Closure Properties of General Grammars Formally Verified

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Closure Properties of General Grammars Formally Verified

- Lean 3
- Mathlib as of 2022-03-15
- 12 500 lines of code
- no dependent types for data
- no effort towards constructivism

Symbols

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inductive symbol (T : Type) (N : Type)
| terminal    : T → symbol
| nonterminal : N → symbol
```

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- We don't explicitly state that `T` and `N` must be finite.
- Only a finite amount of symbols will appear in rewrite rules.

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Rules

```
inductive symbol (T : Type) (N : Type)
| terminal    : T → symbol
| nonterminal : N → symbol

structure grule (T : Type) (N : Type) :=
(input_L : list (symbol T N))
(input_N : N)
(input_R : list (symbol T N))
(output_string : list (symbol T N))
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Grammars

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(input_N : N)
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structure grammar (T : Type) :=
(nt : Type)
(initial : nt)
(rules : list (grule T nt))
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Grammar transformations

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structure grammar (T : Type) :=
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(initial : nt)
(rules : list (grule T nt))

def grammar_transforms (g : grammar T)
  (w₁ w₂ : list (symbol T g.nt)) :
  Prop :=
∃ r : grule T g.nt,
  r ∈ g.rules                                ∧
  ∃ u v : list (symbol T g.nt),
    w₁ = u ++ r.input_L
           ++ [symbol.nonterminal r.input_N]
           ++ r.input_R ++ v                  ∧
    w₂ = u ++ r.output_string ++ v
```

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Grammar derivations

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           ++ r.input_R ++ v                ∧
    w₂ = u ++ r.output_string ++ v

def grammar_derives (g : grammar T) :
  list (symbol T g.nt) → list (symbol T g.nt)
  → Prop :=
relation.refl_trans_gen (grammar_transforms g)
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Words generated by a grammar

```
def grammar_derives (g : grammar T) :
  list (symbol T g.nt) → list (symbol T g.nt)
  → Prop :=
relation.refl_trans_gen (grammar_transforms g)

def grammar_generates (g : grammar T)
  (w : list T) : Prop :=
grammar_derives g
  [symbol.nonterminal g.initial]
  (list.map symbol.terminal w)
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def language (T : Type) : Type :=
set (list T)
```

Language of a grammar

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def grammar_generates (g : grammar T)
  (w : list T) : Prop :=
grammar_derives g
  [symbol.nonterminal g.initial]
  (list.map symbol.terminal w)

def language (T : Type) : Type :=
set (list T)

def grammar_language (g : grammar T) :
  language T :=
set_of (grammar_generates g)
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Type-0 languages

```
def language (T : Type) : Type :=
set (list T)

def grammar_language (g : grammar T) :
  language T :=
set_of (grammar_generates g)

def is_T0 (L : language T) : Prop :=
∃ g : grammar T, grammar_language g = L
```

Union of languages

```
def set.union (s₁ s₂ : set T) : set T :=
{a | a ∈ s₁ ∨ a ∈ s₂}

instance : language.has_add (language T) :=
⟨set.union⟩
```

Union of languages

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{a | a ∈ s₁ ∨ a ∈ s₂}

instance : language.has_add (language T) :=
⟨set.union⟩

theorem T0_of_T0_u_T0 (L₁ L₂ : language T) :
  is_T0 L₁ ∧ is_T0 L₂ → is_T0 (L₁ + L₂)
```

Proof of closure of type-0 language (under union)

- given g_1 and g_2 $\rightsquigarrow$ construct g

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- prove, for any word,
  - generated by g_1 or g_2 $\Longrightarrow$ can be generated by g

Proof of closure of type-0 language (under union)

- given g_1 and g_2 $\rightsquigarrow$ construct g
- prove, for any word,
  - generated by g_1 or g_2 $\implies$ can be generated by g
  - generated by g $\implies$ can be generated by g_1 or g_2

Reversal of a language

```
def reverse_lang (L : language T) :
  language T :=
λ w : list T, w.reverse ∈ L
```

Reversal of a language

```
def reverse_lang (L : language T) :
  language T :=
λ w : list T, w.reverse ∈ L

theorem T0_of_reverse_T0 (L : language T) :
is_T0 L → is_T0 (reverse_lang L)
```

Concatenation of languages

```
def set.image2 (f : α → β → γ)
  (s : set α) (t : set β) : set γ :=
{c | ∃ a b, a ∈ s ∧ b ∈ t ∧ f a b = c}

instance : language.has_mul (language T) :=
⟨set.image2 (++)⟩
```

Concatenation of languages

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def set.image2 (f : α → β → γ)
  (s : set α) (t : set β) : set γ :=
{c | ∃ a b, a ∈ s ∧ b ∈ t ∧ f a b = c}

instance : language.has_mul (language T) :=
⟨set.image2 (++)⟩

theorem T0_of_T0_c_T0 (L₁ L₂ : language T) :
  is_T0 L₁ ∧ is_T0 L₂ → is_T0 (L₁ * L₂)
```

Kleene star of a language

```
def language.star (L : language T) :
  language T :=
{x | ∃ S : list (list T), x = S.join    ∧
                          ∀ y ∈ S, y ∈ L}
```

Kleene star of a language

```
def language.star (L : language T) :
  language T :=
{x | ∃ S : list (list T), x = S.join    ∧
                              ∀ y ∈ S, y ∈ L}

theorem T0_of_star_T0 (L : language T) :
  is_T0 L → is_T0 L.star
```

Related work — Chomsky hierarchy's lower levels

- Finite automata and regular expressions (in many languages)
- Context-free grammars
  - (1992) Carlson et al. (Mizar)
  - (2007) Minamide (Isabelle)
  - (2010) Barthwal and Norrish (HOL4)
  - (2015) Firsov and Uustalu (Agda)
  - (2019) Ramos (Coq)

Related work — Turing-complete models

- Turing machines
  - (2001) Chen and Nakamura (Mizar)
  - (2012) Asperti and Ricciotti (Matita)
  - (2013) Xu et al. (Isabelle)
  - (2019) Carneiro (Lean)
  - (2020) Forster et al. (Coq)
  - (2023) Balbach (Isabelle)
- Random access machines
  - (2003) Coen (Coq)
- Lambda calculus
  - (2017–2021) Forster, Kunze, and their colleagues (Coq)
- Partial recursive functions
  - (2011) Norrish (HOL4)
  - (2019) Carneiro (Lean)

Future work

Equivalence