

# Trigonometry and Derivatives

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# Chapter 1

## 1.1 Sine

$$\begin{aligned}f(x) &= \sin(x) \\ -1 &\leq \sin(x) \leq 1 \\ \sin(0) &= 0\end{aligned}$$

For all integer multiples  $\pi$ ,  $\sin()$  attains 0

$$\sin(k\pi) = 0, \text{ Where } k \text{ is an integer}$$

The graph of sine is periodic with a period of  $2\pi$ , meaning it repeats itself every interval of  $2\pi$

### 1.1.1 Derivative of $\sin(x)$

**Definition 1.1.1:**  $\sin(x)$

if  $y = \sin(x)$

$$y' = \cos(x)$$

**Proof:**

$$\begin{aligned}
y' &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \frac{\sin(x+h) - \sin(x)}{h} \\
&\therefore \frac{\sin(x+h) - \sin(x)}{h} \\
&= \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
&= \frac{\sin(\cos(h) - 1) + \cos(x)\sin(h)}{h} \\
y' &= \lim_{h \rightarrow 0} \frac{\sin(\cos(h) - 1) + \cos(x)\sin(h)}{h} \\
y' &= \lim_{h \rightarrow 0} \frac{\sin(\cos(h) - 1)}{h} + \frac{\cos(x)\sin(h)}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sin(\cos(h) - 1)}{h} + \lim_{h \rightarrow 0} \frac{\cos(x)\sin(h)}{h} \\
&\therefore \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} (f(x) + g(x)) \\
&= \sin(x) \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h}
\end{aligned}$$

$$\therefore \lim_{x \rightarrow a} (kf(x)) = k \lim_{x \rightarrow a} f(x)$$

$$\begin{aligned}
&\lim_{h \rightarrow 0^-} \frac{\cos(h) - 1}{h} = 0 \\
&\lim_{h \rightarrow 0^+} \frac{\cos(h) - 1}{h} = 0 \\
&\therefore \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} = 0 \\
&= \sin(x) \times 0 + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\
&= \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\
&\lim_{h \rightarrow 0^-} \frac{\sin(h)}{h} = 1 \\
&\lim_{h \rightarrow 0^+} \frac{\sin(h)}{h} = 1 \\
&\therefore \lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1 \\
&\therefore \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} = \cos(x) \times 1
\end{aligned}$$

$$= \cos(x)$$

## 1.2 Cosine

$$\begin{aligned} f(x) &= \cos(x) \\ -1 &\leq \cos(x) \leq 1 \\ \cos(0) &= 1 \end{aligned}$$

## 1.3 Tangent

$$\begin{aligned} f(x) &= \tan(x) \\ -1 &\leq \tan(x) \leq 1 \\ \tan(0) &= 0 \end{aligned}$$

### 1.3.1 Vertical Asymptote

**Note:-**

A vertical line  $(x, 0)$  where the values of a function rise or fall infinitely

The line  $x = a$  is a vertical asymptote of  $f(x)$  if

$$\lim_{x \rightarrow a} f(x) = \pm\infty$$

The zero points of  $\cos(x)$  create a vertical asymptote in relation to  $\tan(x)$

### 1.3.2 Derivative of $\tan(x)$

**Definition 1.3.1:**  $\tan(x)$

if  $y = \tan(x)$

$$y' = \sec^2(x)$$

*Proof:*

$$y = \tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$y' = \frac{\cos(x)(\cos(x)) - \sin(x)(-\sin(x))}{(\cos(x))^2}$$

$$\because y' = \frac{v \times u' - u \times v'}{v^2}$$

$$y' = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$$y' = \frac{\cos^2(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)}$$

$$y' = 1 + \left(\frac{\sin(x)}{\cos(x)}\right)^2$$

$$y' = 1 + \tan^2(x)$$

$$= \sec^2(x)$$

⊙

## 1.4 Secant

$$f(x) = \sec(x)$$

$$1 \leq \sec(x) \leq 1$$

$$\sec(0) = 1$$

### 1.4.1 Derivative of $\sec(x)$

**Definition 1.4.1:**  $\sec(x)$

if  $y = \sec(x)$

$$y' = \sec(x) \tan(x)$$

# Chapter 2

## 2.1 Inverse Trigonometric Functions

In the case where

$$f[g(x)] = x$$

and

$$g[f(x)] = x$$

When can say the function  $f$  is the inverse of function  $g$ , due to the function  $g$  being able to extract the original input of the function  $f$ .

Therefore the derivative of the inverse function  $y = \sin^{-1}(x)$  is as follows:

$$\begin{aligned}\sin(y) &= x \\ \frac{d}{dx} \sin(y) &= \frac{d}{dx} x \\ \cos(y) \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos(y)}\end{aligned}$$

Using the identity  $\cos^2(y) + \sin^2(y) = 1$

$$\cos(x) = \sqrt{1 - \sin^2(x)}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2(y)}}$$

And since  $x = \sin(y)$

$$\sin^2(y) = x^2$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

**Example 2.1.1** (Find the derivative of  $\cos^{-1}(x)$ )

$$y = \cos^{-1}(x)$$

$$\cos(y) = x$$

$$-\sin(y) \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\sin(y)}$$

Using the identity  $\cos^2(y) + \sin^2(y) = 1$

$$\sin(y) = \sqrt{1 - \cos^2(y)}$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - \cos^2(y)}}$$

And since  $x = \cos(y)$

$$\cos^2(y) = x^2$$

$$= \frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$$

**Example 2.1.2** (Find the derivative of  $\tan^{-1}(x)$ )

$$\tan(y) = x$$

$$\sec^2(y) \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)}$$

From the identity  $1 + \tan^2(y) = \sec^2(y)$

$$\frac{dy}{dx} = \frac{1}{1 + \tan^2(y)}$$

And since  $x = \tan(y)$

$$\tan^2(y) = x^2$$

$$= \frac{dy}{dx} = \frac{1}{1 + x^2}$$



### 2.1.1 Questions

#### Question 1

$$y = \sin^{-1}(5x + 9)$$

*Solution:*

$$\sin(y) = 5x + 9$$

$$\cos(y) \frac{dy}{dx} = 5$$

$$\frac{dy}{dx} = \frac{5}{\cos(y)}$$

Using the identity  $\cos^2(y) + \sin^2(y) = 1$

$$\cos(y) = \sqrt{1 - \sin^2(y)}$$

$$\frac{dy}{dx} = \frac{5}{\sqrt{1 - \sin^2(y)}}$$

And since  $x = \sin(y)$

$$\frac{dy}{dx} = \frac{5}{\sqrt{1 - (5x + 9)^2}}$$

#### Question 2

$$y = \sin^{-1}(x)$$

*Solution:*

$$\sin(y) = x$$

$$\cos(y) \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)}$$

Using the identity  $\cos^2(y) + \sin^2(y) = 1$

$$\cos(y) = \sqrt{1 - \sin^2(y)}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2(y)}}$$

And since  $x = \sin(y)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

#### Question 3

$$y = \cos^{-1}(x)$$

***Solution:***

$$\cos(y) = x$$

$$-\sin(y) \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\sin(y)}$$

Using the identity  $\cos^2(y) + \sin^2(y) = 1$

$$\sin(y) = \sqrt{1 - \cos^2(y)}$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - \cos^2(y)}}$$

And since  $x = \cos(y)$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$$

# Chapter 3

## 3.1 Relationships

$$\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$$
$$\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$$

## 3.2 Identities

### 3.2.1 Reciprocal Identities

$$\sin(\theta) = \frac{1}{\csc(\theta)} \quad \text{or} \quad \csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\cos(\theta) = \frac{1}{\sec(\theta)} \quad \text{or} \quad \sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\tan(\theta) = \frac{1}{\cot(\theta)} \quad \text{or} \quad \cot(\theta) = \frac{1}{\tan(\theta)}$$

### 3.2.2 Pythagorean Identities

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

$$\csc^2(\theta) = 1 + \cot^2(\theta)$$

### 3.2.3 Ratio Identities

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

### 3.2.4 Sum and Difference of Angles

$$\sin(\alpha + \beta) = \sin(\alpha) \times \cos(\beta) + \cos(\alpha) \times \sin(\beta)$$

$$\sin(\alpha - \beta) = \sin(\alpha) \times \cos(\beta) - \cos(\alpha) \times \sin(\beta)$$

$$\cos(\alpha + \beta) = \cos(\alpha) \times \cos(\beta) - \sin(\alpha) \times \sin(\beta)$$

$$\cos(\alpha - \beta) = \cos(\alpha) \times \cos(\beta) + \sin(\alpha) \times \sin(\beta)$$

$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha) \times \tan(\beta)}$$

$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha) \times \tan(\beta)}$$

### 3.2.5 Double Angles

$$\sin(2\theta) = 2 \sin(\theta) \cos(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

$$= 2 \cos^2(\theta) - 1$$

$$= 1 - 2 \sin^2(\theta)$$

$$\tan(2\theta) = (2 \tan(\theta)) / (1 - \tan^2(\theta))$$