

Lecture 2 – ME6402, Spring 2025

Essentially Nonlinear Phenomena

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January 9, 2025



Goals of Lecture 2

- ▶ List several phenomena unique to systems that are not linear
- ▶ Phase portraits in the plane

Additional Reading

- ▶ Khalil, Chapter 2

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Essentially Nonlinear Phenomena

① Finite Escape Time

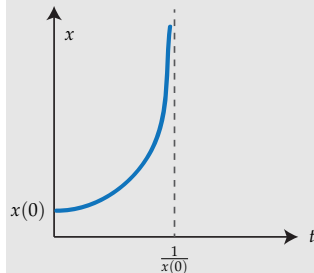
Example: $\dot{x} = x^2$

$$\frac{d}{dt}x^{-1} = -x^{-2}\dot{x} = -1$$

$$\Rightarrow \frac{1}{x(t)} - \frac{1}{x(0)} = -t$$

$$\Rightarrow x(t) = \frac{1}{\frac{1}{x(0)} - t}$$

For linear systems, $x(t) \rightarrow \infty$ cannot happen in finite time.



Essentially Nonlinear Phenomena

② Multiple Isolated Equilibria

Linear systems: either unique equilibrium or a continuum

Pendulum: two isolated equilibria (one stable, one unstable)

“Multi-stable” systems: two or more stable equilibria

Essentially Nonlinear Phenomena: Ex. of Multiple Isolated Equilibria

Example: bistable switch

$$\dot{x}_1 = -ax_1 + x_2 \quad x_1 : \text{concentration of protein}$$

$$\dot{x}_2 = \frac{x_1^2}{1+x_1^2} - bx_2 \quad x_2 : \text{concentration of mRNA}$$

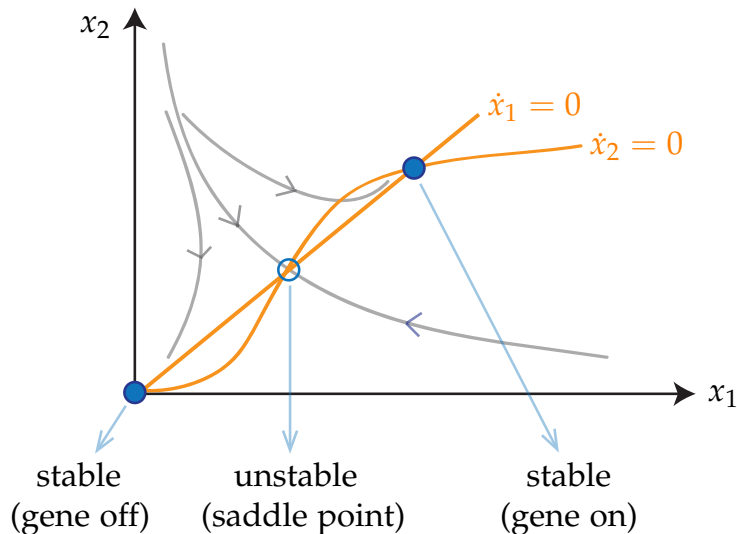
$a > 0$, $b > 0$ are constants. State space: $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$.

This model describes a positive feedback where the protein encoded by a gene stimulates more transcription via the term

$$\frac{x_1^2}{1+x_1^2}.$$

Single equilibrium at the origin when $ab > 0.5$. If $ab < 0.5$, the line where $\dot{x}_1 = 0$ intersects the sigmoidal curve where $\dot{x}_2 = 0$ at two other points, giving rise to a total of three equilibria.

Essentially Nonlinear Phenomena: Ex. of Multiple Isolated Equilibria (cont.)



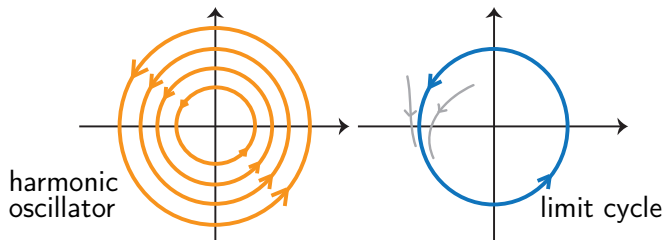
$$\begin{aligned}\dot{x}_1 &= -ax_1 + x_2 \\ \dot{x}_2 &= \frac{x_1^2}{1+x_1^2} - bx_2\end{aligned}$$

Essentially Nonlinear Phenomena

- ③ Limit cycles: Linear oscillators exhibit a continuum of periodic orbits; e.g., every circle is a periodic orbit for $\dot{x} = Ax$ where

$$A = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix} \quad (\lambda_{1,2} = \mp j\beta).$$

In contrast, a limit cycle is an isolated periodic orbit and can occur only in nonlinear systems.

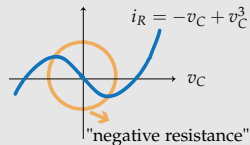
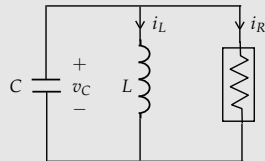
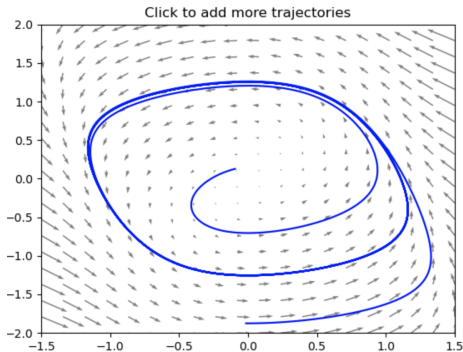


Essentially Nonlinear Phenomena: Example of Limit Cycle

Example: van der Pol oscillator

$$C\dot{v}_C = -i_L + v_C - v_C^3$$

$$L\dot{i}_L = v_C$$



Essentially Nonlinear Phenomena

- ④ Chaos: Irregular oscillations, never exactly repeating.

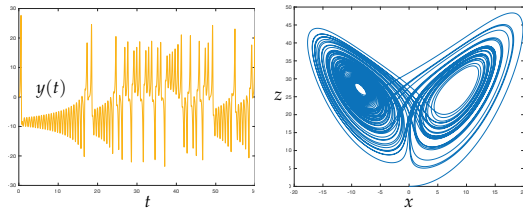
Example: Lorenz system (derived by Ed Lorenz in 1963 as a simplified model of convection rolls in the atmosphere):

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - y - xz$$

$$\dot{z} = xy - bz.$$

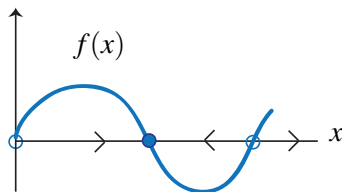
Chaotic behavior with $\sigma = 10$, $b = 8/3$, $r = 28$:



Essentially Nonlinear Phenomena: Chaos (cont.)

- For continuous-time, time-invariant systems, $n \geq 3$ state variables required for chaos.

$n = 1$: $x(t)$ monotone in t , no oscillations:



$n = 2$: Poincaré-Bendixson Theorem (to be studied in Lecture 4) guarantees regular behavior.

Essentially Nonlinear Phenomena: Chaos (cont.)

- ▶ Poincaré-Bendixson does not apply to time-varying systems and $n \geq 2$ is enough for chaos (for Van der Pol oscillator can exhibit chaos).
- ▶ For discrete-time systems, $n = 1$ is enough (we will see an example in Lecture 6).

Planar (Second Order) Dynamical Systems: Linear

Phase Portraits of Linear Systems: $\dot{x} = Ax$

- ▶ Distinct real eigenvalues

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

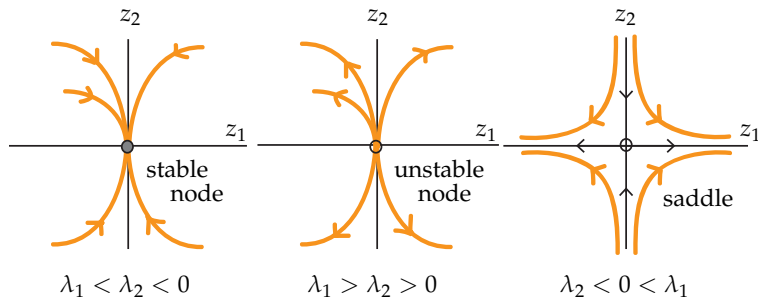
In $z = T^{-1}x$ coordinates:

$$\dot{z}_1 = \lambda_1 z_1, \quad \dot{z}_2 = \lambda_2 z_2.$$

- ▶ Chapter 2 in both Sastry and Khalil

Phase Portraits of Linear Systems, Real Eigenvalues (cont.)

The equilibrium is called a *node* when λ_1 and λ_2 have the same sign (*stable* node when negative and *unstable* when positive). It is called a *saddle point* when λ_1 and λ_2 have opposite signs.



Phase Portraits of Linear Systems, Complex Eigenvalues

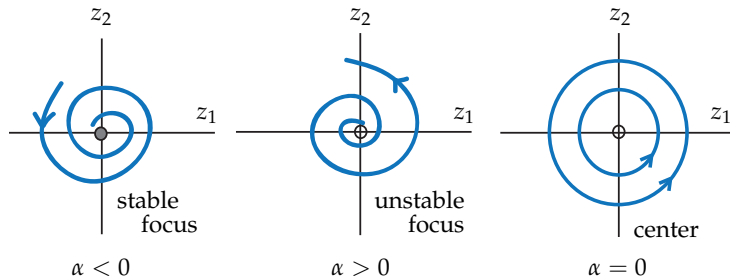
- Complex eigenvalues: $\lambda_{1,2} = \alpha \mp j\beta$

$$T^{-1}AT = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

$$\begin{aligned} \dot{z}_1 &= \alpha z_1 - \beta z_2 \\ \dot{z}_2 &= \alpha z_2 + \beta z_1 \end{aligned} \quad \begin{array}{c} \rightarrow \\ \rightarrow \end{array} \quad \begin{array}{c} \text{polar coordinates} \\ \text{polar coordinates} \end{array} \quad \begin{array}{c} \rightarrow \\ \rightarrow \end{array} \quad \begin{array}{c} \dot{r} = \alpha r \\ \dot{\theta} = \beta \end{array}$$

Phase Portraits of Linear Systems, Complex Eigenvalues (cont.)

Complex eigenvalues:



The phase portraits above assume $\beta > 0$ so that the direction of rotation is counter-clockwise: $\dot{\theta} = \beta > 0$.

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