Theorem 1. *Recursively-enumerable languages are closed under the Kleene star.*

Proof. Let $\mathbf{L} \subseteq \mathbf{T}^*$ be a language generated by grammar $\mathbf{G} = (\mathbf{T}, \mathbf{N}, S, \mathbf{P})$. We construct a grammar $\mathbf{G}^* = (\mathbf{T}, \mathbf{N}^*, Z, \mathbf{P}^*)$ to generate the language $\mathbf{L}^*$.

$$\mathbf{N}^* = \mathbf{N} \cup \{Z, \#, R\}$$

We expanded the set of nonterminals with a new starting symbol, a delimiter, and a symbol that denotes final rewriting.

$$\mathbf{P}^* = \mathbf{P} \cup \{Z \to ZS\#, Z \to R\#, R\# \to R, R\# \to \epsilon\} \cup \{Rt \to tR \mid t \in \mathbf{T}\}$$

The proof continues below.

Lemma 2. *If $\alpha \in (\mathbf{T} \cup \mathbf{N})^*$ can be derived by $\mathbf{G}^*$, then one of the following holds.*

- $(\exists x_1, x_2, \ldots, x_m \in (\mathbf{T} \cup \mathbf{N})^*)$
  $(((\forall i \in [m])(S \Rightarrow^* x_i)) \wedge (\alpha = Zx_1\#x_2\# \ldots x_m\#))$
- $(\exists x_1, x_2, \ldots, x_m \in (\mathbf{T} \cup \mathbf{N})^*)$
  $(((\forall i \in [m])(S \Rightarrow^* x_i)) \wedge (\alpha = R\#x_1\#x_2\# \ldots x_m\#))$
- $(\exists w_1, w_2, \ldots, w_n \in \mathbf{L})(\exists \beta \in \mathbf{T}^*)(\exists \gamma, x_1, x_2, \ldots, x_m \in (\mathbf{T} \cup \mathbf{N})^*)$
  $((S \Rightarrow^* \beta\gamma) \wedge ((\forall i \in [m])(S \Rightarrow^* x_i)) \wedge$
  $(\alpha = w_1 w_2 \ldots w_n \beta R \gamma \#x_1\#x_2\# \ldots x_m\#))$
- $\alpha \in \mathbf{L}^*$
- $(\exists \sigma \in (\mathbf{T} \cup \mathbf{N})^*)(\alpha = \sigma R)$
  (This case happens if the rule $R\# \to R$ is used in the final position (where $R\# \to \epsilon$ should be used instead). R in the final position prevents the derivation from terminating.)
- $(\exists \omega \in (\mathbf{T} \cup \mathbf{N} \cup \{\#\})^*)(\alpha = \omega\#)$
  (This case happens if the rule $R\# \to \epsilon$ is used too early (that is, anywhere but the final # position). The derivation will never terminate in this case.)

Proof. Induction on $\mathbf{G}^*$ derivation steps. Base case $\alpha = Z$ holds by $m = 0$.
Now assume $Z \Rightarrow^* \alpha \Rightarrow \alpha'$ and proceed by case analysis.

- $(\exists x_1, x_2, \ldots, x_m \in (\mathbf{T} \cup \mathbf{N})^*)$
  $(((\forall i \in [m])(S \Rightarrow^* x_i)) \wedge (\alpha = Zx_1\#x_2\# \ldots x_m\#))$

  - ❏ If $\alpha \Rightarrow \alpha'$ used a rule from $\mathbf{P}$, it could be applied only in some x_i. Hence $S \Rightarrow^* x_i \Rightarrow x_i'$ so the same condition holds after replacing x_i by x_i'.
  - ❏ If $\alpha \Rightarrow \alpha'$ used the rule $Z \to ZS\#$, it was applied at the beginning of α. Therefore, we put $m' := m + 1$, $x_1' := S$, and increase all indices by one, that is, $x_2' := x_1,\ x_3' := x_2,\ \ldots,\ x_{m'}' := x_m$. The same condition holds.
  - ❏ If $\alpha \Rightarrow \alpha'$ used the rule $Z \to R\#$, we keep all variables the same and the second condition holds.
  - ❏ The rules $R\# \to R$, $R\# \to \epsilon$, and $Rt \to tR$ are not applicable (α does not contain R).

- $(\exists x_1, x_2, \ldots, x_m \in (\mathbf{T} \cup \mathbf{N})^*)$
  $(((\forall i \in [m])(S \Rightarrow^* x_i)) \wedge (\alpha = R\#x_1\#x_2\# \ldots x_m\#))$

  - ❏ If $\alpha \Rightarrow \alpha'$ used a rule from $\mathbf{P}$, it could be applied only in some x_i. Hence $S \Rightarrow^* x_i \Rightarrow x_i'$ so the same condition holds after replacing x_i by x_i'.
  - ❏ The rules $Z \to ZS\#$ and $Z \to R\#$ are not applicable (α does not contain Z).
  - ❏ If $\alpha \Rightarrow \alpha'$ used the rule $R\# \to R$, it was applied at the beginning of α. If $m = 0$, the fifth condition holds (dead end). Otherwise, we put $m' := m - 1 \geq 0$, $\gamma := x_1$, and decrease all indices by one, that is, $x_1' := x_2,\ x_2' := x_3,\ \ldots,\ x_{m'}' := x_m$. As there is nothing before the nonterminal R, we put $n := 0$ and $\beta := \epsilon$. Now, the third condition holds.
  - ❏ If $\alpha \Rightarrow \alpha'$ used the rule $R\# \to \epsilon$ then; if $m = 0$, we obtain the empty word (which belongs to $\mathbf{L}^*$, satisfying the fourth condition); if $m > 0$, the last condition holds (because $\#$ stayed at the end of α', at the same time R disappeared, and Z did not appear).
  - ❏ The rule $Rt \to tR$ is not applicable (the only R in α is immediately followed by $\#$).

- $(\exists w_1, w_2, \dots, w_n \in \mathbf{L})(\exists \beta \in \mathbf{T}^*)(\exists \gamma, x_1, x_2, \dots, x_m \in (\mathbf{T} \cup \mathbf{N})^*)$
  $((S \Rightarrow^* \beta\gamma) \ \wedge ((\forall i \in [m])(S \Rightarrow^* x_i)) \ \wedge$
  $(\alpha = w_1 w_2 \dots w_n \, \beta R \gamma \, \# x_1 \# x_2 \# \dots x_m \#))$

  ❑ If $\alpha \Rightarrow \alpha'$ used a rule from $\mathbf{P}$, it could be applied in γ or in some x_i. In the first case, $\gamma \Rightarrow \gamma'$ implies $\beta\gamma \Rightarrow \beta\gamma'$, hence $S \Rightarrow^* \beta\gamma \Rightarrow \beta'\gamma'$. In the remaining cases, we observe $S \Rightarrow^* x_i \Rightarrow x_i'$ as we did at the beginning of our case analysis. As a result, the same condition still holds.

  ❑ The rules $Z \to ZS\#$ and $Z \to R\#$ are not applicable (α does not contain Z).

  ❑ If $\alpha \Rightarrow \alpha'$ used the rule $R\# \to R$, γ must have been empty. If $m = 0$, the fifth condition holds (dead end). Otherwise, we put $n' := n + 1$, $w_{n'} := \beta$, $\beta' := \epsilon$, $\gamma' := x_1$, $m' := m - 1$, and decrease indices of x_i by one, that is, $x_1' := x_2$, $x_2' := x_3$, $\dots$, $x_{m'}' := x_m$. Since $w_{n'} = \beta = \beta\gamma \in \mathbf{T}^*$ and $S \Rightarrow^* \beta\gamma$, we have $w_{n'} \in \mathbf{L}$. The same condition holds.

  ❑ If $\alpha \Rightarrow \alpha'$ used the rule $R\# \to \epsilon$, γ must have been empty. If $m = 0$, we get $\alpha = w_1 w_2 \dots w_n \beta$ and $\beta \in \mathbf{L}$, hence the fourth condition $\alpha' \in \mathbf{L}^*$ gets satisfied. If $m > 0$, the last condition now holds (because $\#$ stayed at the end of α', at the same time R disappeared, and Z did not appear).

  ❑ If $\alpha \Rightarrow \alpha'$ used a rule of the form $Rt \to tR$ ($t \in \mathbf{T}$), we have $\delta \in (\mathbf{T} \cup \mathbf{N})^*$ such that $\gamma = t\delta$. We put $\beta' := \beta t$ and $\gamma' := \delta$. As $\beta\gamma = \beta t \delta = \beta'\gamma'$, the same condition holds.

- $\alpha \in \mathbf{L}^*$

  ❑ No rule is applicable (α contains only terminals). The step $\alpha \Rightarrow \alpha'$ did not happen.

- $(\exists \sigma \in (\mathbf{T} \cup \mathbf{N})^*)(\alpha = \sigma R)$

  ❑ No matter which rule was applied, it happened within σ. No rule could match the final R. The same condition holds for $\alpha' = \sigma' R$.

- $(\exists \omega \in (\mathbf{T} \cup \mathbf{N} \cup \{\#\})^*)(\alpha = \omega\#)$

  ❏ If $\alpha \Rightarrow \alpha'$ used a rule from $\mathbf{P}$, the same condition still holds because the nonterminal $\#$ is not on the left side of any rule from $\mathbf{P}$ and neither Z nor R is on the right side of any rule from $\mathbf{P}$.

  ❏ The rules $Z \to ZS\#$, $Z \to R\#$, $R\# \to R$, $R\# \to \epsilon$, and $Rt \to tR$ are not applicable (α contains neither Z nor R).

□

Lemma 3. *Let $w_1, w_2, \ldots, w_n \in \mathbf{L}$. Then $\mathbf{G}^*$ can derive $Zw_1\#w_2\#\ldots w_n\#$.*

Proof. Induction on n. The base case holds because $Z \Rightarrow^* Z$ in zero steps.

Now assume $Z \Rightarrow^* Zw_1\#w_2\#\ldots w_n\#$ and $S \Rightarrow^* w_{n+1}$. We start with the rule $Z \to ZS\#$. We observe $ZS\# \Rightarrow^* Zw_1\#w_2\#\ldots w_n\#S\#$ and $Zw_1\#w_2\#\ldots w_n\#S\# \Rightarrow^* Zw_1\#w_2\#\ldots w_n\#w_{n+1}\#$. From transitivity, we obtain $Z \Rightarrow^* Zw_1\#w_2\#\ldots w_n\#w_{n+1}\#$. □

Proof of Theorem 1. We need to show that the language of $\mathbf{G}^*$ equals $\mathbf{L}^*$.

For "$\subseteq \mathbf{L}^*$", use Lemma 2 and observe that, if $\mathbf{G}^*$ generates $\alpha \in \mathbf{T}^*$, then $\alpha \in \mathbf{L}^*$ because all the remaining options require α to contain a nonterminal.

For "$\supseteq \mathbf{L}^*$", use Lemma 3. If $w \in \mathbf{L}^*$, there are $w_1, w_2, \ldots, w_n \in \mathbf{L}$ such that $w_1w_2\ldots w_n = w$. We see $Zw_1\#w_2\#\ldots w_n\# \Rightarrow R\#w_1\#w_2\#\ldots w_n\#$. Since all w_i are made of terminals only, by repeated application of $R\# \to R$ and $Rt \to tR$ (for all $t \in \mathbf{T}$) we get $R\#w_1\#w_2\#\ldots w_n\# \Rightarrow^* w_1w_2\ldots w_nR\#$. Finally, we get $w_1w_2\ldots w_nR\# \Rightarrow w_1w_2\ldots w_n = w$ using the rule $R\# \to \epsilon$. We conclude that $\mathbf{G}^*$ generates w. □