Magma equations

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2025-01-28

Mathoverflow question

Is there an identity between the commutative identity and the constant identity?

Ask Question

Asked 1 year, 6 months ago Modified 2 months ago Viewed 3k times

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I asked this on Math Stack Exchange, but it didn't get a single answer. So, I am now asking it here. Let our signature be that of a single binary operation $+$. I define the constant identity to be $x+y=z+w$. The commutative identity is, of course, the well-known identity $x+y=y+x$. I wonder, is there an identity strictly between those two, meaning, is there a single identity E such that the constant identity implies E, but not conversely, and E implies the commutative identity, but not conversely?

Mathoverflow answer — before the project started

Yes:

$$(x+x)+y=y+x$$

The constant identity implies this because both sides are $+$es. This does not imply the constant identity because it is true about any set with an operation that is commutative, associative, and idempotent (meaning $x+x=x$ for all x), the smallest nontrivial example is $\{x,y\}$ where $x+x=x+y=y+x=x$ and $y+y=y$.

This implies commutativity. $(x+x)+(x+x)=(x+x)+x=x+x$, so $x+x$ is idempotent. $(x+x)+y=((x+x)+(x+x))+y=y+(x+x)$, so $x+x$ commutes with everything. $(x+x)+(y+y)=(y+y)+x=x+y$, and $(x+x)+(y+y)=(y+y)+(x+x)$ because $x+x$ commutes with everything, so $x+y=y+x$, so $+$ is commutative.

This is not implied by commutativity, because (for example) addition of natural numbers is commutative but does not satisfy this identity.

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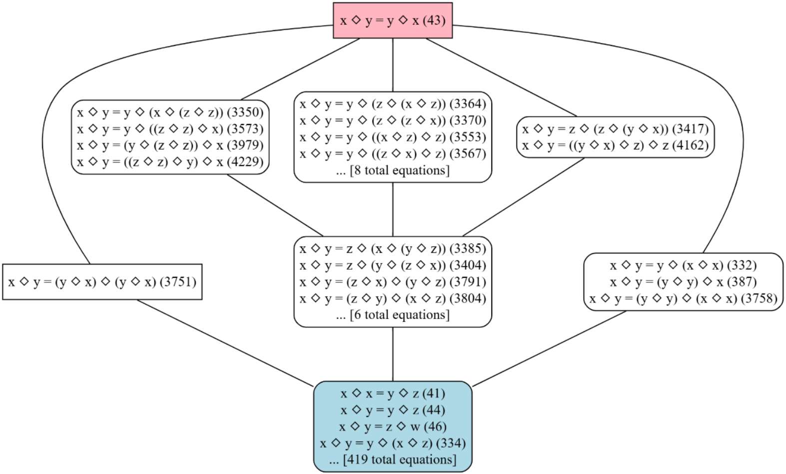
answered Jul 16, 2023 at 23:50

paste bee

Mathoverflow answer — after the project started



The commutative law (43) is the red box on top; the constant law (46) is in the blue box on the bottom

Announcement

Home About Career advice On writing Books Applets Mastodon+

A pilot project in universal algebra to explore new ways to collaborate and use machine assistance?

25 September, 2024 in math.RA, polymath | Tags: Artificial Intelligence, Equational Theory Project, machine assisted proof, universal algebra | by Terence Tao

Traditionally, mathematics research projects are conducted by a small number (typically one to five) of expert mathematicians, each of which are familiar enough with all aspects of the project that they can verify each other's contributions. It has been challenging to organize mathematical projects at larger scales, and particularly those that involve contributions from the general public, due to the need to verify all of the contributions; a single error in one component of a mathematical argument could invalidate the entire project. Furthermore, the sophistication of a typical math project is such that it would not be realistic to expect a member of the public, with say an undergraduate level of mathematics education, to contribute in a meaningful way to many such projects.

Announcement

Proposition 1 Equation4 implies Equation7.

Proof: Suppose that G obeys Equation4, thus

$$(x \circ x) \circ y = y \circ x \qquad (1)$$

for all $x, y \in G$. Specializing to $y = x \circ x$, we conclude

$$(x \circ x) \circ (x \circ x) = (x \circ x) \circ x$$

and hence by another application of (1) we see that $x \circ x$ is idempotent:

$$(x \circ x) \circ (x \circ x) = x \circ x. \qquad (2)$$

Now, replacing x by $x \circ x$ in (1) and then using (2), we see that

$$(x \circ x) \circ y = y \circ (x \circ x),$$

so in particular $x \circ x$ commutes with $y \circ y$:

$$(x \circ x) \circ (y \circ y) = (y \circ y) \circ (x \circ x). \qquad (3)$$

Also, from two applications (1) one has

$$(x \circ x) \circ (y \circ y) = (y \circ y) \circ x = x \circ y.$$

Thus (3) simplifies to $x \circ y = y \circ x$, which is Equation7. $\square$

Announcement

```
import Mathlib.Tactic

universe u

class Magma (α : Type u) where
  op : α → α → α

infixl:65 " ∘ "   => Magma.op

def Equation4 (G: Type*) [Magma G] := ∀ x y : G, (x ∘ x) ∘ y = y ∘ x

def Equation7 (G: Type*) [Magma G] := ∀ x y : G, x ∘ y = y ∘ x

theorem Equation4_implies_Equation7 (G: Type*) [Magma G] (h: Equation4 G) : Equation7 G := by
  have idem (x : G) : (x ∘ x) ∘ (x ∘ x) = (x ∘ x) := by
    rw [h x (x ∘ x), h x x]
  have comm (x y : G) : (x ∘ x) ∘ y = y ∘ (x ∘ x) := by
    rw [<-idem x, h (x ∘ x) y, idem x]
  have op_idem (x y : G) : (x ∘ x) ∘ (y ∘ y) = x ∘ y := by
    rw [h x (y ∘ y), h y x]
  intro x y
  rw [<- op_idem x y, comm x (y ∘ y), op_idem y x]
```

Easy implication

Prove that

$$\forall x, \forall y, \quad x = x \diamond y$$

implies

$$\forall x, \forall y, \forall z, \forall w, \quad x \diamond y = (x \diamond z) \diamond (w \diamond y)$$

.

Easy implication

Prove that

$$\forall x, \forall y, \quad x = x \diamond y$$

implies

$$\forall x, \forall y, \forall z, \forall w, \quad x \diamond y = (x \diamond z) \diamond (w \diamond y)$$

.

Solution:

$$x \diamond y = x = x \diamond z = (x \diamond z) \diamond (w \diamond y)$$

Easy countexexample

Show that

$$\textit{todo}$$

does not imply

$$\textit{todo}$$

.

Easy countexexample

Show that

$$todo$$

does not imply

$$todo$$

.

Solution:

$$todo$$

Why equations?

Why magmas?

Scope of the project

Lean and manual vs automated contributions

Implication graph completion

- If $P \implies Q$ and $Q \implies R$, then $P \implies R$.
- If $P \implies Q$ and $P \not\Longrightarrow R$, then $Q \not\Longrightarrow R$.
- If $Q \implies R$ and $P \not\Longrightarrow R$, then $P \not\Longrightarrow Q$.

Constructivism?

Dashboard

Equation explorer

Free magmas [for equation]

Let X be a set of generators:

Free magmas [for equation]

Let X be a set of generators:

- Free magma —

Free magmas [for equation]

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X

Free magmas [for equation]

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma —

Free magmas [for equation]

Let X be a set of generators:

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- Free commutative magma — unordered rooted full binary trees with leaves labeled by X

Free magmas [for equation]

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- Free commutative magma — unordered rooted full binary trees with leaves labeled by X
- Free semigroup —

Free magmas [for equation]

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma — unordered rooted full binary trees with leaves labeled by X
- Free semigroup — nonempty words over X

Free magmas [for equation]

Let X be a set of generators:

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If X is a singleton:

Free magmas [for equation]

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- Free magma —

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- Free semigroup — nonempty words over X

If X is a singleton:

- Free magma — ordered rooted full binary trees

Free magmas [for equation]

Let X be a set of generators:

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If X is a singleton:

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- Free commutative magma —

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- Free commutative magma — unordered rooted full binary trees

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If X is a singleton:

- Free magma — ordered rooted full binary trees
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- Free semigroup —

Free magmas [for equation]

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma — unordered rooted full binary trees with leaves labeled by X
- Free semigroup — nonempty words over X

If X is a singleton:

- Free magma — ordered rooted full binary trees
- Free commutative magma — unordered rooted full binary trees
- Free semigroup — positive integers

Free magmas — beyond single equations

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma — unordered rooted full binary trees with leaves labeled by X
- Free semigroup — nonempty words over X

Free magmas — beyond single equations

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma — unordered rooted full binary trees with leaves labeled by X
- Free semigroup — nonempty words over X
- Free commutative semigroup —

Free magmas — beyond single equations

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma — unordered rooted full binary trees with leaves labeled by X
- Free semigroup — nonempty words over X
- Free commutative semigroup — nonempty multisets

Free magmas — beyond single equations

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
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- Free commutative semigroup — nonempty multisets
- Free commutative idempotent semigroup —

Free magmas — beyond single equations

Let X be a set of generators:

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- Free semigroup — nonempty words over X
- Free commutative semigroup — nonempty multisets
- Free commutative idempotent semigroup — nonempty sets

Difficult proof

Metatheorems

Counterexample constructions

- Finite magmas
- Linear models
- Translation-invariant models
- Twisting semigroups
- Greedy constructions
- Ad-hoc modifications
- Combinations of the above

We want to build a magma operation $\diamond\colon M \times M \to M$ that obeys one equation E but not another E'.
We one can first consider *partial magma operations* $\diamond\colon \Omega \to M$ defined on some $\Omega \subseteq M \times M$.
We say that a partial operation $\diamond'\colon \Omega' \to M$ *extends* $\diamond\colon \Omega \to M$ iff:

- $\Omega \subseteq \Omega'$
- $(x, y) \in \Omega \implies x \diamond y = x \diamond' y$

Given a sequence $\diamond_n\colon \Omega_n \to M$ of partial operations, each of which is an extension of the previous, we can define the *direct limit* $\diamond_\infty\colon \bigcup_n \Omega_n \to M$ to be the partial operation defined by $x \diamond_\infty y := x \diamond_n y$ whenever $(x, y) \in \Omega_n$.

Greedy constructions

Abstract greedy algorithm

Let E and E' be equations. Let Γ be a first-order theory regarding a partial magma operations $\diamond \colon \Omega \to M$ on a carrier M. Assume the following:

- (Seed) There exists a finitely supported partial magma operation $\diamond_0 \colon \Omega_0 \to M$ satisfying Γ that contradicts E', in the sense that there is some assignment of variables in E' in M such that both sides of E' are defined using $\diamond_0$ but not equal to each other.
- (Soundness) If $\diamond_n \colon \Omega_n \to M$ is a sequence of partial magma operations obeying Γ with each $\diamond_{n+1}$ an extension of $\diamond_n$, and the direct limit $\diamond_\infty$ is total, then this limit obeys E.
- (Greedy extension) If $\diamond \colon \Omega \to M$ is a finitely supported partial magma operation obeying Γ, and $a, b \in M$, then there exists a finitely supported extension $\diamond' \colon \Omega' \to M'$ of $\diamond$ to a possibly larger carrier M' such that $a \diamond' b$ is defined.

Then $E \not\models E'$.

Greedy constructions

Trial and error

1. Start with a minimal rule set Γ that has just enough axioms to imply the soundness property for the given hypothesis E.
2. Attempt to establish the greedy extension property for this rule set by setting $a \diamond' b$ equal to a new element $c \notin M$, and then defining additional values of $\diamond'$ as necessary to recover the axioms of Γ'.
3. If this can be done in all cases, then locate a seed $\diamond_0$ refuting the given target E', stop.
4. If there is an obstruction (often due to a collision in which a given operation $x \diamond' y$ is required to equal two different values), add one or more rules to Γ to avoid this obstruction, and return to Step 2.

Greedy constructions

Example (E73 does not imply E4380)

Show that the equation

$$x = y \diamond (y \diamond (x \diamond y))$$

does not imply the equation

$$x \diamond (x \diamond x) = (x \diamond x) \diamond x$$

.

Greedy constructions

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- $y \diamond (x \diamond y) = d \implies y \diamond d = x$

Greedy constructions

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.

- $y \diamond (x \diamond y) = d \implies y \diamond d = x$
- $x \diamond y = z \diamond y \implies x = z$

Greedy constructions

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.

- $y \diamond (x \diamond y) = d \implies y \diamond d = x$
- $x \diamond y = z \diamond y \implies x = z$
- $x \diamond y \neq y$

Finite magmas

Lessons learnt