Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

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Contents

Chapter 1

Sets

Definition 1.0.1: Set

An unordered collection of objects, called *elements* or *members* of the set. A set contains elements and, we can denote this as $a \in A$ where a is an element of the set A, or $a \notin A$, where a is not an element of the set A.

There are several ways to describe a set:

Roster notation $\{1, 2, 3, 4, 5\}$

Set-Builder notation Where all the elements of a set are described by a property they satisfy.i.e. The set O of all odd positive numbers less than 10 can be expressed as $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$ or specifying the domain of discourse, $O = \{x \in \mathbb{Z}^+ \mid x \text{ is odd and } x < 10\}$, or the set of all positive rational numbers $\mathbb{Q}^+$ can be expressed as $\mathbb{Q}^+ = \left\{x \in \mathbb{R} \mid x\frac{p}{q}, \text{ for some positive integers } q \text{ and } p\right\}$

Definition 1.0.2: Equality of Sets

Two sets A and B are equal if and only if they have the same elements. Therefore, $\forall x\,(x \in A \leftrightarrow x \in B)$, We write $A = B$ if this is the case.

Definition 1.0.3: Empty / Null Set

A set with no elements, denoted by $\emptyset$ or $\{\}$

Definition 1.0.4: Singleton Set

A set with exactly one element, denoted by $\{a\}$. The set $\{\emptyset\}$ is a singleton set as it is a set with one element, the empty set.

1.0.1 Set Definitions

1.0.1.1 Natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

1.0.1.2 Integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

1.0.1.3 Positive Integers

$$\mathbb{Z}^{+} = \{1, 2, 3, 4, 5, \ldots\}$$

1.0.1.4 Rational numbers

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$$

1.0.1.5 Irrational Numbers

$$\mathbb{I} = \{x \mid x \text{ is a number that cannot be expressed as a fraction}\}$$

1.0.1.6 Real numbers

$$\mathbb{R} = \{x \mid x \text{ is a point on the number line}\}$$

Or

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$$

1.0.1.7 Positive Real numbers

$$\mathbb{R}^{+} = \{x \in \mathbb{R} \mid x > 0\}$$

1.0.1.8 Complex numbers

$$\mathbb{C} = \left\{a + bi \mid a, b \in \mathbb{R} \text{ and } i^2 = -1\right\}$$

1.0.2 Venn Diagrams

Definition 1.0.5: Universal Set

The set of all objects under consideration, denoted by U.

Sets can be graphically represented using Venn diagrams. A Venn diagram is a collection of simple closed curves, especially circles, that represent sets. In Venn diagrams the universal set U which contains all the objects under consideration is represented by a rectangle, and the sets are represented by circles within the rectangle, with points inside the circles representing elements of the sets.

1.0.3 Subsets

Definition 1.0.6: Subset

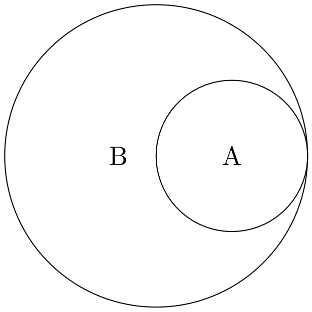
A set A is a *subset* of a set B if and only if every element of A is also an element of B. Denoted by $A \subseteq B$.

We see that $A \subseteq B$ if and only if

$$\forall x \, (x \in A \longrightarrow x \in B)$$

Is true. I.e. If $x \in A$, then $x \in B$. To disprove this we need to show that $\exists x \, (x \in A \land x \notin B)$

Shown graphically:



Example 1.0.1

The set of integers with squares less than 100 is not a subset of the set of nonnegative integers because -1 is in the former set [as $(-1)^2 < 10$], but not the later set. The set of people who have taken discrete mathematics at your school is not a subset of the set of all computer science majors at your school if there is at least one student who has taken discrete mathematics who is not a computer science major.

Theorem 1.0.1

For every set S

1. $\emptyset \subseteq S$
2. $S \subseteq S$

1. ***Proof:*** We will prove that $\emptyset \subseteq S$, using a vacuous proof
   Let S be a set.
   To show $\emptyset \subseteq S$ we must show that $\forall x\, (x \in \emptyset \longrightarrow x \in S)$ is T.
   Because $\emptyset$ contains no elements $x \in \emptyset$ is always $F/$
   This follows that the implication $x \in \emptyset \longrightarrow x \in S$ is always T
   Hence $\emptyset \subseteq S$

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Definition 1.0.7: Proper subset

A set A is *proper subset* of a set B if and only if every element of A is also an element of B and $A \neq B$. Denoted by $A \subset B$. I.e.

$$\exists x\, (x \notin A \land x \in B) \land \forall x\, (x \in A \longrightarrow x \in B)$$

Is T.

Definition 1.0.8: Further Equality

Two sets A and B are equal if $A \subseteq B \land B \subseteq A$ is T. I.e. $A = \{\emptyset, \{a\}, \{a\}, \{b\}, \{a, b\}\}$ and $B = \{x \mid x \text{ is a subset of the set } \{a, b\}\}$ are equal.

1.0.4 Cardinality

Definition 1.0.9: Cardinality

The number of distinct elements n in a set A. Denoted by $|A| = n$. Where n is a non-negative integer, we say that A is a finite set.

Definition 1.0.10: Infinite set

A set A is infinite if it is not finite. I.e. $|A| = \infty$

1.0.5 Power Set

Definition 1.0.11: Power Set

A set containing all the subsets of a given set A. Denoted by $\mathcal{P}(A)$. If a set has n distinct elements, then the cardinality of the power set is 2^n.

Example 1.0.2

Question 1

What is the power set of the set $\{0, 1, 2\}$

Solution:

$$\mathcal{P}(\{0,1,2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, , \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$$

Example 1.0.3

Question 2

What is the power set of $\emptyset$

Solution:

$$\mathcal{P}(\emptyset) = \{\emptyset\}$$

Question 3

What is the power set of $\{\emptyset\}$

Solution:

$$\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$$

1.0.6 N-Tuples

Definition 1.0.12: Ordered N-Tuple

N-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, $\ldots$, and a_n as its nth element.

Two n-tuples are equal if an only if each corresponding pair of their elements is equal, i.e. $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ are equal if and only if $a_i = b_i$, for $i = 1, 2, \ldots, n$.

Ordered 2-tuples are called *ordered pairs*. The ordered pairs, (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

1.0.7 Cartesian Products

Definition 1.0.13: Cartesian Product

Let A and B be sets. The *Cartesian Product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. I.e.

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

The number of items in the Cartesian product of two sets is the product of the cardinality of each set.

Example 1.0.4

Question 4

What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$

Solution:

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

Question 5

Show that the Cartesian product $B \times A$ is not equal to the Cartesian product $A \times B$.

Solution:

$$B \times A = \{(a, 1)(a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$$

$\therefore A \times B \neq B \times A$

Definition 1.0.14: Cartesian Product of more than two sets

The Cartesian product of the sets $A_1, A_2, \ldots, A_n$,denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$. I.e.

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$$

Example 1.0.5

Question 6

What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, $C = \{0, 1, 2\}$.

Solution:

$$A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}$$

We use the notation A^2 to denote $A \times A$, the Cartesian product of A and itself. Therefore

$$A^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for } i = 1, 2, \ldots, n\}$$

Example 1.0.6

Suppose $A = \{1, 2\}$.
It follows $A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$,
and $A^3 = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$

Example 1.0.7

Question 7

What are the ordered pairs in the less than or equal to relation, which contains, (a, b) if $a \leqslant b$, on the set $\{0, 1, 2, 3\}$

Solution: Let R be the relation on the set $\{0, 1, 2, 3\}$, if $a \leqslant b$.

$$R = \{(0,0), (1,1), (2,2), (3,3), (0,1), (0,2), (0,3), (1,2), (1,3), (2,3)\}$$

1.0.8 Set Notation with Quantifiers

We can restrict the domain of a quantifier to a set, I.e. Where S is a set $\forall x \in S\,(P(x))$, denotes the universal quantification of $P(x)$ for all elements in the set S. Which is shorthand for $\forall x\,(x \in S \longrightarrow P(x))$

Example 1.0.8

$\forall x \in \mathbb{R}\left(x^2 \geqslant 0\right)$ means "the square of any real number is greater than or equal to 0".
$\exists x \in \mathbb{Z}\left(x^2 = 1\right)$ means "there exists an integer whose square is 1"

1.0.9 Truth Sets and Quantifiers

Definition 1.0.15: Truth Set

For a predicate P the truth set of P is the set of all elements in the domain of discourse that make P true. I.e. let S be a set. The truth set of $P(x)$ is

$$\{x \in S \mid P(x)\}$$

Example 1.0.9

Question 8

What are the truth set of the predicates $P(x)$, $Q(x)$, and $R(x)$, where the domain is the set of integers, and $P(x)\colon\ |x| = 1$, $Q(x)\colon\ x^2 = 2$, and $R(x)\colon\ |x| = x$

Solution:
The truth set of P is $\{x \in \mathbb{Z} \mid |x| = 1\}$
The truth set of Q is $\left\{x \in Z \mid x^2 = 2\right\}$
The truth set of R is $\{x \in \mathbb{Z} \mid |x| = x\}$

Note:-

$\forall x P(x)$ is T over the domain $\mathbb{U}$ if and only if the truth set of P is $\mathbb{U}$.
$\exists x P(x)$ is T over the domain $\mathbb{U}$ if and only if the truth set of P is not empty.

Chapter 2

Set Operations

2.1 Set Operations

2.1.1 Union

Definition 2.1.1: Union

Let A and B be sets. The *union* of A and B, denoted by $A \cup B$, is the set of all elements that are either in A or in B or in both. I.e.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

2.1.2 Intersection

Definition 2.1.2: Intersection

Let A and B be sets. The *intersection* of A and B, denoted by $A \cap B$, is the set of all elements that are in both A and B. I.e.

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

2.1.3 Complement

Definition 2.1.3: Complement

Let A be a set. The *complement* of the set A (with respect to $\mathbb{U}$), denoted by $\overline{A}$ is the set $\mathbb{U} - A$. I.e.

$$\overline{A} = \{x \in \mathbb{U} \mid x \notin A\}$$

2.1.4 Difference

Definition 2.1.4: Difference

Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set of all elements that are in A but not in B. I.e.

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

Or

$$A - B = A \cap \overline{B}$$

2.1.5 Symmetric Difference

Definition 2.1.5: Symmetric Difference

Let A and B be sets. The *symmetric difference* of A and B, denoted by $A \oplus B$, is the set of all elements that are in exactly one of A and B. I.e.

$$A \oplus B = (A - B) \cup (B - A)$$

Example 2.1.1

Question 9

$$\begin{aligned} \mathbb{U} &= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \\ A &= \{1, 2, 3, 4, 5\} \\ B &= (4, 5, 6, 7, 8) \end{aligned}$$

What is $A \oplus B$

Solution:

$$A \oplus B = \{1, 2, 3, 6, 7, 8\}$$

2.1.6 The Cardinality of the Union of Two Sets

The cardinality of the union of two sets A and B is given by

$$|A \cup B| = |A| + |B| - |A \cap B|$$

2.2 Set Identities

2.2.1 Identity Laws

$$\begin{aligned} A \cap \mathbb{U} &= A \\ A \cup \emptyset &= A \end{aligned}$$

2.2.2 Domination Laws

$$\begin{aligned} A \cup \mathbb{U} &= \mathbb{U} \\ A \cap \emptyset &= \emptyset \end{aligned}$$

2.2.3 Idempotent Laws

$$\begin{aligned} A \cup A &= A \\ A \cap A &= A \end{aligned}$$

2.2.4 Commutative Laws

$$\begin{aligned} A \cup B &= B \cup A \\ A \cap B &= B \cap A \end{aligned}$$

2.2.5 Associative Laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

2.2.6 Distributive Laws

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup) = (A \cap B) \cup (A \cap C)$$

2.2.7 De Morgan's Laws

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

2.3 Exercises

Question 10

List the members of these sets

1. $\{x \mid x \text{ is a real number such that } x^2 = 1\}$
2.
3.
4.
5. $\{x \mid x \text{ is an integer such that } x^2 = 2\}$

Solution:

1. $\{-1, 1\}$
2.
3.
4.
5. $\emptyset$