Topics Covered:

- Rotation Matrices
- Change of Coordinate Frames
- Multiple Displacements

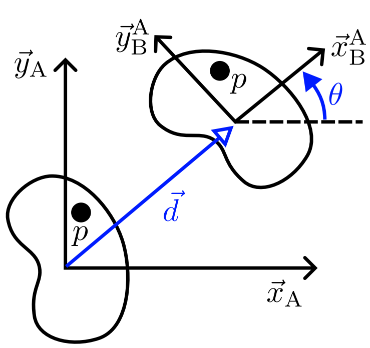
Additional Reading:

- Lynch, K.M. and Park, F.C. Modern Robotics: Section 3.2
- Craig, J.J. Introduction to Robotics: 2.3
- Murray et al. A Mathematical Introduction to Robotic Manipulation: Chapter 2, Section 2.1

Review

Last class we went over how to describe the configuration of a rigid body. We determined that we need 3 degrees of freedom to describe planar transformations: x, y, and θ.

We ended the last class with the 3DOF example:



Configuration of a rigid body:

$$g = (x, y, \theta)^\top \qquad \text{(Vector Notation)}$$

or equivalently:

$$g = (x, y, R(\theta))$$
$$g = (\vec{d}, R(\theta))$$
$$g = (\vec{d}, \theta)$$

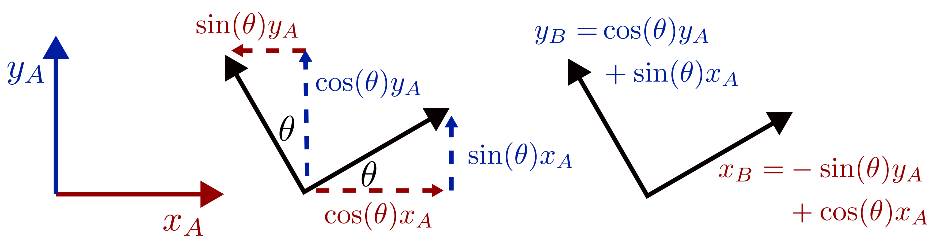
Note: $R(\theta) := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

In todays class, we will go over how to represent both configurations and transformations using rotation matrices.

Rotation Matrices

While it is technically simple and sufficient to represent the configuration of a planar body using only position and orientation of the body frame with respect to the fixed frame, this will become cumbersome when we move to three-dimensional space where we need a set of three angles to describe orientation. However, in three-dimensional space, it will be more straightforward to express the directions of the coordinate axes of the body frame in terms of coefficients of the coordinate axes of the reference frame.

This is illustrated in the following diagram:



Then, we can convert this coordinate axes transformation into matrix form, where the following:

$$\vec{x}_B^A = \cos(\theta)\vec{x}_A - \sin(\theta)\vec{y}_A$$
$$\vec{y}_B^A = \sin(\theta)\vec{x}_A + \cos(\theta)\vec{y}_A$$

in matrix form is written as:

$$\begin{bmatrix} \vec{x}_B^A \\ \vec{y}_B^A \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{R(\theta)} \begin{bmatrix} \vec{x}_A \\ \vec{y}_A \end{bmatrix}$$

We can use this same technique to transform points on a rigid body. Consider some point $p \in \mathbb{R}^2$ on our rigid body, which can be represented by the column vector:

$$p = \begin{bmatrix} p_x \\ p_y \end{bmatrix},$$

We can transform the coordinate frame of this point using our expression from before:

$$\vec{p}_{\mathrm{B}}^{\mathrm{A}} = \vec{d} + R(\theta)\vec{p}_{\mathrm{A}}$$

There's two additional observations to make. The first is that the coordinates $(\vec{d}, R(\theta))$ can be used to describe either a configuration or a transformation.

- If $\vec{d}_{\mathrm{BC}} = 0$, then the transformation is a pure rotation.

- If $\theta_{\text{BC}} = 0$ or $R(\theta_{\text{AB}}) = \mathbb{1}$, then the transformation is a pure translation.

The second observation is that we can translate between a matrix representation of our rotations and a rotation angle using the relationship:

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \qquad \theta = atan2(R_{21}, R_{11})$$

Properties of Rotation Matrices

A rotation matrix is a special type of matrix that encodes a rotation in space, transforming vectors without changing their length or the angles between them. These matrices form a *group* under multiplication, meaning that combining rotations, doing nothing, or undoing a rotation all result in another valid rotation matrix. We will unpack this statement in this section.

Overall, rotation matrices have two important properties:

- $R^\top R = I$ (unit axes are orthogonal)
- $\det R = 1$ (rotations follow the right-hand rule, i.e. $\vec{x} \times \vec{y} = \vec{z}$)

The first property stems from two conditions placed on our derivation of the rotation matrix: a unit norm condition specifying that the coordinate axes are represented by unit vectors, and an orthogonality condition on the unit axes. Together, these four constraints (2 for the unit norm axes and 2 for the orthogonality conditions) can be represented by the single constraint $R^\top R = I$.

The second property is a result of the fact that the rotation matrix is right hande (i.e., $\vec{x} \times \vec{y} = \vec{z}$).

Due to these properties, rotation matrices can be mathematically classified as a *special orthogonal group SO(2)*. This is called a “group” because it satisfies the properties required of a mathematical group.

Definition: Group

A group is a set of elements $G = \{a, b, c, \ldots\}$ with a binary operation $\cdot$ that satisfies the following properties:

closure	$a \cdot b \in G$ for all $a, b \in G$
associativity	$(a \cdot b) \cdot c = a \cdot (b \cdot c))$
identity element exists	there is an $I \in G$ such that $a \cdot I = I \cdot a = a$ for each $a \in G$
inverse exists	for each $a \in G$ there exists $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = I$

For the special orthogonal group $SO(2)$, the group operation is matrix multiplication, and the following group properties are:

closure	$R_1R_2 \in SO(2)$
associativity	$(R_1R_2)R_3 = R_1(R_2R_3)$
identity element exists	$I \in SO(2)$
inverse exists	matrix inverse $R^{-1} = R^\top$

Proofs of these properties are as follows.

Properties of the special orthogonal group $SO(2)$:

Proposition 1. *The inverse of a rotation matrix $R \in SO(2)$ is also a rotation matrix, and it is equal to the transpose of R, i.e., $R^{-1} = R^\top$.*

Proof. The condition $R^\top R = I$ implies that $R^\top = R^{-1}$ and $RR^\top = I$. Since $\det R^\top = \det R = 1$, $R^\top$ is also a rotation matrix. □

Proposition 2. *The product of two rotation matrices is a rotation matrix*

Proof. Given $R_1, R_2 \in SO(2)$, their product R_1R_2 satisfies $(R_1R_2)^\top R_1R_2 = R_2^\top R_1^\top R_1R_2 = R_2^\top R_2 = I$. Also, $\det R_1R_2 = \det R_1 \cdot \det R_2 = 1$. Thus, R_1R_2 satisfies both conditions for a rotation matrix. □

Proposition 3. *Multiplication of rotation matrices is associative, i.e., for $R_1, R_2, R_3 \in SO(2)$, $(R_1R_2)R_3 = R_1(R_2R_3)$.*

Proof. Matrix multiplications are associative. □

Proposition 4. *For any vector $x \in \mathbb{R}^2$ and $R \in SO(2)$, the vector $y = Rx$ has the same length as x.*

Proof. This follows from:

$$\|y\|^2 = y^\top y = (Rx)^\top Rx = x^\top R^\top Rx = x^\top x = \|x\|^2$$

□

Later in the course we will see that these same properties hold for the special group of rotation matrices for three-dimensional transformations, called the *special orthogonal group* $SO(3)$. The only property that differs for rotation matrices in higher do,emsopms os that the rotation matrices are not generally commutative (i.e., $R_1R_2 \neq R_2R_1$).

Uses of Rotation Matrices

There are three major uses for a rotation matrix R:

1. to represent an orientation: R_{AB} represents the orientation of frame B w.r.t. frame A
2. to rotate a vector or a frame: Given some rotation matrix R, $p' = Rp$
3. to change the reference frame in which a vector or a frame is represented

Notation

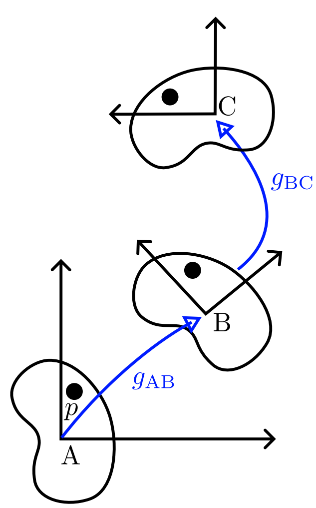
For the sake of consistent notation moving forward, we will define and use the following:

- $\hat{x}_A$: the unit vector in the x direction of frame A
- $\hat{y}_A$: the unit vector in the y direction of frame A
- g_{AB}: the transformation from frame A to frame B
- $\vec{d}^A_{AB}$: the displacement from frame A to frame B in reference to frame A
- $R(\theta_{AB})$: the rotation matrix that rotates a point by θ_{AB} radians, with θ_{AB} being the angle from frame A to frame B
- $\vec{p}^{\text{A}}_{\text{B}}$: a point p in frame B that is represented in frame A
- $\vec{v}$: general vector notation for some variable v

Multiple Displacements

Now, what about multiple displacements?

Consider the following example:



So far, we have only considered the transformation of a point from one frame to another. For example, in this example:

$$p_C \text{ in frame B is } p^B_C = \vec{d}^B_{BC} + R(\theta_{BC})\vec{p}$$
$$p_B \text{ in frame A is } p^A_B = \vec{d}^A_{AB} + R(\theta_{AB})\vec{p}$$

But what if we want to find p_C in frame A? We can use the following relationship:

$$\begin{aligned} p_C^A &= \vec{d}_{AB}^A + R(\theta_{AB}) p_C^B \\ &= \vec{d}_{AB}^A + R(\theta_{AB}) \left(\vec{d}_{BC}^B + R(\theta_{BC}) \vec{p} \right) \\ &= \vec{d}_{AB}^A + R(\theta_{AB}) \vec{d}_{BC}^B + R(\theta_{AB}) R(\theta_{BC}) \vec{p} \end{aligned}$$

We can write this multiple transformation in vector coordinate form as:

$$\begin{aligned} g_{AC} &= g_{AB} \cdot g_{BC} \\ &= (\vec{d}_{AB}^A, R(\theta_{AB})) \cdot (\vec{d}_{BC}^B, R(\theta_{BC})) \\ &= (\underbrace{\vec{d}_{AB}^A + R(\theta_{AB}) \vec{d}_{BC}^B}_{\vec{d}_{AC}^A}, \underbrace{R(\theta_{AB}) R(\theta_{BC})}_{R(\theta_{AC})}) \end{aligned}$$