Orthogonality and Least Squares

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Contents

Chapter 1

Inner Product, Length and Orthogonality

1.1 Inner Product

Definition 1.1.1: Inner / Dot Product

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, then we regard $\mathbf{u}$ and $\mathbf{v}$ as $n \times 1$ matrices. The transpose of $\mathbf{u}^T$ is a $1 \times n$ matrix, and the matrix product $\mathbf{u}^T\mathbf{v}$ is a 1×1 matrix, a scalar. This scalar is called the *inner / dot product* of $\mathbf{u}$ and $\mathbf{v}$ which can also be referred to as:

$$\mathbf{u} \cdot \mathbf{v}$$

Which breaks down into:

$$\mathbf{u}^T \times \mathbf{v}$$

When $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, is then defined as:

$$\begin{bmatrix} u_1 & u_2 & \dots u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \dots + u_nv_n$$

Example 1.1.1

Question 1

Compute $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$ for $\mathbf{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$

Solution:

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^T \times \mathbf{v} = \begin{bmatrix} 2 & -5 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$$
$$= 3\,(2) + (-5)\,(2) + (-1)\,(-3)$$
$$= -1$$

$$\mathbf{v}\cdot\mathbf{u} = \mathbf{v}^T \times \mathbf{u} = \begin{bmatrix} 3 & 2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$$
$$= 2\,(3) + 2\,(-5) + (-3)\,(-1)$$
$$= -1$$

Theorem 1.1.1 Axioms of Inner / Dot products

Let $\mathbf{u}$ and $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

1. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$
2. $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$
3. $(c\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(c\mathbf{v})$
4. $\mathbf{u}\cdot\mathbf{u} \geqslant 0$, and $\mathbf{u}\cdot\mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

1.2 Length of a Vector

Definition 1.2.1: Length of a Vector

If $\mathbf{v}$ is in $\mathbb{R}^n$, with entries $v_1, \ldots, v_n$, then the square root of $\mathbf{v}\cdot\mathbf{v}$ is defined because $\mathbf{v}\cdot\mathbf{v}$ is non-negative. Therefore the *length / norm* of $\mathbf{v}$ is the non-negative scalar $||\mathbf{v}||$, defined:

$$||\mathbf{v}|| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2} \text{ and } ||\mathbf{v}||^2 = \mathbf{v}\cdot\mathbf{v}$$

And similarly for any scalar c, the length of $c\mathbf{v}$ is $|c|$ times the length of $\mathbf{v}$, i.e:

$$||c\mathbf{v}|| = |c| \times ||\mathbf{v}||$$

Definition 1.2.2: Unit Vector

A vector whose length is 1. If we divide a non zero vector by it's length, i.e. multiply by $\frac{1}{||\mathbf{v}||}$, we obtain a unit vector $\mathbf{u}$. This process of creating a unit vector $\mathbf{u}$ from $\mathbf{v}$ can be called *normalizing* $\mathbf{v}$, and the resulting $\mathbf{u}$ is in the same direction as $\mathbf{v}$

1.3 Distance in $\mathbb{R}^n$

Definition 1.3.1: Distance between two vectors

For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the *distance between* $\mathbf{u}$ and $\mathbf{v}$, expressed as $\text{dist}(\mathbf{u}, \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$:

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

Then defined:

$$\begin{aligned}\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| &= \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})} \\ &= \sqrt{(u_1 - v_1)^2 + \ldots + (u_n - v_n)^2}\end{aligned}$$

In $\mathbb{R}^2$ and $\mathbb{R}^3$, this is basically the same as the Euclidean distance between two points.

Example 1.3.1

Question 2

Compute the distance between the vectors $\mathbf{u} = (7, 1)$ and $\mathbf{v} = (3, 2)$

Solution:

$$\begin{aligned}\text{dist}(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\| \\ &= \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})} \\ &= \sqrt{\begin{bmatrix} 4 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \end{bmatrix}} \\ &= \sqrt{4^2 + (-1)^2} \\ &= \sqrt{17}\end{aligned}$$

1.4 Orthogonal Vectors

Consider $\mathbb{R}^2$ and $\mathbb{R}^3$ and two lines through the origin determined by vectors $\mathbf{u}$ and $\mathbf{v}$. These lines are geometrically perpendicular if and only if the distance from $\mathbf{u}$ to $\mathbf{v}$ is the same as the distance from $\mathbf{u}$ to $-\mathbf{v}$. This is equivalent to saying the squares of the distances are the same. Therefore:

$$\begin{aligned}[\text{dist}(\mathbf{u}, -\mathbf{v})]^2 &= \|\mathbf{u} - (-\mathbf{v})\|^2 = \|\mathbf{u} + \mathbf{v}\|^2 \\ &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot (\mathbf{u} + \mathbf{v}) + \mathbf{v} \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}^2\| + \|\mathbf{v}\|^2 + 2\mathbf{u} \cdot \mathbf{v}\end{aligned}$$

And then $\text{dist}(\mathbf{u}, \mathbf{v})$:

$$[\text{dist}(\mathbf{u}, \mathbf{v})] = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}$$

This shows that the two squared distances are only equal if and only if $2\mathbf{u}\cdot\mathbf{v} = -2\mathbf{u}\cdot\mathbf{v}$, which happens if and only if $\mathbf{u}\cdot\mathbf{v} = 0$

Definition 1.4.1: Orthogonality

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are orthogonal, to each other, if $\mathbf{u}\cdot\mathbf{v} = 0$

This then confirms that the zero vector $\mathbf{0}$ is orthogonal to every vector in $\mathbb{R}^n$, since $\mathbf{0}^T\mathbf{v} = 0$ for every $\mathbf{v}$.

Theorem 1.4.1 The Pythagorean Theorem

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in $\mathbb{R}^n$, then:

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

1.5 Exercises

Question 3

Let $\mathbf{a} = \begin{bmatrix}-2\\1\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}-3\\1\end{bmatrix}$. Compute $\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}$ and $\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}$

Solution:

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= (-2)(-3)+1\\ &= 7\\ \mathbf{a}\cdot\mathbf{a} &= (-2)^2+1\\ &= 5\\ \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}} &= \frac{7}{5}\end{aligned}$$

$$\begin{aligned}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} &= \frac{7}{5}\begin{bmatrix}-2\\1\end{bmatrix}\\ &= \begin{bmatrix}-2.8\\ \frac{7}{5}\end{bmatrix}\end{aligned}$$

Question 4

Let $\mathbf{c} = \begin{bmatrix}\frac{4}{3}\\-1\\\frac{2}{3}\end{bmatrix}$ and $\mathbf{d} = \begin{bmatrix}5\\6\\-1\end{bmatrix}$.

1. Find a unit vector $\mathbf{u}$ in the direction of $\mathbf{c}$
2. Show that $\mathbf{d}$ is orthogonal to $\mathbf{c}$.
3. Use the results of parts (1) and (2) to explain why d must be orthogonal to the unit vector $\mathbf{u}$

Solution:

1.

$$\begin{aligned}
\|\mathbf{c}\| &= \sqrt{\mathbf{c}\cdot\mathbf{c}} \\
&= \sqrt{\left(\frac{4}{3}\right)^2 + (-1)^2 + \left(\frac{2}{3}\right)^2} \\
&= \frac{\sqrt{29}}{3} \\
\mathbf{u} &= \frac{1}{\frac{\sqrt{29}}{3}} \begin{bmatrix} \frac{4}{3} \\ -1 \\ \frac{2}{3} \end{bmatrix} \\
&= \frac{3\sqrt{29}}{29} \begin{bmatrix} \frac{4}{3} \\ -1 \\ \frac{2}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{4\sqrt{29}}{29} \\ \frac{3\sqrt{29}}{29} \\ \frac{2\sqrt{29}}{29} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\|\mathbf{u}\| &= \sqrt{\mathbf{u}\cdot\mathbf{u}} \\
&= \sqrt{\left(\frac{4\sqrt{29}}{29}\right)^2 + \left(\frac{3\sqrt{29}}{29}\right)^2 + \left(\frac{2\sqrt{29}}{29}\right)^2} \\
&= 1
\end{aligned}$$

2. If $\mathbf{d}$ is orthogonal to $\mathbf{c}$ then $\mathbf{d}\cdot\mathbf{c} = 0$

$$\begin{aligned}
\mathbf{d}\cdot\mathbf{c} &= \mathbf{d}^T \times \mathbf{c} \\
&= \begin{bmatrix} 5 & 6 & -1 \end{bmatrix} \begin{bmatrix} \frac{4}{3} \\ -1 \\ \frac{2}{3} \end{bmatrix} \\
&= 5\left(\frac{4}{3}\right) + 6(-1) - 1\left(\frac{2}{3}\right) \\
&= \frac{20}{3} - 6 - \frac{2}{3} \\
&= 0
\end{aligned}$$

$\therefore$ $\mathbf{c}$ and $\mathbf{d}$ are orthogonal to each other.

3. $\mathbf{d}$ is orthogonal to the unit vector $\mathbf{u}$ because $\mathbf{d}$ is orthogonal to $\mathbf{c}$ of which $\mathbf{u}$ is a scalar multiple of. I.e $\mathbf{u}$ is in the form $k\mathbf{c}$ for some k and:

$$\mathbf{d}\cdot\mathbf{u} = \mathbf{d}\cdot(k\mathbf{c}) = k(\mathbf{d}\cdot c) = k(0) = 0$$

Chapter 2

Orthogonal Sets

Definition 2.0.1: Orthogonal Set

If $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal set of non-zero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

Chapter 3

Orthogonal Projections

Definition 3.0.1: Orthogonal Projection

Let W be a subspace of $\mathbb{R}^n$, and let $\mathbf{y}$ be in $\mathbb{R}^n$. The *orthogonal projection* of $\mathbf{y}$ onto W, denoted $\text{proj}_W\mathbf{y}$, is the closest point in W to $\mathbf{y}$. This point is obtained by adding the orthogonal projection of $\mathbf{y}$ onto the orthogonal complement of W to the orthogonal projection of $\mathbf{y}$ onto W.

Example 3.0.1

Question 5

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_5\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^5$, and let

$$\mathbf{y} = c_1\mathbf{u}_1 + \ldots + c_5\mathbf{u}_5$$

Consider the subspace $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, and write $\mathbf{y}$ as the sum of a vector $\mathbf{z}_1$ in W and a vector $\mathbf{z}_2$ in $W^\perp$

Solution:

Theorem 3.0.1 The Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form:

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^\perp$. In fact, if $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$, is any orthogonal basis of W, then:

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \ldots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p$$

and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$

Example 3.0.2

Question 6

Let $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Observe that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W

Solution: The orthogonal projection of $\mathbf{y}$ onto W is:

$$\begin{aligned}
\hat{\mathbf{y}} &= \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 \\
&= \frac{9}{30}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{5} \\ \frac{3}{2} \\ -\frac{3}{10} \end{bmatrix} + \begin{bmatrix} -1 \\ \frac{3}{6} \\ \frac{3}{6} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{2}{5} \\ 2 \\ \frac{1}{5} \end{bmatrix}
\end{aligned}$$

So:

$$\begin{aligned}
\mathbf{z} &= \mathbf{y} - \hat{\mathbf{y}} \\
&= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -\frac{2}{5} \\ 2 \\ \frac{1}{5} \end{bmatrix} \\
&= \begin{bmatrix} \frac{7}{5} \\ 0 \\ \frac{14}{5} \end{bmatrix}
\end{aligned}$$

$\mathbf{z}$ is orthogonal to W due to 3, so $\mathbf{y}$ can be expressed as:

$$\mathbf{y} = \begin{bmatrix} -\dfrac{2}{5} \\ 2 \\ \dfrac{1}{5} \end{bmatrix} + \begin{bmatrix} \dfrac{7}{5} \\ 0 \\ \dfrac{14}{5} \end{bmatrix}$$

3.1 Properties of Orthogonal Projections

If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal basis for W and if $\mathbf{y}$ happens to be in W, then the formula for $\text{proj}_W \mathbf{y}$ is exactly the same as the representation of $\mathbf{y}$ in terms of the basis. In this case, $\text{proj}_W \mathbf{y} = \mathbf{y}$.

Theorem 3.1.1

If $\mathbf{y}$ is in $W = \text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$, then $\text{proj}_W \mathbf{y} = \mathbf{y}$

3.1.1 Exercises

Question 7

The distance from a point $\mathbf{y}$ in $\mathbb{R}^n$ to a subspace W is defined as the distance from $\mathbf{y}$ to the nearest point in W. Find the distance from $\mathbf{y}$ to $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, where

$$\mathbf{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Solution:

$$\begin{aligned}
\hat{\mathbf{y}} &= \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \\
&= \frac{15}{30}\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + -\frac{21}{6}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{5}{2} \\ -1 \\ \frac{15}{30} \end{bmatrix} + \begin{bmatrix} -\frac{21}{6} \\ -7 \\ \frac{21}{6} \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix} \\
\mathbf{z} &= \mathbf{y} - \hat{\mathbf{y}} \\
&= \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix} - \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} \\
||\mathbf{z}|| &= \sqrt{3^2 + 6^2} \\
&= \sqrt{45}
\end{aligned}$$

Chapter 4

Exercises

Question 8

Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. Then express $\mathbf{x}$ as a linear combination of the $\mathbf{u}s$

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \text{ and } \mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$$

Solution: For the basis be orthogonal $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, $\mathbf{u}_2 \cdot \mathbf{u}_3 = 0$, and $\mathbf{u}_1 \cdot \mathbf{u}_3 = 0$.

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{u}_2 &= \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \\ &= 3(2) + (-3)(2) + 0(-1) \\ &= 6 - 6 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{u}_2 \cdot \mathbf{u}_3 &= \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \\ &= 2(1) + 2(1) + (-1)(4) \\ &= 2 + 2 - 4 \\ &= 0 \\ \mathbf{u}_3 \cdot \mathbf{u}_1 &= \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \\ &= 1(3) + 1(-3) + 4(0) \\ &= 3 - 3 \\ &= 0 \end{aligned}$$

Therefore $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. To express $\mathbf{x}$ as a linear combination of the $\mathbf{u}s$:

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$$

$$\begin{bmatrix} 3 & 2 & 1 & 5 \\ -3 & 2 & 1 & -3 \\ 0 & -1 & 4 & 1 \end{bmatrix}$$

$$-1R_1 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 3 & 2 & 1 & 5 \\ 0 & -4 & -2 & -2 \\ 0 & -1 & 4 & 1 \end{bmatrix}$$

$$\frac{1}{4}R_2 - R_3 \longrightarrow R_3$$

$$\begin{bmatrix} 3 & 2 & 1 & 5 \\ 0 & -4 & -2 & -2 \\ 0 & 0 & \frac{-9}{2} & \frac{-3}{2} \end{bmatrix}$$

$$\frac{-1}{2}R_2 - R_1 \longrightarrow R_1$$

$$\begin{bmatrix} -3 & 0 & 0 & -4 \\ 0 & -4 & -2 & -2 \\ 0 & 0 & \frac{-9}{2} & \frac{-3}{2} \end{bmatrix}$$

$$\frac{4}{9}R_3 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix} -3 & 0 & 0 & -4 \\ 0 & 4 & 0 & \frac{4}{3} \\ 0 & 0 & \frac{-9}{2} & \frac{-3}{2} \end{bmatrix}$$

$$\frac{-1}{3}R_1 \longrightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 4 & 0 & \frac{4}{3} \\ 0 & 0 & \frac{-9}{2} & \frac{-3}{2} \end{bmatrix}$$

$$\frac{1}{4}R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{-9}{2} & \frac{-3}{2} \end{bmatrix}$$

$$\frac{-2}{9}R_3 \longrightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

$$c_1 = \frac{4}{3}$$

$$c_2 = \frac{1}{3}$$

$$c_3 = \frac{1}{3}$$

$$\mathbf{x} = \frac{4}{3}\begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

Question 9

Let W be the subspace spanned by the $\mathbf{u}$s, and write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

$$\mathbf{y} = \begin{bmatrix}1\\3\\5\end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix}1\\3\\-2\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}5\\1\\4\end{bmatrix}$$

Solution: We can express $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W using the orthogonal decomposition theorem, finding the orthogonal projection of $\mathbf{y}$ onto W, and subtracting that from $\mathbf{y}$ to find the vector orthogonal to W.

$$\begin{aligned}
\hat{\mathbf{y}} &= \frac{0}{14}\begin{bmatrix}1\\3\\-2\end{bmatrix} + \frac{28}{42}\begin{bmatrix}5\\1\\4\end{bmatrix}\\
&= \begin{bmatrix}\frac{10}{3}\\\frac{2}{3}\\\frac{8}{3}\end{bmatrix}\\
\mathbf{z} &= \mathbf{y} - \hat{\mathbf{y}}\\
&= \begin{bmatrix}1\\3\\5\end{bmatrix} - \begin{bmatrix}\frac{10}{3}\\\frac{2}{3}\\\frac{8}{3}\end{bmatrix}\\
\mathbf{z} &= \begin{bmatrix}-\frac{7}{3}\\\frac{7}{3}\\\frac{7}{3}\end{bmatrix}\\
\mathbf{y} &= \mathbf{z} + \hat{\mathbf{y}}\\
&= \begin{bmatrix}-\frac{7}{3}\\\frac{7}{3}\\\frac{7}{3}\end{bmatrix} + \begin{bmatrix}\frac{10}{3}\\\frac{2}{3}\\\frac{8}{3}\end{bmatrix}
\end{aligned}$$