Optimization

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Contents

Chapter 1 **Maximum and Minimum** — Page 2

1.1 Maximize 2
1.2 Minimize 2
1.3 Local/Relative Maximum/Minimum (Optimum) 2
1.4 Global/Absolute Maximum/Minimum (Optima) 3
Conditions for finding the Absolute Optima easily — 3

Chapter 2 **Concavity** — Page 4

2.1 Inflection 4
2.2 Curvature 4
Concave Up — 4 • Concave Down — 4
2.3 Questions 5

Chapter 1

Maximum and Minimum

Note:-

Local - Subsection of range
Global - Whole range

1.1 Maximize

$$R(x) = 45 - \frac{x^2}{3},\ 0 \leqslant x \leqslant 1$$

Find all local maximum <u>values</u>
Find the global maximum <u>value</u>

1.2 Minimize

- Local Minimum - Minimum in specified range
- Global Minimum - Overall minimum

1.3 Local/Relative Maximum/Minimum (Optimum)

- Find the critical values of the function:

  Stationary points, i.e. $f'(?) = 0$

  Undefined points, i.e. $f'(?) = \emptyset$

- Assess them for potential local maximum/minimum:

  Find the first derivative, input values from the left and right of the critical points and check the change in signs:

  $+$ to $-$: Maximum

  $-$ to $+$: Minimum

  Find second derivative, input the critical values and check the sign:

  $-$: Maximum

  $+$: Minimum

1.4 Global/Absolute Maximum/Minimum (Optima)

To find the Absolute Optima of a function whose domain is unrestricted:

$$\lim_{x\to\infty} f(x) \quad \lim_{x\to-\infty} f(x)$$

1.4.1 Conditions for finding the Absolute Optima easily

1. Closed Domain, i.e. $[x_1, x_2]$
2. Function is continuous for the duration of the closed domai

Theorem 1.4.1 Extreme Value Theorem

If a real valued function f is continuous on the closed interval $[a, b]$, the f must attain a maximum and minimum at least once.

$$f(c) \geqslant f(x) \geqslant F(d)$$
$$\forall x \in [a, b]$$

Where $f(c)$ is the function's minimum value and $F(d)$ is the function's maximum value.

Example 1.4.1

$$f(x) = x^3 \text{ on } [-1, 10]$$

- $f(x)$ is continuous due to it being a polynomial
- The function's domain is closed due to the end values being included in the domain

By EVT(1.4.1) $f(x)$ must attain absolute maximum and minimum at least once on the interval. Possibly at:

1. End points of the domain
2. Critical values of $f(x)$

$$\begin{aligned} f(x) &= x^3 \\ f'(x) &= 3x^2 \\ 0 &= 3x^2 \\ \frac{0}{3} &= x^2 \\ 0 &= x \end{aligned}$$

$$\begin{aligned} f(-1) &= -1 \\ f(10) &= 1000 \end{aligned}$$

$$\begin{aligned} \therefore \text{Absolute Maximum is } 1000 \\ \text{Absolute Minimum is } -1 \end{aligned}$$

Chapter 2

Concavity

Let f be a function that is differentiable over an open interval I

- If f' is increasing over I, we say f is concave up over I, i.e. $f'' > 0$
- If f' is decreasing over I, we say f is concave down over I, i.e $f'' < 0$

2.1 Inflection

A point where a function switches concavity, i.e:

$$\begin{gathered} f''(x^-) = +\text{ve} \quad \text{to} \quad f''(x^+) = -\text{ve} \\ \text{or} \\ f''(x^-) = -\text{ve} \quad \text{to} \quad f''(x^+) = +\text{ve} \end{gathered}$$

2.2 Curvature

2.2.1 Concave Up

The <u>cave</u> is facing up

2.2.2 Concave Down

The <u>cave</u> is facing down

2.3 Questions

Question 1

A closed box with a square base is to contain 252 cubic feet. The bottom costs \$ 5 per square foot, the top costs \$2 per square foot, and the sides costs \$ 3 per square foot. Find the dimensions that minimize the cost.

Solution:

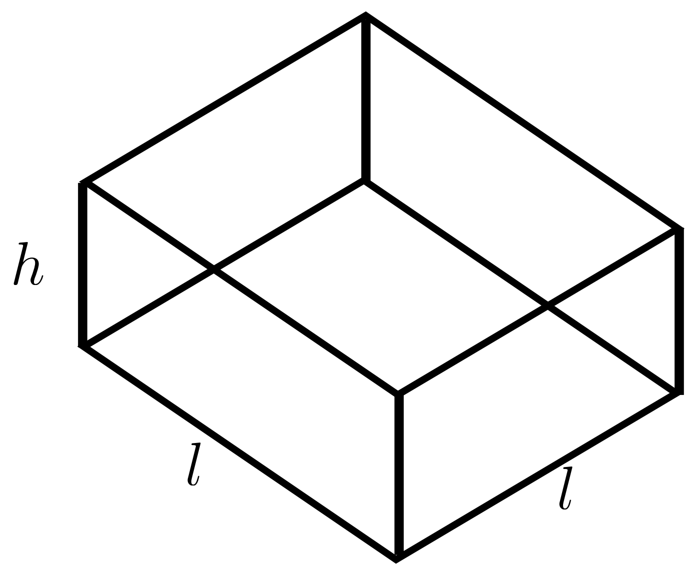

Figure 2.1: Box

$$
\begin{aligned}
V &= L \times B \times H \\
V &= L \times L \times H \\
V &= 252 \\
\therefore 252 &= L^2 H
\end{aligned}
$$

$$
\begin{aligned}
\text{Cost of the top (CT)} &= 2L^2 \\
\text{Cost of the bottom (CB)} &= 5L^2 \\
\text{Cost of one side (CS)} &= 3HL \\
\text{Total cost (TC)} &= CT + CB + 4(CS)
\end{aligned}
$$

$$
\begin{aligned}
TC &= 2L^2 + 4(3HL) + 5L^2 \\
H &= \frac{252}{L^2} \\
TC &= 7L^2 + 12(\frac{252}{L^2})L \\
TC &= 7L^2 + \frac{3024}{L}
\end{aligned}
$$

$$
\begin{aligned}
TC' &= 14L - \frac{3024}{L^2} \\
0 &= 14L^2 - \frac{3024}{L^2} \\
0 &= 14L^3 - 3024 \\
216 &= L^3 \\
6 &= L
\end{aligned}
$$

$$
\begin{aligned}
TC'' &= 14 + \frac{6048}{L^3} \\
TC''(6) &= 42 \quad \therefore \text{Minimum}
\end{aligned}
$$

$$
\begin{aligned}
H &= \frac{252}{(6)^2} \\
H &= 7
\end{aligned}
$$

$\therefore$ at a width of 6 ft and a height of 7 ft the total cost is minimized

Question 2

A wire 16 ft long has to be formed into a rectangle. What dimensions should the rectangle have to maximize area?

Solution:

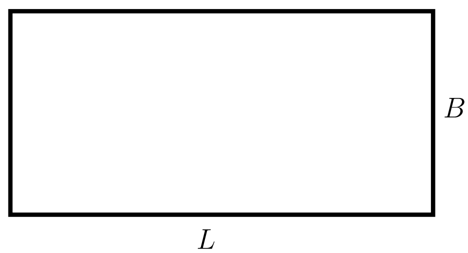

Figure 2.2: Square

$$
\begin{aligned}
A &= L \times B \\
2L + 2B &= 16 \\
2(L + B) &= 16 \\
L + B &= 8 \\
B &= 8 - L
\end{aligned}
$$

$$
\begin{aligned}
A &= L(8 - L) \\
A &= 8L - L^2 \\
A' &= -2L + 8 \\
0 &= -2L + 8 \\
L &= 4
\end{aligned}
$$

$$
\begin{aligned}
A'' &= -2 \\
\therefore L &= 4 \text{ Maximum}
\end{aligned}
$$

$$
\begin{aligned}
B &= 8 - 4 \\
B &= 4
\end{aligned}
$$

$\therefore$ at a length of 4 ft and a width of 4 ft the area is maximized