

Probability Theory and Example

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Contents

Chapter 1

Measure Theory

1.1 Probability Spaces

1.1.1. Let $\Omega = \mathbb{R}$, $\mathcal{F} =$ all subsets so that A or A^c is countable, $P(A) = 0$ in the first case and $= 1$ in the second. Show that (Ω, \mathcal{F}, P) is a probability space.

sol. i) \mathcal{F} is a σ -algebra on \mathbb{R} .

$\emptyset \in \mathcal{F}$ since \emptyset is countable.

By definition, \mathcal{F} is closed under complementations.

Countable union of countable sets is countable. Union of countable sets and uncountable sets is uncountable. Thus, \mathcal{F} is closed under countable union.

ii) P is a probability measure.

$P(\emptyset) = 0$ since \emptyset is countable. By definition, for any set A , $P(A) \geq 0$.

Countable union of countable sets is countable. Union of countable sets and uncountable sets is uncountable. Thus, P has the countable additivity property.

If A is countable, then A^c is uncountable since Ω is uncountable. Thus,

$$P(\Omega) = P(A) + P(A^c) = 1$$

By (i) and (ii), (Ω, \mathcal{F}, P) is a probability space. □

1.1.2. Recall the definition of \mathcal{S}_d from Example 1.1.5. Show that $\sigma(\mathcal{S}_d) = \mathcal{R}^d$, the Borel subsets of \mathbb{R}^d

sol. i) $d = 1$. In \mathbb{R} , any open set can be represented by countable union of open intervals. Thus, we need to show that any open interval can be represented by elements of \mathcal{S} . Let $-\infty < a < b < \infty$.

$$(a, b) = (-\infty, a]^c \cup (\cup_i (-\infty, b - 1/n])$$

If $a = -\infty$, then (a, b) can be represented by the second term. If $b = \infty$, then (a, b) can be represented by the first term. Thus, $\sigma(\mathcal{S}) = \sigma(\mathcal{R}) = \mathcal{R}$.

ii) $d \geq 2$. \mathcal{S}_d is a finite cartesian product of \mathcal{S} . Similarly, $\sigma(\mathcal{S}_d) = \sigma(\mathcal{R}_d) = \mathcal{R}_d$. □

1.1.3. A σ -field F is said to be countably generated if there is a countable collection $C \subset F$ so that $\sigma(C) = F$. Show that \mathcal{R}^d is countably generated.

sol. Let C be the collection that contains all sets of the form

$$[q_1, \infty) \times \cdots \times [q_d, \infty), (q_1, \dots, q_d) \in \mathbb{Q}^d$$

Then, C is countable, since it is finite union of countable sets. And $\sigma(C) = \mathcal{R}^d$ as presented by previous exercise. □

1.1.4. (i) Show that if $F_1 \subset F_2 \subset \dots$ are σ -algebras, then $\cup_i F_i$ is an algebra. (ii) Give an example to show that $\cup_i F_i$ need not be a σ -algebra.

sol.

(i) By definition, $\cup_i F_i$ is not empty. Choose $x \in \cup_i F_i$. Then, there exists F_x such that $x \in F_x$. Therefore, $x^c \in F_x \subset \cup_i F_i$. Thus, $\cup_i F_i$ is closed under complementations.

Choose, $x \in F_i, y \in F_j$ and suppose $i \leq j$. Then $x \in F_j$. Thus, $x \cup y \in F_j \Rightarrow x \cup y \in \cup_i F_i$. Thus, $\cup_i F_i$ is closed under union.

(ii) Let $F_i = \sigma(\{\{1\}, \{2\}, \dots, \{n\}\})$. Let $A = \{\{n\} : n = 3k | k = 1, 2, 3, \dots\}$. Then for all i , $A \notin F_i$. Thus, $A \notin \cup_i F_i$. However, A can be represented by countable union. Therefore, $\cup_i F_i$ is not a σ -algebra.

□

1.1.5. A set $A \subset \{1, 2, \dots\}$ is said to have asymptotic density θ if

$$\lim_{n \rightarrow \infty} |A \cap \{1, 2, \dots, n\}|/n = \theta$$

Let \mathcal{A} be the collection of sets for which the asymptotic density exists. Is \mathcal{A} a σ -algebra? an algebra?

sol. Let A be the set of even numbers. Next, we construct a set B in the following way: we begin with $\{2, 3\}$ and starting with $k = 2$, take all even numbers $2^k < n \leq (3/2) \times 2^k$, and all odd numbers $(3/2) \times 2^k < n \leq 2^{k+1}$. Then, the asymptotic density of B is 0.5. However, the asymptotic density $A \cap B$ does not exist.

When $n = (3/2) \times 2^k$, then the density is $1/3$. When $n = 2^{k+1}$, then the density is $1/4$.

Thus, \mathcal{A} is not closed under intersection. \mathcal{A} is neither σ -algebra nor algebra

□

1.2 Distributions

1.2.1. Suppose X and Y are random variables on (Ω, \mathcal{F}, P) and let $A \in \mathcal{F}$. Show that if we let $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$, then Z is a random variable.

sol. Let the Borel set B which satisfies that

$$\text{if } \omega \in A, \text{ then } Z(\omega) \in B$$

For arbitrary Borel set S , $S = (S \cap B) \cup (S \cap B^c)$. Since S is Borel set, $\{\omega : X(\omega) \in S\}, \{\omega : Y(\omega) \in S\} \in \mathcal{F}$.

$$\begin{aligned} \{\omega : Z(\omega) \in S\} &= \{\omega : Z(\omega) \in (S \cap B)\} \cup \{\omega : Z(\omega) \in (S \cap B^c)\} \\ &= \{\omega : X(\omega) \in (S \cap B)\} \cup \{\omega : Y(\omega) \in (S \cap B^c)\} \end{aligned}$$

Since \mathcal{F} is closed on set operation, $\{\omega : Z(\omega) \in S\} \in \mathcal{F}$.

□

1.2.2. Let χ have the standard normal distribution. Use Theorem 1.2.6 to get upper and lower bounds on $P(\chi \geq 4)$.

sol.

$$\begin{aligned} P(\chi \geq 4) &= (2\pi)^{-1} \int_4^\infty \exp(-y^2/2) dy \leq (8\pi)^{-1} \exp(-8) \\ &= (2\pi)^{-1} \int_4^\infty \exp(-y^2/2) dy \geq (15/128\pi) \exp(-8) \end{aligned}$$

□

1.2.3. Show that a distribution function has at most countably many discontinuities.

sol. Let D be the set of discontinuity points. Choose $x, y \in D$. Then we can choose rational number $q_x \in (F(x-), F(x+))$. Since F is increasing, if $x \neq y$, then $q_x \neq q_y$. Thus $x \rightarrow q_x$ is one-to-one function. Since \mathbb{Q} is countable, D is at most countable.

□

1.2.4. Show that if $F(x) = P(X \leq x)$ is continuous then $Y = F(X)$ has a uniform distribution on $(0, 1)$, that is, if $y \in [0, 1]$, $P(Y \leq y) = y$.

sol.

$$\begin{aligned}
\{\omega|Y(\omega) \leq y\} &= \{\omega|F(X(\omega)) \leq y\} \\
&= \{\omega|X(\omega) \leq k\} \quad k = \inf\{x|F(x) \geq y\} \\
P(\{\omega|Y(\omega) \leq y\}) &= P(\{\omega|X(\omega) \leq k\}) \\
&= P(X \leq k) = y
\end{aligned}$$

□

1.2.5. Suppose X has continuous density f , $P(\alpha \leq X \leq \beta) = 1$ and g is a function that is strictly increasing and differentiable on (α, β) . Then $g(X)$ has density $f(g^{-1}(y))/g'(g^{-1}(y))$ for $y \in (g(\alpha), g(\beta))$ and 0 otherwise. When $g(x) = ax + b$ with $a > 0$, $g^{-1}(y) = (y - b)/a$ so the answer is $(1/a)f((y - b)/a)$.

1.2.6. Suppose X has a normal distribution. Use the previous exercise to compute the density of $\exp(X)$.

1.3 Random Variables

1.3.1. Show that if \mathcal{A} generates \mathcal{S} , then $X^{-1}(A) \equiv \{X \in A\} : A \in \mathcal{A}$ generates $\sigma(X) = \{X \in B\} : B \in \mathcal{S}\}$.

sol. Let $A_1, A_2, \dots \in \mathcal{A}$. Since $\bigcup_i A_i \in \mathcal{S}$,

$$\bigcup_i \{X \in A_i\} = \{X \in \bigcup_i A_i\} \in \sigma(X)$$

□

1.3.2. Prove Theorem 1.3.6 when $n = 2$ by checking $\{X_1 + X_2 < x\} \in \mathcal{F}$.

sol.

$$\{X_1 + X_2 < x\} =$$

□

1.3.3. Show that if f is continuous and $X_n \rightarrow X$ almost surely then $f(X_n) \rightarrow f(X)$ almost surely.

sol. Let $\Omega_0 = \{\omega : \lim X_n(\omega) = X(\omega)\}$ and $\Omega_f = \{\omega : \lim f(X_n(\omega)) = f(X(\omega))\}$.

$$\begin{aligned}
\omega \in \Omega_0 &\Rightarrow \lim X_n(\omega) = X(\omega) \\
&\Rightarrow \lim f(X_n(\omega)) = f(X(\omega)) \quad \because f \text{ is continuous} \\
&\Rightarrow \omega \in \Omega_f \\
&\Rightarrow 1 = P(\Omega_0) \leq P(\Omega_f) \leq 1
\end{aligned}$$

□

1.3.4. (i) Show that a continuous function from $\mathcal{R}^d \rightarrow \mathcal{R}$ is a measurable map from $(\mathbb{R}^d, \mathcal{R}^d)$ to $(\mathbb{R}, \mathcal{R})$.
(ii) Show that \mathcal{R}^d is the smallest σ -field that makes all the continuous functions measurable.

sol.

- (i) \mathcal{R} is σ -field which is generated by all open sets. If f is continuous mapping and image is open, then inverse-image is also open. Thus, f is a measurable map.
- (ii) Let \mathcal{S} be the smallest σ -field that makes all the continuous functions measurable. Since \mathcal{R} is the smallest σ -field which is generated by all open sets, \mathcal{S} is generated by all open sets in \mathbb{R}^d .

□

1.3.5. A function f is said to be lower semicontinuous or l.s.c. if

$$\liminf_{y \rightarrow x} f(y) \geq f(x)$$

and upper semicontinuous (u.s.c.) if $-f$ is l.s.c.. Show that f is l.s.c. if and only if $\{x : f(x) \leq a\}$ is closed for each $a \in \mathbb{R}$ and conclude that semicontinuous functions are measurable.

sol.

if

only if

□

1.4 Integration

1.4.1. Show that if $f \geq 0$ and $\int f d\mu = 0$ then $f = 0$ a.e.

sol. Let N be the set that satisfies $x \in N \Rightarrow f(x) = 0$.

$$\begin{aligned} \int f d\mu &= \int_N f d\mu + \int_{N^c} f d\mu \\ &= \int_{N^c} f d\mu = 0 \\ &\Rightarrow \mu(N^c) = 0 \end{aligned}$$

□