ME 6402 – Lecture 27

FINAL EXAM REVIEW

April 17 2025

Overview:

- Backstepping
- Feedback Linearization
- Normal Form and Zero Dynamics
- Control Lyapunov functions
- Control Barrier functions

Additional Reading:

- Khalil Chapter 14.3 (Backstepping)
- Khalil Chapter 13 (Feedback Linearization of SISO Systems)
- Sastry Chapter 9.3 (Feedback Linearization of MIMO Systems)
- E. Sontag, 1983 (Control Lyapunov Functions)
- A. Ames et al. 2019 (Control Barrier Functions)

Backstepping (Lecture 13)

Backstepping is a specific control design technique for a certain class of systems. The basic idea of backstepping is that we can stabilize the system

$$\begin{aligned}\dot{x}_1 &= F(x_1) + G(x_1)x_2 \\ \dot{x}_2 &= u\end{aligned}$$

through the coordinate shift $z = x_2 - k(x_1)$, where $k(x_1)$ is a function that would result in stable dynamics for the x_1 subsystem. This effectively shifts the equilibrium point for our x_1 system and allows us to render it stable through the coordinate shift. Then, we can stabilize the remaining $\dot{x}_2$ dynamics by choosing u such that $\dot{z}$ is also a stable subsystem.

An example of backstepping is the following (Example 14.8 from Khalil):

Example 1: Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 &= u\end{aligned}$$

The $\dot{x}_1$ dynamics can be stabilized through the "control law"

$$x_2 = k(x_1) = -x_1^2 - x_1$$

Applying this control would yield the system:

$$\dot{x}_1 = -x_1^3 - x_1$$

which is stable. Notably, the $-x_1^3$ term provides an additional damping stabilization term compared to only having $\dot{x}_1 = -x_1$.

Next, we will shift our system to effectively *enforce* this control law when we drive the system to zero. This is done through the coordinate shift $z = x_2 - k(x_1)$, which results in the shifted system:

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + (z + k(x_1)) = -x_1^3 - x_1 + z \\ \dot{z} &= u + \dot{k}\end{aligned}$$

where $\dot{k} = (-2x_1 - 1)\dot{x}_1 = -(2x_1 + 1)(-x_1^3 - x_1 + z)$.

The main idea of backstepping is that we can then prove stability of the x_1 dynamics using the Lyapunov function $V(x_1) = \frac{1}{2}x_1^2$ and then construct an augmented Lyapunov function to construct a control law that would also stabilize the x_2 dynamics:

$$V_+ = V(\eta) + \frac{1}{2}z^2$$

This results in the control law:

$$u = \dot{k} - \frac{\partial V}{\partial x_1}G(x_1) - Kz$$

For our example, $\frac{\partial V}{\partial x_1} = x_1$ and $G(x_1) = 1$. Taking $K = 1$ for simplicity, we get the final control law:

$$\begin{aligned}u &= \dot{k} - \frac{\partial V}{\partial x_1}G(x_1) - z \\ &= -(2x_1 + 1)(-x_1^3 - x_1 + z) - x_1 - z\end{aligned}$$

If instead, we had had a system where $\dot{x}_2$ was control affine, the approach would be mostly the same. A specific example is as follows.

Example 2:

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 &= x_2 + 2u\end{aligned}$$

We can convert the form to our previous form using the same approach as with input-output linearization:

$$u = \frac{1}{2}(-x_2 + v)$$

This results in the system:

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 &= v\end{aligned}$$

Thus, following the same procedure as in Example 1, we would obtain:

$$v = -(2x_1 + 1)(-x_1^3 - x_1 + z) - x_1 - z$$

Plugging this back into our control law, we get:

$$u = \frac{1}{2}\left(-x_2 - (2x_1 + 1)(-x_1^3 - x_1 + z) - x_1 - z\right)$$

Note: While it's possible to recursively perform backstepping across multiple states, this typically results in very ugly and complex control laws and thus will likely not appear on the final exam.

Feedback Linearization (Lectures 16-19)

Relative Degree

Definition: *Relative Degree for SISO.* A SISO system has relative degree r if, in a neighborhood of the equilibrium:

$$\begin{aligned} &L_g L_f^{i-1} h(x) = 0, \quad i = 1, 2, \ldots, r-1 \\ &L_g L_f^{r-1} h(x) \neq 0 \end{aligned}$$

Informally, this is the same as saying that "A SISO system has relative degree r if the input does not appear until the r-th derivative of the output $h(x)$".

Definition: *Relative Degree for MIMO.* A MIMO system has relative degree r_i for each output $h_i(x)$ if the i-th output needs to be differentiated r_i times before *some* input appears.

Definition: *Vector Relative Degree for MIMO.* A MIMO system has vector relative degree $r = \{r_1, \ldots, r_m\}$ if the matrix $A(x)$ is nonsingular:

$$A(x) = \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \cdots & L_{g_1} L_f^{r_1-1} h_1(x) \\ \vdots & \ddots & \vdots \\ L_{g_m} L_f^{r_m-1} h_m(x) & \cdots & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix}$$

Example 3: Consider the system:

$$\begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= x_2 + u \\ y &= x_1 \end{aligned}$$

The system does not have a well-defined relative degree because $\dot{y} = \dot{x}_1 = x_1 = y$. Thus the input u will *never* appear.

Example 4: Consider the system:

$$\begin{aligned}\dot{x}_1 &= -x_1 + \frac{2+x_3^2}{1+x_3^2}u \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_1 x_3 + u \\ y &= x_2\end{aligned}$$

The system has relative degree 2 because:

$$\begin{aligned}\dot{y} &= \dot{x}_2 = x_3 \\ \ddot{y} &= \dot{x}_3 = x_1 x_3 + u\end{aligned}$$

Notably, the relative degree is well-defined for all $x \in \mathbb{R}^3$.

Example 5: Consider the system (it is the controlled van der Pol equation):

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \varepsilon(1 - x_1^2)x_2 + u \\ y &= x_2\end{aligned}$$

The system has relative degree 1 because $\dot{y} = \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u$. This is also well-defined for all $x \in \mathbb{R}^2$.

Example 6: Consider the MIMO system:

$$\begin{aligned}\dot{x}_1 &= \cos(x_3)u_1 \\ \dot{x}_2 &= \sin(x_3)u_1 \\ \dot{x}_3 &= u_2 \\ y_1 &= x_1 \\ y_2 &= x_2\end{aligned}$$

The system has relative degree $r_1 = r_2 = 1$ because

$$\begin{aligned}\dot{y}_1 &= \dot{x}_1 = \cos(x_3)u_1 \\ \dot{y}_2 &= \dot{x}_2 = \sin(x_3)u_1\end{aligned}$$

To check if the system has a valid vector relative degree, we need to check if the matrix $A(x)$ is nonsingular. Explicitly, this matrix is:

$$A := \begin{bmatrix}\cos(x_3) & 0 \\ \sin(x_3) & 0\end{bmatrix}$$

This matrix is NOT nonsingular, so it does not have a valid vector relative degree. This means that we could not perform feedback linearization on this system. Instead, we would need to perform dynamic extension:

$$\begin{aligned}
\dot{x}_1 &= x_4 \cos(x_3) \\
\dot{x}_2 &= x_4 \sin(x_3) \\
\dot{x}_3 &= u_2 \\
\dot{x}_4 &= u_1 \\
y_1 &= x_1 \\
y_2 &= x_2
\end{aligned}$$

This would result in each output having relative degree 2, with the derivatives being:

$$\begin{aligned}
\dot{y}_1 &= \dot{x}_1 = x_4 \cos(x_3) \\
\ddot{y}_1 &= u_1 \cos(x_3) - x_4 \sin(x_3) u_2
\end{aligned}$$

$$\begin{aligned}
\dot{y}_2 &= \dot{x}_2 = x_4 \sin(x_3) \\
\ddot{y}_2 &= u_1 \sin(x_3) + x_4 \cos(x_3) u_2
\end{aligned}$$

Thus, the A matrix is now:

$$A := \begin{bmatrix} \cos(x_3) & -x_4 \sin(x_3) \\ \sin(x_3) & x_4 \cos(x_3) \end{bmatrix}$$

This matrix is only singular when $x_4 = 0$, so for any state such that $x_4 \neq 0$, the system has a valid vector relative degree $r = \{2, 2\}$.

Input-Output Linearization

If a system has a well-defined realtive degree (or a valid vector relative degree for MIMO systems) then it is input-output linearizable. Explicitly, this feedback linearizing control law is:

$$u = \frac{1}{L_g L_f^{r-1} h(x)} \left(-L_f^r h(x) + v \right)$$

or

$$u = A^{-1}(-B + v)$$

You can always think of this as the latter if you rearrange the system to be in the form:

$$y^{(r)} = B + Au$$

By selecting the auxiliary control law

$$v = -k_1 y - k_2 \dot{y} - \cdots - k_r y^{(r-1)} \tag{1}$$

we can transform our input-output system to be:

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \\ \vdots \\ y^{(r)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -k_1 & -k_2 & -k_3 & \cdots & -k_r \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \ddots \\ y^{(r-1)} \end{bmatrix}$$

Full-State Feedback Linearization

If $r = n$, then there exists a diffeomorphism that transforms the system into the linear system

$$\dot{\eta} = A\eta$$

with the transformation being:

$$x \mapsto \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix}$$

We have a theorem to verify when a system is provably full-state feedback linearizable. This theorem also provides us with tools to know how to select the output $h(x)$ such that the system is full-state feedback linearizable.

Theorem: Full-State Feedback Linearizable. *The system $\dot{x} = f(x) + g(x)u$ is full-state feedback linearizable around x_0 if and only if the following two conditions hold:*

- *C1)* $\begin{bmatrix} g(x_0) & \mathrm{ad}_f\, g(x_0) & \ldots & \mathrm{ad}_f^{n-1}\, g(x_0) \end{bmatrix}$ *has rank n.*
- *C2) The distribution* $\Delta(x) = \mathrm{span}\{g(x), \mathrm{ad}_f\, g(x), \ldots, \mathrm{ad}_f^{n-2}\, g(x)\}$ *is involutive in a neighborhood of x_0.*

Importantly, by the Frobenius theorem, a nonsingular distribution is involutive if and only if it is completely integrable, which gives us the condition that there must exist a function $h(x)$ such that:

$$\frac{\partial h}{\partial x} f_j = 0$$

where f_j represents each element in the span of the associated distribution Δ.

Example 7: Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &\sin(x_1) + u\end{aligned}$$

First, to calculate the adjoint elements:

$$\begin{aligned}g(x) &= \begin{bmatrix}0\\1\end{bmatrix} \\ \text{ad}_f\, g(x) = [f,g] &= \frac{\partial g}{\partial x}f(x) - \frac{\partial f}{\partial x}g(x) = -\begin{bmatrix}0 & 1\\ \cos(x_1) & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}\end{aligned}$$

Thus, the matrix of condition 1 is:

$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

which is full rank.

Second, we need to find an output $h(x)$ such that

$$\frac{\partial h}{\partial x}g(x) = 0 \implies \frac{\partial h}{\partial x}\begin{bmatrix}0\\1\end{bmatrix} = 0$$

This condition is satisfied for $h(x) = x_1$.

We can double check this by computing the relative degree associated with $h(x) = x_1$:

$$\begin{aligned}\dot{y} &= \dot{x}_1 = x_2 \\ \ddot{y} &= \dot{x}_2 = \sin(x_1) + u\end{aligned}$$

Normal Form

If the system is *not* full-state feedback linearizable, the system will have *zero dynamics*. The zero dynamics are those that remain when the feedback linearizing control law is applied (with $v = 0$) and the outputs are consequently driven to zero.

Example 8: Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_3^2 + u \\ \dot{x}_3 &= -x_3 + x_1 \\ y &= x_1\end{aligned}$$

First, we analyze the relative degree of the system:

$$\begin{aligned}\dot{y} &= \dot{x}_1 = x_2 \\ \ddot{y} &= \dot{x}_2 = -x_1 + x_3^2 + u\end{aligned}$$

Thus, the system has relative degree $r = 2$. The associated outputs are $y = x_1$ and $\dot{y} = x_2$. The feedback control law is:

$$u = x_1 - x_3^2 + v$$

The zero dynamics can then be derived as:

$$\begin{aligned}\dot{x}_1 &= 0 \\ \dot{x}_2 = 0 + x_3^2 + (0 - x_3^2 + 0) &= 0 \\ \dot{x}_3 &= -x_3 + 0\end{aligned}$$

Thus, the zero dynamics are $\dot{x}_3 = -x_3$.

To derive the zero dynamic coordinate transformation, we must find the transformation z such that z is independent of the outputs, and $\dot{z}$ does not contain u. This is done by ensuring that $\nabla z \cdot g(x) = 0$.

Example 8 continued: The zero dynamic coordinates associated with our previous example can be derived by finding z to satisfy:

$$\frac{\partial z}{\partial x}\begin{bmatrix}0\\1\\0\end{bmatrix} = 0 \quad\implies\quad z = x_3$$

Thus, our transformation to normal form is:

$$T : x \mapsto \begin{bmatrix}\eta_1\\ \eta_2\\ z\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

Note: This example is trivial since the normal form is already decomposed as exactly our system state...

The full normal form dynamics are:

$$\begin{bmatrix}\dot{\eta}_1\\ \dot{\eta}_2\\ \dot{z}\end{bmatrix} = \begin{bmatrix}\eta_2\\ v\\ -z\end{bmatrix}$$

We can check whether this map is a diffeomorphism (with a smooth inverse) by if its Jacobian has full rank.

Note: You should check the Jacobian if a question asks you to "specify the region over which the transformation to Normal Form is valid"

Example 8 continued: The Jacobian of the transformation is:

$$DT = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since this is full rank, our transformation is a diffeomorphism for all $x \in \mathbb{R}^3$.

Control Lyapunov functions (Lecture 20)

Definition: *Control Lyapunov Function.* A positive definite function $V(x)$ is a (global) control Lyapunov function for the system $\dot{x} = f(x) + g(x)u$ if $\forall x \neq 0$, $\exists u$ such that:

$$\dot{V}(x) = \frac{\partial V}{\partial x}\left(f(x) + g(x)u\right) < 0$$

One approach is to use Sontag's formula which is a closed-form solution to our inequality condition:

$$u = \begin{cases} -\left(\left(\frac{\partial V}{\partial x} f\right) + \sqrt{\left(\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4 \right)} \right) \Big/ \left(\frac{\partial V}{\partial x} g\right) & \text{if } \frac{\partial V}{\partial x} g \neq 0 \\ 0 & \text{if } \frac{\partial V}{\partial x} g = 0 \end{cases}$$

The alternative appraoch is to use convex optimization to solve the problem:

$$\begin{aligned} u^* = \underset{\mu}{\text{minimize}} \quad & \|\mu\|^2 \\ \text{subject to} \quad & L_f V(x) + L_g V(x)\mu < 0 \end{aligned}$$

Control Barrier functions (Lectures 23-25)

The summary of control Lyapunov functions compared to control barrier functions is:

$$\underbrace{\dot{V} \leq -\alpha(V(x))}_{\text{Stability}} \quad \text{versus} \quad \underbrace{\dot{h} \geq -\alpha(h(x))}_{\text{Safety}}$$

Definition: *Barrier Function.*A function h with $\mathcal{C} = \{x \mid h(x) \geq 0\}$ is a barrier function for $\dot{x} = f(x)$ if there exists a locally Lipschitz function $\alpha : \mathbb{R} \to \mathbb{R}$ satisfying $\alpha(0) = 0$ such that

$$\dot{h}(x) \geq -\alpha(h(x)), \quad \text{for all } x \in \mathbb{R}^n$$

Definition: *Control Barrier Function.*A function h with $\mathcal{C} = \{x \mid h(x) \geq 0\}$ is a control barrier function for $\dot{x} = f(x) + g(x)u$ if there exists a locally Lipschitz function $\alpha : \mathbb{R} \to \mathbb{R}$ satisfying $\alpha(0) = 0$ such that

$$\sup_{u \in \mathbb{R}^m} \dot{h}(x) \geq -\alpha(h(x)), \quad \text{for all } x \in \mathbb{R}^n$$

As with control Lyapunov functions, we can use either a closed-form expression or convex optimization to find an input that satisfies our inequality condition. The closed-form expression is:

$$u = \begin{cases} 0 & \textit{if } L_f h + \alpha(h(x)) \geq 0 \\ \frac{-(L_f h + \alpha(h(x))) L_g h^T}{\|L_g h\|^2} & \textit{otherwise} \end{cases}$$

The convex optimization approach can take many forms, but we discussed two main ones. The minimum effort control barrier function is:

$$\begin{aligned} u^* = \underset{\mu}{\text{minimize}} \ & \|\mu\|^2 \\ \text{subject to} \quad & L_f h(x) + L_g h(x)\mu \geq -\alpha(h(x)) \end{aligned}$$

The minimally-invasive control barrier function is:

$$\begin{aligned} u^* = \underset{\mu}{\text{minimize}} \ & \|\mu - k(x)\|^2 \\ \text{subject to} \quad & L_f h(x) + L_g h(x)\mu \geq -\alpha(h(x)) \end{aligned}$$

Lastly, if $L_g h(x) \equiv 0$, we will need to instead use a higher-order barrier function.

Example 9: Consider the system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -x_1^2 + u \end{aligned}$$

Synthesize a control barrier function to keep the state x_1 below a threshold of 2.

This desired behavior can be encoded by the function $h(x) = 2 - x_1 \geq 0$. This is associated with the safe setlength

$$\mathcal{C} = \{x \in \mathbb{R}^3 \mid h(x) = 2 - x_1 \geq 0\}$$

Taking the derivative, we get:

$$\dot{h} = -\dot{x}_1 = -x_2 \qquad \Longrightarrow \qquad L_f h = -x_2, \ L_g h = 0$$

Thus, this is an invalid control barrier function because $L_g h \equiv 0$.

Instead, we will need to use a higher-order barrier function of the form:

$$\Psi(x) := \dot{h}(x) + \alpha(h(x))$$

which is associated with its own safe set

$$\mathcal{C}_1 = \{x \in \mathbb{R}^3 \mid \Psi(x) \geq 0\}$$

To check if this higher order barrier function is valid. While doing this, we will assume $\alpha(s) = \gamma_1 s$ for simplicity.

$$\begin{aligned} \dot{\Psi} &= \ddot{h}(x) + \alpha'(h(x))\dot{h} \\ &= -\dot{x}_2 + \gamma_1(-x_2) \\ &= -x_3 - \gamma_1 x_2 \end{aligned}$$

Since this is *still* not valid, we will need to take another higher-order derivative:

$$\Psi_2(x) = \dot{\Psi}(x) + \alpha_2(\Psi(x))$$

which is associated with the safe set

$$\mathcal{C}_2 = \{x \in \mathbb{R}^3 \mid \Psi_2(x) \geq 0\}$$

To check if this higher order barrier function is valid, we need to again check whether $L_g \Psi_2 \neq 0$. Again, we will assume $\alpha_2(s) = \gamma_2 s$ for simplicity.

$$\begin{aligned} \dot{\Psi}_2 &= \ddot{\Psi}(x) + \alpha_2'(\Psi(x))\dot{\Psi} \\ &= (-\dot{x}_3 - \gamma_1 \dot{x}_2) + \gamma_2(-x_3 - \gamma_1 x_2) \\ &= -(-x_1^2 + u) - \gamma_1 x_3 + \gamma_2(-x_3 - \gamma_1 x_2) \end{aligned}$$

Here, $L_g \Psi_2 = -1$ which means that the higher-order barrier function is valid everywhere. Finally, we will enforce this higher-order barrier function by finding u such that:

$$L_f \Psi_2(x) + L_g \Psi_2(x) u \geq -\alpha_2(\Psi_2(x))$$