Computer Science Track Core Course

Linear programming — recitation 1

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ISTA
Kolmogorov group

2023-05-11

We are given an input $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$ and we search for $\mathbf{x} \in \mathbb{Q}^n$ that optimizes...

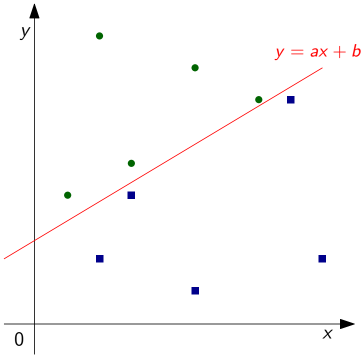
$$\begin{array}{lr} \text{minimize} & c'\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \geq b \end{array} \qquad \begin{array}{lr} \text{maximize} & c'\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq b \end{array}$$

$$\begin{array}{lr} \text{minimize} & c'\mathbf{x} \\ \text{s.t.} & A\mathbf{x} = b \\ \text{where} & \mathbf{x} \geq \mathbf{0} \end{array} \qquad \begin{array}{lr} \text{maximize} & c'\mathbf{x} \\ \text{s.t.} & A\mathbf{x} = b \\ \text{where} & \mathbf{x} \geq \mathbf{0} \end{array}$$

Example of LP

Separating points

We are searching for a line $y = ax + b$ that separates "disks" (top) from "squares" (bottom).



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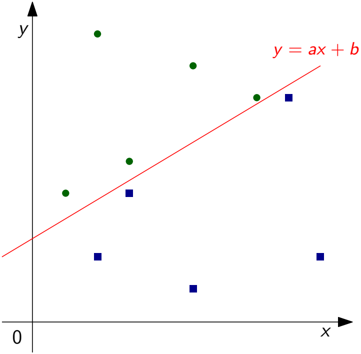

Variables:

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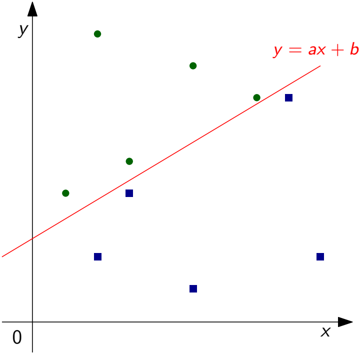

Variables: $a \lessgtr 0,\ b \lessgtr 0$

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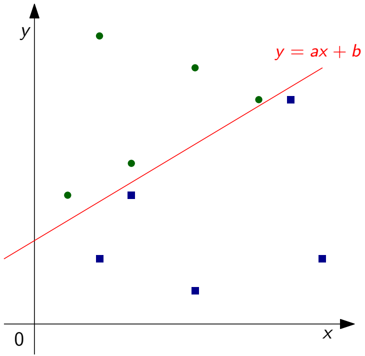

Variables: $a \lesseqgtr 0$, $b \lesseqgtr 0$

Point (x_1, y_1) above the line:

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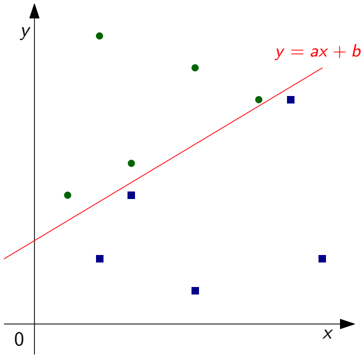

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Point (x_1, y_1) above the line: $x_1 \cdot a + b \leq y_1$ "disk"

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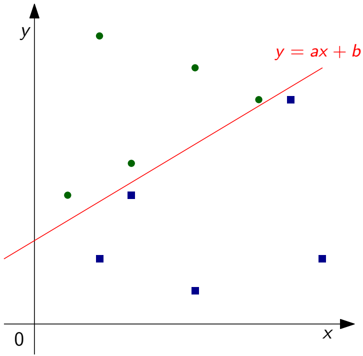

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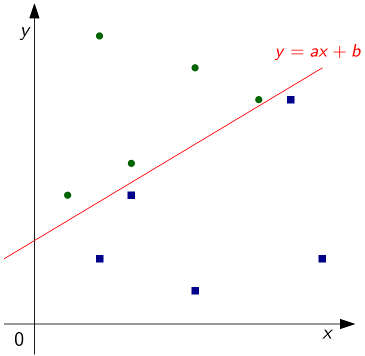

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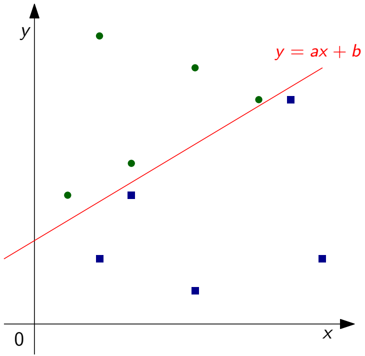

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Objective:

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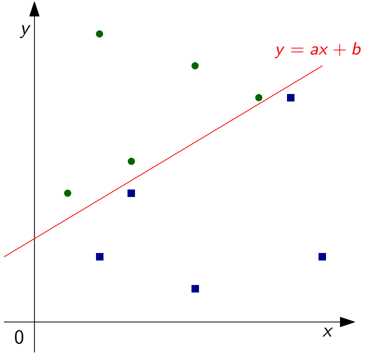

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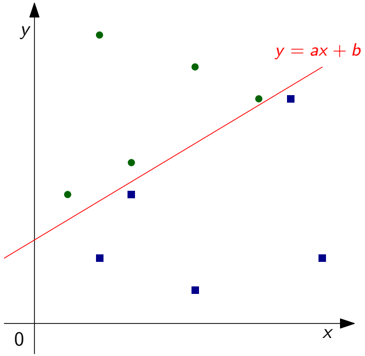
Point (x_2, y_2) below the line: $x_2 \cdot a + b \geq y_2$ "square"

Objective: minimize 0

Example of LP

Separating points strictly

Now, we are searching for a line $y = ax + b$ that separates them strictly.



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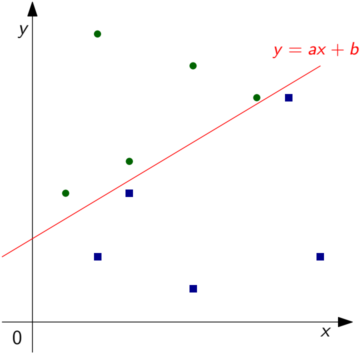

Variables: $a \lessgtr 0$, $b \lessgtr 0$, $d \geq 0$

Point (x_1, y_1) above the line: $x_1 \cdot a + b + d \leq y_1$

Point (x_2, y_2) below the line: $x_2 \cdot a + b - d \geq y_2$

Objective: maximize d

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Example of ILP

Separating points partially

We are searching for a line $y = ax + b$ that separates given points — "disks" (top) from "squares" (bottom) — with as few exceptions as possible.

Let us ignore all vertical and nearly-vertical solutions.

M is a very large integer.

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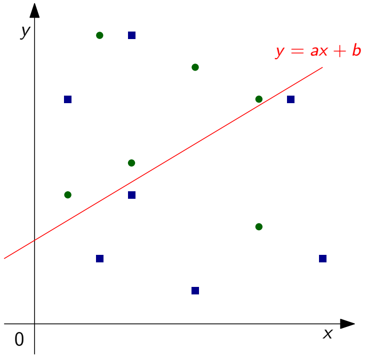

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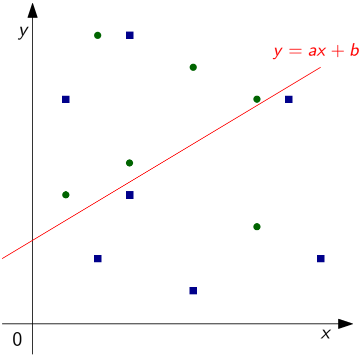

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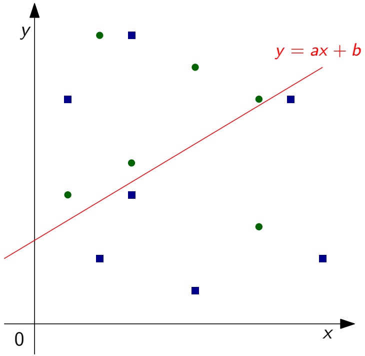

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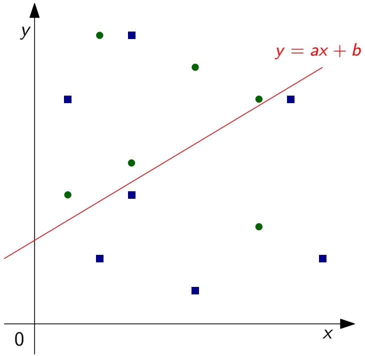

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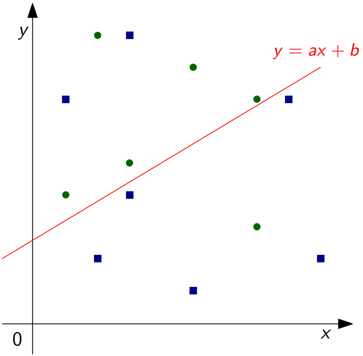

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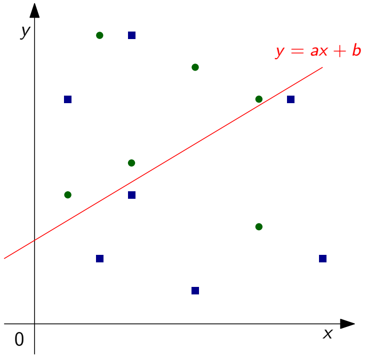

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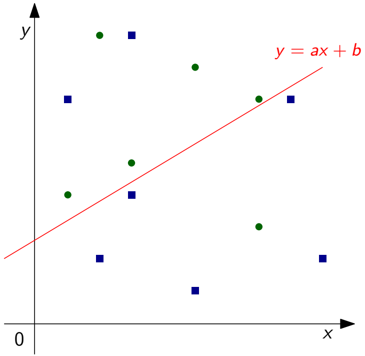

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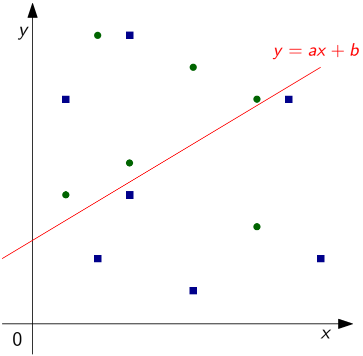

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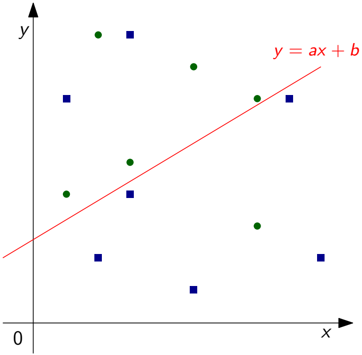

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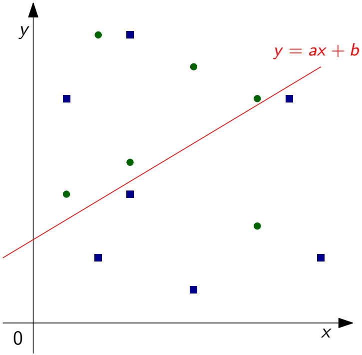

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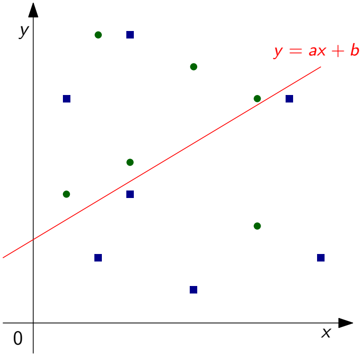

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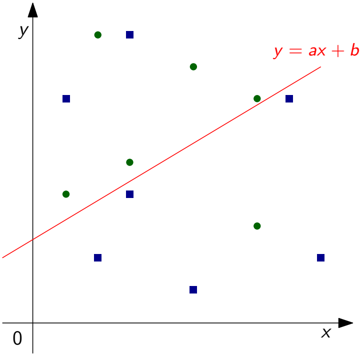

Unfortunately, there is no general algorithm for solving ILP in polynomial time (we don't know any such algorithm — but most importantly, it cannot exist unless P = NP).

This particular problem, however, can be solved in quadratic time. Consider all pairs of points.

Example of LP duality

From minimization to maximization

Primal problem:

$$
\begin{array}{lrcrcl}
\text{minimize} & 6x_1 & + & 6x_2 & & \\
\text{s.t.} & 2x_1 & + & x_2 & \geq & 4 \\
 & x_1 & + & 2x_2 & \geq & 5 \\
\text{where} & x_1, x_2 & \geq & 0 & &
\end{array}
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Dual problem:

$$\begin{array}{llcccc} \text{maximize} & 4y_1 & + & 5y_2 & & \\ \text{s.t.} & 2y_1 & + & y_2 & \leq & 6 \\ & y_1 & + & 2y_2 & \leq & 6 \\ \text{where} & y_1, y_2 & \geq & 0 & & \end{array}$$

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Dual of the dual problem:

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Example of LP duality

Comparison

Primal problem:

$$\begin{array}{lrcrcl} \text{minimize} & 6x_1 & + & 6x_2 & & \\ \text{s.t.} & 2x_1 & + & x_2 & \geq & 4 \\ & x_1 & + & 2x_2 & \geq & 5 \\ \text{where} & x_1, x_2 & \geq & 0 & & \end{array}$$

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We are given $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$ on the input.
We search for either $\mathbf{x} \in \mathbb{Q}^n$ or $\mathbf{y} \in \mathbb{Q}^m$.

Primal problem:

$$\begin{array}{lr} \text{minimize} & c'\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \geq b \\ \text{where} & \mathbf{x} \lessgtr \mathbf{0} \end{array}$$

Dual problem:

$$\begin{array}{lr} \text{maximize} & b'\mathbf{y} \\ \text{s.t.} & A^\top \mathbf{y} = c \\ \text{where} & \mathbf{y} \geq \mathbf{0} \end{array}$$

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Combinations of vectors (linear, affine, convex)

Let V be a vector space over F. Consider vectors $\mathbf{x_1}, \ldots, \mathbf{x_n} \in V$.

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- We say that $\mathbf{y} \in V$ is a linear combination of $\mathbf{x_1}, \ldots, \mathbf{x_n}$ if:

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$$\exists\, \alpha_1, \dots, \alpha_n \in F : \quad \sum_{i=1}^{n} \alpha_i = 1 \;\wedge\; \sum_{i=1}^{n} \alpha_i \mathbf{x_i} = \mathbf{y}$$

Combinations of vectors (linear, affine, convex)

Definition

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- Suppose now that F is a totally ordered field.
  Denote the set of $\varphi \in F$ such that $\varphi \geq 0$ by the symbol F_0^+.
  We say that $\mathbf{y} \in V$ is a convex combination of $\mathbf{x_1}, \ldots, \mathbf{x_n}$ if:

$$\exists\, \alpha_1, \ldots, \alpha_n \in F_0^+ : \quad \sum_{i=1}^{n} \alpha_i = 1 \;\wedge\; \sum_{i=1}^{n} \alpha_i \mathbf{x_i} = \mathbf{y}$$

Let $d \in \mathbb{Z}$ such that $d \geq 2$.
What is the maximum possible amount (largest set) of:

- Linearly independent vectors in $\mathbb{Z}_2^d$ over $\mathbb{Z}_2$?
- Affinely independent vectors in $\mathbb{C}^d$ over $\mathbb{C}$?
- Convexly independent vectors in $\mathbb{Q}^d$ over $\mathbb{Q}$?

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Combinations of vectors (linear, affine, convex)

Quiz

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  ∞

$$\{\text{convex combin.}\} \subseteq \{\text{affine combin.}\} \subseteq \{\text{linear combin.}\}$$

Combinations of vectors (linear, affine, convex)

Basic properties

$$\{\text{convex combin.}\} \subseteq \{\text{affine combin.}\} \subseteq \{\text{linear combin.}\}$$

Let V be a vector space and $X \subseteq V$.

- X is convexly dependent. $\implies$ X is affinely dependent. $\implies$ X is linearly dependent.
- X is linearly independent. $\implies$ X is affinely independent. $\implies$ X is convexly independent.

$\{$convex combin.$\} \subseteq \{$affine combin.$\} \subseteq \{$linear combin.$\}$

Let V be a vector space and $X \subseteq V$.

- X is convexly dependent. $\implies$ X is affinely dependent. $\implies$ X is linearly dependent.
- X is linearly independent. $\implies$ X is affinely independent. $\implies$ X is convexly independent.

Let us have a matrix $A \in F^{m \times n}$ and a vector $b \in F^m$.
We search for a solution $x \in F^n$.

- $Ax = 0$: Any linear combination of solutions is a solution.
- $Ax = b$: Any affine combination of solutions is a solution.
- $Ax \leq b$: Any convex combination of solutions is a solution.
  In this example, F must be a totally ordered field.