Chapter 1

Measure Theory

1.1 Probability Spaces

1.1.1. *Let $\Omega = \mathbb{R}, \mathcal{F} =$ all subsets so that A or A^c is countable, $P(A) = 0$ in the first case and $= 1$ in the second. Show that $(\Omega, \mathcal{F}, P)$ is a probability space.*

sol. i) $\mathcal{F}$ is a $\sigma-$algebra on $\mathbb{R}$.
$\emptyset \in \mathcal{F}$ since $\emptyset$ is countable.
By definition, $\mathcal{F}$ is closed under complementations.
Countable union of countable sets is countable. Union of countable sets and uncountable sets is uncountable. Thus, $\mathcal{F}$ is closed under countable union.
ii) P is a probability measure.
$P(\emptyset) = 0$ since $\emptyset$ is countable. By definition, for any set A, $P(A) \geq 0$.
Countable union of countable sets is countable. Union of countable sets and uncountable sets is uncountable. Thus, P has the countable additivity property.
If A is countable, then A^c is uncountable since Ω is uncountable. Thus,

$$P(\Omega) = P(A) + P(A^c) = 1$$

By (i) and (ii), $(\Omega, \mathcal{F}, P)$ is a probability space. □

1.1.2. *Recall the definition of $\mathcal{S}_d$ from Example 1.1.5. Show that $\sigma(\mathcal{S}_d) = \mathcal{R}^d$, the Borel subsets of $\mathbb{R}^d$*

sol. i) $d = 1$. In $\mathbb{R}$, any open set can be represented by countable union of open intervals. Thus, we need to show that any open interval can be represented by elements of $\mathcal{S}$. Let $-\infty < a < b < \infty$.

$$(a, b) = (-\infty, a]^c \cup (\cup_i(-\infty, b - 1/n])$$

If $a = -\infty$, then (a, b) can be represented by the second term. If $b = \infty$, then (a, b) can be represented by the first term. Thus, $\sigma(\mathcal{S}) = \sigma(\mathcal{R}) = \mathcal{R}$.
ii) $d \geq 2$. $\mathcal{S}_d$ is a finite cartesian product of $\mathcal{S}$. Similarly, $\sigma(\mathcal{S}_d) = \sigma(\mathcal{R}_d) = \mathcal{R}_d$. □

1.1.3. *A σ-field F is said to be countably generated if there is a countable collection $C \subset F$ so that $\sigma(C) = F$. Show that $\mathcal{R}^d$ is countably generated.*

sol. Let C be the collection that contains all sets of the form

$$[q_1, \infty) \times \cdots \times [q_d, \infty), (q_1, \ldots, q_d) \in \mathbb{Q}^d$$

Then, C is countable, since it is finite union of countable sets. And $\sigma(C) = \mathcal{R}^d$ as presented by previous exercise. □

1.1.4. *(i) Show that if $F_1 \subset F_2 \subset \ldots$ are σ-algebras, then $\cup_i F_i$ is an algebra. (ii) Give an example to show that $\cup_i F_i$ need not be a σ-algebra.*

sol.

(i) By definition, $\cup_i F_i$ is not empty. Choose $x \in \cup_i F_i$. Then, there exists F_x such that $x \in F_x$. Therefore, $x^c \in F_x \subset \cup_i F_i$. Thus, $\cup_i F_i$ is closed under complementations.

Choose, $x \in F_i, y \in F_j$ and suppose $i \leq j$. Then $x \in F_j$. Thus, $x \cup y \in F_j \Rightarrow x \cup y \in \cup_i F_i$. Thus, $\cup_i F_i$ is closed under union.

(ii) Let $F_i = \sigma(\{\{1\}, \{2\}, \cdots, \{n\}\})$. Let $A = \{\{n\} : n = 3k | k = 1, 2, 3, \cdots\}$. Then for all i, $A \notin F_i$. Thus, $A \notin \cup_i F_i$. However, A can be represented by countable union. Therefore, $\cup_i F_i$ is not a $\sigma-$algebra.

□

1.1.5. *A set $A \subset \{1, 2, \ldots\}$ is said to have asymptotic density θ if*

$$\lim_{n \to \infty} |A \cap \{1, 2, \cdots, n\}| / n = \theta$$

Let $\mathcal{A}$ be the collection of sets for which the asymptotic density exists. Is $\mathcal{A}$ a σ-algebra? an algebra?

sol. Let A be the set of even numbers. Next, we construct a set B in the following way: we begin with $\{2, 3\}$ and starting with $k = 2$, take all even numbers $2^k < n \leq (3/2) \times 2^k$, and all odd numbers$(3/2) \times 2^k < n \leq 2^{k+1}$. Then, the asymptotic density of B is 0.5. However, the asymptotic density $A \cap B$ does not exists.

When $n = (3/2) \times 2^k$, then the density is 1/3. When $n = 2^{k+1}$, then the density is 1/4.

Thus, $\mathcal{A}$ is not closed under intersection. $\mathcal{A}$ is neither $\sigma-$algebra nor algebra □

1.2 Distributions

1.2.1. *Suppose X and Y are random variables on $(\Omega, \mathcal{F}, P)$ and let $A \in \mathcal{F}$. Show that if we let $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$, then Z is a random variable.*

sol. Let the Borel set B which satisfies that

$$\text{if } \omega \in A, \text{ then } Z(\omega) \in B$$

For arbitrary Borel set S, $S = (S \cap B) \cup (S \cap B^c)$. Since S is Borel set, $\{\omega : X(\omega) \in S\}, \{\omega : Y(\omega) \in S\} \in \mathcal{F}$.

$$\begin{aligned} \{\omega : Z(\omega) \in S\} &= \{\omega : Z(\omega) \in (S \cap B)\} \cup \{\omega : Z(\omega) \in (S \cap B^c)\} \\ &= \{\omega : X(\omega) \in (S \cap B)\} \cup \{\omega : Y(\omega) \in (S \cap B^c)\} \end{aligned}$$

Since $\mathcal{F}$ is closed on set operation, $\{\omega : Z(\omega) \in S\} \in \mathcal{F}$. □

1.2.2. *Let χ have the standard normal distribution. Use Theorem 1.2.6 to get upper and lower bounds on $P(\chi \geq 4)$.*

sol.

$$\begin{aligned} P(\chi \geq 4) = (2\pi)^{-1} \int_4^\infty \exp(-y^2/2) dy &\leq (8\pi)^{-1} \exp(-8) \\ (2\pi)^{-1} \int_4^\infty \exp(-y^2/2) dy &\geq (15/128\pi) \exp(-8) \end{aligned}$$

□

1.2.3. *Show that a distribution function has at most countably many discontinuities.*

sol. Let D be the set of discontinuity points. Choose $x, y \in D$. Then we can choose rational number $q_x \in (F(x-), F(x+))$. Since F is increasing, if $x \neq y$, then $q_x \neq q_y$. Thus $x \to q_x$ is one-to-one function. Since $\mathbb{Q}$ is countable, D is at most countable. □

1.2.4. *Show that if $F(x) = P(X \leq x)$ is continuous then $Y = F(X)$ has a uniform distribution on (0,1), that is, if $y \in [0, 1]$, $P(Y \leq y) = y$.*

sol.

$$\begin{aligned}\{\omega|Y(\omega)\le y\} &= \{\omega|F(X(\omega))\le y\}\\ &= \{\omega|X(\omega)\le k\}\quad k=\inf\{x|F(x)\ge y\}\\ P(\{\omega|Y(\omega)\le y\}) &= P(\{\omega|X(\omega)\le k\})\\ &= P(X\le k)=y\end{aligned}$$

□

1.2.5. *Suppose X has continuous density f, $P(\alpha \le X \le \beta) = 1$ and g is a function that is strictly increasing and differentiable on (α, β). Then $g(X)$ has density $f(g^{-1}(y))/g'(g^{-1}(y))$ for $y \in (g(\alpha), g(\beta))$ and 0 otherwise. When $g(x) = ax + b$ with $a > 0, g^{-1}(y) = (y - b)/a$ so the answer is $(1/a)f((y - b)/a)$.*

sol. Since g is strictly increasing g^{-1} exists.

$$\begin{aligned}P(g(X)\le y) &= P(X\le g^{-1}(y)) = \int_{\alpha}^{g^{-1}(y)} f(x)dx\\ \frac{d}{dy}P(g(X)\le y) &= f(g^{-1}(y))\frac{d}{dy}g^{-1}(y) = f(g^{-1}(y))/g'(g^{-1}(y))\end{aligned}$$

□

1.2.6. *Suppose X has a normal distribution. Use the previous exercise to compute the density of* $\exp(X)$.

sol. Let $g(x) = \exp(x)$. Then $g^{-1}(x) = \log(x)$

$$\begin{aligned}f(g^{-1}(x))/g'(g^{-1}(x)) &= f(\log(x))/\exp(\log(x))\\ &= f(\log(x))/x = \frac{1}{x\sqrt{2\pi}}\exp(-\log(x)^2/2),\ x>0\end{aligned}$$

□

1.3 Random Variables

1.3.1. *Show that if $\mathcal{A}$ generates $\mathcal{S}$, then $X^{-1}(\mathcal{A}) \equiv \{\{X \in A\} : A \in \mathcal{A}\}$ generates $\sigma(X) = \{\{X \in B\} : B \in \mathcal{S}\}$.*

sol. By assumption, $\mathcal{A} \subset \mathcal{S}$. Thus, $X^{-1}(\mathcal{A}) \subset X^{-1}(\mathcal{S})$. Therefore, $\sigma(X^{-1}(\mathcal{A})) \subset \sigma(X^{-1}(\mathcal{S})) = \sigma(X)$.

Choose a set $B \in \mathcal{S}$. Since $\mathcal{A}$ generates $\mathcal{S}$, B can be represented by sets of $\mathcal{A}$. Thus, $\{X \in B\} \in X^{-1}(\mathcal{A})$. Therefore, $\sigma(X) \in X^{-1}(\mathcal{A})$. □

1.3.2. *Prove Theorem 1.3.6 when $n = 2$ by checking $\{X_1 + X_2 < x\} \in \mathcal{F}$.*

sol.

$$\{X_1+X_2<x\} = \bigcup_{q\in\mathbb{Q}}\{X_1<q\}\times\{X_2<x-q\}$$

Both $\{X_1 < q\}$ and $\{X_2 < x - q\}$ are open sets. Thus, $\{X_1 + X_2 < x\} \in \mathcal{R}^2$ since it is represented by countable union. □

1.3.3. *Show that if f is continuous and $X_n \to X$ almost surely then $f(X_n) \to f(X)$ almost surely.*

sol. Let $\Omega_0 = \{\omega : \lim X_n(\omega) = X(\omega)\}$ and $\Omega_f = \{\omega : \lim f(X_n(\omega)) = f(X(\omega))\}$.

$$\begin{aligned}\omega\in\Omega_0 &\Rightarrow \lim X_n(\omega) = X(\omega)\\ &\Rightarrow \lim f(X_n(\omega)) = f(X(\omega))\quad \because f \text{ is continuous}\\ &\Rightarrow \omega\in\Omega_f\\ &\Rightarrow 1 = P(\Omega_0)\le P(\Omega_f)\le 1\end{aligned}$$

□

1.3.4. *(i) Show that a continuous function from $\mathcal{R}^d \to \mathcal{R}$ is a measurable map from $(\mathbb{R}^d, \mathcal{R}^d)$ to $(\mathbb{R}, \mathcal{R})$. (ii) Show that $\mathcal{R}^d$ is the smallest σ-field that makes all the continuous functions measurable.*

sol.

(i) $\mathcal{R}$ is σ-field which is generated by all open sets. If f is continuous mapping and image is open, then inverse-image is also open. Thus, f is a measurable map.

(ii) Let $\mathcal{S}$ be the smallest $\sigma-$field that makes all the continuous functions measurable. Since $\mathcal{R}$ is the smallest $\sigma-$field which is generated by all open sets, $\mathcal{S}$ is generated by all open sets in $\mathbb{R}^d$.

□

1.3.5. *A function f is said to be lower semicontinuous or l.s.c. if*

$$\liminf_{y \to x} f(y) \geq f(x)$$

and upper semicontinuous (u.s.c.) if $-f$ is l.s.c.. Show that f is l.s.c. if and only if $\{x : f(x) \leq a\}$ is closed for each $a \in \mathbb{R}$ and conclude that semicontinuous functions are measurable.

sol. Define $E_a := \{x : f(x) \leq a\}$

if Choose x. Let $a = \liminf_{y \to x} f(y)$. Then, by definition, $a \leq f(x)$. There is the sequence $\{x_n\}$ such that $x_n \to x, f(x_n) \to a$ as $n \to \infty$ For any $b > a$, there exists N_b such that for all $n > N_b$, $x_n \in E_b$. Since E_b is closed, the limit point $x \in E_b$ for all $b > a$. Thus, $x \in E_a$. Since $f(x) \geq a$ and $f(x) \leq a$, $f(x) = a = \liminf_{y \to x} f(y)$.

only if Construct a sequence $\{x_n\}$ in E_a which converges to x. Since f is l.s.c

$$f(x) \leq \liminf_{y \to x} f(y) \leq a$$

Thus, $x \in E_a$. Since an arbitrary limit point is in E_a, E_a is closed.

□

1.4 Integration

1.4.1. *Show that if $f \geq 0$ and $\int f d\mu = 0$ then $f = 0$ a.e.*

sol. Let N be the set that satisfies $x \in N \Rightarrow f(x) = 0$.

$$\begin{aligned} \int f d\mu &= \int_N f d\mu + \int_{N^c} f d\mu \\ &= \int_{N^c} f d\mu = 0 \\ &\Rightarrow \mu(N^c) = 0 \end{aligned}$$

□

1.4.2. *Let $f \geq 0$ and $E_{n,m} = \{x : m/2^n \leq f(x) < (m+1)/2^n\}$. As $n \uparrow \infty$,*

$$\sum_{m=1}^{\infty} \frac{m}{2^n} \mu(E_{n,m}) \uparrow \int f d\mu$$

sol. Let $f_n = \sum_{m=1}^{\infty} 1_{E_{n,m}}$. Then, for any n, $f_n < f$. Thus, $\int f d\mu$ is an upper bound of $\int f_n d\mu$. Since $\int f_n d\mu$ is increasing and bounded, it converges. By definition,

$$\int f - f_n d\mu \leq \int 1/2^n d\mu \to 0 \text{ as } n \to \infty$$

Thus, $\int f_n d\mu$ converges to $\int f d\mu$ □

1.4.3. *Let g be an integrable function on $\mathbb{R}$ and $\epsilon > 0$. (i) Use the definition of the integral to conclude there is a simple function $\varphi = \sum_k b_k 1_{A_k}$ with $\int |g - \varphi| dx < \epsilon$. (ii) Use Exercise A.2.1 to approximate the A_k by finite unions of intervals to get a step function*

$$q = \sum_{j=1}^{k} c_j 1_{(a_{j-1}, a_j)}$$

with $a_0 < a_1 < \cdots < a_k$, so that $\int |\varphi - q| < \epsilon$. (iii) Round the corners of q to get a continuous function r so that $\int |q - r| dx <$ epsilon. (iv) To make a continuous function replace each $c_j 1_{(a_{j-1}, a_j)}$ by a function that is 0 $(a_{j-1}, a_j)^c$, c_j on $[a_{j-1} + \delta - j, a_j - \delta_j]$, and linear otherwise. If the δ_j are small enough and we let $r(x) = \sum_{j=1}^{k} r_j(x)$ then

$$\int |q(x) - r(x)| d\mu = \sum_{j=1}^{k} \delta_j c_j < \epsilon$$

1.4.4. *Prove the Rimann-Lebesgue lemma. If g is integrable then*

$$\lim_{n \to \infty} \int g(x) \cos nx dx = 0$$

sol. Let ϵ be an arbitrary positive number. We can find a step function φ such that $\int |g - \varphi| dx < \epsilon/2$.

$$\begin{aligned}
\int \varphi(x) \cos(nx) dx &= \sum_{i=1}^{m} c_i \int_{a_i}^{b_i} \cos(nx) dx \\
&= \sum_{i=1}^{m} c_i/n \int_{na_i}^{nb_i} \cos(y) dy \\
&= \sum_{i=1}^{m} c_i/n(\sin(nb_i) - \sin(na_i)) \to 0
\end{aligned}$$

Thus, we can choose N which satisfies that for all $m \geq N$, $\int |\varphi(x) \cos(nx)| dx < \epsilon/2$. Then,

$$\begin{aligned}
|\int g(x) \cos(nx) dx| &\leq \int |g(x) \cos(nx)| dx \\
&\leq \int |g(x) - \varphi(x)| |\cos(nx)| dx + \int |\varphi(x) \cos(nx)| dx \\
&\leq \epsilon/2 + \epsilon/2 = \epsilon
\end{aligned}$$

Since ϵ is arbitrary, it converges to 0. □

1.5 Properties of the integral

1.5.1. *Let $\|f\|_\infty = \inf\{M : \mu(\{x : |f(x)| > M\}) = 0\}$. Prove that*

$$\int |fg| d\mu \leq \|f\|_1 \|g\|_\infty$$

sol. By definition, $g \leq \|g\|_\infty$ a.e.. Thus,

$$\int |fg| d\mu \leq \int |f| \|g\|_\infty d\mu = \|g\|_\infty \int |f| d\mu = \|f\|_1 \|g\|_\infty$$

□

1.5.2. *Show that if μ is a probability measure then*

$$\|f\|_\infty = \lim_{p \to \infty} \|f\|_p$$

sol. Since μ is a probability measure, $\mu(\Omega) = 1$. Suppose two positive numbers, $n < m$. Let $\varphi(x) := x^{m/n}$ By Jensen's inequality,

$$\|f\|_n^m = \varphi(\int |f|^n d\mu) \leq \int \varphi(|f|^n) d\mu = \|f\|_m^m$$

Thus, $\{\|f\|_n\}$ is an increasing sequence.

$$\|f\|_n^n = \int |f|^n d\mu \leq \int (\|f\|_\infty^n) d\mu = \|f\|_\infty^n$$

Thus, $\|f\|_\infty$ is an upper bound. Therefore, the sequence converges and $\lim_{p\to\infty} \|f\|_p \leq \|f\|_\infty$.

Let ϵ be the arbitrary number between 0 and $\|f\|_\infty$. Then, there exist a set M such that $|f| > \|f\|_\infty - \epsilon$ for all $x \in M$.

$$\begin{aligned}
\int |f|^p d\mu \geq \int_M |f|^p d\mu &\geq (\|f\|_\infty - \epsilon)^p \mu(M) \\
\Rightarrow \|f\|_p &\geq (\|f\|_\infty - \epsilon)\mu(M)^{1/p} \\
\Rightarrow \lim_{p\to\infty} \|f\|_p &\geq \|f\|_\infty - \epsilon \\
\Rightarrow \lim_{p\to\infty} \|f\|_p &\geq \|f\|
\end{aligned}$$

Thus, the sequences converges to $\|f\|_\infty$. □

1.5.3 (Minkowski's inequality)**.** *(i) Suppose $p \in (1,\infty)$. The inequality $|f+g|^p \leq 2^p(|f|^p + |g|^p)$ shows that if $\|f\|^p$ and $\|g\|^p$ are $< \infty$ then $\|f+g\|_p < \infty$. Apply Holder's inequality to $|f||f+g|^{p-1}$ and $|g||f+g|^{p-1}$ to show $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ (ii) Show that the last result remains true when $p = 1$ or $p = \infty$*

sol.

(i) By Holder's inequality,

$$\begin{aligned}
\int |f||f+g|^{p-1} d\mu &\leq \|f\|_p \|(f+g)^{p-1}\|_{p/(p-1)} \\
&= (\int |f|^p d\mu)^{1/p} (\int |f+g|^p d\mu)^{(p-1)/p}
\end{aligned}$$

$$\text{Similarly, } \int |g||f+g|^{p-1} d\mu \leq (\int |g|^p d\mu)^{1/p} (\int |f+g|^p d\mu)^{(p-1)/p}$$

$$\begin{aligned}
\int |f+g|^p d\mu &= \int (|f|+|g|)|f+g|^{p-1} d\mu \leq (\|f\|_p + \|g\|_p)\|f+g\|_p^{p-1} \\
\Rightarrow \|f+g\|_p &\leq (\|f\|_p + \|g\|_p)
\end{aligned}$$

(ii) When $p = 1$, it is trivial.

Let the set $E = \{M : \mu(\{x : |f(x) + g(x)| > M\}) = 0\}$. By triangle inequality,

$$|f+g| \leq |f| + |g| \leq \|f\|_\infty + \|g\|_\infty \ a.e.$$

Thus,

$$\mu(\{x : |f(x)+g(x)| > \|f\|_\infty + \|g\|_\infty\}) = 0$$

Since $\|f\|_\infty + \|g\|_\infty \in E$, $\|f\|_\infty + \|g\|_\infty \geq \inf E = \|f+g\|_\infty$

□

1.5.4. *If f is integrable and E_m are disjoint sets with union E then*

$$\sum_{m=0}^{\infty} \int_{E_m} f d\mu = \int_E f d\mu$$

So if $f \geq 0$, then $\nu(E) = \int_E f d\mu$ defines a measure.

sol. If $E = \emptyset$, $\mu(E) = 0$. Since $f \geq 0$, for any set E, $\nu(E) \geq \nu(\emptyset) = 0$.

If E_i are disjoint sets, then

$$\mu(\cup_i E_i) = \int_{\cup_i E_i} f d\mu = \sum_i \int_{E_i} f d\mu = \sum_i \nu(E_i)$$

Thus, ν is a measure. □

1.5.5. *If $g_n \uparrow g$ and $\int g_1^- d\mu < \infty$ then $\int g_n d\mu \uparrow \int g d\mu$.*

sol. Since $g_n = g_n^+ - g_n^-$ and $g_n \uparrow g$, $g_n^+ \uparrow g^+$ and $g_n^- \downarrow g^-$. By monotone convergence theorem, $\int g_n^+ d\mu \uparrow \int g^+ d\mu$. Since g_1^- is integrable, by dominated convergence theorem, $\int g_n^- d\mu \to \int g^- d\mu$. Thus, $-\int g_n^- d\mu \uparrow -\int g^- d\mu$.

$$\therefore \int g_n d\mu = \int g_n^+ - g_n^- d\mu \uparrow \int g d\mu$$

□

1.5.6. *If $g_m \geq 0$ then $\int \sum_{m=0}^\infty g_m d\mu = \sum_{m=0}^\infty \int g_m d\mu$.*

sol. Define $f_n := \sum_{m=0}^n g_m$ and $f := \sum_{m=0}^\infty g_m$. Then $f_n \uparrow f$. By result of using previous example, $\int f_n d\mu \uparrow \int f d\mu$. Thus,

$$\begin{aligned}\int \sum_{m=0}^\infty g_m d\mu &= \int \lim_{n\to\infty} f_n d\mu \\ &= \int f d\mu \\ &= \int \sum_{m=0}^\infty g_m d\mu\end{aligned}$$

□

1.5.7. *Let $f \geq 0$. (i) Show that $\int f \wedge n \ d\mu \uparrow \int f d\mu$ as $n \to \infty$. (ii) Use (i) to conclude that if g is integrable and $\epsilon > 0$ then we can pick $\delta > 0$ so that $\mu(A) < \delta$ implies $\int_A |g| d\mu < \epsilon$.*

sol.

(i) Define $E_n := \{x : f(x) > n\}$. Then, for $n \leq m$

$$\begin{aligned}&f \wedge m - f \wedge n = (f - n)1_{E_n \cap E_m^c} + (m - n)1_{E_m} \geq 0 \\ &\lim_{n\to\infty} f \wedge n = f\end{aligned}$$

By definition $f_n \geq 0$. Thus, by monotone convergence theorem, $\int f \wedge n \ d\mu \uparrow \int f d\mu$.

(ii) By results of (i), we can find the positive N for any positive $\epsilon/2$, $\mu(\{x : |g(x)| > N\}) < \epsilon$.

$$\begin{aligned}\int_A |g| d\mu &\leq \int_{\{x:|g(x)|>N\}} |g| d\mu + \int_{\{x\in A:|g(x)|\leq N\}} |g| d\mu \\ &\leq \epsilon/2 + N\mu(A)\end{aligned}$$

Thus, if we set $\delta := \epsilon/(2N)$, then $\mu(A) < \delta$ implies $\int_A |g| d\mu < \epsilon$.

□

1.5.8. *Show that if f is integrable on $[a, b]$, $g(x) = \int_{[a,x]} f(y)dy$ is continuous on (a, b).*

sol. Since f is integrable, there exists a positive number M such that $|f| < M$. For $s, t \in (a, b), s \leq t$

$$\begin{aligned}-M(t - s) < g(t) - g(s) = \int_s^t f(y)dy < M(t - s) \\ \Rightarrow |g(t) - g(s)| < M(t - s)\end{aligned}$$

Thus, g is Lipschitz continuous. Thus, g is continuous. □

1.5.9. *Show that if f has $\|f\|_p = (\int |f|^p d\mu)^{1/p} < \infty$, then there are simple functions φ_n so that $\|\varphi_n - f\|_p \to 0$.*

sol. For f, we can construct sequence of step functions $\{\varphi_n\}$ such that $\varphi_n \uparrow |f|$. By triangle inequailty, $|f - \varphi_n| \le |f| + |\varphi_n|$. Thus, for all n

$$|f - \varphi_n|^p \le (|f| + |\varphi_n|)^p \le (2|f|)^p$$

By dominated convergence theorem,

$$\begin{aligned} \lim_{n\to\infty} \int |f - \varphi_n|^p &= \int \lim_{n\to\infty} |f - \varphi_n|^p = 0 \\ \Rightarrow \|f - \varphi_n\|_p &= 0 \end{aligned}$$

□

1.5.10. *Show that if $\sum_n \int |f_n| d\mu < \infty$, then $\sum_n \int f_n d\mu = \int \sum_n f_n d\mu$.*

sol.

$$\begin{aligned} \sum_n \int |f_n| d\mu &= \sum_n \int f_n^+ d\mu + \sum_n \int f_n^- d\mu \\ &\Rightarrow \sum_n \int f_n^+ d\mu < \infty, \sum_n \int f_n^- d\mu < \infty \end{aligned}$$

By the result of (1.5.6),

$$\begin{aligned} \sum_n \int f_n^+ d\mu &= \int \sum_n f_n^+ d\mu \\ \sum_n \int f_n^- d\mu &= \int \sum_n f_n^- d\mu \\ \sum_n \int f_n d\mu &= \sum_n \int f_n^+ - f_n^- d\mu \\ &= \int \sum_n f_n^+ d\mu - \int \sum_n f_n^- d\mu \\ &= \int \sum_n f_n d\mu \end{aligned}$$

□

1.6 Expected Value

1.6.1. *Suppose φ is strictly convex, i.e., $>$ holds for $\lambda \in (0, 1)$. Show that, under the assumptions of Theorem 1.6.2, $\varphi(EX) = E(\varphi X)$ implies $X = EX$ a.s.*

sol. Let $t = EX$. Since φ is strictly convex, we can find a which satisfies, for all x, $\varphi(x) \ge \varphi(t) + a(x - t)$. Thus $E\varphi(X) \ge \varphi(t) + E[a(X - t)] = \varphi(EX)$. If we add assumption $\varphi(EX) = E(\varphi X)$, then we can conclude $a(X - t) = 0$ a.e. Thus, $X = t = EX$ a.e. □

1.6.2. *Suppose $\varphi : \mathbb{R}^n \to \mathbb{R}$ is convex. Imitate the proof of Theorem 1.5.1 to show*

$$E\varphi(X_1, ..., X_n) \ge \varphi(EX_1, ..., EX_n)$$

provided $E|E\varphi(X_1, ..., X_n)| < \infty$ and $E|X_i| < \infty$ for all i.

1.6.3 (Chebyshev's inequality is and is not sharp)**.** *(i) Show that Theorem 1.6.4 is sharp by showing that if $0 < b \le a$ are fixed there is an X with $EX^2 = b^2$ for which $P(|X| \ge a) = b^2/a^2$. (ii) Show that Theorem 1.6.4 is not sharp by showing that if X has $0 < EX^2 < \infty$ then*

$$\lim_{a\to\infty} a^2 P(|X| \ge a)/EX^2 = 0$$

1.6.4 (One-sided Chebyshev bound). *(i) Let $a > b > 0, 0 < p < 1$, and let X have $P(X = a) = p$ and $P(X = -b) = 1 - p$. Apply Theorem 1.6.4 to $\varphi(x) = (x + b)^2$ and conclude that if Y is any random variable with $EY = EX$ and $Var(Y) = Var(X)$, then $P(Y \geq a) \leq p$ and equality holds when $Y = X$. (ii) Suppose $EY = 0$, $Var(Y) = \sigma^2$, and $a > 0$. Show that $P(Y \geq a) \leq \sigma^2/(a^2 + \sigma^2)$, and there is a Y for which equality holds.*

1.6.5 (Two nonexistent lower bounds). *Show that : (i) if $\epsilon > 0$, $\inf\{P(|X| > \epsilon) : EX = 0, Var(X) = 1\} = 0$. (ii) if $y \geq 1, \sigma^2 \in (0, \infty)$, $\inf\{P(|X| > y) : EX = 1, Var(X) = \sigma^2\} = 0$*

sol. (i)

$$P(|X| > \epsilon) \leq E(|X|)/\epsilon$$

□

1.6.6 (A useful lower bound). *Let $Y \geq 0$ with $EY^2 < \infty$. Apply the Cauchy-Schwarz inequality to $Y 1_{Y>0}$ and conclude*

$$P(Y > 0) \geq (EY)^2/EY^2$$

sol.

$$\begin{aligned}
E[Y 1_{Y>0}] &\leq \sqrt{EY^2 E[1_{Y>0}]} \\
EY - 0 \cdot P(Y = 0) &\leq \sqrt{EY^2 P(Y > 0)} \\
(EY)^2 &\leq EY^2 P(Y > 0) \\
P(Y > 0) &\geq (EY)^2/EY^2
\end{aligned}$$

□

1.6.7. *Let $\Omega = (0, 1)$ equipped with the Borel sets and Lebesgue measure. Let $\alpha \in (1, 2)$ and $X_n = n^{\alpha} 1_{(1/(n+1),1/n)} \to 0$ a.s. Show that Theorem 1.6.8 can be applied with $h(x) = x$ and $g(x) = |x|^{2/\alpha}$, but the X_n are not dominated by an integrable function.*

sol.

1. $g \geq 0$, $\because 2/\alpha > 1$, $g(x) \to \infty$ as $|x| \to \infty$
2. $|h(x)|/g(x) = |x|^{(\alpha-2)/2}$, $\because (\alpha - 2) < 1$, $|h(x)|/g(x) \to 0$ as $|x| \to \infty$.
3. $Eg_n(X) = n^2 \int 1_{(1/n+1,1/n)} d\mu = n/(n + 1) < 1$.

Thus, $Eh(X_n) \to Eh(X)$. However, there is no integrable function which dominates X_n because there is no upper bound for $\sup_n |X_n|$. □

1.6.8. *Suppose that the probability measure μ has $\mu(A) = \int_A f(x)dx$ for all $A \in \mathcal{R}$. Use the proof technique of Theorem 1.6.9 to show that for any g with $g \geq 0$ or $\int |g(x)|\mu(dx) < \infty$ we have*

$$\int g(x)\mu(dx) = \int g(x)f(x)dx$$

sol.

1. Indicator function
   Let $g = c1_A$. Then,
   $$\int g(x)\mu(dx) = \int_A c\mu(dx) = c\mu(A) = \int_A cf(x)dx = \int g(x)f(x)dx$$
2. Simple function
   Let $g = \sum_{i=1}^n c_i 1_{A_i}$. Then, by case 1,
   $$\int g(x)\mu(dx) = \sum_{i=1}^n c_i\mu(A_i) = \int g(x)f(x)dx$$

3. Nonnegative function

   There is a sequence of simple functions $\{g_n\}$ which satisfies that $g_n \uparrow g$. By monotone convergence theorem,

$$\int g(x)\mu(dx) = \lim_{n\to\infty}\int g_n(x)\mu(dx) = \lim_{n\to\infty}\int g_n(x)f(x)dx = \int g(x)f(x)dx$$

   The proof ends if $g \geq 0$.

4. Integrable function(g is integrable)

   Let $g = g^+ - g^-$. Then

$$\begin{aligned}
&\int g^+(x)\mu(dx) = \int g^+(x)f(x)dx, \int g^-(x)\mu(dx) = \int g^-(x)f(x)dx \\
\Rightarrow &\int g(x)\mu(dx) = \int g^+(x)\mu(dx) - \int g^-(x)\mu(dx) \\
&= \int g^+(x)f(x)dx - \int g^-(x)f(x)dx = \int g(x)f(x)dx
\end{aligned}$$

□

1.6.9 (Inclusion-exclusion formula)**.** *Let $A_1, A_2, ..., A_n$ be events and $A = \cup_{i=1}^n A_i$. Prove that $1_A = 1 - \prod_{i=1}^n (1 - 1_{A_i})$. Expand out the right hand side, then take expected value to conclude*

$$\begin{aligned}
P(\cup_{i=1}^n A_i) =& \sum_{i=1}^n P(A_i) - \sum_{i<j} P(A_i \cap A_j) \\
&+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - ... + (-1)^{n-1} P(\cap_{i=1}^n A_i)
\end{aligned}$$

sol.

$$\begin{aligned}
&\omega \in A \Leftrightarrow \omega \in \cup_{i=1}^n A_i \Leftrightarrow \omega \notin \cap_{i=1}^n A_i^C \\
\Rightarrow &1_A(\omega) = 1 - \prod_{i=1}^n (1 - 1_{A_i}(\omega))
\end{aligned}$$

□

1.6.10.

1.6.11. *If $E|X|^k < \infty$ then for $0 < j < k$, $E|X|^j < \infty$, and furthermore*

$$E|X|^j \leq (E|X|^k)^{j/k}$$

sol. Define $\varphi(x) = x^{k/j}$. Then φ is a convex function. By Jensen's inequality

$$\begin{aligned}
(E|X|^j)^{k/j} = \varphi(E|X|^j) \leq E\varphi(|X|^j) = E|X|^k \\
\Rightarrow E|X|^j \leq (E|X|^k)^{j/k}
\end{aligned}$$

□

1.6.12. *Apply Jensen's inequality with $\varphi(x) = e^x$ and $P(X = \log y_m) = p(m)$ to conclude that if $\sum_{m=1}^n p(m) = 1$ and $p(m), y_m > 0$ then*

$$\sum_{m=1}^n p(m)y_m \geq \prod_{m=1}^n y_m^{p(m)}$$

When $p(m) = 1/n$, this says the arithmetic mean exceeds the geometric mean.

sol.

$$E\varphi(X) = \sum_{n=1}^{m} p(m)\exp(\log y_m) = \sum_{n=1}^{m} p(m) y_m$$
$$\varphi(EX) = \exp(\sum_{n=1}^{m} p(m)\log y_m) = \prod_{n=1}^{m} y_m^{p(m)}$$

By Jensen's inequality,

$$\sum_{m=1}^{n} p(m) y_m = E\varphi(X) \geq \varphi(EX) = \prod_{m=1}^{n} y_m^{p(m)}$$

If $p(m) = 1/n$, the left hand becomes arithmetic mean and the right hand becomes geometric mean. □

1.6.13. *If $EX_1^- < \infty$ and $X_n \uparrow X$ then $EX_n \uparrow EX$.*

sol. Same with Exercise 1.5.5. We use the probability measure in this exercise. □

1.6.14. *Let $X \geq 0$ but do NOT assume $E(1/X) < \infty$. Show*

$$\lim_{y\to\infty} yE(1/X; X > y) = 0, \quad \lim_{y\downarrow 0} yE(1/X; X > y) = 0$$

sol.

$$\begin{aligned} 0 \leq yE(1/X; X > y) &= \int_{\{\omega : X(\omega) > y\}} y/X d\mu \\ &\leq \int_{\{\omega : X(\omega) > y\}} 1 d\mu \\ &= \mu(\{\omega : X(\omega) > y\}) \\ \Rightarrow \lim_{y\to\infty} yE(1/X; X > y) &\leq \lim_{y\to\infty} \mu(\{\omega : X(\omega) > y\}) = 0 \end{aligned}$$

□

1.6.15. *If $X_n \geq 0$ then $E(\sum_{n=0}^{\infty} X_n) = \sum_{n=0}^{\infty} EX_n$.*

sol. Same with Exercise 1.5.10. We use the probability measure in this exercise. □

1.6.16. *If X is integrable and A_n are disjoint sets with union A then*

$$\sum_{n=0}^{\infty} E(X; A_n) = E(X; A)$$

i.e., the sum converges absolutely and has the value on the right.

sol. Define $Y_n := \sum_{i=0}^{n} X1_{A_i}$ and $Y := X1_A$. Then, $Y_n \to Y$ and $|Y| \leq |X|$. Since X is integrable, Y is also integrable. For any $\omega \in \cup_{i=0}^{n} A_i$, $Y_n(\omega) = Y(\omega)$. Thus, $|Y_n| \leq |Y|$. By dominated convergence theorem,

$$\lim_{n\to\infty} EY_n \to EY$$

$$\begin{aligned} \lim_{n\to\infty} EY_n &= \lim_{n\to\infty} E(X; A_n) \\ EY &= E(X; A) \end{aligned}$$

□

1.7 Product Measures, Fubini's Theorem

1.7.1. *If* $\int_X \int_Y |f(x,y)|\mu_2(dy)\mu_1(dx) < \infty$ *then*

$$\int_X \int_Y |f(x,y)|\mu_2(dy)\mu_1(dx) = \int_{X\times Y} f d(\mu_1 \times \mu_2) = \int_Y \int_X |f(x,y)|\mu_1(dx)\mu_2(dy)$$

Corollary. *Let* $X = \{1, 2, ...\}$, $\mathcal{A} =$ *all subsets of* X, *and* $\mu_1 =$ *counting measure. If* $\sum_n \int |f_n| d\mu < \infty$ *then* $\sum_n \int f_n d\mu = \int \sum_n f_n d\mu$.

1.7.2. *Let* $g \geq 0$ *be a measurable function on* $(X, \mathcal{A}, \mu)$. *Use Theorem 1.7.2 to conclude that*

$$\int_X g d\mu = (\mu \times \lambda)(\{(x,y) : 0 \leq y < g(x)\}) = \int_0^\infty \mu(\{x : g(x) > y\})dy$$

$$\begin{aligned}
\int_X g d\mu &= \int_X \int_{\mathbb{R}} 1_{0\leq y<g(x)} dy d\mu \\
&= (\mu \times \lambda)(\{(x,y) : 0 \leq y < g(x)\} \\
&= \int_{\mathbb{R}} \int_X 1_{g(x)>y\geq 0} d\mu dy \\
&= \int_{\mathbb{R}} \mu(\{x : g(x) > y \geq 0\})dy \\
&= \int_0^\infty \mu(\{x : g(x) > y\})dy
\end{aligned}$$

1.7.3. *Let* F, G *be Stieltjes measure functions and let* μ, ν *be the corresponding measures on* $(\mathbb{R}, \mathcal{R})$. *Show that*

1.7.4. *Let* μ *be a finite measure on* $\mathbb{R}$ *and* $F(X) = \mu((-\infty, x])$. *Show that*

$$\int (F(x + c) - F(x))dx = c\mu(\mathbb{R})$$

sol.

$$\begin{aligned}
\int (F(x + c) - F(x))dx &= \int \mu((x, x + c])dx \\
&= \int \int 1_{(x,x+c]}(y) d\mu dx \\
&= \int \int 1_{[y-c,y)}(x) dx d\mu \\
&= \int c d\mu = c\mu(\mathbb{R})
\end{aligned}$$

□

1.7.5. *Show that* $e^{-xy} \sin x$ *is integrable in the strip* $0 < x < a$, $0 < y$. *Perform the double integral in the two orders to get:*

$$\int_0^a \frac{\sin x}{x} dx = \arctan(a) - (\cos a)\int_0^\infty \frac{e^{-ay}}{1+y^2}dy - (\sin a)\int_0^\infty \frac{ye^{-ay}}{1+y^2}dy$$

and replace $1 + y^2$ *by* 1 *to conclude* $|\int_0^a (\sin x)/x dx - \arctan(a)| \leq 2/a$ *for* $a \geq 1$.