

#### Definition 2.1.2: Minterm

A product of literals in which each variable appears exactly once. I.e. the minterm of boolean variables  $x_1, x_2, \dots, x_n$  is a boolean product  $y_1 \cdot y_2 \cdot \dots \cdot y_n$ , where

$$y_i = x_i \text{ or } y_i = \overline{x_i}$$

I.e.  $y_1 \cdot y_2 \cdot \dots \cdot y_n$  is a minterm in of the variables  $x_1, x_2, \dots, x_n$

#### 2.1.0.1 Sum of Products / Disjunctive normal form (DNF)

Form a product (using logical and) term for each row in the truth table where the function is 1. Then sum (using logical or) all the terms together.

$x$	$y$	$z$	$F(x, y, z)$	$G(x, y, z)$
1	1	1	0	0
1	1	0	0	1
1	0	1	1	0
1	0	0	0	0
0	1	1	0	0
0	1	0	0	1
0	0	1	0	0
0	0	0	0	0

#### Example 2.1.1

##### Question 8

Find Boolean expressions that represent the functions, using the truth table above.

1.  $F(x, y, z)$
2.  $G(x, y, z)$

**Solution:**

1. First we look for the rows where  $F$  is 1. There is only one row, row 3. Then we determine the minterm for this row which is  $x\bar{y}z$ . Then we boolean sum all the found minterms to derive the function's boolean expression but since there is only one minterm the result is simply

$$F(x, y, z) = x\bar{y}z$$

2. We repeat the same process for the function  $G$ , and as there are two rows where  $G$  is 1 we will have two minterms,  $xy\bar{z}$  and  $\bar{x}y\bar{z}$ , making the boolean expression

$$G(x, y, z) = xy\bar{z} + \bar{x}y\bar{z}$$

**Example 2.1.2****Question 9**

Find the sum-of-products of the expansion for the function  $F(x, y, z) = (x + y)\bar{z}$

**Solution:**

$$\begin{aligned}
 F(x, y, z) &= (x + y)\bar{z} \\
 &= x\bar{z} + y\bar{z} && \text{By Second Distributive Law} \\
 &= x1\bar{z} + y1\bar{z} && \text{By Second Identity Law} \\
 &= x(y + \bar{y})\bar{z} + y(x + \bar{x})\bar{z} && \text{By First Unit Property Law} \\
 &= xy\bar{z} + x\bar{y}\bar{z} + xy\bar{z} + \bar{x}y\bar{z} && \text{By Second Distributive Law} \\
 &= xy\bar{z} + x\bar{y}\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z} && \text{By Second Commutative Law} \\
 &= xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z} && \text{By First Idempotent Law}
 \end{aligned}$$

$$\therefore F(x, y, z) = xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z}$$

**2.1.1 Product of Sums Expansion / Conductive Normal Form (CNF)**

A product of sums expansion is the dual of a sum of product expansion.

**Example 2.1.3**

$F(x, y, z) = xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z}$  can be expressed as a product of sums expansion

$$F(x, y, z) = (x + y + \bar{z}) \cdot (x + \bar{y} + \bar{z}) \cdot (\bar{x} + y + \bar{z})$$

**2.2 Exercises****Question 10**

Use truth tables to prove the domination laws for Boolean.

**Solution:** Conclusion:  $x + 1 = 1$  from column 2 and 4 and  $x \cdot 0 = 0$  from column 3 and 5.

$x$	1	0	$x + 1$	$x \cdot 0$
1	1	0	1	0
0	1	0	1	0

### Question 11

The Boolean operator  $\oplus$ , called **XOR** is defined by  $1 \oplus 1 = 0$ ,  $1 \oplus 0 = 1$

1.  $x \oplus x$
2.  $x \oplus \bar{x}$

**Solution:**

1.

$$x \oplus x$$

When  $x = 1$

$$1 \oplus 1 = 0$$

When  $x = 0$

$$0 \oplus 0 = 0$$

$$x \oplus x = 0$$

2.

$$x \oplus \bar{x}$$

When  $x = 1$

$$1 \oplus \bar{1}$$

$$1 \oplus 0 = 1$$

When  $x = 0$

$$0 \oplus \bar{0}$$

$$0 \oplus 1$$

$$0 \oplus 1 = 1$$

$$x \oplus \bar{x} = 1$$

### Question 12

Prove the absorption law  $x + xy = x$  using the other boolean identities

**Solution:**

$$\begin{aligned} x + xy &= x \cdot 1 + xy \\ &= x(1 + y) \\ &= x \cdot 1 \\ &= x \end{aligned}$$

By Second Identity Law  
By Second Distributive Law  
By First Domination Law  
By Second Identity Law

$$\begin{aligned}
 x(x+y) &= (x+0)(x+y) \\
 &= x+0 \cdot y \\
 &= x+0 \\
 &= x
 \end{aligned}$$

By First Identity Law  
 By First Distributive Law  
 By Second Domination Law  
 By First Identity Law

### Question 13

Find the sum of products expansion of these Boolean functions

1.  $F(x, y) = x + y$
2.  $F(x, y) = xy$
3.  $F(x, y) = 1$
4.  $F(x, y) = y$

**Solution:**

1.

$$\begin{aligned}
 F(x, y) &= x + y \\
 &= x \cdot 1 + y \cdot 1 \\
 &= x \cdot (y + \bar{y}) + y \cdot (x + \bar{x}) \\
 &= xy + x\bar{y} + xy + \bar{x}y \\
 &= xy + xy + x\bar{y} + \bar{x}y \\
 &= xy + x\bar{y} + \bar{x}y
 \end{aligned}$$

By Second Identity Law  
 By Unit Property  
 By Second Distributive Law  
 By First Commutative Law  
 By First Idempotent Law

2.

$$\begin{aligned}
 F(x, y) &= xy \\
 &= x(y + y) \\
 &= xy
 \end{aligned}$$

By First Idempotent Law

3.

$$\begin{aligned}
 F(x, y) &= 1 \\
 &= x + \bar{x} \\
 &= x \cdot 1 + \bar{x} \cdot 1 \\
 &= x \cdot (y + \bar{y}) + \bar{x} \cdot (y + \bar{y}) \\
 &= xy + x\bar{y} + \bar{x}y + \bar{x}\bar{y}
 \end{aligned}$$

By Unit Property  
 By Second Identity Law  
 By Unit Property  
 By Second Distributive Law

4.

$$\begin{aligned}
 F(x, y) &= y \\
 &= y + y && \text{By First Idempotent Law} \\
 &= y \cdot 1 + y \cdot 1 && \text{By Second Identity Law} \\
 &= y \cdot (x + 1) + y \cdot (x + 1) && \text{By First Domination Law} \\
 &= xy + y + xy + y && \text{By Second Distributive Law} \\
 &= xy + xy + y + y && \text{By First Commutative Law} \\
 &= xy + y && \text{By First Idempotent Law} \\
 &= xy + y \cdot 1 && \text{By Second Identity Law} \\
 &= xy + y \cdot (x + \bar{x}) && \text{By Unit Property Law} \\
 &= xy + xy + y\bar{x} && \text{By Second Distributive Law} \\
 &= xy + y\bar{x} && \text{By First Idempotent Law}
 \end{aligned}$$

#### Question 14

Find the sum of products and the product of sums expansion of the Boolean function  $F(x, y, z)$  that equals 1 if and only if

1.  $xy = 0$
2.  $x + y = 0$
3.  $xyz = 0$

**Solution:**

$x$	$y$	$z$	$xy$	$F(x, y, z)$	$x + y$	$F(x, y, z)$	$xyz$	$F(x, y, z)$
1	1	1	1	0	1	0	1	0
1	1	0	1	0	1	0	0	1
1	0	0	0	1	1	0	0	1
1	0	1	0	1	1	0	0	1
0	1	1	0	1	1	0	0	1
0	0	0	0	1	0	1	0	1
0	0	1	0	1	0	1	0	1
0	1	0	0	1	1	0	0	1

1. DNF -  $x\bar{y}\bar{z} + x\bar{y}z + \bar{x}yz + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}y\bar{z}$   
 CNF -  $(\bar{x} + \bar{y} + \bar{z}) \cdot (\bar{x} + \bar{y} + z)$
2. DNF -  $\bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$   
 CNF -  $(x + y + z) \cdot (x + y + \bar{z}) \cdot (x + \bar{y} + \bar{z}) \cdot (x + \bar{y} + z) \cdot (\bar{x} + y + z) \cdot (\bar{x} + y + \bar{z})$

## 2.3 Boolean Circuits

$$\overline{(\bar{x}yz)} (\bar{x} + y + \bar{z})$$

## Chapter 3

## Exercises

### Question 15

1. Use a truth table to express the values of the Boolean function  $F(x, y, z) = x + yz$
2. Using the result from (1) write  $F(x, y, z)$  in conjunctive normal form
3. Using the result from (1) write  $F(x, y, z)$  in disjunctive normal form
4. Using Boolean identities, write  $F(x, y, z)$  in disjunctive normal form

**Solution:**

$x$	$y$	$z$	$yz$	$x + yz$	$F(x, y, z)$
1	1	1	1	1	1
1	1	0	0	1	1
1	0	1	0	1	1
1	0	0	0	1	1
0	1	1	1	1	1
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0

2.

$$F(x, y, z) = (x + \bar{y} + z) \cdot (x + y + \bar{z}) \cdot (x + y + z)$$

3.

$$F(x, y, z) = xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz$$

4.

$$\begin{aligned}
F(x, y, z) &= x + yz \\
&= x1 + y1z && \text{Second Identity Law} \\
&= x \cdot (y + \bar{y}) + yz1 && \text{Unit Property} \\
&= x \cdot (y + \bar{y}) + yz \cdot (x + 1) && \text{First Domination Law} \\
&= xy + x\bar{y} + yzx + yz1 && \text{Second Distributive Law} \\
&= xy + x\bar{y} + yzx + yz \cdot (x + \bar{x}) && \text{Unit Property} \\
&= xy + x\bar{y} + yzx + yzx + yz\bar{x} && \text{Second Distributive Law} \\
&= xy1 + x\bar{y}1 + yzx + yzx + yz\bar{x} && \text{Second Identity Law} \\
&= xy \cdot (z + \bar{z}) + x\bar{y} \cdot (z + \bar{z}) + yzx + yzx + yz\bar{x} && \text{Unit Property} \\
&= xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + xyz + xyz + \bar{x}yz && \text{Second Distributive Law} \\
&= xyz + xyz + x\bar{y}z + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz && \text{First Commutative Law} \\
&= xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz
\end{aligned}$$

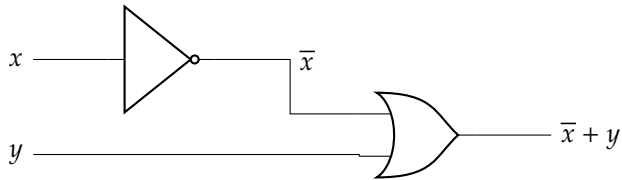
Hence  $F(x, y, z) = xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz$

### Question 16

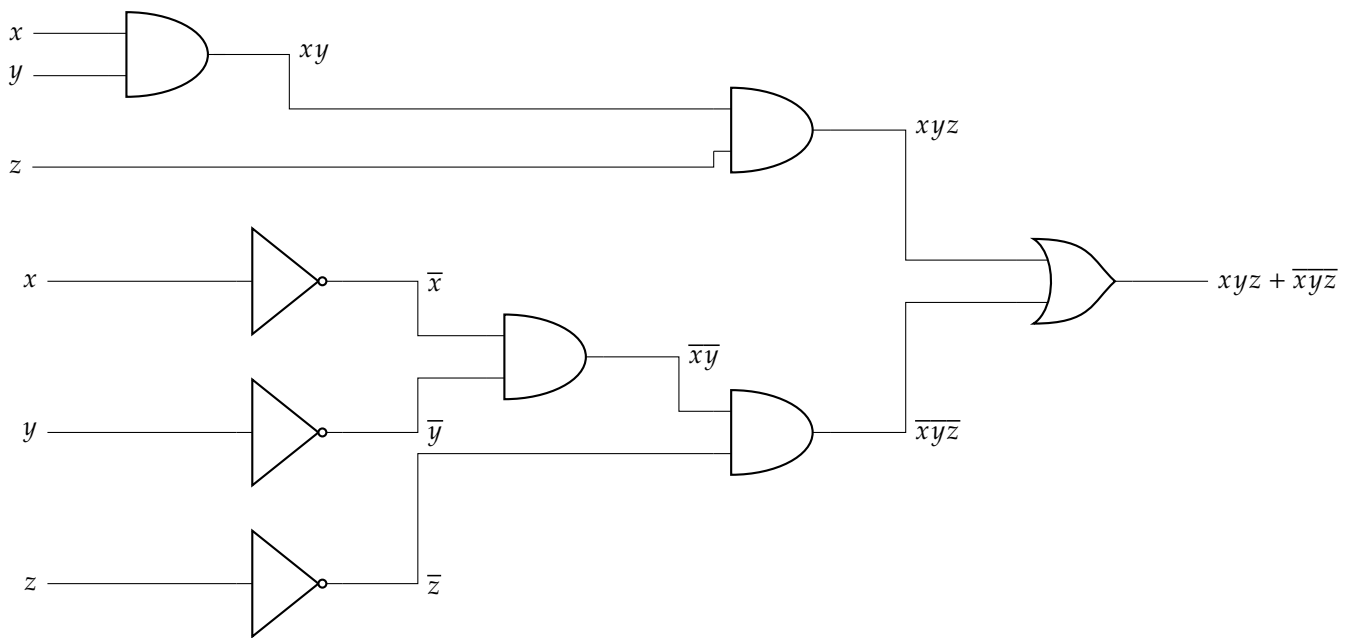
Construct circuits from inverters, AND gates, and OR gates to produce these outputs.

1.  $\bar{x} + y$
2.  $xyz + \bar{x}y\bar{z}$

**Solution:**



1.



2.