Farkas established that a system of linear inequalities has a solution if and only if we cannot obtain a contradiction by taking a nonnegative linear combination of the inequalities. We state and formally prove several Farkas-like theorems in Lean 4. Furthermore, we consider the rational numbers extended with two special elements denoted by $\bot$ and $\top$ where $\bot$ is below every rational number and $\top$ is above every rational number. We define $\bot + a = \bot = a + \bot$ of all a and we define $\top + b = \top = b + \top$ for all $b \neq \bot$. For multiplication, we define $\bot \cdot c = \bot = c \cdot \bot$ for every $c \geq 0$ but $\top \cdot d = \top = d \cdot \top$ only for $d > 0$ because $\top \cdot 0 = 0 = 0 \cdot \top$. We extend certain versions of the Farkas-like theorems about matrices to a setting where coefficients are extended rationals and variables are nonnegative rationals.

1 Results

Let I and J be finite types. Let F be a linearly ordered field. Let A be a matrix of type $(I \times J) \to F$. Let b be a vector of type $I \to F$.

Theorem (equalityFrobenius): Exactly one of the following exists:

- vector $x : J \to F$ such that $A \cdot x = b$
- vector $y : I \to F$ such that $A^T \cdot y = 0$ and $b \cdot y \neq 0$

Theorem (equalityFarkas): Exactly one of the following exists:

- nonnegative vector $x : J \to F$ such that $A \cdot x = b$
- vector $y : I \to F$ such that $A^T \cdot y \geq 0$ and $b \cdot y < 0$

Theorem (equalityFarkas_neg): Exactly one of the following exists:

- nonnegative vector $x : J \to F$ such that $A \cdot x = b$
- vector $y : I \to F$ such that $(-A^T) \cdot y \leq 0$ and $b \cdot y < 0$

Theorem (inequalityFarkas_neg): Exactly one of the following exists:

- nonnegative vector $x : J \to F$ such that $A \cdot x \leq b$
- nonnegative vector $y : I \to F$ such that $(-A^T) \cdot y \leq 0$ and $b \cdot y < 0$

Theorem (inequalityFarkas): Exactly one of the following exists:

- nonnegative vector $x : J \to F$ such that $A \cdot x \leq b$
- nonnegative vector $y : I \to F$ such that $A^T \cdot y \geq 0$ and $b \cdot y < 0$

Theorem (coordinateFarkas): Let I be any type. Let J be a finite type. Let R be a linearly ordered division ring. Let A be an R-linear map from from $(I \to R)$ to $(J \to R)$. Let b be an R-linear map from from $(I \to R)$ to R. Exactly one of the following exists:

- nonnegative vector $x : J \to R$ such that, for all $w : I \to R$, we have $\sum_{j:J} (A\ w)_j \cdot x_j = b\ w$
- vector $y : I \to R$ such that $A\ y \leq 0$ and $b\ y < 0$

Theorem (scalarFarkas): Let J be a finite type. Let R be a linearly ordered division ring. Let W be an R-module. Let A be an R-linear map from from W to $(J \to R)$. Let b be an R-linear map from from W to R. Exactly one of the following exists:

- nonnegative vector $x : J \to R$ such that, for all w in W, we have $\sum_{j:J} (A\ w)_j \cdot x_j = b\ w$
- vector $y : W$ such that $A\ y \leq 0$ and $b\ y < 0$

Theorem (fintypeFarkasBartl): Let J be a finite type. Let R be a linearly ordered division ring. Let W be an R-module. Let V be a linearly ordered R-module[1]. Let A be an R-linear map from from W to $(J \to R)$. Let b be an R-linear map from from W to V. Exactly one of the following exists:

- nonnegative vector $x : J \to V$ such that, for all w in W, we have $\sum_{j:J} (A\ w)_j \cdot x_j = b\ w$
- vector $y : W$ such that $A\ y \leq 0$ and $b\ y < 0$

Definition: Let F be a linearly ordered field. We define an **extended** linearly ordered field F_∞ as $F \cup \{\bot, \top\}$ with the following properties. Let p and q be numbers from F. We have $\bot < p < \top$. Addition and scalar action on F_∞ is defined as follows:

$+$	$\bot$	q	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
p	$\bot$	$p+q$	$\top$
$\top$	$\bot$	$\top$	$\top$

$\bullet$	$\bot$	q	$\top$
0	$\bot$	0	0
$p > 0$	$\bot$	$p \cdot q$	$\top$

[1] We furthermore require PosSMulMono R V.

Theorem (extendedFarkas): Let I and J be finite types. Let F be a linearly ordered field. Let A be a matrix of type $(I \times J) \to F_\infty$. Let b be a vector of type $I \to F_\infty$. Assume that A does not have $\bot$ and $\top$ in the same row. Assume that A does not have $\bot$ and $\top$ in the same column. Assume that A does not have $\top$ in any row where b has $\top$. Assume that A does not have $\bot$ in any row where b has $\bot$. Exactly one of the following exists:

- nonnegative vector $x : J \to F$ such that $A \cdot x \le b$
- nonnegative vector $y : I \to F$ such that $(-A^T) \cdot y \le 0$ and $b \cdot y < 0$

TODO everything about linear programming.

2 Preliminaries

We distinguish two types of vectors; implicit vectors and explicit vectors. Implicit vectors are members of a vector space; they don't have any internal structure. Explicit vectors are functions from coordinates to values. The set of coordinates needn't be ordered. Matrices live next to explicit vectors. They are also functions; they take a row index and a column index and they output a value at the given spot. Neither the row indices nor the column vertices are required to form an ordered set. That's why multiplication between matrices and vectors is defined only in structures where addition forms a commutative semigroup. Consider the following example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} ? \\ \_ \end{pmatrix}$$

We don't know whether the value at the question mark is equal to $(1 \cdot 7 + 2 \cdot 8) + 3 \cdot 9$ or to $(2 \cdot 8 + 1 \cdot 7) + 3 \cdot 9$ or to any other ordering of summands. This is why commutativity of addition is necessary for the definition to be valid. On the other hand, we don't assume any property of multiplication in the definition of multiplication between matrices and vectors; they don't even have to be of the same type; we only require the elements of the vector to have an action on the elements of the matrix.

TODO.

3 Proving the Farkas-Bartl theorem

TODO.

4 Inequalities and linear programming

TODO.

5 Proving the extended strong duality

TODO.