

Boolean Algebra

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Chapter 1

Boolean Functions

1.1 Introduction

Boolean Algebra provides the operations and rules for working with the set $\{0, 1\}$. The three operations that will be discussed are the:

- Boolean sum (**OR**) - $0 + 1 = 1$
- Boolean product (**AND**) - $0 \cdot 1 = 0$
- Complementation (**NOT**) - $\bar{0} = 1$

1.1.1 Boolean Product (AND)

Definition 1.1.1: Boolean Product

The Boolean product of two variables x and y is denoted by $x \cdot y$ and is defined by the following values:

$$1 \cdot 1 = 1$$

$$1 \cdot 0 = 0$$

$$0 \cdot 1 = 0$$

$$0 \cdot 0 = 0$$

1.1.2 Boolean Sum (OR)

Definition 1.1.2: Boolean Sum

The Boolean sum of two variables x and y is denoted by $x + y$ and is defined by the following values:

$$1 + 1 = 1$$

$$1 + 0 = 1$$

$$0 + 1 = 1$$

$$0 + 0 = 0$$

1.1.3 Complementation (NOT)

Definition 1.1.3: Complementation

The complement of a variable x is denoted by \bar{x} and is defined by the following values:

$$\bar{1} = 0$$

$$\bar{0} = 1$$

Example 1.1.1

Question 1

Find the value of $1 \cdot 0 + \overline{(0 + 1)}$

Solution:

$$\begin{aligned} 1 \cdot 0 + \overline{(0 + 1)} &= 1 \cdot 0 + \bar{1} \\ &= 0 + \bar{1} \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Example 1.1.2

Question 2

Translate $1 \cdot 0 + \overline{(0 + 1)} = 0$, into a logical equivalence.

Solution:

$$T \wedge F \vee \neg (F \vee T) \equiv F$$

1.2 Boolean Expressions and Functions

Let $B = \{0, 1\}$, then $B^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in B \text{ for } 1 \leq i \leq n\}$ is the set of all possible n -tuples of 0's and 1's. The variable x is a *Boolean variable*.

Definition 1.2.1: Boolean variable

A variable that can take on the values 0 or 1.

Definition 1.2.2: Boolean Function

A function $f : B^n \rightarrow B$ is called a *Boolean function* of degree n . I.e. takes n inputs and returns a single output.

Example 1.2.1

The function $F(x, y) = x$ from the set of ordered pairs of Boolean variables to the set $\{0, 1\}$, has a degree of 2.

1.2.1 Complement of a Boolean function

Definition 1.2.3: Complement of a Boolean function

The complement of a Boolean function F is denoted by \bar{F} and is defined by:

$$\bar{F}(x_1, x_2, \dots, x_n) = \overline{F(x_1, x_2, \dots, x_n)}$$

1.3 Boolean Identities

1.3.1 Law of Double Complement

$$\overline{\bar{x}} = x$$

1.3.2 Idempotent Laws

$$x + x = x$$

$$x \cdot x = x$$

1.3.3 Identity Laws

$$x + 0 = x$$

$$x \cdot 1 = x$$

1.3.4 Domination Laws

$$x + 1 = 1$$

$$x \cdot 0 = 0$$

1.3.5 Commutative Laws

$$x + y = y + x$$

$$xy = yx$$

1.3.6 Associative Laws

$$x + (y + z) = (x + y) + z$$

$$x(yz) = (xy)z$$

1.3.7 Distributive Laws

$$x + yz = (x + y)(x + z)$$

$$x(y + z) = xy + xz$$

1.3.8 De Morgan's Laws

$$\overline{(xy)} = \bar{x} + \bar{y}$$

$$\overline{(x + y)} = \bar{x} \cdot \bar{y}$$

1.3.9 Absorption Laws

$$\begin{aligned}x + xy &= x \\x(x + y) &= x\end{aligned}$$

1.3.10 Unit Property

$$x + \bar{x} = 1$$

1.3.11 Zero Property

$$x\bar{x} = 0$$

1.4 Duality

Definition 1.4.1: Dual

The dual of a Boolean expression is obtained by replacing the **AND** operation with **OR** and the **OR** operation with **AND**, and interchanging 1s and 0s.

Example 1.4.1

Question 3

Find the duals of $x(y + 0)$ and $\bar{x} \cdot 1 + (\bar{y} + z)$

Solution:

$$x(y + 0) = x + (y \cdot 1)$$

$$\bar{x} \cdot 1 + (\bar{y} + z) = (\bar{x} + 0) \cdot (\bar{y}z)$$

The dual of a boolean function F is the function representing the dual of the expression representing F , denoted by F^d

Definition 1.4.2: Duality Principle

An identity between functions represented by boolean expressions remains valid when the duals of both sides of the expression are taken.

Example 1.4.2

Question 4

Construct an identity from the absorption law $x(x + y) = x$ by taking duals

Solution:

$$x(x+y) = x$$

Let $F(x, y) = x(x+y)$ and $G(x) = x$

$$F(x, y) = G(x)$$

$$F^d(x, y) = G^d(x)$$

$$F^d(x, y) = x + xy$$

$$G^d(x) = x$$

$$x + xy = x$$

1.5 Exercises

Question 5

Show that these identities hold

1. $x \oplus y = (x+y)(xy)$
2. $x \oplus y = (x\bar{y}) + (\bar{x}y)$

Solution:

x	y	\bar{x}	\bar{y}	$x+y$	$\overline{(xy)}$	$x\bar{y}$	$\bar{x}y$	$x \oplus y$	$(x+y)\overline{(xy)}$	$(x\bar{y}) + (\bar{x}y)$
1	1	0	0	1	0	0	0	0	0	0
1	0	0	1	1	1	1	0	1	1	1
0	0	1	1	0	1	0	1	0	0	0
0	1	1	0	1	1	0	0	1	1	1

Question 6

Show that you obtain De Morgan's Laws for propositions when you transform De Morgan's Laws for boolean algebra into logical equivalences

Solution:

$$\overline{(xy)} = \bar{x} + \bar{y}$$

Let x be p and y be q

$$\neg(p \wedge q) = \neg p \vee \neg q$$

$$\overline{(x+y)} = \bar{x} \cdot \bar{y}$$

$$\neg(p \vee q) = \neg p \wedge \neg q$$

Question 7

Show that in a boolean algebra, the Idempotent laws $x \vee x = x$ and $x \wedge x = x$ hold for every element x

Solution:

$$\begin{aligned}x &= x \vee 0 \\&= x \vee (x \wedge \neg x) \\&= (x \vee x) \wedge (x \vee \neg x) \\&= (x \vee x) \wedge 1 \\&= x \vee x\end{aligned}$$

By First Identity Law
By Second Complement Law
By First Distributive Law
By First Complement Law
By Second Identity Law

Chapter 2

Representing Boolean Functions

2.1 Sum of Products Expansion

Definition 2.1.1: Literal

A variable or its complement.

Definition 2.1.2: Minterm

A product of literals in which each variable appears exactly once. I.e. the minterm of boolean variables x_1, x_2, \dots, x_n is a boolean product $y_1 \cdot y_2 \cdot \dots \cdot y_n$, where

$$y_i = x_i \text{ or } y_i = \overline{x_i}$$

I.e. $y_1 \cdot y_2 \cdot \dots \cdot y_n$ is a minterm in of the variables x_1, x_2, \dots, x_n

2.1.0.1 Sum of Products / Disjunctive normal form (DNF)

Form a product (using logical and) term for each row in the truth table where the function is 1. Then sum (using logical or) all the terms together.

x	y	z	$F(x, y, z)$	$G(x, y, z)$
1	1	1	0	0
1	1	0	0	1
1	0	1	1	0
1	0	0	0	0
0	1	1	0	0
0	1	0	0	1
0	0	1	0	0
0	0	0	0	0

Example 2.1.1

Question 8

Find Boolean expressions that represent the functions, using the truth table above.

1. $F(x, y, z)$
2. $G(x, y, z)$

Solution:

1. First we look for the rows where F is 1. There is only one row, row 3. Then we determine the minterm for this row which is $x\bar{y}z$. Then we boolean sum all the found minterms to derive the function's boolean expression but since there is only one minterm the result is simply

$$F(x, y, z) = x\bar{y}z$$

2. We repeat the same process for the function G , and as there are two rows where G is 1 we will have two minterms, $xy\bar{z}$ and $\bar{x}y\bar{z}$, making the boolean expression

$$G(x, y, z) = xy\bar{z} + \bar{x}y\bar{z}$$

Example 2.1.2**Question 9**

Find the sum-of-products of the expansion for the function $F(x, y, z) = (x + y)\bar{z}$

Solution:

$$\begin{aligned}
 F(x, y, z) &= (x + y)\bar{z} \\
 &= x\bar{z} + y\bar{z} && \text{By Second Distributive Law} \\
 &= x1\bar{z} + y1\bar{z} && \text{By Second Identity Law} \\
 &= x(y + \bar{y})\bar{z} + y(x + \bar{x})\bar{z} && \text{By First Unit Property Law} \\
 &= xy\bar{z} + x\bar{y}\bar{z} + xy\bar{z} + \bar{x}y\bar{z} && \text{By Second Distributive Law} \\
 &= xy\bar{z} + x\bar{y}\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z} && \text{By Second Commutative Law} \\
 &= xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z} && \text{By First Idempotent Law}
 \end{aligned}$$

$$\therefore F(x, y, z) = xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z}$$

2.1.1 Product of Sums Expansion / Conductive Normal Form (CNF)

A product of sums expansion is the dual of a sum of product expansion.

Example 2.1.3

$F(x, y, z) = xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z}$ can be expressed as a product of sums expansion

$$F(x, y, z) = (x + y + \bar{z}) \cdot (x + \bar{y} + \bar{z}) \cdot (\bar{x} + y + \bar{z})$$

2.2 Exercises**Question 10**

Use truth tables to prove the domination laws for Boolean.

Solution: Conclusion: $x + 1 = 1$ from column 2 and 4 and $x \cdot 0 = 0$ from column 3 and 5.

x	1	0	$x + 1$	$x \cdot 0$
1	1	0	1	0
0	1	0	1	0

Question 11

The Boolean operator \oplus , called **XOR** is defined by $1 \oplus 1 = 0$, $1 \oplus 0 = 1$

1. $x \oplus x$
2. $x \oplus \bar{x}$

Solution:

1.

$$x \oplus x$$

When $x = 1$

$$1 \oplus 1 = 0$$

When $x = 0$

$$0 \oplus 0 = 0$$

$$x \oplus x = 0$$

2.

$$x \oplus \bar{x}$$

When $x = 1$

$$1 \oplus \bar{1}$$

$$1 \oplus 0 = 1$$

When $x = 0$

$$0 \oplus \bar{0}$$

$$0 \oplus 1$$

$$0 \oplus 1 = 1$$

$$x \oplus \bar{x} = 1$$

Question 12

Prove the absorption law $x + xy = x$ using the other boolean identities

Solution:

$$x + xy = x \cdot 1 + xy$$

$$= x(1 + y)$$

$$= x \cdot 1$$

$$= x$$

By Second Identity Law

By Second Distributive Law

By First Domination Law

By Second Identity Law

$$\begin{aligned}
 x(x+y) &= (x+0)(x+y) \\
 &= x+0 \cdot y \\
 &= x+0 \\
 &= x
 \end{aligned}$$

By First Identity Law
 By First Distributive Law
 By Second Domination Law
 By First Identity Law

Question 13

Find the sum of products expansion of these Boolean functions

1. $F(x, y) = x + y$
2. $F(x, y) = xy$
3. $F(x, y) = 1$
4. $F(x, y) = y$

Solution:

1.

$$\begin{aligned}
 F(x, y) &= x + y \\
 &= x \cdot 1 + y \cdot 1 \\
 &= x \cdot (y + \bar{y}) + y \cdot (x + \bar{x}) \\
 &= xy + x\bar{y} + xy + \bar{x}y \\
 &= xy + x\bar{y} + x\bar{y} + \bar{x}y \\
 &= xy + x\bar{y} + \bar{x}y
 \end{aligned}$$

By Second Identity Law
 By Unit Property
 By Second Distributive Law
 By First Commutative Law
 By First Idempotent Law

2.

$$\begin{aligned}
 F(x, y) &= xy \\
 &= x(y + y) \\
 &= xy
 \end{aligned}$$

By First Idempotent Law

3.

$$\begin{aligned}
 F(x, y) &= 1 \\
 &= x + \bar{x} \\
 &= x \cdot 1 + \bar{x} \cdot 1 \\
 &= x \cdot (y + \bar{y}) + \bar{x} \cdot (y + \bar{y}) \\
 &= xy + x\bar{y} + \bar{x}y + \bar{x}\bar{y}
 \end{aligned}$$

By Unit Property
 By Second Identity Law
 By Unit Property
 By Second Distributive Law

4.

$$\begin{aligned}
 F(x, y) &= y \\
 &= y + y && \text{By First Idempotent Law} \\
 &= y \cdot 1 + y \cdot 1 && \text{By Second Identity Law} \\
 &= y \cdot (x + 1) + y \cdot (x + 1) && \text{By First Domination Law} \\
 &= xy + y + xy + y && \text{By Second Distributive Law} \\
 &= xy + xy + y + y && \text{By First Commutative Law} \\
 &= xy + y && \text{By First Idempotent Law} \\
 &= xy + y \cdot 1 && \text{By Second Identity Law} \\
 &= xy + y \cdot (x + \bar{x}) && \text{By Unit Property Law} \\
 &= xy + xy + y\bar{x} && \text{By Second Distributive Law} \\
 &= xy + y\bar{x} && \text{By First Idempotent Law}
 \end{aligned}$$

Question 14

Find the sum of products and the product of sums expansion of the Boolean function $F(x, y, z)$ that equals 1 if and only if

1. $xy = 0$
2. $x + y = 0$
3. $xyz = 0$

Solution:

x	y	z	xy	$F(x, y, z)$	$x + y$	$F(x, y, z)$	xyz	$F(x, y, z)$
1	1	1	1	0	1	0	1	0
1	1	0	1	0	1	0	0	1
1	0	0	0	1	1	0	0	1
1	0	1	0	1	1	0	0	1
0	1	1	0	1	1	0	0	1
0	0	0	0	1	0	1	0	1
0	0	1	0	1	0	1	0	1
0	1	0	0	1	1	0	0	1

1. DNF - $x\bar{y}\bar{z} + x\bar{y}z + \bar{x}yz + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}y\bar{z}$
 CNF - $(\bar{x} + \bar{y} + \bar{z}) \cdot (\bar{x} + \bar{y} + z)$
2. DNF - $\bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$
 CNF - $(x + y + z) \cdot (x + y + \bar{z}) \cdot (x + \bar{y} + \bar{z}) \cdot (x + \bar{y} + z) \cdot (\bar{x} + y + z) \cdot (\bar{x} + y + \bar{z})$

2.3 Boolean Circuits

$$\overline{(\bar{x}yz)} (\bar{x} + y + \bar{z})$$

Chapter 3

Exercises

Question 15

1. Use a truth table to express the values of the Boolean function $F(x, y, z) = x + yz$
2. Using the result from (1) write $F(x, y, z)$ in conjunctive normal form
3. Using the result from (1) write $F(x, y, z)$ in disjunctive normal form
4. Using Boolean identities, write $F(x, y, z)$ in disjunctive normal form

Solution:

x	y	z	yz	$x + yz$	$F(x, y, z)$
1	1	1	1	1	1
1	1	0	0	1	1
1	0	1	0	1	1
1	0	0	0	1	1
0	1	1	1	1	1
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0

2.

$$F(x, y, z) = (x + \bar{y} + z) \cdot (x + y + \bar{z}) \cdot (x + y + z)$$

3.

$$F(x, y, z) = xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz$$

4.

$$\begin{aligned}
 F(x, y, z) &= x + yz \\
 &= x1 + y1z && \text{Second Identity Law} \\
 &= x \cdot (y + \bar{y}) + yz1 && \text{Unit Property} \\
 &= x \cdot (y + \bar{y}) + yz \cdot (x + 1) && \text{First Domination Law} \\
 &= xy + x\bar{y} + yzx + yz1 && \text{Second Distributive Law} \\
 &= xy + x\bar{y} + yzx + yz \cdot (x + \bar{x}) && \text{Unit Property} \\
 &= xy + x\bar{y} + yzx + yzx + yz\bar{x} && \text{Second Distributive Law} \\
 &= xy1 + x\bar{y}1 + yzx + yzx + yz\bar{x} && \text{Second Identity Law} \\
 &= xy \cdot (z + \bar{z}) + x\bar{y} \cdot (z + \bar{z}) + yzx + yzx + yz\bar{x} && \text{Unit Property} \\
 &= xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + xyz + xyz + \bar{x}yz && \text{Second Distributive Law} \\
 &= xyz + xyz + xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz && \text{First Commutative Law} \\
 &= xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz
 \end{aligned}$$

Hence $F(x, y, z) = xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz$

Question 16

Construct circuits from inverters, AND gates, and OR gates to produce these outputs.

1. $\bar{x} + y$
2. $xyz + \bar{x}\bar{y}\bar{z}$

Solution: