Magma equations

Matthew Bolan, Jose Brox, Mario Carneiro, Martin Dvořák,
Andrés Goens, Harald Husum, Zoltan Kocsis, Alex Meiburg,
Pietro Monticone, David Renshaw, Cody Roux,
Jérémy Scanvic, Shreyas Srinivas, Anand Rao Tadipatri,
Terence Tao, Vlad Tsyrklevich, Daniel Weber, Fan Zheng,
et al.

2025-01-28

Frame Title

Implication graph completion

- If $P \implies Q$ and $Q \implies R$, then $P \implies R$.
- If $P \implies Q$ and $P \not\Longrightarrow R$, then $Q \not\Longrightarrow R$.
- If $Q \implies R$ and $P \not\Longrightarrow R$, then $P \not\Longrightarrow Q$.

Frame Title

Counterexample constructions

- Finite magmas
- Linear models
- Translation-invariant models
- Twisting semigroups
- Greedy constructions
- Ad-hoc modifications
- Combinations of the above

Greedy constructions

Preliminaries

We want to build a magma operation $\diamond\colon M \times M \to M$ that obeys one equation E but not another E'.
We one can first consider *partial magma operations* $\diamond\colon \Omega \to M$ defined on some $\Omega \subseteq M \times M$.
We say that a partial operation $\diamond'\colon \Omega' \to M$ *extends* $\diamond\colon \Omega \to M$ iff:

- $\Omega \subseteq \Omega'$
- $(x, y) \in \Omega \implies x \diamond y = x \diamond' y$

Given a sequence $\diamond_n\colon \Omega_n \to M$ of partial operations, each of which is an extension of the previous, we can define the *direct limit* $\diamond_\infty\colon \bigcup_n \Omega_n \to M$ to be the partial operation defined by $x \diamond_\infty y := x \diamond_n y$ whenever $(x, y) \in \Omega_n$.

Greedy constructions

Abstract greedy algorithm

Let E and E' be equations. Let Γ be a first-order theory regarding a partial magma operations $\diamond\colon \Omega \to M$ on a carrier M. Assume the following:

- (Seed) There exists a finitely supported partial magma operation $\diamond_0\colon \Omega_0 \to M$ satisfying Γ that contradicts E', in the sense that there is some assignment of variables in E' in M such that both sides of E' are defined using $\diamond_0$ but not equal to each other.
- (Soundness) If $\diamond_n\colon \Omega_n \to M$ is a sequence of partial magma operations obeying Γ with each $\diamond_{n+1}$ an extension of $\diamond_n$, and the direct limit $\diamond_\infty$ is total, then this limit obeys E.
- (Greedy extension) If $\diamond\colon \Omega \to M$ is a finitely supported partial magma operation obeying Γ, and $a, b \in M$, then there exists a finitely supported extension $\diamond'\colon \Omega' \to M'$ of $\diamond$ to a possibly larger carrier M' such that $a \diamond' b$ is defined.

Then $E \not\models E'$.

Greedy constructions

Trial and error

1. Start with a minimal rule set Γ that has just enough axioms to imply the soundness property for the given hypothesis E.
2. Attempt to establish the greedy extension property for this rule set by setting $a \diamond' b$ equal to a new element $c \notin M$, and then defining additional values of $\diamond'$ as necessary to recover the axioms of Γ'.
3. If this can be done in all cases, then locate a seed $\diamond_0$ refuting the given target E', stop.
4. If there is an obstruction (often due to a collision in which a given operation $x \diamond' y$ is required to equal two different values), add one or more rules to Γ to avoid this obstruction, and return to Step 2.

Greedy constructions

Example (E73 does not imply E4380)

Prove that the equation

$$x = y \diamond (y \diamond (x \diamond y))$$

does not imply the equation

$$x \diamond (x \diamond x) = (x \diamond x) \diamond x$$

.

Greedy constructions

Example (E73 does not imply E4380)

Prove that the equation

$$x = y \diamond (y \diamond (x \diamond y))$$

does not imply the equation

$$x \diamond (x \diamond x) = (x \diamond x) \diamond x$$

.

- $y \diamond (x \diamond y) = d \implies y \diamond d = x$

Greedy constructions

Example (E73 does not imply E4380)

Prove that the equation

$$x = y \diamond (y \diamond (x \diamond y))$$

does not imply the equation

$$x \diamond (x \diamond x) = (x \diamond x) \diamond x$$

.

- $y \diamond (x \diamond y) = d \implies y \diamond d = x$
- $x \diamond y = z \diamond y \implies x = z$

Greedy constructions

Example (E73 does not imply E4380)

Prove that the equation

$$x = y \diamond (y \diamond (x \diamond y))$$

does not imply the equation

$$x \diamond (x \diamond x) = (x \diamond x) \diamond x$$

.

- $y \diamond (x \diamond y) = d \implies y \diamond d = x$
- $x \diamond y = z \diamond y \implies x = z$
- $x \diamond y \neq y$