

Interpreting Derivatives

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Chapter 1

1.1 Instantaneous Rate of Change

In the case where f is a function of x $f'(x)$ measures the instantaneous rate of change of f with respect to x .

Example 1.1.1

The term widget is an economic term for a generic unit of manufacturing output. Suppose a company produces widgets and knows that the market supports a price of \$10 per unit. Let $P(n)$ give the profit, in dollars, earned by manufacturing and selling n widgets. The company likely cannot make a (positive) profit making just one widget; the start-up costs will likely exceed \$10. Mathematically, we would write this as $P(1) < 0$.

What do $P(1000) = 500$ and $P'(1000) = 0.25$ mean?. Approximate $P(1100)$

Solution:

The equation $P(1000) = 500$ means that selling 1,000 widgets returns a profit of \$500.

We interpret $P'(1000) = 0.25$ as meaning that the profit is increasing at the rate of \$0.25 per widget (the units are "dollars per widget").

Since we have no other information to use, our best approximation for $P(1100)$ is:

$$\begin{aligned} P(1100) &\approx P(1000) + P'(1000) \times 100 \\ &= P(1000) + P'(1000) \times 100 \\ &= 500 + 0.25 \times 100 \\ &= 525 \end{aligned}$$

We approximate that selling 1,100 widgets returns a profit of \$525

1.2 The Slope of the Tangent Line

We can measure the instantaneous rate of change at a given x value c of a non-linear function by computing $f'(c)$. We can determine the behaviour of the function f by observing the slopes of its tangent lines.

1.3 Increasing and Decreasing Functions

1.3.1 Increasing Functions

$f(x)$ is increasing whenever $x_1 < x_2$ and $f(x_1) < f(x_2)$, I.e as you go up the x axis the y or function values increase.

$f(x)$ is increasing if the slope on any point on its graph is positive throughout the function's entire domain.

1.3.2 Decreasing Functions

$f(x)$ is decreasing whenever $x_1 < x_2$ and $f(x_1) > f(x_2)$, .I.e as you go up the x axis the y or function values decrease

$f(x)$ is increasing if the slope on any point on it's graph is negative throughout the function's entire domain.

1.3.3 Critical Points

- Points where the gradient is equal to 0, i.e. $f'(x) = 0$
- Points where the gradient does not exist, i.e. $f'(x) = \emptyset$

Example 1.3.1 ($t\sqrt[3]{t^2 - 4}$)

$$\begin{aligned}
g(t) &= t\sqrt[3]{t^2 - 4} \\
g(t) &= t(t^2 - 4)^{\frac{1}{3}} \\
g'(t) &= (1)(t^2 - 4)^{\frac{1}{3}} + \left(\frac{1}{3}\right)(2t)(t^2 - 4)^{-\frac{2}{3}} \\
g'(t) &= (t^2 - 4)^{\frac{1}{3}} + \frac{2}{3}t^2(t^2 - 4)^{-\frac{2}{3}} \\
g'(t) &= (t^2 - 4)^{\frac{1}{3}} + \frac{2t^2}{3(t^2 - 4)^{\frac{2}{3}}} \\
g'(t) &= \frac{(t^2 - 4)^{\frac{1}{3}}}{1} + \frac{2t^2}{3(t^2 - 4)^{\frac{2}{3}}} \\
g'(t) &= \frac{3(t^2 - 4) + t^2}{3(t^2 - 4)^{\frac{2}{3}}} \\
0 &= \frac{3(t^2 - 4) + t^2}{3(t^2 - 4)^{\frac{2}{3}}} \\
0 &= 3t^2 - 12 + t^2 \\
0 &= 5t^2 - 12 \\
12 &= 5t^2 \\
\frac{12}{5} &= t^2 \\
\pm\sqrt{\frac{12}{5}} &= t \\
3(t^2 - 4)^{\frac{2}{3}} &= 0 \\
(t^2 - 4)^{\frac{2}{3}} &= 0 \\
t &= \pm 2
\end{aligned}$$

Interval	Test Value	Slope _{$g'(x)$}
$x < -2$	-3	+
$-2 < x < -\sqrt{\frac{12}{5}}$	-1.7	+
$-\sqrt{\frac{12}{5}} < x < \sqrt{\frac{12}{5}}$	0	-
$\sqrt{\frac{12}{5}} < x < 2$	2	+
$x > 2$	7	+

$$\therefore \text{When } g'(x) = 0, x = -\sqrt{\frac{12}{5}}, x = \sqrt{\frac{12}{5}}$$

$$\therefore \text{Increasing } (-\infty, -2), (2, \infty), (-2, -\sqrt{\frac{12}{5}}), (\sqrt{\frac{12}{5}}, 2)$$

$$\text{Decreasing } (-\sqrt{\frac{12}{5}}, \sqrt{\frac{12}{5}})$$