Lecture 18 – ME6402, Spring 2025
Full-State Feedback Linearization

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Goals of Lecture 18

- Introduce full-state feedback
- Define a few basic concepts from differential geometry
- Frobenius Theorem

Additional Reading

- Khalil, Chapter 13
- Sastry, Chapter 9

Full-State Feedback Linearization

The system $\dot{x} = f(x) + g(x)u$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, is (full state) feedback linearizable if a function $h : \mathbb{R}^n \mapsto \mathbb{R}$ exists such that the relative degree from u to $y = h(x)$ is n.

Since $r = n$, the normal form in Lecture 17 has no zero dynamics and

$$x \to \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_n \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix}$$

is a diffeomorphism that transforms the system to the form on next slide

Full-State Feedback Linearization (cont)

$$
\begin{aligned}
\dot{\zeta}_1 &= \zeta_2 \\
\dot{\zeta}_2 &= \zeta_3 \\
&\vdots \\
\dot{\zeta}_n &= L_f^n h(x) + L_g L_f^{n-1} h(x) u.
\end{aligned}
$$

Then, the feedback linearizing controller

$$
u = \frac{1}{L_g L_f^{n-1} h(x)} \left(-L_f^n h(x) + v \right), \quad v = -k_1 \zeta_1 \cdots - k_n \zeta_n,
$$

yields the closed-loop system:

$$
\dot{\zeta} = A\zeta \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 & 0 & \dots & \\ 0 & 0 & 1 & \dots & \\ & & & \ddots & \\ & & & & 1 \\ -k_1 & -k_2 & -k_3 & \dots & -k_n \end{bmatrix}.
$$

- The system $\dot{x} = f(x) + g(x)u$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, is (full state) feedback linearizable if a function $h : \mathbb{R}^n \mapsto \mathbb{R}$ exists such that the relative degree from u to $y = h(x)$ is n.

$$
x \rightarrow \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_n \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix}
$$

Example

Example:

$$
\begin{aligned}
\dot{x}_1 &= x_2 + 2x_1^2 \\
\dot{x}_2 &= x_3 + u \\
\dot{x}_3 &= x_1 - x_3
\end{aligned}
$$

The choice $y = x_3$ gives relative degree $r = n = 3$.

Let $\zeta_1 = x_3$, $\zeta_2 = \dot{x}_3 = x_1 - x_3$, $\zeta_3 = \ddot{x}_3 = \dot{x}_1 - \dot{x}_3 = x_2 + 2x_1^2 - x_1 + x_3$.

$$
\begin{aligned}
\dot{\zeta}_1 &= \zeta_2 \\
\dot{\zeta}_2 &= \zeta_3 \\
\dot{\zeta}_3 &= (4x_1 - 1)(x_2 + 2x_1^2) + x_1 + u.
\end{aligned}
$$

Feedback linearizing controller:

$$
u = -(4x_1 - 1)(x_2 + 2x_1^2) - x_1 - k_1\zeta_1 - k_2\zeta_2 - k_3\zeta_3.
$$

Summary

Summary so far:

I/O Linearization:
- suitable for tracking
- output y is an intrinsic physical variable

Full state linearization:
- set point stabilization
- output is not intrinsic, selected to enable a linearizing change of variables.

Remaining question:

- When is a system feedback linearizable, *i.e.*, how do we know whether a relative degree $r = n$ output exists?

Basic Definitions from Differential Geometry

Definition: The Lie bracket of two vector fields f and g is a new vector field defined as:

$$[f,g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x).$$

Note:

1. $[f,g] = -[g,f]$,
2. $[f,f] = 0$,
3. If f, g are constant then $[f,g] = 0$.

Notation for repeated applications:

$$[f,[f,g]] = \mathrm{ad}_f^2 g, \quad [f,[f,[f,g]]] = \mathrm{ad}_f^3 g, \quad \cdots$$
$$\mathrm{ad}_f^0 g(x) \triangleq g(x), \quad \mathrm{ad}_f^k g \triangleq [f, \mathrm{ad}_f^{k-1} g] \quad k = 1,2,3,\ldots$$

Distributions

Definition: Given vector fields $f_1, \ldots, f_k$, a distribution Δ is defined as $\Delta(x) = \mathrm{span}\{f_1(x), \ldots, f_k(x)\}$.

$f \in \Delta$ means that there exist scalar functions $\alpha_i(x)$ such that

$$f(x) = \alpha_1(x) f_1(x) + \cdots + \alpha_k(x) f_k(x).$$

Definition: Δ is said to be nonsingular if $f_1(x), \ldots, f_k(x)$ are linearly independent for all x.

Definition: Δ is said to be involutive if

$$g_1 \in \Delta, g_2 \in \Delta \implies [g_1, g_2] \in \Delta$$

that is, Δ is closed under the Lie bracket operation.

Involutive Distributions

Proposition: $\Delta = \text{span}\{f_1, \ldots, f_k\}$ is involutive if and only if

$$[f_i, f_j] \in \Delta \quad 1 \leq i, j \leq k.$$

Example 1: $\Delta = \text{span}\{f_1, \ldots, f_k\}$ where $f_1, \ldots, f_k$ are constant vectors

Example 2: a single vector field $f(x)$ is involutive since $[f, f] = 0 \in \Delta$

Completely Integrable

Definition: A nonsingular k-dimensional distribution

$$\Delta(x) = \text{span}\{f_1(x), \ldots, f_k(x)\} \quad x \in \mathbb{R}^n$$

is said to be completely integrable if there exist $n-k$ functions

$$\phi_1(x), \ldots, \phi_{n-k}(x)$$

such that

$$\frac{\partial \phi_i}{\partial x} f_j(x) = 0 \quad i = 1, \ldots, n-k, \quad j = 1, \ldots, k$$

and $d\phi_i(x) := \dfrac{\partial \phi_i}{\partial x}$, $\; i = 1, \ldots, n-k$, are linearly independent.

Example

Example 3: If $f_1, \ldots, f_k$ are linearly independent constant vectors, then we can find $n-k$ independent row vectors $T_1, \ldots, T_{n-k}$ s.t.

$$T_i[f_1 \ldots f_k] = 0.$$

Therefore, $\Delta = \mathrm{span}\{f_1, \ldots, f_k\}$ is completely integrable and

$$\phi_i(x) = T_i x, \quad i = 1, \ldots, n-k.$$

Frobenius Theorem

Frobenius Theorem: A nonsingular distribution is completely integrable if and only if it is involutive.

Example 3 above is a special case since Δ is involutive by Example 1.

Example 3: If $f_1, \ldots, f_k$ are linearly independent constant vectors, then we can find $n-k$ independent row vectors $T_1, \ldots, T_{n-k}$ s.t.

$$T_i[f_1 \ldots f_k] = 0.$$

Therefore, $\Delta = \text{span}\{f_1, \ldots, f_k\}$ is completely integrable and

$$\phi_i(x) = T_i x, \quad i = 1, \ldots, n-k.$$

Back to (Full State) Feedback Linearization

Recall: $\dot{x} = f(x) + g(x)u$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ is feedback linearizable if we can find an output $y = h(x)$ such that relative degree $r = n$.

How do we determine if a relative degree $r = n$ output exists?

$$L_g h(x) = L_g L_f h(x) = \cdots = L_g L_f^{n-2} h(x) = 0 \text{ in a nbhd of } x_0$$

$$L_g L_f^{n-1} h(x_0) \neq 0.$$

Back to (Full State) Feedback Linearization

Proposition: (2)-(3) are equivalent to:

$$L_g h(x) = L_{\mathrm{ad}_f g} h(x) = \cdots = L_{\mathrm{ad}_f^{n-2} g} h(x) = 0 \text{ in a nbhd of } x_0 \quad (1)$$

$$L_{\mathrm{ad}_f^{n-1} g} h(x_0) \neq 0.$$

The advantage of (1) over (2) is that it has the form:

$$\frac{\partial h}{\partial x}\left[g(x) \quad \mathrm{ad}_f\, g(x) \quad \ldots \quad \mathrm{ad}_f^{n-2} g(x)\right] = 0$$

which is amenable to the Frobenius Theorem.

$$L_g h(x) = L_g L_f h(x) =$$
$$\cdots = L_g L_f^{n-2} h(x) = 0$$
$$\text{in a nbhd of } x_0 \quad (2)$$
$$L_g L_f^{n-1} h(x_0) \neq 0. \quad (3)$$

- Proposition follows from equation on future slide with $j = 0$

Necessary and Sufficient Conditions for Feedback Linearization

Theorem: $\dot{x} = f(x) + g(x)u$ is feedback linearizable around x_0 if and only if the following two conditions hold:

C1) $[g(x_0) \quad \mathrm{ad}_f\, g(x_0) \quad \dots \quad \mathrm{ad}_f^{n-1} g(x_0)]$ has rank n

C2) $\Delta(x) = \mathrm{span}\{g(x), \mathrm{ad}_f\, g(x), \dots, \mathrm{ad}_f^{n-2} g(x)\}$ is involutive in a neighborhood of x_0.

Necessary and Sufficient Conditions for Feedback Linearization (proof)

Proof: (if) Given C1 and C2 show that there exists $h(x)$ satisfying (4)-(5).

$\Delta(x)$ is nonsingular by C1 and involutive by C2. Thus, by the Frobenius Theorem, there exists $h(x)$ satisfying (4) and $dh(x) \neq 0$.

To prove (5) suppose, to the contrary, $L_{\mathrm{ad}_f^{n-1} g} h(x_0) = 0$. This implies

$$dh(x_0) \underbrace{[g(x_0) \quad \mathrm{ad}_f\, g(x_0) \quad \dots \quad \mathrm{ad}_f^{n-1} g(x_0)]}_{\text{nonsingular by (C1)}} = 0.$$

Thus $dh(x_0) = 0$, a contradiction.

Theorem: $\dot{x} = f(x) + g(x)u$ is feedback linearizable around x_0 if and only if the following two conditions hold:

C1) $[g(x_0) \quad \mathrm{ad}_f\, g(x_0) \quad \dots \quad \mathrm{ad}_f^{n-1} g(x_0)]$ has rank n

C2) $\Delta(x) = \mathrm{span}\{g(x), \mathrm{ad}_f\, g(x), \dots, \mathrm{ad}_f^{n-2} g(x)\}$ is involutive in a neighborhood of x_0.

- Alternative equations for feedback linearization from proposition:

$$L_g h(x) = L_{\mathrm{ad}_f g} h(x) = \cdots = L_{\mathrm{ad}_f^{n-2} g} h(x) = 0 \quad \text{in a nbhd of } x_0 \tag{4}$$

$$L_{\mathrm{ad}_f^{n-1} g} h(x_0) \neq 0. \tag{5}$$

Necessary and Sufficient Conditions for Feedback Linearization (proof)

(only if) Given that $y = h(x)$ with $r = n$ exists, that is (7)-(8) hold, show that C1 and C2 are true.

We will use the following fact which holds when $r = n$:

$$L_{\mathrm{ad}_f^i g} L_f^j h(x) = \begin{cases} 0 & \text{if } i + j \leq n-2 \\ (-1)^{n-1-j} L_g L_f^{n-1} h(x) \neq 0 & \text{if } i + j = n-1. \end{cases}$$

Define the matrix

$$M = \begin{bmatrix} dh \\ dL_f h \\ \vdots \\ dL_f^{n-1} h \end{bmatrix} \begin{bmatrix} g & -\mathrm{ad}_f g & \mathrm{ad}_f^2 g & \dots & (-1)^{n-1} \mathrm{ad}_f^{n-1} g \end{bmatrix} \tag{6}$$

and note that the (k, ℓ) entry is:

$$\begin{aligned} M_{k\ell} &= dL_f^{k-1} h(x) (-1)^{\ell-1} \mathrm{ad}_f^{\ell-1} g(x) \\ &= (-1)^{\ell-1} L_{\mathrm{ad}_f^{\ell-1} g} L_f^{k-1} h(x). \end{aligned}$$

Theorem: $\dot{x} = f(x) + g(x)u$ is feedback linearizable around x_0 if and only if the following two conditions hold:

C1) $[g(x_0) \;\; \mathrm{ad}_f g(x_0) \;\; \dots \;\; \mathrm{ad}_f^{n-1} g(x_0)]$ has rank n

C2) $\Delta(x) = \mathrm{span}\{g(x), \mathrm{ad}_f g(x), \dots, \mathrm{ad}_f^{n-2} g(x)\}$ is involutive in a neighborhood of x_0.

$$L_g h(x) = L_{\mathrm{ad}_f g} h(x) = \cdots = L_{\mathrm{ad}_f^{n-2} g} h(x) = 0$$

in a nbhd of x_0 (7)

$$L_{\mathrm{ad}_f^{n-1} g} h(x_0) \neq 0. \tag{8}$$

- For the fact, see, *e.g.*, Khalil, Lemma C.8

Necessary and Sufficient Conditions for Feedback Linearization (proof)

(only if cont.)

Then, from (9):

$$M_{k\ell} = \begin{cases} 0 & \ell + k \leq n \\ \neq 0 & \ell + k = n+1. \end{cases}$$

Since the diagonal entries are nonzero, M has rank n and thus the factor

$$\left[\begin{array}{cccccc} g & -\mathrm{ad}_f\, g & \mathrm{ad}_f^2\, g & \ldots & (-1)^{n-1}\,\mathrm{ad}_f^{n-1} g \end{array}\right]$$

in (6) must have rank n as well. Thus (C1) follows.

This also implies $\Delta(x)$ is nonsingular; thus, by the Frobenius Thm,

complete integrability $\equiv$ involutivity.

$\Delta(x)$ is completely integrable since $h(x)$ satisfying (7) exists by assumption; thus, we conclude involutivity (C2). □

Theorem: $\dot{x} = f(x) + g(x)u$ is feedback linearizable around x_0 if and only if the following two conditions hold:

C1) $[g(x_0) \;\; \mathrm{ad}_f\, g(x_0) \;\; \ldots \;\; \mathrm{ad}_f^{n-1} g(x_0)]$ has rank n

C2) $\Delta(x) = \mathrm{span}\{g(x), \mathrm{ad}_f\, g(x), \ldots, \mathrm{ad}_f^{n-2} g(x)\}$ is involutive in a neighborhood of x_0.

$$L_{\mathrm{ad}_f^i g} L_f^j h(x) = \begin{cases} 0 & \text{if } i+j \leq n-2 \\ (-1)^{n-1-j} L_g L_f^{n-1} h(x) \neq 0 & \\ & \text{if } i+j = n-1. \end{cases} \tag{9}$$

- Form of M:

$$\begin{bmatrix} 0 & 0 & \cdots & \star \\ 0 & & / & \vdots \\ \vdots & \star & & \vdots \\ \star & \cdots & \cdots & \star \end{bmatrix}$$

Example

$$\begin{aligned}\dot{x}_1 &= x_2 + 2x_1^2 \\ \dot{x}_2 &= x_3 + u \\ \dot{x}_3 &= x_1 - x_3\end{aligned}$$

Feedback linearizability was shown earlier by inspection: $y = x_3$ gives relative degree $= 3$. Verify with the theorem above:

$$f(x) = \qquad\qquad g(x) =$$

$$[f,g](x) = \qquad\qquad [f,[f,g]](x) =$$

Example

$$\begin{aligned}\dot{x}_1 &= x_2 + 2x_1^2 \\ \dot{x}_2 &= x_3 + u \\ \dot{x}_3 &= x_1 - x_3\end{aligned}$$

Feedback linearizability was shown earlier by inspection: $y = x_3$ gives relative degree $= 3$. Verify with the theorem above:

$$f(x) = \begin{bmatrix} x_2 + 2x_1^2 \\ x_3 \\ x_1 - x_3 \end{bmatrix} \quad g(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$[f,g](x) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad [f,[f,g]](x) = \begin{bmatrix} 4x_1 \\ 0 \\ 1 \end{bmatrix}$$

Example (cont.)

Conditions of the theorem:

Theorem: $\dot{x} = f(x) + g(x)u$ is feedback linearizable around x_0 if and only if the following two conditions hold:

C1) $[g(x_0) \quad \mathrm{ad}_f\, g(x_0) \quad \ldots \quad \mathrm{ad}_f^{n-1} g(x_0)]$ has rank n

C2) $\Delta(x) = \mathrm{span}\{g(x), \mathrm{ad}_f\, g(x), \ldots, \mathrm{ad}_f^{n-2} g(x)\}$ is involutive in a neighborhood of x_0.

- Example:

$$\begin{aligned}\dot{x}_1 &= x_2 + 2x_1^2 \\ \dot{x}_2 &= x_3 + u \\ \dot{x}_3 &= x_1 - x_3\end{aligned}$$

$$f(x) = \begin{bmatrix} x_2 + 2x_1^2 \\ x_3 \\ x_1 - x_3 \end{bmatrix} \quad g(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$[f,g](x) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad [f,[f,g]](x) = \begin{bmatrix} 4x_1 \\ 0 \\ 1 \end{bmatrix}$$

Example (cont.)

Conditions of the theorem:

1. $$\begin{bmatrix} 0 & -1 & 4x_1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ full rank

2. $$\Delta = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$ involutive

$$\frac{\partial h}{\partial x} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$ satisfied by $h(x) = x_3$.

Theorem: $\dot{x} = f(x) + g(x)u$ is feedback linearizable around x_0 if and only if the following two conditions hold:

C1) $[g(x_0) \;\; \mathrm{ad}_f\, g(x_0) \;\; \ldots \;\; \mathrm{ad}_f^{n-1} g(x_0)]$ has rank n

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- Example:

$$\begin{aligned} \dot{x}_1 &= x_2 + 2x_1^2 \\ \dot{x}_2 &= x_3 + u \\ \dot{x}_3 &= x_1 - x_3 \end{aligned}$$

$$f(x) = \begin{bmatrix} x_2 + 2x_1^2 \\ x_3 \\ x_1 - x_3 \end{bmatrix} \quad g(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$[f,g](x) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad [f,[f,g]](x) = \begin{bmatrix} 4x_1 \\ 0 \\ 1 \end{bmatrix}$$