Stan Functions Reference

Version 2.28

Stan Development Team

Contents

Overview

This is the reference for the functions defined in the Stan math library and available in the Stan programming language.

The Stan project comprises a domain-specific language for probabilistic programming, a differentiable mathematics and probability library, algorithms for Bayesian posterior inference and posterior analysis, along with interfaces and analysis tools in all of the popular data analysis languages.

In addition to this reference manual, there is a user's guide and a language reference manual for the Stan language and algorithms. The *Stan User's Guide* provides example models and programming techniques for coding statistical models in Stan. The *Stan Reference Manual* specifies the Stan programming language and inference algorithms.

There is also a separate installation and getting started guide for each of the Stan interfaces (R, Python, Julia, Stata, MATLAB, Mathematica, and command line).

Interfaces and platforms

Stan runs under Windows, Mac OS X, and Linux.

Stan uses a domain-specific programming language that is portable across data analysis languages. Stan has interfaces for R, Python, Julia, MATLAB, Mathematica, Stata, and the command line, as well as an alternative language interface in Scala. See the web site https://mc-stan.org for interface-specific links and getting started instructions

Web site

The official resource for all things related to Stan is the web site:

https://mc-stan.org

The web site links to all of the packages comprising Stan for both users and developers. This is the place to get started with Stan. Find the interface in the language you want to use and follow the download, installation, and getting started instructions.

GitHub organization

Stan's source code and much of the developer process is hosted on GitHub. Stan's organization is:

https://github.com/stan-dev

Each package has its own repository within the `stan-dev` organization. The web site is also hosted and managed through GitHub. This is the place to peruse the source code, request features, and report bugs. Much of the ongoing design discussion is hosted on the GitHub Wiki.

Forums

Stan hosts message boards for discussing all things related to Stan.

https://discourse.mc-stan.org

This is the place to ask questions about Stan, including modeling, programming, and installation.

Licensing

- *Computer code:* BSD 3-clause license

The core C++ code underlying Stan, including the math library, language, and inference algorithms, is licensed under the BSD 3-clause licensed as detailed in each repository and on the web site along with the distribution links.

- *Logo:* Stan logo usage guidelines

Acknowledgements

The Stan project could not exist without the generous grant funding of many grant agencies to the participants in the project. For more details of direct funding for the project, see the web site and project pages of the Stan developers.

The Stan project could also not exist without the generous contributions of its users in reporting and in many cases fixing bugs in the code and its documentation. We used to try to list all of those who contributed patches and bug reports for the manual here, but when that number passed into the hundreds, it became too difficult to manage

reliably. Instead, we will defer to GitHub (link above), where all contributions to the project are made and tracked.

Finally, we should all thank the Stan developers, without whom this project could not exist. We used to try and list the developers here, but like the bug reporters, once the list grew into the dozens, it became difficult to track. Instead, we will defer to the Stan web page and GitHub itself for a list of core developers and all developer contributions respectively.

Built-In Functions

1. Void Functions

Stan does not technically support functions that do not return values. It does support two types of statements, one printing and one for rejecting outputs.

Although `print` and `reject` appear to have the syntax of functions, they are actually special kinds of statements with slightly different form and behavior than other functions. First, they are the constructs that allow a variable number of arguments. Second, they are the the only constructs to accept string literals (e.g., `"hello world"`) as arguments. Third, they have no effect on the log density function and operate solely through side effects.

The special keyword `void` is used for their return type because they behave like variadic functions with void return type, even though they are special kinds of statements.

1.1. Print statement

Printing has no effect on the model's log probability function. Its sole purpose is the side effect (i.e., an effect not represented in a return value) of arguments being printed to whatever the standard output stream is connected to (e.g., the terminal in command-line Stan or the R console in RStan).

`void` **`print`**`(T1 x1,..., TN xN)`
Print the values denoted by the arguments x1 through xN on the output message stream. There are no spaces between items in the print, but a line feed (LF; Unicode U+000A; C++ literal `'\n'`) is inserted at the end of the printed line. The types `T1` through `TN` can be any of Stan's built-in numerical types or double quoted strings of characters (bytes).
Available since 2.1

1.2. Reject statement

The reject statement has the same syntax as the print statement, accepting an arbitrary number of arguments of any type (including string literals). The effect of executing a reject statement is to throw an exception internally that terminates the current iteration with a rejection (the behavior of which will depend on the algorithmic context in which it occurs).

`void `**`reject`**`(T1 x1,..., TN xN)`

Reject the current iteration and print the values denoted by the arguments x1 through xN on the output message stream. There are no spaces between items in the print, but a line feed (LF; Unicode U+000A; C++ literal `'\n'`) is inserted at the end of the printed line. The types `T1` through `TN` can be any of Stan's built-in numerical types or double quoted strings of characters (bytes).

Available since 2.18

2. Integer-Valued Basic Functions

This chapter describes Stan's built-in function that take various types of arguments and return results of type integer.

2.1. Integer-valued arithmetic operators

Stan's arithmetic is based on standard double-precision C++ integer and floating-point arithmetic. If the arguments to an arithmetic operator are both integers, as in 2 + 2, integer arithmetic is used. If one argument is an integer and the other a floating-point value, as in 2.0 + 2 and 2 + 2.0, then the integer is promoted to a floating point value and floating-point arithmetic is used.

Integer arithmetic behaves slightly differently than floating point arithmetic. The first difference is how overflow is treated. If the sum or product of two integers overflows the maximum integer representable, the result is an undesirable wraparound behavior at the bit level. If the integers were first promoted to real numbers, they would not overflow a floating-point representation. There are no extra checks in Stan to flag overflows, so it is up to the user to make sure it does not occur.

Secondly, because the set of integers is not closed under division and there is no special infinite value for integers, integer division implicitly rounds the result. If both arguments are positive, the result is rounded down. For example, 1 / 2 evaluates to 0 and 5 / 3 evaluates to 1.

If one of the integer arguments to division is negative, the latest C++ specification (C++11), requires rounding toward zero. This would have 1 / 2 and -1 / 2 evaluate to 0, -7 / 2 evaluate to -3, and 7 / 2 evaluate to 3. Before the C++11 specification, the behavior was platform dependent, allowing rounding up or down. All compilers recent enough to be able to deal with Stan's templating should follow the C++11 specification, but it may be worth testing if you are not sure and plan to use integer division with negative values.

Unlike floating point division, where 1.0 / 0.0 produces the special positive infinite value, integer division by zero, as in 1 / 0, has undefined behavior in the C++ standard. For example, the clang++ compiler on Mac OS X returns 3764, whereas the g++ compiler throws an exception and aborts the program with a warning. As with overflow, it is up to the user to make sure integer divide-by-zero does not occur.

Binary infix operators

Operators are described using the C++ syntax. For instance, the binary operator of addition, written X + Y, would have the Stan signature int operator+(int, int) indicating it takes two real arguments and returns a real value. As noted previously, the value of integer division is platform-dependent when rounding is platform dependent before C++11; the descriptions below provide the C++11 definition.

int **operator+**(int x, int y)
The sum of the addends x and y

$$\text{operator+}(x, y) = (x + y)$$

Available since 2.0

int **operator-**(int x, int y)
The difference between the minuend x and subtrahend y

$$\text{operator-}(x, y) = (x - y)$$

Available since 2.0

int **operator***(int x, int y)
The product of the factors x and y

$$\text{operator*}(x, y) = (x \times y)$$

Available since 2.0

int **operator/**(int x, int y)
The integer quotient of the dividend x and divisor y

$$\text{operator/}(x, y) = \begin{cases} \lfloor x/y \rfloor & \text{if } x/y \geq 0 \\ -\lfloor \text{floor}(-x/y) \rfloor & \text{if } x/y < 0. \end{cases}$$

deprecated; - use operator%/% instead.
Available since 2.0, deprecated in 2.24

int **operator%/%**(int x, int y)
The integer quotient of the dividend x and divisor y

$$\text{operator%/%}(x, y) = \begin{cases} \lfloor x/y \rfloor & \text{if } x/y \geq 0 \\ -\lfloor \text{floor}(-x/y) \rfloor & \text{if } x/y < 0. \end{cases}$$

Available since 2.24

int **operator%**(int x, int y)
x modulo y, which is the positive remainder after dividing x by y. If both x and y are
non-negative, so is the result; otherwise, the sign of the result is platform dependent.

$$\operatorname{operator\%}(x, y) \;=\; x \bmod y \;=\; x - y * \lfloor x/y \rfloor$$

Available since 2.13

Unary prefix operators

int **operator-**(int x)
The negation of the subtrahend x [operator-(x) = -x
Available since 2.0

int **operator+**(int x)
This is a no-op.

$$\operatorname{operator+}(x) = x$$

Available since 2.0

2.2.   Absolute functions

R **abs**(T x)
absolute value of x
Available since 2.0

int **int_step**(int x)

int **int_step**(real x)
Return the step function of x as an integer,

$$\operatorname{int\_step}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \text{ or } x \text{ is } NaN \end{cases}$$

Warning: int_step(0) and int_step(NaN) return 0 whereas step(0) and
step(NaN) return 1.

See the warning in section step functions about the dangers of step functions applied to anything other than data.
Available since 2.0

2.3. Bound functions

int **min**(int x, int y)
Return the minimum of x and y.

$$\min(x, y) = \begin{cases} x & \text{if } x < y \\ y & \text{otherwise} \end{cases}$$

Available since 2.0

int **max**(int x, int y)
Return the maximum of x and y.

$$\max(x, y) = \begin{cases} x & \text{if } x > y \\ y & \text{otherwise} \end{cases}$$

Available since 2.0

2.4. Size functions

int **size**(int x)

int **size**(real x)

Return the size of x which for scalar-valued x is 1
Available since 2.26

3.   Real-Valued Basic Functions

This chapter describes built-in functions that take zero or more real or integer arguments and return real values.

3.1.   Vectorization of real-valued functions

Although listed in this chapter, many of Stan's built-in functions are vectorized so that they may be applied to any argument type. The vectorized form of these functions is not any faster than writing an explicit loop that iterates over the elements applying the function—it's just easier to read and write and less error prone.

Unary function vectorization

Many of Stan's unary functions can be applied to any argument type. For example, the exponential function, exp, can be applied to `real` arguments or arrays of `real` arguments. Other than for integer arguments, the result type is the same as the argument type, including dimensionality and size. Integer arguments are first promoted to real values, but the result will still have the same dimensionality and size as the argument.

Real and real array arguments

When applied to a simple real value, the result is a real value. When applied to arrays, vectorized functions like exp() are defined elementwise. For example,

```
// declare some variables for arguments
real x0;
array[5] real x1;
array[4, 7] real x2;
// ...
// declare some variables for results
real y0;
array[5] real y1;
array[4, 7] real y2;
// ...
// calculate and assign results
```

```
y0 = exp(x0);
y1 = exp(x1);
y2 = exp(x2);
```

When exp is applied to an array, it applies elementwise. For example, the statement above,

```
y2 = exp(x2);
```

produces the same result for y2 as the explicit loop

```
for (i in 1:4) {
  for (j in 1:7) {
    y2[i, j] = exp(x2[i, j]);
  }
}
```

Vector and matrix arguments

Vectorized functions also apply elementwise to vectors and matrices. For example,

```
vector[5] xv;
row_vector[7] xrv;
matrix[10, 20] xm;

vector[5] yv;
row_vector[7] yrv;
matrix[10, 20] ym;

yv = exp(xv);
yrv = exp(xrv);
ym = exp(xm);
```

Arrays of vectors and matrices work the same way. For example,

```
array[12] matrix[17, 93] u;

array[12] matrix[17, 93] z;

z = exp(u);
```

After this has been executed, `z[i, j, k]` will be equal to `exp(u[i, j, k])`.

Integer and integer array arguments

Integer arguments are promoted to real values in vectorized unary functions. Thus if
n is of type `int`, `exp(n)` is of type `real`. Arrays work the same way, so that if n2 is a
one dimensional array of integers, then `exp(n2)` will be a one-dimensional array of
reals with the same number of elements as n2. For example,

```
array[23] int n1;
array[23] real z1;
z1 = exp(n1);
```

It would be illegal to try to assign `exp(n1)` to an array of integers; the return type is
a real array.

Binary function vectorization

Like the unary functions, many of Stan's binary functions have been vectorized, and
can be applied elementwise to combinations of both scalars or container types.

Scalar and scalar array arguments

When applied to two scalar values, the result is a scalar value. When applied to two
arrays, or combination of a scalar value and an array, vectorized functions like `pow()`
are defined elementwise. For example,

```
// declare some variables for arguments
real x00;
real x01;
array[5] real x10;
array[5]real x11;
array[4, 7] real x20;
array[4, 7] real x21;
// ...
// declare some variables for results
real y0;
array[5] real y1;
array[4, 7] real y2;
// ...
```

```
// calculate and assign results
y0 = pow(x00, x01);
y1 = pow(x10, x11);
y2 = pow(x20, x21);
```

When pow is applied to two arrays, it applies elementwise. For example, the statement above,

```
y2 = pow(x20, x21);
```

produces the same result for y2 as the explicit loop

```
for (i in 1:4) {
  for (j in 1:7) {
    y2[i, j] = pow(x20[i, j], x21[i, j]);
  }
}
```

Alternatively, if a combination of an array and a scalar are provided, the scalar value is broadcast to be applied to each value of the array. For example, the following statement:

```
y2 = pow(x20, x00);
```

produces the same result for y2 as the explicit loop:

```
for (i in 1:4) {
  for (j in 1:7) {
    y2[i, j] = pow(x20[i, j], x00);
  }
}
```

Vector and matrix arguments

Vectorized binary functions also apply elementwise to vectors and matrices, and to combinations of these with scalar values. For example,

```
real x00;
vector[5] xv00;
vector[5] xv01;
row_vector[7] xrv;
matrix[10, 20] xm;
```

```
vector[5] yv;
row_vector[7] yrv;
matrix[10, 20] ym;

yv = pow(xv00, xv01);
yrv = pow(xrv, x00);
ym = pow(x00, xm);
```

Arrays of vectors and matrices work the same way. For example,

```
array[12] matrix[17, 93] u;

array[12] matrix[17, 93] z;

z = pow(u, x00);
```

After this has been executed, `z[i, j, k]` will be equal to `pow(u[i, j, k], x00)`.

Input & return types

Vectorised binary functions require that both inputs, unless one is a real, be containers of the same type and size. For example, the following statements are legal:

```
vector[5] xv;
row_vector[7] xrv;
matrix[10, 20] xm;

vector[5] yv = pow(xv, xv)
row_vector[7] yrv = pow(xrv, xrv)
matrix[10, 20] = pow(xm, xm)
```

But the following statements are not:

```
vector[5] xv;
vector[7] xv2;
row_vector[5] xrv;

// Cannot mix different types
vector[5] yv = pow(xv, xrv)
```

```
// Cannot mix different sizes of the same type
vector[5] yv = pow(xv, xv2)
```

While the vectorized binary functions generally require the same input types, the only exception to this is for binary functions that require one input to be an integer and the other to be a real (e.g., `bessel_first_kind`). For these functions, one argument can be a container of any type while the other can be an integer array, as long as the dimensions of both are the same. For example, the following statements are legal:

```
vector[5] xv;
matrix[5, 5] xm;
array[5] int xi;
array[5, 5] int xii;

vector[5] yv = bessel_first_kind(xi, xv);
matrix[5, 5] ym = bessel_first_kind(xii, xm);
```

Whereas these are not:

```
vector[5] xv;
matrix[5, 5] xm;
array[7] int xi;

// Dimensions of containers do not match
vector[5] yv = bessel_first_kind(xi, xv);

// Function requires first argument be an integer type
matrix[5, 5] ym = bessel_first_kind(xm, xm);
```

3.2. Mathematical constants

Constants are represented as functions with no arguments and must be called as such. For instance, the mathematical constant π must be written in a Stan program as `pi()`.

```
real pi()
```
π, the ratio of a circle's circumference to its diameter
Available since 2.0

```
real e()
```

e, the base of the natural logarithm
Available since 2.0

```
real sqrt2()
```
The square root of 2
Available since 2.0

```
real log2()
```
The natural logarithm of 2
Available since 2.0

```
real log10()
```
The natural logarithm of 10
Available since 2.0

3.3. Special values

```
real not_a_number()
```
Not-a-number, a special non-finite real value returned to signal an error
Available since 2.0

```
real positive_infinity()
```
Positive infinity, a special non-finite real value larger than all finite numbers
Available since 2.0

```
real negative_infinity()
```
Negative infinity, a special non-finite real value smaller than all finite numbers
Available since 2.0

```
real machine_precision()
```
The smallest number x such that $(x + 1) \neq 1$ in floating-point arithmetic on the current hardware platform
Available since 2.0

3.4. Log probability function

The basic purpose of a Stan program is to compute a log probability function and its derivatives. The log probability function in a Stan model outputs the log density on the unconstrained scale. A log probability accumulator starts at zero and is then incremented in various ways by a Stan program. The variables are first transformed from unconstrained to constrained, and the log Jacobian determinant added to the log probability accumulator. Then the model block is executed on the constrained parameters, with each sampling statement (~) and log probability

increment statement (`increment_log_prob`) adding to the accumulator. At the end of the model block execution, the value of the log probability accumulator is the log probability value returned by the Stan program.

Stan provides a special built-in function `target()` that takes no arguments and returns the current value of the log probability accumulator.[1] This function is primarily useful for debugging purposes, where for instance, it may be used with a print statement to display the log probability accumulator at various stages of execution to see where it becomes ill defined.

real **`target`**`()`
Return the current value of the log probability accumulator.
Available since 2.10

real **`get_lp`**`()`
Return the current value of the log probability accumulator; **deprecated**; - use `target()` instead.
Available since 2.5

Both `target` and the deprecated `get_lp` act like other functions ending in `_lp`, meaning that they may only may only be used in the model block.

3.5. Logical functions

Like C++, BUGS, and R, Stan uses 0 to encode false, and 1 to encode true. Stan supports the usual boolean comparison operations and boolean operators. These all have the same syntax and precedence as in C++; for the full list of operators and precedences, see the reference manual.

Comparison operators

All comparison operators return boolean values, either 0 or 1. Each operator has two signatures, one for integer comparisons and one for floating-point comparisons. Comparing an integer and real value is carried out by first promoting the integer value.

int **`operator<`**`(int x, int y)`

int **`operator<`**`(real x, real y)`

---
[1]This function used to be called `get_lp()`, but that name has been deprecated; using it will print a warning. The function `get_lp()` will be removed in a future release.

Return 1 if x is less than y and 0 otherwise.

$$\text{operator}{<}(x, y) = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{otherwise} \end{cases}$$

Available since 2.0

```
int operator<=(int x, int y)
```

```
int operator<=(real x, real y)
```
Return 1 if x is less than or equal y and 0 otherwise.

$$\text{operator}{<}{=}(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Available since 2.0

```
int operator>(int x, int y)
```

```
int operator>(real x, real y)
```
Return 1 if x is greater than y and 0 otherwise.

$$\text{operator}{>} = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{otherwise} \end{cases}$$

Available since 2.0

```
int operator>=(int x, int y)
```

```
int operator>=(real x, real y)
```
Return 1 if x is greater than or equal to y and 0 otherwise.

$$\text{operator}{>}{=} = \begin{cases} 1 & \text{if } x \geq y \\ 0 & \text{otherwise} \end{cases}$$

Available since 2.0

```
int operator==(int x, int y)
```

```
int operator==(real x, real y)
```
Return 1 if x is equal to y and 0 otherwise.

$$\text{operator==}(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

Available since 2.0

```
int operator!=(int x, int y)
```

```
int operator!=(real x, real y)
```
Return 1 if x is not equal to y and 0 otherwise.

$$\text{operator!=}(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

Available since 2.0

Boolean operators

Boolean operators return either 0 for false or 1 for true. Inputs may be any real or integer values, with non-zero values being treated as true and zero values treated as false. These operators have the usual precedences, with negation (not) binding the most tightly, conjunction the next and disjunction the weakest; all of the operators bind more tightly than the comparisons. Thus an expression such as !a && b is interpreted as (!a) && b, and a < b || c >= d && e != f as (a < b) || (((c >= d) && (e != f))).

```
int operator!(int x)
```

```
int operator!(real x)
```
Return 1 if x is zero and 0 otherwise.

$$\text{operator!}(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Available since 2.0

```
int operator&&(int x, int y)
```

```
int operator&&(real x, real y)
```
Return 1 if x is unequal to 0 and y is unequal to 0.

$$\text{operator\&\&}(x, y) = \begin{cases} 1 & \text{if } x \neq 0 \text{ and } y \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Available since 2.0

```
int operator||(int x, int y)
```

```
int operator||(real x, real y)
```
Return 1 if x is unequal to 0 or y is unequal to 0.

$$\text{operator||}(x, y) = \begin{cases} 1 & \text{if } x \neq 0 \text{ or } y \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Available since 2.0

Boolean operator short circuiting

Like in C++, the boolean operators && and || and are implemented to short circuit directly to a return value after evaluating the first argument if it is sufficient to resolve the result. In evaluating a || b, if a evaluates to a value other than zero, the expression returns the value 1 without evaluating the expression b. Similarly, evaluating a && b first evaluates a, and if the result is zero, returns 0 without evaluating b.

Logical functions

The logical functions introduce conditional behavior functionally and are primarily provided for compatibility with BUGS and JAGS.

```
real step(real x)
```
Return 1 if x is positive and 0 otherwise.

$$\text{step}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{otherwise} \end{cases}$$

Warning: int_step(0) and int_step(NaN) return 0 whereas step(0) and step(NaN) return 1.

The step function is often used in BUGS to perform conditional operations. For instance, `step(a-b)` evaluates to 1 if `a` is greater than `b` and evaluates to 0 otherwise. `step` is a step-like functions; see the warning in section step functions applied to expressions dependent on parameters.
Available since 2.0

`int is_inf(real x)`
Return 1 if x is infinite (positive or negative) and 0 otherwise.
Available since 2.5

`int is_nan(real x)`
Return 1 if x is NaN and 0 otherwise.
Available since 2.5

Care must be taken because both of these indicator functions are step-like and thus can cause discontinuities in gradients when applied to parameters; see section step-like functions for details.

3.6. Real-valued arithmetic operators

The arithmetic operators are presented using C++ notation. For instance `operator+(x,y)` refers to the binary addition operator and `operator-(x)` to the unary negation operator. In Stan programs, these are written using the usual infix and prefix notations as `x + y` and `-x`, respectively.

Binary infix operators

`real operator+(real x, real y)`
Return the sum of x and y.

$$(x + y) = \text{operator+}(x, y) = x + y$$

Available since 2.0

`real operator-(real x, real y)`
Return the difference between x and y.

$$(x - y) = \text{operator-}(x, y) = x - y$$

Available since 2.0

```
real operator*(real x, real y)
```
Return the product of x and y.

$$(x * y) = \text{operator*}(x, y) = xy$$

Available since 2.0

```
real operator/(real x, real y)
```
Return the quotient of x and y.

$$(x/y) = \text{operator/}(x, y) = \frac{x}{y}$$

Available since 2.0

```
real operator^(real x, real y)
```
Return x raised to the power of y.

$$(x^\wedge y) = \text{operator}^\wedge(x, y) = x^y$$

Available since 2.5

Unary prefix operators

```
real operator-(real x)
```
Return the negation of the subtrahend x.

$$\text{operator-}(x) = (-x)$$

Available since 2.0

```
real operator+(real x)
```
Return the value of x.

$$\text{operator+}(x) = x$$

Available since 2.0

3.7. Step-like functions

Warning: *These functions can seriously hinder sampling and optimization efficiency for gradient-based methods (e.g., NUTS, HMC, BFGS) if applied to parameters (including transformed parameters and local variables in the transformed parameters or model block). The problem is that they break gradients due to discontinuities coupled with zero gradients elsewhere. They do not hinder sampling when used in the data, transformed data, or generated quantities blocks.*

Absolute value functions

R `fabs`(T x)
absolute value of x
Available since 2.0, vectorized in 2.13

real `fdim`(real x, real y)
Return the positive difference between x and y, which is x - y if x is greater than y and 0 otherwise; see warning above.

$$\text{fdim}(x, y) = \begin{cases} x - y & \text{if } x \geq y \\ 0 & \text{otherwise} \end{cases}$$

Available since 2.0

R `fdim`(T1 x, T2 y)
Vectorized implementation of the `fdim` function
Available since 2.25

Bounds functions

real `fmin`(real x, real y)
Return the minimum of x and y; see warning above.

$$\text{fmin}(x, y) = \begin{cases} x & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

Available since 2.0

R `fmin`(T1 x, T2 y)
Vectorized implementation of the `fmin` function
Available since 2.25

real **fmax**(real x, real y)
Return the maximum of x and y; see warning above.

$$\text{fmax}(x, y) = \begin{cases} x & \text{if } x \geq y \\ y & \text{otherwise} \end{cases}$$

Available since 2.0

R **fmax**(T1 x, T2 y)
Vectorized implementation of the **fmax** function
Available since 2.25

Arithmetic functions

real **fmod**(real x, real y)
Return the real value remainder after dividing x by y; see warning above.

$$\text{fmod}(x, y) = x - \left\lfloor \frac{x}{y} \right\rfloor y$$

The operator $\lfloor u \rfloor$ is the floor operation; see below.
Available since 2.0

R **fmod**(T1 x, T2 y)
Vectorized implementation of the **fmod** function
Available since 2.25

Rounding functions

Warning: Rounding functions convert real values to integers. Because the output is
an integer, any gradient information resulting from functions applied to the integer
is not passed to the real value it was derived from. With MCMC sampling using HMC
or NUTS, the MCMC acceptance procedure will correct for any error due to poor
gradient calculations, but the result is likely to be reduced acceptance probabilities
and less efficient sampling.

The rounding functions cannot be used as indices to arrays because they return real
values. Stan may introduce integer-valued versions of these in the future, but as of
now, there is no good workaround.

R **floor**(T x)
floor of x, which is the largest integer less than or equal to x, converted to a real

value; see warning at start of section step-like functions
Available since 2.0, vectorized in 2.13

R `ceil(T x)`
ceiling of x, which is the smallest integer greater than or equal to x, converted to a
real value; see warning at start of section step-like functions
Available since 2.0, vectorized in 2.13

R `round(T x)`
nearest integer to x, converted to a real value; see warning at start of section step-like
functions
Available since 2.0, vectorized in 2.13

R `trunc(T x)`
integer nearest to but no larger in magnitude than x, converted to a double value;
see warning at start of section step-like functions
Available since 2.0, vectorized in 2.13

3.8.   Power and logarithm functions

R `sqrt(T x)`
square root of x
Available since 2.0, vectorized in 2.13

R `cbrt(T x)`
cube root of x
Available since 2.0, vectorized in 2.13

R `square(T x)`
square of x
Available since 2.0, vectorized in 2.13

R `exp(T x)`
natural exponential of x
Available since 2.0, vectorized in 2.13

R `exp2(T x)`
base-2 exponential of x
Available since 2.0, vectorized in 2.13

R `log(T x)`
natural logarithm of x
Available since 2.0, vectorized in 2.13

R `log2`(T x)
base-2 logarithm of x
Available since 2.0, vectorized in 2.13

R `log10`(T x)
base-10 logarithm of x
Available since 2.0, vectorized in 2.13

real `pow`(real x, real y)
Return x raised to the power of y.

$$\text{pow}(x, y) = x^y$$

Available since 2.0

R `pow`(T1 x, T2 y)
Vectorized implementation of the `pow` function
Available since 2.25

R `inv`(T x)
inverse of x
Available since 2.0, vectorized in 2.13

R `inv_sqrt`(T x)
inverse of the square root of x
Available since 2.0, vectorized in 2.13

R `inv_square`(T x)
inverse of the square of x
Available since 2.0, vectorized in 2.13

3.9.   Trigonometric functions

real `hypot`(real x, real y)
Return the length of the hypotenuse of a right triangle with sides of length x and y.

$$\text{hypot}(x, y) = \begin{cases} \sqrt{x^2 + y^2} & \text{if } x, y \geq 0 \\ \text{NaN} & \text{otherwise} \end{cases}$$

Available since 2.0

R `hypot`(T1 x, T2 y)
Vectorized implementation of the `hypot` function
Available since 2.25

`R cos(T x)`
cosine of the angle x (in radians)
Available since 2.0, vectorized in 2.13

`R sin(T x)`
sine of the angle x (in radians)
Available since 2.0, vectorized in 2.13

`R tan(T x)`
tangent of the angle x (in radians)
Available since 2.0, vectorized in 2.13

`R acos(T x)`
principal arc (inverse) cosine (in radians) of x
Available since 2.0, vectorized in 2.13

`R asin(T x)`
principal arc (inverse) sine (in radians) of x
Available since 2.0

`R atan(T x)`
principal arc (inverse) tangent (in radians) of x, with values from $-\pi/2$ to $\pi/2$
Available since 2.0, vectorized in 2.13

`real atan2(real y, real x)`
Return the principal arc (inverse) tangent (in radians) of y divided by x,

$$\text{atan2}(y, x) = \arctan\left(\frac{y}{x}\right)$$

Available since 2.0, vectorized in 2.13

3.10.   Hyperbolic trigonometric functions

`R cosh(T x)`
hyperbolic cosine of x (in radians)
Available since 2.0, vectorized in 2.13

`R sinh(T x)`
hyperbolic sine of x (in radians)
Available since 2.0, vectorized in 2.13

`R tanh(T x)`
hyperbolic tangent of x (in radians)
Available since 2.0, vectorized in 2.13

R `acosh`(T x)
inverse hyperbolic cosine (in radians)
Available since 2.0, vectorized in 2.13

R `asinh`(T x)
inverse hyperbolic cosine (in radians)
Available since 2.0, vectorized in 2.13

R `atanh`(T x)
inverse hyperbolic tangent (in radians) of x
Available since 2.0, vectorized in 2.13

3.11.   Link functions

The following functions are commonly used as link functions in generalized linear models. The function Φ is also commonly used as a link function (see section probability-related functions).

R `logit`(T x)
log odds, or logit, function applied to x
Available since 2.0, vectorized in 2.13

R `inv_logit`(T x)
logistic sigmoid function applied to x
Available since 2.0, vectorized in 2.13

R `inv_cloglog`(T x)
inverse of the complementary log-log function applied to x
Available since 2.0, vectorized in 2.13

3.12.   Probability-related functions

Normal cumulative distribution functions

The error function `erf` is related to the standard normal cumulative distribution function Φ by scaling. See section normal distribution for the general normal cumulative distribution function (and its complement).

R `erf`(T x)
error function, also known as the Gauss error function, of x
Available since 2.0, vectorized in 2.13

R `erfc`(T x)
complementary error function of x

Available since 2.0, vectorized in 2.13

R `Phi`(T x)
standard normal cumulative distribution function of x
Available since 2.0, vectorized in 2.13

R `inv_Phi`(T x)
Return the value of the inverse standard normal cdf Φ^{-1} at the specified quantile x. The details of the algorithm can be found in (Wichura 1988). Quantile arguments below 1e-16 are untested; quantiles above 0.999999999 result in increasingly large errors.
Available since 2.0, vectorized in 2.13

R `Phi_approx`(T x)
fast approximation of the unit (may replace `Phi` for probit regression with maximum absolute error of 0.00014, see (Bowling et al. 2009) for details)
Available since 2.0, vectorized in 2.13

Other probability-related functions

real `binary_log_loss`(int y, real y_hat)
Return the log loss function for for predicting $\hat{y} \in [0,1]$ for boolean outcome $y \in \{0,1\}$.

$$\text{binary\_log\_loss}(y, \hat{y}) = \begin{cases} -\log \hat{y} & \text{if } y = 0 \\ -\log(1 - \hat{y}) & \text{otherwise} \end{cases}$$

Available since 2.0

R `binary_log_loss`(T1 x, T2 y)
Vectorized implementation of the `binary_log_loss` function
Available since 2.25

real `owens_t`(real h, real a)
Return the Owen's T function for the probability of the event $X > h$ and $0 < Y < aX$ where X and Y are independent standard normal random variables.

$$\text{owens\_t}(h, a) = \frac{1}{2\pi} \int_0^a \frac{\exp(-\frac{1}{2}h^2(1 + x^2))}{1 + x^2} dx$$

Available since 2.25

R `owens_t`(T1 x, T2 y)
Vectorized implementation of the `owens_t` function

Available since 2.25

3.13. Combinatorial functions

`real beta(real alpha, real beta)`
Return the beta function applied to alpha and beta. The beta function, $\mathrm{B}(\alpha, \beta)$,
computes the normalizing constant for the beta distribution, and is defined for $\alpha > 0$
and $\beta > 0$. See section appendix for definition of $\mathrm{B}(\alpha, \beta)$.
Available since 2.25

`R beta(T1 x, T2 y)`
Vectorized implementation of the `beta` function
Available since 2.25

`real inc_beta(real alpha, real beta, real x)`
Return the regularized incomplete beta function up to x applied to alpha and beta.
See section appendix for a definition.
Available since 2.10

`real lbeta(real alpha, real beta)`
Return the natural logarithm of the beta function applied to alpha and beta. The
beta function, $\mathrm{B}(\alpha, \beta)$, computes the normalizing constant for the beta distribution,
and is defined for $\alpha > 0$ and $\beta > 0$.

$$\mathrm{lbeta}(\alpha, \beta) = \log \Gamma(a) + \log \Gamma(b) - \log \Gamma(a + b)$$

See section appendix for definition of $\mathrm{B}(\alpha, \beta)$.
Available since 2.0

`R lbeta(T1 x, T2 y)`
Vectorized implementation of the `lbeta` function
Available since 2.25

`R tgamma(T x)`
gamma function applied to x. The gamma function is the generalization of the
factorial function to continuous variables, defined so that $\Gamma(n + 1) = n!$. See for a
full definition of $\Gamma(x)$. The function is defined for positive numbers and non-integral
negative numbers,
Available since 2.0, vectorized in 2.13

`R lgamma(T x)`
natural logarithm of the gamma function applied to x,
Available since 2.0, vectorized in 2.15

R `digamma`(T x)
digamma function applied to x. The digamma function is the derivative of the natural logarithm of the Gamma function. The function is defined for positive numbers and non-integral negative numbers
Available since 2.0, vectorized in 2.13

R `trigamma`(T x)
trigamma function applied to x. The trigamma function is the second derivative of the natural logarithm of the Gamma function
Available since 2.0, vectorized in 2.13

real `lmgamma`(int n, real x)
Return the natural logarithm of the multivariate gamma function Γ_n with n dimensions applied to x.

$$\text{lmgamma}(n, x) = \begin{cases} \frac{n(n-1)}{4} \log \pi + \sum_{j=1}^{n} \log \Gamma \left(x + \frac{1-j}{2} \right) & \text{if } x \notin \{\ldots, -3, -2, -1, 0\} \\ \text{error} & \text{otherwise} \end{cases}$$

Available since 2.0

R `lmgamma`(T1 x, T2 y)
Vectorized implementation of the `lmgamma` function
Available since 2.25

real `gamma_p`(real a, real z)
Return the normalized lower incomplete gamma function of a and z defined for positive a and nonnegative z.

$$\text{gamma\_p}(a, z) = \begin{cases} \frac{1}{\Gamma(a)} \int_0^z t^{a-1} e^{-t} dt & \text{if } a > 0, z \geq 0 \\ \text{error} & \text{otherwise} \end{cases}$$

Available since 2.0

R `gamma_p`(T1 x, T2 y)
Vectorized implementation of the `gamma_p` function
Available since 2.25

real `gamma_q`(real a, real z)
Return the normalized upper incomplete gamma function of a and z defined for positive a and nonnegative z.

$$\text{gamma\_q}(a, z) = \begin{cases} \frac{1}{\Gamma(a)} \int_z^\infty t^{a-1} e^{-t} dt & \text{if } a > 0, z \geq 0 \\ \text{error} & \text{otherwise} \end{cases}$$

Available since 2.0

R `gamma_q`(T1 x, T2 y)
Vectorized implementation of the `gamma_q` function
Available since 2.25

real `binomial_coefficient_log`(real x, real y)
Warning: This function is deprecated and should be replaced with `lchoose`. Return the natural logarithm of the binomial coefficient of x and y. For non-negative integer inputs, the binomial coefficient function is written as $\binom{x}{y}$ and pronounced "x choose y." This function generalizes to real numbers using the gamma function. For $0 \le y \le x$,

$$\text{binomial\_coefficient\_log}(x, y) = \log \Gamma(x+1) - \log \Gamma(y+1) - \log \Gamma(x - y + 1).$$

Available since 2.0, deprecated since 2.10

R `binomial_coefficient_log`(T1 x, T2 y)
Vectorized implementation of the `binomial_coefficient_log` function
Available since 2.25

int `choose`(int x, int y)
Return the binomial coefficient of x and y. For non-negative integer inputs, the binomial coefficient function is written as $\binom{x}{y}$ and pronounced "x choose y." In its the antilog of the `lchoose` function but returns an integer rather than a real number with no non-zero decimal places. For $0 \le y \le x$, the binomial coefficient function can be defined via the factorial function

$$\text{choose}(x, y) = \frac{x!}{(y!)\,(x-y)!}.$$

Available since 2.14

R `choose`(T1 x, T2 y)
Vectorized implementation of the `choose` function
Available since 2.25

real `bessel_first_kind`(int v, real x)
Return the Bessel function of the first kind with order v applied to x.

$$\text{bessel\_first\_kind}(v, x) = J_v(x),$$

where

$$J_v(x) = \left(\frac{1}{2}x\right)^v \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{4}x^2\right)^k}{k!\,\Gamma(v+k+1)}$$

Available since 2.5

R `bessel_first_kind`(T1 x, T2 y)
Vectorized implementation of the `bessel_first_kind` function
Available since 2.25

real `bessel_second_kind`(int v, real x)
Return the Bessel function of the second kind with order v applied to x defined for
positive x and v. For $x, v > 0$,

$$\text{bessel\_second\_kind}(v, x) = \begin{cases} Y_v(x) & \text{if } x > 0 \\ \text{error} & \text{otherwise} \end{cases}$$

where

$$Y_v(x) = \frac{J_v(x)\cos(v\pi) - J_{-v}(x)}{\sin(v\pi)}$$

Available since 2.5

R `bessel_second_kind`(T1 x, T2 y)
Vectorized implementation of the `bessel_second_kind` function
Available since 2.25

real `modified_bessel_first_kind`(int v, real z)
Return the modified Bessel function of the first kind with order v applied to z defined
for all z and integer v.

$$\text{modified\_bessel\_first\_kind}(v, z) = I_v(z)$$

where

$$I_v(z) = \left(\frac{1}{2}z\right)^v \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k!\,\Gamma(v+k+1)}$$

Available since 2.1

R `modified_bessel_first_kind`(T1 x, T2 y)
Vectorized implementation of the `modified_bessel_first_kind` function
Available since 2.25

real `log_modified_bessel_first_kind`(real v, real z)
Return the log of the modified Bessel function of the first kind. v does not have to be
an integer.
Available since 2.26

R `log_modified_bessel_first_kind`(T1 x, T2 y)
Vectorized implementation of the `log_modified_bessel_first_kind` function
Available since 2.26

real `modified_bessel_second_kind`(int v, real z)
Return the modified Bessel function of the second kind with order v applied to z
defined for positive z and integer v.

$$\text{modified\_bessel\_second\_kind}(v, z) = \begin{cases} K_v(z) & \text{if } z > 0 \\ \text{error} & \text{if } z \leq 0 \end{cases}$$

where

$$K_v(z) = \frac{\pi}{2} \cdot \frac{I_{-v}(z) - I_v(z)}{\sin(v\pi)}$$

Available since 2.1

R `modified_bessel_second_kind`(T1 x, T2 y)
Vectorized implementation of the `modified_bessel_second_kind` function
Available since 2.25

real `falling_factorial`(real x, real n)
Return the falling factorial of x with power n defined for positive x and real n.

$$\text{falling\_factorial}(x, n) = \begin{cases} (x)_n & \text{if } x > 0 \\ \text{error} & \text{if } x \leq 0 \end{cases}$$

where

$$(x)_n = \frac{\Gamma(x + 1)}{\Gamma(x - n + 1)}$$

Available since 2.0

R `falling_factorial`(T1 x, T2 y)
Vectorized implementation of the `falling_factorial` function
Available since 2.25

real `lchoose`(real x, real y)
Return the natural logarithm of the generalized binomial coefficient of x and y. For

non-negative integer inputs, the binomial coefficient function is written as $\binom{x}{y}$ and pronounced "x choose y." This function generalizes to real numbers using the gamma function. For $0 \leq y \leq x$,

$$\text{binomial\_coefficient\_log}(x, y) = \log \Gamma(x + 1) - \log \Gamma(y + 1) - \log \Gamma(x - y + 1).$$

Available since 2.10

`real log_falling_factorial(real x, real n)`
Return the log of the falling factorial of x with power n defined for positive x and real n.

$$\text{log\_falling\_factorial}(x, n) = \begin{cases} \log(x)_n & \text{if } x > 0 \\ \text{error} & \text{if } x \leq 0 \end{cases}$$

Available since 2.0

`real rising_factorial(real x, int n)`
Return the rising factorial of x with power n defined for positive x and integer n.

$$\text{rising\_factorial}(x, n) = \begin{cases} x^{(n)} & \text{if } x > 0 \\ \text{error} & \text{if } x \leq 0 \end{cases}$$

where

$$x^{(n)} = \frac{\Gamma(x + n)}{\Gamma(x)}$$

Available since 2.20

`R rising_factorial(T1 x, T2 y)`
Vectorized implementation of the `rising_factorial` function
Available since 2.25

`real log_rising_factorial(real x, real n)`
Return the log of the rising factorial of x with power n defined for positive x and real n.

$$\text{log\_rising\_factorial}(x, n) = \begin{cases} \log x^{(n)} & \text{if } x > 0 \\ \text{error} & \text{if } x \leq 0 \end{cases}$$

Available since 2.0

`R log_rising_factorial(T1 x, T2 y)`
Vectorized implementation of the `log_rising_factorial` function
Available since 2.25

3.14.   Composed functions

The functions in this section are equivalent in theory to combinations of other functions. In practice, they are implemented to be more efficient and more numerically stable than defining them directly using more basic Stan functions.

R **expm1**(T x)
natural exponential of x minus 1
Available since 2.0, vectorized in 2.13

real **fma**(real x, real y, real z)
Return z plus the result of x multiplied by y.

$$\text{fma}(x, y, z) = (x \times y) + z$$

Available since 2.0

real **multiply_log**(real x, real y)
Warning: This function is deprecated and should be replaced with `lmultiply`.
Return the product of x and the natural logarithm of y.

$$\text{multiply\_log}(x, y) = \begin{cases} 0 & \text{if } x = y = 0 \\ x \log y & \text{if } x, y \neq 0 \\ \text{NaN} & \text{otherwise} \end{cases}$$

Available since 2.0, deprecated in 2.10

R **multiply_log**(T1 x, T2 y)
Vectorized implementation of the `multiply_log` function
Available since 2.25

real **ldexp**(real x, int y)
Return the product of x and two raised to the y power.

$$\text{ldexp}(x, y) = x2^y$$

Available since 2.25

R **ldexp**(T1 x, T2 y)
Vectorized implementation of the `ldexp` function
Available since 2.25

`real lmultiply(real x, real y)`
Return the product of x and the natural logarithm of y.

$$\mathrm{lmultiply}(x, y) = \begin{cases} 0 & \text{if } x = y = 0 \\ x \log y & \text{if } x, y \neq 0 \\ \text{NaN} & \text{otherwise} \end{cases}$$

Available since 2.10

`R lmultiply(T1 x, T2 y)`
Vectorized implementation of the `lmultiply` function
Available since 2.25

`R log1p(T x)`
natural logarithm of 1 plus x
Available since 2.0, vectorized in 2.13

`R log1m(T x)`
natural logarithm of 1 minus x
Available since 2.0, vectorized in 2.13

`R log1p_exp(T x)`
natural logarithm of one plus the natural exponentiation of x
Available since 2.0, vectorized in 2.13

`R log1m_exp(T x)`
logarithm of one minus the natural exponentiation of x
Available since 2.0, vectorized in 2.13

`real log_diff_exp(real x, real y)`
Return the natural logarithm of the difference of the natural exponentiation of x and
the natural exponentiation of y.

$$\mathrm{log\_diff\_exp}(x, y) = \begin{cases} \log(\exp(x) - \exp(y)) & \text{if } x > y \\ \text{NaN} & \text{otherwise} \end{cases}$$

Available since 2.0

`R log_diff_exp(T1 x, T2 y)`
Vectorized implementation of the `log_diff_exp` function
Available since 2.25

`real log_mix(real theta, real lp1, real lp2)`
Return the log mixture of the log densities lp1 and lp2 with mixing proportion theta,
defined by

$$\log\_mix(\theta, \lambda_1, \lambda_2) \;=\; \log(\theta \exp(\lambda_1) + (1 - \theta) \exp(\lambda_2))$$
$$=\; \log\_sum\_exp(\log(\theta) + \lambda_1, \; \log(1 - \theta) + \lambda_2).$$

Available since 2.6

`real log_sum_exp(real x, real y)`
Return the natural logarithm of the sum of the natural exponentiation of x and the
natural exponentiation of y.

$$\log\_sum\_exp(x, y) = \log(\exp(x) + \exp(y))$$

Available since 2.0

`R log_inv_logit(T x)`
natural logarithm of the inverse logit function of x
Available since 2.0, vectorized in 2.13

`R log_inv_logit_diff(T1 x, T2 y)`
natural logarithm of the difference of the inverse logit function of x and the inverse
logit function of y
Available since 2.25

`R log1m_inv_logit(T x)`
natural logarithm of 1 minus the inverse logit function of x
Available since 2.0, vectorized in 2.13

3.15.   Special functions

`R lambert_w0(T x)`

Implementation of the W_0 branch of the Lambert W function, i.e., solution to the
function $W_0(x) \exp^{W_0(x)} = x$
Available since 2.25

`R lambert_wm1(T x)`
Implementation of the W_{-1} branch of the Lambert W function, i.e., solution to the
function $W_{-1}(x) \exp^{W_{-1}(x)} = x$
Available since 2.25

4. Complex-Valued Basic Functions

This chapter describes built-in functions that operate on complex numbers, either as an argument type or a return type. This includes the arithmetic operators generalized to complex numbers.

4.1. Complex assignment and promotion

Just as integers may be assigned to real variables, real variables may be assigned to complex numbers, with the result being a zero imaginary component.

```
int n = 5;       // n = 5
real x = a;      // x = 5.0
complex z1 = n;  // z = 5.0 + 0.0i
complex z2 = x;  // z = 5.0 + 0.0i
```

Complex function arguments

Function arguments of type `int` or `real` may be promoted to type `complex`. The complex version of functions in this chapter are only used if one of the arguments is complex. For example, if z is complex, then `pow(z, 2)` will call the complex verison of the power function and the integer 2 will be promoted to a complex number with a real component of 2 and an imaginary component of 0. The same goes for binary operators like addition and subtraction, where `z + 2` will be legal and produce a complex result. Functions such as `arg` and `conj` that are only available for complex numbers can accept integer or real arguments, promoting them to `complex` before applying the function.

4.2. Complex constructors and accessors

Complex constructors

Variables and constants of type `complex` are constructed from zero, one, or two real numbers.

```
complex z1 = to_complex();       // z1 = 0.0 + 0.0i
real re = -2.9;
complex z2 = to_complex(re);     // z2 = -2.9 + 0.0i
```

```
real im = 1.3;
complex z3 = to_complex(re, im);   // z3 = -2.9 + 1.3i
```

complex **to_complex**()
Return complex number with real part 0.0 and imaginary part 0.0.
Available since 2.28

complex **to_complex**(real re)
Return complex number with real part `re` and imaginary part 0.0.
Available since 2.28

complex **to_complex**(real re, real im)
Return complex number with real part `re` and imaginary part `im`.
Available since 2.28

Complex accessors

Given a complex number, its real and imaginary parts can be extracted with the
following functions.

real **get_real**(complex z)
Return the real part of the complex number `z`.
Available since 2.28

real **get_imag**(complex z)
Return the imaginary part of the complex number `z`.
Available since 2.28

4.3.   Complex arithmetic operators

The arithmetic operators have the same precedence for complex and real arguments.
The complex form of an operator will be selected if at least one of its argument is of
type `complex`. If there are two arguments and only one is of type `complex`, then the
other will be promoted to type `complex` before performing the operation.

Unary operators

complex **operator+**(complex z)
Return the complex argument `z`,

$$+z = z.$$

Available since 2.28

```
complex operator-(complex z)
```
Return the negation of the complex argument z, which for $z = x + yi$ is

$$-z = -x - yi.$$

Available since 2.28

Binary operators

```
complex operator+(complex x, complex y)
```
Return the sum of x and y,

$$(x + y) = \text{operator}+(x, y) = x + y.$$

Available since 2.28

```
complex operator-(complex x, complex y)
```
Return the difference between x and y,

$$(x - y) = \text{operator-}(x, y) = x - y.$$

Available since 2.28

```
complex operator*(complex x, complex y)
```
Return the product of x and y,

$$(x * y) = \text{operator*}(x, y) = x \times y.$$

Available since 2.28

```
complex operator/(complex x, complex y)
```
Return the quotient of x and y,

$$(x/y) = \text{operator/}(x, y) = \frac{x}{y}$$

Available since 2.28

```
complex operator^(complex x, complex y)
```
Return x raised to the power of y,

$$(x^\wedge y) = \text{operator}^\wedge(x, y) = \exp(y \log(x)).$$

Available since 2.28

4.4. Complex comparison operators

Complex numbers are equal if and only if both their real and imaginary components are equal. That is, the conditional

```
z1 == z2
```

is equivalent to

```
get_real(z1) == get_real(z2) && get_imag(z1) == get_imag(z2)
```

As with other complex functions, if one of the arguments is of type `real` or `int`, it will be promoted to type `complex` before comparison. For example, if z is of type `complex`, then `z == 0` will be true if z has real component equal to 0.0 and complex component equal to 0.0.

Warning: As with real values, it is usually a mistake to compare complex numbers for equality because their parts are implemented using floating-point arithmetic, which suffers from precision errors, rendering algebraically equivalent expressions not equal after evaluation.

int **operator==**(complex x, complex y)
Return 1 if x is equal to y and 0 otherwise,

$$(x == y) \; = \; \text{operator==}(x, y) \; = \; \begin{cases} 1 & \text{if } x = y, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Available since 2.28

int **operator!=**(complex x, complex y)
Return 1 if x is not equal to y and 0 otherwise,

$$(x \, != y) \; = \; \text{operator!=}(x, y) \; = \; \begin{cases} 1 & \text{if } x \neq y, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Available since 2.28

4.5. Complex (compound) assignment operators

The assignment operator only serves as a component in the assignment statement and is thus not technically a function in the Stan language. With that caveat, it is documented here for completeness.

Assignment of complex numbers works elementwise. If an expression of type `int` or `real` is assigned to a complex number, it will be promoted before assignment as if calling `to_complex()`, so that the imaginary component is 0.0.

`void` **`operator=`**`(complex x, complex y)`
`y = x;` assigns a (copy of) the value of `y` to `x`.
Available since 2.28

`void` **`operator+=`**`(complex x, complex y)`
`x += y;` is equivalent to `x = x + y;`.
Available since 2.28

`void` **`operator-=`**`(complex x, complex y)`
`x -= y;` is equivalent to `x = x - y;`.
Available since 2.28

`void` **`operator*=`**`(complex x, complex y)`
`x *= y;` is equivalent to `x = x * y;`.
Available since 2.28

`void` **`operator/=`**`(complex x, complex y)`
`x /= y;` is equivalent to `x = x / y;`.
Available since 2.28

4.6. Complex special functions

The following functions are specific to complex numbers other than absolute value, which has a specific meaning for complex numbers.

`real` **`abs`**`(complex z)`
Return the absolute value of z, also known as the modulus or magnitude, which for $z = x + yi$ is

$$\mathrm{abs}(z) = \sqrt{x^2 + y^2}.$$

Available since 2.28

`real` **`arg`**`(complex z)`
Return the phase angle (in radians) of z, which for $z = x + yi$ is

$$\arg(z) = \mathrm{atan2}(y, x) = \mathrm{atan}(y/x).$$

Available since 2.28

`real `**`norm`**`(complex z)`
Return the Euclidean norm of z, which is its absolute value squared, and which for $z = x + yi$ is
$$\text{norm}(z) = \text{abs}^2(z) = x^2 + y^2.$$

Available since 2.28

`complex `**`conj`**`(complex z)`
Return the complex conjugate of z, which negates the imaginary component, so that if $z = x + yi$,
$$\text{conj}(z) = x - yi.$$

Available since 2.28

`complex `**`proj`**`(complex z)`
Return the projection of z onto the Riemann sphere, which for $z = x + yi$ is

$$\text{proj}(z) = \begin{cases} z & \text{if } z \text{ is finite, and} \\ 0 + \text{sign}(y)i & \text{otherwise,} \end{cases}$$

where $\text{sign}(y)$ is -1 if y is negative and 1 otherwise.
Available since 2.28

`complex `**`polar`**`(real r, real theta)`
Return the complex number with magnitude (absolute value) `r` and phase angle `theta`.
Available since 2.28

4.7. Complex exponential and power functions

The exponential, log, and power functions may be supplied with complex arguments with specialized meanings that generalize their real counterparts. These versions are only called when the argument is complex.

`complex `**`exp`**`(complex z)`
Return the complex natural exponential of z, which for $z = x + yi$ is

$$\exp z = \exp(x)\text{cis}(y) = \exp(x)(\cos(y) + i\sin(y)).$$

Available since 2.28

`complex `**`log`**`(complex z)`
Return the complex natural logarithm of z, which for $z = \text{polar}(r, \theta)$ is

$$\log z = \log r + \theta i.$$

Available since 2.28

`complex `**`log10`**`(complex z)`
Return the complex common logarithm of z,

$$\log_{10} z = \frac{\log z}{\log 10}.$$

Available since 2.28

`complex `**`pow`**`(complex x, complex y)`
Return x raised to the power of y,

$$\text{pow}(x, y) = \exp(y \, \log(x)).$$

Available since 2.28

`complex `**`sqrt`**`(complex x)`
Return the complex square root of x with branch cut along the negative real axis.
For finite inputs, the result will be in the right half-plane.
Available since 2.28

4.8. Complex trigonometric functions

The standard trigonometric functions are supported for complex numbers.

`complex `**`cos`**`(complex z)`
Return the complex cosine of z, which is

$$\cos(z) = \cosh(z \, i) = \frac{\exp(z \, i) + \exp(-z \, i)}{2}.$$

Available since 2.28

`complex `**`sin`**`(complex z)`
Return the complex sine of z,

$$\sin(z) = -\sinh(z \, i) \, i = \frac{\exp(z \, i) - \exp(-z \, i)}{2 \, i}.$$

Available since 2.28

complex **tan**(complex z)
Return the complex tangent of z,

$$\tan(z) = -\tanh(z\,i)\,i = \frac{(\exp(-z\,i) - \exp(z\,i))\,i}{\exp(-z\,i) + \exp(z\,i)}.$$

Available since 2.28

complex **acos**(complex z)
Return the complex arc (inverse) cosine of z,

$$\mathrm{acos}(z) = \frac{1}{2}\pi + \log(z\,i + \sqrt{1 - z^2})\,i.$$

Available since 2.28

complex **asin**(complex z)
Return the complex arc (inverse) sine of z,

$$\mathrm{asin}(z) = -\log(z\,i + \sqrt{1 - z^2})\,i.$$

Available since 2.28

complex **atan**(complex z)
Return the complex arc (inverse) tangent of z,

$$\mathrm{atan}(z) = -\frac{1}{2}(\log(1 - z\,i) - \log(1 + z\,i))\,i.$$

Available since 2.28

4.9.   Complex hyperbolic trigonometric functions

The standard hyperbolic trigonometric functions are supported for complex numbers.

complex **cosh**(complex z)
Return the complex hyperbolic cosine of z,

$$\cosh(z) = \frac{\exp(z) + \exp(-z)}{2}.$$

Available since 2.28

```
complex sinh(complex z)
```
Return the complex hyperbolic sine of z,

$$\sinh(z) = \frac{\exp(z) - \exp(-z)}{2}.$$

Available since 2.28

```
complex tanh(complex z)
```
Return the complex hyperbolic tangent of z,

$$\tanh(z) \;=\; \frac{\sinh(z)}{\cosh(z)} \;=\; \frac{\exp(z) - \exp(-z)}{\exp(z) + \exp(-z)}.$$

Available since 2.28

```
complex acosh(complex z)
```
Return the complex hyperbolic arc (inverse) cosine of z,

$$\mathrm{acosh}(z) = \log(z + \sqrt{(z + 1)(z - 1)}).$$

Available since 2.28

```
complex asinh(complex z)
```
Return the complex hyperbolic arc (inverse) sine of z,

$$\mathrm{asinh}(z) = \log(z + \sqrt{1 + z^2}).$$

Available since 2.28

```
complex atanh(complex z)
```
Return the complex hyperbolic arc (inverse) tangent of z,

$$\mathrm{atanh}(z) = \frac{\log(1 + z) - \log(1 - z)}{2}.$$

Available since 2.28

5.   Array Operations

5.1.   Reductions

The following operations take arrays as input and produce single output values. The boundary values for size 0 arrays are the unit with respect to the combination operation (min, max, sum, or product).

Minimum and maximum

`real min(array[] real x)`
The minimum value in x, or $+\infty$ if x is size 0.
Available since 2.0

`int min(array[] int x)`
The minimum value in x, or error if x is size 0.
Available since 2.0

`real max(array[] real x)`
The maximum value in x, or $-\infty$ if x is size 0.
Available since 2.0

`int max(array[] int x)`
The maximum value in x, or error if x is size 0.
Available since 2.0

Sum, product, and log sum of exp

`int sum(array[] int x)`
The sum of the elements in x, defined for x of size N by

$$\mathrm{sum}(x) = \begin{cases} \sum_{n=1}^{N} x_n & \text{if } N > 0 \\ 0 & \text{if } N = 0 \end{cases}$$

Available since 2.1

`real sum(array[] real x)`
The sum of the elements in x; see definition above.
Available since 2.0

```
real prod(array[] real x)
```
The product of the elements in x, or 1 if x is size 0.
Available since 2.0

```
real prod(array[] int x)
```
The product of the elements in x,

$$
\text{product}(x) = \begin{cases} \prod_{n=1}^{N} x_n & \text{if } N > 0 \\ 1 & \text{if } N = 0 \end{cases}
$$

Available since 2.0

```
real log_sum_exp(array[] real x)
```
The natural logarithm of the sum of the exponentials of the elements in x, or $-\infty$ if the array is empty.
Available since 2.0

Sample mean, variance, and standard deviation

The sample mean, variance, and standard deviation are calculated in the usual way. For i.i.d. draws from a distribution of finite mean, the sample mean is an unbiased estimate of the mean of the distribution. Similarly, for i.i.d. draws from a distribution of finite variance, the sample variance is an unbiased estimate of the variance.[1] The sample deviation is defined as the square root of the sample deviation, but is not unbiased.

```
real mean(array[] real x)
```
The sample mean of the elements in x. For an array x of size $N > 0$,

$$
\text{mean}(x) \;=\; \bar{x} \;=\; \frac{1}{N} \sum_{n=1}^{N} x_n.
$$

It is an error to the call the mean function with an array of size 0.
Available since 2.0

```
real variance(array[] real x)
```
The sample variance of the elements in x. For $N > 0$,

$$
\text{variance}(x) \;=\; \begin{cases} \frac{1}{N-1} \sum_{n=1}^{N} (x_n - \bar{x})^2 & \text{if } N > 1 \\ 0 & \text{if } N = 1 \end{cases}
$$

---

[1]Dividing by N rather than $(N-1)$ produces a maximum likelihood estimate of variance, which is biased to underestimate variance.

It is an error to call the `variance` function with an array of size 0.
Available since 2.0

`real sd(array[] real x)`
The sample standard deviation of elements in x.

$$\mathrm{sd}(x) = \begin{cases} \sqrt{\mathrm{variance}(x)} & \text{if } N > 1 \\ 0 & \text{if } N = 0 \end{cases}$$

It is an error to call the `sd` function with an array of size 0.
Available since 2.0

Euclidean distance and squared distance

`real distance(vector x, vector y)`
The Euclidean distance between x and y, defined by

$$\mathrm{distance}(x, y) = \sqrt{\sum_{n=1}^{N}(x_n - y_n)^2}$$

where N is the size of x and y. It is an error to call `distance` with arguments of unequal size.
Available since 2.2

`real distance(vector x, row_vector y)`
The Euclidean distance between x and y
Available since 2.2

`real distance(row_vector x, vector y)`
The Euclidean distance between x and y
Available since 2.2

`real distance(row_vector x, row_vector y)`
The Euclidean distance between x and y
Available since 2.2

`real squared_distance(vector x, vector y)`
The squared Euclidean distance between x and y, defined by

$$\mathrm{squared\_distance}(x, y) = \mathrm{distance}(x, y)^2 = \sum_{n=1}^{N}(x_n - y_n)^2,$$

where N is the size of x and y. It is an error to call `squared_distance` with arguments of unequal size.
Available since 2.7

real **squared_distance**(vector x, row_vector y)
The squared Euclidean distance between x and y
Available since 2.26

real **squared_distance**(row_vector x, vector y)
The squared Euclidean distance between x and y
Available since 2.26

real **squared_distance**(row_vector x, row_vector y)
The Euclidean distance between x and y
Available since 2.26

Quantile

Produces sample quantiles corresponding to the given probabilities. The smallest observation corresponds to a probability of 0 and the largest to a probability of 1.

Implements algorithm 7 from Hyndman, R. J. and Fan, Y., Sample quantiles in Statistical Packages (R's default quantile function).

real **quantile**(data array[] real x, data real p)
The p-th quantile of x
Available since 2.27

array[] real **quantile**(data array[] real x, data array[] real p)
An array containing the quantiles of x given by the array of probabilities p
Available since 2.27

5.2. Array size and dimension function

The size of an array or matrix can be obtained using the `dims()` function. The `dims()` function is defined to take an argument consisting of any variable with up to 8 array dimensions (and up to 2 additional matrix dimensions) and returns an array of integers with the dimensions. For example, if two variables are declared as follows,

```
array[7, 8, 9] real x;
array[7] matrix[8, 9] y;
```

then calling `dims(x)` or `dims(y)` returns an integer array of size 3 containing the elements 7, 8, and 9 in that order.

The `size()` function extracts the number of elements in an array. This is just the top-level elements, so if the array is declared as

```
array[M, N] real a;
```

the size of a is M.

The function `num_elements`, on the other hand, measures all of the elements, so that the array a above has $M \times N$ elements.

The specialized functions `rows()` and `cols()` should be used to extract the dimensions of vectors and matrices.

`array[] int dims(T x)`
Return an integer array containing the dimensions of x; the type of the argument T can be any Stan type with up to 8 array dimensions.
Available since 2.0

`int num_elements(array[] T x)`
Return the total number of elements in the array x including all elements in contained arrays, vectors, and matrices. T can be any array type. For example, if x is of type `array[4, 3] real` then `num_elements(x)` is 12, and if y is declared as `array[5] matrix[3, 4] y`, then `size(y)` evaluates to 60.
Available since 2.5

`int size(array[] T x)`
Return the number of elements in the array x; the type of the array T can be any type, but the size is just the size of the top level array, not the total number of elements contained. For example, if x is of type `array[4, 3] real` then `size(x)` is 4.
Available since 2.0

5.3. Array broadcasting

The following operations create arrays by repeating elements to fill an array of a specified size. These operations work for all input types T, including reals, integers, vectors, row vectors, matrices, or arrays.

`array[] T rep_array(T x, int n)`
Return the n array with every entry assigned to x.
Available since 2.0

`array [,] T rep_array(T x, int m, int n)`
Return the m by n array with every entry assigned to x.
Available since 2.0

`array[„] T rep_array(T x, int k, int m, int n)`

Return the k by m by n array with every entry assigned to x.
Available since 2.0

For example, `rep_array(1.0,5)` produces a real array (type `array[] real`) of size
5 with all values set to 1.0. On the other hand, `rep_array(1,5)` produces an
integer array (type `array[] int`) of size 5 with all values set to 1. This distinction
is important because it is not possible to assign an integer array to a real array.
For example, the following example contrasts legal with illegal array creation and
assignment

```
array[5] real y;
array[5] int x;

x = rep_array(1, 5);     // ok
y = rep_array(1.0, 5);   // ok

x = rep_array(1.0, 5);   // illegal
y = rep_array(1, 5);     // illegal

x = y;                   // illegal
y = x;                   // illegal
```

If the value being repeated v is a vector (i.e., T is `vector`), then `rep_array(v, 27)`
is a size 27 array consisting of 27 copies of the vector v.

```
vector[5] v;
array[3] vector[5] a;

a = rep_array(v, 3);  // fill a with copies of v
a[2, 4] = 9.0;        // v[4], a[1, 4], a[3, 4] unchanged
```

If the type T of x is itself an array type, then the result will be an array with one,
two, or three added dimensions, depending on which of the `rep_array` functions is
called. For instance, consider the following legal code snippet.

```
array[5, 6] real a;
array[3, 4, 5, 6] real b;

b = rep_array(a, 3, 4); //  make (3 x 4) copies of a
b[1, 1, 1, 1] = 27.9;   //  a[1, 1] unchanged
```

After the assignment to b, the value for b[j, k, m, n] is equal to a[m, n] where it

is defined, for j in 1:3, k in 1:4, m in 1:5, and n in 1:6.

5.4.   Array concatenation

T `append_array`(T x, T y)
Return the concatenation of two arrays in the order of the arguments. T must be an
N-dimensional array of any Stan type (with a maximum N of 7). All dimensions but
the first must match.
Available since 2.18

For example, the following code appends two three dimensional arrays of matrices
together. Note that all dimensions except the first match. Any mismatches will cause
an error to be thrown.

```
array[2, 1, 7] matrix[4, 6] x1;
array[3, 1, 7] matrix[4, 6] x2;
array[5, 1, 7] matrix[4, 6] x3;

x3 = append_array(x1, x2);
```

5.5.   Sorting functions

Sorting can be used to sort values or the indices of those values in either ascending
or descending order. For example, if v is declared as a real array of size 3, with
values

$$v = (1, -10.3, 20.987),$$

then the various sort routines produce

$$
\begin{aligned}
\text{sort\_asc}(v) &= (-10.3, 1, 20.987) \\
\text{sort\_desc}(v) &= (20.987, 1, -10.3) \\
\text{sort\_indices\_asc}(v) &= (2, 1, 3) \\
\text{sort\_indices\_desc}(v) &= (3, 1, 2)
\end{aligned}
$$

array[] real **sort_asc**(array[] real v)
Sort the elements of v in ascending order
Available since 2.0

array[] int **sort_asc**(array[] int v)
Sort the elements of v in ascending order
Available since 2.0

```
array[] real sort_desc(array[] real v)
```
Sort the elements of v in descending order
Available since 2.0

```
array[] int sort_desc(array[] int v)
```
Sort the elements of v in descending order
Available since 2.0

```
array[] int sort_indices_asc(array[] real v)
```
Return an array of indices between 1 and the size of v, sorted to index v in ascending order.
Available since 2.3

```
array[] int sort_indices_asc(array[] int v)
```
Return an array of indices between 1 and the size of v, sorted to index v in ascending order.
Available since 2.3

```
array[] int sort_indices_desc(array[] real v)
```
Return an array of indices between 1 and the size of v, sorted to index v in descending order.
Available since 2.3

```
array[] int sort_indices_desc(array[] int v)
```
Return an array of indices between 1 and the size of v, sorted to index v in descending order.
Available since 2.3

```
int rank(array[] real v, int s)
```
Number of components of v less than v[s]
Available since 2.0

```
int rank(array[] int v, int s)
```
Number of components of v less than v[s]
Available since 2.0

5.6.   Reversing functions

Stan provides functions to create a new array by reversing the order of elements in an existing array. For example, if v is declared as a real array of size 3, with values

$$v = (1, -10.3, 20.987),$$

then

$$\text{reverse}(v) = (20.987, \, -10.3, \, 1).$$

`array[] T reverse(array[] T v)`
Return a new array containing the elements of the argument in reverse order.
Available since 2.23

6. Matrix Operations

6.1. Integer-valued matrix size functions

`int num_elements(vector x)`
The total number of elements in the vector x (same as function `rows`)
Available since 2.5

`int num_elements(row_vector x)`
The total number of elements in the vector x (same as function `cols`)
Available since 2.5

`int num_elements(matrix x)`
The total number of elements in the matrix x. For example, if x is a 5×3 matrix, then `num_elements(x)` is 15
Available since 2.5

`int rows(vector x)`
The number of rows in the vector x
Available since 2.0

`int rows(row_vector x)`
The number of rows in the row vector x, namely 1
Available since 2.0

`int rows(matrix x)`
The number of rows in the matrix x
Available since 2.0

`int cols(vector x)`
The number of columns in the vector x, namely 1
Available since 2.0

`int cols(row_vector x)`
The number of columns in the row vector x
Available since 2.0

`int cols(matrix x)`
The number of columns in the matrix x
Available since 2.0

```
int size(vector x)
```
The size of x, i.e., the number of elements
Available since 2.26

```
int size(row_vector x)
```
The size of x, i.e., the number of elements
Available since 2.26

```
int size(matrix x)
```
The size of the matrix x. For example, if x is a 5×3 matrix, then `size(x)` is 15
Available since 2.26

6.2.   Matrix arithmetic operators

Stan supports the basic matrix operations using infix, prefix and postfix operations.
This section lists the operations supported by Stan along with their argument and
result types.

Negation prefix operators

```
vector operator-(vector x)
```
The negation of the vector x.
Available since 2.0

```
row_vector operator-(row_vector x)
```
The negation of the row vector x.
Available since 2.0

```
matrix operator-(matrix x)
```
The negation of the matrix x.
Available since 2.0

Infix matrix operators

```
vector operator+(vector x, vector y)
```
The sum of the vectors x and y.
Available since 2.0

```
row_vector operator+(row_vector x, row_vector y)
```
The sum of the row vectors x and y.
Available since 2.0

```
matrix operator+(matrix x, matrix y)
```
The sum of the matrices x and y

Available since 2.0

`vector` **`operator-`**`(vector x, vector y)`
The difference between the vectors x and y.
Available since 2.0

`row_vector` **`operator-`**`(row_vector x, row_vector y)`
The difference between the row vectors x and y
Available since 2.0

`matrix` **`operator-`**`(matrix x, matrix y)`
The difference between the matrices x and y
Available since 2.0

`vector` **`operator*`**`(real x, vector y)`
The product of the scalar x and vector y
Available since 2.0

`row_vector` **`operator*`**`(real x, row_vector y)`
The product of the scalar x and the row vector y
Available since 2.0

`matrix` **`operator*`**`(real x, matrix y)`
The product of the scalar x and the matrix y
Available since 2.0

`vector` **`operator*`**`(vector x, real y)`
The product of the scalar y and vector x
Available since 2.0

`matrix` **`operator*`**`(vector x, row_vector y)`
The product of the vector x and row vector y
Available since 2.0

`row_vector` **`operator*`**`(row_vector x, real y)`
The product of the scalar y and row vector x
Available since 2.0

`real` **`operator*`**`(row_vector x, vector y)`
The product of the row vector x and vector y
Available since 2.0

`row_vector` **`operator*`**`(row_vector x, matrix y)`
The product of the row vector x and matrix y
Available since 2.0

`matrix` **`operator*`**`(matrix x, real y)`
The product of the scalar y and matrix x
Available since 2.0

`vector` **`operator*`**`(matrix x, vector y)`
The product of the matrix x and vector y
Available since 2.0

`matrix` **`operator*`**`(matrix x, matrix y)`
The product of the matrices x and y
Available since 2.0

Broadcast infix operators

`vector` **`operator+`**`(vector x, real y)`
The result of adding y to every entry in the vector x
Available since 2.0

`vector` **`operator+`**`(real x, vector y)`
The result of adding x to every entry in the vector y
Available since 2.0

`row_vector` **`operator+`**`(row_vector x, real y)`
The result of adding y to every entry in the row vector x
Available since 2.0

`row_vector` **`operator+`**`(real x, row_vector y)`
The result of adding x to every entry in the row vector y
Available since 2.0

`matrix` **`operator+`**`(matrix x, real y)`
The result of adding y to every entry in the matrix x
Available since 2.0

`matrix` **`operator+`**`(real x, matrix y)`
The result of adding x to every entry in the matrix y
Available since 2.0

`vector` **`operator-`**`(vector x, real y)`
The result of subtracting y from every entry in the vector x
Available since 2.0

`vector` **`operator-`**`(real x, vector y)`
The result of adding x to every entry in the negation of the vector y

Available since 2.0

`row_vector` **operator-**`(row_vector x, real y)`
The result of subtracting y from every entry in the row vector x
Available since 2.0

`row_vector` **operator-**`(real x, row_vector y)`
The result of adding x to every entry in the negation of the row vector y
Available since 2.0

`matrix` **operator-**`(matrix x, real y)`
The result of subtracting y from every entry in the matrix x
Available since 2.0

`matrix` **operator-**`(real x, matrix y)`
The result of adding x to every entry in negation of the matrix y
Available since 2.0

`vector` **operator/**`(vector x, real y)`
The result of dividing each entry in the vector x by y
Available since 2.0

`row_vector` **operator/**`(row_vector x, real y)`
The result of dividing each entry in the row vector x by y
Available since 2.0

`matrix` **operator/**`(matrix x, real y)`
The result of dividing each entry in the matrix x by y
Available since 2.0

6.3. Transposition operator

Matrix transposition is represented using a postfix operator.

`matrix` **operator'**`(matrix x)`
The transpose of the matrix x, written as x'
Available since 2.0

`row_vector` **operator'**`(vector x)`
The transpose of the vector x, written as x'
Available since 2.0

`vector` **operator'**`(row_vector x)`
The transpose of the row vector x, written as x'
Available since 2.0

6.4.   Elementwise functions

Elementwise functions apply a function to each element of a vector or matrix, returning a result of the same shape as the argument. There are many functions that are vectorized in addition to the ad hoc cases listed in this section; see section function vectorization for the general cases.

vector **operator.***(vector x, vector y)
The elementwise product of y and x
Available since 2.0

row_vector **operator.***(row_vector x, row_vector y)
The elementwise product of y and x
Available since 2.0

matrix **operator.***(matrix x, matrix y)
The elementwise product of y and x
Available since 2.0

vector **operator./**(vector x, vector y)
The elementwise quotient of y and x
Available since 2.0

vector **operator./**(vector x, real y)
The elementwise quotient of y and x
Available since 2.4

vector **operator./**(real x, vector y)
The elementwise quotient of y and x
Available since 2.4

row_vector **operator./**(row_vector x, row_vector y)
The elementwise quotient of y and x
Available since 2.0

row_vector **operator./**(row_vector x, real y)
The elementwise quotient of y and x
Available since 2.4

row_vector **operator./**(real x, row_vector y)
The elementwise quotient of y and x
Available since 2.4

matrix **operator./**(matrix x, matrix y)
The elementwise quotient of y and x

Available since 2.0

`matrix` **operator./**`(matrix x, real y)`
The elementwise quotient of y and x
Available since 2.4

`matrix` **operator./**`(real x, matrix y)`
The elementwise quotient of y and x
Available since 2.4

`vector` **operator.^**`(vector x, vector y)`
The elementwise power of y and x
Available since 2.24

`vector` **operator.^**`(vector x, real y)`
The elementwise power of y and x
Available since 2.24

`vector` **operator.^**`(real x, vector y)`
The elementwise power of y and x
Available since 2.24

`row_vector` **operator.^**`(row_vector x, row_vector y)`
The elementwise power of y and x
Available since 2.24

`row_vector` **operator.^**`(row_vector x, real y)`
The elementwise power of y and x
Available since 2.24

`row_vector` **operator.^**`(real x, row_vector y)`
The elementwise power of y and x
Available since 2.24

`matrix` **operator.^**`(matrix x, matrix y)`
The elementwise power of y and x
Available since 2.24

`matrix` **operator.^**`(matrix x, real y)`
The elementwise power of y and x
Available since 2.24

`matrix` **operator.^**`(real x, matrix y)`
The elementwise power of y and x
Available since 2.24

6.5. Dot products and specialized products

`real` **`dot_product`**`(vector x, vector y)`
The dot product of x and y
Available since 2.0

`real` **`dot_product`**`(vector x, row_vector y)`
The dot product of x and y
Available since 2.0

`real` **`dot_product`**`(row_vector x, vector y)`
The dot product of x and y
Available since 2.0

`real` **`dot_product`**`(row_vector x, row_vector y)`
The dot product of x and y
Available since 2.0

`row_vector` **`columns_dot_product`**`(vector x, vector y)`
The dot product of the columns of x and y
Available since 2.0

`row_vector` **`columns_dot_product`**`(row_vector x, row_vector y)`
The dot product of the columns of x and y
Available since 2.0

`row_vector` **`columns_dot_product`**`(matrix x, matrix y)`
The dot product of the columns of x and y
Available since 2.0

`vector` **`rows_dot_product`**`(vector x, vector y)`
The dot product of the rows of x and y
Available since 2.0

`vector` **`rows_dot_product`**`(row_vector x, row_vector y)`
The dot product of the rows of x and y
Available since 2.0

`vector` **`rows_dot_product`**`(matrix x, matrix y)`
The dot product of the rows of x and y
Available since 2.0

`real` **`dot_self`**`(vector x)`
The dot product of the vector x with itself
Available since 2.0

```
real dot_self(row_vector x)
```
The dot product of the row vector x with itself
Available since 2.0

```
row_vector columns_dot_self(vector x)
```
The dot product of the columns of x with themselves
Available since 2.0

```
row_vector columns_dot_self(row_vector x)
```
The dot product of the columns of x with themselves
Available since 2.0

```
row_vector columns_dot_self(matrix x)
```
The dot product of the columns of x with themselves
Available since 2.0

```
vector rows_dot_self(vector x)
```
The dot product of the rows of x with themselves
Available since 2.0

```
vector rows_dot_self(row_vector x)
```
The dot product of the rows of x with themselves
Available since 2.0

```
vector rows_dot_self(matrix x)
```
The dot product of the rows of x with themselves
Available since 2.0

Specialized products

```
matrix tcrossprod(matrix x)
```
The product of x postmultiplied by its own transpose, similar to the tcrossprod(x) function in R. The result is a symmetric matrix $x\,x^\top$.
Available since 2.0

```
matrix crossprod(matrix x)
```
The product of x premultiplied by its own transpose, similar to the crossprod(x) function in R. The result is a symmetric matrix $x^\top x$.
Available since 2.0

The following functions all provide shorthand forms for common expressions, which are also much more efficient.

```
matrix quad_form(matrix A, matrix B)
```

The quadratic form, i.e., `B' * A * B`.
Available since 2.0

real **quad_form**(matrix A, vector B)
The quadratic form, i.e., `B' * A * B`.
Available since 2.0

matrix **quad_form_diag**(matrix m, vector v)
The quadratic form using the column vector v as a diagonal matrix, i.e.,
`diag_matrix(v) * m * diag_matrix(v)`.
Available since 2.3

matrix **quad_form_diag**(matrix m, row_vector rv)
The quadratic form using the row vector rv as a diagonal matrix, i.e.,
`diag_matrix(rv) * m * diag_matrix(rv)`.
Available since 2.3

matrix **quad_form_sym**(matrix A, matrix B)
Similarly to quad_form, gives `B' * A * B`, but additionally checks if A is symmetric
and ensures that the result is also symmetric.
Available since 2.3

real **quad_form_sym**(matrix A, vector B)
Similarly to quad_form, gives `B' * A * B`, but additionally checks if A is symmetric
and ensures that the result is also symmetric.
Available since 2.3

real **trace_quad_form**(matrix A, matrix B)
The trace of the quadratic form, i.e., `trace(B' * A * B)`.
Available since 2.0

real **trace_gen_quad_form**(matrix D,matrix A, matrix B)
The trace of a generalized quadratic form, i.e., `trace(D * B' * A * B)`.
Available since 2.0

matrix **multiply_lower_tri_self_transpose**(matrix x)
The product of the lower triangular portion of x (including the diagonal) times its
own transpose; that is, if L is a matrix of the same dimensions as x with L(m,n) equal
to x(m,n) for $n \leq m$ and L(m,n) equal to 0 if $n > m$, the result is the symmetric
matrix $LL^\top$. This is a specialization of tcrossprod(x) for lower-triangular matrices.
The input matrix does not need to be square.
Available since 2.0

matrix **diag_pre_multiply**(vector v, matrix m)

Return the product of the diagonal matrix formed from the vector v and the matrix
m, i.e., `diag_matrix(v) * m`.
Available since 2.0

`matrix` **`diag_pre_multiply`**`(row_vector rv, matrix m)`
Return the product of the diagonal matrix formed from the vector rv and the matrix
m, i.e., `diag_matrix(rv) * m`.
Available since 2.0

`matrix` **`diag_post_multiply`**`(matrix m, vector v)`
Return the product of the matrix m and the diagonal matrix formed from the vector
v, i.e., `m * diag_matrix(v)`.
Available since 2.0

`matrix` **`diag_post_multiply`**`(matrix m, row_vector rv)`
Return the product of the matrix m and the diagonal matrix formed from the the row
vector rv, i.e., `m * diag_matrix(rv)`.
Available since 2.0

6.6. Reductions

Log sum of exponents

`real` **`log_sum_exp`**`(vector x)`
The natural logarithm of the sum of the exponentials of the elements in x
Available since 2.0

`real` **`log_sum_exp`**`(row_vector x)`
The natural logarithm of the sum of the exponentials of the elements in x
Available since 2.0

`real` **`log_sum_exp`**`(matrix x)`
The natural logarithm of the sum of the exponentials of the elements in x
Available since 2.0

Minimum and maximum

`real` **`min`**`(vector x)`
The minimum value in x, or $+\infty$ if x is empty
Available since 2.0

`real` **`min`**`(row_vector x)`
The minimum value in x, or $+\infty$ if x is empty
Available since 2.0

```
real min(matrix x)
```
The minimum value in x, or $+\infty$ if x is empty
Available since 2.0

```
real max(vector x)
```
The maximum value in x, or $-\infty$ if x is empty
Available since 2.0

```
real max(row_vector x)
```
The maximum value in x, or $-\infty$ if x is empty
Available since 2.0

```
real max(matrix x)
```
The maximum value in x, or $-\infty$ if x is empty
Available since 2.0

Sums and products

```
real sum(vector x)
```
The sum of the values in x, or 0 if x is empty
Available since 2.0

```
real sum(row_vector x)
```
The sum of the values in x, or 0 if x is empty
Available since 2.0

```
real sum(matrix x)
```
The sum of the values in x, or 0 if x is empty
Available since 2.0

```
real prod(vector x)
```
The product of the values in x, or 1 if x is empty
Available since 2.0

```
real prod(row_vector x)
```
The product of the values in x, or 1 if x is empty
Available since 2.0

```
real prod(matrix x)
```
The product of the values in x, or 1 if x is empty
Available since 2.0

Sample moments

Full definitions are provided for sample moments in section array reductions.

real **mean**(vector x)
The sample mean of the values in x; see section array reductions for details.
Available since 2.0

real **mean**(row_vector x)
The sample mean of the values in x; see section array reductions for details.
Available since 2.0

real **mean**(matrix x)
The sample mean of the values in x; see section array reductions for details.
Available since 2.0

real **variance**(vector x)
The sample variance of the values in x; see section array reductions for details.
Available since 2.0

real **variance**(row_vector x)
The sample variance of the values in x; see section array reductions for details.
Available since 2.0

real **variance**(matrix x)
The sample variance of the values in x; see section array reductions for details.
Available since 2.0

real **sd**(vector x)
The sample standard deviation of the values in x; see section array reductions for
details.
Available since 2.0

real **sd**(row_vector x)
The sample standard deviation of the values in x; see section array reductions for
details.
Available since 2.0

real **sd**(matrix x)
The sample standard deviation of the values in x; see section array reductions for
details.
Available since 2.0

Quantile

Produces sample quantiles corresponding to the given probabilities. The smallest observation corresponds to a probability of 0 and the largest to a probability of 1.

Implements algorithm 7 from Hyndman, R. J. and Fan, Y., Sample quantiles in Statistical Packages (R's default quantile function).

`real` **`quantile`**`(data vector x, data real p)`
The p-th quantile of x
Available since 2.27

`array[] real` **`quantile`**`(data vector x, data array[] real p)`
An array containing the quantiles of x given by the array of probabilities p
Available since 2.27

`real` **`quantile`**`(data row_vector x, data real p)`
The p-th quantile of x
Available since 2.27

`array[] real` **`quantile`**`(data row_vector x, data array[] real p)`
An array containing the quantiles of x given by the array of probabilities p
Available since 2.27

6.7.   Broadcast functions

The following broadcast functions allow vectors, row vectors and matrices to be created by copying a single element into all of their cells. Matrices may also be created by stacking copies of row vectors vertically or stacking copies of column vectors horizontally.

`vector` **`rep_vector`**`(real x, int m)`
Return the size m (column) vector consisting of copies of x.
Available since 2.0

`row_vector` **`rep_row_vector`**`(real x, int n)`
Return the size n row vector consisting of copies of x.
Available since 2.0

`matrix` **`rep_matrix`**`(real x, int m, int n)`
Return the m by n matrix consisting of copies of x.
Available since 2.0

`matrix` **`rep_matrix`**`(vector v, int n)`
Return the m by n matrix consisting of n copies of the (column) vector v of size m.

Available since 2.0

matrix **rep_matrix**(row_vector rv, int m)
Return the m by n matrix consisting of m copies of the row vector rv of size n.
Available since 2.0

Unlike the situation with array broadcasting (see section array broadcasting), where there is a distinction between integer and real arguments, the following two statements produce the same result for vector broadcasting; row vector and matrix broadcasting behave similarly.

```
vector[3] x;
x = rep_vector(1, 3);
x = rep_vector(1.0, 3);
```

There are no integer vector or matrix types, so integer values are automatically promoted.

Symmetrization

matrix **symmetrize_from_lower_tri**(matrix A)

Construct a symmetric matrix from the lower triangle of A.
Available since 2.26

6.8. Diagonal matrix functions

matrix **add_diag**(matrix m, row_vector d)
Add row_vector d to the diagonal of matrix m.
Available since 2.21

matrix **add_diag**(matrix m, vector d)
Add vector d to the diagonal of matrix m.
Available since 2.21

matrix **add_diag**(matrix m, real d)
Add scalar d to every diagonal element of matrix m.
Available since 2.21

vector **diagonal**(matrix x)
The diagonal of the matrix x
Available since 2.0

matrix **diag_matrix**(vector x)
The diagonal matrix with diagonal x
Available since 2.0

Although the diag_matrix function is available, it is unlikely to ever show up in an efficient Stan program. For example, rather than converting a diagonal to a full matrix for use as a covariance matrix,

```
y ~ multi_normal(mu, diag_matrix(square(sigma)));
```

it is much more efficient to just use a univariate normal, which produces the same density,

```
y ~ normal(mu, sigma);
```

Rather than writing m * diag_matrix(v) where m is a matrix and v is a vector, it is much more efficient to write diag_post_multiply(m, v) (and similarly for pre-multiplication). By the same token, it is better to use quad_form_diag(m, v) rather than quad_form(m, diag_matrix(v)).

matrix **identity_matrix**(int k)
Create an identity matrix of size $k \times k$
Available since 2.26

6.9.   Container construction functions

array[] real **linspaced_array**(int n, data real lower, data real upper)
Create a real array of length n of equidistantly-spaced elements between lower and upper
Available since 2.24

array[] real **linspaced_int_array**(int n, int lower, int upper)
Create a regularly spaced, increasing integer array of length n between lower and upper, inclusively. If (upper - lower) / (n - 1) is less than one, repeat each output (n - 1) / (upper - lower) times. If neither (upper - lower) / (n - 1) or (n - 1) / (upper - lower) are integers, upper is reduced until one of these is true.
Available since 2.26

vector **linspaced_vector**(int n, data real lower, data real upper)
Create an n-dimensional vector of equidistantly-spaced elements between lower and upper
Available since 2.24

`row_vector` **`linspaced_row_vector`**`(int n, data real lower, data real upper)`
Create an n-dimensional row-vector of equidistantly-spaced elements between `lower` and upper
Available since 2.24

`array[] int` **`one_hot_int_array`**`(int n, int k)`
Create a one-hot encoded int array of length n with `array[k] = 1`
Available since 2.26

`array[] real` **`one_hot_array`**`(int n, int k)`
Create a one-hot encoded real array of length n with `array[k] = 1`
Available since 2.24

`vector` **`one_hot_vector`**`(int n, int k)`
Create an n-dimensional one-hot encoded vector with `vector[k] = 1`
Available since 2.24

`row_vector` **`one_hot_row_vector`**`(int n, int k)`
Create an n-dimensional one-hot encoded row-vector with `row_vector[k] = 1`
Available since 2.24

`array[] int` **`ones_int_array`**`(int n)`
Create an int array of length n of all ones
Available since 2.26

`array[] real` **`ones_array`**`(int n)`
Create a real array of length n of all ones
Available since 2.26

`vector` **`ones_vector`**`(int n)`
Create an n-dimensional vector of all ones
Available since 2.26

`row_vector` **`ones_row_vector`**`(int n)`
Create an n-dimensional row-vector of all ones
Available since 2.26

`array[] int` **`zeros_int_array`**`(int n)`
Create an int array of length n of all zeros
Available since 2.26

`array[] real` **`zeros_array`**`(int n)`
Create a real array of length n of all zeros

Available since 2.24

vector **zeros_vector**(int n)
Create an n-dimensional vector of all zeros
Available since 2.24

row_vector **zeros_row_vector**(int n)
Create an n-dimensional row-vector of all zeros
Available since 2.24

vector **uniform_simplex**(int n)
Create an n-dimensional simplex with elements vector[i] = 1 / n for all $i \in 1, \ldots, n$
Available since 2.24

6.10.   Slicing and blocking functions

Stan provides several functions for generating slices or blocks or diagonal entries for matrices.

Columns and rows

vector **col**(matrix x, int n)
The n-th column of matrix x
Available since 2.0

row_vector **row**(matrix x, int m)
The m-th row of matrix x
Available since 2.0

The row function is special in that it may be used as an lvalue in an assignment statement (i.e., something to which a value may be assigned). The row function is also special in that the indexing notation x[m] is just an alternative way of writing row(x,m). The col function may **not**, be used as an lvalue, nor is there an indexing based shorthand for it.

Block operations

Matrix slicing operations

Block operations may be used to extract a sub-block of a matrix.

matrix **block**(matrix x, int i, int j, int n_rows, int n_cols)
Return the submatrix of x that starts at row i and column j and extends n_rows rows

and n_cols columns.
Available since 2.0

The sub-row and sub-column operations may be used to extract a slice of row or
column from a matrix

`vector` **`sub_col`**`(matrix x, int i, int j, int n_rows)`
Return the sub-column of x that starts at row i and column j and extends n_rows
rows and 1 column.
Available since 2.0

`row_vector` **`sub_row`**`(matrix x, int i, int j, int n_cols)`
Return the sub-row of x that starts at row i and column j and extends 1 row and
n_cols columns.
Available since 2.0

Vector and array slicing operations

The head operation extracts the first n elements of a vector and the tail operation
the last. The segment operation extracts an arbitrary subvector.

`vector` **`head`**`(vector v, int n)`
Return the vector consisting of the first n elements of v.
Available since 2.0

`row_vector` **`head`**`(row_vector rv, int n)`
Return the row vector consisting of the first n elements of rv.
Available since 2.0

`array[] T` **`head`**`(array[] T sv, int n)`
Return the array consisting of the first n elements of sv; applies to up to three-
dimensional arrays containing any type of elements `T`.
Available since 2.0

`vector` **`tail`**`(vector v, int n)`
Return the vector consisting of the last n elements of v.
Available since 2.0

`row_vector` **`tail`**`(row_vector rv, int n)`
Return the row vector consisting of the last n elements of rv.
Available since 2.0

`array[] T tail(array[] T sv, int n)`
Return the array consisting of the last n elements of sv; applies to up to three-dimensional arrays containing any type of elements T.
Available since 2.0

`vector segment(vector v, int i, int n)`
Return the vector consisting of the n elements of v starting at i; i.e., elements i through through i + n - 1.
Available since 2.0

`row_vector segment(row_vector rv, int i, int n)`
Return the row vector consisting of the n elements of rv starting at i; i.e., elements i through through i + n - 1.
Available since 2.10

`array[] T segment(array[] T sv, int i, int n)`
Return the array consisting of the n elements of sv starting at i; i.e., elements i through through i + n - 1. Applies to up to three-dimensional arrays containing any type of elements T.
Available since 2.0

6.11. Matrix concatenation

Stan's matrix concatenation operations `append_col` and `append_row` are like the operations `cbind` and `rbind` in R.

Horizontal concatenation

`matrix append_col(matrix x, matrix y)`
Combine matrices x and y by columns. The matrices must have the same number of rows.
Available since 2.5

`matrix append_col(matrix x, vector y)`
Combine matrix x and vector y by columns. The matrix and the vector must have the same number of rows.
Available since 2.5

`matrix append_col(vector x, matrix y)`
Combine vector x and matrix y by columns. The vector and the matrix must have the same number of rows.
Available since 2.5

```
matrix append_col(vector x, vector y)
```
Combine vectors x and y by columns. The vectors must have the same number of rows.
Available since 2.5

```
row_vector append_col(row_vector x, row_vector y)
```
Combine row vectors x and y of any size into another row vector.
Available since 2.5

```
row_vector append_col(real x, row_vector y)
```
Append x to the front of y, returning another row vector.
Available since 2.12

```
row_vector append_col(row_vector x, real y)
```
Append y to the end of x, returning another row vector.
Available since 2.12

Vertical concatenation

```
matrix append_row(matrix x, matrix y)
```
Combine matrices x and y by rows. The matrices must have the same number of columns.
Available since 2.5

```
matrix append_row(matrix x, row_vector y)
```
Combine matrix x and row vector y by rows. The matrix and the row vector must have the same number of columns.
Available since 2.5

```
matrix append_row(row_vector x, matrix y)
```
Combine row vector x and matrix y by rows. The row vector and the matrix must have the same number of columns.
Available since 2.5

```
matrix append_row(row_vector x, row_vector y)
```
Combine row vectors x and y by row. The row vectors must have the same number of columns.
Available since 2.5

```
vector append_row(vector x, vector y)
```
Concatenate vectors x and y of any size into another vector.
Available since 2.5

`vector` **`append_row`**`(real x, vector y)`
Append x to the top of y, returning another vector.
Available since 2.12

`vector` **`append_row`**`(vector x, real y)`
Append y to the bottom of x, returning another vector.
Available since 2.12

6.12.   Special matrix functions

Softmax

The softmax function maps[1] $y \in \mathbb{R}^K$ to the K-simplex by

$$\text{softmax}(y) = \frac{\exp(y)}{\sum_{k=1}^K \exp(y_k)},$$

where $\exp(y)$ is the componentwise exponentiation of y. Softmax is usually calculated on the log scale,

$$\log \text{softmax}(y) \quad = \quad y - \log \sum_{k=1}^K \exp(y_k)$$

$$= \quad y - \log\_sum\_exp(y).$$

where the vector y minus the scalar $\log\_sum\_exp(y)$ subtracts the scalar from each component of y.

Stan provides the following functions for softmax and its log.

`vector` **`softmax`**`(vector x)`
The softmax of x
Available since 2.0

`vector` **`log_softmax`**`(vector x)`
The natural logarithm of the softmax of x
Available since 2.0

---

[1]The softmax function is so called because in the limit as $y_n \to \infty$ with y_m for $m \neq n$ held constant, the result tends toward the "one-hot" vector θ with $\theta_n = 1$ and $\theta_m = 0$ for $m \neq n$, thus providing a "soft" version of the maximum function.

Cumulative sums

The cumulative sum of a sequence $x_1, \ldots, x_N$ is the sequence $y_1, \ldots, y_N$, where

$$y_n = \sum_{m=1}^{n} x_m.$$

`array[] real` **`cumulative_sum`**`(array[] real x)`
The cumulative sum of x
Available since 2.0

`vector` **`cumulative_sum`**`(vector v)`
The cumulative sum of v
Available since 2.0

`row_vector` **`cumulative_sum`**`(row_vector rv)`
The cumulative sum of rv
Available since 2.0

6.13. Covariance functions

Exponentiated quadratic covariance function

The exponentiated quadratic kernel defines the covariance between $f(x_i)$ and $f(x_j)$ where $f : \mathbb{R}^D \mapsto \mathbb{R}$ as a function of the squared Euclidian distance between $x_i \in \mathbb{R}^D$ and $x_j \in \mathbb{R}^D$:

$$\text{cov}(f(x_i), f(x_j)) = k(x_i, x_j) = \alpha^2 \exp\left(-\frac{1}{2\rho^2} \sum_{d=1}^{D} (x_{i,d} - x_{j,d})^2\right)$$

with α and ρ constrained to be positive.

There are two variants of the exponentiated quadratic covariance function in Stan. One builds a covariance matrix, $K \in \mathbb{R}^{N \times N}$ for $x_1, \ldots, x_N$, where $K_{i,j} = k(x_i, x_j)$, which is necessarily symmetric and positive semidefinite by construction. There is a second variant of the exponentiated quadratic covariance function that builds a $K \in \mathbb{R}^{N \times M}$ covariance matrix for $x_1, \ldots, x_N$ and $x'_1, \ldots, x'_M$, where $x_i \in \mathbb{R}^D$ and $x'_i \in \mathbb{R}^D$ and $K_{i,j} = k(x_i, x'_j)$.

`matrix` **`cov_exp_quad`**`(row_vectors x, real alpha, real rho)`
The covariance matrix with an exponentiated quadratic kernel of x.
Available since 2.16

`matrix` **`cov_exp_quad`**`(vectors x, real alpha, real rho)`
The covariance matrix with an exponentiated quadratic kernel of x.
Available since 2.16

`matrix` **`cov_exp_quad`**`(array[] real x, real alpha, real rho)`
The covariance matrix with an exponentiated quadratic kernel of x.
Available since 2.16

`matrix` **`cov_exp_quad`**`(row_vectors x1, row_vectors x2, real alpha, real rho)`
The covariance matrix with an exponentiated quadratic kernel of x1 and x2.
Available since 2.18

`matrix` **`cov_exp_quad`**`(vectors x1, vectors x2, real alpha, real rho)`
The covariance matrix with an exponentiated quadratic kernel of x1 and x2.
Available since 2.18

`matrix` **`cov_exp_quad`**`(array[] real x1, array[] real x2, real alpha, real rho)`
The covariance matrix with an exponentiated quadratic kernel of x1 and x2.
Available since 2.18

6.14. Linear algebra functions and solvers

Matrix division operators and functions

In general, it is much more efficient and also more arithmetically stable to use matrix division than to multiply by an inverse. There are specialized forms for lower triangular matrices and for symmetric, positive-definite matrices.

Matrix division operators

`row_vector` **`operator/`**`(row_vector b, matrix A)`
The right division of b by A; equivalently `b * inverse(A)`
Available since 2.0

`matrix` **`operator/`**`(matrix B, matrix A)`
The right division of B by A; equivalently `B * inverse(A)`
Available since 2.5

`vector` **`operator\`**`(matrix A, vector b)`
The left division of A by b; equivalently `inverse(A) * b`

Available since 2.18

`matrix` **`operator\`**`(matrix A, matrix B)`
The left division of A by B; equivalently `inverse(A) * B`
Available since 2.18

Lower-triangular matrix division functions

There are four division functions which use lower triangular views of a matrix. The lower triangular view of a matrix tri(A) is used in the definitions and defined by

$$\text{tri}(A)[m, n] = \begin{cases} A[m, n] & \text{if } m \geq n, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

When a lower triangular view of a matrix is used, the elements above the diagonal are ignored.

`vector` **`mdivide_left_tri_low`**`(matrix A, vector b)`
The left division of b by a lower-triangular view of A; algebraically equivalent to the less efficient and stable form `inverse(tri(A)) * b`, where `tri(A)` is the lower-triangular portion of A with the above-diagonal entries set to zero.
Available since 2.12

`matrix` **`mdivide_left_tri_low`**`(matrix A, matrix B)`
The left division of B by a triangular view of A; algebraically equivalent to the less efficient and stable form `inverse(tri(A)) * B`, where `tri(A)` is the lower-triangular portion of A with the above-diagonal entries set to zero.
Available since 2.5

`row_vector` **`mdivide_right_tri_low`**`(row_vector b, matrix A)`
The right division of b by a triangular view of A; algebraically equivalent to the less efficient and stable form `b * inverse(tri(A))`, where `tri(A)` is the lower-triangular portion of A with the above-diagonal entries set to zero.
Available since 2.12

`matrix` **`mdivide_right_tri_low`**`(matrix B, matrix A)`
The right division of B by a triangular view of A; algebraically equivalent to the less efficient and stable form `B * inverse(tri(A))`, where `tri(A)` is the lower-triangular portion of A with the above-diagonal entries set to zero.
Available since 2.5

Symmetric positive-definite matrix division functions

There are four division functions which are specialized for efficiency and stability for symmetric positive-definite matrix dividends. If the matrix dividend argument is not symmetric and positive definite, these will reject and print warnings.

matrix **mdivide_left_spd**(matrix A, vector b)
The left division of b by the symmetric, positive-definite matrix A; algebraically equivalent to the less efficient and stable form inverse(A) * b.
Available since 2.12

vector **mdivide_left_spd**(matrix A, matrix B)
The left division of B by the symmetric, positive-definite matrix A; algebraically equivalent to the less efficient and stable form inverse(A) * B.
Available since 2.12

row_vector **mdivide_right_spd**(row_vector b, matrix A)
The right division of b by the symmetric, positive-definite matrix A; algebraically equivalent to the less efficient and stable form b *inverse(A).
Available since 2.12

matrix **mdivide_right_spd**(matrix B, matrix A)
The right division of B by the symmetric, positive-definite matrix A; algebraically equivalent to the less efficient and stable form B * inverse(A).
Available since 2.12

Matrix exponential

The exponential of the matrix A is formally defined by the convergent power series:

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

matrix **matrix_exp**(matrix A)
The matrix exponential of A
Available since 2.13

matrix **matrix_exp_multiply**(matrix A, matrix B)
The multiplication of matrix exponential of A and matrix B; algebraically equivalent to the less efficient form matrix_exp(A) * B.
Available since 2.18

matrix **scale_matrix_exp_multiply**(real t, matrix A, matrix B)
The multiplication of matrix exponential of tA and matrix B; algebraically equivalent

to the less efficient form `matrix_exp(t * A) * B`.
Available since 2.18

Matrix power

Returns the nth power of the specific matrix:

$$M^n = M_1 * ... * M_n$$

matrix **matrix_power**(matrix A, int B)
Matrix A raised to the power B.
Available since 2.24

Linear algebra functions

Trace

real **trace**(matrix A)
The trace of A, or 0 if A is empty; A is not required to be diagonal
Available since 2.0

Determinants

real **determinant**(matrix A)
The determinant of A
Available since 2.0

real **log_determinant**(matrix A)
The log of the absolute value of the determinant of A
Available since 2.0

Inverses

It is almost never a good idea to use matrix inverses directly because they are both
inefficient and arithmetically unstable compared to the alternatives. Rather than
inverting a matrix m and post-multiplying by a vector or matrix a, as in `inverse(m)`
`* a`, it is better to code this using matrix division, as in `m \ a`. The pre-multiplication
case is similar, with `b * inverse(m)` being more efficiently coded as as `b / m`. There
are also useful special cases for triangular and symmetric, positive-definite matrices
that use more efficient solvers.

Warning: The function `inv(m)` is the elementwise inverse function, which returns 1
/ `m[i, j]` for each element.

matrix **inverse**(matrix A)
Compute the inverse of A
Available since 2.0

matrix **inverse_spd**(matrix A)
Compute the inverse of A where A is symmetric, positive definite. This version
is faster and more arithmetically stable when the input is symmetric and positive
definite.
Available since 2.0

matrix **chol2inv**(matrix L)
Compute the inverse of the matrix whose cholesky factorization is L. That is, for
$A = LL^T$, return A^{-1}.
Available since 2.26

Generalized Inverse

The generalized inverse M^+ of a matrix M is a matrix that satisfies $MM^+M = M$.
For an invertible, square matrix M, M^+ is equivalent to M^{-1}. The dimensions of
M^+ are equivalent to the dimensions of M^T. The generalized inverse exists for any
matrix, so the M may be singular or less than full rank.

Even though the generalized inverse exists for any arbitrary matrix, the derivatives
of this function only exist on matrices of locally constant rank (Golub and Pereyra
1973), meaning, the derivatives do not exist if small perturbations make the matrix
change rank. For example, considered the rank of the matrix A as a function of ϵ:

$$A = \begin{pmatrix} 1 + \epsilon & 2 & 1 \\ 2 & 4 & 2 \end{pmatrix}$$

When $\epsilon = 0$, A is rank 1 because the second row is twice the first (and so there
is only one linearly independent row). If $\epsilon \neq 0$, the rows are no longer linearly
dependent, and the matrix is rank 2. This matrix does not have locally constant rank
at $\epsilon = 0$, and so the derivatives do not exist at zero. Because HMC depends on the
derivatives existing, this lack of differentiability creates undefined behavior.

matrix **generalized_inverse**(matrix A)
The generalized inverse of A

Available since 2.26

Eigendecomposition

vector **eigenvalues_sym**(matrix A)
The vector of eigenvalues of a symmetric matrix A in ascending order
Available since 2.0

matrix **eigenvectors_sym**(matrix A)
The matrix with the (column) eigenvectors of symmetric matrix A in the same order as returned by the function eigenvalues_sym
Available since 2.0

Because multiplying an eigenvector by -1 results in an eigenvector, eigenvectors returned by a decomposition are only identified up to a sign change. In order to compare the eigenvectors produced by Stan's eigendecomposition to others, signs may need to be normalized in some way, such as by fixing the sign of a component, or doing comparisons allowing a multiplication by -1.

The condition number of a symmetric matrix is defined to be the ratio of the largest eigenvalue to the smallest eigenvalue. Large condition numbers lead to difficulty in numerical algorithms such as computing inverses, and thus known as "ill conditioned." The ratio can even be infinite in the case of singular matrices (i.e., those with eigenvalues of 0).

QR decomposition

matrix **qr_thin_Q**(matrix A)
The orthogonal matrix in the thin QR decomposition of A, which implies that the resulting matrix has the same dimensions as A
Available since 2.18

matrix **qr_thin_R**(matrix A)
The upper triangular matrix in the thin QR decomposition of A, which implies that the resulting matrix is square with the same number of columns as A
Available since 2.18

matrix **qr_Q**(matrix A)
The orthogonal matrix in the fat QR decomposition of A, which implies that the resulting matrix is square with the same number of rows as A
Available since 2.3

matrix `qr_R(matrix A)`
The upper trapezoidal matrix in the fat QR decomposition of A, which implies that
the resulting matrix will be rectangular with the same dimensions as A
Available since 2.3

The thin QR decomposition is always preferable because it will consume much less
memory when the input matrix is large than will the fat QR decomposition. Both
versions of the decomposition represent the input matrix as

$$A = Q R.$$

Multiplying a column of an orthogonal matrix by -1 still results in an orthogonal
matrix, and you can multiply the corresponding row of the upper trapezoidal matrix
by -1 without changing the product. Thus, Stan adopts the normalization that
the diagonal elements of the upper trapezoidal matrix are strictly positive and
the columns of the orthogonal matrix are reflected if necessary. Also, these QR
decomposition algorithms do not utilize pivoting and thus may be numerically
unstable on input matrices that have less than full rank.

Cholesky decomposition

Every symmetric, positive-definite matrix (such as a correlation or covariance matrix)
has a Cholesky decomposition. If Σ is a symmetric, positive-definite matrix, its
Cholesky decomposition is the lower-triangular vector L such that

$$\Sigma = L L^\top.$$

matrix **cholesky_decompose**(matrix A)
The lower-triangular Cholesky factor of the symmetric positive-definite matrix A
Available since 2.0

Singular value decomposition

The matrix A can be decomposed into a diagonal matrix of singular values, D, and
matrices of its left and right singular vectors, U and V,

$$A = UDV^T.$$

The matrices of singular vectors here are thin. That is for an N by P input A,
$M = min(N, P)$, U is size N by M and V is size P by M.

vector **singular_values**(matrix A)
The singular values of A in descending order
Available since 2.0

vector **svd_U**(matrix A)
The left-singular vectors of A
Available since 2.26

vector **svd_V**(matrix A)
The right-singular vectors of A
Available since 2.26

6.15.   Sort functions

See the sorting functions section for examples of how the functions work.

vector **sort_asc**(vector v)
Sort the elements of v in ascending order
Available since 2.0

row_vector **sort_asc**(row_vector v)
Sort the elements of v in ascending order
Available since 2.0

vector **sort_desc**(vector v)
Sort the elements of v in descending order
Available since 2.0

row_vector **sort_desc**(row_vector v)
Sort the elements of v in descending order
Available since 2.0

array[] int **sort_indices_asc**(vector v)
Return an array of indices between 1 and the size of v, sorted to index v in ascending
order.
Available since 2.3

array[] int **sort_indices_asc**(row_vector v)
Return an array of indices between 1 and the size of v, sorted to index v in ascending
order.
Available since 2.3

array[] int **sort_indices_desc**(vector v)
Return an array of indices between 1 and the size of v, sorted to index v in descending

order.
Available since 2.3

`array[] int` **`sort_indices_desc`**`(row_vector v)`
Return an array of indices between 1 and the size of v, sorted to index v in descending
order.
Available since 2.3

`int` **`rank`**`(vector v, int s)`
Number of components of v less than v[s]
Available since 2.0

`int` **`rank`**`(row_vector v, int s)`
Number of components of v less than v[s]
Available since 2.0

6.16.   Reverse functions

`vector` **`reverse`**`(vector v)`
Return a new vector containing the elements of the argument in reverse order.
Available since 2.23

`row_vector` **`reverse`**`(row_vector v)`
Return a new row vector containing the elements of the argument in reverse order.
Available since 2.23

7.   Sparse Matrix Operations

For sparse matrices, for which many elements are zero, it is more efficient to use specialized representations to save memory and speed up matrix arithmetic (including derivative calculations). Given Stan's implementation, there is substantial space (memory) savings by using sparse matrices. Because of the ease of optimizing dense matrix operations, speed improvements only arise at 90% or even greater sparsity; below that level, dense matrices are faster but use more memory.

Because of this speedup and space savings, it may even be useful to read in a dense matrix and convert it to a sparse matrix before multiplying it by a vector. This chapter covers a very specific form of sparsity consisting of a sparse matrix multiplied by a dense vector.

7.1.   Compressed row storage

Sparse matrices are represented in Stan using compressed row storage (CSR). For example, the matrix

$$A = \begin{bmatrix} 19 & 27 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 52 \\ 81 & 0 & 95 & 33 \end{bmatrix}$$

is translated into a vector of the non-zero real values, read by row from the matrix A,

$$w(A) = \begin{bmatrix} 19 & 27 & 52 & 81 & 95 & 33 \end{bmatrix}^\top,$$

an array of integer column indices for the values,

$$v(A) = \begin{bmatrix} 1 & 2 & 4 & 1 & 3 & 4 \end{bmatrix},$$

and an array of integer indices indicating where in $w(A)$ a given row's values start,

$$u(A) = \begin{bmatrix} 1 & 3 & 3 & 4 & 7 \end{bmatrix},$$

with a padded value at the end to guarantee that

$$u(A)[n+1] - u(A)[n]$$

is the number of non-zero elements in row n of the matrix (here 2, 0, 1, and 3). Note that because the second row has no non-zero elements both the second and third

elements of $u(A)$ correspond to the third element of $w(A)$, which is 52. The values $(w(A), v(A), u(A))$ are sufficient to reconstruct A.

The values are structured so that there is a real value and integer column index for each non-zero entry in the array, plus one integer for each row of the matrix, plus one for padding. There is also underlying storage for internal container pointers and sizes. The total memory usage is roughly $12K + M$ bytes plus a small constant overhead, which is often considerably fewer bytes than the $M \times N$ required to store a dense matrix. Even more importantly, zero values do not introduce derivatives under multiplication or addition, so many storage and evaluation steps are saved when sparse matrices are multiplied.

7.2.   Conversion functions

Conversion functions between dense and sparse matrices are provided.

Dense to sparse conversion

Converting a dense matrix m to a sparse representation produces a vector w and two integer arrays, u and v.

vector **csr_extract_w**(matrix a)
Return non-zero values in matrix a; see section compressed row storage.
Available since 2.8

array[] int **csr_extract_v**(matrix a)
Return column indices for values in csr_extract_w(a); see compressed row storage.
Available since 2.8

array[] int **csr_extract_u**(matrix a)
Return array of row starting indices for entries in csr_extract_w(a) followed by the size of csr_extract_w(a) plus one; see section compressed row storage.
Available since 2.8

Sparse to dense conversion

To convert a sparse matrix representation to a dense matrix, there is a single function.

matrix   **csr_to_dense_matrix**(int m, int n, vector w, array[] int v, array[] int u)
Return dense m × n matrix with non-zero matrix entries w, column indices v, and row starting indices u; the vector w and array v must be the same size (corresponding to the total number of nonzero entries in the matrix), array v must have index values

bounded by m, array u must have length equal to m + 1 and contain index values bounded by the number of nonzeros (except for the last entry, which must be equal to the number of nonzeros plus one). See section compressed row storage for more details.

Available since 2.10

7.3.   Sparse matrix arithmetic

Sparse matrix multiplication

The only supported operation is the multiplication of a sparse matrix A and a dense vector b to produce a dense vector $A\,b$. Multiplying a dense row vector b and a sparse matrix A can be coded using transposition as

$$b\,A = (A^\top\,b^\top)^\top,$$

but care must be taken to represent $A^\top$ rather than A as a sparse matrix.

vector **csr_matrix_times_vector**(int m, int n, vector w, array[] int v, array[] int u, vector b)
Multiply the m × n matrix represented by values w, column indices v, and row start indices u by the vector b; see compressed row storage.

Available since 2.18

8. Mixed Operations

These functions perform conversions between Stan containers matrix, vector, row vector and arrays.

`matrix` **`to_matrix`**`(matrix m)`
Return the matrix m itself.
Available since 2.3

`matrix` **`to_matrix`**`(vector v)`
Convert the column vector v to a `size(v)` by 1 matrix.
Available since 2.3

`matrix` **`to_matrix`**`(row_vector v)`
Convert the row vector v to a 1 by `size(v)` matrix.
Available since 2.3

`matrix` **`to_matrix`**`(matrix m, int m, int n)`
Convert a matrix m to a matrix with m rows and n columns filled in column-major order.
Available since 2.15

`matrix` **`to_matrix`**`(vector v, int m, int n)`
Convert a vector v to a matrix with m rows and n columns filled in column-major order.
Available since 2.15

`matrix` **`to_matrix`**`(row_vector v, int m, int n)`
Convert a row_vector a to a matrix with m rows and n columns filled in column-major order.
Available since 2.15

`matrix` **`to_matrix`**`(matrix m, int m, int n, int col_major)`
Convert a matrix m to a matrix with m rows and n columns filled in row-major order if col_major equals 0 (otherwise, they get filled in column-major order).
Available since 2.15

`matrix` **`to_matrix`**`(vector v, int m, int n, int col_major)`
Convert a vector v to a matrix with m rows and n columns filled in row-major order

if col_major equals 0 (otherwise, they get filled in column-major order).
Available since 2.15

`matrix` **`to_matrix`**`(row_vector v, int m, int n, int col_major)`
Convert a row_vector a to a matrix with m rows and n columns filled in row-major
order if col_major equals 0 (otherwise, they get filled in column-major order).
Available since 2.15

`matrix` **`to_matrix`**`(array[] real a, int m, int n)`
Convert a one-dimensional array a to a matrix with m rows and n columns filled in
column-major order.
Available since 2.15

`matrix` **`to_matrix`**`(array[] int a, int m, int n)`
Convert a one-dimensional array a to a matrix with m rows and n columns filled in
column-major order.
Available since 2.15

`matrix` **`to_matrix`**`(array[] real a, int m, int n, int col_major)`
Convert a one-dimensional array a to a matrix with m rows and n columns filled in
row-major order if col_major equals 0 (otherwise, they get filled in column-major
order).
Available since 2.15

`matrix` **`to_matrix`**`(array[] int a, int m, int n, int col_major)`
Convert a one-dimensional array a to a matrix with m rows and n columns filled in
row-major order if col_major equals 0 (otherwise, they get filled in column-major
order).
Available since 2.15

`matrix` **`to_matrix`**`(array[,] real a)`
Convert the two dimensional array a to a matrix with the same dimensions and
indexing order.
Available since 2.3

`matrix` **`to_matrix`**`(array[,] int a)`
Convert the two dimensional array a to a matrix with the same dimensions and
indexing order. If any of the dimensions of a are zero, the result will be a 0×0
matrix.
Available since 2.3

`vector` **`to_vector`**`(matrix m)`
Convert the matrix m to a column vector in column-major order.

Available since 2.0

`vector` **`to_vector`**`(vector v)`
Return the column vector v itself.
Available since 2.3

`vector` **`to_vector`**`(row_vector v)`
Convert the row vector v to a column vector.
Available since 2.3

`vector` **`to_vector`**`(array[] real a)`
Convert the one-dimensional array a to a column vector.
Available since 2.3

`vector` **`to_vector`**`(array[] int a)`
Convert the one-dimensional integer array a to a column vector.
Available since 2.3

`row_vector` **`to_row_vector`**`(matrix m)`
Convert the matrix m to a row vector in column-major order.
Available since 2.3

`row_vector` **`to_row_vector`**`(vector v)`
Convert the column vector v to a row vector.
Available since 2.3

`row_vector` **`to_row_vector`**`(row_vector v)`
Return the row vector v itself.
Available since 2.3

`row_vector` **`to_row_vector`**`(array[] real a)`
Convert the one-dimensional array a to a row vector.
Available since 2.3

`row_vector` **`to_row_vector`**`(array[] int a)`
Convert the one-dimensional array a to a row vector.
Available since 2.3

`array[,] real` **`to_array_2d`**`(matrix m)`
Convert the matrix m to a two dimensional array with the same dimensions and indexing order.
Available since 2.3

`array[] real` **`to_array_1d`**`(vector v)`
Convert the column vector v to a one-dimensional array.

Available since 2.3

```
array[] real to_array_1d(row_vector v)
```
Convert the row vector v to a one-dimensional array.
Available since 2.3

```
array[] real to_array_1d(matrix m)
```
Convert the matrix m to a one-dimensional array in column-major order.
Available since 2.3

```
array[] real to_array_1d(array[...] real a)
```
Convert the array a (of any dimension up to 10) to a one-dimensional array in row-major order.
Available since 2.3

```
array[] int to_array_1d(array[...] int a)
```
Convert the array a (of any dimension up to 10) to a one-dimensional array in row-major order.
Available since 2.3

9.   Compound Arithmetic and Assignment

Compound arithmetic and assignment statements combine an arithmetic operation and assignment,

```
x = x op y;
```

replacing them with the compound form

```
x op= y;
```

For example, `x = x + 1` may be replaced with `x += 1`.

The signatures of the supported compound arithmetic and assignment operations are as follows.

9.1.   Compound addition and assignment

void **operator+=**(int x, int y)
x += y is equivalent to x = x + y.
Available since 2.17

void **operator+=**(real x, real y)
x += y is equivalent to x = x + y.
Available since 2.17

void **operator+=**(vector x, real y)
x += y is equivalent to x = x + y.
Available since 2.17

void **operator+=**(row_vector x, real y)
x += y is equivalent to x = x + y.
Available since 2.17

void **operator+=**(matrix x, real y)
x += y is equivalent to x = x + y.
Available since 2.17

void **operator+=**(vector x, vector y)
x += y is equivalent to x = x + y.
Available since 2.17

```
void operator+=(row_vector x, row_vector y)
```
x += y is equivalent to x = x + y.
Available since 2.17

```
void operator+=(matrix x, matrix y)
```
x += y is equivalent to x = x + y.
Available since 2.17

9.2. Compound subtraction and assignment

```
void operator-=(int x, int y)
```
x -= y is equivalent to x = x - y.
Available since 2.17

```
void operator-=(real x, real y)
```
x -= y is equivalent to x = x - y.
Available since 2.17

```
void operator-=(vector x, real y)
```
x -= y is equivalent to x = x - y.
Available since 2.17

```
void operator-=(row_vector x, real y)
```
x -= y is equivalent to x = x - y.
Available since 2.17

```
void operator-=(matrix x, real y)
```
x -= y is equivalent to x = x - y.
Available since 2.17

```
void operator-=(vector x, vector y)
```
x -= y is equivalent to x = x - y.
Available since 2.17

```
void operator-=(row_vector x, row_vector y)
```
x -= y is equivalent to x = x - y.
Available since 2.17

```
void operator-=(matrix x, matrix y)
```
x -= y is equivalent to x = x - y.
Available since 2.17

9.3.   Compound multiplication and assignment

void **operator*=**(int x, int y)
x *= y is equivalent to x = x * y.
Available since 2.17

void **operator*=**(real x, real y)
x *= y is equivalent to x = x * y.
Available since 2.17

void **operator*=**(vector x, real y)
x *= y is equivalent to x = x * y.
Available since 2.17

void **operator*=**(row_vector x, real y)
x *= y is equivalent to x = x * y.
Available since 2.17

void **operator*=**(matrix x, real y)
x *= y is equivalent to x = x * y.
Available since 2.17

void **operator*=**(row_vector x, matrix y)
x *= y is equivalent to x = x * y.
Available since 2.17

void **operator*=**(matrix x, matrix y)
x *= y is equivalent to x = x * y.
Available since 2.17

9.4.   Compound division and assignment

void **operator/=**(int x, int y)
x /= y is equivalent to x = x / y.
Available since 2.17

void **operator/=**(real x, real y)
x /= y is equivalent to x = x / y.
Available since 2.17

void **operator/=**(vector x, real y)
x /= y is equivalent to x = x / y.
Available since 2.17

```
void operator/=(row_vector x, real y)
```
x /= y is equivalent to x = x / y.
Available since 2.17

```
void operator/=(matrix x, real y)
```
x /= y is equivalent to x = x / y.
Available since 2.17

9.5. Compound elementwise multiplication and assignment

```
void operator.*=(vector x, vector y)
```
x .*= y is equivalent to x = x .* y.
Available since 2.17

```
void operator.*=(row_vector x, row_vector y)
```
x .*= y is equivalent to x = x .* y.
Available since 2.17

```
void operator.*=(matrix x, matrix y)
```
x .*= y is equivalent to x = x .* y.
Available since 2.17

9.6. Compound elementwise division and assignment

```
void operator./=(vector x, vector y)
```
x ./= y is equivalent to x = x ./ y.
Available since 2.17

```
void operator./=(row_vector x, row_vector y)
```
x ./= y is equivalent to x = x ./ y.
Available since 2.17

```
void operator./=(matrix x, matrix y)
```
x ./= y is equivalent to x = x ./ y.
Available since 2.17

```
void operator./=(vector x, real y)
```
x ./= y is equivalent to x = x ./ y.
Available since 2.17

```
void operator./=(row_vector x, real y)
```
x ./= y is equivalent to x = x ./ y.
Available since 2.17

```
void operator./=(matrix x, real y)
```
x ./= y is equivalent to x = x ./ y.
Available since 2.17

10.   Higher-Order Functions

Stan provides a few higher-order functions that act on other functions. In all cases, the function arguments to the higher-order functions are defined as functions within the Stan language and passed by name to the higher-order functions.

10.1.   Algebraic equation solver

Stan provides two built-in algebraic equation solvers, respectively based on Powell's and Newton's methods. The Newton method constitutes a more recent addition to Stan; its use is recommended for most problems. Although they look like other function applications, algebraic solvers are special in two ways.

First, an algebraic solver is a higher-order function, i.e. it takes another function as one of its arguments. Other functions in Stan which share this feature are the ordinary differential equation solvers (see section Ordinary Differential Equation (ODE) Solvers). Ordinary Stan functions do not allow functions as arguments.

Second, some of the arguments of the algebraic solvers are restricted to data only expressions. These expressions must not contain variables other than those declared in the data or transformed data blocks. Ordinary Stan functions place no restriction on the origin of variables in their argument expressions.

Specifying an algebraic equation as a function

An algebraic system is specified as an ordinary function in Stan within the function block. The algebraic system function must have this signature:

```
vector algebra_system(vector y, vector theta,
                      data array[] real x_r, array[] int x_i)
```

The algebraic system function should return the value of the algebraic function which goes to 0, when we plug in the solution to the algebraic system.

The argument of this function are:

- y, the unknowns we wish to solve for

- $theta$, parameter values used to evaluate the algebraic system

- x_r, data values used to evaluate the algebraic system

- `x_i`, integer data used to evaluate the algebraic system

The algebraic system function separates parameter values, `theta`, from data values, `x_r`, for efficiency in propagating the derivatives through the algebraic system.

Call to the algebraic solver

```
vector algebra_solver(function algebra_system, vector y_guess, vector
theta, data array[] real x_r, array[] int x_i)
```
Solves the algebraic system, given an initial guess, using the Powell hybrid algorithm.
Available since 2.17

```
vector algebra_solver(function algebra_system, vector y_guess, vector
theta, data array[] real x_r, array[] int x_i, data real rel_tol,
data real f_tol, int max_steps)
```
Solves the algebraic system, given an initial guess, using the Powell hybrid algorithm with additional control parameters for the solver.
Available since 2.17

Note: In future releases, the function `algebra_solver` will be deprecated and replaced with `algebra_solver_powell`.

```
vector algebra_solver_newton(function algebra_system, vector y_guess,
vector theta, data array[] real x_r, array[] int x_i)
```
Solves the algebraic system, given an initial guess, using Newton's method.
Available since 2.24

```
vector algebra_solver_newton(function algebra_system, vector y_guess,
vector theta, data array[] real x_r, array[] int x_i, data real
rel_tol, data real f_tol, int max_steps)
```
Solves the algebraic system, given an initial guess, using Newton's method with additional control parameters for the solver.
Available since 2.24

Arguments to the algebraic solver

The arguments to the algebraic solvers are as follows:

- *algebra_system*: function literal referring to a function specifying the system of algebraic equations with signature (vector, vector, array[] real, array[] int):vector. The arguments represent (1) unknowns, (2) parameters, (3) real data, and (4) integer data, and the return value contains the

value of the algebraic function, which goes to 0 when we plug in the solution to the algebraic system,

- *y_guess*: initial guess for the solution, type `vector`,

- *theta*: parameters only, type `vector`,

- *x_r*: real data only, type `array[] real`, and

- *x_i*: integer data only, type `array[] int`.

For more fine-grained control of the algebraic solver, these parameters can also be provided:

- *rel_tol*: relative tolerance for the algebraic solver, type `real`, data only,

- *function_tol*: function tolerance for the algebraic solver, type `real`, data only,

- *max_num_steps*: maximum number of steps to take in the algebraic solver, type `int`, data only.

Return value

The return value for the algebraic solver is an object of type `vector`, with values which, when plugged in as y make the algebraic function go to 0.

Sizes and parallel arrays

Certain sizes have to be consistent. The initial guess, return value of the solver, and return value of the algebraic function must all be the same size.

The parameters, real data, and integer data will be passed from the solver directly to the system function.

Algorithmic details

Stan offers two algebraic solvers: `algebra_solver` and `algebra_solver_newton`. `algebra_solver` is baed on the Powell hybrid method (Powell 1970), which in turn uses first-order derivatives. The Stan code builds on the implementation of the hybrid solver in the unsupported module for nonlinear optimization problems of the

Eigen library (Guennebaud, Jacob, et al. 2010). This solver is in turn based on the
algorithm developed for the package MINPACK-1 (Jorge J. More 1980).

`algebra_solver_newton`, uses Newton's method, also a first-order derivative based
numerical solver. The Stan code builds on the implementation in KINSOL from
the SUNDIALS suite (Hindmarsh et al. 2005). For many problems, we find that
`algebra_solver_newton` is faster than Powell's method. If however Newton's
method performs poorly, either failing to or requiring an excessively long time
to converge, the user should be prepared to switch to `algebra_solver`.

For both solvers, the Jacobian of the solution with respect to auxiliary parameters
is computed using the implicit function theorem. Intermediate Jacobians (of the
algebraic function's output with respect to the unknowns y and with respect to the
auxiliary parameters theta) are computed using Stan's automatic differentiation.

10.2. Ordinary differential equation (ODE) solvers

Stan provides several higher order functions for solving initial value problems
specified as Ordinary Differential Equations (ODEs).

Solving an initial value ODE means given a set of differential equations $y'(t, \theta) = f(t, y, \theta)$ and initial conditions $y(t_0, \theta)$, solving for y at a sequence of times $t_0 < t_1 \leq t_2, \cdots \leq t_n$. $f(t, y, \theta)$ is referred to here as the ODE system function.

$f(t, y, \theta)$ will be defined as a function with a certain signature and provided along
with the initial conditions and output times to one of the ODE solver functions.

To make it easier to write ODEs, the solve functions take extra arguments that are
passed along unmodified to the user-supplied system function. Because there can be
any number of these arguments and they can be of different types, they are denoted
below as . . . . The types of the arguments represented by . . . in the ODE solve
function call must match the types of the arguments represented by . . . in the
user-supplied system function.

Non-stiff solver

```
array[] vector     ode_rk45(function ode, vector initial_state, real
initial_time, array[] real times, ...)
```
Solves the ODE system for the times provided using the Dormand-Prince algorithm,
a 4th/5th order Runge-Kutta method.
Available since 2.24

```
array[] vector     ode_rk45_tol(function ode, vector initial_state,
```

```
real initial_time, array[] real times, data real rel_tol, data real
abs_tol, int max_num_steps, ...)
```
Solves the ODE system for the times provided using the Dormand-Prince algorithm, a 4th/5th order Runge-Kutta method with additional control parameters for the solver.
Available since 2.24

```
array[] vector      ode_ckrk(function ode, vector initial_state, real
initial_time, array[] real times, ...)
```
Solves the ODE system for the times provided using the Cash-Karp algorithm, a 4th/5th order explicit Runge-Kutta method.
Available since 2.27

```
array[] vector      ode_ckrk_tol(function ode, vector initial_state,
real initial_time, array[] real times, data real rel_tol, data real
abs_tol, int max_num_steps, ...)
```
Solves the ODE system for the times provided using the Cash-Karp algorithm, a 4th/5th order explicit Runge-Kutta method with additional control parameters for the solver.
Available since 2.27

```
array[] vector      ode_adams(function ode, vector initial_state, real
initial_time, array[] real times, ...)
```
Solves the ODE system for the times provided using the Adams-Moulton method.
Available since 2.24

```
array[] vector      ode_adams_tol(function ode, vector initial_state,
real initial_time, array[] real times, data real rel_tol, data real
abs_tol, int max_num_steps, ...)
```
Solves the ODE system for the times provided using the Adams-Moulton method with additional control parameters for the solver.
Available since 2.24

Stiff solver

```
array[] vector      ode_bdf(function ode, vector initial_state, real
initial_time, array[] real times, ...)
```
Solves the ODE system for the times provided using the backward differentiation formula (BDF) method.
Available since 2.24

```
array[] vector   ode_bdf_tol(function ode, vector initial_state, real
```

```
initial_time, array[] real times, data real rel_tol, data real
abs_tol, int max_num_steps, ...)
```
Solves the ODE system for the times provided using the backward differentiation
formula (BDF) method with additional control parameters for the solver.
Available since 2.24

Adjoint solver

```
array[] vector          ode_adjoint_tol_ctl(function ode, vector
initial_state, real initial_time, array[] real times, data
real rel_tol_forward, data vector abs_tol_forward, data real
rel_tol_backward, data vector abs_tol_backward, int max_num_steps,
int num_steps_between_checkpoints, int interpolation_polynomial, int
solver_forward, int solver_backward, ...)
```

Solves the ODE system for the times provided using the adjoint ODE solver method
from CVODES. The adjoint ODE solver requires a checkpointed forward in time ODE
integration, a backwards in time integration that makes uses of an interpolated ver-
sion of the forward solution, and the solution of a quadrature problem (the number
of which depends on the number of parameters passed to the solve). The tolerances
and numeric methods used for the forward solve, backward solve, quadratures, and
interpolation can all be configured.
Available since 2.27

ODE system function

The first argument to one of the ODE solvers is always the ODE system function. The
ODE system function must have a `vector` return type, and the first two arguments
must be a `real` and `vector` in that order. These two arguments are followed by the
variadic arguments that are passed through from the ODE solve function call:

```
vector ode(real time, vector state, ...)
```

The ODE system function should return the derivative of the state with respect to
time at the time and state provided. The length of the returned vector must match
the length of the state input into the function.

The arguments to this function are:

- *time*, the time to evaluate the ODE system
- *state*, the state of the ODE system at the time specified

- *. . .*, sequence of arguments passed unmodified from the ODE solve function call. The types here must match the types in the *. . .* arguments of the ODE solve function call.

Arguments to the ODE solvers

The arguments to the ODE solvers in both the stiff and non-stiff solvers are the same. The arguments to the adjoint ODE solver are different; see Arguments to the adjoint ODE solvers.

- *ode* : ODE system function,

- *initial_state* : initial state, type vector,

- *initial_time* : initial time, type real,

- *times* : solution times, type array[] real,

- *. . .* : sequence of arguments that will be passed through unmodified to the ODE system function. The types here must match the types in the *. . .* arguments of the ODE system function.

For the versions of the ode solver functions ending in _tol, these three parameters must be provided after times and before the *. . .* arguments:

- data *rel_tol* : relative tolerance for the ODE solver, type real, data only,

- data *abs_tol* : absolute tolerance for the ODE solver, type real, data only, and

- *max_num_steps* : maximum number of steps to take between output times in the ODE solver, type int, data only.

Because the tolerances are data arguments, they must be defined in either the data or transformed data blocks. They cannot be parameters, transformed parameters or functions of parameters or transformed parameters.

Arguments to the adjoint ODE solver

The arguments to the adjoint ODE solver are different from those for the other functions (for those see Arguments to the adjoint ODE solvers).

- *ode* : ODE system function,

- *initial_state* : initial state, type vector,

- *initial_time* : initial time, type real,

- *times*: solution times, type `array[] real`,

- data *rel_tol_forward*: Relative tolerance for forward solve, type `real`, data only,

- data *abs_tol_forward*: Absolute tolerance vector for each state for forward solve, type `vector`, data only,

- data *rel_tol_backward*: Relative tolerance for backward solve, type `real`, data only,

- data *abs_tol_backward*: Absolute tolerance vector for each state for backward solve, type `vector`, data only,

- data *rel_tol_quadrature*: Relative tolerance for backward quadrature, type `real`, data only,

- data *abs_tol_quadrature*: Absolute tolerance for backward quadrature, type `real`, data only,

- data *max_num_steps*: Maximum number of time-steps to take in integrating the ODE solution between output time points for forward and backward solve, type `int`, data only,

- *num_steps_between_checkpoints*: number of steps between checkpointing forward solution, type `int`, data only,

- *interpolation_polynomial*: can be 1 for hermite or 2 for polynomial interpolation method of CVODES, type `int`, data only,

- *solver_forward*: solver used for forward ODE problem: 1=Adams (non-stiff), 2=BDF (stiff), type `int`, data only,

- *solver_backward*: solver used for backward ODE problem: 1=Adams (non-stiff), 2=BDF (stiff), type `int`, data only.

- `...` : sequence of arguments that will be passed through unmodified to the ODE system function. The types here must match the types in the `...` arguments of the ODE system function.

Because the tolerances are `data` arguments, they must be defined in either the data or transformed data blocks. They cannot be parameters, transformed parameters or functions of parameters or transformed parameters.

Return values

The return value for the ODE solvers is an array of vectors (type `array[] vector`), one vector representing the state of the system at every time in specified in the `times` argument.

Array and vector sizes

The sizes must match, and in particular, the following groups are of the same size:

- state variables passed into the system function, derivatives returned by the system function, initial state passed into the solver, and length of each vector in the output,

- number of solution times and number of vectors in the output,

10.3. 1D integrator

Stan provides a built-in mechanism to perform 1D integration of a function via quadrature methods.

It operates similarly to the algebraic solver and the ordinary differential equations solver in that it allows as an argument a function.

Like both of those utilities, some of the arguments are limited to data only expressions. These expressions must not contain variables other than those declared in the data or transformed data blocks.

Specifying an integrand as a function

Performing a 1D integration requires the integrand to be specified somehow. This is done by defining a function in the Stan functions block with the special signature:

```
real integrand(real x, real xc, array[] real theta,
               array[] real x_r, array[] int x_i)
```

The function should return the value of the integrand evaluated at the point x.

The argument of this function are:

- *x*, the independent variable being integrated over

- *xc*, a high precision version of the distance from x to the nearest endpoint in a definite integral (for more into see section Precision Loss).

- *theta*, parameter values used to evaluate the integral
- *x_r*, data values used to evaluate the integral
- *x_i*, integer data used to evaluate the integral

Like algebraic solver and the differential equations solver, the 1D integrator separates parameter values, theta, from data values, x_r.

Call to the 1D integrator

```
real integrate_1d (function integrand, real a, real b, array[] real
theta, array[] real x_r, array[] int x_i)
```
Integrates the integrand from a to b.
Available since 2.23

```
real integrate_1d (function integrand, real a, real b, array[] real
theta, array[] real x_r, array[] int x_i, real relative_tolerance)
```
Integrates the integrand from a to b with the given relative tolerance.
Available since 2.23

Arguments to the 1D integrator

The arguments to the 1D integrator are as follows:

- *integrand*: function literal referring to a function specifying the integrand with signature (real, real, array[] real, array[] real, array[] int):real The arguments represent
    - (1) where integrand is evaluated,
    - (2) distance from evaluation point to integration limit for definite integrals,
    - (3) parameters,
    - (4) real data
    - (5) integer data, and the return value is the integrand evaluated at the given point,
- *a*: left limit of integration, may be negative infinity, type real,
- *b*: right limit of integration, may be positive infinity, type real,
- *theta*: parameters only, type array[] real,
- *x_r*: real data only, type array[] real,
- *x_i*: integer data only, type array[] int.

A `relative_tolerance` argument can optionally be provided for more control over the algorithm:

- *relative_tolerance*: relative tolerance for the 1d integrator, type `real`, data only.

Return value

The return value for the 1D integrator is a `real`, the value of the integral.

Zero-crossing integrals

For numeric stability, integrals on the (possibly infinite) interval (a, b) that cross zero are split into two integrals, one from $(a, 0)$ and one from $(0, b)$. Each integral is separately integrated to the given `relative_tolerance`.

Precision loss near limits of integration in definite integrals

When integrating certain definite integrals, there can be significant precision loss in evaluating the integrand near the endpoints. This has to do with the breakdown in precision of double precision floating point values when adding or subtracting a small number from a number much larger than it in magnitude (for instance, `1.0 - x`). `xc` (as passed to the integrand) is a high-precision version of the distance between `x` and the definite integral endpoints and can be used to address this issue. More information (and an example where this is useful) is given in the User's Guide. For zero crossing integrals, `xc` will be a high precision version of the distance to the endpoints of the two smaller integrals. For any integral with an endpoint at negative infinity or positive infinity, `xc` is set to `NaN`.

Algorithmic details

Internally the 1D integrator uses the double-exponential methods in the Boost 1D quadrature library. Boost in turn makes use of quadrature methods developed in (Takahasi and Mori 1974), (Mori 1978), (Bailey, Jeyabalan, and Li 2005), and (Tanaka et al. 2009).

The gradients of the integral are computed in accordance with the Leibniz integral rule. Gradients of the integrand are computed internally with Stan's automatic differentiation.

10.4.   Reduce-sum function

Stan provides a higher-order reduce function for summation. A function which returns a scalar g: U -> real is mapped to every element of a list of type array[] U, { x1, x2, ... } and all the results are accumulated,

```
g(x1) + g(x2) + ...
```

For efficiency reasons the reduce function doesn't work with the element-wise evaluated function g itself, but instead works through evaluating partial sums, f: array[] U -> real, where:

```
f({ x1 }) = g(x1)
f({ x1, x2 }) = g(x1) + g(x2)
f({ x1, x2, ... }) = g(x1) + g(x2) + ...
```

Mathematically the summation reduction is associative and forming arbitrary partial sums in an arbritrary order will not change the result. However, floating point numerics on computers only have a limited precision such that associativity does not hold exactly. This implies that the order of summation determines the exact numerical result. For this reason, the higher-order reduce function is available in two variants:

- reduce_sum: Automatically choose partial sums partitioning based on a dynamic scheduling algorithm.
- reduce_sum_static: Compute the same sum as reduce_sum, but partition the input in the same way for given data set (in reduce_sum this partitioning might change depending on computer load). This should result in stable numerical evaluations.

Specifying the reduce-sum function

The higher-order reduce function takes a partial sum function f, an array argument x (with one array element for each term in the sum), a recommended grainsize, and a set of shared arguments. This representation allows parallelization of the resultant sum.

```
real reduce_sum(F f, array[] T x, int grainsize, T1 s1, T2 s2, ...)
real reduce_sum_static(F f, array[] T x, int grainsize, T1 s1, T2 s2,
```

...)

Returns the equivalent of `f(x, 1, size(x), s1, s2, ...)`, but computes the result in parallel by breaking the array x into independent partial sums. `s1, s2, ...` are shared between all terms in the sum.
Available since 2.23

- *f*: function literal referring to a function specifying the partial sum operation. Refer to the partial sum function.
- *x*: array of T, one for each term of the reduction, T can be any type,
- *grainsize*: For `reduce_sum`, grainsize is the recommended size of the partial sum (grainsize = 1 means pick totally automatically). For `reduce_sum_static`, grainsize determines the maximum size of the partial sums, type `int`,
- *s1*: first (optional) shared argument, type T1, where T1 can be any type
- *s2*: second (optional) shared argument, type T2, where T2 can be any type,
- ...: remainder of shared arguments, each of which can be any type.

The partial sum function

The partial sum function must have the following signature where the type T, and the types of all the shared arguments (T1, T2, ...) match those of the original `reduce_sum` (`reduce_sum_static`) call.

```
(array[] T x_subset, int start, int end, T1 s1, T2 s2, ...):real
```

The partial sum function returns the sum of the `start` to `end` terms (inclusive) of the overall calculations. The arguments to the partial sum function are:

- *x_subset*, the subset of x a given partial sum is responsible for computing, type `array[] T`, where T matches the type of x in `reduce_sum` (`reduce_sum_static`)
- *start*, the index of the first term of the partial sum, type `int`
- *end*, the index of the last term of the partial sum (inclusive), type `int`
- *s1*, first shared argument, type T1, matching type of s1 in `reduce_sum` (`reduce_sum_static`)
- *s2*, second shared argument, type T2, matching type of s2 in `reduce_sum` (`reduce_sum_static`)

- . . ., remainder of shared arguments, with types matching those in `reduce_sum` (`reduce_sum_static`)

10.5.   Map-rect function

Stan provides a higher-order map function. This allows map-reduce functionality to be coded in Stan as described in the user's guide.

Specifying the mapped function

The function being mapped must have a signature identical to that of the function `f` in the following declaration.

```
vector f(vector phi, vector theta,
         data array[] real x_r, data array[] int x_i);
```

The map function returns the sequence of results for the particular shard being evaluated. The arguments to the mapped function are:

- *phi*, the sequence of parameters shared across shards

- *theta*, the sequence of parameters specific to this shard

- *x_r*, sequence of real-valued data

- *x_i*, sequence of integer data

All input for the mapped function must be packed into these sequences and all output from the mapped function must be packed into a single vector. The vector of output from each mapped function is concatenated into the final result.

Rectangular map

The rectangular map function operates on rectangular (not ragged) data structures, with parallel data structures for job-specific parameters, job-specific real data, and job-specific integer data.

`vector` **`map_rect`**`(F f, vector phi, array[] vector theta, data array[,] real x_r, data array[,] int x_i)`
Return the concatenation of the results of applying the function f, of type (`vector`, `vector`, `array[] real`, `array[] int`):`vector` elementwise, i.e., `f(phi, theta[n], x_r[n], x_i[n])` for each n in 1:N, where N is the size of the parallel arrays of job-specific/local parameters theta, real data x_r, and integer

data x_r. The shared/global parameters phi are passed to each invocation of f.
Available since 2.18

11. Deprecated Functions

This appendix lists currently deprecated functionality along with how to replace it. These deprecated features are likely to be removed in the future.

11.1. integrate_ode_rk45, integrate_ode_adams, integrate_ode_bdf ODE integrators

These ODE integrator functions have been replaced by those described in:

Specifying an ordinary differential equation as a function

A system of ODEs is specified as an ordinary function in Stan within the functions block. The ODE system function must have this function signature:

```
array[] real ode(real time, array[] real state, array[] real theta,
                  array[] real x_r, array[] int x_i);
```

The ODE system function should return the derivative of the state with respect to time at the time provided. The length of the returned real array must match the length of the state input into the function.

The arguments to this function are:

- *time*, the time to evaluate the ODE system
- *state*, the state of the ODE system at the time specified
- *theta*, parameter values used to evaluate the ODE system
- *x_r*, data values used to evaluate the ODE system
- *x_i*, integer data values used to evaluate the ODE system.

The ODE system function separates parameter values, *theta*, from data values, *x_r*, for efficiency in computing the gradients of the ODE.

Non-stiff solver

```
array[,] real          integrate_ode_rk45(function ode, array[] real
initial_state, real initial_time, array[] real times, array[] real
theta, array[] real x_r, array[] int x_i)
```

Solves the ODE system for the times provided using the Dormand-Prince algorithm, a 4th/5th order Runge-Kutta method.
Available since 2.10, deprecated in 2.24

```
array[,] real          integrate_ode_rk45(function ode, array[] real
initial_state, real initial_time, array[] real times, array[]
real theta, array[] real x_r, array[] int x_i, real rel_tol, real
abs_tol, int max_num_steps)
```
Solves the ODE system for the times provided using the Dormand-Prince algorithm, a 4th/5th order Runge-Kutta method with additional control parameters for the solver.
Available since 2.10, deprecated in 2.24

```
array[,] real integrate_ode(function ode, array[] real initial_state,
real initial_time, array[] real times, array[] real theta, array[]
real x_r, array[] int x_i)
```
Solves the ODE system for the times provided using the Dormand-Prince algorithm, a 4th/5th order Runge-Kutta method.
Available since 2.10, deprecated in 2.24

```
array[,] real          integrate_ode_adams(function ode, array[] real
initial_state, real initial_time, array[] real times, array[] real
theta, data array[] real x_r, data array[] int x_i)
```
Solves the ODE system for the times provided using the Adams-Moulton method.
Available since 2.23, deprecated in 2.24

```
array[,] real          integrate_ode_adams(function ode, array[] real
initial_state, real initial_time, array[] real times, array[]
real theta, data array[] real x_r, data array[] int x_i, data real
rel_tol, data real abs_tol, data int max_num_steps)
```
Solves the ODE system for the times provided using the Adams-Moulton method with additional control parameters for the solver.
Available since 2.23, deprecated in 2.24

Stiff solver

```
array[,] real          integrate_ode_bdf(function ode, array[] real
initial_state, real initial_time, array[] real times, array[] real
theta, data array[] real x_r, data array[] int x_i)
```
Solves the ODE system for the times provided using the backward differentiation formula (BDF) method.
Available since 2.10, deprecated in 2.24

```
array[,] real            integrate_ode_bdf(function ode, array[] real
initial_state, real initial_time, array[] real times, array[]
real theta, data array[] real x_r, data array[] int x_i, data real
rel_tol, data real abs_tol, data int max_num_steps)
```
Solves the ODE system for the times provided using the backward differentiation
formula (BDF) method with additional control parameters for the solver.
Available since 2.10, deprecated in 2.24

Arguments to the ODE solvers

The arguments to the ODE solvers in both the stiff and non-stiff cases are as follows.

- *ode*: function literal referring to a function specifying the system of differential
  equations with signature:

```
(real, array[] real, array[] real, data array[] real, data array[] int):arra
```

The arguments represent (1) time, (2) system state, (3) parameters, (4) real data,
and (5) integer data, and the return value contains the derivatives with respect to
time of the state,

- *initial_state*: initial state, type array[] real,

- *initial_time*: initial time, type int or real,

- *times*: solution times, type array[] real,

- *theta*: parameters, type array[] real,

- data *x_r*: real data, type array[] real, data only, and

- data *x_i*: integer data, type array[] int, data only.

For more fine-grained control of the ODE solvers, these parameters can also be
provided:

- data *rel_tol*: relative tolerance for the ODE solver, type real, data only,

- data *abs_tol*: absolute tolerance for the ODE solver, type real, data only,
  and

- data *max_num_steps*: maximum number of steps to take in the ODE solver,
  type int, data only.

Return values

The return value for the ODE solvers is an array of type `array[,] real`, with values consisting of solutions at the specified times.

Sizes and parallel arrays

The sizes must match, and in particular, the following groups are of the same size:

- state variables passed into the system function, derivatives returned by the system function, initial state passed into the solver, and rows of the return value of the solver,

- solution times and number of rows of the return value of the solver,

- parameters, real data and integer data passed to the solver will be passed to the system function

12. Conventions for Probability Functions

Functions associated with distributions are set up to follow the same naming conventions for both built-in distributions and for user-defined distributions.

12.1. Suffix marks type of function

The suffix is determined by the type of function according to the following table.

function	outcome	suffix
log probability mass function	discrete	`_lpmf`
log probability density function	continuous	`_lpdf`
log cumulative distribution function	any	`_lcdf`
log complementary cumulative distribution function	any	`_lccdf`
random number generator	any	`_rng`

For example, `normal_lpdf` is the log of the normal probability density function (pdf) and `bernoulli_lpmf` is the log of the bernoulli probability mass function (pmf). The log of the corresponding cumulative distribution functions (cdf) use the same suffix, `normal_lcdf` and `bernoulli_lcdf`.

12.2. Argument order and the vertical bar

Each probability function has a specific outcome value and a number of parameters. Following conditional probability notation, probability density and mass functions use a vertical bar to separate the outcome from the parameters of the distribution. For example, `normal_lpdf(y | mu, sigma)` returns the value of mathematical formula $\log \text{Normal}(y \,|\, \mu, \sigma)$. Cumulative distribution functions separate the outcome from the parameters in the same way (e.g., `normal_lcdf(y_low | mu, sigma)`

12.3. Sampling notation

The notation

```
y ~ normal(mu, sigma);
```

provides the same (proportional) contribution to the model log density as the explicit target density increment,

```
target += normal_lpdf(y | mu, sigma);
```

In both cases, the effect is to add terms to the target log density. The only difference is that the example with the sampling (~) notation drops all additive constants in the log density; the constants are not necessary for any of Stan's sampling, approximation, or optimization algorithms.

12.4. Finite inputs

All of the distribution functions are configured to throw exceptions (effectively rejecting samples or optimization steps) when they are supplied with non-finite arguments. The two cases of non-finite arguments are the infinite values and not-a-number value—these are standard in floating-point arithmetic.

12.5. Boundary conditions

Many distributions are defined with support or constraints on parameters forming an open interval. For example, the normal density function accepts a scale parameter $\sigma > 0$. If $\sigma = 0$, the probability function will throw an exception.

This is true even for (complementary) cumulative distribution functions, which will throw exceptions when given input that is out of the support.

12.6. Pseudorandom number generators

For most of the probability functions, there is a matching pseudorandom number generator (PRNG) with the suffix `_rng`. For example, the function `normal_rng(real, real)` accepts two real arguments, an unconstrained location μ and positive scale $\sigma > 0$, and returns an unconstrained pseudorandom value drawn from $\text{Normal}(\mu, \sigma)$. There are also vectorized forms of random number generators which return more than one random variate at a time.

Restricted to transformed data and generated quantities

Unlike regular functions, the PRNG functions may only be used in the transformed data or generated quantities blocks.

Limited vectorization

Unlike the probability functions, only some of the PRNG functions are vectorized.

12.7. Cumulative distribution functions

For most of the univariate probability functions, there is a corresponding cumulative distribution function, log cumulative distribution function, and log complementary cumulative distribution function.

For a univariate random variable Y with probability function $p_Y(y \mid \theta)$, the cumulative distribution function (CDF) F_Y is defined by

$$F_Y(y) \;=\; \Pr[Y \leq y] \;=\; \int_{-\infty}^{y} p(y \mid \theta) \, \mathrm{d}y.$$

The complementary cumulative distribution function (CCDF) is defined as

$$\Pr[Y > y] \;=\; 1 - F_Y(y).$$

The reason to use CCDFs instead of CDFs in floating-point arithmetic is that it is possible to represent numbers very close to 0 (the closest you can get is roughly 10^{-300}), but not numbers very close to 1 (the closest you can get is roughly $1 - 10^{-15}$).

In Stan, there is a cumulative distribution function for each probability function. For instance, `normal_cdf(y, mu, sigma)` is defined by

$$\int_{-\infty}^{y} \mathrm{Normal}(y \mid \mu, \sigma) \, \mathrm{d}y.$$

There are also log forms of the CDF and CCDF for most univariate distributions. For example, `normal_lcdf(y | mu, sigma)` is defined by

$$\log \left(\int_{-\infty}^{y} \mathrm{Normal}(y \mid \mu, \sigma) \, \mathrm{d}y \right)$$

and `normal_lccdf(y | mu, sigma)` is defined by

$$\log \left(1 - \int_{-\infty}^{y} \mathrm{Normal}(y \mid \mu, \sigma) \, \mathrm{d}y \right).$$

12.8. Vectorization

Stan's univariate log probability functions, including the log density functions, log mass functions, log CDFs, and log CCDFs, all support vectorized function application, with results defined to be the sum of the elementwise application of the function. Some of the PRNG functions support vectorization, see section vectorized PRNG functions for more details.

In all cases, matrix operations are at least as fast and usually faster than loops and vectorized log probability functions are faster than their equivalent form defined with loops. This isn't because loops are slow in Stan, but because more efficient automatic differentiation can be used. The efficiency comes from the fact that a vectorized log probability function only introduces one new node into the expression graph, thus reducing the number of virtual function calls required to compute gradients in C++, as well as from allowing caching of repeated computations.

Stan also overloads the multivariate normal distribution, including the Cholesky-factor form, allowing arrays of row vectors or vectors for the variate and location parameter. This is a huge savings in speed because the work required to solve the linear system for the covariance matrix is only done once.

Stan also overloads some scalar functions, such as `log` and `exp`, to apply to vectors (arrays) and return vectors (arrays). These vectorizations are defined elementwise and unlike the probability functions, provide only minimal efficiency speedups over repeated application and assignment in a loop.

Vectorized function signatures

Vectorized scalar arguments

The normal probability function is specified with the signature

```
normal_lpdf(reals | reals, reals);
```

The pseudotype `reals` is used to indicate that an argument position may be vectorized. Argument positions declared as `reals` may be filled with a real, a one-dimensional array, a vector, or a row-vector. If there is more than one array or vector argument, their types can be anything but their size must match. For instance, it is legal to use `normal_lpdf(row_vector | vector, real)` as long as the vector and row vector have the same size.

Vectorized vector and row vector arguments

The multivariate normal distribution accepting vector or array of vector arguments is written as

```
multi_normal_lpdf(vectors | vectors, matrix);
```

These arguments may be row vectors, column vectors, or arrays of row vectors or column vectors.

Vectorized integer arguments

The pseudotype `ints` is used for vectorized integer arguments. Where it appears either an integer or array of integers may be used.

Evaluating vectorized log probability functions

The result of a vectorized log probability function is equivalent to the sum of the evaluations on each element. Any non-vector argument, namely `real` or `int`, is repeated. For instance, if `y` is a vector of size `N`, `mu` is a vector of size `N`, and `sigma` is a scalar, then

```
ll = normal_lpdf(y | mu, sigma);
```

is just a more efficient way to write

```
ll = 0;
for (n in 1:N) {
  ll = ll + normal_lpdf(y[n] | mu[n], sigma);
}
```

With the same arguments, the vectorized sampling statement

```
y ~ normal(mu, sigma);
```

has the same effect on the total log probability as

```
for (n in 1:N) {
  y[n] ~ normal(mu[n], sigma);
}
```

Evaluating vectorized PRNG functions

Some PRNG functions accept sequences as well as scalars as arguments. Such functions are indicated by argument pseudotypes `reals` or `ints`. In cases of sequence arguments, the output will also be a sequence. For example, the following is allowed in the transformed data and generated quantities blocks.

```
vector[3] mu = // ...
array[3] real x = normal_rng(mu, 3);
```

Argument types

In the case of PRNG functions, arguments marked `ints` may be integers or integer arrays, whereas arguments marked `reals` may be integers or reals, integer or real arrays, vectors, or row vectors.

pseudotype	allowable PRNG arguments
ints	int, array[] int
reals	int, array[] int, real, array[] real, vector, row_vector

Dimension matching

In general, if there are multiple non-scalar arguments, they must all have the same dimensions, but need not have the same type. For example, the `normal_rng` function may be called with one vector argument and one real array argument as long as they have the same number of elements.

```
vector[3] mu = // ...
array[3] real sigma = // ...
array[3] real x = normal_rng(mu, sigma);
```

Return type

The result of a vectorized PRNG function depends on the size of the arguments and the distribution's support. If all arguments are scalars, then the return type is a scalar. For a continuous distribution, if there are any non-scalar arguments, the return type is a real array (`array[] real`) matching the size of any of the non-scalar arguments, as all non-scalar arguments must have matching size. Discrete distributions return `ints` and continuous distributions return `reals`, each of appropriate size. The symbol R denotes such a return type.

Discrete Distributions

13. Binary Distributions

Binary probability distributions have support on $\{0, 1\}$, where 1 represents the value true and 0 the value false.

13.1. Bernoulli distribution

Probability mass function

If $\theta \in [0, 1]$, then for $y \in \{0, 1\}$,

$$\text{Bernoulli}(y \mid \theta) = \left\{ \begin{array}{ll} \theta & \text{if } y = 1, \text{ and} \\ 1 - \theta & \text{if } y = 0. \end{array} \right.$$

Sampling statement

`y ~ bernoulli(theta)`

Increment target log probability density with `bernoulli_lupmf(y | theta)`.
Available since 2.0

Stan Functions

real **bernoulli_lpmf**(ints y | reals theta)
The log Bernoulli probability mass of y given chance of success `theta`
Available since 2.12

real **bernoulli_lupmf**(ints y | reals theta)
The log Bernoulli probability mass of y given chance of success theta dropping constant additive terms
Available since 2.25

real **bernoulli_cdf**(ints y, reals theta)
The Bernoulli cumulative distribution function of y given chance of success `theta`
Available since 2.0

real **bernoulli_lcdf**(ints y | reals theta)
The log of the Bernoulli cumulative distribution function of y given chance of success
`theta`
Available since 2.12

real **bernoulli_lccdf**(ints y | reals theta)
The log of the Bernoulli complementary cumulative distribution function of y given chance of success `theta`
Available since 2.12

R **bernoulli_rng**(reals theta)
Generate a Bernoulli variate with chance of success `theta`; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.
Available since 2.18

13.2.   Bernoulli distribution, logit parameterization

Stan also supplies a direct parameterization in terms of a logit-transformed chance-of-success parameter. This parameterization is more numerically stable if the chance-of-success parameter is on the logit scale, as with the linear predictor in a logistic regression.

Probability mass function

If $\alpha \in \mathbb{R}$, then for $y \in \{0, 1\}$,

$$\text{BernoulliLogit}(y \mid \alpha) = \text{Bernoulli}(y | \text{logit}^{-1}(\alpha)) = \begin{cases} \text{logit}^{-1}(\alpha) & \text{if } y = 1, \text{ and} \\ 1 - \text{logit}^{-1}(\alpha) & \text{if } y = 0. \end{cases}$$

Sampling statement

y ~ **bernoulli_logit**(alpha)

Increment target log probability density with `bernoulli_logit_lupmf(y | alpha)`.
Available since 2.0

Stan Functions

real **bernoulli_logit_lpmf**(ints y | reals alpha)
The log Bernoulli probability mass of y given chance of success `inv_logit(alpha)`
Available since 2.12

real **bernoulli_logit_lupmf**(ints y | reals alpha)
The log Bernoulli probability mass of y given chance of success `inv_logit(alpha)` dropping constant additive terms
Available since 2.25

R **bernoulli_logit_rng**(reals alpha)
Generate a Bernoulli variate with chance of success $\text{logit}^{-1}(\alpha)$; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.
Available since 2.18

13.3. Bernoulli-logit generalized linear model (Logistic Regression)

Stan also supplies a single function for a generalized linear model with Bernoulli likelihood and logit link function, i.e. a function for a logistic regression. This provides a more efficient implementation of logistic regression than a manually written regression in terms of a Bernoulli likelihood and matrix multiplication.

Probability mass function

If $x \in \mathbb{R}^{n \cdot m}, \alpha \in \mathbb{R}^n, \beta \in \mathbb{R}^m$, then for $y \in \{0, 1\}^n$,

$$\text{BernoulliLogitGLM}(y \mid x, \alpha, \beta) = \prod_{1 \leq i \leq n} \text{Bernoulli}(y_i \mid \text{logit}^{-1}(\alpha_i + x_i \cdot \beta))$$

$$= \prod_{1 \leq i \leq n} \left\{ \begin{array}{ll} \text{logit}^{-1}(\alpha_i + \sum_{1 \leq j \leq m} x_{ij} \cdot \beta_j) & \text{if } y_i = 1, \text{ and} \\ 1 - \text{logit}^{-1}(\alpha_i + \sum_{1 \leq j \leq m} x_{ij} \cdot \beta_j) & \text{if } y_i = 0. \end{array} \right.$$

Sampling statement

y ~ **bernoulli_logit_glm**(x, alpha, beta)

Increment target log probability density with bernoulli_logit_glm_lupmf(y | x, alpha, beta).
Available since 2.25

Stan Functions

real   **bernoulli_logit_glm_lpmf**(int y | matrix x, real alpha, vector beta)
The log Bernoulli probability mass of y given chance of success inv_logit(alpha + x * beta).
Available since 2.23

real   **bernoulli_logit_glm_lupmf**(int y | matrix x, real alpha, vector beta)

The log Bernoulli probability mass of y given chance of success inv_logit(alpha +
x * beta) dropping constant additive terms.
Available since 2.25

```
real  bernoulli_logit_glm_lpmf(int y | matrix x, vector alpha, vector
beta)
```
The log Bernoulli probability mass of y given chance of success inv_logit(alpha +
x * beta).
Available since 2.23

```
real bernoulli_logit_glm_lupmf(int y | matrix x, vector alpha, vector
beta)
```
The log Bernoulli probability mass of y given chance of success inv_logit(alpha +
x * beta) dropping constant additive terms.
Available since 2.25

```
real     bernoulli_logit_glm_lpmf(array[] int y | row_vector x, real
alpha, vector beta)
```
The log Bernoulli probability mass of y given chance of success inv_logit(alpha +
x * beta).
Available since 2.23

```
real     bernoulli_logit_glm_lupmf(array[] int y | row_vector x, real
alpha, vector beta)
```
The log Bernoulli probability mass of y given chance of success inv_logit(alpha +
x * beta) dropping constant additive terms.
Available since 2.25

```
real    bernoulli_logit_glm_lpmf(array[] int y | row_vector x, vector
alpha, vector beta)
```
The log Bernoulli probability mass of y given chance of success inv_logit(alpha +
x * beta).
Available since 2.23

```
real   bernoulli_logit_glm_lupmf(array[] int y | row_vector x, vector
alpha, vector beta)
```
The log Bernoulli probability mass of y given chance of success inv_logit(alpha +
x * beta) dropping constant additive terms.
Available since 2.25

```
real   bernoulli_logit_glm_lpmf(array[] int y | matrix x, real alpha,
vector beta)
```
The log Bernoulli probability mass of y given chance of success inv_logit(alpha +

```
x * beta).
```
Available since 2.18

real **bernoulli_logit_glm_lupmf**(array[] int y | matrix x, real alpha, vector beta)
The log Bernoulli probability mass of y given chance of success inv_logit(alpha + x * beta) dropping constant additive terms.
Available since 2.25

real **bernoulli_logit_glm_lpmf**(array[] int y | matrix x, vector alpha, vector beta)
The log Bernoulli probability mass of y given chance of success inv_logit(alpha + x * beta).
Available since 2.18

real **bernoulli_logit_glm_lupmf**(array[] int y | matrix x, vector alpha, vector beta)
The log Bernoulli probability mass of y given chance of success inv_logit(alpha + x * beta) dropping constant additive terms.
Available since 2.25

14. Bounded Discrete Distributions

Bounded discrete probability functions have support on $\{0, \ldots, N\}$ for some upper bound N.

14.1. Binomial distribution

Probability mass function

Suppose $N \in \mathbb{N}$ and $\theta \in [0, 1]$, and $n \in \{0, \ldots, N\}$.

$$\mathrm{Binomial}(n \mid N, \theta) = \binom{N}{n} \theta^n (1 - \theta)^{N-n}.$$

Log probability mass function

$$
\begin{aligned}
\log \mathrm{Binomial}(n \mid N, \theta) \;=\; & \log \Gamma(N + 1) - \log \Gamma(n + 1) - \log \Gamma(N - n + 1) \\
& + n \log \theta + (N - n) \log(1 - \theta),
\end{aligned}
$$

Gradient of log probability mass function

$$\frac{\partial}{\partial \theta} \log \mathrm{Binomial}(n \mid N, \theta) = \frac{n}{\theta} - \frac{N - n}{1 - \theta}$$

Sampling statement

`n ~ binomial(N, theta)`

Increment target log probability density with `binomial_lupmf(n | N, theta)`.
Available since 2.0

Stan functions

real **binomial_lpmf**(ints n | ints N, reals theta)
The log binomial probability mass of n successes in N trials given chance of success theta
Available since 2.12

```
real binomial_lupmf(ints n | ints N, reals theta)
```
The log binomial probability mass of n successes in N trials given chance of success theta dropping constant additive terms
Available since 2.25

```
real binomial_cdf(ints n, ints N, reals theta)
```
The binomial cumulative distribution function of n successes in N trials given chance of success theta
Available since 2.0

```
real binomial_lcdf(ints n | ints N, reals theta)
```
The log of the binomial cumulative distribution function of n successes in N trials given chance of success theta
Available since 2.12

```
real binomial_lccdf(ints n | ints N, reals theta)
```
The log of the binomial complementary cumulative distribution function of n successes in N trials given chance of success theta
Available since 2.12

```
R binomial_rng(ints N, reals theta)
```
Generate a binomial variate with N trials and chance of success theta; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.
Available since 2.18

14.2. Binomial distribution, logit parameterization

Stan also provides a version of the binomial probability mass function distribution with the chance of success parameterized on the unconstrained logistic scale.

Probability mass function

Suppose $N \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and $n \in \{0, \ldots, N\}$. Then

$$
\begin{aligned}
\text{BinomialLogit}(n \mid N, \alpha) &= \text{Binomial}(n \mid N, \text{logit}^{-1}(\alpha)) \\
&= \binom{N}{n} \left(\text{logit}^{-1}(\alpha)\right)^n \left(1 - \text{logit}^{-1}(\alpha)\right)^{N-n}.
\end{aligned}
$$

Log probability mass function

$$
\begin{aligned}
\log \mathrm{BinomialLogit}(n \mid N, \alpha) \;=\; & \log \Gamma(N+1) - \log \Gamma(n+1) - \log \Gamma(N-n+1) \\
& + n \log \mathrm{logit}^{-1}(\alpha) + (N-n) \log \left(1 - \mathrm{logit}^{-1}(\alpha)\right),
\end{aligned}
$$

Gradient of log probability mass function

$$
\frac{\partial}{\partial \alpha} \log \mathrm{BinomialLogit}(n \mid N, \alpha) = \frac{n}{\mathrm{logit}^{-1}(-\alpha)} - \frac{N-n}{\mathrm{logit}^{-1}(\alpha)}
$$

Sampling statement

`n ~ binomial_logit(N, alpha)`

Increment target log probability density with `binomial_logit_lupmf(n | N, alpha)`.
Available since 2.0

Stan functions

real **binomial_logit_lpmf**(ints n | ints N, reals alpha)
The log binomial probability mass of n successes in N trials given logit-scaled chance of success alpha
Available since 2.12

real **binomial_logit_lupmf**(ints n | ints N, reals alpha)
The log binomial probability mass of n successes in N trials given logit-scaled chance of success alpha dropping constant additive terms
Available since 2.25

14.3.   Beta-binomial distribution

Probability mass function

If $N \in \mathbb{N}$, $\alpha \in \mathbb{R}^+$, and $\beta \in \mathbb{R}^+$, then for $n \in 0, \ldots, N$,

$$
\mathrm{BetaBinomial}(n \mid N, \alpha, \beta) = \binom{N}{n} \frac{\mathrm{B}(n+\alpha, N-n+\beta)}{\mathrm{B}(\alpha, \beta)},
$$

where the beta function $\mathrm{B}(u, v)$ is defined for $u \in \mathbb{R}^+$ and $v \in \mathbb{R}^+$ by

$$\mathrm{B}(u, v) = \frac{\Gamma(u) \, \Gamma(v)}{\Gamma(u + v)}.$$

Sampling statement

`n ~ beta_binomial(N, alpha, beta)`

Increment target log probability density with `beta_binomial_lupmf(n | N, alpha, beta)`.
Available since 2.0

Stan functions

real **beta_binomial_lpmf**(ints n | ints N, reals alpha, reals beta)
The log beta-binomial probability mass of n successes in N trials given prior success count (plus one) of alpha and prior failure count (plus one) of beta
Available since 2.12

real **beta_binomial_lupmf**(ints n | ints N, reals alpha, reals beta)
The log beta-binomial probability mass of n successes in N trials given prior success count (plus one) of alpha and prior failure count (plus one) of beta dropping constant additive terms
Available since 2.25

real **beta_binomial_cdf**(ints n, ints N, reals alpha, reals beta)
The beta-binomial cumulative distribution function of n successes in N trials given prior success count (plus one) of alpha and prior failure count (plus one) of beta
Available since 2.0

real **beta_binomial_lcdf**(ints n | ints N, reals alpha, reals beta)
The log of the beta-binomial cumulative distribution function of n successes in N trials given prior success count (plus one) of alpha and prior failure count (plus one) of beta
Available since 2.12

real **beta_binomial_lccdf**(ints n | ints N, reals alpha, reals beta)
The log of the beta-binomial complementary cumulative distribution function of n successes in N trials given prior success count (plus one) of alpha and prior failure count (plus one) of beta
Available since 2.12

R **beta_binomial_rng**(ints N, reals alpha, reals beta)
Generate a beta-binomial variate with N trials, prior success count (plus one) of
alpha, and prior failure count (plus one) of beta; may only be used in transformed
data and generated quantities blocks. For a description of argument and return
types, see section vectorized PRNG functions.
Available since 2.18

14.4.   Hypergeometric distribution

Probability mass function

If $a \in \mathbb{N}$, $b \in \mathbb{N}$, and $N \in \{0, \ldots, a+b\}$, then for $n \in \{\max(0, N-b), \ldots, \min(a, N)\}$,

$$\text{Hypergeometric}(n \mid N, a, b) = \frac{\binom{a}{n}\binom{b}{N-n}}{\binom{a+b}{N}}.$$

Sampling statement

n ~ **hypergeometric**(N, a, b)

Increment target log probability density with hypergeometric_lupmf(n | N, a,
b).
Available since 2.0

Stan functions

real **hypergeometric_lpmf**(int n | int N, int a, int b)
The log hypergeometric probability mass of n successes in N trials given total success
count of a and total failure count of b
Available since 2.12

real **hypergeometric_lupmf**(int n | int N, int a, int b)
The log hypergeometric probability mass of n successes in N trials given total success
count of a and total failure count of b dropping constant additive terms
Available since 2.25

int **hypergeometric_rng**(int N, int a, int b)
Generate a hypergeometric variate with N trials, total success count of a, and total
failure count of b; may only be used in transformed data and generated quantities
blocks
Available since 2.18

14.5. Categorical distribution

Probability mass functions

If $N \in \mathbb{N}$, $N > 0$, and if $\theta \in \mathbb{R}^N$ forms an N-simplex (i.e., has nonnegative entries summing to one), then for $y \in \{1, \ldots, N\}$,

$$\text{Categorical}(y \mid \theta) = \theta_y.$$

In addition, Stan provides a log-odds scaled categorical distribution,

$$\text{CategoricalLogit}(y \mid \beta) = \text{Categorical}(y \mid \text{softmax}(\beta)).$$

See the definition of softmax for the definition of the softmax function.

Sampling statement

`y ~ categorical(theta)`

Increment target log probability density with `categorical_lupmf(y | theta)` dropping constant additive terms.
Available since 2.0

Sampling statement

`y ~ categorical_logit(beta)`

Increment target log probability density with `categorical_logit_lupmf(y | beta)`.
Available since 2.4

Stan functions

All of the categorical distributions are vectorized so that the outcome y can be a single integer (type `int`) or an array of integers (type `array[] int`).

`real categorical_lpmf(ints y | vector theta)`
The log categorical probability mass function with outcome(s) y in $1 : N$ given N-vector of outcome probabilities theta. The parameter theta must have non-negative entries that sum to one, but it need not be a variable declared as a simplex.
Available since 2.12

`real categorical_lupmf(ints y | vector theta)`
The log categorical probability mass function with outcome(s) y in $1 : N$ given N-vector of outcome probabilities theta dropping constant additive terms. The

parameter theta must have non-negative entries that sum to one, but it need not be a variable declared as a simplex.
Available since 2.25

real **categorical_logit_lpmf**(ints y | vector beta)
The log categorical probability mass function with outcome(s) y in $1 : N$ given log-odds of outcomes beta.
Available since 2.12

real **categorical_logit_lupmf**(ints y | vector beta)
The log categorical probability mass function with outcome(s) y in $1 : N$ given log-odds of outcomes beta dropping constant additive terms.
Available since 2.25

int **categorical_rng**(vector theta)
Generate a categorical variate with N-simplex distribution parameter theta; may only be used in transformed data and generated quantities blocks
Available since 2.0

int **categorical_logit_rng**(vector beta)
Generate a categorical variate with outcome in range $1 : N$ from log-odds vector beta; may only be used in transformed data and generated quantities blocks
Available since 2.16

14.6. Categorical logit generalized linear model (softmax regression)

Stan also supplies a single function for a generalized linear model with categorical likelihood and logit link function, i.e. a function for a softmax regression. This provides a more efficient implementation of softmax regression than a manually written regression in terms of a Categorical likelihood and matrix multiplication.

Note that the implementation does not put any restrictions on the coefficient matrix β. It is up to the user to use a reference category, a suitable prior or some other means of identifiability. See Multi-logit in the Stan User's Guide.

Probability mass functions

If $N, M, K \in \mathbb{N}, \ N, M, K > 0$, and if $x \in \mathbb{R}^{M \times K}, \alpha \in \mathbb{R}^N, \beta \in \mathbb{R}^{K \cdot N}$, then for $y \in \{1, \ldots, N\}^M$,

$$\text{CategoricalLogitGLM}(y \mid x, \alpha, \beta) = \prod_{1 \leq i \leq M} \text{CategoricalLogit}(y_i \mid \alpha + x_i \cdot \beta) = \prod_{1 \leq i \leq M} \text{Categorica}$$

See the definition of softmax for the definition of the softmax function.

Sampling statement

`y ~ categorical_logit_glm(x, alpha, beta)`

Increment target log probability density with `categorical_logit_glm_lupmf(y | x, alpha, beta)`.
Available since 2.23

Stan functions

real **categorical_logit_glm_lpmf**(int y | row_vector x, vector alpha, matrix beta)
The log categorical probability mass function with outcome y in $1 : N$ given N-vector of log-odds of outcomes `alpha + x * beta`.
Available since 2.23

real **categorical_logit_glm_lupmf**(int y | row_vector x, vector alpha, matrix beta)
The log categorical probability mass function with outcome y in $1 : N$ given N-vector of log-odds of outcomes `alpha + x * beta` dropping constant additive terms.
Available since 2.25

real   **categorical_logit_glm_lpmf**(int y | matrix x, vector alpha, matrix beta)
The log categorical probability mass function with outcomes y in $1 : N$ given N-vector of log-odds of outcomes `alpha + x * beta`.
Available since 2.23

real   **categorical_logit_glm_lupmf**(int y | matrix x, vector alpha, matrix beta)
The log categorical probability mass function with outcomes y in $1 : N$ given N-vector of log-odds of outcomes `alpha + x * beta` dropping constant additive terms.
Available since 2.25

real **categorical_logit_glm_lpmf**(array[] int y | row_vector x, vector alpha, matrix beta)
The log categorical probability mass function with outcomes y in $1 : N$ given N-vector of log-odds of outcomes `alpha + x * beta`.
Available since 2.23

real **categorical_logit_glm_lupmf**(array[] int y | row_vector x, vector alpha, matrix beta)

The log categorical probability mass function with outcomes y in $1 : N$ given N-vector of log-odds of outcomes alpha + x * beta dropping constant additive terms.
Available since 2.25

real   **categorical_logit_glm_lpmf**(array[] int y | matrix x, vector alpha, matrix beta)

The log categorical probability mass function with outcomes y in $1 : N$ given N-vector of log-odds of outcomes alpha + x * beta.
Available since 2.23

real   **categorical_logit_glm_lupmf**(array[] int y | matrix x, vector alpha, matrix beta)

The log categorical probability mass function with outcomes y in $1 : N$ given N-vector of log-odds of outcomes alpha + x * beta dropping constant additive terms.
Available since 2.25

14.7.   Discrete range distribution

Probability mass functions

If $l, u \in \mathbb{Z}$ are lower and upper bounds ($l \leq u$), then for any integer $y \in \{l, \ldots, u\}$,

$$\text{DiscreteRange}(y \mid l, u) = \frac{1}{u - l + 1}.$$

Sampling statement

y ~ **discrete_range**(l, u)

Increment the target log probability density with discrete_range_lupmf(y | l, u) dropping constant additive terms.
Available since 2.26

Stan functions

All of the discrete range distributions are vectorized so that the outcome y and the bounds l, u can be a single integer (type int) or an array of integers (type array[] int).

```
real discrete_range_lpmf(ints y | ints l, ints u)
```
The log probability mass function with outcome(s) y in $l : u$.
Available since 2.26

```
real discrete_range_lupmf(ints y | ints l, ints u)
```
The log probability mass function with outcome(s) y in $l : u$ dropping constant additive terms.
Available since 2.26

```
real discrete_range_cdf(ints y, ints l, ints u)
```
The discrete range cumulative distribution function for the given y, lower and upper bounds.
Available since 2.26

```
real discrete_range_lcdf(ints y | ints l, ints u)
```
The log of the discrete range cumulative distribution function for the given y, lower and upper bounds.
Available since 2.26

```
real discrete_range_lccdf(ints y | ints l, ints u)
```
The log of the discrete range complementary cumulative distribution function for the given y, lower and upper bounds.
Available since 2.26

```
int discrete_range_rng(ints l, ints u)
```
Generate a discrete variate between the given lower and upper bounds; may only be used in transformed data and generated quantities blocks.
Available since 2.26

14.8.   Ordered logistic distribution

Probability mass function

If $K \in \mathbb{N}$ with $K > 2$, $c \in \mathbb{R}^{K-1}$ such that $c_k < c_{k+1}$ for $k \in \{1, \ldots, K-2\}$, and $\eta \in \mathbb{R}$, then for $k \in \{1, \ldots, K\}$,

$$
\text{OrderedLogistic}(k \mid \eta, c) = \begin{cases} 1 - \text{logit}^{-1}(\eta - c_1) & \text{if } k = 1, \\ \text{logit}^{-1}(\eta - c_{k-1}) - \text{logit}^{-1}(\eta - c_k) & \text{if } 1 < k < K, \text{and} \\ \text{logit}^{-1}(\eta - c_{K-1}) - 0 & \text{if } k = K. \end{cases}
$$

The $k = K$ case is written with the redundant subtraction of zero to illustrate the parallelism of the cases; the $k = 1$ and $k = K$ edge cases can be subsumed into the

general definition by setting $c_0 = -\infty$ and $c_K = +\infty$ with $\mathrm{logit}^{-1}(-\infty) = 0$ and $\mathrm{logit}^{-1}(\infty) = 1$.

Sampling statement

`k ~ ordered_logistic(eta, c)`

Increment target log probability density with `ordered_logistic_lupmf(k | eta, c)`.
Available since 2.0

Stan functions

real **ordered_logistic_lpmf**(ints k | vector eta, vectors c)
The log ordered logistic probability mass of k given linear predictors `eta`, and cutpoints `c`.
Available since 2.18

real **ordered_logistic_lupmf**(ints k | vector eta, vectors c)
The log ordered logistic probability mass of k given linear predictors `eta`, and cutpoints `c` dropping constant additive terms.
Available since 2.25

int **ordered_logistic_rng**(real eta, vector c)
Generate an ordered logistic variate with linear predictor `eta` and cutpoints `c`; may only be used in transformed data and generated quantities blocks
Available since 2.0

14.9. Ordered logistic generalized linear model (ordinal regression)

Probability mass function

If $N, M, K \in \mathbb{N}$ with $N, M > 0$, $K > 2$, $c \in \mathbb{R}^{K-1}$ such that $c_k < c_{k+1}$ for $k \in \{1, \ldots, K-2\}$, and $x \in \mathbb{R}^{N \times M}, \beta \in \mathbb{R}^M$, then for $y \in \{1, \ldots, K\}^N$,

$$
\mathrm{OrderedLogisticGLM}(y \mid x, \beta, c) = \prod_{1 \leq i \leq N} \mathrm{OrderedLogistic}(y_i \mid x_i \cdot \beta, c) = \prod_{1 \leq i \leq N} \left\{ \begin{array}{l} 1 - \mathrm{logit}^{-1} \\ \mathrm{logit}^{-1}(x_i \\ \mathrm{logit}^{-1}(x_i \end{array} \right.
$$

The $k = K$ case is written with the redundant subtraction of zero to illustrate the parallelism of the cases; the $y = 1$ and $y = K$ edge cases can be subsumed into the general definition by setting $c_0 = -\infty$ and $c_K = +\infty$ with $\text{logit}^{-1}(-\infty) = 0$ and $\text{logit}^{-1}(\infty) = 1$.

Sampling statement

```
y ~ ordered_logistic_glm(x, beta, c)
```

Increment target log probability density with `ordered_logistic_lupmf(y | x, beta, c)`.
Available since 2.23

Stan functions

real `ordered_logistic_glm_lpmf`(int y | row_vector x, vector beta, vector c)
The log ordered logistic probability mass of y, given linear predictors x * beta, and cutpoints c. The cutpoints c must be ordered.
Available since 2.23

real `ordered_logistic_glm_lupmf`(int y | row_vector x, vector beta, vector c)
The log ordered logistic probability mass of y, given linear predictors x * beta, and cutpoints c dropping constant additive terms. The cutpoints c must be ordered.
Available since 2.25

real `ordered_logistic_glm_lpmf`(int y | matrix x, vector beta, vector c)
The log ordered logistic probability mass of y, given linear predictors x * beta, and cutpoints c. The cutpoints c must be ordered.
Available since 2.23

real `ordered_logistic_glm_lupmf`(int y | matrix x, vector beta, vector c)
The log ordered logistic probability mass of y, given linear predictors x * beta, and cutpoints c dropping constant additive terms. The cutpoints c must be ordered.
Available since 2.25

real `ordered_logistic_glm_lpmf`(array[] int y | row_vector x, vector beta, vector c)
The log ordered logistic probability mass of y, given linear predictors x * beta, and

cutpoints c. The cutpoints `c` must be ordered.
Available since 2.23

real **ordered_logistic_glm_lupmf**(array[] int y | row_vector x, vector
beta, vector c)
The log ordered logistic probability mass of y, given linear predictors x * beta, and
cutpoints c dropping constant additive terms. The cutpoints `c` must be ordered.
Available since 2.25

real **ordered_logistic_glm_lpmf**(array[] int y | matrix x, vector beta,
vector c)
The log ordered logistic probability mass of y, given linear predictors x * beta, and
cutpoints c. The cutpoints `c` must be ordered.
Available since 2.23

real   **ordered_logistic_glm_lupmf**(array[] int y | matrix x, vector
beta, vector c)
The log ordered logistic probability mass of y, given linear predictors x * beta, and
cutpoints c dropping constant additive terms. The cutpoints `c` must be ordered.
Available since 2.25

14.10. Ordered probit distribution

Probability mass function

If $K \in \mathbb{N}$ with $K > 2$, $c \in \mathbb{R}^{K-1}$ such that $c_k < c_{k+1}$ for $k \in \{1, \ldots, K-2\}$, and
$\eta \in \mathbb{R}$, then for $k \in \{1, \ldots, K\}$,

$$\text{OrderedProbit}(k \mid \eta, c) = \begin{cases} 1 - \Phi(\eta - c_1) & \text{if } k = 1, \\ \Phi(\eta - c_{k-1}) - \Phi(\eta - c_k) & \text{if } 1 < k < K, \text{and} \\ \Phi(\eta - c_{K-1}) - 0 & \text{if } k = K. \end{cases}$$

The $k = K$ case is written with the redundant subtraction of zero to illustrate the
parallelism of the cases; the $k = 1$ and $k = K$ edge cases can be subsumed into
the general definition by setting $c_0 = -\infty$ and $c_K = +\infty$ with $\Phi(-\infty) = 0$ and
$\Phi(\infty) = 1$.

Sampling statement

k ~ **ordered_probit**(eta, c)

Increment target log probability density with ordered_probit_lupmf(k | eta,
c).

Available since 2.19

Stan functions

`real` **`ordered_probit_lpmf`**`(ints k | vector eta, vectors c)`
The log ordered probit probability mass of k given linear predictors eta, and cutpoints c.
Available since 2.18

`real` **`ordered_probit_lupmf`**`(ints k | vector eta, vectors c)`
The log ordered probit probability mass of k given linear predictors eta, and cutpoints c dropping constant additive terms.
Available since 2.25

`int` **`ordered_probit_rng`**`(real eta, vector c)`
Generate an ordered probit variate with linear predictor eta and cutpoints c; may only be used in transformed data and generated quantities blocks
Available since 2.18

15. Unbounded Discrete Distributions

The unbounded discrete distributions have support over the natural numbers (i.e., the non-negative integers).

15.1. Negative binomial distribution

For the negative binomial distribution Stan uses the parameterization described in Gelman et al. (2013). For alternative parameterizations, see section negative binomial glm.

Probability mass function

If $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}^+$, then for $n \in \mathbb{N}$,

$$\text{NegBinomial}(n \mid \alpha, \beta) = \binom{n + \alpha - 1}{\alpha - 1} \left(\frac{\beta}{\beta + 1}\right)^\alpha \left(\frac{1}{\beta + 1}\right)^n.$$

The mean and variance of a random variable $n \sim \text{NegBinomial}(\alpha, \beta)$ are given by

$$\mathbb{E}[n] = \frac{\alpha}{\beta} \quad \text{and} \quad \text{Var}[n] = \frac{\alpha}{\beta^2}(\beta + 1).$$

Sampling statement

`n ~ `**`neg_binomial`**`(alpha, beta)`

Increment target log probability density with `neg_binomial_lupmf(n | alpha, beta)`.
Available since 2.0

Stan functions

real **`neg_binomial_lpmf`**`(ints n | reals alpha, reals beta)`
The log negative binomial probability mass of n given shape `alpha` and inverse scale `beta`
Available since 2.12

real **`neg_binomial_lupmf`**`(ints n | reals alpha, reals beta)`
The log negative binomial probability mass of n given shape `alpha` and inverse scale

beta dropping constant additive terms
Available since 2.25

real **neg_binomial_cdf**(ints n, reals alpha, reals beta)
The negative binomial cumulative distribution function of n given shape alpha and inverse scale beta
Available since 2.0

real **neg_binomial_lcdf**(ints n | reals alpha, reals beta)
The log of the negative binomial cumulative distribution function of n given shape alpha and inverse scale beta
Available since 2.12

real **neg_binomial_lccdf**(ints n | reals alpha, reals beta)
The log of the negative binomial complementary cumulative distribution function of n given shape alpha and inverse scale beta
Available since 2.12

R **neg_binomial_rng**(reals alpha, reals beta)
Generate a negative binomial variate with shape alpha and inverse scale beta; may only be used in transformed data and generated quantities blocks. alpha / beta must be less than 2^{29}. For a description of argument and return types, see section vectorized function signatures.
Available since 2.18

15.2. Negative binomial distribution (alternative parameterization)

Stan also provides an alternative parameterization of the negative binomial distribution directly using a mean (i.e., location) parameter and a parameter that controls overdispersion relative to the square of the mean. Section combinatorial functions, below, provides a second alternative parameterization directly in terms of the log mean.

Probability mass function

The first parameterization is for $\mu \in \mathbb{R}^+$ and $\phi \in \mathbb{R}^+$, which for $n \in \mathbb{N}$ is defined as

$$\text{NegBinomial2}(n \,|\, \mu, \phi) = \binom{n + \phi - 1}{n} \left(\frac{\mu}{\mu + \phi}\right)^n \left(\frac{\phi}{\mu + \phi}\right)^\phi.$$

The mean and variance of a random variable $n \sim \text{NegBinomial2}(n \mid \mu, \phi)$ are

$$\mathbb{E}[n] = \mu \quad \text{and} \quad \text{Var}[n] = \mu + \frac{\mu^2}{\phi}.$$

Recall that $\text{Poisson}(\mu)$ has variance μ, so $\mu^2/\phi > 0$ is the additional variance of the negative binomial above that of the Poisson with mean μ. So the inverse of parameter ϕ controls the overdispersion, scaled by the square of the mean, μ^2.

Sampling statement

`n ~ `**`neg_binomial_2`**`(mu, phi)`

Increment target log probability density with neg_binomial_2_lupmf(n | mu, phi).
Available since 2.3

Stan functions

real **neg_binomial_2_lpmf**(ints n | reals mu, reals phi)
The negative binomial probability mass of n given location mu and precision phi.
Available since 2.20

real **neg_binomial_2_lupmf**(ints n | reals mu, reals phi)
The negative binomial probability mass of n given location mu and precision phi dropping constant additive terms.
Available since 2.25

real **neg_binomial_2_cdf**(ints n, reals mu, reals phi)
The negative binomial cumulative distribution function of n given location mu and precision phi.
Available since 2.6

real **neg_binomial_2_lcdf**(ints n | reals mu, reals phi)
The log of the negative binomial cumulative distribution function of n given location mu and precision phi.
Available since 2.12

real **neg_binomial_2_lccdf**(ints n | reals mu, reals phi)
The log of the negative binomial complementary cumulative distribution function of n given location mu and precision phi.
Available since 2.12

R **neg_binomial_2_rng**(reals mu, reals phi)
Generate a negative binomial variate with location mu and precision phi; may only

be used in transformed data and generated quantities blocks. `mu` must be less than 2^{29}. For a description of argument and return types, see section vectorized function signatures.
Available since 2.18

15.3. Negative binomial distribution (log alternative parameterization)

Related to the parameterization in section negative binomial, alternative parameterization, the following parameterization uses a log mean parameter $\eta = \log(\mu)$, defined for $\eta \in \mathbb{R}$, $\phi \in \mathbb{R}^+$, so that for $n \in \mathbb{N}$,

$$\text{NegBinomial2Log}(n \mid \eta, \phi) = \text{NegBinomial2}(n \mid \exp(\eta), \phi).$$

This alternative may be used for sampling, as a function, and for random number generation, but as of yet, there are no CDFs implemented for it. This is especially useful for log-linear negative binomial regressions.

Sampling statement

`n ~ `**`neg_binomial_2_log`**`(eta, phi)`

Increment target log probability density with `neg_binomial_2_log_lupmf(n | eta, phi)`.
Available since 2.3

Stan functions

real **`neg_binomial_2_log_lpmf`**`(ints n | reals eta, reals phi)`
The log negative binomial probability mass of `n` given log-location `eta` and inverse overdispersion parameter `phi`.
Available since 2.20

real **`neg_binomial_2_log_lupmf`**`(ints n | reals eta, reals phi)`
The log negative binomial probability mass of `n` given log-location `eta` and inverse overdispersion parameter `phi` dropping constant additive terms.
Available since 2.25

R **`neg_binomial_2_log_rng`**`(reals eta, reals phi)`
Generate a negative binomial variate with log-location `eta` and inverse overdispersion control `phi`; may only be used in transformed data and generated quantities blocks. `eta` must be less than $29 \log 2$. For a description of argument and return

types, see section vectorized function signatures.
Available since 2.18

15.4. Negative-binomial-2-log generalized linear model (negative binomial regression)

Stan also supplies a single function for a generalized linear model with negative binomial likelihood and log link function, i.e. a function for a negative binomial regression. This provides a more efficient implementation of negative binomial regression than a manually written regression in terms of a negative binomial likelihood and matrix multiplication.

Probability mass function

If $x \in \mathbb{R}^{n \cdot m}, \alpha \in \mathbb{R}^n, \beta \in \mathbb{R}^m, \phi \in \mathbb{R}^+$, then for $y \in \mathbb{N}^n$,

$$\text{NegBinomial2LogGLM}(y \mid x, \alpha, \beta, \phi) = \prod_{1 \leq i \leq n} \text{NegBinomial2}(y_i \mid \exp(\alpha_i + x_i \cdot \beta), \phi).$$

Sampling statement

`y ~ neg_binomial_2_log_glm(x, alpha, beta, phi)`

Increment target log probability density with `neg_binomial_2_log_glm_lupmf(y | x, alpha, beta, phi)`.
Available since 2.19

Stan functions

`real neg_binomial_2_log_glm_lpmf(int y | matrix x, real alpha, vector beta, real phi)`
The log negative binomial probability mass of y given log-location `alpha + x * beta` and inverse overdispersion parameter `phi`.
Available since 2.23

`real neg_binomial_2_log_glm_lupmf(int y | matrix x, real alpha, vector beta, real phi)`
The log negative binomial probability mass of y given log-location `alpha + x * beta` and inverse overdispersion parameter `phi` dropping constant additive terms.
Available since 2.25

`real neg_binomial_2_log_glm_lpmf(int y | matrix x, vector alpha, vector beta, real phi)`

The log negative binomial probability mass of y given log-location `alpha + x * beta` and inverse overdispersion parameter `phi`.
Available since 2.23

real   **neg_binomial_2_log_glm_lupmf**(int y | matrix x, vector alpha,
vector beta, real phi)
The log negative binomial probability mass of y given log-location `alpha + x * beta` and inverse overdispersion parameter `phi` dropping constant additive terms.
Available since 2.25

real   **neg_binomial_2_log_glm_lpmf**(array[] int y | row_vector x, real
alpha, vector beta, real phi)
The log negative binomial probability mass of y given log-location `alpha + x * beta` and inverse overdispersion parameter `phi`.
Available since 2.23

real **neg_binomial_2_log_glm_lupmf**(array[] int y | row_vector x, real
alpha, vector beta, real phi)
The log negative binomial probability mass of y given log-location `alpha + x * beta` and inverse overdispersion parameter `phi` dropping constant additive terms.
Available since 2.25

real **neg_binomial_2_log_glm_lpmf**(array[] int y | row_vector x, vector
alpha, vector beta, real phi)
The log negative binomial probability mass of y given log-location `alpha + x * beta` and inverse overdispersion parameter `phi`.
Available since 2.23

real   **neg_binomial_2_log_glm_lupmf**(array[] int y | row_vector x,
vector alpha, vector beta, real phi)
The log negative binomial probability mass of y given log-location `alpha + x * beta` and inverse overdispersion parameter `phi` dropping constant additive terms.
Available since 2.25

real   **neg_binomial_2_log_glm_lpmf**(array[] int y | matrix x, real
alpha, vector beta, real phi)
The log negative binomial probability mass of y given log-location `alpha + x * beta` and inverse overdispersion parameter `phi`.
Available since 2.18

real   **neg_binomial_2_log_glm_lupmf**(array[] int y | matrix x, real
alpha, vector beta, real phi)
The log negative binomial probability mass of y given log-location `alpha + x *`

beta and inverse overdispersion parameter `phi` dropping constant additive terms.
Available since 2.25

`real     neg_binomial_2_log_glm_lpmf`(array[] int y | matrix x, vector
alpha, vector beta, real phi)
The log negative binomial probability mass of y given log-location `alpha + x *`
`beta` and inverse overdispersion parameter `phi`.
Available since 2.18

`real     neg_binomial_2_log_glm_lupmf`(array[] int y | matrix x, vector
alpha, vector beta, real phi)
The log negative binomial probability mass of y given log-location `alpha + x *`
`beta` and inverse overdispersion parameter `phi` dropping constant additive terms.
Available since 2.25

15.5.   Poisson distribution

Probability mass function

If $\lambda \in \mathbb{R}^+$, then for $n \in \mathbb{N}$,

$$\text{Poisson}(n|\lambda) = \frac{1}{n!}\,\lambda^n\,\exp(-\lambda).$$

Sampling statement

`n ~ poisson(lambda)`

Increment target log probability density with `poisson_lupmf(n | lambda)`.
Available since 2.0

Stan functions

`real poisson_lpmf`(ints n | reals lambda)
The log Poisson probability mass of n given rate lambda
Available since 2.12

`real poisson_lupmf`(ints n | reals lambda)
The log Poisson probability mass of n given rate lambda dropping constant additive
terms
Available since 2.25

`real poisson_cdf`(ints n, reals lambda)
The Poisson cumulative distribution function of n given rate lambda
Available since 2.0

```
real poisson_lcdf(ints n | reals lambda)
```
The log of the Poisson cumulative distribution function of n given rate lambda
Available since 2.12

```
real poisson_lccdf(ints n | reals lambda)
```
The log of the Poisson complementary cumulative distribution function of n given
rate lambda
Available since 2.12

```
R poisson_rng(reals lambda)
```
Generate a Poisson variate with rate lambda; may only be used in transformed data
and generated quantities blocks. lambda must be less than 2^{30}. For a description of
argument and return types, see section vectorized function signatures.
Available since 2.18

15.6. Poisson distribution, log parameterization

Stan also provides a parameterization of the Poisson using the log rate $\alpha = \log \lambda$ as
a parameter. This is useful for log-linear Poisson regressions so that the predictor
does not need to be exponentiated and passed into the standard Poisson probability
function.

Probability mass function

If $\alpha \in \mathbb{R}$, then for $n \in \mathbb{N}$,

$$\text{PoissonLog}(n|\alpha) = \frac{1}{n!} \exp\left(n\alpha - \exp(\alpha)\right).$$

Sampling statement

```
n ~ poisson_log(alpha)
```

Increment target log probability density with `poisson_log_lupmf(n | alpha)`.
Available since 2.0

Stan functions

```
real poisson_log_lpmf(ints n | reals alpha)
```
The log Poisson probability mass of n given log rate alpha
Available since 2.12

```
real poisson_log_lupmf(ints n | reals alpha)
```
The log Poisson probability mass of n given log rate alpha dropping constant additive

terms
Available since 2.25

R `poisson_log_rng`(reals alpha)
Generate a Poisson variate with log rate alpha; may only be used in transformed data
and generated quantities blocks. alpha must be less than $30 \log 2$. For a description
of argument and return types, see section vectorized function signatures.
Available since 2.18

15.7.   Poisson-log generalized linear model (Poisson regression)

Stan also supplies a single function for a generalized linear model with Poisson
likelihood and log link function, i.e. a function for a Poisson regression. This
provides a more efficient implementation of Poisson regression than a manually
written regression in terms of a Poisson likelihood and matrix multiplication.

Probability mass function

If $x \in \mathbb{R}^{n \cdot m}, \alpha \in \mathbb{R}^n, \beta \in \mathbb{R}^m$, then for $y \in \mathbb{N}^n$,

$$\text{PoisonLogGLM}(y|x, \alpha, \beta) = \prod_{1 \le i \le n} \text{Poisson}(y_i| \exp(\alpha_i + x_i \cdot \beta)).$$

Sampling statement

y ~ `poisson_log_glm`(x, alpha, beta)

Increment target log probability density with poisson_log_glm_lupmf(y | x,
alpha, beta).
Available since 2.19

Stan functions

real `poisson_log_glm_lpmf`(int y | matrix x, real alpha, vector beta)
The log Poisson probability mass of y given the log-rate alpha + x * beta.
Available since 2.23

real `poisson_log_glm_lupmf`(int y | matrix x, real alpha, vector beta)
The log Poisson probability mass of y given the log-rate alpha + x * beta dropping
constant additive terms.
Available since 2.25

real   `poisson_log_glm_lpmf`(int y | matrix x, vector alpha, vector
beta)

The log Poisson probability mass of y given the log-rate `alpha + x * beta`.
Available since 2.23

real   **poisson_log_glm_lupmf**(int y | matrix x, vector alpha, vector
beta)
The log Poisson probability mass of y given the log-rate `alpha + x * beta` dropping
constant additive terms.
Available since 2.25

real   **poisson_log_glm_lpmf**(array[] int y | row_vector x, real alpha,
vector beta)
The log Poisson probability mass of y given the log-rate `alpha + x * beta`.
Available since 2.23

real   **poisson_log_glm_lupmf**(array[] int y | row_vector x, real alpha,
vector beta)
The log Poisson probability mass of y given the log-rate `alpha + x * beta` dropping
constant additive terms.
Available since 2.25

real **poisson_log_glm_lpmf**(array[] int y | row_vector x, vector alpha,
vector beta)
The log Poisson probability mass of y given the log-rate `alpha + x * beta`.
Available since 2.23

real   **poisson_log_glm_lupmf**(array[] int y | row_vector x, vector
alpha, vector beta)
The log Poisson probability mass of y given the log-rate `alpha + x * beta` dropping
constant additive terms.
Available since 2.25

real   **poisson_log_glm_lpmf**(array[] int y | matrix x, real alpha,
vector beta)
The log Poisson probability mass of y given the log-rate `alpha + x * beta`.
Available since 2.18

real   **poisson_log_glm_lupmf**(array[] int y | matrix x, real alpha,
vector beta)
The log Poisson probability mass of y given the log-rate `alpha + x * beta` dropping
constant additive terms.
Available since 2.25

real   **poisson_log_glm_lpmf**(array[] int y | matrix x, vector alpha,

vector beta)
The log Poisson probability mass of y given the log-rate `alpha + x * beta`.
Available since 2.18

`real`   **`poisson_log_glm_lupmf`**`(array[] int y | matrix x, vector alpha,`
`vector beta)`
The log Poisson probability mass of y given the log-rate `alpha + x * beta` dropping
constant additive terms.
Available since 2.25

16.   Multivariate Discrete Distributions

The multivariate discrete distributions are over multiple integer values, which are expressed in Stan as arrays.

16.1.   Multinomial distribution

Probability mass function

If $K \in \mathbb{N}$, $N \in \mathbb{N}$, and $\theta \in K$-simplex, then for $y \in \mathbb{N}^K$ such that $\sum_{k=1}^{K} y_k = N$,

$$\text{Multinomial}(y|\theta) = \binom{N}{y_1, \ldots, y_K} \prod_{k=1}^{K} \theta_k^{y_k},$$

where the multinomial coefficient is defined by

$$\binom{N}{y_1, \ldots, y_k} = \frac{N!}{\prod_{k=1}^{K} y_k!}.$$

Sampling statement

`y ~ multinomial(theta)`

Increment target log probability density with `multinomial_lupmf(y | theta)`.
Available since 2.0

Stan functions

real **multinomial_lpmf**(array[] int y | vector theta)
The log multinomial probability mass function with outcome array y of size K given the K-simplex distribution parameter theta and (implicit) total count `N = sum(y)`
Available since 2.12

real **multinomial_lupmf**(array[] int y | vector theta)
The log multinomial probability mass function with outcome array y of size K given the K-simplex distribution parameter theta and (implicit) total count `N = sum(y)` dropping constant additive terms
Available since 2.25

```
array[] int multinomial_rng(vector theta, int N)
```
Generate a multinomial variate with simplex distribution parameter theta and total count N; may only be used in transformed data and generated quantities blocks
Available since 2.8

16.2.   Multinomial distribution, logit parameterization

Stan also provides a version of the multinomial probability mass function distribution with the K-simplex for the event count probabilities per category given on the unconstrained logistic scale.

Probability mass function

If $K \in \mathbb{N}$, $N \in \mathbb{N}$, and $\text{softmax}^{-1}(\theta) \in K$-simplex, then for $y \in \mathbb{N}^K$ such that $\sum_{k=1}^{K} y_k = N$,

$$\text{MultinomialLogit}(y|\theta) = \text{Multinomial}(y|\text{softmax}^{-1}(\theta)) = \binom{N}{y_1, \ldots, y_K} \prod_{k=1}^{K} [\text{softmax}^{-1}(\theta)_k]^y$$

where the multinomial coefficient is defined by

$$\binom{N}{y_1, \ldots, y_k} = \frac{N!}{\prod_{k=1}^{K} y_k!}.$$

Sampling statement

```
y ~ multinomial_logit(theta)
```

Increment target log probability density with `multinomial_logit_lupmf(y | theta)`.
Available since 2.24

Stan functions

```
real multinomial_logit_lpmf(array[] int y | vector theta)
```
The log multinomial probability mass function with outcome array y of size K given the K-simplex distribution parameter $\text{softmax}^{-1}(\theta)$ and (implicit) total count `N = sum(y)`
Available since 2.24

```
real multinomial_logit_lupmf(array[] int y | vector theta)
```
The log multinomial probability mass function with outcome array y of size K given

the K-simplex distribution parameter $\text{softmax}^{-1}(\theta)$ and (implicit) total count `N = sum(y)` dropping constant additive terms
Available since 2.25

`array[] int` **`multinomial_logit_rng`**`(vector theta, int N)`
Generate a multinomial variate with simplex distribution parameter $\text{softmax}^{-1}(\theta)$ and total count N; may only be used in transformed data and generated quantities blocks
Available since 2.24

Continuous Distributions

17. Unbounded Continuous Distributions

The unbounded univariate continuous probability distributions have support on all real numbers.

17.1. Normal distribution

Probability density function

If $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$, then for $y \in \mathbb{R}$,

$$\text{Normal}(y|\mu, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right).$$

Sampling statement

`y ~ normal(mu, sigma)`

Increment target log probability density with `normal_lupdf(y | mu, sigma)`.
Available since 2.0

Stan functions

`real normal_lpdf(reals y | reals mu, reals sigma)`
The log of the normal density of y given location mu and scale sigma
Available since 2.12

`real normal_lupdf(reals y | reals mu, reals sigma)`
The log of the normal density of y given location mu and scale sigma dropping constant additive terms.
Available since 2.25

`real normal_cdf(reals y, reals mu, reals sigma)`
The cumulative normal distribution of y given location mu and scale sigma; normal_cdf will underflow to 0 for $\frac{y-\mu}{\sigma}$ below -37.5 and overflow to 1 for $\frac{y-\mu}{\sigma}$ above 8.25; the function `Phi_approx` is more robust in the tails, but must be scaled and translated for anything other than a standard normal.
Available since 2.0

`real` **`normal_lcdf`**`(reals y | reals mu, reals sigma)`
The log of the cumulative normal distribution of y given location mu and scale sigma;
normal_lcdf will underflow to $-\infty$ for $\frac{y-\mu}{\sigma}$ below -37.5 and overflow to 0 for $\frac{y-\mu}{\sigma}$
above 8.25; `log(Phi_approx(...))` is more robust in the tails, but must be scaled
and translated for anything other than a standard normal.
Available since 2.12

`real` **`normal_lccdf`**`(reals y | reals mu, reals sigma)`
The log of the complementary cumulative normal distribution of y given location mu
and scale sigma; normal_lccdf will overflow to 0 for $\frac{y-\mu}{\sigma}$ below -37.5 and underflow
to $-\infty$ for $\frac{y-\mu}{\sigma}$ above 8.25; `log1m(Phi_approx(...))` is more robust in the tails,
but must be scaled and translated for anything other than a standard normal.
Available since 2.15

`R` **`normal_rng`**`(reals mu, reals sigma)`
Generate a normal variate with location mu and scale sigma; may only be used in
transformed data and generated quantities blocks. For a description of argument
and return types, see section vectorized PRNG functions.
Available since 2.18

Standard normal distribution

The standard normal distribution is so-called because its parameters are the units
for their respective operations—the location (mean) is zero and the scale (standard
deviation) one. The standard normal is parameter-free, and the unit parameters
allow considerable simplification of the expression for the density.

$$\text{StdNormal}(y) \;=\; \text{Normal}(y \mid 0, 1) \;=\; \frac{1}{\sqrt{2\pi}} \, \exp\left(\frac{-y^2}{2}\right).$$

Up to a proportion on the log scale, where Stan computes,

$$\log \text{Normal}(y \mid 0, 1) \;=\; \frac{-y^2}{2} + \text{const.}$$

With no logarithm, no subtraction, and no division by a parameter, the standard
normal log density is much more efficient to compute than the normal log density
with constant location 0 and scale 1.

Sampling statement

`y ~ std_normal()`

Increment target log probability density with `std_normal_lupdf(y)`.
Available since 2.19

Stan functions

real **std_normal_lpdf**(reals y)
The standard normal (location zero, scale one) log probability density of y.
Available since 2.18

real **std_normal_lupdf**(reals y)
The standard normal (location zero, scale one) log probability density of y dropping constant additive terms.
Available since 2.25

real **std_normal_cdf**(reals y)
The cumulative standard normal distribution of y; std_normal_cdf will underflow to 0 for y below -37.5 and overflow to 1 for y above 8.25; the function `Phi_approx` is more robust in the tails.
Available since 2.21

real **std_normal_lcdf**(reals y)
The log of the cumulative standard normal distribution of y; std_normal_lcdf will underflow to $-\infty$ for y below -37.5 and overflow to 0 for y above 8.25; `log(Phi_approx(...))` is more robust in the tails.
Available since 2.21

real **std_normal_lccdf**(reals y)
The log of the complementary cumulative standard normal distribution of y; std_normal_lccdf will overflow to 0 for y below -37.5 and underflow to $-\infty$ for y above 8.25; `log1m(Phi_approx(...))` is more robust in the tails.
Available since 2.21

real **std_normal_rng**()
Generate a normal variate with location zero and scale one; may only be used in transformed data and generated quantities blocks.
Available since 2.21

17.2. Normal-id generalized linear model (linear regression)

Stan also supplies a single function for a generalized linear lodel with normal likelihood and identity link function, i.e. a function for a linear regression. This provides a more efficient implementation of linear regression than a manually written regression in terms of a normal likelihood and matrix multiplication.

Probability distribution function

If $x \in \mathbb{R}^{n \cdot m}, \alpha \in \mathbb{R}^n, \beta \in \mathbb{R}^m, \sigma \in \mathbb{R}^+$, then for $y \in \mathbb{R}^n$,

$$\text{NormalIdGLM}(y|x, \alpha, \beta, \sigma) = \prod_{1 \leq i \leq n} \text{Normal}(y_i|\alpha_i + x_i \cdot \beta, \sigma).$$

Sampling statement

`y ~ normal_id_glm(x, alpha, beta, sigma)`

Increment target log probability density with `normal_id_glm_lupdf(y | x, alpha, beta, sigma)`.
Available since 2.19

Stan functions

real **normal_id_glm_lpdf**(real y | matrix x, real alpha, vector beta, real sigma)
The log normal probability density of y given location `alpha + x * beta` and scale `sigma`.
Available since 2.23

real **normal_id_glm_lupdf**(real y | matrix x, real alpha, vector beta, real sigma)
The log normal probability density of y given location `alpha + x * beta` and scale `sigma` dropping constant additive terms.
Available since 2.25

real **normal_id_glm_lpdf**(real y | matrix x, vector alpha, vector beta, real sigma)
The log normal probability density of y given location `alpha + x * beta` and scale `sigma`.
Available since 2.23

real   **normal_id_glm_lupdf**(real y | matrix x, vector alpha, vector beta, real sigma)
The log normal probability density of y given location `alpha + x * beta` and scale `sigma` dropping constant additive terms.
Available since 2.25

real **normal_id_glm_lpdf**(vector y | row_vector x, real alpha, vector beta, real sigma)

The log normal probability density of y given location `alpha + x * beta` and scale `sigma`.
Available since 2.23

`real` **`normal_id_glm_lupdf`**`(vector y | row_vector x, real alpha, vector beta, real sigma)`
The log normal probability density of y given location `alpha + x * beta` and scale `sigma` dropping constant additive terms.
Available since 2.25

`real` **`normal_id_glm_lpdf`**`(vector y | row_vector x, vector alpha, vector beta, real sigma)`
The log normal probability density of y given location `alpha + x * beta` and scale `sigma`.
Available since 2.23

`real` **`normal_id_glm_lupdf`**`(vector y | row_vector x, vector alpha, vector beta, real sigma)`
The log normal probability density of y given location `alpha + x * beta` and scale `sigma` dropping constant additive terms.
Available since 2.25

`real` **`normal_id_glm_lpdf`**`(vector y | matrix x, real alpha, vector beta, real sigma)`
The log normal probability density of y given location `alpha + x * beta` and scale `sigma`.
Available since 2.20

`real` **`normal_id_glm_lupdf`**`(vector y | matrix x, real alpha, vector beta, real sigma)`
The log normal probability density of y given location `alpha + x * beta` and scale `sigma` dropping constant additive terms.
Available since 2.25

`real` **`normal_id_glm_lpdf`**`(vector y | matrix x, vector alpha, vector beta, real sigma)`
The log normal probability density of y given location `alpha + x * beta` and scale `sigma`.
Available since 2.20

`real` **`normal_id_glm_lupdf`**`(vector y | matrix x, vector alpha, vector beta, real sigma)`
The log normal probability density of y given location `alpha + x * beta` and scale

sigma dropping constant additive terms.
Available since 2.25

17.3.   Exponentially modified normal distribution

Probability density function

If $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$, and $\lambda \in \mathbb{R}^+$, then for $y \in \mathbb{R}$,

$$\text{ExpModNormal}(y|\mu, \sigma, \lambda) = \frac{\lambda}{2}\, \exp\left(\frac{\lambda}{2}\left(2\mu + \lambda\sigma^2 - 2y\right)\right) \text{erfc}\left(\frac{\mu + \lambda\sigma^2 - y}{\sqrt{2}\sigma}\right).$$

Sampling statement

`y ~ exp_mod_normal(mu, sigma, lambda)`

Increment target log probability density with `exp_mod_normal_lupdf(y | mu, sigma, lambda)`.
Available since 2.0

Stan functions

`real        exp_mod_normal_lpdf(reals y | reals mu, reals sigma, reals lambda)`
The log of the exponentially modified normal density of y given location mu, scale sigma, and shape lambda
Available since 2.18

`real        exp_mod_normal_lupdf(reals y | reals mu, reals sigma, reals lambda)`
The log of the exponentially modified normal density of y given location mu, scale sigma, and shape lambda dropping constant additive terms
Available since 2.25

`real exp_mod_normal_cdf(reals y, reals mu, reals sigma, reals lambda)`
The exponentially modified normal cumulative distribution function of y given location mu, scale sigma, and shape lambda
Available since 2.0

`real        exp_mod_normal_lcdf(reals y | reals mu, reals sigma, reals lambda)`
The log of the exponentially modified normal cumulative distribution function of y

given location mu, scale sigma, and shape lambda
Available since 2.18

real   `exp_mod_normal_lccdf`(reals y | reals mu, reals sigma, reals lambda)
The log of the exponentially modified normal complementary cumulative distribution function of y given location mu, scale sigma, and shape lambda
Available since 2.18

R `exp_mod_normal_rng`(reals mu, reals sigma, reals lambda)
Generate a exponentially modified normal variate with location mu, scale sigma, and shape lambda; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.
Available since 2.18

17.4. Skew normal distribution

Probability density function

If $\xi \in \mathbb{R}$, $\omega \in \mathbb{R}^+$, and $\alpha \in \mathbb{R}$, then for $y \in \mathbb{R}$,

$$\text{SkewNormal}(y \mid \xi, \omega, \alpha) = \frac{1}{\omega\sqrt{2\pi}} \, \exp\left(-\frac{1}{2}\left(\frac{y-\xi}{\omega}\right)^2\right)\left(1 + \text{erf}\left(\alpha\left(\frac{y-\xi}{\omega\sqrt{2}}\right)\right)\right).$$

Sampling statement

`y ~ skew_normal(xi, omega, alpha)`

Increment target log probability density with `skew_normal_lupdf(y | xi, omega, alpha)`.
Available since 2.0

Stan functions

real **`skew_normal_lpdf`**(reals y | reals xi, reals omega, reals alpha)
The log of the skew normal density of y given location xi, scale omega, and shape alpha
Available since 2.16

real **`skew_normal_lupdf`**(reals y | reals xi, reals omega, reals alpha)
The log of the skew normal density of y given location xi, scale omega, and shape

alpha dropping constant additive terms
Available since 2.25

real **skew_normal_cdf**(reals y, reals xi, reals omega, reals alpha)
The skew normal distribution function of y given location xi, scale omega, and shape alpha
Available since 2.16

real **skew_normal_lcdf**(reals y | reals xi, reals omega, reals alpha)
The log of the skew normal cumulative distribution function of y given location xi, scale omega, and shape alpha
Available since 2.18

real **skew_normal_lccdf**(reals y | reals xi, reals omega, reals alpha)
The log of the skew normal complementary cumulative distribution function of y given location xi, scale omega, and shape alpha
Available since 2.18

R **skew_normal_rng**(reals xi, reals omega, real alpha)
Generate a skew normal variate with location xi, scale omega, and shape alpha; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.
Available since 2.18

17.5.   Student-t distribution

Probability density function

If $\nu \in \mathbb{R}^+$, $\mu \in \mathbb{R}$, and $\sigma \in \mathbb{R}^+$, then for $y \in \mathbb{R}$,

$$\text{StudentT}(y|\nu,\mu,\sigma) = \frac{\Gamma\left((\nu+1)/2\right)}{\Gamma(\nu/2)} \; \frac{1}{\sqrt{\nu\pi}\;\sigma} \; \left(1 + \frac{1}{\nu}\left(\frac{y-\mu}{\sigma}\right)^2\right)^{-(\nu+1)/2}.$$

Sampling statement

y ~ **student_t**(nu, mu, sigma)

Increment target log probability density with student_t_lupdf(y | nu, mu, sigma).
Available since 2.0

Stan functions

real **student_t_lpdf**(reals y | reals nu, reals mu, reals sigma)
The log of the Student-t density of y given degrees of freedom nu, location mu, and scale sigma
Available since 2.12

real **student_t_lupdf**(reals y | reals nu, reals mu, reals sigma)
The log of the Student-t density of y given degrees of freedom nu, location mu, and scale sigma dropping constant additive terms
Available since 2.25

real **student_t_cdf**(reals y, reals nu, reals mu, reals sigma)
The Student-t cumulative distribution function of y given degrees of freedom nu, location mu, and scale sigma
Available since 2.0

real **student_t_lcdf**(reals y | reals nu, reals mu, reals sigma)
The log of the Student-t cumulative distribution function of y given degrees of freedom nu, location mu, and scale sigma
Available since 2.12

real **student_t_lccdf**(reals y | reals nu, reals mu, reals sigma)
The log of the Student-t complementary cumulative distribution function of y given degrees of freedom nu, location mu, and scale sigma
Available since 2.12

R **student_t_rng**(reals nu, reals mu, reals sigma)
Generate a Student-t variate with degrees of freedom nu, location mu, and scale sigma; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.
Available since 2.18

17.6. Cauchy distribution

Probability density function

If $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$, then for $y \in \mathbb{R}$,

$$\text{Cauchy}(y|\mu, \sigma) = \frac{1}{\pi\sigma} \, \frac{1}{1 + ((y - \mu)/\sigma)^2}.$$

Sampling statement

```
y ~ cauchy(mu, sigma)
```

Increment target log probability density with `cauchy_lupdf(y | mu, sigma)`.
Available since 2.0

Stan functions

real **cauchy_lpdf**(reals y | reals mu, reals sigma)
The log of the Cauchy density of y given location mu and scale sigma
Available since 2.12

real **cauchy_lupdf**(reals y | reals mu, reals sigma)
The log of the Cauchy density of y given location mu and scale sigma dropping
constant additive terms
Available since 2.25

real **cauchy_cdf**(reals y, reals mu, reals sigma)
The Cauchy cumulative distribution function of y given location mu and scale sigma
Available since 2.0

real **cauchy_lcdf**(reals y | reals mu, reals sigma)
The log of the Cauchy cumulative distribution function of y given location mu and
scale sigma
Available since 2.12

real **cauchy_lccdf**(reals y | reals mu, reals sigma)
The log of the Cauchy complementary cumulative distribution function of y given
location mu and scale sigma
Available since 2.12

R **cauchy_rng**(reals mu, reals sigma)
Generate a Cauchy variate with location mu and scale sigma; may only be used in
transformed data and generated quantities blocks. For a description of argument
and return types, see section vectorized PRNG functions.
Available since 2.18

17.7.   Double exponential (Laplace) distribution

Probability density function

If $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$, then for $y \in \mathbb{R}$,

$$\text{DoubleExponential}(y|\mu,\sigma) = \frac{1}{2\sigma} \exp\left(-\frac{|y-\mu|}{\sigma}\right).$$

Note that the double exponential distribution is parameterized in terms of the scale, in contrast to the exponential distribution (see section exponential distribution), which is parameterized in terms of inverse scale.

The double-exponential distribution can be defined as a compound exponential-normal distribution (Ding and Blitzstein 2018). Using the inverse scale parameterization for the exponential distribution, and the standard deviation parameterization for the normal distribution, one can write

$$\alpha \sim \text{Exponential}\left(\frac{1}{2\sigma^2}\right)$$

and

$$\beta \mid \alpha \sim \text{Normal}(\mu, \sqrt{\alpha}),$$

then

$$\beta \sim \text{DoubleExponential}(\mu, \sigma).$$

This may be used to code a non-centered parameterization by taking

$$\beta^{\text{raw}} \sim \text{Normal}(0,1)$$

and defining

$$\beta = \mu + \alpha\,\beta^{\text{raw}}.$$

Sampling statement

`y ~ double_exponential(mu, sigma)`

Increment target log probability density with `double_exponential_lupdf(y | mu, sigma)`.

Available since 2.0

Stan functions

`real` **`double_exponential_lpdf`**`(reals y | reals mu, reals sigma)`
The log of the double exponential density of y given location mu and scale sigma
Available since 2.12

`real` **`double_exponential_lupdf`**`(reals y | reals mu, reals sigma)`
The log of the double exponential density of y given location mu and scale sigma
dropping constant additive terms
Available since 2.25

`real` **`double_exponential_cdf`**`(reals y, reals mu, reals sigma)`
The double exponential cumulative distribution function of y given location mu and
scale sigma
Available since 2.0

`real` **`double_exponential_lcdf`**`(reals y | reals mu, reals sigma)`
The log of the double exponential cumulative distribution function of y given location
mu and scale sigma
Available since 2.12

`real` **`double_exponential_lccdf`**`(reals y | reals mu, reals sigma)`
The log of the double exponential complementary cumulative distribution function
of y given location mu and scale sigma
Available since 2.12

`R` **`double_exponential_rng`**`(reals mu, reals sigma)`
Generate a double exponential variate with location mu and scale sigma; may only
be used in transformed data and generated quantities blocks. For a description of
argument and return types, see section vectorized PRNG functions.
Available since 2.18

17.8.   Logistic distribution

Probability density function

If $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$, then for $y \in \mathbb{R}$,

$$\text{Logistic}(y|\mu, \sigma) = \frac{1}{\sigma} \, \exp\left(-\frac{y-\mu}{\sigma}\right) \, \left(1 + \exp\left(-\frac{y-\mu}{\sigma}\right)\right)^{-2}.$$

Sampling statement

`y ~` **`logistic`**`(mu, sigma)`

Increment target log probability density with `logistic_lupdf(y | mu, sigma)`.
Available since 2.0

Stan functions

`real` **`logistic_lpdf`**`(reals y | reals mu, reals sigma)`
The log of the logistic density of y given location mu and scale sigma
Available since 2.12

`real` **`logistic_lupdf`**`(reals y | reals mu, reals sigma)`
The log of the logistic density of y given location mu and scale sigma dropping
constant additive terms
Available since 2.25

`real` **`logistic_cdf`**`(reals y, reals mu, reals sigma)`
The logistic cumulative distribution function of y given location mu and scale sigma
Available since 2.0

`real` **`logistic_lcdf`**`(reals y | reals mu, reals sigma)`
The log of the logistic cumulative distribution function of y given location mu and
scale sigma
Available since 2.12

`real` **`logistic_lccdf`**`(reals y | reals mu, reals sigma)`
The log of the logistic complementary cumulative distribution function of y given
location mu and scale sigma
Available since 2.12

`R` **`logistic_rng`**`(reals mu, reals sigma)`
Generate a logistic variate with location mu and scale sigma; may only be used in
transformed data and generated quantities blocks. For a description of argument
and return types, see section vectorized PRNG functions.
Available since 2.18

17.9. Gumbel distribution

Probability density function

If $\mu \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$, then for $y \in \mathbb{R}$,

$$\text{Gumbel}(y|\mu, \beta) = \frac{1}{\beta} \, \exp\left(-\frac{y - \mu}{\beta} - \exp\left(-\frac{y - \mu}{\beta}\right)\right).$$

Sampling statement

`y ~ gumbel(mu, beta)`

Increment target log probability density with `gumbel_lupdf(y | mu, beta)`.
Available since 2.0

Stan functions

real **gumbel_lpdf**(reals y | reals mu, reals beta)
The log of the gumbel density of y given location mu and scale beta
Available since 2.12

real **gumbel_lupdf**(reals y | reals mu, reals beta)
The log of the gumbel density of y given location mu and scale beta dropping
constant additive terms
Available since 2.25

real **gumbel_cdf**(reals y, reals mu, reals beta)
The gumbel cumulative distribution function of y given location mu and scale beta
Available since 2.0

real **gumbel_lcdf**(reals y | reals mu, reals beta)
The log of the gumbel cumulative distribution function of y given location mu and
scale beta
Available since 2.12

real **gumbel_lccdf**(reals y | reals mu, reals beta)
The log of the gumbel complementary cumulative distribution function of y given
location mu and scale beta
Available since 2.12

R **gumbel_rng**(reals mu, reals beta)
Generate a gumbel variate with location mu and scale beta; may only be used in
transformed data and generated quantities blocks. For a description of argument
and return types, see section vectorized PRNG functions.
Available since 2.18

17.10.   Skew double exponential distribution

Probability density function

If $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$ and $\tau \in [0, 1]$, then for $y \in \mathbb{R}$,

$$\text{SkewDoubleExponential}(y|\mu, \sigma, \tau) =$$
$$\frac{2\tau(1-\tau)}{\sigma} \exp\left[-\frac{2}{\sigma}\left[(1-\tau)\,I(y < \mu)(\mu - y) + \tau I(y > \mu)(y - \mu)\right]\right]$$

Sampling statement

`y ~ skew_double_exponential(mu, sigma, tau)`

Increment target log probability density with `skew_double_exponential(y | mu, sigma, tau)`
Available since 2.28

Stan functions

`real    skew_double_exponential_lpdf(reals y | reals mu, reals sigma, reals tau)`
The log of the skew double exponential density of y given location mu, scale sigma and skewness tau
Available since 2.28

`real    skew_double_exponential_lupdf(reals y | reals mu, reals sigma, reals tau)`
The log of the skew double exponential density of y given location mu, scale sigma and skewness tau dropping constant additive terms
Available since 2.28

`real     skew_double_exponential_cdf(reals y, reals mu, reals sigma, reals tau)`
The skew double exponential cumulative distribution function of y given location mu, scale sigma and skewness tau
Available since 2.28

`real    skew_double_exponential_lcdf(reals y | reals mu, reals sigma, reals tau)`
The log of the skew double exponential cumulative distribution function of y given location mu, scale sigma and skewness tau
Available since 2.28

real   **skew_double_exponential_lccdf**(reals y | reals mu, reals sigma, reals tau)
The log of the skew double exponential complementary cumulative distribution function of y given location mu, scale sigma and skewness tau
Available since 2.28

R **skew_double_exponential_rng**(reals mu, reals sigma, reals tau)
Generate a skew double exponential variate with location mu, scale sigma and skewness tau; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.
Available since 2.28

18.   Positive Continuous Distributions

The positive continuous probability functions have support on the positive real numbers.

18.1.   Lognormal distribution

Probability density function

If $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$, then for $y \in \mathbb{R}^+$,

$$\text{LogNormal}(y|\mu, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma}\,\frac{1}{y}\,\exp\left(-\frac{1}{2}\,\left(\frac{\log y - \mu}{\sigma}\right)^2\right).$$

Sampling statement

`y ~ lognormal(mu, sigma)`

Increment target log probability density with `lognormal_lupdf(y | mu, sigma)`.
Available since 2.0

Stan functions

real **lognormal_lpdf**(reals y | reals mu, reals sigma)
The log of the lognormal density of y given location mu and scale sigma
Available since 2.12

real **lognormal_lupdf**(reals y | reals mu, reals sigma)
The log of the lognormal density of y given location mu and scale sigma dropping constant additive terms
Available since 2.25

real **lognormal_cdf**(reals y, reals mu, reals sigma)
The cumulative lognormal distribution function of y given location mu and scale sigma
Available since 2.0

real **lognormal_lcdf**(reals y | reals mu, reals sigma)
The log of the lognormal cumulative distribution function of y given location mu

and scale sigma
Available since 2.12

real **lognormal_lccdf**(reals y | reals mu, reals sigma)
The log of the lognormal complementary cumulative distribution function of y given
location mu and scale sigma
Available since 2.12

R **lognormal_rng**(reals mu, reals sigma)
Generate a lognormal variate with location mu and scale sigma; may only be used in
transformed data and generated quantities blocks. For a description of argument
and return types, see section vectorized PRNG functions.
Available since 2.22

18.2. Chi-square distribution

Probability density function

If $\nu \in \mathbb{R}^+$, then for $y \in \mathbb{R}^+$,

$$\text{ChiSquare}(y|\nu) = \frac{2^{-\nu/2}}{\Gamma(\nu/2)}\, y^{\nu/2-1}\, \exp\left(-\frac{1}{2}\, y\right).$$

Sampling statement

y ~ **chi_square**(nu)

Increment target log probability density with chi_square_lupdf(y | nu).
Available since 2.0

Stan functions

real **chi_square_lpdf**(reals y | reals nu)
The log of the Chi-square density of y given degrees of freedom nu
Available since 2.12

real **chi_square_lupdf**(reals y | reals nu)
The log of the Chi-square density of y given degrees of freedom nu dropping constant
additive terms
Available since 2.25

real **chi_square_cdf**(reals y, reals nu)
The Chi-square cumulative distribution function of y given degrees of freedom nu
Available since 2.0

```
real chi_square_lcdf(reals y | reals nu)
```
The log of the Chi-square cumulative distribution function of y given degrees of freedom nu
Available since 2.12

```
real chi_square_lccdf(reals y | reals nu)
```
The log of the complementary Chi-square cumulative distribution function of y given degrees of freedom nu
Available since 2.12

```
R chi_square_rng(reals nu)
```
Generate a Chi-square variate with degrees of freedom nu; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.
Available since 2.18

18.3. Inverse chi-square distribution

Probability density function

If $\nu \in \mathbb{R}^+$, then for $y \in \mathbb{R}^+$,

$$\text{InvChiSquare}(y \mid \nu) = \frac{2^{-\nu/2}}{\Gamma(\nu/2)} \, y^{-\nu/2-1} \, \exp\left(-\frac{1}{2}\frac{1}{y}\right).$$

Sampling statement

```
y ~ inv_chi_square(nu)
```

Increment target log probability density with `inv_chi_square_lupdf(y | nu)`.
Available since 2.0

Stan functions

```
real inv_chi_square_lpdf(reals y | reals nu)
```
The log of the inverse Chi-square density of y given degrees of freedom nu
Available since 2.12

```
real inv_chi_square_lupdf(reals y | reals nu)
```
The log of the inverse Chi-square density of y given degrees of freedom nu dropping constant additive terms
Available since 2.25

real **inv_chi_square_cdf**(reals y, reals nu)
The inverse Chi-squared cumulative distribution function of y given degrees of
freedom nu
Available since 2.0

real **inv_chi_square_lcdf**(reals y | reals nu)
The log of the inverse Chi-squared cumulative distribution function of y given degrees
of freedom nu
Available since 2.12

real **inv_chi_square_lccdf**(reals y | reals nu)
The log of the inverse Chi-squared complementary cumulative distribution function
of y given degrees of freedom nu
Available since 2.12

R **inv_chi_square_rng**(reals nu)
Generate an inverse Chi-squared variate with degrees of freedom nu; may only
be used in transformed data and generated quantities blocks. For a description of
argument and return types, see section vectorized PRNG functions.
Available since 2.18

18.4. Scaled inverse chi-square distribution

Probability density function

If $\nu \in \mathbb{R}^+$ and $\sigma \in \mathbb{R}^+$, then for $y \in \mathbb{R}^+$,

$$\text{ScaledInvChiSquare}(y|\nu, \sigma) = \frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)} \, \sigma^\nu \, y^{-(\nu/2+1)} \, \exp\left(-\frac{1}{2} \, \nu \, \sigma^2 \, \frac{1}{y}\right).$$

Sampling statement

y ~ **scaled_inv_chi_square**(nu, sigma)

Increment target log probability density with scaled_inv_chi_square_lupdf(y |
nu, sigma).
Available since 2.0

Stan functions

real **scaled_inv_chi_square_lpdf**(reals y | reals nu, reals sigma)
The log of the scaled inverse Chi-square density of y given degrees of freedom nu

and scale sigma
Available since 2.12

real **scaled_inv_chi_square_lupdf**(reals y | reals nu, reals sigma)
The log of the scaled inverse Chi-square density of y given degrees of freedom nu
and scale sigma dropping constant additive terms
Available since 2.25

real **scaled_inv_chi_square_cdf**(reals y, reals nu, reals sigma)
The scaled inverse Chi-square cumulative distribution function of y given degrees of
freedom nu and scale sigma
Available since 2.0

real **scaled_inv_chi_square_lcdf**(reals y | reals nu, reals sigma)
The log of the scaled inverse Chi-square cumulative distribution function of y given
degrees of freedom nu and scale sigma
Available since 2.12

real **scaled_inv_chi_square_lccdf**(reals y | reals nu, reals sigma)
The log of the scaled inverse Chi-square complementary cumulative distribution
function of y given degrees of freedom nu and scale sigma
Available since 2.12

R **scaled_inv_chi_square_rng**(reals nu, reals sigma)
Generate a scaled inverse Chi-squared variate with degrees of freedom nu and scale
sigma; may only be used in transformed data and generated quantities blocks. For a
description of argument and return types, see section vectorized PRNG functions.
Available since 2.18

18.5. Exponential distribution

Probability density function

If $\beta \in \mathbb{R}^+$, then for $y \in \mathbb{R}^+$,

$$\text{Exponential}(y|\beta) = \beta \exp(-\beta y).$$

Sampling statement

y ~ **exponential**(beta)

Increment target log probability density with exponential_lupdf(y | beta).
Available since 2.0

Stan functions

`real` **`exponential_lpdf`**`(reals y | reals beta)`
The log of the exponential density of y given inverse scale beta
Available since 2.12

`real` **`exponential_lupdf`**`(reals y | reals beta)`
The log of the exponential density of y given inverse scale beta dropping constant
additive terms
Available since 2.25

`real` **`exponential_cdf`**`(reals y, reals beta)`
The exponential cumulative distribution function of y given inverse scale beta
Available since 2.0

`real` **`exponential_lcdf`**`(reals y | reals beta)`
The log of the exponential cumulative distribution function of y given inverse scale
beta
Available since 2.12

`real` **`exponential_lccdf`**`(reals y | reals beta)`
The log of the exponential complementary cumulative distribution function of y
given inverse scale beta
Available since 2.12

`R` **`exponential_rng`**`(reals beta)`
Generate an exponential variate with inverse scale beta; may only be used in
transformed data and generated quantities blocks. For a description of argument
and return types, see section vectorized PRNG functions.
Available since 2.18

18.6.   Gamma distribution

Probability density function

If $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}^+$, then for $y \in \mathbb{R}^+$,

$$\text{Gamma}(y|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, y^{\alpha-1} \exp(-\beta\, y).$$

Sampling statement

`y ~` **`gamma`**`(alpha, beta)`

Increment target log probability density with `gamma_lupdf(y | alpha, beta)`.
Available since 2.0

Stan functions

real **gamma_lpdf**(reals y | reals alpha, reals beta)
The log of the gamma density of y given shape alpha and inverse scale beta
Available since 2.12

real **gamma_lupdf**(reals y | reals alpha, reals beta)
The log of the gamma density of y given shape alpha and inverse scale beta dropping
constant additive terms
Available since 2.25

real **gamma_cdf**(reals y, reals alpha, reals beta)
The cumulative gamma distribution function of y given shape alpha and inverse
scale beta
Available since 2.0

real **gamma_lcdf**(reals y | reals alpha, reals beta)
The log of the cumulative gamma distribution function of y given shape alpha and
inverse scale beta
Available since 2.12

real **gamma_lccdf**(reals y | reals alpha, reals beta)
The log of the complementary cumulative gamma distribution function of y given
shape alpha and inverse scale beta
Available since 2.12

R **gamma_rng**(reals alpha, reals beta)
Generate a gamma variate with shape alpha and inverse scale beta; may only be used
in transformed data and generated quantities blocks. For a description of argument
and return types, see section vectorized PRNG functions.
Available since 2.18

18.7.   Inverse gamma Distribution

Probability density function

If $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}^+$, then for $y \in \mathbb{R}^+$,

$$\text{InvGamma}(y|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \ y^{-(\alpha+1)} \ \exp\left(-\beta \ \frac{1}{y}\right).$$

Sampling statement

`y ~ inv_gamma(alpha, beta)`

Increment target log probability density with `inv_gamma_lupdf(y | alpha, beta)`.
Available since 2.0

Stan functions

real **inv_gamma_lpdf**`(reals y | reals alpha, reals beta)`
The log of the inverse gamma density of y given shape alpha and scale beta
Available since 2.12

real **inv_gamma_lupdf**`(reals y | reals alpha, reals beta)`
The log of the inverse gamma density of y given shape alpha and scale beta dropping
constant additive terms
Available since 2.25

real **inv_gamma_cdf**`(reals y, reals alpha, reals beta)`
The inverse gamma cumulative distribution function of y given shape alpha and
scale beta
Available since 2.0

real **inv_gamma_lcdf**`(reals y | reals alpha, reals beta)`
The log of the inverse gamma cumulative distribution function of y given shape
alpha and scale beta
Available since 2.12

real **inv_gamma_lccdf**`(reals y | reals alpha, reals beta)`
The log of the inverse gamma complementary cumulative distribution function of y
given shape alpha and scale beta
Available since 2.12

R **inv_gamma_rng**`(reals alpha, reals beta)`
Generate an inverse gamma variate with shape alpha and scale beta; may only
be used in transformed data and generated quantities blocks. For a description of
argument and return types, see section vectorized PRNG functions.
Available since 2.18

18.8.   Weibull distribution

Probability density function

If $\alpha \in \mathbb{R}^+$ and $\sigma \in \mathbb{R}^+$, then for $y \in [0, \infty)$,

$$\text{Weibull}(y|\alpha, \sigma) = \frac{\alpha}{\sigma} \left(\frac{y}{\sigma}\right)^{\alpha-1} \exp\left(-\left(\frac{y}{\sigma}\right)^{\alpha}\right).$$

Note that if $Y \propto \text{Weibull}(\alpha, \sigma)$, then $Y^{-1} \propto \text{Frechet}(\alpha, \sigma^{-1})$.

Sampling statement

`y ~ weibull(alpha, sigma)`

Increment target log probability density with `weibull_lupdf(y | alpha, sigma)`.
Available since 2.0

Stan functions

real **weibull_lpdf**(reals y | reals alpha, reals sigma)
The log of the Weibull density of y given shape alpha and scale sigma
Available since 2.12

real **weibull_lupdf**(reals y | reals alpha, reals sigma)
The log of the Weibull density of y given shape alpha and scale sigma dropping
constant additive terms
Available since 2.25

real **weibull_cdf**(reals y, reals alpha, reals sigma)
The Weibull cumulative distribution function of y given shape alpha and scale sigma
Available since 2.0

real **weibull_lcdf**(reals y | reals alpha, reals sigma)
The log of the Weibull cumulative distribution function of y given shape alpha and
scale sigma
Available since 2.12

real **weibull_lccdf**(reals y | reals alpha, reals sigma)
The log of the Weibull complementary cumulative distribution function of y given
shape alpha and scale sigma
Available since 2.12

R **weibull_rng**(reals alpha, reals sigma)
Generate a weibull variate with shape alpha and scale sigma; may only be used in

transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.
Available since 2.18

18.9.   Frechet distribution

Probability density function

If $\alpha \in \mathbb{R}^+$ and $\sigma \in \mathbb{R}^+$, then for $y \in \mathbb{R}^+$,

$$\text{Frechet}(y|\alpha, \sigma) = \frac{\alpha}{\sigma} \left(\frac{y}{\sigma}\right)^{-\alpha-1} \exp\left(-\left(\frac{y}{\sigma}\right)^{-\alpha}\right).$$

Note that if $Y \propto \text{Frechet}(\alpha, \sigma)$, then $Y^{-1} \propto \text{Weibull}(\alpha, \sigma^{-1})$.

Sampling statement

`y ~ frechet(alpha, sigma)`

Increment target log probability density with `frechet_lupdf(y | alpha, sigma)`.
Available since 2.5

Stan functions

real **frechet_lpdf**(reals y | reals alpha, reals sigma)
The log of the Frechet density of y given shape alpha and scale sigma
Available since 2.12

real **frechet_lupdf**(reals y | reals alpha, reals sigma)
The log of the Frechet density of y given shape alpha and scale sigma dropping constant additive terms
Available since 2.25

real **frechet_cdf**(reals y, reals alpha, reals sigma)
The Frechet cumulative distribution function of y given shape alpha and scale sigma
Available since 2.5

real **frechet_lcdf**(reals y | reals alpha, reals sigma)
The log of the Frechet cumulative distribution function of y given shape alpha and scale sigma
Available since 2.12

real **frechet_lccdf**(reals y | reals alpha, reals sigma)
The log of the Frechet complementary cumulative distribution function of y given

shape alpha and scale sigma
Available since 2.12

R `frechet_rng`(reals alpha, reals sigma)
Generate a Frechet variate with shape alpha and scale sigma; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.
Available since 2.18

18.10. Rayleigh distribution

Probability density function

If $\sigma \in \mathbb{R}^+$, then for $y \in [0, \infty)$,

$$\text{Rayleigh}(y|\sigma) = \frac{y}{\sigma^2} \exp(-y^2/2\sigma^2).$$

Sampling statement

y ~ `rayleigh`(sigma)

Increment target log probability density with `rayleigh_lupdf(y | sigma)`.
Available since 2.0

Stan functions

real `rayleigh_lpdf`(reals y | reals sigma)
The log of the Rayleigh density of y given scale sigma
Available since 2.12

real `rayleigh_lupdf`(reals y | reals sigma)
The log of the Rayleigh density of y given scale sigma dropping constant additive terms
Available since 2.25

real `rayleigh_cdf`(real y, real sigma)
The Rayleigh cumulative distribution of y given scale sigma
Available since 2.0

real `rayleigh_lcdf`(real y | real sigma)
The log of the Rayleigh cumulative distribution of y given scale sigma
Available since 2.12

real **rayleigh_lccdf**(real y | real sigma)
The log of the Rayleigh complementary cumulative distribution of y given scale sigma
Available since 2.12

R **rayleigh_rng**(reals sigma)
Generate a Rayleigh variate with scale sigma; may only be used in generated quantities block. For a description of argument and return types, see section vectorized PRNG functions.
Available since 2.18

19.   Positive Lower-Bounded Distributions

The positive lower-bounded probabilities have support on real values above some positive minimum value.

19.1.   Pareto distribution

Probability density function

If $y_{\min} \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}^+$, then for $y \in \mathbb{R}^+$ with $y \geq y_{\min}$,

$$\text{Pareto}(y|y_{\min}, \alpha) = \frac{\alpha\, y_{\min}^{\alpha}}{y^{\alpha+1}}.$$

Sampling statement

`y ~ pareto(y_min, alpha)`

Increment target log probability density with `pareto_lupdf(y | y_min, alpha)`.
Available since 2.0

Stan functions

real **pareto_lpdf**(reals y | reals y_min, reals alpha)
The log of the Pareto density of y given positive minimum value y_min and shape alpha
Available since 2.12

real **pareto_lupdf**(reals y | reals y_min, reals alpha)
The log of the Pareto density of y given positive minimum value y_min and shape alpha dropping constant additive terms
Available since 2.25

real **pareto_cdf**(reals y, reals y_min, reals alpha)
The Pareto cumulative distribution function of y given positive minimum value y_min and shape alpha
Available since 2.0

real **pareto_lcdf**(reals y | reals y_min, reals alpha)
The log of the Pareto cumulative distribution function of y given positive minimum

value y_min and shape alpha
Available since 2.12

real **pareto_lccdf**(reals y | reals y_min, reals alpha)
The log of the Pareto complementary cumulative distribution function of y given
positive minimum value y_min and shape alpha
Available since 2.12

R **pareto_rng**(reals y_min, reals alpha)
Generate a Pareto variate with positive minimum value y_min and shape alpha; may
only be used in transformed data and generated quantities blocks. For a description
of argument and return types, see section vectorized PRNG functions.
Available since 2.18

19.2.   Pareto type 2 distribution

Probability density function

If $\mu \in \mathbb{R}$, $\lambda \in \mathbb{R}^+$, and $\alpha \in \mathbb{R}^+$, then for $y \geq \mu$,

$$\mathrm{Pareto\_Type\_2}(y|\mu, \lambda, \alpha) = \frac{\alpha}{\lambda}\left(1 + \frac{y - \mu}{\lambda}\right)^{-(\alpha+1)}.$$

Note that the Lomax distribution is a Pareto Type 2 distribution with $\mu = 0$.

Sampling statement

y ~ **pareto_type_2**(mu, lambda, alpha)

Increment target log probability density with pareto_type_2_lupdf(y | mu,
lambda, alpha).
Available since 2.5

Stan functions

real   **pareto_type_2_lpdf**(reals y | reals mu, reals lambda, reals
alpha)
The log of the Pareto Type 2 density of y given location mu, scale lambda, and shape
alpha
Available since 2.18

real   **pareto_type_2_lupdf**(reals y | reals mu, reals lambda, reals
alpha)

The log of the Pareto Type 2 density of y given location mu, scale lambda, and shape alpha dropping constant additive terms
Available since 2.25

real **pareto_type_2_cdf**(reals y, reals mu, reals lambda, reals alpha)
The Pareto Type 2 cumulative distribution function of y given location mu, scale lambda, and shape alpha
Available since 2.5

real   **pareto_type_2_lcdf**(reals y | reals mu, reals lambda, reals alpha)
The log of the Pareto Type 2 cumulative distribution function of y given location mu, scale lambda, and shape alpha
Available since 2.18

real   **pareto_type_2_lccdf**(reals y | reals mu, reals lambda, reals alpha)
The log of the Pareto Type 2 complementary cumulative distribution function of y given location mu, scale lambda, and shape alpha
Available since 2.18

R **pareto_type_2_rng**(reals mu, reals lambda, reals alpha)
Generate a Pareto Type 2 variate with location mu, scale lambda, and shape alpha; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.
Available since 2.18

19.3.   Wiener First Passage Time Distribution

Probability density function

If $\alpha \in \mathbb{R}^+$, $\tau \in \mathbb{R}^+$, $\beta \in [0, 1]$ and $\delta \in \mathbb{R}$, then for $y > \tau$,

$$\text{Wiener}(y|\alpha, \tau, \beta, \delta) = \frac{\alpha^3}{(y - \tau)^{3/2}} \exp\left(-\delta\alpha\beta - \frac{\delta^2(y - \tau)}{2}\right) \sum_{k=-\infty}^{\infty} (2k+\beta)\phi\left(\frac{2k\alpha + \beta}{\sqrt{y - \tau}}\right)$$

where $\phi(x)$ denotes the standard normal density function; see (Feller 1968), (Navarro and Fuss 2009).

Sampling statement

y ~ **wiener**(alpha, tau, beta, delta)

Increment target log probability density with `wiener_lupdf(y | alpha, tau, beta, delta)`.
Available since 2.7

Stan functions

`real  wiener_lpdf(reals y | reals alpha, reals tau, reals beta, reals delta)`
The log of the Wiener first passage time density of y given boundary separation alpha, non-decision time tau, a-priori bias beta and drift rate delta
Available since 2.18

`real wiener_lupdf(reals y | reals alpha, reals tau, reals beta, reals delta)`
The log of the Wiener first passage time density of y given boundary separation alpha, non-decision time tau, a-priori bias beta and drift rate delta dropping constant additive terms
Available since 2.25

boundaries

Stan returns the first passage time of the accumulation process over the upper boundary only. To get the result for the lower boundary, use

$$\text{wiener}(y|\alpha, \tau, 1 - \beta, -\delta)$$

For more details, see the appendix of Vandekerckhove and Wabersich (2014).

20. Continuous Distributions on [0, 1]

The continuous distributions with outcomes in the interval $[0, 1]$ are used to characterized bounded quantities, including probabilities.

20.1. Beta distribution

Probability density function

If $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}^+$, then for $\theta \in (0, 1)$,

$$\text{Beta}(\theta | \alpha, \beta) = \frac{1}{\text{B}(\alpha, \beta)} \, \theta^{\alpha-1} \, (1 - \theta)^{\beta-1},$$

where the beta function $\text{B}()$ is as defined in section combinatorial functions.

Warning: If $\theta = 0$ or $\theta = 1$, then the probability is 0 and the log probability is $-\infty$. Similarly, the distribution requires strictly positive parameters, $\alpha, \beta > 0$.

Sampling statement

`theta ~ beta(alpha, beta)`

Increment target log probability density with `beta_lupdf(theta | alpha, beta)`.
Available since 2.0

Stan functions

`real beta_lpdf(reals theta | reals alpha, reals beta)`
The log of the beta density of `theta` in $[0, 1]$ given positive prior successes (plus one) alpha and prior failures (plus one) beta
Available since 2.12

`real beta_lupdf(reals theta | reals alpha, reals beta)`
The log of the beta density of `theta` in $[0, 1]$ given positive prior successes (plus one) alpha and prior failures (plus one) beta dropping constant additive terms
Available since 2.25

`real beta_cdf(reals theta, reals alpha, reals beta)`
The beta cumulative distribution function of `theta` in $[0, 1]$ given positive prior

successes (plus one) alpha and prior failures (plus one) beta
Available since 2.0

```
real beta_lcdf(reals theta | reals alpha, reals beta)
```
The log of the beta cumulative distribution function of `theta` in $[0, 1]$ given positive prior successes (plus one) alpha and prior failures (plus one) beta
Available since 2.12

```
real beta_lccdf(reals theta | reals alpha, reals beta)
```
The log of the beta complementary cumulative distribution function of `theta` in $[0, 1]$ given positive prior successes (plus one) alpha and prior failures (plus one) beta
Available since 2.12

```
R beta_rng(reals alpha, reals beta)
```
Generate a beta variate with positive prior successes (plus one) alpha and prior failures (plus one) beta; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.
Available since 2.18

20.2. Beta proportion distribution

Probability density function

If $\mu \in (0, 1)$ and $\kappa \in \mathbb{R}^+$, then for $\theta \in (0, 1)$,

$$\text{Beta\_Proportion}(\theta|\mu, \kappa) = \frac{1}{\text{B}(\mu\kappa, (1 - \mu)\kappa)} \theta^{\mu\kappa - 1} (1 - \theta)^{(1-\mu)\kappa - 1},$$

where the beta function $\text{B}()$ is as defined in section combinatorial functions.

Warning: If $\theta = 0$ or $\theta = 1$, then the probability is 0 and the log probability is $-\infty$. Similarly, the distribution requires $\mu \in (0, 1)$ and strictly positive parameter, $\kappa > 0$.

Sampling statement

```
theta ~ beta_proportion(mu, kappa)
```

Increment target log probability density with `beta_proportion_lupdf(theta | mu, kappa)`.
Available since 2.19

Stan functions

real **beta_proportion_lpdf**(reals theta | reals mu, reals kappa)
The log of the beta_proportion density of theta in $(0, 1)$ given mean mu and precision kappa
Available since 2.19

real **beta_proportion_lupdf**(reals theta | reals mu, reals kappa)
The log of the beta_proportion density of theta in $(0, 1)$ given mean mu and precision kappa dropping constant additive terms
Available since 2.25

real **beta_proportion_lcdf**(reals theta | reals mu, reals kappa)
The log of the beta_proportion cumulative distribution function of theta in $(0, 1)$ given mean mu and precision kappa
Available since 2.18

real **beta_proportion_lccdf**(reals theta | reals mu, reals kappa)
The log of the beta_proportion complementary cumulative distribution function of theta in $(0, 1)$ given mean mu and precision kappa
Available since 2.18

R **beta_proportion_rng**(reals mu, reals kappa)
Generate a beta_proportion variate with mean mu and precision kappa; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.
Available since 2.18

21. Circular Distributions

Circular distributions are defined for finite values y in any interval of length 2π.

21.1. Von Mises distribution

Probability density function

If $\mu \in \mathbb{R}$ and $\kappa \in \mathbb{R}^+$, then for $y \in \mathbb{R}$,

$$\text{VonMises}(y|\mu, \kappa) = \frac{\exp(\kappa \cos(y - \mu))}{2\pi I_0(\kappa)}.$$

In order for this density to properly normalize, y must be restricted to some interval $(c, c + 2\pi)$ of length 2π, because

$$\int_c^{c+2\pi} \text{VonMises}(y|\mu, \kappa)dy = 1.$$

Similarly, if μ is a parameter, it will typically be restricted to the same range as y.

If $\kappa > 0$, a von Mises distribution with its 2π interval of support centered around its location μ will have a single mode at μ; for example, restricting y to $(-\pi, \pi)$ and taking $\mu = 0$ leads to a single local optimum at the mode μ. If the location μ is not in the center of the support, the density is circularly translated and there will be a second local maximum at the boundary furthest from the mode. Ideally, the parameterization and support will be set up so that the bulk of the probability mass is in a continuous interval around the mean μ.

For $\kappa = 0$, the Von Mises distribution corresponds to the circular uniform distribution with density $1/(2\pi)$ (independently of the values of y or μ).

Sampling statement

`y ~ von_mises(mu, kappa)`

Increment target log probability density with `von_mises_lupdf(y | mu, kappa)`.
Available since 2.0

Stan functions

R `von_mises_lpdf`(reals y | reals mu, reals kappa)
The log of the von mises density of y given location mu and scale kappa.
Available since 2.18

R `von_mises_lupdf`(reals y | reals mu, reals kappa)
The log of the von mises density of y given location mu and scale kappa dropping constant additive terms.
Available since 2.25

R `von_mises_rng`(reals mu, reals kappa)
Generate a Von Mises variate with location mu and scale kappa (i.e. returns values in the interval $[(\mu \mod 2\pi) - \pi, (\mu \mod 2\pi) + \pi]$); may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.
Available since 2.18

Numerical stability

Evaluating the Von Mises distribution for $\kappa > 100$ is numerically unstable in the current implementation. Nathanael I. Lichti suggested the following workaround on the Stan users group, based on the fact that as $\kappa \to \infty$,

$$\text{VonMises}(y|\mu, \kappa) \to \text{Normal}(\mu, \sqrt{1/\kappa}).$$

The workaround is to replace y ~ `von_mises(mu,kappa)` with

```
if (kappa < 100) {
  y ~ von_mises(mu, kappa);
} else {
  y ~ normal(mu, sqrt(1 / kappa));
}
```

22.   Bounded Continuous Distributions

The bounded continuous probabilities have support on a finite interval of real numbers.

22.1.   Uniform distribution

Probability density function

If $\alpha \in \mathbb{R}$ and $\beta \in (\alpha, \infty)$, then for $y \in [\alpha, \beta]$,

$$\text{Uniform}(y|\alpha, \beta) = \frac{1}{\beta - \alpha}.$$

Sampling statement

`y ~ uniform(alpha, beta)`

Increment target log probability density with `uniform_lupdf(y | alpha, beta)`.
Available since 2.0

Stan functions

real **uniform_lpdf**(reals y | reals alpha, reals beta)
The log of the uniform density of y given lower bound alpha and upper bound beta
Available since 2.12

real **uniform_lupdf**(reals y | reals alpha, reals beta)
The log of the uniform density of y given lower bound alpha and upper bound beta dropping constant additive terms
Available since 2.25

real **uniform_cdf**(reals y, reals alpha, reals beta)
The uniform cumulative distribution function of y given lower bound alpha and upper bound beta
Available since 2.0

real **uniform_lcdf**(reals y | reals alpha, reals beta)
The log of the uniform cumulative distribution function of y given lower bound alpha and upper bound beta
Available since 2.12

`real` **`uniform_lccdf`**`(reals y | reals alpha, reals beta)`
The log of the uniform complementary cumulative distribution function of y given lower bound alpha and upper bound beta
Available since 2.12

`R` **`uniform_rng`**`(reals alpha, reals beta)`
Generate a uniform variate with lower bound alpha and upper bound beta; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.
Available since 2.18

23.   Distributions over Unbounded Vectors

The unbounded vector probability distributions have support on all of $\mathbb{R}^K$ for some fixed K.

23.1.   Multivariate normal distribution

Probability density function

If $K \in \mathbb{N}$, $\mu \in \mathbb{R}^K$, and $\Sigma \in \mathbb{R}^{K \times K}$ is symmetric and positive definite, then for $y \in \mathbb{R}^K$,

$$\text{MultiNormal}(y|\mu, \Sigma) = \frac{1}{(2\pi)^{K/2}} \ \frac{1}{\sqrt{|\Sigma|}} \ \exp\left(-\frac{1}{2}(y - \mu)^\top \Sigma^{-1} (y - \mu)\right),$$

where $|\Sigma|$ is the absolute determinant of Σ.

Sampling statement

`y ~ multi_normal(mu, Sigma)`

Increment target log probability density with `multi_normal_lupdf(y | mu, Sigma)`.
Available since 2.0

Stan functions

The multivariate normal probability function is overloaded to allow the variate vector y and location vector μ to be vectors or row vectors (or to mix the two types). The density function is also vectorized, so it allows arrays of row vectors or vectors as arguments; see section vectorized function signatures for a description of vectorization.

real **multi_normal_lpdf**(vectors y | vectors mu, matrix Sigma)
The log of the multivariate normal density of vector(s) y given location vector(s) mu and covariance matrix Sigma
Available since 2.12

real **multi_normal_lupdf**(vectors y | vectors mu, matrix Sigma)
The log of the multivariate normal density of vector(s) y given location vector(s) mu

and covariance matrix Sigma dropping constant additive terms
Available since 2.25

`real` **`multi_normal_lpdf`**`(vectors y | row_vectors mu, matrix Sigma)`
The log of the multivariate normal density of vector(s) y given location row vector(s) mu and covariance matrix Sigma
Available since 2.12

`real` **`multi_normal_lupdf`**`(vectors y | row_vectors mu, matrix Sigma)`
The log of the multivariate normal density of vector(s) y given location row vector(s) mu and covariance matrix Sigma dropping constant additive terms
Available since 2.25

`real` **`multi_normal_lpdf`**`(row_vectors y | vectors mu, matrix Sigma)`
The log of the multivariate normal density of row vector(s) y given location vector(s) mu and covariance matrix Sigma
Available since 2.12

`real` **`multi_normal_lupdf`**`(row_vectors y | vectors mu, matrix Sigma)`
The log of the multivariate normal density of row vector(s) y given location vector(s) mu and covariance matrix Sigma dropping constant additive terms
Available since 2.25

`real` **`multi_normal_lpdf`**`(row_vectors y | row_vectors mu, matrix Sigma)`
The log of the multivariate normal density of row vector(s) y given location row vector(s) mu and covariance matrix Sigma
Available since 2.12

`real` **`multi_normal_lupdf`**`(row_vectors y | row_vectors mu, matrix Sigma)`
The log of the multivariate normal density of row vector(s) y given location row vector(s) mu and covariance matrix Sigma dropping constant additive terms
Available since 2.25

Although there is a direct multi-normal RNG function, if more than one result is required, it's much more efficient to Cholesky factor the covariance matrix and call `multi_normal_cholesky_rng`; see section multi-variate normal, cholesky parameterization.

`vector` **`multi_normal_rng`**`(vector mu, matrix Sigma)`
Generate a multivariate normal variate with location mu and covariance matrix Sigma; may only be used in transformed data and generated quantities blocks
Available since 2.0

`vector` **`multi_normal_rng`**`(row_vector mu, matrix Sigma)`

Generate a multivariate normal variate with location mu and covariance matrix
Sigma; may only be used in transformed data and generated quantities blocks
Available since 2.18

```
vectors multi_normal_rng(vectors mu, matrix Sigma)
```
Generate an array of multivariate normal variates with locations mu and covariance
matrix Sigma; may only be used in transformed data and generated quantities blocks
Available since 2.18

```
vectors multi_normal_rng(row_vectors mu, matrix Sigma)
```
Generate an array of multivariate normal variates with locations mu and covariance
matrix Sigma; may only be used in transformed data and generated quantities blocks
Available since 2.18

23.2. Multivariate normal distribution, precision parameterization

Probability density function

If $K \in \mathbb{N}$, $\mu \in \mathbb{R}^K$, and $\Omega \in \mathbb{R}^{K \times K}$ is symmetric and positive definite, then for
$y \in \mathbb{R}^K$,
$$\text{MultiNormalPrecision}(y|\mu, \Omega) = \text{MultiNormal}(y|\mu, \Omega^{-1})$$

Sampling statement

```
y ~ multi_normal_prec(mu, Omega)
```

Increment target log probability density with `multi_normal_prec_lupdf(y | mu,
Omega)`.
Available since 2.3

Stan functions

```
real multi_normal_prec_lpdf(vectors y | vectors mu, matrix Omega)
```
The log of the multivariate normal density of vector(s) y given location vector(s) mu
and positive definite precision matrix Omega
Available since 2.18

```
real multi_normal_prec_lupdf(vectors y | vectors mu, matrix Omega)
```
The log of the multivariate normal density of vector(s) y given location vector(s) mu
and positive definite precision matrix Omega dropping constant additive terms
Available since 2.25

`real` **`multi_normal_prec_lpdf`**`(vectors y | row_vectors mu, matrix Omega)`
The log of the multivariate normal density of vector(s) y given location row vector(s) mu and positive definite precision matrix Omega
Available since 2.18

`real` **`multi_normal_prec_lupdf`**`(vectors y | row_vectors mu, matrix Omega)`
The log of the multivariate normal density of vector(s) y given location row vector(s) mu and positive definite precision matrix Omega dropping constant additive terms
Available since 2.25

`real` **`multi_normal_prec_lpdf`**`(row_vectors y | vectors mu, matrix Omega)`
The log of the multivariate normal density of row vector(s) y given location vector(s) mu and positive definite precision matrix Omega
Available since 2.18

`real` **`multi_normal_prec_lupdf`**`(row_vectors y | vectors mu, matrix Omega)`
The log of the multivariate normal density of row vector(s) y given location vector(s) mu and positive definite precision matrix Omega dropping constant additive terms
Available since 2.25

`real` **`multi_normal_prec_lpdf`**`(row_vectors y | row_vectors mu, matrix Omega)`
The log of the multivariate normal density of row vector(s) y given location row vector(s) mu and positive definite precision matrix Omega
Available since 2.18

`real` **`multi_normal_prec_lupdf`**`(row_vectors y | row_vectors mu, matrix Omega)`
The log of the multivariate normal density of row vector(s) y given location row vector(s) mu and positive definite precision matrix Omega dropping constant additive terms
Available since 2.25

23.3.   Multivariate normal distribution, Cholesky parameterization

Probability density function

If $K \in \mathbb{N}$, $\mu \in \mathbb{R}^K$, and $L \in \mathbb{R}^{K \times K}$ is lower triangular and such that $LL^\top$ is positive definite, then for $y \in \mathbb{R}^K$,

$$\text{MultiNormalCholesky}(y|\mu, L) = \text{MultiNormal}(y|\mu, LL^\top).$$

If L is lower triangular and LL^{top} is a $K \times K$ positive definite matrix, then $L_{k,k}$ must be strictly positive for $k \in 1{:}K$. If an L is provided that is not the Cholesky factor of a positive-definite matrix, the probability functions will raise errors.

Sampling statement

`y ~ multi_normal_cholesky(mu, L)`

Increment target log probability density with `multi_normal_cholesky_lupdf(y | mu, L)`.
Available since 2.0

Stan functions

real **multi_normal_cholesky_lpdf**(vectors y | vectors mu, matrix L)
The log of the multivariate normal density of vector(s) y given location vector(s) mu and lower-triangular Cholesky factor of the covariance matrix L
Available since 2.18

real **multi_normal_cholesky_lupdf**(vectors y | vectors mu, matrix L)
The log of the multivariate normal density of vector(s) y given location vector(s) mu and lower-triangular Cholesky factor of the covariance matrix L dropping constant additive terms
Available since 2.25

real **multi_normal_cholesky_lpdf**(vectors y | row_vectors mu, matrix L)
The log of the multivariate normal density of vector(s) y given location row vector(s) mu and lower-triangular Cholesky factor of the covariance matrix L
Available since 2.18

real   **multi_normal_cholesky_lupdf**(vectors y | row_vectors mu, matrix L)
The log of the multivariate normal density of vector(s) y given location row vector(s) mu and lower-triangular Cholesky factor of the covariance matrix L dropping

constant additive terms
Available since 2.25

real **multi_normal_cholesky_lpdf**(row_vectors y | vectors mu, matrix L)
The log of the multivariate normal density of row vector(s) y given location vector(s) mu and lower-triangular Cholesky factor of the covariance matrix L
Available since 2.18

real **multi_normal_cholesky_lupdf**(row_vectors y | vectors mu, matrix L)
The log of the multivariate normal density of row vector(s) y given location vector(s) mu and lower-triangular Cholesky factor of the covariance matrix L dropping constant additive terms
Available since 2.25

real **multi_normal_cholesky_lpdf**(row_vectors y | row_vectors mu, matrix L)
The log of the multivariate normal density of row vector(s) y given location row vector(s) mu and lower-triangular Cholesky factor of the covariance matrix L
Available since 2.18

real **multi_normal_cholesky_lupdf**(row_vectors y | row_vectors mu, matrix L)
The log of the multivariate normal density of row vector(s) y given location row vector(s) mu and lower-triangular Cholesky factor of the covariance matrix L dropping constant additive terms
Available since 2.25

vector **multi_normal_cholesky_rng**(vector mu, matrix L)
Generate a multivariate normal variate with location mu and lower-triangular Cholesky factor of the covariance matrix L; may only be used in transformed data and generated quantities blocks
Available since 2.3

vector **multi_normal_cholesky_rng**(row_vector mu, matrix L)
Generate a multivariate normal variate with location mu and lower-triangular Cholesky factor of the covariance matrix L; may only be used in transformed data and generated quantities blocks
Available since 2.18

vectors **multi_normal_cholesky_rng**(vectors mu, matrix L)
Generate an array of multivariate normal variates with locations mu and lower-triangular Cholesky factor of the covariance matrix L; may only be used in trans-

formed data and generated quantities blocks
Available since 2.18

vectors **multi_normal_cholesky_rng**(row_vectors mu, matrix L)
Generate an array of multivariate normal variates with locations mu and lower-triangular Cholesky factor of the covariance matrix L; may only be used in transformed data and generated quantities blocks
Available since 2.18

23.4. Multivariate Gaussian process distribution

Probability density function

If $K, N \in \mathbb{N}$, $\Sigma \in \mathbb{R}^{N \times N}$ is symmetric, positive definite kernel matrix and $w \in \mathbb{R}^K$ is a vector of positive inverse scales, then for $y \in \mathbb{R}^{K \times N}$,

$$\text{MultiGP}(y|\Sigma, w) = \prod_{i=1}^{K} \text{MultiNormal}(y_i|0, w_i^{-1}\Sigma),$$

where y_i is the ith row of y. This is used to efficiently handle Gaussian Processes with multi-variate outputs where only the output dimensions share a kernel function but vary based on their scale. Note that this function does not take into account the mean prediction.

Sampling statement

y ~ **multi_gp**(Sigma, w)

Increment target log probability density with multi_gp_lupdf(y | Sigma, w).
Available since 2.3

Stan functions

real **multi_gp_lpdf**(matrix y | matrix Sigma, vector w)
The log of the multivariate GP density of matrix y given kernel matrix Sigma and inverses scales w
Available since 2.12

real **multi_gp_lupdf**(matrix y | matrix Sigma, vector w)
The log of the multivariate GP density of matrix y given kernel matrix Sigma and inverses scales w dropping constant additive terms
Available since 2.25

23.5. Multivariate Gaussian process distribution, Cholesky parameterization

Probability density function

If $K, N \in \mathbb{N}$, $L \in \mathbb{R}^{N \times N}$ is lower triangular and such that $LL^\top$ is positive definite kernel matrix (implying $L_{n,n} > 0$ for $n \in 1{:}N$), and $w \in \mathbb{R}^K$ is a vector of positive inverse scales, then for $y \in \mathbb{R}^{K \times N}$,

$$\text{MultiGPCholesky}(y \mid L, w) = \prod_{i=1}^{K} \text{MultiNormal}(y_i | 0, w_i^{-1} LL^\top),$$

where y_i is the ith row of y. This is used to efficiently handle Gaussian Processes with multi-variate outputs where only the output dimensions share a kernel function but vary based on their scale. If the model allows parameterization in terms of Cholesky factor of the kernel matrix, this distribution is also more efficient than MultiGP(). Note that this function does not take into account the mean prediction.

Sampling statement

`y ~ multi_gp_cholesky(L, w)`

Increment target log probability density with `multi_gp_cholesky_lupdf(y | L, w)`.
Available since 2.5

Stan functions

real **multi_gp_cholesky_lpdf**(matrix y | matrix L, vector w)
The log of the multivariate GP density of matrix y given lower-triangular Cholesky factor of the kernel matrix L and inverses scales w
Available since 2.12

real **multi_gp_cholesky_lupdf**(matrix y | matrix L, vector w)
The log of the multivariate GP density of matrix y given lower-triangular Cholesky factor of the kernel matrix L and inverses scales w dropping constant additive terms
Available since 2.25

23.6.   Multivariate Student-t distribution

Probability density function

If $K \in \mathbb{N}$, $\nu \in \mathbb{R}^+$, $\mu \in \mathbb{R}^K$, and $\Sigma \in \mathbb{R}^{K \times K}$ is symmetric and positive definite, then for $y \in \mathbb{R}^K$,

$$\text{MultiStudentT}(y \mid \nu, \mu, \Sigma)$$
$$= \frac{1}{\pi^{K/2}} \frac{1}{\nu^{K/2}} \frac{\Gamma((\nu+K)/2)}{\Gamma(\nu/2)} \frac{1}{\sqrt{|\Sigma|}} \left(1 + \frac{1}{\nu} (y - \mu)^\top \Sigma^{-1} (y - \mu) \right)^{-(\nu+K)/2}.$$

Sampling statement

`y ~ multi_student_t(nu, mu, Sigma)`

Increment target log probability density with `multi_student_t_lupdf(y | nu, mu, Sigma)`.
Available since 2.0

Stan functions

real   **multi_student_t_lpdf**(vectors y | real nu, vectors mu, matrix Sigma)
The log of the multivariate Student-t density of vector(s) y given degrees of freedom nu, location vector(s) mu, and scale matrix Sigma
Available since 2.18

real   **multi_student_t_lupdf**(vectors y | real nu, vectors mu, matrix Sigma)
The log of the multivariate Student-t density of vector(s) y given degrees of freedom nu, location vector(s) mu, and scale matrix Sigma dropping constant additive terms
Available since 2.25

real **multi_student_t_lpdf**(vectors y | real nu, row_vectors mu, matrix Sigma)
The log of the multivariate Student-t density of vector(s) y given degrees of freedom nu, location row vector(s) mu, and scale matrix Sigma
Available since 2.18

real   **multi_student_t_lupdf**(vectors y | real nu, row_vectors mu, matrix Sigma)
The log of the multivariate Student-t density of vector(s) y given degrees of freedom nu, location row vector(s) mu, and scale matrix Sigma dropping constant additive

terms
Available since 2.25

`real` **`multi_student_t_lpdf`**`(row_vectors y | real nu, vectors mu, matrix Sigma)`
The log of the multivariate Student-t density of row vector(s) y given degrees of freedom nu, location vector(s) mu, and scale matrix Sigma
Available since 2.18

`real` **`multi_student_t_lupdf`**`(row_vectors y | real nu, vectors mu, matrix Sigma)`
The log of the multivariate Student-t density of row vector(s) y given degrees of freedom nu, location vector(s) mu, and scale matrix Sigma dropping constant additive terms
Available since 2.25

`real` **`multi_student_t_lpdf`**`(row_vectors y | real nu, row_vectors mu, matrix Sigma)`
The log of the multivariate Student-t density of row vector(s) y given degrees of freedom nu, location row vector(s) mu, and scale matrix Sigma
Available since 2.18

`real` **`multi_student_t_lupdf`**`(row_vectors y | real nu, row_vectors mu, matrix Sigma)`
The log of the multivariate Student-t density of row vector(s) y given degrees of freedom nu, location row vector(s) mu, and scale matrix Sigma dropping constant additive terms
Available since 2.25

`vector` **`multi_student_t_rng`**`(real nu, vector mu, matrix Sigma)`
Generate a multivariate Student-t variate with degrees of freedom nu, location mu, and scale matrix Sigma; may only be used in transformed data and generated quantities blocks
Available since 2.0

`vector` **`multi_student_t_rng`**`(real nu, row_vector mu, matrix Sigma)`
Generate a multivariate Student-t variate with degrees of freedom nu, location mu, and scale matrix Sigma; may only be used in transfomed data and generated quantities blocks
Available since 2.18

`vectors` **`multi_student_t_rng`**`(real nu, vectors mu, matrix Sigma)`
Generate an array of multivariate Student-t variates with degrees of freedom nu,

locations mu, and scale matrix Sigma; may only be used in transformed data and generated quantities blocks
Available since 2.18

`vectors` **`multi_student_t_rng`**`(real nu, row_vectors mu, matrix Sigma)`
Generate an array of multivariate Student-t variates with degrees of freedom nu, locations mu, and scale matrix Sigma; may only be used in transformed data andgenerated quantities blocks
Available since 2.18

23.7.   Gaussian dynamic linear models

A Gaussian Dynamic Linear model is defined as follows, For $t \in 1, \dots, T$,

$$y_t \sim N(F'\theta_t, V)$$
$$\theta_t \sim N(G\theta_{t-1}, W)$$
$$\theta_0 \sim N(m_0, C_0)$$

where y is $n \times T$ matrix where rows are variables and columns are observations. These functions calculate the log-likelihood of the observations marginalizing over the latent states $(p(y|F, G, V, W, m_0, C_0))$. This log-likelihood is a system that is calculated using the Kalman Filter. If V is diagonal, then a more efficient algorithm which sequentially processes observations and avoids a matrix inversions can be used (Durbin and Koopman 2001, sec. 6.4).

Sampling statement

`y ~ ` **`gaussian_dlm_obs`**`(F, G, V, W, m0, C0)`

Increment target log probability density with `gaussian_dlm_obs_lupdf(y | F, G, V, W, m0, C0)`.
Available since 2.0

Stan functions

The following two functions differ in the type of their V, the first taking a full observation covariance matrix V and the second a vector V representing the diagonal of the observation covariance matrix. The sampling statement defined in the previous section works with either type of observation V.

`real    ` **`gaussian_dlm_obs_lpdf`**`(matrix y | matrix F, matrix G, matrix V, matrix W, vector m0, matrix C0)`

The log of the density of the Gaussian Dynamic Linear model with observation matrix y in which rows are variables and columns are observations, design matrix F, transition matrix G, observation covariance matrix V, system covariance matrix W, and the initial state is distributed normal with mean m0 and covariance C0.
Available since 2.12

real **gaussian_dlm_obs_lupdf**(matrix y | matrix F, matrix G, matrix V, matrix W, vector m0, matrix C0)
The log of the density of the Gaussian Dynamic Linear model with observation matrix y in which rows are variables and columns are observations, design matrix F, transition matrix G, observation covariance matrix V, system covariance matrix W, and the initial state is distributed normal with mean m0 and covariance C0. This function drops constant additive terms.
Available since 2.25

real **gaussian_dlm_obs_lpdf**(matrix y | matrix F, matrix G, vector V, matrix W, vector m0, matrix C0)
The log of the density of the Gaussian Dynamic Linear model with observation matrix y in which rows are variables and columns are observations, design matrix F, transition matrix G, observation covariance matrix with diagonal V, system covariance matrix W, and the initial state is distributed normal with mean m0 and covariance C0.
Available since 2.12

real **gaussian_dlm_obs_lupdf**(matrix y | matrix F, matrix G, vector V, matrix W, vector m0, matrix C0)
The log of the density of the Gaussian Dynamic Linear model with observation matrix y in which rows are variables and columns are observations, design matrix F, transition matrix G, observation covariance matrix with diagonal V, system covariance matrix W, and the initial state is distributed normal with mean m0 and covariance C0. This function drops constant additive terms.
Available since 2.25

24.   Simplex Distributions

The simplex probabilities have support on the unit K-simplex for a specified K. A K-dimensional vector θ is a unit K-simplex if $\theta_k \geq 0$ for $k \in \{1, \ldots, K\}$ and $\sum_{k=1}^{K} \theta_k = 1$.

24.1.   Dirichlet distribution

Probability density function

If $K \in \mathbb{N}$ and $\alpha \in (\mathbb{R}^+)^K$, then for $\theta \in K$-simplex,

$$\text{Dirichlet}(\theta | \alpha) = \frac{\Gamma\left(\sum_{k=1}^{K} \alpha_k\right)}{\prod_{k=1}^{K} \Gamma(\alpha_k)} \prod_{k=1}^{K} \theta_k^{\alpha_k - 1}.$$

Warning: If any of the components of θ satisfies $\theta_i = 0$ or $\theta_i = 1$, then the probability is 0 and the log probability is $-\infty$. Similarly, the distribution requires strictly positive parameters, with $\alpha_i > 0$ for each i.

Meaning of Dirichlet parameters

A symmetric Dirichlet prior is $[\alpha, \ldots, \alpha]^\top$. To code this in Stan,

```
data {
  int<lower=1> K;
  real<lower=0> alpha;
}
generated quantities {
  vector[K] theta = dirichlet_rng(rep_vector(alpha, K));
}
```

Taking $K = 10$, here are the first five draws for $\alpha = 1$. For $\alpha = 1$, the distribution is uniform over simplexes.

```
1) 0.17 0.05 0.07 0.17 0.03 0.13 0.03 0.03 0.27 0.05
2) 0.08 0.02 0.12 0.07 0.52 0.01 0.07 0.04 0.01 0.06
3) 0.02 0.03 0.22 0.29 0.17 0.10 0.09 0.00 0.05 0.03
4) 0.04 0.03 0.21 0.13 0.04 0.01 0.10 0.04 0.22 0.18
5) 0.11 0.22 0.02 0.01 0.06 0.18 0.33 0.04 0.01 0.01
```

That does not mean it's uniform over the marginal probabilities of each element. As the size of the simplex grows, the marginal draws become more and more concentrated below (not around) $1/K$. When one component of the simplex is large, the others must all be relatively small to compensate. For example, in a uniform distribution on 10-simplexes, the probability that a component is greater than the mean of $1/10$ is only 39%. Most of the posterior marginal probability mass for each component is in the interval $(0, 0.1)$.

When the α value is small, the draws gravitate to the corners of the simplex. Here are the first five draws for $\alpha = 0.001$.

```
1) 3e-203 0e+00 2e-298 9e-106 1e+000 0e+00 0e+000 1e-047 0e+00 4e-279
2) 1e+000 0e+00 5e-279 2e-014 1e-275 0e+00 3e-285 9e-147 0e+00 0e+000
3) 1e-308 0e+00 1e-213 0e+000 0e+000 8e-75 0e+000 1e+000 4e-58 7e-112
4) 6e-166 5e-65 3e-068 3e-147 0e+000 1e+00 3e-249 0e+000 0e+00 0e+000
5) 2e-091 0e+00 0e+000 0e+000 1e-060 0e+00 4e-312 1e+000 0e+00 0e+000
```

Each row denotes a draw. Each draw has a single value that rounds to one and other values that are very close to zero or rounded down to zero.

As α increases, the draws become increasingly uniform. For $\alpha = 1000$,

```
1) 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10
2) 0.10 0.10 0.09 0.10 0.10 0.10 0.11 0.10 0.10 0.10
3) 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10
4) 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10
5) 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10
```

Sampling statement

```
theta ~ dirichlet(alpha)
```

Increment target log probability density with `dirichlet_lupdf(theta | alpha)`.
Available since 2.0

Stan functions

real **dirichlet_lpdf**(vector theta | vector alpha)
The log of the Dirichlet density for simplex theta given prior counts (plus one) alpha
Available since 2.12

real **dirichlet_lupdf**(vector theta | vector alpha)
The log of the Dirichlet density for simplex theta given prior counts (plus one) alpha

dropping constant additive terms
Available since 2.25

`vector` **`dirichlet_rng`**`(vector alpha)`
Generate a Dirichlet variate with prior counts (plus one) alpha; may only be used in
transformed data and generated quantities blocks
Available since 2.0

25. Correlation Matrix Distributions

The correlation matrix distributions have support on the (Cholesky factors of) correlation matrices. A Cholesky factor L for a $K \times K$ correlation matrix Σ of dimension K has rows of unit length so that the diagonal of $LL^\top$ is the unit K-vector. Even though models are usually conceptualized in terms of correlation matrices, it is better to operationalize them in terms of their Cholesky factors. If you are interested in the posterior distribution of the correlations, you can recover them in the generated quantities block via

```
generated quantities {
  corr_matrix[K] Sigma;
  Sigma = multiply_lower_tri_self_transpose(L);
}
```

25.1. LKJ correlation distribution

Probability density function

For $\eta > 0$, if Σ a positive-definite, symmetric matrix with unit diagonal (i.e., a correlation matrix), then

$$\text{LkjCorr}(\Sigma|\eta) \propto \det{(\Sigma)}^{(\eta-1)}.$$

The expectation is the identity matrix for any positive value of the shape parameter η, which can be interpreted like the shape parameter of a symmetric beta distribution:

- if $\eta = 1$, then the density is uniform over correlation matrices of order K;

- if $\eta > 1$, the identity matrix is the modal correlation matrix, with a sharper peak in the density at the identity matrix for larger η; and

- for $0 < \eta < 1$, the density has a trough at the identity matrix.

- if η were an unknown parameter, the Jeffreys prior is proportional to $\sqrt{2\sum_{k=1}^{K-1}\left(\psi_1\left(\eta + \frac{K-k-1}{2}\right) - 2\psi_1\left(2\eta + K - k - 1\right)\right)}$, where $\psi_1()$ is the trigamma function

See (Lewandowski, Kurowicka, and Joe 2009) for definitions. However, it is much better computationally to work directly with the Cholesky factor of Σ, so this distribution should never be explicitly used in practice.

Sampling statement

`y ~ `**`lkj_corr`**`(eta)`

Increment target log probability density with `lkj_corr_lupdf(y | eta)`.
Available since 2.3

Stan functions

real **`lkj_corr_lpdf`**`(matrix y | real eta)`
The log of the LKJ density for the correlation matrix y given nonnegative shape eta.
`lkj_corr_cholesky_lpdf` is faster, more numerically stable, uses less memory, and
should be preferred to this.
Available since 2.12

real **`lkj_corr_lupdf`**`(matrix y | real eta)`
The log of the LKJ density for the correlation matrix y given nonnegative shape
eta dropping constant additive terms. `lkj_corr_cholesky_lupdf` is faster, more
numerically stable, uses less memory, and should be preferred to this.
Available since 2.25

matrix **`lkj_corr_rng`**`(int K, real eta)`
Generate a LKJ random correlation matrix of order K with shape eta; may only be
used in transformed data and generated quantities blocks
Available since 2.0

25.2. Cholesky LKJ correlation distribution

Stan provides an implicit parameterization of the LKJ correlation matrix density in
terms of its Cholesky factor, which you should use rather than the explicit parameter-
ization in the previous section. For example, if L is a Cholesky factor of a correlation
matrix, then

```
 L ~ lkj_corr_cholesky(2.0); # implies L * L' ~ lkj_corr(2.0);
```

Because Stan requires models to have support on all valid constrained parameters, L
will almost always[1] be a parameter declared with the type of a Cholesky factor for a
correlation matrix; for example,

```
 parameters {   cholesky_factor_corr[K] L;   # rather than corr_matrix[K] Si
```

---

[1]It is possible to build up a valid L within Stan, but that would then require Jacobian adjustments to
imply the intended posterior.

Probability density function

For $\eta > 0$, if L is a $K \times K$ lower-triangular Cholesky factor of a symmetric positive-definite matrix with unit diagonal (i.e., a correlation matrix), then

$$\text{LkjCholesky}(L|\eta) \propto |J| \det(LL^\top)^{(\eta-1)} = \prod_{k=2}^{K} L_{kk}^{K-k+2\eta-2}.$$

See the previous section for details on interpreting the shape parameter η. Note that even if $\eta = 1$, it is still essential to evaluate the density function because the density of L is not constant, regardless of the value of η, even though the density of $LL^\top$ is constant iff $\eta = 1$.

A lower triangular L is a Cholesky factor for a correlation matrix if and only if $L_{k,k} > 0$ for $k \in 1{:}K$ and each row L_k has unit Euclidean length.

Sampling statement

`L ~ lkj_corr_cholesky(eta)`

Increment target log probability density with `lkj_corr_cholesky_lupdf(L | eta)`.
Available since 2.4

Stan functions

real **lkj_corr_cholesky_lpdf**(matrix L | real eta)
The log of the LKJ density for the lower-triangular Cholesky factor L of a correlation matrix given shape eta
Available since 2.12

real **lkj_corr_cholesky_lupdf**(matrix L | real eta)
The log of the LKJ density for the lower-triangular Cholesky factor L of a correlation matrix given shape eta dropping constant additive terms
Available since 2.25

matrix **lkj_corr_cholesky_rng**(int K, real eta)
Generate a random Cholesky factor of a correlation matrix of order K that is distributed LKJ with shape eta; may only be used in transformed data and generated quantities blocks
Available since 2.4

26.   Covariance Matrix Distributions

The covariance matrix distributions have support on symmetric, positive-definite $K \times K$ matrices.

26.1.   Wishart distribution

Probability density function

If $K \in \mathbb{N}$, $\nu \in (K - 1, \infty)$, and $S \in \mathbb{R}^{K \times K}$ is symmetric and positive definite, then for symmetric and positive-definite $W \in \mathbb{R}^{K \times K}$,

$$\text{Wishart}(W|\nu, S) = \frac{1}{2^{\nu K/2}} \ \frac{1}{\Gamma_K\left(\frac{\nu}{2}\right)} \ |S|^{-\nu/2} \ |W|^{(\nu-K-1)/2} \ \exp\left(-\frac{1}{2} \ \text{tr}\left(S^{-1}W\right)\right),$$

where tr() is the matrix trace function, and $\Gamma_K()$ is the multivariate Gamma function,

$$\Gamma_K(x) = \frac{1}{\pi^{K(K-1)/4}} \ \prod_{k=1}^{K} \Gamma\left(x + \frac{1-k}{2}\right).$$

Sampling statement

`W ~ wishart(nu, Sigma)`

Increment target log probability density with `wishart_lupdf(W | nu, Sigma)`.
Available since 2.0

Stan functions

real **wishart_lpdf**(matrix W | real nu, matrix Sigma)
The log of the Wishart density for symmetric and positive-definite matrix W given degrees of freedom nu and symmetric and positive-definite scale matrix Sigma
Available since 2.12

real **wishart_lupdf**(matrix W | real nu, matrix Sigma)
The log of the Wishart density for symmetric and positive-definite matrix W given degrees of freedom nu and symmetric and positive-definite scale matrix Sigma dropping constant additive terms
Available since 2.25

matrix **wishart_rng**(real nu, matrix Sigma)
Generate a Wishart variate with degrees of freedom nu and symmetric and positive-definite scale matrix Sigma; may only be used in transformed data and generated quantities blocks
Available since 2.0

26.2.   Inverse Wishart distribution

Probability density function

If $K \in \mathbb{N}$, $\nu \in (K - 1, \infty)$, and $S \in \mathbb{R}^{K \times K}$ is symmetric and positive definite, then for symmetric and positive-definite $W \in \mathbb{R}^{K \times K}$,

$$\text{InvWishart}(W|\nu, S) = \frac{1}{2^{\nu K/2}} \frac{1}{\Gamma_K\left(\frac{\nu}{2}\right)} |S|^{\nu/2} |W|^{-(\nu+K+1)/2} \exp\left(-\frac{1}{2}\operatorname{tr}(SW^{-1})\right).$$

Sampling statement

W ~ **inv_wishart**(nu, Sigma)

Increment target log probability density with inv_wishart_lupdf(W | nu, Sigma).
Available since 2.0

Stan functions

real **inv_wishart_lpdf**(matrix W | real nu, matrix Sigma)
The log of the inverse Wishart density for symmetric and positive-definite matrix W given degrees of freedom nu and symmetric and positive-definite scale matrix Sigma
Available since 2.12

real **inv_wishart_lupdf**(matrix W | real nu, matrix Sigma)
The log of the inverse Wishart density for symmetric and positive-definite matrix W given degrees of freedom nu and symmetric and positive-definite scale matrix Sigma dropping constant additive terms
Available since 2.25

matrix **inv_wishart_rng**(real nu, matrix Sigma)
Generate an inverse Wishart variate with degrees of freedom nu and symmetric and positive-definite scale matrix Sigma; may only be used in transformed data and generated quantities blocks
Available since 2.0

Additional Distributions

27. Hidden Markov Models

An elementary first-order Hidden Markov model is a probabilistic model over N observations, y_n, and N hidden states, x_n, which can be fully defined by the conditional distributions $p(y_n \mid x_n, \phi)$ and $p(x_n \mid x_{n-1}, \phi)$. Here we make the dependency on additional model parameters, ϕ, explicit. When x is continuous, the user can explicitly encode these distributions in Stan and use Markov chain Monte Carlo to integrate x out.

When each state x takes a value over a discrete and finite set, say $\{1, 2, ..., K\}$, we can take advantage of the dependency structure to marginalize x and compute $p(y \mid \phi)$. We start by defining the conditional observational distribution, stored in a $K \times N$ matrix ω with

$$\omega_{kn} = p(y_n \mid x_n = k, \phi).$$

Next, we introduce the $K \times K$ transition matrix, Γ, with

$$\Gamma_{ij} = p(x_n = j \mid x_{n-1} = i, \phi).$$

Each row defines a probability distribution and must therefore be a simplex (i.e. its components must add to 1). Currently, Stan only supports stationary transitions where a single transition matrix is used for all transitions. Finally we define the initial state K-vector ρ, with

$$\rho_k = p(x_0 = k \mid \phi).$$

The Stan functions that support this type of model are special in that the user does not explicitly pass y and ϕ as arguments. Instead, the user passes $\log \omega$, Γ, and ρ, which in turn depend on y and ϕ.

27.1. Stan functions

real **hmm_marginal**(matrix log_omega, matrix Gamma, vector rho)
Returns the log probability density of y, with x_n integrated out at each iteration.
Available since 2.24

The arguments represent (1) the log density of each output, (2) the transition matrix, and (3) the initial state vector.

- *log_omega*: $\log \omega_{kn} = \log p(y_n \mid x_n = k, \phi)$, log density of each output,

- *Gamma*: $\Gamma_{ij} = p(x_n = j|x_{n-1} = i, \phi)$, the transition matrix,

- *rho*: $\rho_k = p(x_0 = k \mid \phi)$, the initial state probability.

`array[] int`   **`hmm_latent_rng`**`(matrix log_omega, matrix Gamma, vector rho)`
Returns a length N array of integers over $\{1, ..., K\}$, sampled from the joint posterior distribution of the hidden states, $p(x \mid \phi, y)$. May be only used in transformed data and generated quantities.
Available since 2.24

`matrix`   **`hmm_hidden_state_prob`**`(matrix log_omega, matrix Gamma, vector rho)`
Returns the matrix of marginal posterior probabilities of each hidden state value. This will be a $K \times N$ matrix. The nth column is a simplex of probabilities for the nth variable. Moreover, let A be the output. Then $A_{ij} = p(x_j = i \mid \phi, y)$. This function may only be used in transformed data and generated quantities.
Available since 2.24

Appendix

28.   Mathematical Functions

This appendix provides the definition of several mathematical functions used throughout the manual.

28.1.   Beta

The beta function, $B(a, b)$, computes the normalizing constant for the beta distribution, and is defined for $a > 0$ and $b > 0$ by

$$B(a, b) \;=\; \int_0^1 u^{a-1}(1-u)^{b-1}\, du \;=\; \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\,,$$

where $\Gamma(x)$ is the Gamma function.

28.2.   Incomplete beta

The incomplete beta function, $B(x; a, b)$, is defined for $x \in [0, 1]$ and $a, b \geq 0$ such that $a + b \neq 0$ by

$$B(x; a, b) \;=\; \int_0^x u^{a-1}\,(1-u)^{b-1}\, du,$$

where $B(a, b)$ is the beta function defined in appendix. If $x = 1$, the incomplete beta function reduces to the beta function, $B(1; a, b) = B(a, b)$.

The regularized incomplete beta function divides the incomplete beta function by the beta function,

$$I_x(a, b) \;=\; \frac{B(x; a, b)}{B(a, b)}\,.$$

28.3.   Gamma

The gamma function, $\Gamma(x)$, is the generalization of the factorial function to continuous variables, defined so that for positive integers n,

$$\Gamma(n + 1) = n!$$

Generalizing to all positive numbers and non-integer negative numbers,

$$\Gamma(x) = \int_0^\infty u^{x-1} \exp(-u)\, du.$$

28.4. Digamma

The digamma function Ψ is the derivative of the $\log \Gamma$ function,

$$\Psi(u) \;=\; \frac{d}{du} \log \Gamma(u) \;=\; \frac{1}{\Gamma(u)} \frac{d}{du} \Gamma(u).$$

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