Theory on Matroids

April 5, 2024

This document contains definitions and properties that are implemented in the project. The focus is on clarity of statements and lean-friendliness of the proofs.

1 References

- James Oxley, Matroid Theory, Second edition, Oxford University Press, New York, 2011.
- Henning Bruhn, Reinhard Diestel, Matthias Kriesell, Rudi Pendavingh, Paul Wollan, Axioms for infinite matroids, Advances in Mathematics, Volume 239, 2013, Pages 18–46. `https://doi.org/10.1016/j.aim.2013.01.011`

Todo: create bibliography, move to the end of document

2 Matroid

Notation. Given $\mathcal{E} \subseteq 2^E$, we write $\mathcal{E}^{\max}$ for the set of maximal elements of $\mathcal{E}$.

2.1 Definition via Independence Axioms

Definition 2.1. A matroid $\mathcal{M} = (E, \mathcal{I})$ is a pair of a ground set E (finite or infinite) and a set $\mathcal{I} \subseteq 2^E$ satisfying the following independence axioms:

1. $\emptyset \in \mathcal{I}$.
2. If $A \subseteq B$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$.
3. For all $I \in \mathcal{I} \setminus \mathcal{I}^{\max}$ and $I' \in \mathcal{I}^{\max}$, there is an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}$.
4. Whenever $I \subseteq X \subseteq E$ and $I \in \mathcal{I}$, the set $\{I' \in \mathcal{I} \mid I \subseteq I' \subseteq X\}$ has a maximal element.

Todo: more axioms, proof of equivalence following [2]?

3 Functions on Matroids

3.1 Closure

Definition 3.1. The closure operator is the function $2^E \to 2^E$ mapping a set $X \subseteq E$ to the set

$$\operatorname{cl}(X) = X \cup \{x \mid \exists I \subseteq X \colon I \in \mathcal{I} \text{ but } I + x \notin \mathcal{I}\}.$$

Todo: state and prove properties

3.2 Relative Rank

Definition 3.2. The relative rank function is the function $r\colon \left(2^E \times 2^E\right)_{\subseteq} \to \mathbb{N} \cup \{\infty\}$ that maps a pair $A \supseteq B$ of subsets of E to

$$r\left(A \mid B\right) = \max\left\{|I \setminus J| \mid I \supseteq J,\ I \in \mathcal{I} \cap 2^A,\ J \text{ maximal in } \mathcal{I} \cap 2^B\right\}.$$

Todo: state and prove properties

Lemma 3.1. *This maximum is attained and is independent of the choice of J.*

4 Operations on Matroids

4.1 Direct Sum

Definition 4.1. Let $\mathcal{M}_1 = (E_1, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E_2, \mathcal{I}_2)$ be matroids on disjoint sets E_1 and E_2. Let $E = E_1 \cup E_2$ and $\mathcal{I} = \{I_1 \cup I_2 \mid I_1 \in \mathcal{I}_1,\ I_2 \in \mathcal{I}_{\in}\}$. The matroid $\mathcal{M} = (E, \mathcal{I})$ is the direct sum of $\mathcal{M}_1$ and $\mathcal{M}_2$ and is denoted $\mathcal{M}_1 \oplus \mathcal{M}_2$.

Lemma 4.1. *$\mathcal{M}_1 \oplus \mathcal{M}_2$ is indeed a matroid, i.e., $\mathcal{I}$ satisfies the independence axioms.*

Proof.

1. Since $\emptyset \in \mathcal{I}_1$ and $\emptyset \in \mathcal{I}_2$, we have $\emptyset \cup \emptyset = \emptyset \in \mathcal{I}$.
2. Let $A \subseteq B$ and $B \in \mathcal{I}$. Since $B \in \mathcal{I}$, by definition of $\mathcal{I}$ it can be represented as $B = B_1 \cup B_2$ with $B_1 \in \mathcal{I}_1$ and $B_2 \in \mathcal{I}_2$. Let $A_1 = A \cap B_1$ and $A_2 = A \cap B_2$. Since $A_1 \subseteq B_1$ and $B_1 \in \mathcal{I}_1$, by independence axiom 2 in $\mathcal{M}_1$ we have $A_1 \in \mathcal{I}_1$. By repeating this for $\mathcal{M}_2$, we get $A_2 \in \mathcal{I}_2$. Thus, $A = A_1 \cup A_2$ where $A_1 \in \mathcal{I}_1$ and $A_2 \in \mathcal{I}_2$, so $A \in \mathcal{I}$by definition of $\mathcal{I}$.
3. We split the proof into two parts.
   (a) First, let us show that if $I \in \mathcal{I}^{\max}$, then $I \cap E_1 \in \mathcal{I}_1^{\max}$ and $I \cap E_2 \in \mathcal{I}_2^{\max}$. To this end, suppose that $I \cap E_1 \notin \mathcal{I}_1^{\max}$, i.e., there is $x \in E_1 \setminus I$ such that $(I \cap E_1) + x \in \mathcal{I}_1$. But then $(I \cap E_1) + (I \cap E_2) + x \in \mathcal{I}$ by definition of $\mathcal{I}$, which contradicts maximality of I. The proof for $I \cap E_2$ is identical up to swapping the roles of $\mathcal{M}_1$ and $\mathcal{M}_2$.
   (b) Now, let $I \in \mathcal{I} \setminus \mathcal{I}^{\max}$ and $I' \in \mathcal{I}^{\max}$. Let us introduce the following notation: $I_1 = I \cap E_1$, $I_2 = I \cap E_2$, $I_1' = I' \cap E_1$, $I_2' = I' \cap E_2$. From the first part of the proof we know that $I_1' \in \mathcal{I}_1$ and $I_2' \in \mathcal{I}_2$. Additionally, as $I \notin \mathcal{I}^{\max}$, the first part also implies that at least one of the following holds: $I_1 \notin \mathcal{I}_1^{\max}$ or $I_2 \notin \mathcal{I}_2^{\max}$.

      Todo: show: there is an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}$
4. Let $I \subseteq X \subseteq E$ and $I \in \mathcal{I}$. For the sake of deriving a contradiction suppose that the set $T = \{I' \in \mathcal{I} \mid I \subseteq I' \subseteq X\}$ has no maximal element.

   Todo: derive contradiction

□

4.2 Mapping

Todo: definition, proof that result is a matroid

4.3 Union

Todo: definition, proof that result is a matroid

5 Representable Matroids

Todo: definition, proof that result is a matroid

6 Regular Matroids

Todo: definition, proof that result is a matroid

7 Largest Common Independent Set

Lemma 7.1. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be finite matroids with rank functions r_1 and r_2 and a common ground set E. Then*

$$\max\left\{|I| \mid I \in \mathcal{I}_1 \cap \mathcal{I}_2\right\} = \min\left\{r_1(T) + r_2(E - T) \mid T \subseteq E\right\}.$$

Todo: proof? generalize?

8 Seymour's Decomposition Theorem

Todo: statement, proof