

Optimization

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Chapter 1

Maximum and Minimum

Note:-

Local - Subsection of range

Global - Whole range

1.1 Maximize

$$R(x) = 45 - \frac{x^2}{3}, \quad 0 \leq x \leq 1$$

Find all local maximum values

Find the global maximum value

1.2 Minimize

- Local Minimum - Minimum in specified range
- Global Minimum - Overall minimum

1.3 Local/Relative Maximum/Minimum (Optimum)

- Find the critical values of the function:

Stationary points, i.e. $f'(\cdot) = 0$

Undefined points, i.e. $f'(\cdot) = \emptyset$

- Assess them for potential local maximum/minimum:

Find the first derivative, input values from the left and right of the critical points and check the change in signs:

+ to -: Maximum

- to +: Minimum

Find second derivative, input the critical values and check the sign:

-: Maximum

+: Minimum

1.4 Global/Absolute Maximum/Minimum (Optima)

To find the Absolute Optima of a function whose domain is unrestricted:

$$\lim_{x \rightarrow \infty} f(x) \quad \lim_{x \rightarrow -\infty} f(x)$$

1.4.1 Conditions for finding the Absolute Optima easily

1. Closed Domain, i.e. $[x_1, x_2]$
2. Function is continuous for the duration of the closed domain

Theorem 1.4.1 Extreme Value Theorem

If a real valued function f is continuous on the closed interval $[a, b]$, the f must attain a maximum and minimum at least once.

$$f(c) \geq f(x) \geq F(d) \\ \forall x \in [a, b]$$

Where $f(c)$ is the function's minimum value and $F(d)$ is the function's maximum value.

Example 1.4.1

$$f(x) = x^3 \text{ on } [-1, 10]$$

- $f(x)$ is continuous due to it being a polynomial
- The function's domain is closed due to the end values being included in the domain

By EVT(1.4.1) $f(x)$ must attain absolute maximum and minimum at least once on the interval. Possibly at:

1. End points of the domain
2. Critical values of $f(x)$

$$\begin{aligned} f(x) &= x^3 \\ f'(x) &= 3x^2 \\ 0 &= 3x^2 \\ \frac{0}{3} &= x^2 \\ 0 &= x \end{aligned}$$

$$\begin{aligned} f(-1) &= -1 \\ f(10) &= 1000 \end{aligned}$$

\therefore Absolute Maximum is 1000
Absolute Minimum is -1

Chapter 2

Concavity

Let f be a function that is differentiable over an open interval I

- If f' is increasing over I , we say f is concave up over I , i.e. $f'' > 0$
- If f' is decreasing over I , we say f is concave down over I , i.e. $f'' < 0$

2.1 Inflection

A point where a function switches concavity, i.e:

$$\begin{aligned} f''(x^-) = +\text{ve} \text{ to } f''(x^+) = -\text{ve} \\ \text{or} \\ f''(x^-) = -\text{ve} \text{ to } f''(x^+) = +\text{ve} \end{aligned}$$

2.2 Curvature

2.2.1 Concave Up

The cave is facing up

2.2.2 Concave Down

The cave is facing down

2.3 Questions

Question 1

A closed box with a square base is to contain 252 cubic feet. The bottom costs \$ 5 per square foot, the top costs \$2 per square foot, and the sides costs \$ 3 per square foot. Find the dimensions that minimize the cost.

Solution:

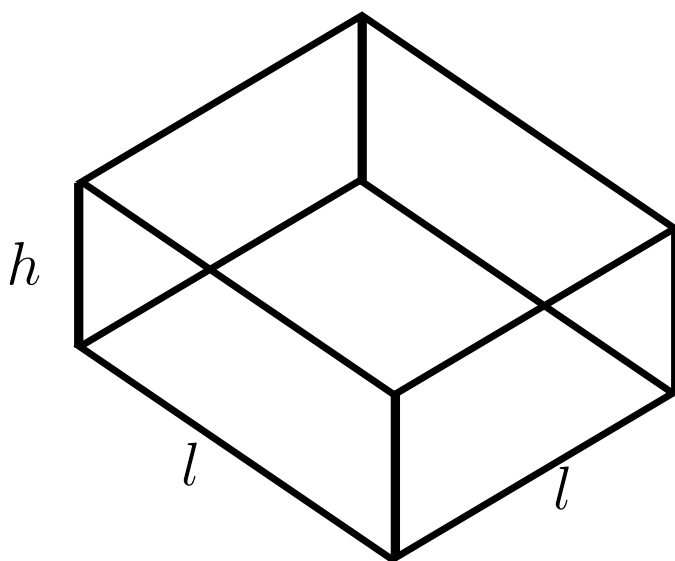


Figure 2.1: Box

$$V = L \times B \times H$$

$$V = L \times L \times H$$

$$V = 252$$

$$\therefore 252 = L^2 H$$

$$\text{Cost of the top (CT)} = 2L^2$$

$$\text{Cost of the bottom (CB)} = 5L^2$$

$$\text{Cost of one side (CS)} = 3HL$$

$$\text{Total cost (TC)} = CT + CB + 4(CS)$$

$$TC = 2L^2 + 4(3HL) + 5L^2$$

$$H = \frac{252}{L^2}$$

$$TC = 7L^2 + 12\left(\frac{252}{L^2}\right)L$$

$$TC = 7L^2 + \frac{3024}{L}$$

$$TC' = 14L - \frac{3024}{L^2}$$

$$0 = 14L^2 - \frac{3024}{L^2}$$

$$0 = 14L^3 - 3024$$

$$216 = L^3$$

$$6 = L$$

$$TC'' = 14 + \frac{6048}{L^3}$$

$$TC''(6) = 42 \therefore \text{Minimum}$$

$$H = \frac{252}{(6)^2}$$

$$H = 7$$

\therefore at a width of 6 ft and a height of 7 ft the total cost is minimized

Question 2

A wire 16 ft long has to be formed into a rectangle. What dimensions should the rectangle have to maximize area?

Solution:

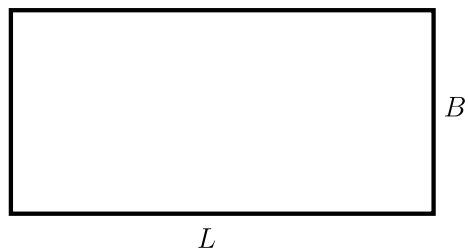


Figure 2.2: Square

$$A = L \times B$$

$$2L + 2B = 16$$

$$2(L + B) = 16$$

$$L + B = 8$$

$$B = 8 - L$$

$$A = L(8 - L)$$

$$A = 8L - L^2$$

$$A' = -2L + 8$$

$$0 = -2L + 8$$

$$L = 4$$

$$A'' = -2$$

$$\therefore L = 4 \text{ Maximum}$$

$$B = 8 - 4$$

$$B = 4$$

\therefore at a length of 4 ft and a width of 4 ft the area is maximized