Computer Science Track Core Course

Linear programming — recitation 2

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2023-05-16

Homework 1

Deadline: Thursday 2023-05-25 at 14:45

Please do not send me scans.

- You can typeset it and send it to my e-mail[1].
- You can bring your hand-written solution to the lecture.

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Linear equalities versus inequalities

Remark about Rouché-Capelli theorem

We have $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Rouché–Capelli theorem in English speaking countries, Italy and Brazil;
Kronecker–Capelli theorem in Austria, Poland, Romania and Russia;
Rouché–Fontené theorem in France;
Rouché–Frobenius theorem in Spain and many countries in Latin America;
Frobenius theorem in the Czech Republic and in Slovakia.

Rouché-Capelli theorem for the system of equalities $A\mathbf{x} = b$ ...
Exactly one of the following statements is true:

- $\exists \mathbf{x} \in \mathbb{R}^n : \quad A\mathbf{x} = b$
- $\exists \mathbf{y} \in \mathbb{R}^m : \quad A^\top \mathbf{y} = \mathbf{0} \ \wedge \ b^\top \mathbf{y} \neq 0$

Rouché-Capelli theorem is typically stated as ...
Let $A \in F^{m \times n}$ be a matrix and $b \in F^m$ be a vector over a field F.
The system of equalities $A\mathbf{x} = b$ is inconsistent (that is, no solution $\mathbf{x} \in F^n$ exists) if and only if $\operatorname{rank}(A) < \operatorname{rank}(A \,|\, b)$.

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Now we fix $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Rouché-Capelli theorem for the system of equalities $A\mathbf{x} = b$ ...
Exactly one of the following statements is true:

- $\exists \mathbf{x} \in \mathbb{R}^n : \quad A\mathbf{x} = b$ (*)
- $\exists \mathbf{y} \in \mathbb{R}^m : \quad A^\top \mathbf{y} = \mathbf{0} \;\wedge\; b^\top \mathbf{y} < 0$ (**)

Farkas' lemma for the system of equalities $A\mathbf{x} = b$ over $\mathbf{x} \geq 0$ ...
Exactly one of the following statements is true:

- $\exists \mathbf{x} \in \mathbb{R}^n : \quad A\mathbf{x} = b \;\wedge\; \mathbf{x} \geq \mathbf{0}$ (*)
- $\exists \mathbf{y} \in \mathbb{R}^m : \quad A^\top \mathbf{y} \geq \mathbf{0} \;\wedge\; b^\top \mathbf{y} < 0$ (**)

Intuition behind the dichotomy: (*) The problem has a solution.
(**) There is a linear combination of rows that provides a contradiction.

Now we fix $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

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Farkas' lemma for the system of inequalities $A\mathbf{x} \leq b$, $\mathbf{x} \lesseqgtr 0$ ...
Exactly one of the following statements is true:

- $\exists \mathbf{x} \in \mathbb{R}^n : \quad A\mathbf{x} \leq b$ (*)
- $\exists \mathbf{y} \in \mathbb{R}^m : \quad A^\top \mathbf{y} = \mathbf{0} \;\wedge\; b^\top \mathbf{y} < 0 \;\wedge\; \mathbf{y} \geq \mathbf{0}$ (**)

Intuition behind the dichotomy: (*) The problem has a solution.
(**) There is a linear/nnlin. combination of rows that provides a contradiction.

Linear equalities versus inequalities

Rouché-Capelli theorem versus Farkas' lemma (III)

Now we fix $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Rouché-Capelli theorem for the system of equalities $A\mathbf{x} = b$ ...
Exactly one of the following statements is true:

- $\exists \mathbf{x} \in \mathbb{R}^n : \quad A\mathbf{x} = b$ (*)
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Farkas' lemma for the system of inequalities $A\mathbf{x} \leq b$, $\mathbf{x} \geq 0$ ...
Exactly one of the following statements is true:

- $\exists \mathbf{x} \in \mathbb{R}^n : \quad A\mathbf{x} \leq b \;\wedge\; \mathbf{x} \geq \mathbf{0}$ (*)
- $\exists \mathbf{y} \in \mathbb{R}^m : \quad A^\top \mathbf{y} \geq \mathbf{0} \;\wedge\; b^\top \mathbf{y} < 0 \;\wedge\; \mathbf{y} \geq \mathbf{0}$ (**)

Intuition behind the dichotomy: (*) The problem has a solution.
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We have $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Exactly one of the following statements is true:

- $\exists \mathbf{x} \in \mathbb{R}^n : \quad A\mathbf{x} = b \ \wedge \ \mathbf{x} \geq \mathbf{0}$
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Farkas' lemmata have many applications.

Their staple use case is the proof of the Strong duality theorem(s).

Strong duality theorem

Let's outline the proof for the bounded case using Farkas' lemma (III)

We have $E \in \mathbb{R}^{m' \times n'}$ and $d \in \mathbb{R}^{m'}$.

Exactly one of the following statements is true:

- $\exists \mathbf{z} \in \mathbb{R}^{n'} : \quad E\mathbf{z} \leq d \quad \wedge \quad \mathbf{z} \geq \mathbf{0}$
- $\exists \mathbf{w} \in \mathbb{R}^{m'} : \quad E^\top \mathbf{w} \geq \mathbf{0} \quad \wedge \quad d^\top \mathbf{w} < 0 \quad \wedge \quad \mathbf{w} \geq \mathbf{0}$

LP duality (III): Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ ...

Primal problem:

$$\begin{array}{lr} \text{minimize} & c^\top \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \geq b \\ \text{where} & \mathbf{x} \geq \mathbf{0} \end{array}$$

Dual problem:

$$\begin{array}{lr} \text{maximize} & b^\top \mathbf{y} \\ \text{s.t.} & A^\top \mathbf{y} \leq c \\ \text{where} & \mathbf{y} \geq \mathbf{0} \end{array}$$

Strong duality theorem

Idea of the proof for the bounded case

We have $E \in \mathbb{R}^{m' \times n'}$ and $d \in \mathbb{R}^{m'}$.

Exactly one of the following statements is true:

- $\exists \mathbf{z} \in \mathbb{R}^{n'} : \quad E\mathbf{z} \leq d \quad \wedge \quad \mathbf{z} \geq \mathbf{0}$
- $\exists \mathbf{w} \in \mathbb{R}^{m'} : \quad E^\top \mathbf{w} \geq \mathbf{0} \quad \wedge \quad d^\top \mathbf{w} < 0 \quad \wedge \quad \mathbf{w} \geq \mathbf{0}$

LP duality (III): Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ ...

Primal problem:

$$\begin{array}{ll} \text{minimize} & c^\top \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \geq b \\ \text{where} & \mathbf{x} \geq \mathbf{0} \end{array}$$

$$E := \begin{pmatrix} A^\top \\ \hdashline -b^\top \end{pmatrix} \qquad d := \begin{pmatrix} c \\ \hdashline -\min c^\top \mathbf{x} \end{pmatrix}$$

Dual problem:

$$\begin{array}{ll} \text{maximize} & b^\top \mathbf{y} \\ \text{s.t.} & A^\top \mathbf{y} \leq c \\ \text{where} & \mathbf{y} \geq \mathbf{0} \end{array}$$