Topics Covered:

- Introduction to Inverse Kinematics
- Geometric Approach to Inverse Kinematics

Additional Reading:

- Craig Chapter 4; LP Chapter 6; MLS Chapter 3, Section 3

Introduction to Inverse Kinematics

Inverse kinematics refers to the process of determining a feasible joint-configuration given a desired end-effector configuration.

- this is a nonlinear problem
- clearly desired end-effector configuration must be in workspace to have a solution (somtimes will have multiple solutions)
- no real general algorithms exist to handle the nonlinear inverse kinematics problem
- problem/manipulator considered solveable if joint-configuration can be determined by an algorithm that allows one to determine all joint variables associated with a given end-effector configuration.
- closed form solutions
  - algebraic (numerical)
  - geometric (analytic)
  - Paden-Kahan
  - Pieper's solution: Closed form solution for a serial 6 DOF in which three consecutive axes intersect at a point (including robots with three consecutive parallel axes since they meet at a point at infinity). Pieper's method applies to the majority of commercially available industrial robots.

We can think of the main approaches using the following table:

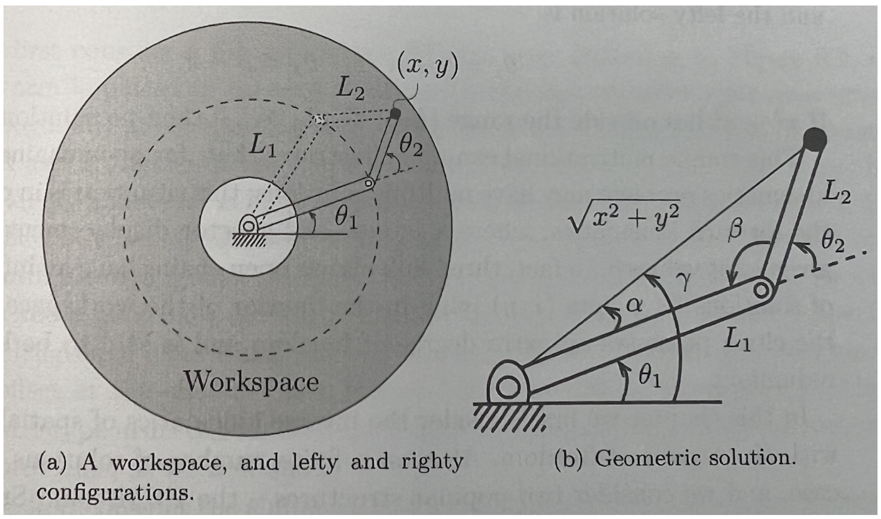
Method	Advantages	Disadvantages
algebraic	systematic and general	hard to understand, hard to use
geometric	easy to understand and apply	not systematic, not always applicable
Paden-Kahan	easy to understand and systematic	not general

The Geometric Approach

- by observation, reduce geometry of the linkage to easily solved sub-problems.
- for example, one technique reduces the linkage geometry to triangles

Example

Many manipulators have the following subproblem, which is usually obtained by projecting the manipulator geometry to 2 coordinate axes.



(a) A workspace, and lefty and righty configurations. (b) Geometric solution.

Here, the forward kinematics are given by:

$$\begin{Bmatrix} x_c \\ y_c \\ \theta_c \end{Bmatrix} = \begin{Bmatrix} l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) \\ \theta_1 + \theta_2 \end{Bmatrix}$$

Assuming $L_1 > L_2$, the set of reachable points (the workspace) is an annulus of inner radius $L1 - L2$ and outer radius $L1 + L2$. Looking at the workspace above, we can visually see that there will be either zero, one, or two solutions depending on whether (x, y) lies on the exterior, boundary, or interior of this annulus, respectively.

When there are two solutions, the elbow joint (second joint) will be either positive or negative. This is sometimes called "lefty" or "righty", or "elbow-up" or "elbow-down" solutions.

Here, orientation is not fully controllable, but consider placement of end-effector in Euclidean space. The manipulator (port) forms a triangle:

CHECK: $l_1 + l_2 < \sqrt{x_e^2 + y_e^2}$

As long as this condition is satisfied, we can use the law of cosines for our inverse kinematics. Recall that the law of cosines is:

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

where a, b, and c are the sides of a triangle and C is the angle between sides a and b (opposite of side c).

In our situation, the law of cosines yields the formula:

$$x^2 + y^2 = l_1^2 + l_2^2 - 2l_1 l_2 \cos(\beta)$$

Thus, rearranging this equation gives us:

$$\beta = \cos^{-1}\left(\frac{l_1^2 + l_2^2 - x^2 - y^2}{2l_1 l_2}\right)$$

Similarly, applyign the law of cosines to α gives us:

$$\alpha = \cos^{-1}\left(\frac{x^2 + y^2 + l_1^2 - l_2^2}{2l_1\sqrt{x^2 + y^2}}\right)$$

However, there is a little problem since inverse of cosine is not unique (there are 2 ways to get x_des and y_des). So there will be only 2 out of 4 possible solutions which actually work. We will find these solutions by verifying the orientation at the end.

Taking γ using the two-argument arctangent function:

$$\gamma = \text{atan2}(y, x),$$

we can solve for the joint angles (assuming the righty-solution) as:

$$\begin{aligned}\theta_1 &= \gamma - \alpha \\ \theta_2 &= \pi - \beta\end{aligned}$$

The lefty solution is:

$$\begin{aligned}\theta_1 &= \gamma + \alpha \\ \theta_2 &= \beta - \pi\end{aligned}$$

Lastly, we solve for orientation as $\theta_1 + \theta_2$. This orientation must be equal to:

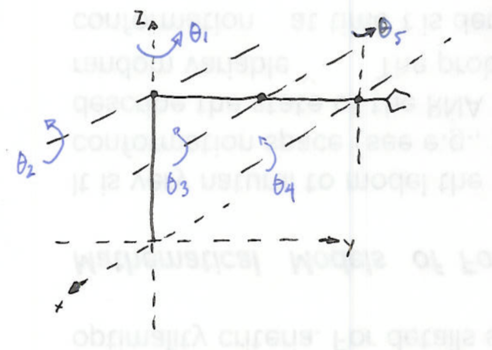
$$\begin{aligned}\theta_1 + \theta_2 &= \gamma - \alpha + \pi - \beta \\ &= \gamma + \alpha + \beta - \pi\end{aligned}$$

We will have to check these conditions to make sure that the solution we found is valid.

But, how can geometric technique be applied to more complicated manipulators? We will break the problem down to:

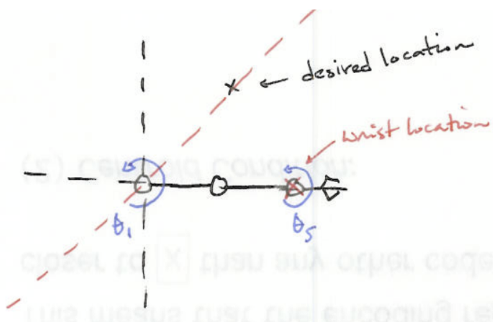
1. position of the wrist
2. orientation of the hand/wrist
3. angle of the first joint

We will assume that (2) is solved enough to give us desired position of the wrist.



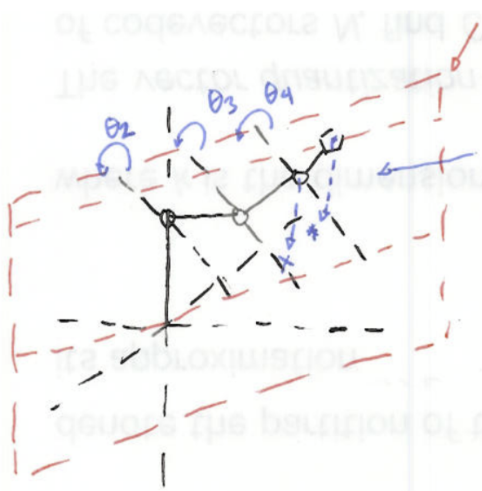
1. has wrist at end for reorientation
2. to place wrist at a particular location, consider the following projections

1. Top View



If we can get the manipulator to lie along the red dotted line, then we can project to this plane since all of the remaining joints work in this plane.
Basically, $\theta_1 = atan(x, y) - \frac{\pi}{2}$ (since reference configuration aligned with y-axis and not x-axis)

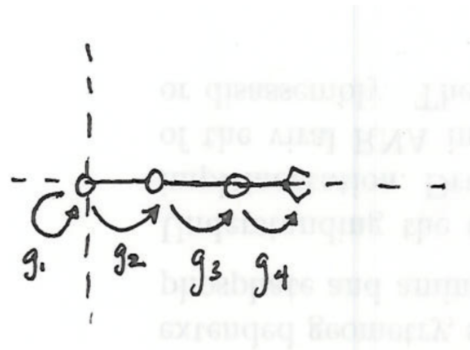
2. Side View



red arrow: plane of projection
blue arrow: end-effector is too high and too far out (same with wrist)

When projected to the plane, the placement of the wrist reduces to the previous subproblem (θ_1, θ_2).

Or, consider the following



here, we can control orientation inside the $\underbrace{\text{dexterous workspace}}_{\text{(discussed this before)}}$

have desired end-effector configuration g_e^*

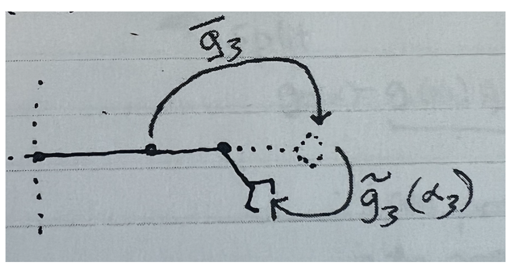
actual $g_e(\vec{\theta}) = g_1(\theta_1)g_2(\theta_2)g_3(\theta_3)g_4$

recall each $g_i(\theta_i) = \underbrace{\bar{g}_i}_{\text{constant}} \underbrace{\tilde{g}_i}_{\text{varies with } \theta - i}$

Using this, we can break our problem into two problems:

$$g_e^* = \underbrace{g_1 g_2 \bar{g}_3}_{\text{moves to wrist}} \underbrace{\tilde{g}_3 g_4}_{\text{wrist to gripper}}$$

where this breakdown is shown by the diagram:

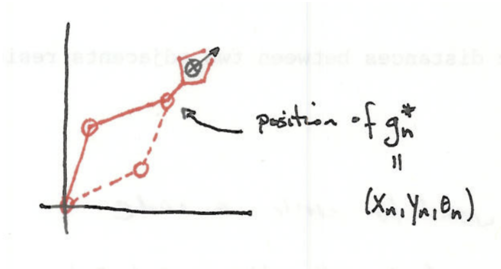


Thus, $g_e = g_1 g_2 \bar{g}_3 \tilde{g}_3 g_4$, where in our case, $\tilde{g}_3$ is just rotation, no transformation. So, we desire a joint configuration $\vec{\theta}^*$ such that

$$\begin{aligned} g_e^* &= g_e(\vec{\theta}^*) \\ &= g_1 g_2 \bar{g}_3 \tilde{g}_3 g_4 \\ g_n^* &= g_e^* g_4^{-1} = g_1 g_2 \bar{g}_3 \tilde{g}_3 \end{aligned}$$

But since $\tilde{g}_3$ is just a rotation and does not change the position, we can ignore it.

So our problem is to identify θ_1 and θ_2 such that the position of $g_1 g_2 \bar{g}_3$ is our desired position.



just need to find θ_1, θ_2 to position $g_1 g_2 g_3$ at (x_n, y_n).

Once we have θ_1 and θ_2, what is θ_3? Well,

$$\begin{aligned} \theta_1 + \theta_2 + \theta_3 &= \theta^* \\ \theta_3 &= \theta^* - \theta_1 - \theta_2 \end{aligned}$$

Summary of the Geometric Approach

1. it's easy because you are working it out intuitively using geometry.
2. not systematic because requires good intuition
3. not always applicable (limited set of "geometric techniques"). Good thing is that design of manipulators can allow for simple/geometric solution.