# Interpreting Derivatives

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# Chapter 1

## 1.1 Instantaneous Rate of Change

In the case where $f$ is a function of $x$ $f'(x)$ measures the instantaneous rate of change of $f$ with respect to $x$.

**Example 1.1.1**

The term widget is an economic term for a generic unit of manufacturing output. Suppose a company produces widgets and knows that the market supports a price of \$10 per unit. Let $P(n)$ give the profit, in dollars, earned by manufacturing and selling n widgets, The company likely cannot make a (positive) profit making just one widget; the start–up costs will likely exceed \$10. Mathematically, we would write this as $P(1) < 0$.
What do $P(1000) = 500$ and $P'(1000) = 0.25$ mean?. Approximate $P(1100)$
***Solution:***
The equation $P(1000) = 500$ means that selling 1,000 widgets returns a profit of \$500.
We interpret $P'(1000) = 0.25$ as meaning that the profit is increasing at the rate of \$0.25 per widget (the units are "dollars per widget").
Since we have no other information to use, out best approximation for $P(1100)$ is:

$$\begin{aligned} P(1100) &\approx P(1000) + P'(1000) \times 100 \\ &= P(1000) + P'(1000) \times 100 \\ &= 500 + 0.25 \times 100 \\ &= 525 \end{aligned}$$

We approximate that selling 1.100 widgets returns a profit of \$525

## 1.2 The Slope of the Tangent Line

We can measure the instantaneous rate of change at a given $x$ value $c$ of a non-linear function by computing $f'(c)$. We can determine the behaviour of the function $f$ by observing the slopes of its tangent lines.

## 1.3 Increasing and Decreasing Functions

### 1.3.1 Increasing Functions

$f(x)$ is increasing whenever $x_1 < x_2$ and $f(x_1) < f(x_2)$, I.e as you go up the $x$ axis the $y$ or function values increase.

$f(x)$ is increasing if the slope on any point on it's graph is positive throughout the function's entire domain.

### 1.3.2 Decreasing Functions

$f(x)$ is decreasing whenever $x_1 < x_2$ and $f(x_1) > f(x_2)$, .I.e as you go up the $x$ axis the $y$ or function values decrease

$f(x)$ is increasing if the slope on any point on it's graph is negative throughout the function's entire domain.

### 1.3.3 Critical Points

- Points where the gradient is equal to 0, i.e. $f'(x) = 0$
- Points where the gradient does not exist, i.e. $f'(x) = \emptyset$

**Example 1.3.1** ($t\sqrt[3]{t^2 - 4}$)

$$
\begin{aligned}
g(t) &= t\sqrt[3]{t^2-4} \\
g(t) &= t(t^2-4)^{\frac{1}{3}} \\
g'(t) &= (1)(t^2-4)^{\frac{1}{3}} + (\frac{1}{3})(2t)(t^2-4)\times(t) \\
g'(t) &= (t^2-4)^{\frac{1}{3}} + \frac{2}{3}t^2(t^2-4)^{-\frac{2}{3}} \\
g'(t) &= (t^2-4)^{\frac{1}{3}} + \frac{2t^2}{3(t^2-4)^{\frac{2}{3}}} \\
g'(t) &= \frac{(t^2-4)^{\frac{1}{3}}}{1} + \frac{2t^2}{3(t^2-4)^{\frac{2}{3}}} \\
g'(t) &= \frac{3(t^2-4)+t^2}{3(t^2-4)^{\frac{2}{3}}} \\
0 &= \frac{3(t^2-4)+t^2}{3(t^2-4)^{\frac{2}{3}}} \\
0 &= 3t^2-12+2t^2 \\
0 &= 5t^2-12 \\
12 &= 5t^2 \\
\frac{12}{5} &= t^2 \\
\pm\sqrt{\frac{12}{5}} &= t \\
3(t^2-4)^{\frac{2}{3}} &= 0 \\
(t^2-4)^{\frac{2}{3}} &= 0 \\
t &= \pm 2
\end{aligned}
$$

| Interval | Test Value | $\text{Slope}_{g'(x)}$ |
|---|---|---|
| $x < -2$ | $-3$ | $+$ |
| $-2 < x < -\sqrt{\frac{12}{5}}$ | $-1.7$ | $+$ |
| $-\sqrt{\frac{12}{5}} < x < \sqrt{\frac{12}{5}}$ | $0$ | $-$ |
| $\sqrt{\frac{12}{5}} < x < 2$ | $2$ | $+$ |
| $x > 2$ | $7$ | $+$ |

$$\therefore \text{When } g'(x) = 0,\ x = -\sqrt{\frac{12}{5}},\ x = \sqrt{\frac{12}{5}}$$

$$\therefore \text{Increasing } (-\infty,-2),\ (2,\infty),\ (-2,-\sqrt{\frac{12}{5}}),\ (\sqrt{\frac{12}{5}},2)$$

$$\text{Decreasing } (-\sqrt{\frac{12}{5}},\sqrt{\frac{12}{5}})$$