Boolean Algebra

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Contents

Chapter 1

Boolean Functions

1.1 Introduction

Boolean Algebra provides the operations and rules for working with the set $\{0, 1\}$. The three operations that will be discussed are the:

- Boolean sum (**OR**) - $0 + 1 = 1$
- Boolean product (**AND**) - $0 \cdot 1 = 0$
- Complementation (**NOT**) - $\overline{0} = 1$

1.1.1 Boolean Product (AND)

Definition 1.1.1: Boolean Product

The Boolean product of two variables x and y is denoted by $x \cdot y$ and is defined by the following values:

$$1 \cdot 1 = 1$$
$$1 \cdot 0 = 0$$
$$0 \cdot 1 = 0$$
$$0 \cdot 0 = 0$$

1.1.2 Boolean Sum (OR)

Definition 1.1.2: Boolean Sum

The Boolean sum of two variables x and y is denoted by $x + y$ and is defined by the following values:

$$1 + 1 = 1$$
$$1 + 0 = 1$$
$$0 + 1 = 1$$
$$0 + 0 = 0$$

1.1.3 Complementation (NOT)

Definition 1.1.3: Complementation

The complement of a variable x is denoted by $\overline{x}$ and is defined by the following values:

$$\overline{1} = 0$$
$$\overline{0} = 1$$

Example 1.1.1

Question 1

Find the value of $1 \cdot 0 + \overline{(0+1)}$

Solution:

$$\begin{aligned} 1 \cdot 0 + \overline{(0+1)} &= 1 \cdot 0 + \overline{1} \\ &= 0 + \overline{1} \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Example 1.1.2

Question 2

Translate $1 \cdot 0 + \overline{(0+1)} = 0$, into a logical equivalence.

Solution:

$$T \wedge F \vee \neg(F \vee T) \equiv F$$

1.2 Boolean Expressions and Functions

Let $B = \{0, 1\}$, then $B^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in B \text{ for } 1 \leqslant i \leqslant n\}$ is the set of all possible n-tuples of 0's and 1's. The variable x is a *Boolean variable*.

Definition 1.2.1: Boolean variable

A variable that can take on the values 0 or 1.

Definition 1.2.2: Boolean Function

A function $f : B^n \to B$ is called a *Boolean function* of degree n. I.e. takes n inputs and returns a single output.

Example 1.2.1

The function $F\left(x, y\right) = x$ from the set of ordered pairs of Boolean variables to the set $\{0, 1\}$, has a degree of 2.

1.2.1 Complement of a Boolean function

Definition 1.2.3: Complement of a Boolean function

The complement of a Boolean function F is denoted by $\overline{F}$ and is defined by:

$$\overline{F}(x_1, x_2, \ldots, x_n) = \overline{F(x_1, x_2, \ldots, x_n)}$$

1.3 Boolean Identities

1.3.1 Law of Double Complement

$$\overline{\overline{x}} = x$$

1.3.2 Idempotent Laws

$$\begin{aligned} x + x &= x \\ x \cdot x &= x \end{aligned}$$

1.3.3 Identity Laws

$$\begin{aligned} x + 0 &= x \\ x \cdot 1 &= x \end{aligned}$$

1.3.4 Domination Laws

$$\begin{aligned} x + 1 &= 1 \\ x \cdot 0 &= 0 \end{aligned}$$

1.3.5 Commutative Laws

$$\begin{aligned} x + y &= y + x \\ xy &= yx \end{aligned}$$

1.3.6 Associative Laws

$$\begin{aligned} x + (y + z) &= (x + y) + z \\ x(yz) &= (xy)z \end{aligned}$$

1.3.7 Distributive Laws

$$\begin{aligned} x + yz &= (x + y)(x + z) \\ x(y + z) &= xy + xz \end{aligned}$$

1.3.8 De Morgan's Laws

$$\begin{aligned} \overline{(xy)} &= \overline{x} + \overline{y} \\ \overline{(x + y)} &= \overline{x} \cdot \overline{y} \end{aligned}$$

1.3.9 Absorption Laws

$$x + xy = x$$
$$x\left(x + y\right) = x$$

1.3.10 Unit Property

$$x + \overline{x} = 1$$

1.3.11 Zero Property

$$x\overline{x} = 0$$

1.4 Duality

Definition 1.4.1: Dual

The dual of a Boolean expression is obtained by replacing the **AND** operation with **OR** and the **OR** operation with **AND**, and interchanging 1s and 0s.

Example 1.4.1

Question 3

Find the duals of $x\left(y + 0\right)$ and $\overline{x} \cdot 1 + \left(\overline{y} + z\right)$

Solution:

$$x\left(y + 0\right) = x + \left(y \cdot 1\right)$$

$$\overline{x} \cdot 1 + \left(\overline{y} + z\right) = \left(\overline{x} + 0\right) \cdot \left(\overline{y}z\right)$$

The dual of a boolean function F is the function representing the dual of the expression representing F, denoted by F^d

Definition 1.4.2: Duality Principle

An identity between functions represented by boolean expressions remains valid when the duals of both sides of the expression are taken.

Example 1.4.2

Question 4

Construct an identity from the absorption law $x\left(x + y\right) = x$ by taking duals

Solution:

$$
\begin{aligned}
x\left(x+y\right) &= x \\
\text{Let } F\left(x,y\right) = x\left(x+y\right) \text{ and } G\left(x\right) &= x \\
\\
F\left(x,y\right) &= G\left(x\right) \\
F^{d}\left(x,y\right) &= G^{d}\left(x\right) \\
F^{d}\left(x,y\right) &= x+xy \\
G^{d}\left(x\right) &= x \\
\\
x+xy &= x
\end{aligned}
$$

1.5 Exercises

Question 5

Show that these identities hold

1. $x \oplus y = \left(x+y\right)\overline{\left(xy\right)}$
2. $x \oplus y = \left(x\overline{y}\right) + \left(\overline{x}y\right)$

Solution:

x	y	$\overline{x}$	$\overline{y}$	$x+y$	$\overline{(xy)}$	$x\overline{y}$	$\overline{x}y$	$x \oplus y$	$(x+y)\overline{(xy)}$	$(x\overline{y})+(\overline{x}y)$
1	1	0	0	1	0	0	0	0	0	0
1	0	0	1	1	1	1	0	1	1	1
0	0	1	1	0	1	0	1	0	0	0
0	1	1	0	1	1	0	0	1	1	1

Question 6

Show that you obtain De Morgan's Laws for propositions when you transform De Morgan's Laws for boolean algebra into logical equivalences

Solution:

$$
\begin{aligned}
\overline{\left(xy\right)} &= \overline{x}+\overline{y} \\
\text{Let } x \text{ be } p \text{ and } y \text{ be } q \\
\neg\left(p \wedge q\right) &= \neg p \vee \neg q \\
\\
\overline{\left(x+y\right)} &= \overline{x}\cdot\overline{y} \\
\neg\left(p \vee q\right) &= \neg p \wedge \neg q
\end{aligned}
$$

Question 7

Show that in a boolean algebra, the Idempotent laws $x \vee v = x$ and $x \wedge x = x$ hold for every element x

Solution:

$$
\begin{aligned}
x &= x \vee 0 && \text{By First Identity Law} \\
&= x \vee (x \wedge \neg x) && \text{By Second Complement Law} \\
&= (x \vee x) \wedge (x \vee \neg x) && \text{By First Distributive Law} \\
&= (x \vee x) \wedge 1 && \text{By First Complement Law} \\
&= x \vee x && \text{By Second Identity Law}
\end{aligned}
$$

Chapter 2

Representing Boolean Functions

2.1 Sum of Products Expansion

Definition 2.1.1: Literal

A variable or its complement.

Definition 2.1.2: Minterm

A product of literals in which each variable appears exactly once. I.e. the minterm of boolean variables $x_1, x_2, \ldots, x_n$ is a boolean product $y_1 \cdot y_2 \cdot \ldots \cdot y_n$, where

$$y_i = x_i \text{ or } y_i = \overline{x_i}$$

I.e. $y_1 \cdot y_2 \cdot \ldots \cdot y_n$ is a minterm in of the variables $x_1, x_2, \ldots, x_n$

2.1.0.1 Sum of Products / Disjunctive normal form (DNF)

Form a product (using logical and) term for each row in the truth table where the function is 1. Then sum (using logical or) all the terms together.

x	y	z	$F(x,y,z)$	$G(x,y,z)$
1	1	1	0	0
1	1	0	0	1
1	0	1	1	0
1	0	0	0	0
0	1	1	0	0
0	1	0	0	1
0	0	1	0	0
0	0	0	0	0

Example 2.1.1

Question 8

Find Boolean expressions that represent the functions, using the truth table above.

1. $F(x,y,z)$
2. $G(x,y,z)$

Solution:

1. First we look for the rows where F is 1. There is only one row, row 3. Then we determine the minterm for this row which is $x\overline{y}z$. Then we boolean sum all the found minterms to derive the function's boolean expression but since there is only one minterm the result is simply

$$F\left(x, y, z\right) = x\overline{y}z$$

2. We repeat the same process for the function G, and as there are two rows where G is 1 we will have two minterms, $xy\overline{z}$ and $\overline{x}y\overline{z}$, making the boolean expression expression

$$G\left(x, y, z\right) = xy\overline{z} + \overline{x}y\overline{z}$$

Example 2.1.2

Question 9

Find the sum-of-products of the expansion for the function $F\left(x, y, z\right) = \left(x + y\right)\overline{z}$

Solution:

$$\begin{aligned} F\left(x, y, z\right) &= \left(x + y\right)\overline{z} & \\ &= x\overline{z} + y\overline{x} & \text{By Second Distributive Law} \\ &= x1\overline{z} + y1\overline{z} & \text{By Second Identity Law} \\ &= x\left(y + \overline{y}\right)\overline{z} + y\left(x + \overline{x}\right)\overline{z} & \text{By First Unit Property Law} \\ &= xy\overline{z} + x\overline{y}\overline{z} + xy\overline{z} + \overline{x}y\overline{z} & \text{By Second Distributive Law} \\ &= xy\overline{z} + xy\overline{z} + x\overline{y}\overline{z} + \overline{x}y\overline{z} & \text{By Second Commutative Law} \\ &= xy\overline{z} + x\overline{y}\overline{z} + \overline{x}y\overline{z} & \text{By First Idempotent Law} \end{aligned}$$

$\therefore F\left(x, y, z\right) = xy\overline{z} + x\overline{y}\overline{z} + \overline{x}y\overline{z}$

2.1.1 Product of Sums Expansion / Conductive Normal Form (CNF)

A product of sums expansion is the dual of a sum of product expansion.

Example 2.1.3

$F\left(x, y, z\right) = xy\overline{z} + x\overline{y}\overline{z} + \overline{x}y\overline{z}$ can be expressed as a product of sums expansion

$$F\left(x, y, z\right) = \left(x + y + \overline{z}\right) \cdot \left(x + \overline{y} + \overline{z}\right) \cdot \left(\overline{x} + y + \overline{z}\right)$$

2.2 Exercises

Question 10

Use truth tables to prove the domination laws for Boolean.

Solution: Conclusion: $x + 1 = 1$ from column 2 and 4 and $x \cdot 0 = 0$ from column 3 and 5.

x	1	0	$x + 1$	$x \cdot 0$
1	1	0	1	0
0	1	0	1	0

Question 11

The Boolean operator $\oplus$, called **XOR** is defined by $1 \oplus 1 = 0$, $1 \oplus 0 = 1$

1. $x \oplus x$
2. $x \oplus \overline{x}$

Solution:

1.

$$
\begin{aligned}
x \oplus x \\
\\
\text{When } x = 1 \\
1 \oplus 1 = 0 \\
\\
\text{When } x = 0 \\
0 \oplus 0 = 0 \\
\\
x \oplus x = 0
\end{aligned}
$$

2.

$$
\begin{aligned}
x \oplus \overline{x} \\
\\
\text{When } x = 1 \\
1 \oplus \overline{1} \\
1 \oplus 0 = 1 \\
\\
\text{When } x = 0 \\
0 \oplus \overline{0} \\
0 \oplus 1 \\
0 \oplus 1 = 1 \\
\\
x \oplus \overline{x} = 1
\end{aligned}
$$

Question 12

Prove the absorption law $x + xy = x$ using the other boolean identities

Solution:

$$
\begin{aligned}
x + xy &= x \cdot 1 + xy && \text{By Second Identity Law} \\
&= x\left(1 + y\right) && \text{By Second Distributive Law} \\
&= x \cdot 1 && \text{By First Domination Law} \\
&= x && \text{By Second Identity Law}
\end{aligned}
$$

$$
\begin{aligned}
x\left(x+y\right) &= (x+0)\left(x+y\right) && \text{By First Identity Law}\\
&= x + 0\cdot y && \text{By First Distributive Law}\\
&= x + 0 && \text{By Second Domination Law}\\
&= x && \text{By First Identity Law}
\end{aligned}
$$

Question 13

Find the sum of products expansion of these Boolean functions

1. $F\left(x,y\right) = x + y$
2. $F\left(x,y\right) = xy$
3. $F\left(x,y\right) = 1$
4. $F\left(x,y\right) = y$

Solution:

1.

$$
\begin{aligned}
F\left(x,y\right) &= x + y\\
&= x\cdot 1 + y\cdot 1 && \text{By Second Identity Law}\\
&= x\cdot\left(y+\overline{y}\right) + y\cdot\left(x+\overline{x}\right) && \text{By Unit Property}\\
&= xy + x\overline{y} + xy + \overline{x}y && \text{By Second Distributive Law}\\
&= xy + xy + x\overline{y} + \overline{x}y && \text{By First Commutative Law}\\
&= xy + x\overline{y} + \overline{x}y && \text{By First Idempotent Law}
\end{aligned}
$$

2.

$$
\begin{aligned}
F\left(x,y\right) &= xy\\
&= x\left(y+y\right) && \text{By First Idempotent Law}\\
&= xy
\end{aligned}
$$

3.

$$
\begin{aligned}
F\left(x,y\right) &= 1\\
&= x + \overline{x} && \text{By Unit Property}\\
&= x\cdot 1 + \overline{x}\cdot 1 && \text{By Second Identity Law}\\
&= x\cdot\left(y+\overline{y}\right) + \overline{x}\cdot\left(y+\overline{y}\right) && \text{By Unit Property}\\
&= xy + x\overline{y} + \overline{x}y + \overline{xy} && \text{By Second Distributive Law}
\end{aligned}
$$

4.

$$\begin{aligned}
F(x,y) &= y \\
&= y + y && \text{By First Idempotent Law} \\
&= y \cdot 1 + y \cdot 1 && \text{By Second Identity Law} \\
&= y \cdot (x+1) + y \cdot (x+1) && \text{By First Domination Law} \\
&= xy + y + xy + y && \text{By Second Distributive Law} \\
&= xy + xy + y + y && \text{By First Commutative Law} \\
&= xy + y && \text{By First Idempotent Law} \\
&= xy + y \cdot 1 && \text{By Second Identity Law} \\
&= xy + y \cdot (x + \overline{x}) && \text{By Unit Property Law} \\
&= xy + xy + y\overline{x} && \text{By Second Distributive Law} \\
&= xy + y\overline{x} && \text{By First Idempotent Law}
\end{aligned}$$

Question 14

Find the sum of products and the product of sums expansion of the Boolean function $F(x,y,z)$ that equals 1 if and only if

1. $xy = 0$
2. $x + y = 0$
3. $xyz = 0$

Solution:

x	y	z	xy	$F(x,y,z)$	$x+y$	$F(x,y,z)$	xyz	$F(x,y,z)$
1	1	1	1	0	1	0	1	0
1	1	0	1	0	1	0	0	1
1	0	0	0	1	1	0	0	1
1	0	1	0	1	1	0	0	1
0	1	1	0	1	1	0	0	1
0	0	0	0	1	0	1	0	1
0	0	1	0	1	0	1	0	1
0	1	0	0	1	1	0	0	1

1. DNF - $x\overline{y}\overline{z} + x\overline{y}z + \overline{x}yz + \overline{x}\overline{y}\overline{z} + \overline{x}\overline{y}z + \overline{x}y\overline{z}$
   CNF - $(\overline{x} + \overline{y} + \overline{z}) \cdot (\overline{x} + \overline{y} + z)$
2. DNF - $\overline{x}\overline{y}\overline{z} + \overline{x}\overline{y}z$
   CNF - $(x+y+z) \cdot (x+y+\overline{z}) \cdot (x+\overline{y}+\overline{z}) \cdot (x+\overline{y}+z) \cdot (\overline{x}+y+z) \cdot (\overline{x}+y+\overline{z})$

2.3 Boolean Circuits

$$\overline{(\overline{x}yz)}\,(\overline{x} + y + \overline{z})$$