Computer Science Track Core Course

Linear programming — part 3

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2021-06-17

We are given an input $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$ (and $c \in \mathbb{Q}^n$ in OPT)
and we search for $\mathbf{x} \in \mathbb{Q}^n$ that satisfies/optimizes...

FEAS:

$$\begin{array}{lr} \text{find any} & \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \geq b \\ \text{where} & \mathbf{x} \geq \mathbf{0} \end{array}$$

OPT:

$$\begin{array}{lr} \text{minimize} & c'\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \geq b \\ \text{where} & \mathbf{x} \geq \mathbf{0} \end{array}$$

We are given an input $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$ (and $c \in \mathbb{Q}^n$ in OPT) and we search for $\mathbf{x} \in \mathbb{Q}^n$ that satisfies/optimizes...

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OPT:

$$\begin{array}{ll} \text{minimize} & c'\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \geq b \\ \text{where} & \mathbf{x} \geq \mathbf{0} \end{array}$$

Reduction FEAS $\rightarrow$ OPT:

We are given an input $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$ (and $c \in \mathbb{Q}^n$ in OPT) and we search for $\mathbf{x} \in \mathbb{Q}^n$ that satisfies/optimizes…

FEAS:

$$\begin{array}{lr} \text{find any} & \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \geq b \\ \text{where} & \mathbf{x} \geq \mathbf{0} \end{array}$$

OPT:

$$\begin{array}{lr} \text{minimize} & c'\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \geq b \\ \text{where} & \mathbf{x} \geq \mathbf{0} \end{array}$$

Reduction FEAS $\rightarrow$ OPT: $c = \mathbf{0}$

Reductions FEAS $\leftrightarrow$ OPT (for input LP in explicit form)

FEAS $\leftarrow$ OPT

We are given an input $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, and $c \in \mathbb{Q}^n$.
We search for $\mathbf{x} \in \mathbb{Q}^n$

Reduction OPT $\rightarrow$ FEAS:

OPT:

$$\begin{array}{lr} \text{minimize} & c'\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \geq b \\ \text{where} & \mathbf{x} \geq \mathbf{0} \end{array}$$

FEAS:

We are given an input $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, and $c \in \mathbb{Q}^n$.
We search for $\mathbf{x} \in \mathbb{Q}^n$ (and auxiliary $\mathbf{y} \in \mathbb{Q}^m$ in FEAS).

Reduction OPT $\rightarrow$ FEAS:

OPT:

$$\begin{array}{lr} \text{minimize} & c'\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \geq b \\ \text{where} & \mathbf{x} \geq \mathbf{0} \end{array}$$

FEAS:

$$\begin{array}{lr} \text{find any} & (\mathbf{x}, \mathbf{y}) \\ \text{s.t.} & A\mathbf{x} \geq b \\ & A^\top \mathbf{y} \leq c \\ & c'\mathbf{x} = b'\mathbf{y} \\ \text{where} & \mathbf{x} \geq \mathbf{0} \\ & \mathbf{y} \geq \mathbf{0} \end{array}$$

Linear equalities versus inequalities

Remark about Rouché-Capelli theorem

We have $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Rouché–Capelli theorem in English speaking countries, Italy and Brazil;
Kronecker–Capelli theorem in Austria, Poland, Romania and Russia;
Rouché–Fontené theorem in France;
Rouché–Frobenius theorem in Spain and many countries in Latin America;
Frobenius theorem in the Czech Republic and in Slovakia.

Rouché-Capelli theorem for the system of equalities $A\mathbf{x} = b$ ...
Exactly one of the following statements is true:

- $\exists \mathbf{x} \in \mathbb{R}^n : \quad A\mathbf{x} = b$
- $\exists \mathbf{y} \in \mathbb{R}^m : \quad A^\top \mathbf{y} = \mathbf{0} \ \wedge \ b^\top \mathbf{y} \neq 0$

Rouché-Capelli theorem is typically stated as ...
Let $A \in F^{m \times n}$ be a matrix and $b \in F^m$ be a vector over a field F.
The system of equalities $A\mathbf{x} = b$ is inconsistent (that is, no solution $\mathbf{x} \in F^n$ exists) if and only if $\operatorname{rank}(A) < \operatorname{rank}(A\,|\,b)$.

Linear equalities versus inequalities

Rouché-Capelli theorem versus Farkas' lemma (I)

Now we fix $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Rouché-Capelli theorem for the system of equalities $A\mathbf{x} = b$ ...
Exactly one of the following statements is true:

- $\exists \mathbf{x} \in \mathbb{R}^n : \quad A\mathbf{x} = b$
- $\exists \mathbf{y} \in \mathbb{R}^m : \quad A^\top \mathbf{y} = \mathbf{0} \;\; \wedge \;\; b^\top \mathbf{y} \neq 0$

Farkas' lemma for the system of (in)equalities $A\mathbf{x} = b$, $\mathbf{x} \geq 0$ ...
Exactly one of the following statements is true:

- $\exists \mathbf{x} \in \mathbb{R}^n : \quad A\mathbf{x} = b \;\; \wedge \;\; \mathbf{x} \geq \mathbf{0}$
- $\exists \mathbf{y} \in \mathbb{R}^m : \quad A^\top \mathbf{y} \geq \mathbf{0} \;\; \wedge \;\; b^\top \mathbf{y} < 0$

Linear equalities versus inequalities

Rouché-Capelli theorem versus Farkas' lemma (II)

Now we fix $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Rouché-Capelli theorem for the system of equalities $A\mathbf{x} = b$ ...
Exactly one of the following statements is true:

- $\exists \mathbf{x} \in \mathbb{R}^n : \quad A\mathbf{x} = b$
- $\exists \mathbf{y} \in \mathbb{R}^m : \quad A^\top \mathbf{y} = \mathbf{0} \quad \wedge \quad b^\top \mathbf{y} \neq 0$

Farkas' lemma for the system of inequalities $A\mathbf{x} \leq b$, $\mathbf{x} \lessgtr 0$ ...
Exactly one of the following statements is true:

- $\exists \mathbf{x} \in \mathbb{R}^n : \quad A\mathbf{x} \leq b$
- $\exists \mathbf{y} \in \mathbb{R}^m : \quad A^\top \mathbf{y} = \mathbf{0} \quad \wedge \quad b^\top \mathbf{y} < 0 \quad \wedge \quad \mathbf{y} \geq \mathbf{0}$

Linear equalities versus inequalities

Rouché-Capelli theorem versus Farkas' lemma (III)

Now we fix $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Rouché-Capelli theorem for the system of equalities $A\mathbf{x} = b$ ...
Exactly one of the following statements is true:

- $\exists \mathbf{x} \in \mathbb{R}^n : \quad A\mathbf{x} = b$
- $\exists \mathbf{y} \in \mathbb{R}^m : \quad A^\top \mathbf{y} = \mathbf{0} \ \wedge \ b^\top \mathbf{y} \neq 0$

Farkas' lemma for the system of inequalities $A\mathbf{x} \leq b$, $\mathbf{x} \geq 0$ ...
Exactly one of the following statements is true:

- $\exists \mathbf{x} \in \mathbb{R}^n : \quad A\mathbf{x} \leq b \ \wedge \ \mathbf{x} \geq \mathbf{0}$
- $\exists \mathbf{y} \in \mathbb{R}^m : \quad A^\top \mathbf{y} \geq \mathbf{0} \ \wedge \ b^\top \mathbf{y} < 0 \ \wedge \ \mathbf{y} \geq \mathbf{0}$

We have $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Farkas' lemma for the system of inequalities $A\mathbf{x} \leq b$, $\mathbf{x} \lesseqgtr 0$ ...
Exactly one of the following statements is true:

- $\exists \mathbf{x} \in \mathbb{R}^n : \quad A\mathbf{x} \leq b$
- $\exists \mathbf{y} \in \mathbb{R}^m : \quad A^\top \mathbf{y} = \mathbf{0} \ \wedge \ b^\top \mathbf{y} < 0 \ \wedge \ \mathbf{y} \geq \mathbf{0}$

Farkas' lemmata are used to prove the Strong duality theorems.

Let us rename variables in the Farkas' Lemma (II) . . .

Exactly one of the following statements is true:

- $\exists \mathbf{z} \in \mathbb{R}^n : \quad E\mathbf{z} \leq d$
- $\exists \mathbf{w} \in \mathbb{R}^m : \quad E^\top \mathbf{w} = \mathbf{0} \ \wedge \ d^\top \mathbf{w} < 0 \ \wedge \ \mathbf{w} \geq \mathbf{0}$

Primal problem:

$$\begin{array}{lr} \text{minimize} & c'\mathbf{x} \\ \text{s.t.} & A\mathbf{x} = b \\ \text{where} & \mathbf{x} \geq \mathbf{0} \end{array}$$

Dual problem:

$$\begin{array}{lr} \text{maximize} & b'\mathbf{y} \\ \text{s.t.} & A^\top \mathbf{y} \leq c \\ \text{where} & \mathbf{y} \lesseqgtr \mathbf{0} \end{array}$$

Linear equalities versus inequalities

Complexity over reals versus integers

Systems of … over … domain	Equalities	Inequalities
Real ($\mathbb{Q}^n$, $\mathbb{R}^n$)		
Integral ($\mathbb{Z}^n$)		

Linear equalities versus inequalities

Complexity over reals versus integers

Systems of . . . over . . . domain	Equalities	Inequalities
Real ($\mathbb{Q}^n$, $\mathbb{R}^n$)	Gaussian elimination	
Integral ($\mathbb{Z}^n$)		

Linear equalities versus inequalities

Complexity over reals versus integers

Systems of ... over ... domain	Equalities	Inequalities
Real ($\mathbb{Q}^n$, $\mathbb{R}^n$)	Gaussian elimination	Ellipsoid method
Integral ($\mathbb{Z}^n$)		

Linear equalities versus inequalities

Complexity over reals versus integers

Systems of ... over ... domain	Equalities	Inequalities
Real ($\mathbb{Q}^n$, $\mathbb{R}^n$)	Gaussian elimination	Ellipsoid method
Integral ($\mathbb{Z}^n$)	Hermite normal form	

Linear equalities versus inequalities

Complexity over reals versus integers

Systems of ... over ... domain	Equalities	Inequalities
Real ($\mathbb{Q}^n$, $\mathbb{R}^n$)	Gaussian elimination	Ellipsoid method
Integral ($\mathbb{Z}^n$)	Hermite normal form	NP-complete

Thanks for your attention!

Questions?