Homework 1

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Question 1

Find the general solution of the system whose augmented matrix is given below

$$\begin{bmatrix} 1 & 0 & -5 & 0 & -8 & 3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 0 & -5 & 0 & -8 & 3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-\frac{8}{1}R_3 - R_1 \to R_1$$

$$\begin{bmatrix} -1 & 0 & 5 & 0 & 0 & -3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-R_1 \to R_1$$

$$\begin{bmatrix} 1 & 0 & -5 & 0 & 0 & 3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - 5x_2 &= 3 \\ x_2 + 4x_3 - x_4 &= 6 \\ x_3 &= x_3 \\ x_5 &= 0 \\ x_4 &= x_4 \end{aligned}$$

$$\begin{aligned} x_1 &= 3 + 5x_2 \\ x_2 &= 6 - 4x_3 + x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \\ x_5 &= 0 \end{aligned}$$

$$\begin{cases} x_1 = 3 + 5x_2 \\ x_2 = 6 - 4x_3 + x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \\ x_5 = 0 \end{cases}$$

Question 2

Choose h and k such that the following system has:

1. No solution
2. A unique solution
3. Many solutions

Give separate answers for each part.

$$\begin{aligned} x_1 - 3x &= 1 \\ 2x_1 + hx_2 &= k \end{aligned}$$

Solution:

1.

$$\begin{bmatrix} 1 & -3 & 1 \\ 2 & h & k \end{bmatrix}$$

$$2R_1 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & (-6-h) & (2-k) \end{bmatrix}$$

$$\begin{aligned} -6 - h &= 0 \\ h &= -6 \end{aligned}$$

$$\begin{aligned} 2 - k &\neq 0 \\ k &\neq 2 \end{aligned}$$

$\therefore$ the system has no solution when $h = -6$ and $k \neq 2$. So for example $h = -6$ and $k = 0$

2.

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & (-6-h) & (2-k) \end{bmatrix}$$

$$\begin{aligned} -6 - h &\neq 0 \\ h &\neq -6 \end{aligned}$$

$$\begin{aligned} 2 - k &\neq 0 \\ k &\neq 2 \end{aligned}$$

$\therefore$ the system has a unique solution when $h \neq -6$ and $k \neq 2$. So for example $h = 0$ and $k = 0$

3.

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & (-6-h) & (2-k) \end{bmatrix}$$

$$\begin{aligned} -6 - h &= 0 \\ h &= -6 \end{aligned}$$

$$\begin{aligned} 2 - k &= 0 \\ k &= 2 \end{aligned}$$

$\therefore$ the system has many solutions when $h = -6$ and $k = 2$.

Question 3

A system of linear equations with fewer equations than unknowns is sometimes called an underdetermined system. Give an example of an inconsistent underdetermined system of two equations in three unknowns.

Solution:

$$
\begin{aligned}
2x_1 + 3x_2 + 4x_3 &= 1 \\
2x_1 + 3x_2 + 4x_3 &= 10
\end{aligned}
$$

Question 4

Determine if $\mathbf{b}$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$$

Solution: To determine if $\mathbf{b}$ is a linear combination of the given vectors I must prove that there exists weights x_1, x_2, x_3 such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_2 = \mathbf{b}$$

Therefore:

$$\begin{aligned} x_1 - 2x_2 - 6x_3 &= 11 \\ 3x_2 + 7x_3 &= -5 \\ x_1 - 2x_2 + 5x_3 &= 9 \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & -2 & -6 & 11 \\ 1 & -2 & 5 & 9 \\ 0 & 3 & 7 & -5 \end{bmatrix}$$

$$R_1 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 0 & -11 & 2 \\ 0 & 3 & 7 & -5 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & -11 & 2 \end{bmatrix}$$

$$\frac{-2}{3} R_2 - R_1 \longrightarrow R_1$$

$$\begin{bmatrix} -1 & 0 & \frac{4}{3} & \frac{-23}{3} \\ 0 & 3 & 7 & -5 \\ 0 & 0 & -11 & 2 \end{bmatrix}$$

$$\frac{-7}{11} R_3 - R_2 \longrightarrow R_2$$

$$\begin{bmatrix} -1 & 0 & \frac{4}{3} & \frac{-23}{3} \\ 0 & -3 & 0 & \frac{41}{11} \\ 0 & 0 & -11 & 2 \end{bmatrix}$$

$$\frac{-4}{33} R_3 - R_1 \longrightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{245}{33} \\ 0 & -3 & 0 & \frac{41}{11} \\ 0 & 0 & -11 & 2 \end{bmatrix}$$

$$-\frac{1}{3} R_2 \longrightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{245}{33} \\ 0 & 1 & 0 & -\frac{41}{33} \\ 0 & 0 & -11 & 2 \end{bmatrix}$$

$$-\frac{1}{11} R_3 \longrightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{245}{33} \\ 0 & 1 & 0 & -\frac{41}{33} \\ 0 & 0 & 1 & -\frac{2}{11} \end{bmatrix}$$

$\therefore$ the vector $\mathbf{b}$ is a linear combination of the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ for weights:

$$x_1 = \frac{245}{33}$$
$$x_2 = -\frac{41}{33}$$
$$x_3 = -\frac{2}{11}$$

Question 5

Show that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution for all possible $\mathbf{b}$ and describe the set of all $\mathbf{b}$ for which the equation does have a solution.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Solution:

The equation $A\mathbf{x} = \mathbf{b}$ is as follows:

$$\begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

resulting in the system:

$$x_1 - 2x_2 - x_3 = b_1$$
$$-2x_1 + 2x_2 = b_1$$
$$4x_1 - x_2 + 3x_2 = b_3$$

Finding the solutions to the system:

$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ -2 & 2 & 0 & b_2 \\ 4 & -1 & 3 & b_3 \end{bmatrix}$$
$$-2R_1 - R_2 \longrightarrow R_2$$
$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & 2 & 2 & -2b_1 - b_2 \\ 4 & -1 & 3 & b_3 \end{bmatrix}$$
$$4R_1 - R_3 \longrightarrow R_3$$
$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & 2 & 2 & -2b_1 - b_2 \\ 0 & -7 & -7 & 4b_1 - b_3 \end{bmatrix}$$
$$-\frac{7}{2}R_2 - R_3 \longrightarrow R_3$$
$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & 2 & 2 & -2b_1 - b_2 \\ 0 & 0 & 0 & 3b_1 + \frac{7}{2}b_2 + b_3 \end{bmatrix}$$

From the echelon form I can see that this system cannot be consistent for all values of $\mathbf{b}$ because not all values of $\mathbf{b}$ will result in the expression $3b_1 + \frac{7}{2}b_2 + b_3$ evaluating to 0.

$$3b_1 + \frac{7}{2}b_2 + b_3 = 0$$

The set of **b** for which the equation does have a solution is the set of all **b** that satisfy the above equation.