Farkas established that a system of linear inequalities has a solution if and only if we cannot obtain a contradiction by taking a nonnegative linear combination of the inequalities. We state and formally prove several Farkas-like theorems in Lean 4. Furthermore, we consider the rational numbers extended with two special elements denoted by $\bot$ and $\top$ where $\bot$ is below every rational number and $\top$ is above every rational number. We define $\bot + a = \bot = a + \bot$ of all a and we define $\top + b = \top = b + \top$ for all $b \neq \bot$. For multiplication, we define $\bot \cdot c = \bot = c \cdot \bot$ for every $c \geq 0$ but $\top \cdot d = \top = d \cdot \top$ only for $d > 0$ because $\top \cdot 0 = 0 = 0 \cdot \top$. We extend certain versions of the Farkas-like theorems about matrices to a setting where coefficients are extended rationals and variables are nonnegative rationals.

1 Introduction

Rouché, Capelli, Kronecker, Fonténe, Frobenius established that a system of linear equalities has a solution if and only if we cannot obtain a contradiction by taking a linear combination of the equalities. One of many ways to state the theorem is to say that, given a matrix $A : (I \times J) \to F$ and a vector $b : I \to F$, where I is the set of row indices, J is the set of column indices, and F is a field, exactly one of the following statements is true:

- $\exists\, x : J \to F,\ A \cdot x = b$
- $\exists\, y : I \to F,\ A^T \cdot y = 0 \land b \cdot y < 0$

A similar characterization exists for systems of linear equalities with nonnegative variables, due to Farkas; a system of linear equalities has a solution if and only if we cannot obtain a contradiction by taking a linear combination of the equalities. From here on, we require F to be a linear-ordered field. Using the same notation as before, we can write that exactly one of the following statements is true:

- $\exists\, x : J \to F,\ A \cdot x = b \land x \geq 0$
- $\exists\, y : I \to F,\ A^T \cdot y \geq 0 \land b \cdot y < 0$

The former branch got stricter by requiring x to be nonnegative, while the latter branch got looser by allowing $\geq$ in place of the original equality. We can also read the statement from bottom up (take A^T in place of A) to obtain a characterizationof systems of linear inequalities with unconstrained variables. We state the same theorem in Lean as follows:

```
theorem equalityFarkas (A : Matrix I J F) (b : I → F) :
    (∃ x : J → F, 0 ≤ x ∧ A *ᵥ x = b) ≠ (∃ y : I → F, 0 ≤ Aᵀ *ᵥ y ∧ b ·ᵥ y < 0)
```

We can do one more step in the direction of considering inequalities and nonnegative variables only. There is a characterization of systems of linear inequalities with nonnegative variables. We can write that exactly one of the following statements is true:

- $\exists\, x : J \to F,\ A \cdot x \leq b \land x \geq 0$
- $\exists\, y : I \to F,\ A^T \cdot y \geq 0 \land b \cdot y < 0 \land y \geq 0$

We state this theorem in Lean as follows:

```
theorem inequalityFarkas (A : Matrix I J F) (b : I → F) :
    (∃ x : J → F, 0 ≤ x ∧ A *ᵥ x ≤ b) ≠ (∃ y : I → F, 0 ≤ y ∧ 0 ≤ Aᵀ *ᵥ y ∧ b ·ᵥ y < 0)
```

2 Preliminaries

We distinguish two types of vectors; implicit vectors and explicit vectors. Implicit vectors are members of a vector space; they don't have any internal structure. Explicit vectors are functions from coordinates to values. The set of coordinates needn't be ordered. Matrices live next to explicit vectors. They are also functions; they take a row index and a column index and they output a value at the given spot. Neither the row indices nor the column vertices are required to form an ordered set. That's why multiplication between matrices and vectors is defined only in structures where addition forms a commutative semigroup. Consider the following example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} ? \\ - \end{pmatrix}$$

We don't know whether the value at the question mark is equal to $(1 \cdot 7 + 2 \cdot 8) + 3 \cdot 9$ or to $(2 \cdot 8 + 1 \cdot 7) + 3 \cdot 9$ or to any other ordering of summands. This is why commutativity of addition is necessary for the definition to be valid. On the other hand, we don't assume any property of multiplication in the definition of multiplication between matrices and vectors; they don't even have to be of the same type; we only require the elements of the vector to have an action on the elements of the matrix.

Let p and q be rational numbers. Addition and multiplication on extended rationals is defined as follows:

+	$\bot$	q	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
p	$\bot$	$p+q$	$\top$
$\top$	$\bot$	$\top$	$\top$

$\bullet$	$\bot$	q	$\top$
0	$\bot$	0	0
$p > 0$	$\bot$	$p \cdot q$	$\top$

TODO

3 Proving the Farkas-Bartl theorem

Let n be a natural number. Let R be a linear-ordered division ring. Let V be a linear-ordered module over R. Let W be a module over R. Let A be a linear (w.r.t. R) map from W to R^n. Let b be a linear (w.r.t. R) map from W to V. We claim that exactly one of the following is true; (1) there is nonnegative x in V^n such that b is equal to the sum of elementwise mutliplications of x by scalars from A, or (2) there exists y in W such that A applied to y gives a nonpositive vector from V and b applied to y gives a positive vector from V.

The part "exists at most one" is straightforward.

Lemma inv_pos_of_pos': If $0 < a$ then $0 < a^{-1}$.

Lemma filter_yielding_singleton_attach_sum: $\sum_{i \in [m]; \neg(i<m)} f\ j \bullet v = f\ m \bullet v$

Lemma impossible_index: Let $i \in [m]$. We cannot have both $\neg(i < m)$ and $i \neq m$.

Lemma sum_nneg_aux: If $x \geq 0$ and $A\ y \leq 0$ then $\sum_{i \in [m]} A\ y\ i \bullet x\ i \leq 0$.

Lemma finishing_piece: Let R be a semiring, V be an additive commutative monoid that is a module over R, W be an additive commutative monoid that is a module over R, m be a natural number, A be an R-linear map from W to $[m+1] \to R$, w be a vector from W, and x be a vector from $[m] \to V$. We claim $\sum_{i \in [m]} A_{<m}\ w\ i \bullet x\ i = \sum_{i \in [m+1]; i<m} A\ w\ i \bullet x\ i$.

TODO.

4 Inequalities and linear programming

TODO.

5 Proving the extended strong duality

TODO.