

In this section, we prove:

```
theorem finFarkasBartl {n : ℕ} [LinearOrderedDivisionRing R]
  [LinearOrderedAddCommGroup V] [Module R V] [PosSMulMono R V] [AddCommGroup W] [Module R W]
  (A : W →l [R] Fin n → R) (b : W →l [R] V) :
  (∃ x : Fin n → V, 0 ≤ x ∧ ∀ w : W, ∑ j : Fin n, A w j • x j = b w) ≠ (∃ y : W, 0 ≤ A y ∧ b y < 0)
```

We first rephrase the goal to:

$$(\exists x : \text{Fin } n \rightarrow V, 0 \leq x \wedge \forall w : W, \sum j : \text{Fin } n, A w j \bullet x j = b w) \leftrightarrow (\forall y : W, 0 \leq A y \rightarrow 0 \leq b y)$$

Implication from left to right is immediately satisfied by the following term:

```
fun ⟨x, hx, hb⟩ y hy => hb y ▷ Finset.sum_nonneg (fun i _ => smul_nonneg (hy i) (hx i))
```

Implication from right to left will be proved by induction on n with generalized A and b . In case $n = 0$ we immediately have:

```
A_tauto (w : W) : 0 ≤ A w
```

We have an assumption:

```
hAb : ∀ y : W, 0 ≤ A y → 0 ≤ b y
```

We set x to be the empty vector family. Now, for every $w : W$, we must prove:

$$\sum j : \text{Fin } 0, A w j \bullet (0 : \text{Fin } 0 \rightarrow V) j = b w$$

We simplify the goal to:

```
0 = b w
```

We utilize that V is ordered and prove the equality as two inequalities. Inequality $0 \leq b w$ is directly satisfied by:

```
hAb w (A_tauto w)
```

Inequality $b w \leq 0$ is easily reduced to:

```
hAb (-w) (A_tauto (-w))
```

The induction step is stated as a lemma:

```
lemma industepFarkasBartl {m : ℕ} [LinearOrderedDivisionRing R]
  [LinearOrderedAddCommGroup V] [Module R V] [PosSMulMono R V] [AddCommGroup W] [Module R W]
  (ih : ∀ A0 : W →l [R] Fin m → R, ∀ b0 : W →l [R] V,
    (∀ y0 : W, 0 ≤ A0 y0 → 0 ≤ b0 y0) →
    (∃ x0 : Fin m → V, 0 ≤ x0 ∧ ∀ w0 : W, ∑ i0 : Fin m, A0 w0 i0 • x0 i0 = b0 w0))
  {A : W →l [R] Fin m.succ → R} {b : W →l [R] V} (hAb : ∀ y : W, 0 ≤ A y → 0 ≤ b y) :
  ∃ x : Fin m.succ → V, 0 ≤ x ∧ ∀ w : W, ∑ i : Fin m.succ, A w i • x i = b w
```

We define

```
a : W →l [R] Fin m → R
```

as the first m rows of A , i.e., A without the last row:

```
a := (fun w : W => fun i : Fin m => A w i)
```

To prove `industepFarkasBartl` we first consider the easy case:

```
is_easy : ∀ y : W, 0 ≤ a y → 0 ≤ b y
```

From `ih a b is_easy` we obtain:

```
x : Fin m → V
hx : 0 ≤ x
hxb : ∀ w0 : W, ∑ i0 : Fin m, a w0 i0 • x i0 = b w0
```

The lemma is satisfied by this vector family:

```
(fun i : Fin m.succ => if hi : i < m then x i else 0)
```

Easy case analysis shows that the vector family is nonnegative. Now we need to prove:

$$\forall w : W, \sum i : \text{Fin } m.succ, A w i \bullet (\text{fun } i : \text{Fin } m.succ \Rightarrow \text{if } hi : i < m \text{ then } x i \text{ else } 0) i = b w$$

We simplify the goal to:

$$\forall w : W, \sum i : \text{Fin } m, A w i \bullet x i = b w$$

This is exactly `hxb`.

Now for the hard case; negation of `is_easy` gives us:

$$\begin{aligned} y' &: W \\ \text{hay}' &: 0 \leq a \cdot y' \\ \text{hby}' &: b \cdot y' < 0 \end{aligned}$$

Let y be flipped and rescaled y' as follows:

$$y : W := (A \cdot y' \cdot m)^{-1} \bullet y'$$

From `hAb` we get:

$$\text{hAy}' : A \cdot y' \cdot m < 0$$

Therefore `hAy'.ne : A · y' · m ≠ 0` implies that y has the property that motivated the rescaling:

$$\text{hAy} : A \cdot y \cdot m = 1$$

From `hAy` we have:

$$\text{hAA} : \forall w : W, A \cdot (w - (A \cdot w \cdot m \bullet y)) \cdot m = 0$$

Using `hAA` and `hAb` we prove:

$$\text{hbA} : \forall w : W, 0 \leq a \cdot (w - (A \cdot w \cdot m \bullet y)) \rightarrow 0 \leq b \cdot (w - (A \cdot w \cdot m \bullet y))$$

From `hbA` we have:

$$\text{hbAb} : \forall w : W, 0 \leq (a - (A \cdot m \bullet a \cdot y)) \cdot w \rightarrow 0 \leq (b - (A \cdot m \bullet b \cdot y)) \cdot w$$

We observe that these two terms (appearing in `hbAb` we just proved) are linear maps:

$$\begin{aligned} (a - (A \cdot m \bullet a \cdot y)) \\ (b - (A \cdot m \bullet b \cdot y)) \end{aligned}$$

Therefore, we can plug them into `ih` and provide `hbAb` as the last argument. We obtain:

$$\begin{aligned} x' &: \text{Fin } m \rightarrow V \\ \text{hx}' &: 0 \leq x' \\ \text{hxb}' &: \forall w_0 : W, \sum i_0 : \text{Fin } m, (a - (A \cdot m \bullet a \cdot y)) \cdot w_0 \cdot i_0 \bullet x' \cdot i_0 = (b - (A \cdot m \bullet b \cdot y)) \cdot w_0 \end{aligned}$$

We claim that our lemma is satisfied by this vector family:

$$(\text{fun } i : \text{Fin } m.\text{succ} \Rightarrow \text{if } h_i : i < m \text{ then } x' \cdot i \text{ else } b \cdot y - \sum j : \text{Fin } m, a \cdot y \cdot j \bullet x' \cdot j)$$

Let us show the nonnegativity first. Nonnegativity of everything except of the last vector follows from `hx'`. From `hAy'` we have:

$$\text{hAy}'' : (A \cdot y' \cdot m)^{-1} \leq 0$$

From `hAy''` with `hay'` we have:

$$\text{hay} : a \cdot y \leq 0$$

From `hAy''` with `hby'` converted to nonstrict inequality we have:

$$\text{hby} : 0 \leq b \cdot y$$

For the nonnegativity of the last vector, we need to prove:

$$\sum j : \text{Fin } m, a \cdot y \cdot j \bullet x' \cdot j \leq b \cdot y$$

It follows from `hay · j` with `hx' · j` and `hby` using basic properties of inequalities. The only remaining task is to show:

$$\forall w : W, \sum i : \text{Fin } m.\text{succ}, (A \cdot w \cdot i \bullet (\text{if } h_i : i < m \text{ then } x' \cdot i \text{ else } b \cdot y - \sum j : \text{Fin } m, a \cdot y \cdot j \bullet x' \cdot j)) = b \cdot w$$

Given general $w : W$ we make a key observation (using `hxb' · w`):

$$\text{haAa} : \sum i : \text{Fin } m, (a \cdot w \cdot i - A \cdot w \cdot m \bullet a \cdot y \cdot i) \bullet x' \cdot i = b \cdot w - A \cdot w \cdot m \bullet b \cdot y$$

With the help of `haAa` we transform the goal to:

$$\sum i : \text{Fin } m.\text{succ}, (A \cdot w \cdot i \bullet (\text{if } h_i : i < m \text{ then } x' \cdot i \text{ else } b \cdot y - \sum j : \text{Fin } m, a \cdot y \cdot j \bullet x' \cdot j)) = \sum i : \text{Fin } m, (a \cdot w \cdot i - A \cdot w \cdot m \bullet a \cdot y \cdot i) \bullet x' \cdot i + A \cdot w \cdot m \bullet b \cdot y$$

We distribute \bullet over `if` so that the goal becomes:

$$\sum i : \text{Fin } m.\text{succ}, (\text{if } h_i : i < m \text{ then } A \cdot w \cdot i \bullet x' \cdot i \text{ else } A \cdot w \cdot i \bullet (b \cdot y - \sum j : \text{Fin } m, a \cdot y \cdot j \bullet x' \cdot j)) = \sum i : \text{Fin } m, (a \cdot w \cdot i - A \cdot w \cdot m \bullet a \cdot y \cdot i) \bullet x' \cdot i + A \cdot w \cdot m \bullet b \cdot y$$

We split the left-hand side into two parts:

$$\sum i : \text{Fin } m, (a \cdot w \cdot i \bullet x' \cdot i) + A \cdot w \cdot m \bullet (b \cdot y - \sum j : \text{Fin } m, a \cdot y \cdot j \bullet x' \cdot j) = \sum i : \text{Fin } m, (a \cdot w \cdot i - A \cdot w \cdot m \bullet a \cdot y \cdot i) \bullet x' \cdot i + A \cdot w \cdot m \bullet b \cdot y$$

The rest is just a manipulation with sums.