1 Generalized determinant

Alexander Barvinok [1] defines a symmetrized determinant of a matrix nxn over a ring (which need not be commutative) as:

$$\mathrm{sdet}_n(X) = \frac{1}{n!} \cdot \sum_{\sigma,\tau \in S_n} \mathrm{sgn}(\sigma) \cdot \mathrm{sgn}(\tau) \cdot \prod_{i=1}^{n} x_{\tau(i),\sigma(i)}$$

I was interested in its properties, so I examined the definition as follows:

$$\mathrm{sdet}_n(X) = \frac{1}{n!} \cdot \sum_{\sigma,\tau \in S_n} \mathrm{sgn}(\sigma) \cdot \mathrm{sgn}(\tau) \cdot \prod_{i=1}^{n} x_{\tau(i),\sigma(i)} =$$

$$= \frac{1}{n!} \cdot \sum_{\sigma,\tau \in S_n} \mathrm{sgn}(\sigma) \cdot \mathrm{sgn}(\tau) \cdot \prod_{i=1}^{n} x_{\tau(i),(\sigma \circ (\tau^{-1} \circ \tau))(i)} =$$

$$= \frac{1}{n!} \cdot \sum_{\tau \in S_n} \mathrm{sgn}(\tau) \cdot \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \cdot \prod_{i=1}^{n} x_{\tau(i),((\sigma \circ \tau^{-1}) \circ \tau)(i)} =$$

$$= \frac{1}{n!} \cdot \sum_{\tau \in S_n} \mathrm{sgn}(\tau) \cdot \sum_{\pi \in (S_n \tau^{-1})} \mathrm{sgn}(\pi \circ \tau) \cdot \prod_{i=1}^{n} x_{\tau(i),(\pi \circ \tau)(i)} =$$

$$= \frac{1}{n!} \cdot \sum_{\tau \in S_n} \mathrm{sgn}(\tau) \cdot \sum_{\pi \in S_n} \mathrm{sgn}(\pi) \cdot \mathrm{sgn}(\tau) \cdot \prod_{i=1}^{n} x_{\tau(i),(\pi \circ \tau)(i)} =$$

$$= \frac{1}{n!} \cdot \sum_{\tau \in S_n} \sum_{\pi \in S_n} \mathrm{sgn}(\pi) \cdot \prod_{i=1}^{n} x_{\tau(i),\pi(\tau(i))} =$$

$$= \sum_{\pi \in S_n} \mathrm{sgn}(\pi) \cdot \ \frac{1}{n!} \cdot \sum_{\tau \in S_n} \prod_{i=1}^{n} x_{\tau(i),\pi(\tau(i))} =$$

$$= \sum_{\pi \in S_n} \mathrm{sgn}(\pi) \cdot \prod^{\text{ave}} (x_{1,\pi(1)}, x_{2,\pi(2)}, \ldots, x_{n,\pi(n)})$$

The average product is defined as the average value of products over all permutations of the factors:

$$\prod^{\text{ave}} (a_1, a_2, \ldots, a_n) = \frac{1}{n!} \cdot \sum_{\nu \in S_n} a_{\nu(1)} \cdot a_{\nu(2)} \cdot \ldots \cdot a_{\nu(n)}$$

2 Comparison

As a result, the only difference between a normal determinant and a symmetrized determinant is the following:

$$\det_n(X) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \cdot x_{1,\pi(1)} \cdot x_{2,\pi(2)} \cdot \ldots \cdot x_{n,\pi(n)}$$

$$\operatorname{sdet}_n(X) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \cdot \prod^{\text{ave}} (x_{1,\pi(1)}, x_{2,\pi(2)}, \ldots, x_{n,\pi(n)})$$

If the ring were commutative, then $\det_n(X)$ would be equal to $\operatorname{sdet}_n(X)$.

Paper [2] shows that calculating the normal determinant (also called Cayley determinant) over a non-commutative ring is NP-hard, but the symmetrized determinant can be always computed efficiently.

3 References

[1] A. Barvinok. New Permanent Estimators via Non-Commutative Determinants. Online: https://arxiv.org/abs/math/0007153
[2] V. Arvind, S. Srinivasan. On the hardness of the noncommutative determinant. Online: https://arxiv.org/abs/0910.2370