Differential Equations

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Chapter 1

Introduction

Definition 1.0.1: Differential Equation

A differential equation is an equation in the form

$$y = f(x)$$

Where $f(x)$ is an unknown function and including one or more of it's derivatives, i.e. $f'(x)$.
A solution to such an equation is a function $f(x)$ that satisfies the differential equation when f and it's derivatives are substituted into the equation.

Question 1

Verify that the function $y = e^{-3x} + 2x + 3$ is a solution to the differential equation $y' + 3y = 6x + 11$

Solution:

$$\begin{aligned} y &= e^{-3x} + 2x + 3 \\ y' &= -3e^{-3x} + 2 \end{aligned}$$

$$\begin{aligned} \text{Let } y' = -3e^{-3x} + 2 \text{ and } y = e^{-3x} + 2x + 3 \\ (-3e^{-3x} + 2) + 3(e^{-3x} + 2x + 3) = 6x + 11 \\ -3e^{-3x} + 2 + 3e^{-3x} + 6x + 9 = 6x + 11 \\ 6x + 11 = 6x + 11 \\ \text{Lhs } = \text{Rhs } \therefore \text{ y is a solution} \end{aligned}$$

Question 2

Verify that $y = 2e^{3x} - 2x - 2$ is a solution to the differential equation $y' - 3y = 6x + 4$

Solution:

$$\begin{aligned} y &= 2e^{3x} - 2x - 2 \\ y' &= 6e^{3x} - 2 \end{aligned}$$

$$\begin{aligned} \text{Let } y' = 6e^{3x} - 2 \text{ and } y &= 2e^{3x} - 2x - 2 \\ (6e^{3x} - 2) - 3(2e^{3x} - 2x - 2) &= 6x + 4 \\ 6e^{3x} - 2 - 6e^{3x} + 6x + 6 &= 6x + 4 \\ 7x + 4 &= 6x + 4 \\ \text{Lhs } &= \text{Rhs } \therefore \text{ y is a solution} \end{aligned}$$

1.1 Order of a differential equation

Definition 1.1.1: Order of a differential equation

The order of a differential equation is the highest order of any derivate of the unknown function that appears in the equation. I.e. the order of the differential equation:

$$x^2y''' - 3xy'' + xy' - 3y = \sin(x)$$

is 3 because the highest order of any derivate of y is y'''

Chapter 2

General and Particular Solutions

Definition 2.0.1: General Solution

The general solution of a differential equation is a solution that contains an arbitrary constant. I.e. the general solution of the differential equation

$$y' = 2x$$

is

$$y = x^2 + c$$

where c is an arbitrary constant.

Definition 2.0.2: Particular Solution

A particular solution of a differential equation is a solution that does not contain an arbitrary constant. I.e. the particular solution of the differential equation

$$y' = 2x$$

is

$$y = x^2 + 1$$

Question 3

Find the particular and general solution to the differential equation $y' = 2x$ passing through the point $(2, 7)$

Solution:

$$
\begin{aligned}
y' &= 2x \\
\frac{dy}{dx} &= 2x \\
\mathrm{d}y &= 2x \;\; \mathrm{d}x \\
\int 1\mathrm{d}y &= \int 2x\mathrm{d}x \\
y &= x^2 + c \\
\text{General equation } y &= x^2 + c
\end{aligned}
$$

$$
\begin{aligned}
(2, 7) \\
7 &= (2)^2 + c \\
c &= 3 \\
\text{Particular equation } y &= x^2 + 3
\end{aligned}
$$

2.1 Initial Value Problems

Definition 2.1.1: Initial Value Problem

An initial value problem is a differential equation with an initial condition. I.e. the initial value problem

$$y' = 2x \text{ with } y(2) = 7$$

Question 4

Verify that the function $y = 2e^{-2t} + e^t$ is a solution to the initial-value problem

$$y' + 2y = 3e^t \text{ with } y(0) = 3$$

Solution:

$$
\begin{aligned}
y &= 2e^{-2t} + e^t \\
y' &= -4e^{-2t} + e^t
\end{aligned}
$$

$$
\begin{aligned}
\text{Let } y' = -4e^{-2t} + e^t \text{ and } y &= 2e^{-2t} + e^t \\
(-4e^{-2t} + e^t) + 2(2e^{-2t} + e^t) &= 3e^t \\
-4e^{-2t} + e^t + 4e^{-2t} + 2e^t &= 3e^t \\
3e^t &= 3e^t \\
\text{Lhs } &= \text{Rhs } \therefore \text{ y is a solution}
\end{aligned}
$$

$$
\begin{aligned}
y(0) &= 2e^0 + e^0 \\
y(0) &= 3 \\
\therefore \text{ y is a solution to the initial value problem}
\end{aligned}
$$

Question 5

Verify that $y = 3e^{2t} + 4\sin(t)$ is a solution to the initial-value problem

$$y' - 2y = 4\cos(t) - 8\sin(t) \text{ with } y(0) = 3$$

Solution:

$$\begin{aligned} y &= 3e^{2t} + 4\sin(t) \\ y' &= 6e^{2t} + 4\cos(t) \end{aligned}$$

$$\begin{aligned} (6e^{2t} + 4\cos(t)) - 2(3e^{2t} + 4\sin(t)) &= 4\cos(t) - 8\sin(t) \\ 6e^{2t} + 4\cos(t) - 6e^{2t} - 8\sin(t) &= 4\cos(t) - 8\sin(t) \\ 4\cos(t) - 8\sin(t) &= 4\cos(t) - 8\sin(t) \\ \text{Lhs } &= \text{Rhs } \therefore \text{ y is a solution} \end{aligned}$$

$$\begin{aligned} y(0) &= 3e^{0} + 4\sin(0) \\ y(0) &= 3 + 0 \\ y(0) &= 3 \\ \therefore & \text{ y is a solution to the initial value problem} \end{aligned}$$

Question 6

Solve the following initial-value problem

$$y' = x^2 - 4x + 3 - 6e^x \text{ with } y(0) = 8$$

Solution:

$$\begin{aligned} y' &= x^2 - 4x + 3 - 6e^x \\ \frac{dy}{dx} &= x^2 - 4x + 3 - 6e^x \\ 1 \ \mathrm{d}y &= (x^2 - 4x + 3 - 6e^x) \ \mathrm{d}x \\ \int 1 \ \mathrm{d}y &= \int (x^2 - 4x + 3 - 6e^x) \ \mathrm{d}x \\ y &= \frac{1}{3}x^3 - 2x^2 + 3x - 6e^x + c \\ y(0) &= \frac{1}{3}(0)^3 - 2(0)^2 + 3(0) - 6e^0 + c \\ 8 &= -6 + c \\ c &= 14 \end{aligned}$$

$$y = \frac{1}{3}x^3 - 2x^2 + 3x - 6e^x + 14$$

Chapter 3

Application

3.1 Physics

In Physics we use the knowledge that the forces acting on an object may result in motion and Newton's second law of motion $F = ma$, where F represents force, m represents mass, and a represents acceleration, to derive an equation that can be solved to find the velocity of an object at a given time.

For instance if we have an object with mass m falling or rising from/to a height, the acceleration due to gravity will be approx. $g = 9.8m/s^2$. Then representing the velocity of the object as $v(t)$, we can represent the object falling as $v(t) < 0$ and rising as $v(t) > 0$.

We can then setup an initial-value problem to find the velocity $v(t)$ at any time t.Therefore using Newton's second law of motion $F = ma$ we can represent acceleration a as the derivate of the object's velocity at a given time $v'(t)$ giving us

$$F = mv'(t)$$

However this force F is the force of gravity acting on the object, therefore again using Newton's second law, we can represent this force as

$$F_g = -mg$$

$-$ negative since the force of gravity always works downwards. Therefore we obtain the equation

$$F = F_g$$

Which then becomes

$$mv'(t) = -mg$$
$$v'(t) = -g$$

With the initial-value being the initial velocity i.e. when $t = 0$ giving us the complete initial-value problem.

$$v'(t) = -g \text{ with } v(0) = v_0$$

Question 7

Suppose a rock falls from rest from a height of 100 meters and the only force acting on it is gravity. Find a equation for the velocity $v(t)$ as a function of time, measured in meters per second.

Solution:

$$v'(t) = -g \text{ with } v(0) = v_0$$
$$\int v'(t)\ \mathrm{d}t = \int -9.8\ \mathrm{d}t v(t) = -9.8t + c$$

$$v(0) = -9.8(0) + c$$
$$v(0) = c$$
$$c = 0$$
$$v(t) = -9.8t$$

Another question to ask is how high the object will be above the earth's surface at a given point in time. Let $s(t)$ represent the height above the Earth's surface of the object. Then using the knowledge that $s(t) = \int v(t)$ / $s'(t) = v(t)$ we can generate an initial-value problem

$$s'(t) = v(t) \text{ with } s(0) = s_0$$

Question 8

A baseball is thrown upward from a height of 3 meters above the Earth's surface with an initial velocity of 10 m/s, and the only force acting on it is gravity. The ball has a mass of 0.15 kg.

1. Find the position $s(t)$ of the baseball at time t.
2. What is its height after 2 seconds?

Solution:

1.

$$s'(t) = v(t)$$
$$\text{Let } v(t) = -9.8t + 10 \text{ and } s(0) = 3$$
$$\int s'() = \int (-9.8t + 10)$$
$$s(t) = -4.9t^2 + 10t + c$$

$$s(0) = -4.9(0)^2 + 10(0) + c$$
$$3 = cs(t) = -4.9t^2 + 10t + 3$$

2.

$$t = 2s(2) = -4.9(2)^2 + 10(2) + 3$$
$$s(2) = -19.6 + 20 + 3$$
$$s(2) = 3.4m$$