Computer Science Track Core Course

Linear programming — part 1

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Linear program

Definitions

We are given an input $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$ and we search for $\mathbf{x} \in \mathbb{Q}^n$ that optimizes...

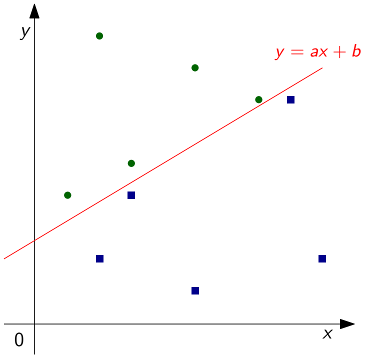
$$\begin{array}{lr} \text{minimize} & c'\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \geq b \end{array} \qquad \begin{array}{lr} \text{maximize} & c'\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq b \end{array}$$

$$\begin{array}{lr} \text{minimize} & c'\mathbf{x} \\ \text{s.t.} & A\mathbf{x} = b \\ \text{where} & \mathbf{x} \geq \mathbf{0} \end{array} \qquad \begin{array}{lr} \text{maximize} & c'\mathbf{x} \\ \text{s.t.} & A\mathbf{x} = b \\ \text{where} & \mathbf{x} \geq \mathbf{0} \end{array}$$

Examples of LP

Separating points

We are searching for a line $y = ax + b$ that separates "disks" (top) from "squares" (bottom).



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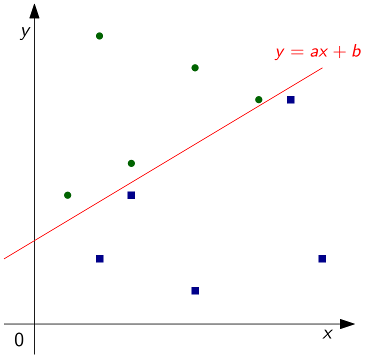

Variables:

Examples of LP

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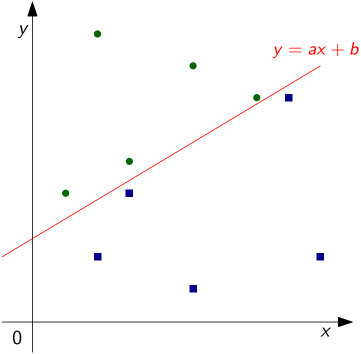

Variables: $a \lesseqgtr 0,\ b \lesseqgtr 0$

Examples of LP

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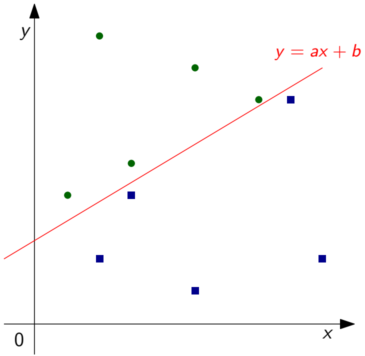

Variables: $a \lessgtr 0$, $b \lessgtr 0$

Point (x_1, y_1) above the line:

Examples of LP

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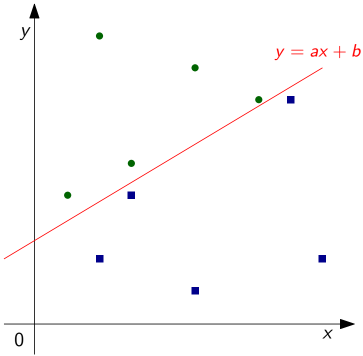

Variables: $a \lessgtr 0,\ b \lessgtr 0$

Point (x_1, y_1) above the line: $x_1 \cdot a + b \leq y_1$ "disk"

Examples of LP

Separating points

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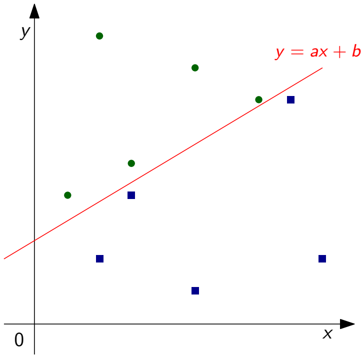

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Point (x_1, y_1) above the line: $x_1 \cdot a + b \leq y_1$ "disk"

Point (x_2, y_2) below the line:

Examples of LP

Separating points

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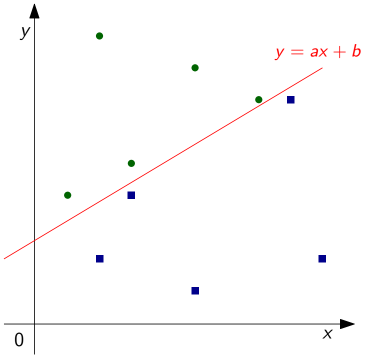

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Point (x_1, y_1) above the line: $x_1 \cdot a + b \leq y_1$ "disk"

Point (x_2, y_2) below the line: $x_2 \cdot a + b \geq y_2$ "square"

Examples of LP

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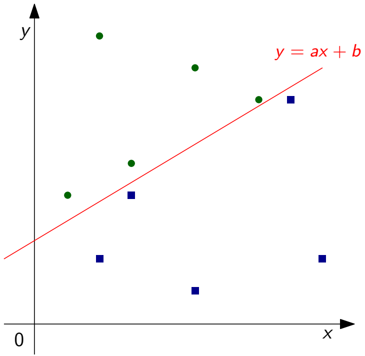
We are searching for a line $y = ax + b$ that separates "disks" (top) from "squares" (bottom).



Variables: $a \lessgtr 0,\ b \lessgtr 0$

Point (x_1, y_1) above the line: $x_1 \cdot a + b \leq y_1$ "disk"

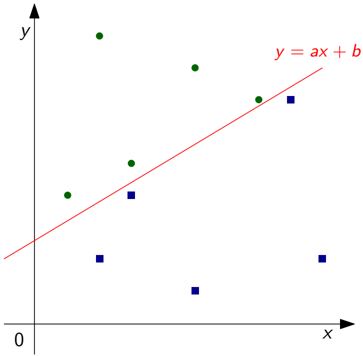
Point (x_2, y_2) below the line: $x_2 \cdot a + b \geq y_2$ "square"

Objective:

Examples of LP

Separating points

We are searching for a line $y = ax + b$ that separates "disks" (top) from "squares" (bottom).



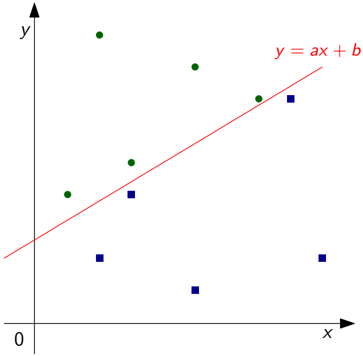
Variables: $a \lesseqgtr 0,\ b \lesseqgtr 0$

Point (x_1, y_1) above the line: $x_1 \cdot a + b \leq y_1$ "disk"

Point (x_2, y_2) below the line: $x_2 \cdot a + b \geq y_2$ "square"

Objective: minimize 0

Now, we are searching for a line $y = ax + b$ that separates them strictly.



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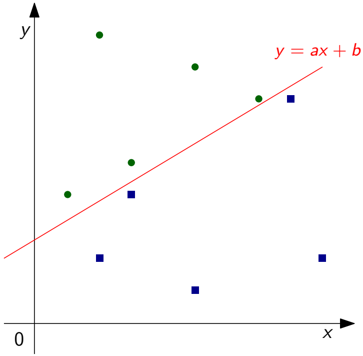

Variables: $a \lesseqgtr 0$, $b \lesseqgtr 0$, $d \geq 0$

Point (x_1, y_1) above the line: $x_1 \cdot a + b + d \leq y_1$

Point (x_2, y_2) below the line: $x_2 \cdot a + b - d \geq y_2$

Objective: maximize d

Example of duality

From minimization to maximization

Primal problem:

$$
\begin{array}{lrcrcl}
\text{minimize} & 6x_1 & + & 6x_2 & & \\
\text{s.t.} & 2x_1 & + & x_2 & \geq & 4 \\
 & x_1 & + & 2x_2 & \geq & 5 \\
\text{where} & x_1, x_2 & \geq & 0 & &
\end{array}
$$

Example of duality

From minimization to maximization

Primal problem:

$$\begin{array}{lcccccc} \text{minimize} & 6x_1 & + & 6x_2 & & \\ \text{s.t.} & 2x_1 & + & x_2 & \geq & 4 \\ & x_1 & + & 2x_2 & \geq & 5 \\ \text{where} & x_1, x_2 & \geq & 0 & & \end{array}$$

Dual problem:

$$\begin{array}{lcccccc} \text{maximize} & 4y_1 & + & 5y_2 & & \\ \text{s.t.} & 2y_1 & + & y_2 & \leq & 6 \\ & y_1 & + & 2y_2 & \leq & 6 \\ \text{where} & y_1, y_2 & \geq & 0 & & \end{array}$$

Example of duality

From maximization to minimization

Dual problem:

$$\begin{array}{lrcrcr} \text{maximize} & 4y_1 & + & 5y_2 & & \\ \text{s.t.} & 2y_1 & + & y_2 & \leq & 6 \\ & y_1 & + & 2y_2 & \leq & 6 \\ \text{where} & y_1, y_2 & \geq & 0 & & \end{array}$$

Dual problem:

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\begin{array}{lrcrcl}
\text{maximize} & 4y_1 & + & 5y_2 & & \\
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 & y_1 & + & 2y_2 & \leq & 6 \\
\text{where} & y_1, y_2 & \geq & 0 & &
\end{array}
$$

Dual of the dual problem:

$$
\begin{array}{lrcrcl}
\text{minimize} & 6z_1 & + & 6z_2 & & \\
\text{s.t.} & 2z_1 & + & z_2 & \geq & 4 \\
 & z_1 & + & 2z_2 & \geq & 5 \\
\text{where} & z_1, z_2 & \geq & 0 & &
\end{array}
$$

Example of duality

Comparison

Primal problem:

$$
\begin{array}{lrcrcl}
\text{minimize} & 6x_1 & + & 6x_2 & & \\
\text{s.t.} & 2x_1 & + & x_2 & \geq & 4 \\
 & x_1 & + & 2x_2 & \geq & 5 \\
\text{where} & x_1, x_2 & \geq & 0 & &
\end{array}
$$

Dual of the dual problem:

$$
\begin{array}{lrcrcl}
\text{minimize} & 6z_1 & + & 6z_2 & & \\
\text{s.t.} & 2z_1 & + & z_2 & \geq & 4 \\
 & z_1 & + & 2z_2 & \geq & 5 \\
\text{where} & z_1, z_2 & \geq & 0 & &
\end{array}
$$

We are given $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$ on the input.
We search for either $\mathbf{x} \in \mathbb{Q}^n$ or $\mathbf{y} \in \mathbb{Q}^m$.

Primal problem:

$$\begin{array}{lr} \text{minimize} & c'\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \geq b \\ \text{where} & \mathbf{x} \lesseqgtr \mathbf{0} \end{array}$$

Dual problem:

$$\begin{array}{lr} \text{maximize} & b'\mathbf{y} \\ \text{s.t.} & A^\top \mathbf{y} = c \\ \text{where} & \mathbf{y} \geq \mathbf{0} \end{array}$$

LP duality

In general (II)

We are given $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$ on the input.
We search for either $\mathbf{x} \in \mathbb{Q}^n$ or $\mathbf{y} \in \mathbb{Q}^m$.

Primal problem:

$$\begin{array}{lr} \text{minimize} & c'\mathbf{x} \\ \text{s.t.} & A\mathbf{x} = b \\ \text{where} & \mathbf{x} \geq \mathbf{0} \end{array}$$

Dual problem:

$$\begin{array}{lr} \text{maximize} & b'\mathbf{y} \\ \text{s.t.} & A^\top\mathbf{y} \leq c \\ \text{where} & \mathbf{y} \lesseqgtr \mathbf{0} \end{array}$$

LP duality

In general (III)

We are given $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$ on the input.
We search for either $\mathbf{x} \in \mathbb{Q}^n$ or $\mathbf{y} \in \mathbb{Q}^m$.

Primal problem:

$$\begin{array}{lr} \text{minimize} & c'\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \geq b \\ \text{where} & \mathbf{x} \geq \mathbf{0} \end{array}$$

Dual problem:

$$\begin{array}{lr} \text{maximize} & b'\mathbf{y} \\ \text{s.t.} & A^\top \mathbf{y} \leq c \\ \text{where} & \mathbf{y} \geq \mathbf{0} \end{array}$$

Write an *infeasible* LP such that its dual is *not unbounded*.

Thanks for your attention!

Questions?