Magma equations

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2025-01-28

Mathoverflow question

Is there an identity between the commutative identity and the constant identity?

Ask Question

Asked 1 year, 6 months ago Modified 2 months ago Viewed 3k times

26

I asked this on Math Stack Exchange, but it didn't get a single answer. So, I am now asking it here. Let our signature be that of a single binary operation $+$. I define the constant identity to be $x + y = z + w$. The commutative identity is, of course, the well-known identity $x + y = y + x$. I wonder, is there an identity strictly between those two, meaning, is there a single identity E such that the constant identity implies E, but not conversely, and E implies the commutative identity, but not conversely?

Mathoverflow answer — before the project started

Yes:

$$(x + x) + y = y + x$$

The constant identity implies this because both sides are $+$es. This does not imply the constant identity because it is true about any set with an operation that is commutative, associative, and idempotent (meaning $x + x = x$ for all x), the smallest nontrivial example is $\{x, y\}$ where $x + x = x + y = y + x = x$ and $y + y = y$.

This implies commutativity. $(x + x) + (x + x) = (x + x) + x = x + x$, so $x + x$ is idempotent. $(x + x) + y = ((x + x) + (x + x)) + y = y + (x + x)$, so $x + x$ commutes with everything. $(x + x) + (y + y) = (y + y) + x = x + y$, and $(x + x) + (y + y) = (y + y) + (x + x)$ because $x + x$ commutes with everything, so $x + y = y + x$, so $+$ is commutative.

This is not implied by commutativity, because (for example) addition of natural numbers is commutative but does not satisfy this identity.

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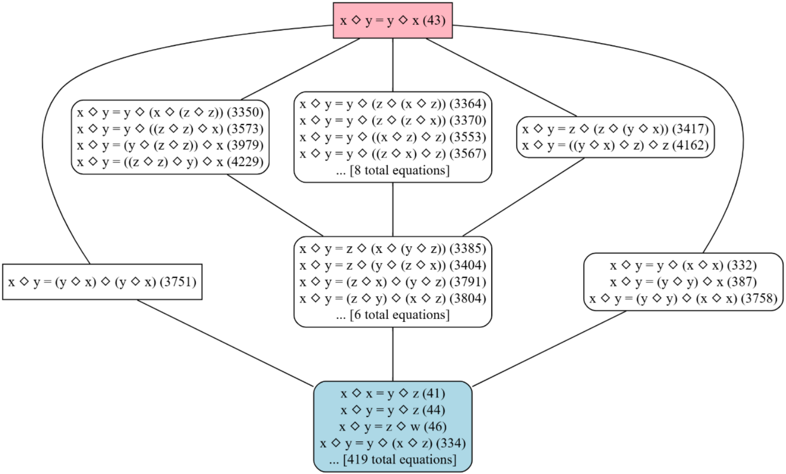
answered Jul 16, 2023 at 23:50


paste bee

Mathoverflow answer — after the project started



The commutative law (43) is the red box on top; the constant law (46) is in the blue box on the bottom

Announcement

Home About Career advice On writing Books Applets Mastodon+

A pilot project in universal algebra to explore new ways to collaborate and use machine assistance?

25 September, 2024 in math.RA, polymath | Tags: Artificial Intelligence, Equational Theory Project, machine assisted proof, universal algebra | by Terence Tao

Traditionally, mathematics research projects are conducted by a small number (typically one to five) of expert mathematicians, each of which are familiar enough with all aspects of the project that they can verify each other's contributions. It has been challenging to organize mathematical projects at larger scales, and particularly those that involve contributions from the general public, due to the need to verify all of the contributions; a single error in one component of a mathematical argument could invalidate the entire project. Furthermore, the sophistication of a typical math project is such that it would not be realistic to expect a member of the public, with say an undergraduate level of mathematics education, to contribute in a meaningful way to many such projects.

Announcement

Proposition 1 Equation4 implies Equation7.

Proof: Suppose that G obeys Equation4, thus

$$(x \circ x) \circ y = y \circ x \qquad (1)$$

for all $x, y \in G$. Specializing to $y = x \circ x$, we conclude

$$(x \circ x) \circ (x \circ x) = (x \circ x) \circ x$$

and hence by another application of (1) we see that $x \circ x$ is idempotent:

$$(x \circ x) \circ (x \circ x) = x \circ x. \qquad (2)$$

Now, replacing x by $x \circ x$ in (1) and then using (2), we see that

$$(x \circ x) \circ y = y \circ (x \circ x),$$

so in particular $x \circ x$ commutes with $y \circ y$:

$$(x \circ x) \circ (y \circ y) = (y \circ y) \circ (x \circ x). \qquad (3)$$

Also, from two applications (1) one has

$$(x \circ x) \circ (y \circ y) = (y \circ y) \circ x = x \circ y.$$

Thus (3) simplifies to $x \circ y = y \circ x$, which is Equation7. $\square$

Announcement

LEAN

```
import Mathlib.Tactic

universe u

class Magma (α : Type u) where
  op : α → α → α

infixl:65 " ∘ "   => Magma.op

def Equation4 (G: Type*) [Magma G] := ∀ x y : G, (x ∘ x) ∘ y = y ∘ x

def Equation7 (G: Type*) [Magma G] := ∀ x y : G, x ∘ y = y ∘ x

theorem Equation4_implies_Equation7 (G: Type*) [Magma G] (h: Equation4 G) : Equation7 G := by
  have idem (x : G) : (x ∘ x) ∘ (x ∘ x) = (x ∘ x) := by
    rw [h x (x ∘ x), h x x]
  have comm (x y : G) : (x ∘ x) ∘ y = y ∘ (x ∘ x) := by
    rw [<-idem x, h (x ∘ x) y, idem x]
  have op_idem (x y : G) : (x ∘ x) ∘ (y ∘ y) = x ∘ y := by
    rw [h x (y ∘ y), h y x]
  intro x y
  rw [<- op_idem x y, comm x (y ∘ y), op_idem y x]
```

Easy implication

Prove that

$$\forall x, \forall y, \quad x = x \diamond y$$

implies

$$\forall x, \forall y, \forall z, \forall w, \quad x \diamond y = (x \diamond z) \diamond (w \diamond y)$$

.

Easy implication

Prove that

$$\forall x, \forall y, \quad x = x \diamond y$$

implies

$$\forall x, \forall y, \forall z, \forall w, \quad x \diamond y = (x \diamond z) \diamond (w \diamond y)$$

.

Solution:

$$x \diamond y = x = x \diamond z = (x \diamond z) \diamond (w \diamond y)$$

Easy counterexample

Show that

$$\textit{todo}$$

does not imply

$$\textit{todo}$$

.

Easy counterexample

Show that

$$todo$$

does not imply

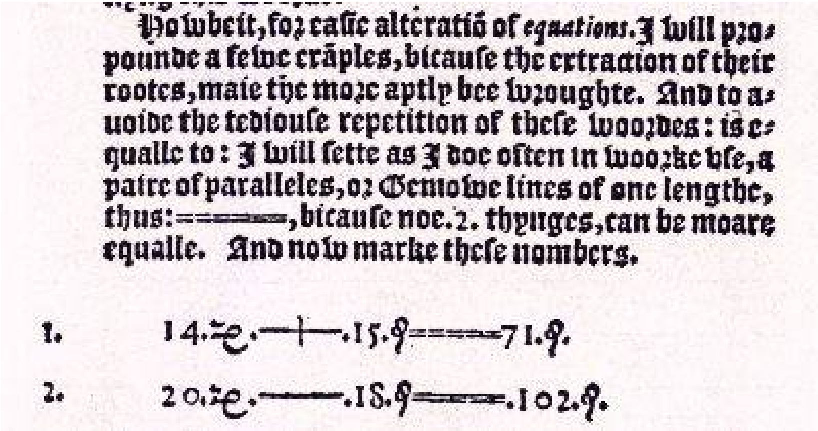
$$todo$$

.

Solution:

$$todo$$

Why equations?


Howbeit, for easie alteratiō of *equations*. I will propounde a fewe exāples, bicause the extraction of their rootes, maie the more aptly bee wroughte. And to auoide the tediouse repetition of these woordes: is equalle to: I will sette as I doe often in woorke use, a paire of paralleles, or Gemowe lines of one lengthe, thus: =====, bicause noe. 2. thynges, can be moare equalle. And now marke these nombers.

1. 14.ꝛ.—+—.15.ϑ=====71.ϑ.

2. 20.ꝛ.———.18.ϑ=====.102.ϑ.

"I will sette as I doe often in woorke use, a paire of paralleles, or twin lines of one lengthe, thus: $=$, bicause noe 2 thynges can be moare equalle."

Robert Recorde's mathematical book The Whetstone of Witte (1557)

Why magmas?

Scope of the project

Order-0 equations:

$$x = x \tag{E1}$$

$$x = y \tag{E2}$$

Order-1 equations e.g.:

$$x = y \diamond z \tag{E7}$$

Order-2 equations e.g.:

$$x \diamond y = y \diamond x \tag{E43}$$

Order-3 equations e.g.:

$$x = (y \diamond x) \diamond (x \diamond z) \tag{E168}$$

Order-4 equations e.g.:

$$(x \diamond y) \diamond z = (u \diamond v) \diamond w \tag{E4694}$$

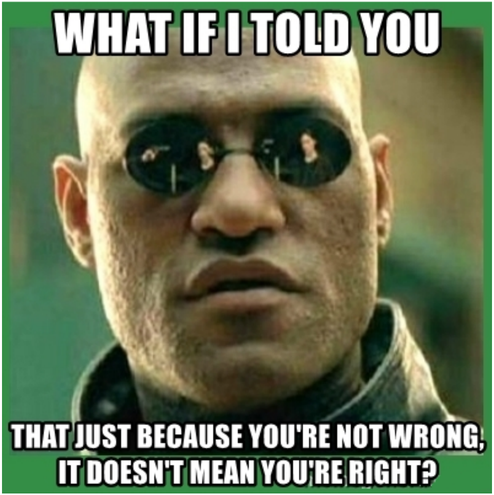
Lean and manual vs automated contributions

Implication graph completion

- If $P \implies Q$ and $Q \implies R$, then $P \implies R$.
- If $P \implies Q$ and $P \not\Longrightarrow R$, then $Q \not\Longrightarrow R$.
- If $Q \implies R$ and $P \not\Longrightarrow R$, then $P \not\Longrightarrow Q$.

Constructivism?

Intuitionists be like:



Dashboard

The implication graph is **99.99986%** complete.

An implication is considered *explicitly true* or *explicitly false* if we have a proof of the corresponding proposition formalised in Lean. It is *implicitly true* or *implicitly false* if the proposition can be derived by taking the reflexive transitive closure of explicitly proven implications.

Our current counts of implications in each of those categories are:

explicitly true	implicitly true	explicitly false	implicitly false	no proof
10,657	8,167,622	586,915	13,268,412	30

The *no proof* column above represents work that we still need to do. Among the *no proof* implications, we have the following conjecture counts:

explicitly true	implicitly true	explicitly false	implicitly false	no conjecture
0	0	9	21	0

The implication graph is **100.00000%** complete if we include conjectures.

Equation explorer

Free magmas [for equation]

Let X be a set of generators:

Free magmas [for equation]

Let X be a set of generators:

- Free magma —

Free magmas [for equation]

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X

Free magmas [for equation]

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma —

Free magmas [for equation]

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma — unordered rooted full binary trees with leaves labeled by X

Free magmas [for equation]

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma — unordered rooted full binary trees with leaves labeled by X
- Free semigroup —

Free magmas [for equation]

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma — unordered rooted full binary trees with leaves labeled by X
- Free semigroup — nonempty words over X

Free magmas [for equation]

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma — unordered rooted full binary trees with leaves labeled by X
- Free semigroup — nonempty words over X

If X is a singleton:

Free magmas [for equation]

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma — unordered rooted full binary trees with leaves labeled by X
- Free semigroup — nonempty words over X

If X is a singleton:

- Free magma —

Free magmas [for equation]

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma — unordered rooted full binary trees with leaves labeled by X
- Free semigroup — nonempty words over X

If X is a singleton:

- Free magma — ordered rooted full binary trees

Free magmas [for equation]

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma — unordered rooted full binary trees with leaves labeled by X
- Free semigroup — nonempty words over X

If X is a singleton:

- Free magma — ordered rooted full binary trees
- Free commutative magma —

Free magmas [for equation]

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma — unordered rooted full binary trees with leaves labeled by X
- Free semigroup — nonempty words over X

If X is a singleton:

- Free magma — ordered rooted full binary trees
- Free commutative magma — unordered rooted full binary trees

Free magmas [for equation]

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma — unordered rooted full binary trees with leaves labeled by X
- Free semigroup — nonempty words over X

If X is a singleton:

- Free magma — ordered rooted full binary trees
- Free commutative magma — unordered rooted full binary trees
- Free semigroup —

Free magmas [for equation]

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma — unordered rooted full binary trees with leaves labeled by X
- Free semigroup — nonempty words over X

If X is a singleton:

- Free magma — ordered rooted full binary trees
- Free commutative magma — unordered rooted full binary trees
- Free semigroup — positive integers

Free magmas — beyond single equations

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma — unordered rooted full binary trees with leaves labeled by X
- Free semigroup — nonempty words over X

Free magmas — beyond single equations

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma — unordered rooted full binary trees with leaves labeled by X
- Free semigroup — nonempty words over X
- Free commutative semigroup —

Free magmas — beyond single equations

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma — unordered rooted full binary trees with leaves labeled by X
- Free semigroup — nonempty words over X
- Free commutative semigroup — nonempty multisets

Free magmas — beyond single equations

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma — unordered rooted full binary trees with leaves labeled by X
- Free semigroup — nonempty words over X
- Free commutative semigroup — nonempty multisets
- Free commutative idempotent semigroup —

Free magmas — beyond single equations

Let X be a set of generators:

- Free magma — ordered rooted full binary trees with leaves labeled by X
- Free commutative magma — unordered rooted full binary trees with leaves labeled by X
- Free semigroup — nonempty words over X
- Free commutative semigroup — nonempty multisets
- Free commutative idempotent semigroup — nonempty sets

Difficult implication

Prove that

$$\forall x, \forall y, \forall z, \quad x = y \diamond ((x \diamond x) \diamond (z \diamond z))$$

implies

$$\forall x_1, \forall y_1, \quad x_1 = y_1$$

.

Difficult implication

Prove that

$$\forall x, \forall y, \forall z, \quad x = y \diamond ((x \diamond x) \diamond (z \diamond z))$$

implies

$$\forall x_1, \forall y_1, \quad x_1 = y_1$$

.

Solution (found by egg):

let $x_2 := x_1 \diamond x_1$
let $x_4 := x_2 \diamond x_2$
let $x_8 := x_4 \diamond x_4$
let $y_2 := y_1 \diamond y_1$
let $y_4 := y_2 \diamond y_2$
let $y_8 := y_4 \diamond y_4$

Difficult implication

$$\texttt{ass} \;:\quad \forall x, \forall y, \forall z, \quad x = y \diamond ((x \diamond x) \diamond (z \diamond z))$$

Solution (found by egg):

$$\begin{aligned}
x_1 &= y_1 \diamond x_4 && \texttt{ass}\ x_1\ y_1\ x_1 \\
&= y_1 \diamond (y_1 \diamond (x_8 \diamond x_8)) && \texttt{ass}\ x_4\ y_1\ x_4 \\
&= y_1 \diamond (y_1 \diamond (x_1 \diamond x_1)) && \texttt{ass}\ x_1\ x_4\ x_1 \\
&= y_1 \diamond (y_8 \diamond (x_1 \diamond x_1)) && \texttt{ass}\ y_1\ y_4\ y_1 \\
&= y_4 && \texttt{ass}\ y_4\ y_1\ x_1 \\
&= y_1 \diamond (y_8 \diamond (y_1 \diamond y_1)) && \texttt{ass}\ y_4\ y_1\ y_1 \\
&= y_1 \diamond (y_1 \diamond (y_1 \diamond y_1)) && \texttt{ass}\ y_1\ y_4\ y_1 \\
&= y_1 \diamond (y_1 \diamond (y_8 \diamond y_8)) && \texttt{ass}\ y_1\ y_4\ y_1 \\
&= y_1 \diamond y_4 && \texttt{ass}\ y_4\ y_1\ y_4 \\
&= y_1 && \texttt{ass}\ y_1\ y_1\ y_1
\end{aligned}$$

Metatheorems

Counterexample constructions

- Finite magmas
- Linear models
- Translation-invariant models
- Twisting semigroups
- Greedy constructions
- Ad-hoc modifications
- Combinations of the above

Greedy constructions

Preliminaries

We want to build a magma operation $\diamond \colon M \times M \to M$ that obeys one equation E but not another E'.

We one can first consider *partial magma operations* $\diamond \colon \Omega \to M$ defined on some $\Omega \subseteq M \times M$.

We say that a partial operation $\diamond' \colon \Omega' \to M$ *extends* $\diamond \colon \Omega \to M$ iff:

- $\Omega \subseteq \Omega'$
- $(x, y) \in \Omega \implies x \diamond y = x \diamond' y$

Given a sequence $\diamond_n \colon \Omega_n \to M$ of partial operations, each of which is an extension of the previous, we can define the *direct limit* $\diamond_\infty \colon \bigcup_n \Omega_n \to M$ to be the partial operation defined by $x \diamond_\infty y := x \diamond_n y$ whenever $(x, y) \in \Omega_n$.

Greedy constructions

Abstract greedy algorithm

Let E and E' be equations. Let Γ be a first-order theory regarding a partial magma operations $\diamond \colon \Omega \to M$ on a carrier M. Assume the following:

- (Seed) There exists a finitely supported partial magma operation $\diamond_0 \colon \Omega_0 \to M$ satisfying Γ that contradicts E', in the sense that there is some assignment of variables in E' in M such that both sides of E' are defined using $\diamond_0$ but not equal to each other.
- (Soundness) If $\diamond_n \colon \Omega_n \to M$ is a sequence of partial magma operations obeying Γ with each $\diamond_{n+1}$ an extension of $\diamond_n$, and the direct limit $\diamond_\infty$ is total, then this limit obeys E.
- (Greedy extension) If $\diamond \colon \Omega \to M$ is a finitely supported partial magma operation obeying Γ, and $a, b \in M$, then there exists a finitely supported extension $\diamond' \colon \Omega' \to M'$ of $\diamond$ to a possibly larger carrier M' such that $a \diamond' b$ is defined.

Then $E \not\models E'$.

Greedy constructions

Trial and error

1. Start with a minimal rule set Γ that has just enough axioms to imply the soundness property for the given hypothesis E.
2. Attempt to establish the greedy extension property for this rule set by setting $a \diamond' b$ equal to a new element $c \notin M$, and then defining additional values of $\diamond'$ as necessary to recover the axioms of Γ'.
3. If this can be done in all cases, then locate a seed $\diamond_0$ refuting the given target E', stop.
4. If there is an obstruction (often due to a collision in which a given operation $x \diamond' y$ is required to equal two different values), add one or more rules to Γ to avoid this obstruction, and return to Step 2.

Greedy constructions

Example (E73 does not imply E4380)

Show that the equation

$$x = y \diamond (y \diamond (x \diamond y))$$

does not imply the equation

$$x \diamond (x \diamond x) = (x \diamond x) \diamond x$$

.

Greedy constructions

Example (E73 does not imply E4380)

Show that the equation

$$x = y \diamond (y \diamond (x \diamond y))$$

does not imply the equation

$$x \diamond (x \diamond x) = (x \diamond x) \diamond x$$

.

- $y \diamond (x \diamond y) = d \implies y \diamond d = x$

Greedy constructions

Example (E73 does not imply E4380)

Show that the equation

$$x = y \diamond (y \diamond (x \diamond y))$$

does not imply the equation

$$x \diamond (x \diamond x) = (x \diamond x) \diamond x$$

.

- $y \diamond (x \diamond y) = d \implies y \diamond d = x$
- $x \diamond y = z \diamond y \implies x = z$

Greedy constructions

Example (E73 does not imply E4380)

Show that the equation

$$x = y \diamond (y \diamond (x \diamond y))$$

does not imply the equation

$$x \diamond (x \diamond x) = (x \diamond x) \diamond x$$

.

- $y \diamond (x \diamond y) = d \implies y \diamond d = x$
- $x \diamond y = z \diamond y \implies x = z$
- $x \diamond y \neq y$

Finite magmas

Finite graph

Some implications are true specifically only for finite magmas.

The finite implication graph is **99.99999%** complete.

explicitly true	implicitly true	explicitly false	implicitly false	no proof
10,750	8,168,349	586,220	13,268,315	2

The finite implication graph is **99.99999%** complete if we include conjectures.

explicitly true	implicitly true	explicitly false	implicitly false	no conjecture
0	0	0	0	2

About the project

The Atlantic

We're Entering Uncharted Territory for Math

Terence Tao, the world's greatest living mathematician, has a vision for AI.

By Matteo Wong

About the project

Substack

On Math Platform

MICHAEL BUCKO

OKT. 05, 2024

Teilen

I was thinking a combination of *mathematics as a collaborative game (simulations)* and *engineering mathematics.* A platform with the map of mathematics as well as the pull request capability — that could take an idea (or a group of ideas and insights), that kind of seed, modeled as an e-graph (or something similar), given the mathlib-like math knowledge base context, to something that gets formalized, and is fully reusable. With enough compute, one could get agents to work on that thing in the background — it'd become a small island (rather than only a point) in that special embedding space.

Lessons learnt