Computer Science Track Core Course

Linear programming — part 2

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Linear program
Definitions

We are given an input $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$ and we search for $\mathbf{x} \in \mathbb{Q}^n$ that optimizes...

$$\begin{array}{ll} \text{minimize} & c'\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \geq b \end{array} \qquad \begin{array}{ll} \text{maximize} & c'\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq b \end{array}$$

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Integer linear program

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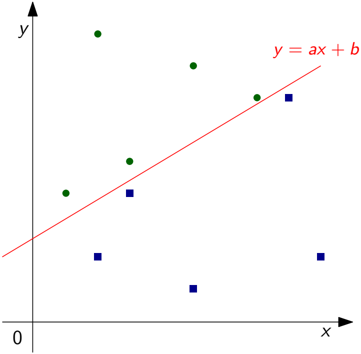
Example of LP

Separating points

We are searching for a line $y = ax + b$ that separates "disks" (top) from "squares" (bottom).



Variables: $a \lesseqgtr 0$, $b \lesseqgtr 0$

Point (x_1, y_1) above the line: $x_1 \cdot a + b \leq y_1$ "disk"

Point (x_2, y_2) below the line: $x_2 \cdot a + b \geq y_2$ "square"

Objective: minimize 0

Example of ILP

Separating points partially

We are searching for a line $y = ax + b$ that separates given points — "disks" (top) from "squares" (bottom) — with as few exceptions as possible.

Let us ignore all vertical and nearly-vertical solutions.

M is a very large integer.

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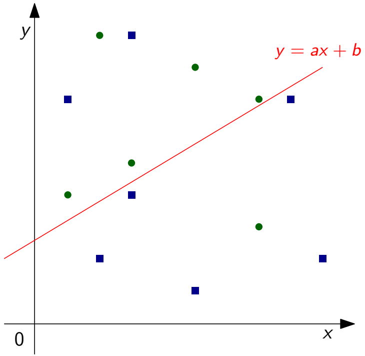

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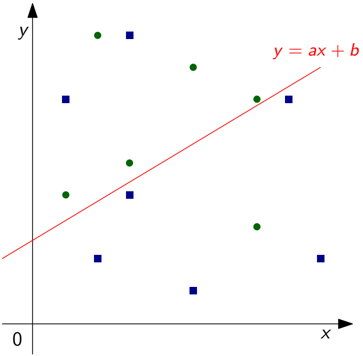

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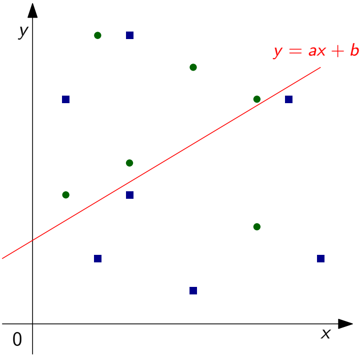

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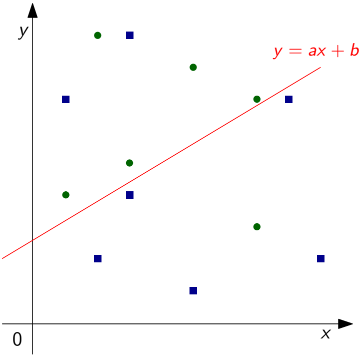

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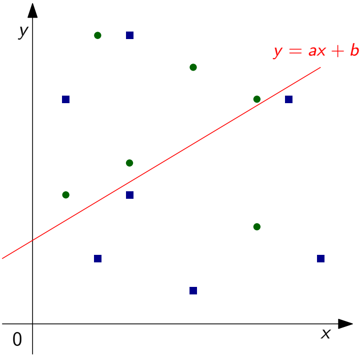

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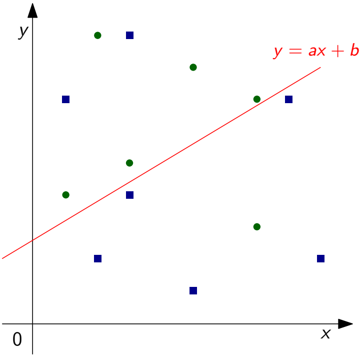

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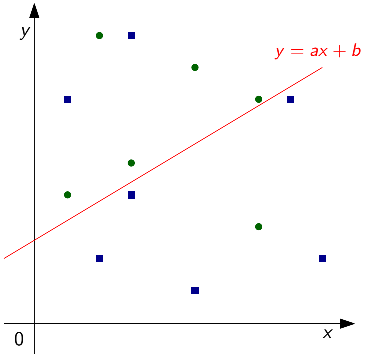

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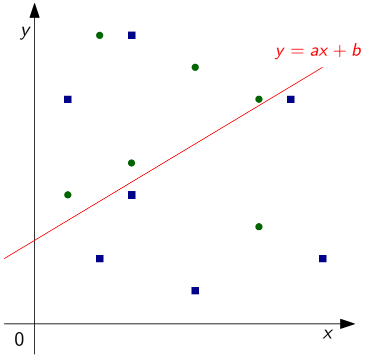

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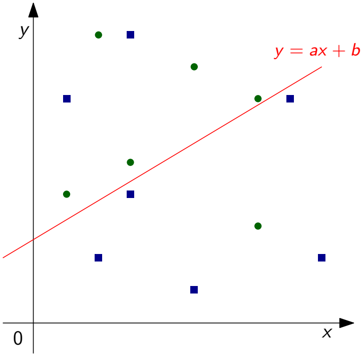

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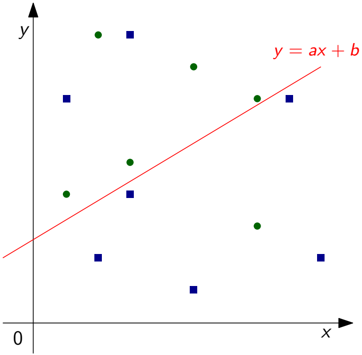

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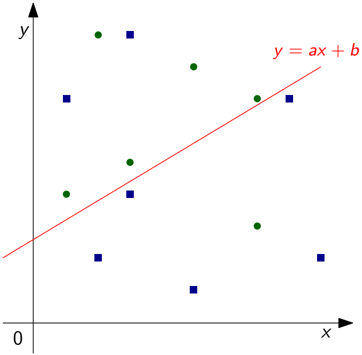

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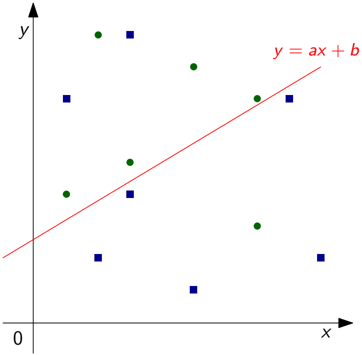

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Does it mean, however, that our problem cannot be solved in polynomial time?

Let V be a vector space over F. Consider vectors $\mathbf{x_1}, \ldots, \mathbf{x_n} \in V$.

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- We say that $\mathbf{y} \in V$ is a linear combination of $\mathbf{x_1}, \ldots, \mathbf{x_n}$ if:

$$\exists\, \alpha_i, \ldots, \alpha_n \in F : \quad \sum_{i=1}^{n} \alpha_i \mathbf{x_i} = \mathbf{y}$$

Combinations of vectors (linear, affine, convex)

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- Suppose now that F is a totally ordered field.
  Denote the set of $\varphi \in F$ such that $\varphi \geq 0$ by the symbol F_0^+.
  We say that $\mathbf{y} \in V$ is a convex combination of $\mathbf{x_1}, \ldots, \mathbf{x_n}$ if:

$$\exists\, \alpha_i, \ldots, \alpha_n \in F_0^+ : \quad \sum_{i=1}^{n} \alpha_i = 1 \wedge \sum_{i=1}^{n} \alpha_i \mathbf{x_i} = \mathbf{y}$$

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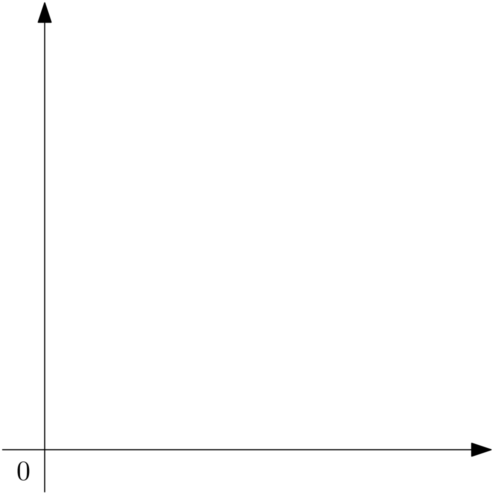

Let $d \in \mathbb{Z}$ such that $d \geq 2$.
What is the maximum possible amount (largest set) of:

- Linearly independent vectors in $\mathbb{Z}_2^d$ over $\mathbb{Z}_2$?
- Affinely independent vectors in $\mathbb{C}^d$ over $\mathbb{C}$?
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Combinations of vectors (linear, affine, convex)

Quiz

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- Affinely independent vectors in $\mathbb{C}^d$ over $\mathbb{R}$?
- Linearly independent vectors in $\mathbb{R}^d$ over $\mathbb{Q}$?

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- Convexly independent vectors in $\mathbb{Q}^d$ over $\mathbb{Q}$?
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- Affinely independent vectors in $\mathbb{C}^d$ over $\mathbb{R}$?
  $2d+1$
- Linearly independent vectors in $\mathbb{R}^d$ over $\mathbb{Q}$?

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Let V be a vector space and $X \subseteq V$.

- X is convexly dependent. $\implies$ X is affinely dependent. $\implies$ X is linearly dependent.
- X is linearly independent. $\implies$ X is affinely independent. $\implies$ X is convexly independent.

$$\{\text{convex combin.}\} \subseteq \{\text{affine combin.}\} \subseteq \{\text{linear combin.}\}$$

Let V be a vector space and $X \subseteq V$.

- X is convexly dependent. $\implies$ X is affinely dependent. $\implies$ X is linearly dependent.
- X is linearly independent. $\implies$ X is affinely independent. $\implies$ X is convexly independent.

Let us have a matrix $A \in F^{m\times n}$ and a vector $b \in F^m$.
We search for a solution $x \in F^n$.

- $Ax = 0$: Any linear combination of solutions is a solution.
- $Ax = b$: Any affine combination of solutions is a solution.
- $Ax \le b$: Any convex combination of solutions is a solution.
  In this example, F must be a totally ordered field.

Convex hull

Definition and exercises

Def. Let V be a vector space over a totally ordered field. Consider a set $X \subseteq V$. We define a convex hull of X as:

$$\mathrm{conv}(X) = \{\mathbf{y} \in V \mid \mathbf{y} \text{ is a convex combin. of some } \mathbf{x_1}, \mathbf{x_2}, \ldots \in X\}$$

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Ex. Prove this set identity:

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Ex. Let $v \in V$. If $M \subseteq V$, we define $M + v$ as $\{m + v \mid m \in M\}$. Prove this set identity:

$$\mathrm{conv}(X) + v = \mathrm{conv}(X + v)$$

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Ex. Express a d-dimensional simplex (triangle, tetrahedron,...) as:

- A convex hull of $d + 1$ points.
- An intersection of $d + 1$ closed half-spaces.

Polyhedral vertices

Equivalent formulations

Def. Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ where $m \geq n$. Consider a set $P \subseteq \mathbb{R}^n$ defined by $P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq b\}$, that is, a polyhedron.
Let $\mathbf{p} \in P$. The following conditions are equivalent formulations of $\mathbf{p}$ being a vertex of the polyhedron P.

- No vector $\mathbf{y} \neq \mathbf{0}$ satisfies both $\mathbf{p} + \mathbf{y} \in P$ and $\mathbf{p} - \mathbf{y} \in P$.
- We have $\mathbf{p} \notin \text{conv}(P \setminus \{\mathbf{p}\})$.
- There exists a hyperplane H of dimension $n - 1$ such that $P \cap H = \{\mathbf{p}\}$.
  Recall that $H = \{\mathbf{x} \in \mathbb{R}^n \mid h'\mathbf{x} = r\}$ for some $h \in \mathbb{R}^n$, $r \in \mathbb{R}$.
- There is a cost vector $c \in \mathbb{R}^n$ such that $\mathbf{p}$ is the unique maximum of the corresponding cost function, that is, $\forall \mathbf{x} \in (P \setminus \{\mathbf{p}\})$ we have $c'\mathbf{p} > c'\mathbf{x}$.
- There are n linearly independent constraints tight $(=)$ at $\mathbf{p}$.

Ex. Prove their equivalence.

Thanks for your attention!

Questions?