Theory on Matroids

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This document contains definitions and properties that are implemented in the project. The focus is on clarity of statements and lean-friendliness of the proofs.

1 References

- James Oxley, Matroid Theory, Second edition, Oxford University Press, New York, 2011.
- Henning Bruhn, Reinhard Diestel, Matthias Kriesell, Rudi Pendavingh, Paul Wollan, Axioms for infinite matroids, Advances in Mathematics, Volume 239, 2013, Pages 18–46. `https://doi.org/10.1016/j.aim.2013.01.011`

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2 Matroid

Notation. Given $\mathcal{E} \subseteq 2^E$, we write $\mathcal{E}^{\max}$ for the set of maximal elements of $\mathcal{E}$.

2.1 Definition via Independence Axioms

Definition 2.1. A *matroid* $\mathcal{M} = (E, \mathcal{I})$ is a pair of a ground set E (finite or infinite) and a set $\mathcal{I} \subseteq 2^E$ satisfying the following independence axioms:

1. $\emptyset \in \mathcal{I}$.
2. If $A \subseteq B$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$.
3. For all $I \in \mathcal{I} \setminus \mathcal{I}^{\max}$ and $I' \in \mathcal{I}^{\max}$, there is an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}$.
4. Whenever $I \subseteq X \subseteq E$ and $I \in \mathcal{I}$, the set $\{I' \in \mathcal{I} \mid I \subseteq I' \subseteq X\}$ has a maximal element.

Todo: more axioms, proof of equivalence following [2]?

3 Functions on Matroids

3.1 Closure

Definition 3.1. The *closure operator* is the function $2^E \to 2^E$ mapping a set $X \subseteq E$ to the set

$$\mathrm{cl}\,(X) = X \cup \{x \in E \mid \exists I \subseteq X \text{ s.t. } : I \in \mathcal{I} \text{ but } I + x \notin \mathcal{I}\}\,.$$

Todo: state and prove properties

3.2 Relative Rank

Definition 3.2. The *relative rank function* is the function $r\colon \left(2^E \times 2^E\right)_{\subseteq} \to \mathbb{N} \cup \{\infty\}$ that maps a pair $A \supseteq B$ of subsets of E to

$$r\left(A \mid B\right) = \max\left\{|I \setminus J| \mid I \supseteq J,\ I \in \mathcal{I} \cap 2^A,\ J \text{ maximal in } \mathcal{I} \cap 2^B\right\}.$$

Todo: state and prove properties

Lemma 3.1. *This maximum is attained and is independent of the choice of J.*

4 Operations on Matroids

4.1 Direct Sum

Definition 4.1. Let $\mathcal{M}_1 = (E_1, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E_2, \mathcal{I}_2)$ be matroids on disjoint sets E_1 and E_2. Let $E = E_1 \cup E_2$ and $\mathcal{I} = \{I_1 \cup I_2 \mid I_1 \in \mathcal{I}_1,\ I_2 \in \mathcal{I}_2\}$. The *direct sum* of $\mathcal{M}_1$ and $\mathcal{M}_2$ is the matroid $\mathcal{M}_1 \oplus \mathcal{M}_2 = (E, \mathcal{I})$.

Lemma 4.1. *$\mathcal{M}_1 \oplus \mathcal{M}_2$ is indeed a matroid, i.e., $\mathcal{I}$ satisfies the independence axioms.*

Proof. First, we list several propositions that will be useful when checking the axioms.

1. If $A, B \in E_1 \cup E_2$ where $E_1 \cap E_2 = \emptyset$, then $A \subseteq B$ if and only if $A \cap E_1 \subseteq B \cap E_1$ and $A \cap E_2 \subseteq B \cap E_2$. Moreover, if $A \subsetneq B$, then $A \cap E_1 \subsetneq B \cap E_1$ or $A \cap E_2 \subsetneq B \cap E_2$. This holds by set theory.

2. $I \in \mathcal{I}$ if and only if $I \cap E_1 \in \mathcal{I}_1$ and $I \cap E_2 \in \mathcal{I}_2$. Indeed, this follows from the definition of $\mathcal{I}$ and the fact that $E_1 \cap E_2 = \emptyset$. need more details?

3. $I \in \mathcal{I}^{\max}$ if and only if $I \cap E_1 \in \mathcal{I}_1^{\max}$ and $I \cap E_2 \in \mathcal{I}_2^{\max}$. To prove the forward direction, first note that $I \cap E_1 \in \mathcal{I}_1$ and $I \cap E_2 \in \mathcal{I}_2$ by proposition 2. For the sake of deriving a contradiction, suppose that $I \cap E_1 \notin \mathcal{I}_1^{\max}$, i.e., there is $x \in E_1 \setminus I$ such that $(I \cap E_1) + x \in \mathcal{I}_1$. But then $(I \cap E_1) + x + (I \cap E_2) \in \mathcal{I}$ by definition of $\mathcal{I}$, which contradicts maximality of I. Thus, $I \cap E_1 \in \mathcal{I}_1^{\max}$, as desired. We can show $I \cap E_2 \in \mathcal{I}_2^{\max}$ similarly by swapping the roles of $\mathcal{M}_1$ and $\mathcal{M}_2$. To show the converse by contraposition, suppose we have $I \notin \mathcal{I}^{\max}$. Then there exists $I' \in \mathcal{I}$ such that $I \subsetneq I'$. By proposition 1, we have $I \cap E_1 \subsetneq I' \cap E_1$ or $I \cap E_2 \subsetneq I' \cap E_2$. Since $I \cap E_1, I' \cap E_1 \in \mathcal{I}_1$ and $I \cap E_2, I' \cap E_2 \in \mathcal{I}_2$ by proposition 2, this means that $I \cap E_1 \notin \mathcal{I}_1^{\max}$ or $I \cap E_2 \notin \mathcal{I}_2^{\max}$.

4. Let $T = \{I' \in \mathcal{I} \mid I \subseteq I' \subseteq X\}$, $T_1 = \{I'' \in \mathcal{I} \mid I_1 \subseteq I'' \subseteq X_1\}$, $T_2 = \{I'' \in \mathcal{I} \mid I_1 \subseteq I'' \subseteq X_1\}$. If $I' \in T$, then $I' = I_1' \cup I_2'$ where $I_1' \in T_1$ and $I_2' \in T_2$. Indeed, by definition of $\mathcal{I}$ we have $I' = I_1' \cup I_2'$ where $I_1' \in \mathcal{I}_1$ and $I_2' \in \mathcal{I}_2$. By applying proposition 1 to I and I' and to I' and X, we obtain $I_1 \subseteq I_1' \subseteq X_1$ and $I_2 \subseteq I_2' \subseteq X_2$.

Now we are ready to prove the independence axioms.

1. Since $\emptyset \in \mathcal{I}_1$ and $\emptyset \in \mathcal{I}_2$, we have $\emptyset \cup \emptyset = \emptyset \in \mathcal{I}$.

2. Let $A \subseteq B$ and $B \in \mathcal{I}$. By combining proposition 1 with proposition 2, we get $A \cap E_1 \subseteq B \cap E_1 \in \mathcal{I}$ and $A \cap E_2 \subseteq B \cap E_2 \in \mathcal{I}$. By independence axiom 2 for $\mathcal{M}_1$ and $\mathcal{M}_2$, this implies $A \cap E_1 \in \mathcal{I}$ and $A \cap E_2 \in \mathcal{I}$, so $A \in \mathcal{I}$ by proposition 2.

3. Let $I \in \mathcal{I} \setminus \mathcal{I}^{\max}$ and $I' \in \mathcal{I}^{\max}$. Our goal is to show that there exists $x \in I' \setminus I$ such that $I + x \in \mathcal{I}$. To simplify notation, let $I_1 = I \cap E_1$, $I_2 = I \cap E_2$, $I_1' = I' \cap E_1$, $I_2' = I' \cap E_2$. By proposition 3 applied to I', we have $I_1' \in \mathcal{I}_1^{\max}$ and $I_2' \in \mathcal{I}_2^{\max}$, while by the same proposition applied to I, we know that $I_1 \notin \mathcal{I}_1^{\max}$ or $I_2 \notin \mathcal{I}_2^{\max}$. We assume that $I_1 \notin \mathcal{I}_1^{\max}$, as the argument in case $I_2 \notin \mathcal{I}_2^{\max}$ is the same up to swapping $\mathcal{M}_1$ and $\mathcal{M}_2$. Since $I_1 \in \mathcal{I}_1$ by proposition 2, we can apply independence axiom 3 for $\mathcal{M}_1$ to I_1 and I_1', which yields $x \in I_1' \setminus I_1$ such that $I_1 + x \in \mathcal{I}_1$. Clearly, $x \in I' \setminus I$: on the one hand, we have $x \in I_1' \subseteq I'$, and on the other hand, $x \notin I$ follows from $x \notin I_1$ (by construction of x) and purely set-theoretic facts ($x \in I_1' \subseteq E_1$, $I_2 \subseteq E_2$, $E_1 \cap E_2 = \emptyset$, $I = I_1 \cup I_2$). Additionally, $I_1 + x \in \mathcal{I}_1$ together with $I_2 \in \mathcal{I}_2$ implies $I + x = I_1 + x + I_2 \in \mathcal{I}$. Thus, element x satisfies both the desired properties.

4. Let $I \subseteq X \subseteq E$ and $I \in \mathcal{I}$. For the sake of deriving a contradiction, assume that the set $T = \{I' \in \mathcal{I} \mid I \subseteq I' \subseteq X\}$ has no maximal element. Let us introduce the following notation: $I_1 = I \cap E_1$, $I_2 = I \cap E_2$, $X_1 = X \cap E_1$, $X_2 = X \cap E_2$. By independence axiom 4 for $\mathcal{M}_1$ and $\mathcal{M}_2$, we know that $T_1 = \{I'' \in \mathcal{I} \mid I_1 \subseteq I'' \subseteq X_1\}$ and $T_2 = \{I'' \in \mathcal{I} \mid I_2 \subseteq I'' \subseteq X_2\}$ each have a maximal element, which we denote by S_1 and S_2, respectively. Consider the set $S = S_1 \cup S_2$ and observe that $S \in T$. Indeed, $S \in \mathcal{I}$ follows from $S = S_1 \cup S_2$, $S_1 \in \mathcal{I}_1$ (as $S_1 \in T_1$), and $S_2 \in \mathcal{I}_2$ (as $S_2 \in T_2$), while applying proposition 1 to I and S and to S and X yields $I \subseteq S \subseteq X$. Since by our assumption T has no maximal element, there exists $S' \in T$ such that $S \subsetneq S'$. Let $S'_1 = S' \cap E_1$ and $S'_2 = S' \cap E_2$ and note that $S'_1 \in \mathcal{I}_1$ and $S'_2 \in \mathcal{I}_2$ by proposition 2. Applying proposition 4 to S' yields $S'_1 \in T_1$ and $S'_2 \in T_2$. By maximality of S_1 and S_2 in T_1 and T_2, respectively, we have $S'_1 \subseteq S_1$ and $S'_2 \subseteq S_2$. However, this implies $S' = S'_1 \cup S'_2 \subseteq S_1 \cup S_2 = S$, which contradicts $S \subsetneq S'$ from earlier.

□

Questions about lean optimizations: maximality properties are automatically preserved under finite unions?

4.2 Mapping

Lemma 4.2. *Let $f\colon E' \to E$ be a set mapping and let $\mathcal{M}' = (E', \mathcal{I}')$ be a matroid. Let $\mathcal{I} \subseteq 2^{E'}$ with $I \in \mathcal{I}$ if and only if $\exists I' \in \mathcal{I}'$ with $f(I') = I$. Then $\mathcal{M} = (E, \mathcal{I})$ is a matroid.*

Proof. First, we list several propositions that will be useful when checking the axioms.

1. For any $S' \subseteq E'$ and $T \subseteq E$, we have $f\left(S \cap f^{-1}(T)\right) = f(S) \cap T$.
2. For any $S' \subseteq E'$, we have $S' \subseteq f^{-1}(f(S'))$.
3. For any $S \subseteq T \subseteq E$, we have $f^{-1}(S) \subseteq f^{-1}(T)$.
4. For any $S' \subseteq T' \subseteq E'$, we have $f(S') \subseteq f(T')$.

Now we are ready to prove the independence axioms.

1. Observe that $\emptyset \in \mathcal{I}$, since $\emptyset \in \mathcal{I}'$ and $f(\emptyset) = \emptyset$.
2. Let $A \subseteq B \subseteq E$ where $B \in \mathcal{I}$. Then $B = f(B')$ for some $B' \in \mathcal{I}'$. Let $A' = f^{-1}(A) \cap B'$. Note that $A' \subseteq B'$ by construction, so by independence axiom 2 for $\mathcal{M}'$ we have $A' \in \mathcal{I}'$. Moreover, by combining proposition 1 with definitions of A', B', A, and B, we get:
$$f(A') = f\left(f^{-1}(A) \cap B'\right) = A \cap f(B') = A \cap B = A.$$
Thus, $A' \in \mathcal{I}'$ satisfies $A = f(A')$, so $A \in \mathcal{I}$ by construction of $\mathcal{I}$.
3. Let $I \in \mathcal{I} \setminus \mathcal{I}^{\max}$ and $J \in \mathcal{I}^{\max}$.

   Todo: there is an $x \in J \setminus I$ such that $I + x \in \mathcal{I}$.
4. Let $I \subseteq X \subseteq E$ where $I \in \mathcal{I}$ and let $T = \{J \in \mathcal{I} \mid I \subseteq J \subseteq X\}$. Let $I' \in \mathcal{I}'$ be such that $f(I') = I$ (from construction of $\mathcal{I}$) and let $X' = f^{-1}(X)$. Observe that $I' \subseteq X'$, since
$$I' \subseteq f^{-1}(f(I')) = f^{-1}(I) \subseteq f^{-1}(X) = X',$$
where the containments hold by propositions 2 and 3, respectively. Let $T' = \{J' \in \mathcal{I}' \mid I' \subseteq J' \subseteq X'\}$ and let S' be its maximal element from independence axiom 4 for $\mathcal{M}'$. Observe that $S = f(S') \in T$. Indeed, by construction $S' \in \mathcal{I}'$ and $I' \subseteq S' \subseteq X'$, so $S = f(S') \in \mathcal{I}$ holds by construction of $\mathcal{I}$ and $I \subseteq S \subseteq X$ by proposition 4. For the sake of deriving a contradiction, suppose that T has no maximal element. In particular, there exists $Q \in T$ such that $S \subsetneq Q$. Since $Q \in T$, there exists $Q' \in \mathcal{I}'$ such that $f(Q') = Q$. Since $I \subseteq Q \subseteq X$, by proposition 3 we have
$$f^{-1}(I) \subseteq f^{-1}(Q) \subseteq f^{-1}(X).$$
By proposition 2, $I' \subseteq f^{-1}(I)$, $S' \subseteq f^{-1}(S)$, $Q' \subseteq f^{-1}(Q)$. By construction of S', $I' \subseteq S'$.

   Todo: T has a maximal element.

□

4.3 Union

Todo: definition, proof that result is a matroid

5 Representable Matroids

Todo: definition, proof that result is a matroid

6 Regular Matroids

Todo: definition, proof that result is a matroid

7 Largest Common Independent Set

Lemma 7.1. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be finite matroids with rank functions r_1 and $r_2$and a common ground set E. Then*

$$\max\left\{|I| \mid I \in \mathcal{I}_1 \cap \mathcal{I}_2\right\} = \min\left\{r_1\left(T\right) + r_2\left(E - T\right) \mid T \subseteq E\right\}.$$

Todo: proof? generalize?

8 Seymour's Decomposition Theorem

Todo: statement, proof