

# Orthogonality and Least Squares

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# Chapter 1

## Inner Product, Length and Orthogonality

### 1.1 Inner Product

#### Definition 1.1.1: Inner / Dot Product

If  $\mathbf{u}$  and  $\mathbf{v}$  are vectors in  $\mathbb{R}^n$ , then we regard  $\mathbf{u}$  and  $\mathbf{v}$  as  $n \times 1$  matrices. The transpose of  $\mathbf{u}^T$  is a  $1 \times n$  matrix, and the matrix product  $\mathbf{u}^T \mathbf{v}$  is a  $1 \times 1$  matrix, a scalar. This scalar is called the *inner / dot product* of  $\mathbf{u}$  and  $\mathbf{v}$  which can also be referred to as:

$$\mathbf{u} \cdot \mathbf{v}$$

Which breaks down into:

$$\mathbf{u}^T \times \mathbf{v}$$

When  $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ , is then defined as:

$$\begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

#### Example 1.1.1

##### Question 1

Compute  $\mathbf{u} \cdot \mathbf{v}$  and  $\mathbf{v} \cdot \mathbf{u}$  for  $\mathbf{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$

**Solution:**

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= \mathbf{u}^T \times \mathbf{v} = \begin{bmatrix} 2 & -5 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix} \\ &= 3(2) + (-5)(2) + (-1)(-3) \\ &= -1\end{aligned}$$

$$\begin{aligned}\mathbf{v} \cdot \mathbf{u} &= \mathbf{v}^T \times \mathbf{u} = \begin{bmatrix} 3 & 2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} \\ &= 2(3) + 2(-5) + (-3)(-1) \\ &= -1\end{aligned}$$

### Theorem 1.1.1 Axioms of Inner / Dot products

Let  $\mathbf{u}$  and  $\mathbf{v}$ , and  $\mathbf{w}$  be vectors in  $\mathbb{R}^n$ , and let  $c$  be a scalar. Then

1.  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2.  $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
3.  $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$
4.  $\mathbf{u} \cdot \mathbf{u} \geq 0$ , and  $\mathbf{u} \cdot \mathbf{u} = 0$  if and only if  $\mathbf{u} = \mathbf{0}$

## 1.2 Length of a Vector

### Definition 1.2.1: Length of a Vector

If  $\mathbf{v}$  is in  $\mathbb{R}^n$ , with entries  $v_1, \dots, v_n$ , then the square root of  $\mathbf{v} \cdot \mathbf{v}$  is defined because  $\mathbf{v} \cdot \mathbf{v}$  is non-negative. Therefore the *length / norm* of  $\mathbf{v}$  is the non-negative scalar  $\|\mathbf{v}\|$ , defined:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2} \text{ and } \|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$$

And similarly for any scalar  $c$ , the length of  $c\mathbf{v}$  is  $|c|$  times the length of  $\mathbf{v}$ , i.e:

$$\|c\mathbf{v}\| = |c| \times \|\mathbf{v}\|$$

### Definition 1.2.2: Unit Vector

A vector whose length is 1. If we divide a non zero vector by it's length, i.e. multiply by  $\frac{1}{\|\mathbf{v}\|}$ , we obtain a unit vector  $\mathbf{u}$ . This process of creating a unit vector  $\mathbf{u}$  from  $\mathbf{v}$  can be called *normalizing*  $\mathbf{v}$ , and the resulting  $\mathbf{u}$  is in the same direction as  $\mathbf{v}$

## 1.3 Distance in $\mathbb{R}^n$

### Definition 1.3.1: Distance between two vectors

For  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbb{R}^n$ , the *distance between  $\mathbf{u}$  and  $\mathbf{v}$* , expressed as  $\text{dist}(\mathbf{u}, \mathbf{v})$ , is the length of the vector  $\mathbf{u} - \mathbf{v}$ :

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

Then defined:

$$\begin{aligned}\text{dist}(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\| = \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})} \\ &= \sqrt{(u_1 - v_1)^2 + \dots + (u_n - v_n)^2}\end{aligned}$$

In  $\mathbb{R}^2$  and  $\mathbb{R}^3$ , this is basically the same as the Euclidean distance between two points.

### Example 1.3.1

#### Question 2

Compute the distance between the vectors  $\mathbf{u} = (7, 1)$  and  $\mathbf{v} = (3, 2)$

**Solution:**

$$\begin{aligned}\text{dist}(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\| \\ &= \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})} \\ &= \sqrt{\begin{bmatrix} 4 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \end{bmatrix}} \\ &= \sqrt{4^2 + (-1)^2} \\ &= \sqrt{17}\end{aligned}$$

## 1.4 Orthogonal Vectors

Consider  $\mathbb{R}^2$  and  $\mathbb{R}^3$  and two lines through the origin determined by vectors  $\mathbf{u}$  and  $\mathbf{v}$ . These lines are geometrically perpendicular if and only if the distance from  $\mathbf{u}$  to  $\mathbf{v}$  is the same as the distance from  $\mathbf{u}$  to  $-\mathbf{v}$ . This is equivalent to saying the squares of the distances are the same. Therefore:

$$\begin{aligned}[\text{dist}(\mathbf{u}, -\mathbf{v})]^2 &= \|\mathbf{u} - (-\mathbf{v})\|^2 = \|\mathbf{u} + \mathbf{v}\|^2 \\ &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot (\mathbf{u} + \mathbf{v}) + \mathbf{v} \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u} \cdot \mathbf{v}\end{aligned}$$

And then  $\text{dist}(\mathbf{u}, \mathbf{v})$ :

$$[\text{dist}(\mathbf{u}, \mathbf{v})]^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}$$

This shows that the two squared distances are only equal if and only if  $2\mathbf{u} \cdot \mathbf{v} = -2\mathbf{u} \cdot \mathbf{v}$ , which happens if and only if  $\mathbf{u} \cdot \mathbf{v} = 0$

#### Definition 1.4.1: Orthogonality

Two vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbb{R}^n$  are orthogonal, to each other, if  $\mathbf{u} \cdot \mathbf{v} = 0$

This then confirms that the zero vector  $\mathbf{0}$  is orthogonal to every vector in  $\mathbb{R}^n$ , since  $\mathbf{0}^T \mathbf{v} = 0$  for every  $\mathbf{v}$ .

#### Theorem 1.4.1 The Pythagorean Theorem

If  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal vectors in  $\mathbb{R}^n$ , then:

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

## 1.5 Exercises

### Question 3

Let  $\mathbf{a} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$  and  $\mathbf{b} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ . Compute  $\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}$  and  $\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a}$

**Solution:**

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (-2)(-3) + 1 \\ &= 7 \end{aligned}$$

$$\begin{aligned} \mathbf{a} \cdot \mathbf{a} &= (-2)^2 + 1 \\ &= 5 \end{aligned}$$

$$\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} = \frac{7}{5}$$

$$\begin{aligned} \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a} &= \frac{7}{5} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -2.8 \\ \frac{7}{5} \end{bmatrix} \end{aligned}$$

### Question 4

Let  $\mathbf{c} = \begin{bmatrix} \frac{4}{3} \\ -1 \\ \frac{2}{3} \end{bmatrix}$  and  $\mathbf{d} = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}$ .

1. Find a unit vector  $\mathbf{u}$  in the direction of  $\mathbf{c}$
2. Show that  $\mathbf{d}$  is orthogonal to  $\mathbf{c}$ .
3. Use the results of parts (1) and (2) to explain why  $\mathbf{d}$  must be orthogonal to the unit vector  $\mathbf{u}$

**Solution:**

1.

$$\begin{aligned}
 \|\mathbf{c}\| &= \sqrt{\mathbf{c} \cdot \mathbf{c}} \\
 &= \sqrt{\left(\frac{4}{3}\right)^2 + (-1)^2 + \left(\frac{2}{3}\right)^2} \\
 &= \frac{\sqrt{29}}{3} \\
 \mathbf{u} &= \frac{1}{\frac{\sqrt{29}}{3}} \begin{bmatrix} \frac{4}{3} \\ -1 \\ \frac{2}{3} \end{bmatrix} \\
 &= \frac{3\sqrt{29}}{29} \begin{bmatrix} \frac{4}{3} \\ -1 \\ \frac{2}{3} \end{bmatrix} \\
 &= \begin{bmatrix} \frac{4\sqrt{29}}{29} \\ \frac{3\sqrt{29}}{29} \\ \frac{2\sqrt{29}}{29} \end{bmatrix} \\
 \|\mathbf{u}\| &= \sqrt{\mathbf{u} \cdot \mathbf{u}} \\
 &= \sqrt{\left(\frac{4\sqrt{29}}{29}\right)^2 + \left(\frac{3\sqrt{29}}{29}\right)^2 + \left(\frac{2\sqrt{29}}{29}\right)^2} \\
 &= 1
 \end{aligned}$$

2. If  $\mathbf{d}$  is orthogonal to  $\mathbf{c}$  then  $\mathbf{d} \cdot \mathbf{c} = 0$

$$\begin{aligned}
 \mathbf{d} \cdot \mathbf{c} &= \mathbf{d}^T \times \mathbf{c} \\
 &= \begin{bmatrix} 5 & 6 & -1 \end{bmatrix} \begin{bmatrix} \frac{4}{3} \\ -1 \\ \frac{2}{3} \end{bmatrix} \\
 &= 5\left(\frac{4}{3}\right) + 6(-1) - 1\left(\frac{2}{3}\right) \\
 &= \frac{20}{3} - 6 - \frac{2}{3} \\
 &= 0
 \end{aligned}$$

$\therefore \mathbf{c}$  and  $\mathbf{d}$  are orthogonal to each other.

3.  $\mathbf{d}$  is orthogonal to the unit vector  $\mathbf{u}$  because  $\mathbf{d}$  is orthogonal to  $\mathbf{c}$  of which  $\mathbf{u}$  is a scalar multiple of. I.e  $\mathbf{u}$  is in the form  $k\mathbf{c}$  for some  $k$  and:

$$\mathbf{d} \cdot \mathbf{u} = \mathbf{d} \cdot (k\mathbf{c}) = k(\mathbf{d} \cdot \mathbf{c}) = k(0) = 0$$

## Chapter 2

# Orthogonal Sets

### Definition 2.0.1: Orthogonal Set

If  $S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$  is an orthogonal set of non-zero vectors in  $\mathbb{R}^n$ , then  $S$  is linearly independent and hence is a basis for the subspace spanned by  $S$ .

### Definition 2.0.2: Orthogonal Basis

An orthogonal basis for a subspace  $W$  of  $\mathbb{R}^n$  is a basis for  $W$  that is also an orthogonal set.

### Theorem 2.0.1

Let  $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$  be an orthogonal basis for a subspace  $W$  of  $\mathbb{R}^n$ . For each  $\mathbf{y}$  in  $W$ , the weights in the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + \dots + c_p\mathbf{u}_p$$

are given by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \quad (j = 1, \dots, p)$$

### Example 2.0.1

#### Question 5

The set  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ , where

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -\frac{1}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix}$$

is an orthogonal basis for  $\mathbb{R}^3$ . Express the vector  $\mathbf{y} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$  as a linear combination of the vectors in  $S$



***Solution:***

$$\begin{aligned} \mathbf{y} &= \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 + \frac{\mathbf{y} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3} \mathbf{u}_3 \\ &= \frac{11}{11} \mathbf{u}_1 + -\frac{12}{6} \mathbf{u}_2 + -\frac{33}{2} \mathbf{u}_3 \\ &= \mathbf{u}_1 - 2\mathbf{u}_2 - 2\mathbf{u}_3 \end{aligned}$$

## Chapter 3

# Orthogonal Projections

### Definition 3.0.1: Orthogonal Projection

Let  $W$  be a subspace of  $\mathbb{R}^n$ , and let  $\mathbf{y}$  be in  $\mathbb{R}^n$ . The *orthogonal projection* of  $\mathbf{y}$  onto  $W$ , denoted  $\text{proj}_W \mathbf{y}$ , is the closest point in  $W$  to  $\mathbf{y}$ . This point is obtained by adding the orthogonal projection of  $\mathbf{y}$  onto the orthogonal complement of  $W$  to the orthogonal projection of  $\mathbf{y}$  onto  $W$ .

### Example 3.0.1

#### Question 6

Let  $\{\mathbf{u}_1, \dots, \mathbf{u}_5\}$  be an orthogonal basis for a subspace  $W$  of  $\mathbb{R}^5$ , and let

$$\mathbf{y} = c_1 \mathbf{u}_1 + \dots + c_5 \mathbf{u}_5$$

Consider the subspace  $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ , and write  $\mathbf{y}$  as the sum of a vector  $\mathbf{z}_1$  in  $W$  and a vector  $\mathbf{z}_2$  in  $W^\perp$

**Solution:**

$$\mathbf{z}_1 = \mathbf{y} - \mathbf{z}_2$$

$$\mathbf{y} = \mathbf{z}_2 + \mathbf{z}_1$$

$$\mathbf{z}_1 = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2$$

$$\mathbf{z}_2 = c_3 \mathbf{u}_3 + c_4 \mathbf{u}_4 + c_5 \mathbf{u}_5$$

$$\mathbf{y} = \mathbf{z}_1 + \mathbf{z}_2 = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + c_3 \mathbf{u}_3 + c_4 \mathbf{u}_4 + c_5 \mathbf{u}_5$$

### Theorem 3.0.1 The Orthogonal Decomposition Theorem

Let  $W$  be a subspace of  $\mathbb{R}^n$ . Then each  $\mathbf{y}$  in  $\mathbb{R}^n$  can be written uniquely in the form:

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where  $\hat{\mathbf{y}}$  is in  $W$  and  $\mathbf{z}$  is in  $W^\perp$ . In fact, if  $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$  is any orthogonal basis of  $W$ , then:

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \dots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p} \mathbf{u}_p$$

and  $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$

### Example 3.0.2

#### Question 7

Let  $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ ,  $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ , and  $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ . Observe that  $\{\mathbf{u}_1, \mathbf{u}_2\}$  is an orthogonal basis for  $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ . Write  $\mathbf{y}$  as the sum of a vector in  $W$  and a vector orthogonal to  $W$

**Solution:** The orthogonal projection of  $\mathbf{y}$  onto  $W$  is:

$$\begin{aligned}\hat{\mathbf{y}} &= \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 \\ &= \frac{9}{30} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{5} \\ \frac{3}{2} \\ -\frac{1}{10} \end{bmatrix} + \begin{bmatrix} -1 \\ \frac{1}{2} \\ \frac{1}{6} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{2}{5} \\ 2 \\ \frac{1}{5} \end{bmatrix}\end{aligned}$$

So:

$$\begin{aligned}\mathbf{z} &= \mathbf{y} - \hat{\mathbf{y}} \\ &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -\frac{2}{5} \\ 2 \\ \frac{1}{5} \end{bmatrix} \\ &= \begin{bmatrix} \frac{7}{5} \\ 0 \\ \frac{14}{5} \end{bmatrix}\end{aligned}$$

$\mathbf{z}$  is orthogonal to  $W$  due to 3, so  $\mathbf{y}$  can be expressed as:

$$\mathbf{y} = \begin{bmatrix} -\frac{2}{5} \\ 2 \\ \frac{1}{5} \end{bmatrix} + \begin{bmatrix} \frac{7}{5} \\ 0 \\ \frac{14}{5} \end{bmatrix}$$

## 3.1 Properties of Orthogonal Projections

If  $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$  is an orthogonal basis for  $W$  and if  $\mathbf{y}$  happens to be in  $W$ , then the formula for  $\text{proj}_W \mathbf{y}$  is exactly the same as the representation of  $\mathbf{y}$  in terms of the basis. In this case,  $\text{proj}_W \mathbf{y} = \mathbf{y}$ .

### Theorem 3.1.1

If  $\mathbf{y}$  is in  $W = \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ , then  $\text{proj}_W \mathbf{y} = \mathbf{y}$

This leads to the next theorem:

**Theorem 3.1.2 The Best Approximation Theorem**

Let  $W$  be a subspace of  $\mathbb{R}^n$ , let  $\mathbf{y}$  be any vector in  $\mathbb{R}^n$ , and let  $\hat{\mathbf{y}}$  be an orthogonal projection of  $\mathbf{y}$  onto  $W$ . Then  $\hat{\mathbf{y}}$  is the closest point in  $W$  to  $\mathbf{y}$ , in the sense that

$$\|\mathbf{y} - \hat{\mathbf{y}}\| < \|\mathbf{y} - \mathbf{v}\|$$

For all  $\mathbf{v}$  in  $W$  distinct from  $\hat{\mathbf{y}}$

**Definition 3.1.1: Orthonormality**

A set of vectors is orthonormal if each of them are orthogonal to each other and have a length of 1.

**Theorem 3.1.3 Orthonormal Basis**

If  $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$  is an Orthonormal basis for a subspace  $W$  of  $\mathbb{R}^n$ , then

$$\text{proj}_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1) \mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2) \mathbf{u}_2 + \dots + (\mathbf{y} \cdot \mathbf{u}_p) \mathbf{u}_p$$

If  $U = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \dots \quad \mathbf{u}_p]$ , then

$$\text{proj}_W \mathbf{y} = UU^T \mathbf{y} \quad \forall \mathbf{y} \in \mathbb{R}^n$$

**3.1.1 Exercises****Question 8**

The distance from a point  $\mathbf{y}$  in  $\mathbb{R}^n$  to a subspace  $W$  is defined as the distance from  $\mathbf{y}$  to the nearest point in  $W$ . Find the distance from  $\mathbf{y}$  to  $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ , where

$$\mathbf{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

**Solution:**

$$\begin{aligned}\hat{\mathbf{y}} &= \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 \\&= \frac{15}{30} \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + -\frac{21}{6} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \\&= \begin{bmatrix} \frac{5}{2} \\ -1 \\ \frac{15}{30} \end{bmatrix} + \begin{bmatrix} -\frac{21}{6} \\ -7 \\ \frac{21}{6} \end{bmatrix} \\&= \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{z} &= \mathbf{y} - \hat{\mathbf{y}} \\&= \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix} - \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix} \\&= \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\|\mathbf{z}\| &= \sqrt{3^2 + 6^2} \\&= \sqrt{45}\end{aligned}$$

## Chapter 4

# The Gram-Schmidt Process

The Gram-Schmidt process is a simple algorithm for producing an orthogonal or orthonormal basis for any non-zero subspace of  $\mathbb{R}^n$ .

### Example 4.0.1

#### Question 9

Let  $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$  and  $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ , and  $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ . Then  $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$  is clearly linearly independent and thus a basis for a subspace  $W$  of  $\mathbb{R}^4$ . Construct an orthogonal basis for  $W$ .

#### Solution:

**Step 1** Let  $\mathbf{v}_1 = \mathbf{x}_1$  and  $W_1 = \text{Span}\{\mathbf{x}_1\} = \text{Span}\{\mathbf{v}_1\}$

**Step 2** Let  $\mathbf{v}_2$  be the vector produced by subtracting from  $\mathbf{x}_2$  its projection onto the subspace  $W_1$ . That is, let

$$\begin{aligned}\mathbf{v}_2 &= \mathbf{x}_2 - \text{proj}_{W_1} \mathbf{x}_2 \\ &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1\end{aligned}$$

Since  $\mathbf{v}_1 = \mathbf{x}_1$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

$\mathbf{v}_2$  is the component of  $\mathbf{x}_2$  orthogonal to  $\mathbf{x}_1$ , and  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is an orthogonal basis for the subspace  $W_2$  spanned by  $\mathbf{x}_1$  and  $\mathbf{x}_2$

**Step 2' (Optional)** If possible scale  $\mathbf{v}_2$  to simplify future calculations. Since  $\mathbf{v}_2$  has fractional entries, it is convenient to scale it by a factor of 4 and replace  $\{\mathbf{v}_1, \mathbf{v}_2\}$  by the orthogonal basis

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2' = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

**Step 3** Let  $\mathbf{v}_3$  be the vector produced by subtracting from  $\mathbf{x}_3$  its projection onto the subspace  $W_2$ , using the

orthogonal basis  $\{\mathbf{v}_1, \mathbf{v}'_2\}$  to compute this projection to  $W_2$ :

$$\text{proj}_{W_2} \mathbf{x}_3 = \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 + \frac{\mathbf{x}_3 \cdot \mathbf{v}'_2}{\mathbf{v}'_2 \cdot \mathbf{v}'_2} \mathbf{v}'_2$$

$$\begin{aligned} \mathbf{x}_3 \cdot \mathbf{v}_1 &= 0 + 0 + 1 + 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_1 &= 1 + 1 + 1 + 1 \\ &= 4 \end{aligned}$$

$$\begin{aligned} \mathbf{x}_3 \cdot \mathbf{v}'_2 &= 0 + 0 + 1 + 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \mathbf{v}'_2 \cdot \mathbf{v}'_2 &= 9 + 1 + 1 + 1 \\ &= 12 \end{aligned}$$

$$= \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{2}{12} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{4} \\ \frac{2}{4} \\ \frac{2}{4} \\ \frac{2}{4} \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \text{proj}_{W_2} \mathbf{x}_3$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

Therefore the orthogonal basis for  $W$  is  $\{\mathbf{v}_1, \mathbf{v}'_2, \mathbf{v}_3\}$ , where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}'_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

**Theorem 4.0.1** The Gram-Schmidt Process

Given a basis  $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$  for a non-zero subspace  $W$  of  $\mathbb{R}^n$ , define

$$\begin{aligned}\mathbf{v}_1 &= \mathbf{x}_1 \\ \mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 \\ \mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 \\ &\vdots \\ \mathbf{v}_p &= \mathbf{x}_p - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 - \dots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}} \mathbf{v}_{p-1}\end{aligned}$$

Then  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is an orthogonal basis for  $W$ . In addition

$$\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\} \quad \text{for } 1 \leq k \leq p$$

## 4.1 Orthonormal Bases

An orthonormal basis is constructed easily from an orthogonal basis  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ , by normalizing all the  $\mathbf{v}_k$ .

**Example 4.1.1**

Given the constructed basis

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

An orthonormal basis is

$$\begin{aligned}\mathbf{u}_1 &= \frac{1}{\|\mathbf{v}_1\|} \mathbf{v}_1 = \frac{1}{\sqrt{45}} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix} \\ \mathbf{u}_2 &= \frac{1}{\|\mathbf{v}_2\|} \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\end{aligned}$$

### 4.1.1 Exercises

**Question 10**

Let  $A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ . Find an orthonormal basis for the column space of  $A$ .



**Solution:**

$$\begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

$$R_1 - R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & -5 & 6 & 0 \\ 1 & 4 & 2 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$R_1 - R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & -5 & 6 & 0 \\ 0 & -5 & 2 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$R_1 - R_4 \rightarrow R_4$$

$$\begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & -5 & 6 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$$R_2 - R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & -5 & 6 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$$R_3 - R_4 \rightarrow R_4$$

$$\begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & -5 & 6 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{5}R_2 - R_1 \rightarrow R_1$$

$$\begin{bmatrix} -1 & 0 & \frac{-14}{5} & 0 \\ 0 & -5 & 6 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{3}{2}R_3 - R_2 \rightarrow R_2$$

$$\begin{bmatrix} -1 & 0 & \frac{-14}{5} & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{-7}{10}R_3 - R_1 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{5}R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{4}R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Therefore Col } A = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} \right\}$$

Let  $\mathbf{v}_1 = \mathbf{x}_1$  and  $W_1 = \text{Span}\{\mathbf{v}_1\}$

$$\begin{aligned}
 \mathbf{v}_2 &= \mathbf{x}_2 - \text{proj}_{W_1} \mathbf{x}_2 \\
 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 \\
 &= \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} - \frac{6}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
 &= \begin{bmatrix} -\frac{5}{2} \\ \frac{5}{2} \\ \frac{5}{2} \\ -\frac{5}{2} \end{bmatrix} \\
 \mathbf{v}'_2 &= \begin{bmatrix} -5 \\ 5 \\ 5 \\ -5 \end{bmatrix}
 \end{aligned}$$

Now let  $W_2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}'_2\}$

$$\begin{aligned}
 \mathbf{v}_3 &= \mathbf{x}_3 - \text{proj}_{W_2} \mathbf{x}_3 \\
 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}'_2}{\mathbf{v}'_2 \cdot \mathbf{v}'_2} \mathbf{v}'_2 \\
 &= \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \frac{4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left(-\frac{20}{100}\right) \begin{bmatrix} -5 \\ 5 \\ 5 \\ -5 \end{bmatrix} \\
 &= \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}
 \end{aligned}$$

Therefore the orthogonal basis for the column space of  $a$  is  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 5 \\ 5 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix} \right\}$ . The orthonormal basis is

$$\begin{aligned}
 \mathbf{u}_1 &= \frac{1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \\
 \mathbf{u}_2 &= \frac{1}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{100}} \begin{bmatrix} -5 \\ 5 \\ 5 \\ -5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \\
 \mathbf{u}_3 &= \frac{1}{\|\mathbf{v}_3\|} = \frac{1}{\sqrt{16}} \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}
 \end{aligned}$$

Therefore the orthonormal basis for the column space of  $A$  is  $\left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \right\}$

## Chapter 5

## Exercises

### Question 11

Show that  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  is an orthogonal basis for  $\mathbb{R}^3$ . Then express  $\mathbf{x}$  as a linear combination of the  $\mathbf{u}$ s

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \text{ and } \mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$$

**Solution:** For the basis to be orthogonal  $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$ ,  $\mathbf{u}_2 \cdot \mathbf{u}_3 = 0$ , and  $\mathbf{u}_1 \cdot \mathbf{u}_3 = 0$ .

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{u}_2 &= \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \\ &= 3(2) + (-3)(2) + 0(-1) \\ &= 6 - 6 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{u}_2 \cdot \mathbf{u}_3 &= \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \\ &= 2(1) + 2(1) + (-1)(4) \\ &= 2 + 2 - 4 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{u}_3 \cdot \mathbf{u}_1 &= \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \\ &= 1(3) + 1(-3) + 4(0) \\ &= 3 - 3 \\ &= 0 \end{aligned}$$

Therefore  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  is an orthogonal basis for  $\mathbb{R}^3$ . To express  $\mathbf{x}$  as a linear combination of the  $\mathbf{u}$ s:

$$\mathbf{x} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + c_3 \mathbf{u}_3$$

$$\begin{bmatrix} 3 & 2 & 1 & 5 \\ -3 & 2 & 1 & -3 \\ 0 & -1 & 4 & 1 \end{bmatrix}$$

$$-1R_1 - R_2 \rightarrow R_2$$

$$\begin{bmatrix} 3 & 2 & 1 & 5 \\ 0 & -4 & -2 & -2 \\ 0 & -1 & 4 & 1 \end{bmatrix}$$

$$\frac{1}{4}R_2 - R_3 \rightarrow R_3$$

$$\begin{bmatrix} 3 & 2 & 1 & 5 \\ 0 & -4 & -2 & -2 \\ 0 & 0 & \frac{-9}{2} & \frac{-3}{2} \end{bmatrix}$$

$$\frac{-1}{2}R_2 - R_1 \rightarrow R_1$$

$$\begin{bmatrix} -3 & 0 & 0 & -4 \\ 0 & -4 & -2 & -2 \\ 0 & 0 & \frac{-9}{2} & \frac{-3}{2} \end{bmatrix}$$

$$\frac{4}{9}R_3 - R_2 \rightarrow R_2$$

$$\begin{bmatrix} -3 & 0 & 0 & -4 \\ 0 & 4 & 0 & \frac{4}{3} \\ 0 & 0 & \frac{-9}{2} & \frac{-3}{2} \end{bmatrix}$$

$$\frac{-1}{3}R_1 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 4 & 0 & \frac{4}{3} \\ 0 & 0 & \frac{-9}{2} & \frac{-3}{2} \end{bmatrix}$$

$$\frac{1}{4}R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{-9}{2} & \frac{-3}{2} \end{bmatrix}$$

$$\frac{-2}{9}R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

$$c_1 = \frac{4}{3}$$

$$c_2 = \frac{1}{3}$$

$$c_3 = \frac{1}{3}$$

$$\mathbf{x} = \frac{4}{3} \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

### Question 12

Let  $W$  be the subspace spanned by the  $\mathbf{u}$ s, and write  $\mathbf{y}$  as the sum of a vector in  $W$  and a vector orthogonal to  $W$ .

$$\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

**Solution:** We can express  $\mathbf{y}$  as the sum of a vector in  $W$  and a vector orthogonal to  $W$  using the orthogonal decomposition theorem, finding the orthogonal projection of  $\mathbf{y}$  onto  $W$ , and subtracting that from  $\mathbf{y}$  to find the vector orthogonal to  $W$ .

$$\hat{\mathbf{y}} = \frac{0}{14} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \frac{28}{42} \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{10}{3} \\ \frac{3}{3} \\ \frac{8}{3} \end{bmatrix}$$

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$$

$$= \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} \frac{10}{3} \\ \frac{3}{3} \\ \frac{8}{3} \end{bmatrix}$$

$$\mathbf{z} = \begin{bmatrix} -\frac{7}{3} \\ \frac{7}{3} \\ \frac{7}{3} \end{bmatrix}$$

$$\mathbf{y} = \mathbf{z} + \hat{\mathbf{y}}$$

$$= \begin{bmatrix} -\frac{7}{3} \\ \frac{7}{3} \\ \frac{7}{3} \end{bmatrix} + \begin{bmatrix} \frac{10}{3} \\ \frac{3}{3} \\ \frac{8}{3} \end{bmatrix}$$