Lecture 16 – ME6402, Spring 2025
Feedback Linearization

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March 4, 2025

Goals of Lecture 16

- Relative degree
- Input-output linearization
- Zero dynamics

Additional Reading

- Khalil Chapter 13
- Sastry Chapter 9

Relative Degree

Today: Relative degree, input-output linearization, zero dynamics

Consider the single-input single-output (SISO) nonlinear system:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x). \end{aligned} \tag{1}$$

Relative degree (informal definition): Number of times we need to take the time derivative of the output to see the input:

$$\dot{y} = \underbrace{\frac{\partial h}{\partial x} f(x)}_{=: L_f h(x)} + \underbrace{\frac{\partial h}{\partial x} g(x)}_{=: L_g h(x)} u$$

- $L_f h$ is called the *Lie derivative* of h along the vector field f

Relative Degree (cont.)

If $L_g h(x) \neq 0$ in an open set containing the equilibrium, then the relative degree is equal to 1. If $L_g h(x) \equiv 0$, continue taking derivatives:

$$\ddot{y} = \underbrace{L_f L_f h(x)}_{=: L_f^2 h(x)} + L_g L_f h(x) u.$$

If $L_g L_f h(x) \neq 0$, then relative degree is 2. If $L_g L_f h(x) \equiv 0$, continue.

Relative Degree (cont.)

Definition: The system (2) has *relative degree* r if, in a neighbourhood of the equilibrium,

$$L_g L_f^{i-1} h(x) = 0 \quad i = 1, 2, \ldots, r-1$$
$$L_g L_f^{r-1} h(x) \neq 0.$$

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x). \end{aligned} \tag{2}$$

Example 1

The system

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1^3 + u \\
y &= x_1
\end{aligned}
$$

has relative degree

Example 1

The system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1^3 + u$$
$$y = x_1$$

has relative degree = 2.

Example 2

SISO linear system:

$$\dot{x} = Ax + Bu \quad y = Cx$$

$L_g h(x) = CB, \quad L_g L_f h(x) = CAB, \quad \dots\ , \quad L_g L_f^{r-1} = CA^{r-1}B.$

- $CB \neq 0 \;\Rightarrow\;$ relative degree $= 1$
- $CB = 0, \quad CAB \neq 0 \;\Rightarrow\;$ relative degree $= 2$
- $CB = \cdots = CA^{r-2}B = 0, \quad CA^{r-1}B \neq 0 \;\Rightarrow\;$ relative degree $= r$

The parameters $CA^{i-1}B \;\; i = 1, 2, 3, \dots$ are called *Markov parameters* and are invariant under similarity transformations.

Example 3

$$\dot{x}_1 = x_2 + x_3^3 \qquad\qquad y = x_1$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = u$$

Example 3

$$\begin{aligned} \dot{x}_1 &= x_2 + x_3^3 & y &= x_1 \\ \dot{x}_2 &= x_3 & \dot{y} &= \dot{x}_1 = x_2 + x_3^3 \\ \dot{x}_3 &= u & \ddot{y} &= \dot{x}_2 + 3x_3^2\dot{x}_3 = x_3 + 3x_3^2 u \end{aligned}$$

$L_g L_f h(x) = 3x_3^2 = 0$ when $x_3 = 0$, and $\neq 0$ elsewhere. Thus, this system does not have a well-defined relative degree around $x = 0$.

Input-Output Linearization

If a system has a well-defined relative degree then it is input-output linearizable:

$$y^{(r)} = L_f^r h(x) + \underbrace{L_g L_f^{r-1} h(x)}_{\neq 0} u$$

Apply preliminary feedback:

$$\boxed{u = \frac{1}{L_g L_f^{r-1} h(x)} \left(-L_f^r h(x) + v \right)} \tag{3}$$

where v is a new input to be designed.

Input-Output Linearization (cont.)

Then, $y^{(r)} = v$ is a linear system in the form of an integrator chain:

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= \zeta_3 \\ &\vdots \\ \dot{\zeta}_r &= v \end{aligned}$$

where $\zeta_1 =: y = h(x)$, $\zeta_2 =: \dot{y} = L_f h(x)$, $\ldots$, $\zeta_r =: y^{(r-1)} = L_f^{r-1} h(x)$.

- $$y^{(r)} = L_f^r h(x) + \underbrace{L_g L_f^{r-1} h(x)}_{\neq 0} u$$
- $u = \dfrac{1}{L_g L_f^{r-1} h(x)} \cdot \left(-L_f^r h(x) + v \right)$

Input-Output Linearization (cont.)

To ensure $y(t) \to 0$ as $t \to \infty$, apply the feedback:

$$\boxed{\begin{aligned} v &= -k_1\zeta_1 - k_2\zeta_2 - \cdots - k_r\zeta_r \\ &= -k_1 h(x) - k_2 L_f h(x) - \cdots - k_r L_f^{r-1} h(x) \end{aligned}} \tag{4}$$

where $k_1, \ldots, k_r$ are such that $s^r + k_r s^{r-1} + \cdots + k_2 s + k_1$ has all roots in the open left half-plane.

- $$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= \zeta_3 \\ &\vdots \\ \dot{\zeta}_r &= v \end{aligned}$$

Zero Dynamics

Does the controller (5)-(6) achieve asymptotic stability of $x=0$?

Not necessarily! It renders the $(n-r)$-dimensional manifold:

$$h(x) = L_f h(x) = \cdots = L_f^{r-1} h(x) = 0$$

invariant and attractive.

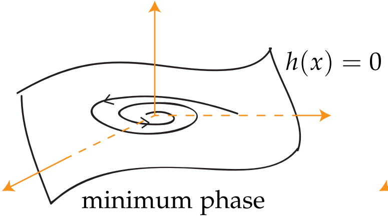
- The dynamics restricted to this manifold are called zero dynamics and determine whether or not $x=0$ is stable.
- If the origin of the zero dynamics is asymptotically stable, the system is called minimum phase. If unstable, it is called nonminimum phase.

$$u = \frac{1}{L_g L_f^{r-1} h(x)} \left(-L_f^r h(x) + v \right) \tag{5}$$

$$\begin{aligned} v &= -k_1 \zeta_1 - k_2 \zeta_2 - \cdots - k_r \zeta_r \\ &= -k_1 h(x) - k_2 L_f h(x) - \\ &\qquad \cdots - k_r L_f^{r-1} h(x) \end{aligned} \tag{6}$$

Zero Dynamics (cont.)

Example: $n = 3, \quad r = 1$



Finding the Zero Dynamics

Set $y = \dot{y} = \cdots = y^{(r-1)} = 0$ and substitute (5) with $v = 0$, that is:

$$u^* = \frac{-L_f^r h(x)}{L_g L_f^{r-1} h(x)}.$$

The remaining dynamical equations describe the zero dynamics.

Finding the Zero Dynamics: Example

Example:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \alpha x_3 + u \\
\dot{x}_3 &= \beta x_3 - u \\
y &= x_1
\end{aligned}
\tag{7}
$$

This system has relative degree 2. With $x_1 = x_2 = 0$ and $u^* = -\alpha x_3$, the remaining dynamical equation is

$$\dot{x}_3 = (\alpha + \beta) x_3.$$

Thus this system is minimum phase if $\alpha + \beta < 0$.

Zero Dynamics of a Linear System

For a linear SISO system, *relative degree* is the difference between the degrees of the denominator and the numerator of the transfer function, and *zeros* are the roots of the numerator. The definitions of relative degree and zero dynamics above generalize these concepts to nonlinear systems.

As an example, the transfer function for (8) is

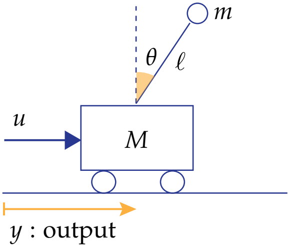
$$\frac{s-(\alpha+\beta)}{s^2(s-\beta)},$$

which has relative degree two and a zero at $s=\alpha+\beta$ as expected.

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \alpha x_3 + u\\ \dot{x}_3 &= \beta x_3 - u\\ y &= x_1\end{aligned} \tag{8}$$

Example

Example: Cart/Pole



$$\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2 \ell \sin\theta - g\sin\theta\cos\theta\right)$$

$$\ddot{\theta} = \frac{1}{\ell(\frac{M}{m} + \sin^2\theta)}\left(-\frac{u}{m}\cos\theta - \dot{\theta}^2\ell\cos\theta\sin\theta + \frac{M+m}{m} g\sin\theta\right)$$

Relative degree $= 2$.

Example (cont.)

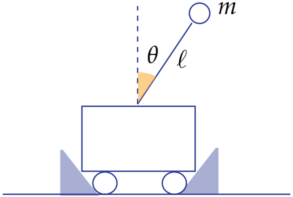
To find the zero dynamics, substitute $y = \dot{y} = 0$, and

$$u^* = -m(\dot{\theta}^2 \ell \sin\theta - g\sin\theta\cos\theta)$$

in the $\ddot{\theta}$ equation:

$$\boxed{\ddot{\theta} = \frac{g}{\ell}\sin\theta.}$$

Same as the dynamics of the pole when the cart is held still:



Nonminimum phase because $\theta = 0$ is unstable for the zero dynamics.

$$\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2\theta} \cdot \left(\frac{u}{m} + \dot{\theta}^2 \ell \sin\theta - g\sin\theta\cos\theta\right)$$

$$\ddot{\theta} = \frac{1}{\ell(\frac{M}{m} + \sin^2\theta)} \cdot \left(-\frac{u}{m}\cos\theta - \dot{\theta}^2 \ell\cos\theta\sin\theta + \frac{M+m}{m} g\sin\theta\right)$$