# Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Madiba Hudson-Quansah

# Contents

# Chapter 1

# Sets

> **Definition 1.0.1: Set**
>
> An unordered collection of objects, called *elements* or *members* of the set. A set contains elements and, we can denote this as $a \in A$ where $a$ is an element of the set $A$, or $a \notin A$, where $a$ is not an element of the set $A$.

There are several ways to describe a set:

**Roster notation** $\{1, 2, 3, 4, 5\}$

**Set-Builder notation** Where all the elements of a set are described by a property they satisfy.i.e. The set $O$ of all odd positive numbers less than 10 can be expressed as $O = \{x \mid x$ is an odd positive integer less than 10$\}$ or specifying the domain of discourse, $O = \{x \in \mathbb{Z}^+ \mid x$ is odd and $x < 10\}$, or the set of all positive rational numbers $\mathbb{Q}^+$ can be expressed as $\mathbb{Q}^+ = \left\{x \in \mathbb{R} \mid x = \frac{p}{q}, \text{ for some positive integers } q \text{ and } p\right\}$

> **Definition 1.0.2: Equality of Sets**
>
> Two sets $A$ and $B$ are equal if and only if they have the same elements. Therefore, $\forall x\, (x \in A \leftrightarrow x \in B)$, We write $A = B$ if this is the case.

> **Definition 1.0.3: Empty / Null Set**
>
> A set with no elements, denoted by $\emptyset$ or $\{\}$. Can be expressed as $\{x \mid F\}$

> **Definition 1.0.4: Singleton Set**
>
> A set with exactly one element, denoted by $\{a\}$. The set $\{\emptyset\}$ is a singleton set as it is a set with one element, the empty set.

## 1.0.1 Set Definitions

### 1.0.1.1 Natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

### 1.0.1.2 Integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

#### 1.0.1.3  Positive Integers

$$\mathbb{Z}^+ = \{1, 2, 3, 4, 5, \ldots\}$$

#### 1.0.1.4  Rational numbers

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$$

#### 1.0.1.5  Irrational Numbers

$$\mathbb{I} = \{x \mid x \text{ is a number that cannot be expressed as a fraction}\}$$

#### 1.0.1.6  Real numbers

$$\mathbb{R} = \{x \mid x \text{ is a point on the number line}\}$$

Or

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$$

#### 1.0.1.7  Positive Real numbers

$$\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$$

#### 1.0.1.8  Complex numbers

$$\mathbb{C} = \left\{ a + bi \mid a, b \in \mathbb{R} \text{ and } i^2 = -1 \right\}$$

### 1.0.2  Venn Diagrams

> **Definition 1.0.5: Universal Set**
>
> The set of all objects under consideration, denoted by $U$. Can be expressed as $\{x \mid T\}$

Sets can be graphically represented using Venn diagrams. A Venn diagram is a collection of simple closed curves, especially circles, that represent sets. In Venn diagrams the universal set $U$ which contains all the objects under consideration is represented by a rectangle, and the sets are represented by circles within the rectangle, with points inside the circles representing elements of the sets.

### 1.0.3  Subsets

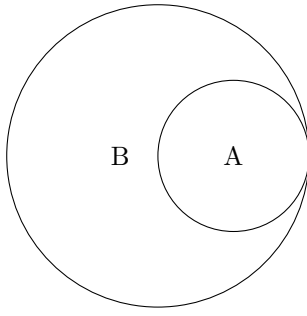> **Definition 1.0.6: Subset**
>
> A set $A$ is a *subset* of a set $B$ if and only if every element of $A$ is also an element of $B$. Denoted by $A \subseteq B$.

We see that $A \subseteq B$ if and only if

$$\forall x \, (x \in A \rightarrow x \in B)$$

Is true. I.e. If $x \in A$, then $x \in B$. To disprove this we need to show that $\exists x \, (x \in A \wedge x \notin B)$

Shown graphically:

**Example 1.0.1**

The set of integers with squares less than 100 is not a subset of the set of nonnegative integers because -1 is in the former set [as $(-1)^2 < 10$], but not the later set. The set of people who have taken discrete mathematics at your school is not a subset of the set of all computer science majors at your school if there is at least one student who has taken discrete mathematics who is not a computer science major.

**Theorem 1.0.1**

For every set $S$

1. $\emptyset \subseteq S$
2. $S \subseteq S$

1. **Proof:** We will prove that $\emptyset \subseteq S$, using a vacuous proof

   Let $S$ be a set.

   To show $\emptyset \subseteq S$ we must show that $\forall x \, (x \in \emptyset \rightarrow x \in S)$ is $T$.

   Because $\emptyset$ contains no elements $x \in \emptyset$ is always $F$

   This follows that the implication $x \in \emptyset \rightarrow x \in S$ is always $T$

   Hence $\emptyset \subseteq S$

   ☺

2. **Proof:** We will prove that $S \subseteq S$, using a direct proof

   Let $S$ be a set

   To show $S \subseteq S$ we must show that $\forall x \, (x \in S \rightarrow x \in S)$ is $T$

   Assume $x \in S$

   Because $x \in S$ is always $T$, the implication $x \in S \rightarrow x \in S$ is always $T$

   $\therefore \forall x \, (x \in S \rightarrow x \in S)$ is $T$

   Hence $S \subseteq S$

   ☺

**Definition 1.0.7: Proper subset**

A set $A$ is *proper subset* of a set $B$ if and only if every element of $A$ is also an element of $B$ and $A \neq B$. Denoted by $A \subset B$. I.e.

$$\exists x \, (x \notin A \wedge x \in B) \wedge \forall x \, (x \in A \rightarrow x \in B)$$

Is $T$.

**Definition 1.0.8: Further Equality**

Two sets $A$ and $B$ are equal if $A \subseteq B \wedge B \subseteq A$ is $T$. I.e. $A = \{\emptyset, \{a\}, \{a\}, \{b\}, \{a, b\}\}$ and $B = \{x \mid x \text{ is a subset of the set } \{a, b\}\}$ are equal.

### 1.0.4 Cardinality

> **Definition 1.0.9: Cardinality**
>
> The number of distinct elements $n$ in a set $A$. Denoted by $|A| = n$. Where $n$ is a non-negative integer, we say that $A$ is a finite set.

> **Definition 1.0.10: Infinite set**
>
> A set $A$ is infinite if it is not finite. I.e. $|A| = \infty$

### 1.0.5 Power Set

> **Definition 1.0.11: Power Set**
>
> A set containing all the subsets of a given set $A$. Denoted by $\mathcal{P}(A)$. If a set has $n$ distinct elements, then the cardinality of the power set is $2^n$.

---

**Example 1.0.2**

> **Question 1**
>
> What is the power set of the set $\{0, 1, 2\}$

*Solution:*
$$\mathcal{P}(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, , \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$$

---

**Example 1.0.3**

> **Question 2**
>
> What is the power set of $\emptyset$

*Solution:*
$$\mathcal{P}(\emptyset) = \{\emptyset\}$$

> **Question 3**
>
> What is the power set of $\{\emptyset\}$

*Solution:*
$$\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$$

---

### 1.0.6 N-Tuples

> **Definition 1.0.12: Ordered N-Tuple**
>
> N-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has $a_1$ as its first element, $a_2$ as its second element, $\ldots$, and $a_n$ as its $n$th element.

Two n-tuples are equal if an only if each corresponding pair of their elements is equal, i.e. $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ are equal if and only if $a_i = b_i$, for $i = 1, 2, \ldots, n$.

Ordered 2-tuples are called *ordered pairs*. The ordered pairs, $(a, b)$ and $(c, d)$ are equal if and only if $a = c$ and $b = d$.

## 1.0.7 Cartesian Products

---
**Definition 1.0.13: Cartesian Product**

Let $A$ and $B$ be sets. The *Cartesian Product* of $A$ and $B$, denoted by $A \times B$, is the set of all ordered pairs $(a, b)$, where $a \in A$ and $b \in B$. I.e.

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

The number of items in the Cartesian product of two sets is the product of the cardinality of each set.

---
**Example 1.0.4**

**Question 4**

What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$

*Solution:*
$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

**Question 5**

Show that the Cartesian product $B \times A$ is not equal to the Cartesian product $A \times B$.

*Solution:*
$$B \times A = \{(a, 1)(a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$$

$\therefore A \times B \neq B \times A$

---
**Definition 1.0.14: Cartesian Product of more than two sets**

The Cartesian product of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of ordered $n$-tuples $(a_1, a_2, \ldots, a_n)$, where $a_i$ belongs to $A_i$ for $i = 1, 2, \ldots, n$. I.e.

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$$

---
**Example 1.0.5**

**Question 6**

What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, $C = \{0, 1, 2\}$.

*Solution:*

$A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}$

---
We use the notation $A^2$ to denote $A \times A$, the Cartesian product of $A$ and itself. Therefore

$$A^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for } i = 1, 2, \ldots, n\}$$

---
**Example 1.0.6**

Suppose $A = \{1, 2\}$.

---

It follows $A^2 = \{(1,1),(1,2),(2,1),(2,2)\}$,
and $A^3 = \{(1,1,1),(1,1,2),(1,2,1),(1,2,2),(2,1,1),(2,1,2),(2,2,1),(2,2,2)\}$

## Example 1.0.7

**Question 7**

What are the ordered pairs in the less than or equal to relation, which contains, $(a,b)$ if $a \leqslant b$, on the set $\{0,1,2,3\}$

***Solution:*** Let $R$ be the relation on the set $\{0,1,2,3\}$, if $a \leqslant b$.

$$R = \{(0,0),(1,1),(2,2),(3,3),(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)\}$$

### 1.0.8 Set Notation with Quantifiers

We can restrict the domain of a quantifier to a set, I.e. Where $S$ is a set $\forall x \in S (P(x))$, denotes the universal quantification of $P(x)$ for all elements in the set $S$. Which is shorthand for $\forall x (x \in S \rightarrow P(x))$

## Example 1.0.8

$\forall x \in \mathbb{R} \left( x^2 \geqslant 0 \right)$ means "the square of any real number is greater than or equal to 0".
$\exists x \in \mathbb{Z} \left( x^2 = 1 \right)$ means "there exists an integer whose square is 1"

### 1.0.9 Truth Sets and Quantifiers

**Definition 1.0.15: Truth Set**

For a predicate $P$ the truth set of $P$ is the set of all elements in the domain of discourse that make $P$ true. I.e. let $S$ be a set. The truth set of $P(x)$ is

$$\{x \in S \mid P(x)\}$$

## Example 1.0.9

**Question 8**

What are the truth set of the predicates $P(x)$, $Q(x)$, and $R(x)$, where the domain is the set of integers, and $P(x)$: $|x| = 1$, $Q(x)$: $x^2 = 2$, and $R(x)$: $|x| = x$

***Solution:***
The truth set of $P$ is $\{x \in \mathbb{Z} \mid |x| = 1\}$
The truth set of $Q$ is $\{x \in \mathbb{Z} \mid x^2 = 2\}$
The truth set of $R$ is $\{x \in \mathbb{Z} \mid |x| = x\}$

**Note:-**

$\forall x P(x)$ is $T$ over the domain $U$ if and only if the truth set of $P$ is $U$.
$\exists x P(x)$ is $T$ over the domain $U$ if and only if the truth set of $P$ is not empty.

## 1.1 Exercises

**Question 9**

List the members of these sets
1. $\{x \mid x$ is the square of an integer and $x < 100\}$
2. $\{x \mid x$ is an integer such that $x^2 = 2\}$

*Solution:*

1. $\{1, 4, 9, 16, 25, 36, 49, 64, 81\}$

2. $\emptyset$

**Question 10**

Use set builder notation to describe the following sets
1. $\{-3, -2, -1, 0, 1, 2, 3\}$
2. $\{m, n, o, p\}$

*Solution:*

1. $\{x \mid -3 \leqslant x \leqslant 3\}$

2. $\{x \mid x$ is a letter in the word monopoly excluding "l" and "y" $\}$

**Question 11**

Suppose that $A = \{2, 4, 6\}$, $B = \{2, 6\}$, $C = \{4, 6\}$ and $D = \{4, 6, 8\}$. Determine which of these sets are subsets of which other sets.

*Solution:*

$$B \subseteq A$$
$$C \subseteq A$$
$$C \subseteq D$$

**Question 12**

Suppose that $A$, $B$, $C$, are sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$

*Solution:*

$$A \subseteq B \text{ means } \forall x \, (x \in A \rightarrow x \in B)$$
$$B \subseteq C \text{ means } \forall x \, (x \in B \rightarrow x \in C)$$
$$A \subseteq C \text{ means } \forall x \, (x \in A \rightarrow x \in C)$$

$$\forall x \, (x \in A \rightarrow x \in B)$$
$$\forall x \, (x \in B \rightarrow x \in C)$$
$$\therefore \overline{\forall x \, (x \in A \rightarrow x \in C)}$$

| | Steps | Reasons |
|---|---|---|
| 1 | $\forall x\, (x \in A \rightarrow x \in B)$ | Premise 1 |
| 2 | $\forall x\, (x \in B \rightarrow x \in C)$ | Premise 2 |
| 3 | $x \in A \rightarrow x \in B$ | Universal Instantiation of 1 |
| 4 | $x \in B \rightarrow x \in C$ | Universal Instantiation of 2 |
| 5 | $x \in A \rightarrow x \in C$ | By Hypothetical Syllogism of 3 and 4 |
| 6 | $\forall x\, (x \in A \rightarrow x \in C)$ | Universal generalization of 5 |

---

### Question 13

Find the power set of each of these sets, where $a$ and $b$ are distinct elements.
1. $\{a\}$
2. $\{a, b\}$
3. $\{\varnothing, \{\varnothing\}\}$

*Solution:*

1. $\mathcal{P}\,(\{a\}) = \{\varnothing, \{a\}\}$

2. $\mathcal{P}(\{a, b\}) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$

3. $\mathcal{P}(\{\varnothing, \{\varnothing\}\}) = \{\varnothing, \{\varnothing\}\, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$

# Chapter 2

# Set Operations

## 2.1 Set Operations

### 2.1.1 Union

**Definition 2.1.1: Union**

Let $A$ and $B$ be sets. The *union* of $A$ and $B$, denoted by $A \cup B$, is the set of all elements that are either in $A$ or in $B$ or in both. I.e.

$$A \cup B = \{x \mid x \in A \lor x \in B\}$$

### 2.1.2 Intersection

**Definition 2.1.2: Intersection**

Let $A$ and $B$ be sets. The *intersection* of $A$ and $B$, denoted by $A \cap B$, is the set of all elements that are in both $A$ and $B$. I.e.

$$A \cap B = \{x \mid x \in A \land x \in B\}$$

### 2.1.3 Complement

**Definition 2.1.3: Complement**

Let $A$ be a set. The *complement* of the set A (with respect to $U$), denoted by $\overline{A}$ is the set $U - A$. I.e.

$$\overline{A} = \{x \in U \mid x \notin A\}$$

### 2.1.4 Difference

**Definition 2.1.4: Difference**

Let $A$ and $B$ be sets. The *difference* of $A$ and $B$, denoted by $A - B$, is the set of all elements that are in $A$ but not in $B$. I.e.

$$A - B = \{x \mid x \in A \land x \notin B\}$$

Or

$$A - B = A \cap \overline{B}$$

## 2.1.5 Symmetric Difference

> **Definition 2.1.5: Symmetric Difference**
>
> Let $A$ and $B$ be sets. The *symmetric difference* of $A$ and $B$, denoted by $A \oplus B$, is the set of all elements that are in exactly one of $A$ and $B$. I.e.
>
> $$A \oplus B = (A - B) \cup (B - A)$$

**Example 2.1.1**

**Question 14**

$$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$
$$A = \{1, 2, 3, 4, 5\}$$
$$B = (4, 5, 6, 7, 8)$$

What is $A \oplus B$

*Solution:*
$$A \oplus B = \{1, 2, 3, 6, 7, 8\}$$

## 2.1.6 The Cardinality of the Union of Two Sets

The cardinality of the union of two sets $A$ and $B$ is given by

$$|A \cup B| = |A| + |B| - |A \cap B|$$

# 2.2 Set Identities

## 2.2.1 Identity Laws

$$A \cap U = A$$
$$A \cup \emptyset = A$$

## 2.2.2 Domination Laws

$$A \cup U = U$$
$$A \cap \emptyset = \emptyset$$

## 2.2.3 Idempotent Laws

$$A \cup A = A$$
$$A \cap A = A$$

## 2.2.4 Complementation Law

$$\overline{\left(\overline{A}\right)} = A$$

### 2.2.5 Commutative Laws

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

### 2.2.6 Associative Laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

### 2.2.7 Distributive Laws

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

### 2.2.8 De Morgan's Laws

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

### 2.2.9 Absorption Laws

$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$

### 2.2.10 Complement Laws

$$A \cup \overline{A} = U$$
$$A \cap \overline{A} = \emptyset$$

### 2.2.11 Proving Set Identities

There are different ways to prove set identities, these include:

- Proving each set is a subset of the other
- Using set builder notation and propositional logic
- Using Membership tables

> **Definition 2.2.1: Membership Table**
>
> A table that shows the truth value of a predicate for all possible combinations of truth values of its variables.

> **Example 2.2.1**

Prove that
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Using propositional logic:

**Proof:** We prove this identity by showing that each set is a subset of the other. I.e.

$$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B} \wedge \overline{A} \cap \overline{B} \subseteq \overline{A \cap B}$$

$$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B} \text{ means } \forall x \left( x \in \overline{A \cap B} \rightarrow x \in \overline{A} \cup \overline{B} \right)$$

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cap B} \text{ means } \forall x \left( x \in \overline{A} \cup \overline{B} \rightarrow x \in \overline{A \cap B} \right)$$

Assume that $x \in \overline{A \cap B}$

| | |
|---|---|
| $x \in \overline{A \cap B}$ | Assumption |
| $x \notin A \cap B$ | Definition of Complement |
| $\neg (x \in A \cap B)$ | Definition of $\notin$ |
| $\neg (x \in A \wedge x \in B)$ | Definition of intersection |
| $\neg (x \in A) \vee \neg (x \in B)$ | By First De Moragn's Law for propositional logic |
| $x \notin A \vee x \notin B$ | Definition of Complement |
| $x \in \overline{A} \cup \overline{B}$ | Definition of union |

Then we assume $x \in \overline{A} \cup \overline{B}$

| | |
|---|---|
| $x \in \overline{A} \cup \overline{B}$ | Assumption |
| $x \notin A \vee x \notin B$ | Definition of union |
| $\neg (x \in A) \vee \neg (x \in B)$ | Definition of Complement |
| $\neg (x \in A \wedge x \in B)$ | By Second De Morgan's Law for propositional logic |
| $x \notin A \wedge x \notin B$ | Definition of Complement |
| $x \notin A \cap B$ | Definition of intersection |
| $x \in \overline{A \cap B}$ | Definition of Complement |

☺

Using set builder notation

| | |
|---|---|
| $\overline{A \cap B} = \{x \mid x \notin A \cap B\}$ | Definition of Complement |
| $= \{x \mid \neg (x \in (A \cap B))\}$ | Definition of $\notin$ |
| $= \{x \mid \neg (x \in A \wedge x \in B)\}$ | Definition of Intersection |
| $= \{x \mid \neg (x \in A) \vee \neg (x \in B)\}$ | By First De Morgan's Law for propositional logic |
| $= \{x \mid x \notin A \vee x \notin B\}$ | Definition of Complement |
| $= \{x \mid x \in \overline{A} \cup \overline{B}\}$ | Definition of union |
| $= \overline{A} \cup \overline{B}$ | |

## 2.3 Generalized Unions and Intersections

### Definition 2.3.1: Generalized Union

The union of a collection of sets that contains those elements that are members of at least one set in the collection. Denoted by

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$$

Where $A_1 \cup A_2 \cup \ldots A_n$ is the union of sets $A_1, A_2, \ldots, A_n$

### Definition 2.3.2: Generalized Intersection

The intersection of a collection of sets that contains those elements that are members of all the sets in the collection. Denoted by

$$A_1 \cap A_2 \cap \ldots A_n = \bigcap_{i=1}^{n} A_i$$

Where $A_1 \cap A_2 \cap \ldots \cap A_n$ is the intersection of sets $A_1, A_2, \ldots, A_n$

### Example 2.3.1

For $i = 1, 2, \ldots$, let $A_i = \{i, i+1, i+2, \ldots\}$. Then.

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\}$$

and

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2\} = A_n$$

We can extend this notation to other families of sets I.e.

$$A_1 \cup A_2 \cup \ldots \cup A_n \cup \ldots = \bigcup_{i=1}^{\infty} A_i$$

Denotes the union of the sets $A_1, A_2, \ldots, A_n, \ldots$, and the intersection of these sets is denoted by

$$A_1 \cap A_2 \cap \ldots \cap A_n \cap \ldots = \bigcap_{i=1}^{\infty} A_i$$

Generally when $I$ is set, the notations $\bigcap_{i \in I} A_i$ and $\bigcup_{i \in I} A_i$ are used to denote the intersection and union of the sets $A_i$ for $i \in I$, respectively, where

$$\bigcap_{i \in I} A_i = \{x \mid \forall i \in I \, (x \in A_i)\}$$

and

$$\bigcup_{i \in I} A_i = \{x \mid \exists i \in I \, (x \in A_i)\}$$

### Example 2.3.2

Suppose $A_i = \{1, 2, 3, \ldots, i\}$ for $i = \{1, 2, 3, \ldots\}$ Then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \{1, 2, 3, \ldots, i\} = \mathbb{Z}^+$$

and

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} \{1, 2, 3, \ldots, i\} = \{1\}$$

## 2.4  Exercises

### Question 16

List the members of these sets
1. $\left\{x \mid x \text{ is a real number such that } x^2 = 1\right\}$
2. $\{x \mid x \text{ is a positive integer less than } 12\}$
3. $\{x \mid x \text{ is the square of an integer and } x < 100\}$
4. $\left\{x \mid x \text{ is an integer such that } x^2 = 2\right\}$

**Solution:**

1. $\{-1, 1\}$

2. $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

3. $\{0, 1, 4, 9, 16, 25, 36, 49, 64, 81\}$

4. $\varnothing$

### Question 17

Use set builder notation to show that:
1. $A \cup U = U$
2. $A \cap \varnothing = \varnothing$
3. $A \cup \overline{A} = U$
4. $A \cap \overline{A} = \varnothing$

**Solution:**

1.

$$
\begin{aligned}
A \cup U &= \{x \mid x \in A \cup U\} && \text{Set builder notation} \\
&= \{x \mid x \in A \vee x \in U\} && \text{Definition of Union} \\
&= \{x \mid x \in A \vee T\} && \text{Definition of Universal Set} \\
&= \{x \mid T\} && \text{By First Domination law for propositional logic} \\
&= U && \text{Definition of Universal Set}
\end{aligned}
$$

2.

$$A \cap \emptyset = \{x \mid x \in A \cap \emptyset\} \qquad \text{Set builder notation}$$
$$= \{x \mid x \in A \wedge x \in \emptyset\} \qquad \text{Definition of Intersection}$$
$$= \{x \mid x \in A \wedge F\} \qquad \text{Definition of Empty set}$$
$$= \{x \mid F\} \qquad \text{By Second Domination Law for propositional logic}$$
$$= \emptyset \qquad \text{Definition of Empty set}$$

3.

$$A \cup \overline{A} = \{x \mid x \in A \cap \overline{A}\} \qquad \text{Set builder notation}$$
$$= \{x \mid x \in A \vee x \in \overline{A}\} \qquad \text{Definition of Union}$$
$$= \{x \mid x \in A \vee x \notin A\} \qquad \text{Definition of Complement}$$
$$= \{x \mid x \in A \vee \neg(x \in A)\} \qquad \text{Definition of Complement}$$
$$= \{x \mid T\} \qquad \text{By First Negation Law for propositional logic}$$
$$= U \qquad \text{Definition of Universal set}$$

4.

$$A \cap \overline{A} = \{x \mid x \in A \cap \overline{A}\} \qquad \text{Set builder notation}$$
$$= \{x \mid x \in A \wedge x \in \overline{A}\} \qquad \text{Definition of intersection}$$
$$= \{x \mid x \in A \wedge (x \notin A)\} \qquad \text{Definition of Complement}$$
$$= \{x \mid x \in A \wedge \neg(x \in A)\} \qquad \text{Definition of Complement}$$
$$= \{x \mid F\} \qquad \text{By Second Negation law of propositional logic}$$
$$= \emptyset \qquad \text{Definition of } \emptyset$$

## Question 18

Let $A$ and $B$ be sets. Show that
1. $(A \cap B) \subseteq A$
2. $A \subseteq (A \cup B)$
3. $A - B \subseteq A$
4. $A \cap (B - A) = \emptyset$
5. $A \cup (B - A) = A \cup B$

**Solution:**

1.

$$(A \cap B) \subseteq A \text{ means } \forall x \, (x \in (A \cap B) \rightarrow x \in A)$$

Assume $x \in (A \cap B)$

|   | Steps | Reasons |
|---|-------|---------|
| 1 | $x \in A \cap B$ | Assumption |
| 2 | $x \in A \wedge x \in B$ | Definition of intersection |
| 3 | $x \in A$ | Simplification of 2 |

$\therefore \ x \in (A \cap B) \rightarrow x \in A$
Conclusion: $(A \cap B) \subseteq A$

2. **Proof:**

$$A \subseteq (A \cup B) \text{ means } \forall x\,(x \in A \to x \in A \cup B)$$

|   | Steps | Reasons |
|---|-------|---------|
| 1 | $x \in A$ | Premise |
| 2 | $x \in A \land x \in B$ | Addition |
| 3 | $x \in A \cup B$ | Definition of Union |

$\therefore\ x \in A \to x \in A \cup B$
Conclusion: $A \subseteq A \cup B$ ☺

3. **Proof:** $A - B \subseteq A$ means $\forall x\,(x \in A - B \to x \in A)$
Assume $x \in A - B$

|   | Steps | Reasons |
|---|-------|---------|
| 1 | $x \in A - B$ | Assumption |
| 2 | $x \in A \land x \notin B$ | Definition of Difference |
| 3 | $x \in A$ | By Simplification on 2 |

$\therefore\ x \in A - B \to x \in A$
Hence $A - B \subseteq A$ ☺

4.

$$
\begin{aligned}
A \cap (B - A) &= A \cap \left(B \cap \overline{A}\right) && \text{Definition of Difference} \\
&= B \cap \left(A \cap \overline{A}\right) && \text{By Second Associative Law} \\
&= B \cap \varnothing && \text{By Second Complement Law} \\
&= \varnothing && \text{By Second Domination Law}
\end{aligned}
$$

Hence $A \cap (B - A) = \varnothing$

5.

$$
\begin{aligned}
A \cup (B - A) &= A \cup \left(B \cap \overline{A}\right) && \text{Definition of Difference} \\
&= (A \cup B) \cap \left(A \cup \overline{A}\right) && \text{By First Distributive Law} \\
&= (A \cup B) \cap U && \text{By First Complement Law} \\
&= (A \cap U) \cup (B \cap U) && \text{By Second Distributive Law} \\
&= A \cup B && \text{By First Identity Law}
\end{aligned}
$$

Hence $A \cup (B - A) = A \cup B$

---

**Question 19**

Show that if $A$ is a subset of a universal set $U$, then
1. $A \oplus A = \varnothing$
2. $A \oplus \varnothing = A$
3. $A \oplus U = \overline{A}$
4. $A \oplus \overline{A} = U$

**Solution:**

1.

$$\begin{aligned}
A \oplus A &= \{x \mid A \oplus A\} && \text{Set Builder Notation} \\
&= \{x \mid x \in (A - A) \cup (A - A)\} && \text{Definition of Symmetric Difference} \\
&= \{x \mid (x \in A - A) \vee (x \in A - A)\} && \text{Definition of Union} \\
&= \{x \mid x \in A - A\} && \text{By Idempotent Law for propositional logic} \\
&= \{x \mid (x \in A) \wedge (x \notin A)\} && \text{Definition of Difference} \\
&= \{x \mid (x \in A) \wedge \neg (x \in A)\} && \text{Definition of Complement} \\
&= \{x \mid F\} && \text{By First Negation Law for propositional logic} \\
&= \varnothing
\end{aligned}$$

Hence $A \oplus A = \varnothing$

2.

$$\begin{aligned}
A \oplus \varnothing &= \{x \mid A \oplus \varnothing\} && \text{Set Builder Notation} \\
&= \{x \mid x \in (A - \varnothing) \cup (\varnothing - A)\} && \text{Definition of Symmetric Difference} \\
&= \{x \mid x \in (A - \varnothing) \vee x \in (\varnothing - A)\} && \text{Definition of Union} \\
&= \{x \mid (x \in A \wedge x \notin \varnothing) \vee (x \in \varnothing \wedge x \notin A)\} && \text{Definition of Difference} \\
&= \{x \mid (x \in A \wedge \neg(x \in \varnothing)) \vee (x \in \varnothing \wedge \neg(x \in A))\} && \text{Definition of Complement} \\
&= \{x \mid (x \in A \wedge \neg(F)) \vee (F \wedge \neg(x \in A))\} && \text{Definition of } \in \\
&= \{x \mid (x \in A \wedge T) \vee (F)\} && \text{By Second Domination Law for propositional logic} \\
&= \{x \mid x \in A \vee F\} && \text{By First Identity Law for propositional logic} \\
&= \{x \mid x \in A\} && \text{By Second Identity Law for propositional logic} \\
&= A && \text{Definition of set } A
\end{aligned}$$

By set identities:

$$\begin{aligned}
A \oplus \varnothing &= (A - \varnothing) \cup (\varnothing - A) && \text{Definition of Symmetric Difference} \\
&= \left(A \cap \overline{\varnothing}\right) \cup \left(\varnothing \cap \overline{A}\right) && \text{Definition of Difference} \\
&= (A \cap U) \cup \left(\varnothing \cap \overline{A}\right) && \text{Complementation of an Empty set} \\
&= (A) \cup \left(\varnothing \cap \overline{A}\right) && \text{By First Identity Law} \\
&= A \cup \left(\overline{A} \cap \varnothing\right) && \text{By Second Commutative Law} \\
&= A \cup \varnothing && \text{By Second Domination Law} \\
&= A && \text{By Second Identity Law}
\end{aligned}$$

Hence $A \oplus \varnothing = A$

3.

$$A \oplus U = (A - U) \cup (U - A) \qquad \text{Definition of Symmetric Difference}$$
$$= \left(A \cap \overline{U}\right) \cup \left(U \cap \overline{A}\right) \qquad \text{Definition of Difference}$$
$$= (A \cap \varnothing) \cup \left(U \cap \overline{A}\right) \qquad \text{Complementation of the Universal set}$$
$$= \varnothing \cup \left(U \cap \overline{A}\right) \qquad \text{By Second Domination Law}$$
$$= \varnothing \cup \left(\overline{A} \cap U\right) \qquad \text{By Second Commutative Law}$$
$$= \varnothing \cup \overline{A} \qquad \text{By First Identity Law}$$
$$= \overline{A} \qquad \text{By Second Identity Law}$$

Hence $A \oplus U = \overline{A}$

4.

$$A \oplus \overline{A} = \left(A - \overline{A}\right) \cup \left(\overline{A} - A\right) \qquad \text{Definition of Symmetric Difference}$$
$$= \left(A \cap \overline{\overline{A}}\right) \cup \left(\overline{A} \cap \overline{A}\right) \qquad \text{Definition of Difference}$$
$$= (A \cap A) \cup \left(\overline{A} \cap \overline{A}\right) \qquad \text{By Complementation Law}$$
$$= A \cup \overline{A} \qquad \text{By Second Idempotent Law}$$
$$= U \qquad \text{By First Complement Law}$$

Hence $A \oplus \overline{A} = U$

## Question 20

Find two sets $A$ and $B$ such that $A \in B$ and $A \subseteq B$

*Solution:*

$$\text{Let } A = \varnothing$$
$$B = \{\varnothing\}$$

## Question 21

Find the power set of each of these sets, where $a$ and $b$ are distinct elements.
1. $\{a, b\}$
2. $\{\varnothing, \{\varnothing\}\}$

*Solution:*

1.

$$\mathcal{P}\left(\{a, b\}\right) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$$

2.

$$\mathcal{P}\left(\{\varnothing, \{\varnothing\}\}\right) = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$$

## Question 22

Find the truth set of each of these predicates where the domain is the set of integers
1. $Q(x)$: $x^2 = 2$
2. $R(x)$: $x < x^2$

***Solution:***

1. $\emptyset$

2. $\{x \in \mathbb{Z} \mid x \neq 0 \wedge x \neq 1\}$

## Question 23

Let $A$ and $B$ be sets. Show that
1. $A \subseteq (A \cup B)$
2. $A - B \subseteq A$
3. $A \cap (B - A) = \emptyset$

***Solution:***

1. ***Proof:***

$$A \subseteq (A \cup B) \text{ means } \forall x \, (x \in A \rightarrow x \in A \cup B)$$

Assume $x \in A$

| | Steps | Reasons |
|---|---|---|
| 1 | $x \in A$ | Assumption |
| 2 | $x \in A \vee x \in B$ | Addition on 1 |
| 3 | $x \in A \cup B$ | Definition of Union |

$\therefore x \in A \rightarrow x \in A \cup B$
Hence $A \subseteq (A \cup B)$ ☺

2. ***Proof:***

$$A - B \subseteq A \text{ means } \forall x \, (x \in A - B \rightarrow x \in A)$$

Assume $x \in A - B$

| | Steps | Reasons |
|---|---|---|
| 1 | $x \in A - B$ | Assumption |
| 2 | $x \in A \wedge x \notin B$ | Definition of Difference |
| 3 | $x \in A$ | Simplification of 2 |

$\therefore x \in A - B \rightarrow x \in A$
Hence $A - B \subseteq A$ ☺

3.

$$
\begin{aligned}
A \cap (B - A) &= A \cap \left( B \cap \overline{A} \right) && \text{Definition of Difference} \\
&= B \cap \left( A \cap \overline{A} \right) && \text{By Second Distributive Law} \\
&= B \cap \emptyset && \text{By Second Complement Law} \\
&= \emptyset && \text{By Second Identity Law}
\end{aligned}
$$

Hence $A \cap (B - A) = \emptyset$