Dimensional reduced modelling of the heart

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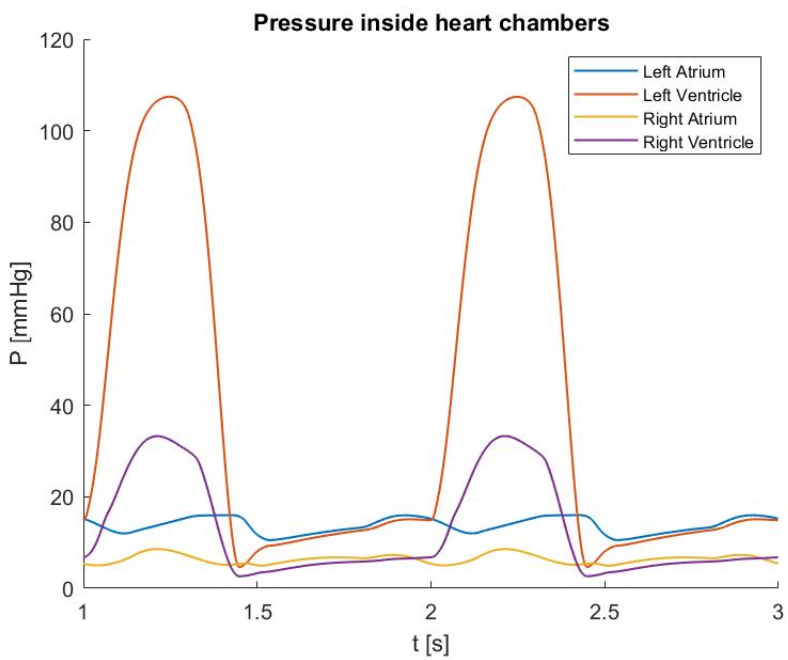
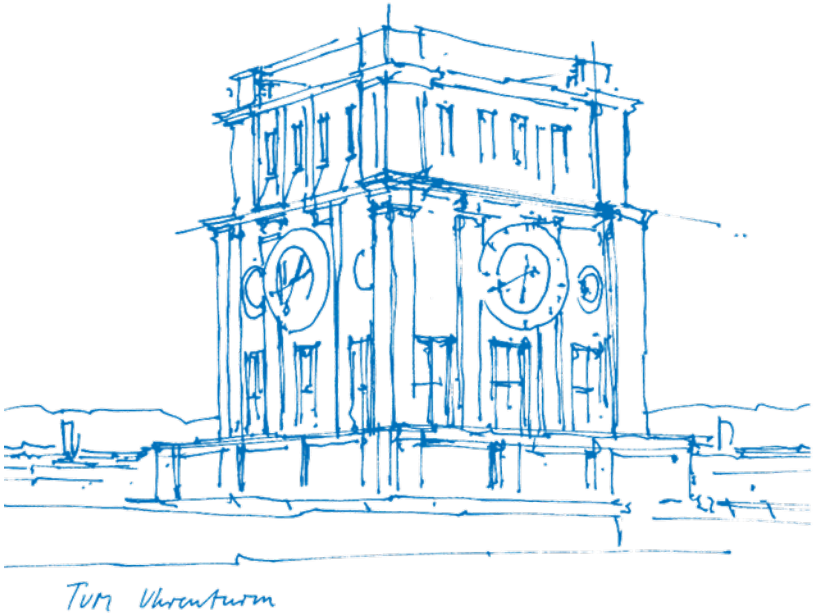




Medical background

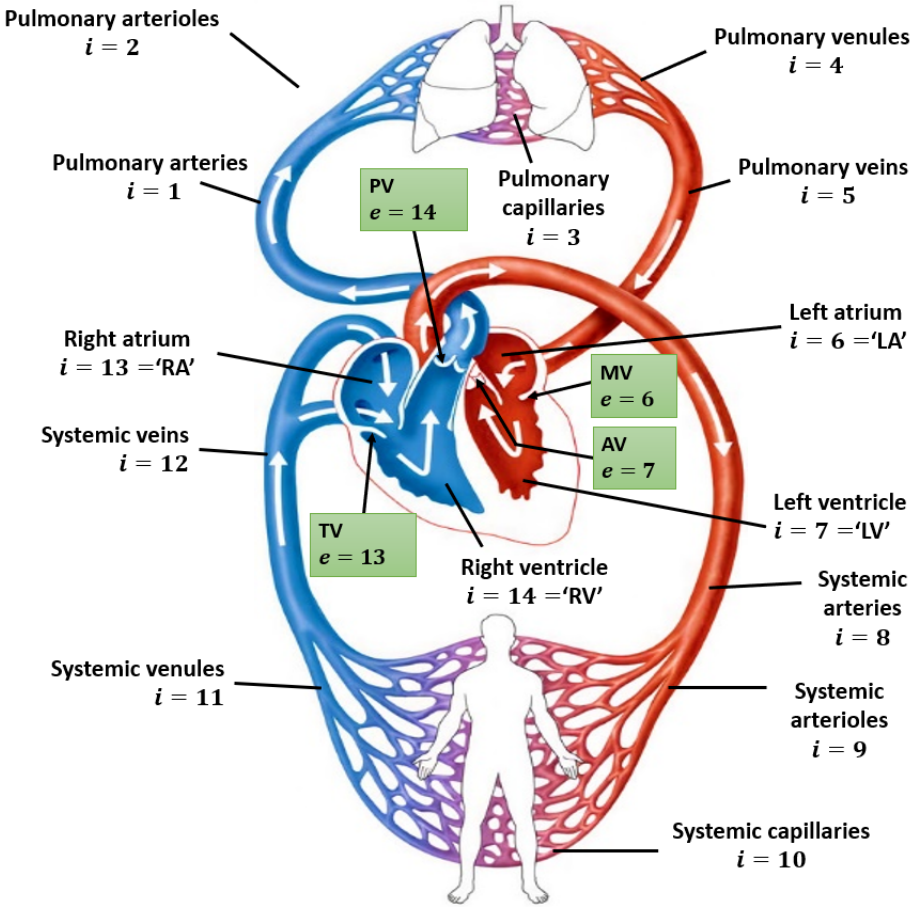


Figure: Schematic view of the cardiovascular system

Blood circulation compared to RLC circuit

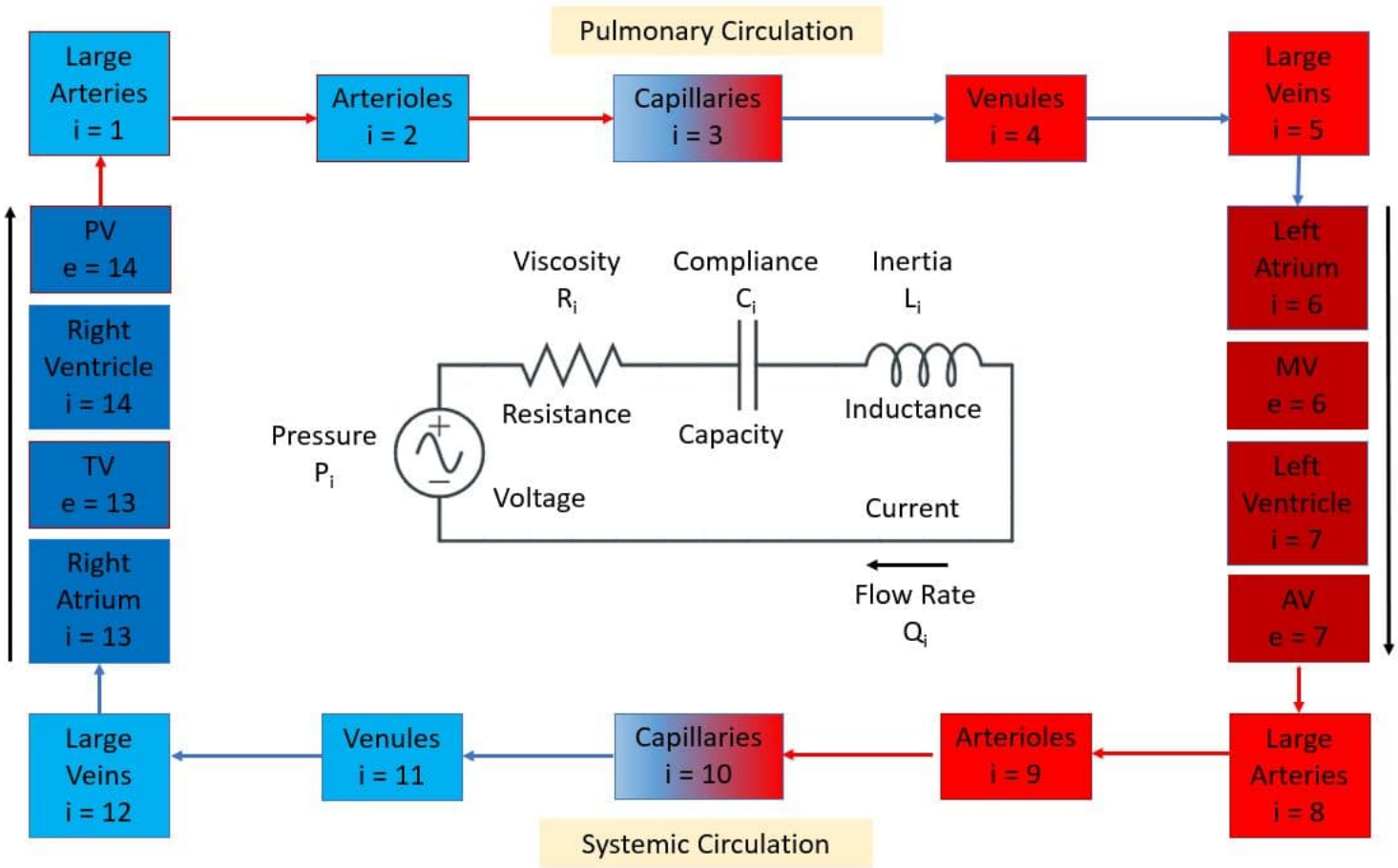


Figure: Compartment model compared to RLC circuit

Physical meaning

Circulation	Circuit	Meaning
Pressure	Voltage	Potential difference that pushes current through loop
Viscosity	Resistance	Resistance to flow
Compliance	Capacity	Ability to store charge
Inertia	Inductance	Tendency to oppose change in flow
Flow rate	Current	Stream of particles moving through the loop

Table: Physical meaning of variables

Elasticity $\neq$ compliance!

Example: Ohm's Law for Voltage

$$V = IR$$

Elasticity functions for heart cavities

Given T, the duration of a heart beat, and $t \equiv t(modT)$
Elasticity function of compartment i: $E_i(t) = E_{A,i} e_i(t(modT)) + E_{B_i}$, with the amplitude $E_{A,i}$ and the basis $E_{B,i}$.
Ventricles:

$$e_i(t) = \begin{cases} \frac{1}{2}\left[1 - cos(\frac{\pi t}{T_{cp,i}})\right], & \text{if } 0 \leq t \leq T_{cp,i} \\ \frac{1}{2}\left[1 + cos(\frac{\pi(t - T_{cp,i})}{T_{rp,i}})\right], & \text{if } T_{cp,i} < t \leq T_{cp,i} + T_{rp,i} \\ 0, & \text{if } T_{cp,i} + T_{rp,i} < t \leq T \end{cases}$$

Atria:

$$e_i(t) = \begin{cases} \frac{1}{2}\left[1 + cos(\frac{\pi(t + T - t_{r,i})}{T_{rp,i}})\right], & \text{if } 0 \leq t \leq t_{r,i} + T_{rp,i} - T \\ 0, & \text{if } t_{r,i} + T_{rp,i} - T < t \leq t_{c,i} \\ \frac{1}{2}\left[1 - cos(\frac{\pi(t - t_{c,i})}{T_{cp,i}})\right], & \text{if } t_{c,i} < t \leq t_{c,i} + T_{cp,i} \\ \frac{1}{2}\left[1 + cos(\frac{\pi(t - t_{r,i})}{T_{rp,i}})\right], & \text{if } t_{c,i} + T_{cp,i} < t \leq T \end{cases}$$

- $T_{cp,i}$ and $T_{rp,i}$ duration of contraction and relaxation for the ventricles.
- $t_{c,i}$ and $t_{r,i}$ is the start of contraction and relaxation.

A closer look to the heart

- Blood enters and leaves heart chambers through valves
- Contraction and relaxation of compartments: systole and diastole
- Pressure gradient: high to low
- Elasticity influences pumping power

Task: finding a suitable model

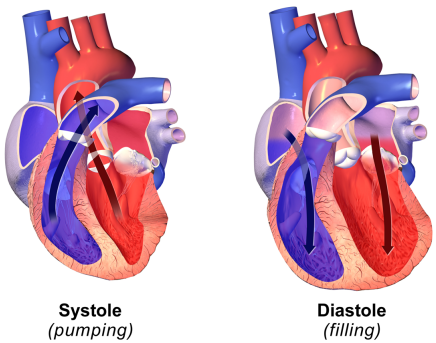


Figure: Systolic and diastolic phases

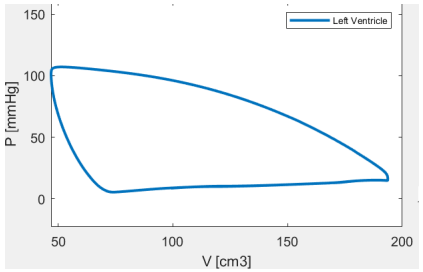


Figure: Pumping power depending on elasticity

Motivation

Why is it a sound comparison?

- Closed loop $\Longrightarrow$ Conservation
- Pressure follows same structure as Kirchhoff's law
- Valves act as diodes
- Use of 0D model instead of 3D: simple and effective

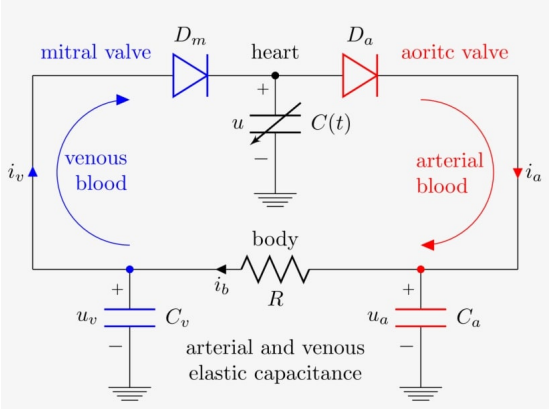


Figure: Comparison valves - diodes

Model variables

46 equations:

- 14 ODEs for the **volume** $V_i(t)$
- 14 ODEs for the **flow rate** $Q_e(t)$
- 4 ODEs for the **opening coefficient** $O_e(t)$ of the valve at edge e
- 14 equations for the **pressure** $P_i(t)$

Opening coefficient bounded, $O_e(t) \in [0, 1] \ \ \forall t$

Indexes:

- Compartments $i \in \{1, ..., 14\}$
- Edges $e \in \{1, ..., 14\}$.

Ordinary Differential Equations for the model

Volume, Flow rate and Opening coefficients follow:

$$\dot{V}_i(t) = Q_{in}(t) - Q_{out}(t) \qquad \text{for each compartment } i \text{ and its adjacent edges} \tag{1}$$

$$L_e\dot{Q}_e(t) = P_i(t) - P_{i+1}(t) - R_eQ_e(t) - B_eQ_e(t)|Q_e(t)| \qquad \forall e \text{ edge adjacent to the heart} \tag{2}$$

$$L_e\dot{Q}_e(t) = P_i(t) - P_{i+1}(t) - R_eQ_e(t) \qquad \forall e \text{ not connected to any heart chamber} \tag{3}$$

$$C_{d,e}\dot{O}_e(t) = 0.5\left[1 + \tanh\left(C_{v,e}(P_i(t) - P_{i+1}(t))\right)\right] - O_e(t) \qquad \text{for each heart valve in edge } e \tag{4}$$

Post-processing for the valves:

$$Q_e(t) = O_e(t)Q_e(t) \qquad \text{for each heart valve in edge } e \tag{5}$$

Where L_e is the inertia, R_e, the viscosity and B_e, the turbulence produced by the valve at edge e.

$C_{d,i}$, $C_{v,i}$ are the Dynamic Characteristics and Ejection Velocity for compartment i.

Algebraic Equations for Pressure

The Pressure follows an algebraic equation:

$$[1 - 0.0005(Q_{e-1}(t) - Q_e(t))]\, P_i(t) = E_i(t)(V_i(t) - V_{0,i}) \qquad \text{for each heart chamber } i \text{ and edges} \tag{6}$$

$$C_i P_i(t) = V_i(t) - V_{0,i} \qquad \text{for each compartment } i \text{ outside heart} \tag{7}$$

Where C_i is the compliance, $E_i(t)$ the elasticity function and $V_{0,i}$ the dead volume of compartment i.

Formulation of Runge-Kutta Method

At each iteration Runge Kutta requires a given solution at a point and a stepsize.

In our context, we will use the formulation:

- The numerical solution at time t is denoted by $\eta(t)$.
- The time-step is denoted by dt.
- The starting time point where we have the solution is t_{old}.
- The new time point is $t_{new} = t_{old} + dt$.

Explicit Runge-Kutta methods

- We choose explicit over implicit for computational efficiency.
- We need a balance between precision and computational cost: Runge-Kutta fourth order method.

$$
\begin{aligned}
RK_{step1} &= F(\eta(t_{old}));\\
RK_{step2} &= F(\eta(t_{old}) + \frac{dt}{2} \cdot RK_{step1});\\
RK_{step3} &= F(\eta(t_{old}) + \frac{dt}{2} \cdot RK_{step2});\\
RK_{step4} &= F(\eta(t_{old}) + dt \cdot RK_{step3});\\
\eta(t_{new}) &= \eta(t_{old}) + \frac{dt}{6} \cdot (RK_{step1} + 2 \cdot RK_{step2} + 2 \cdot RK_{step3} + RK_{step4});
\end{aligned}
$$

Where F is the R.H.S. of the ODE system.

Which issues do we need to address?

1. Smoothness. $L_e\dot{Q}_e(t) = P_i(t) - P_{i+1}(t) - R_eQ_e(t) - B_eQ_e(t)|Q_e(t)|$ is only piece-wise smooth.
2. We need to keep track of the local discretisation error.

Implementation of a Root-finding method

Overview of the implementation:

1. Compute solution at t_{new}
2. Detect change of sign at any of the heart flow rates between t_{old} and t_{new}.
3. Perform Andersson-Björck Method (Regula Falsi).
5. For the new time-step suggest observed average.

Which issues do we need to address?

a. Discard the cases where the old or new flow rate is too small.

b. Choosing a tolerance. It should be coherent with the tolerance for the local discretization error.

Andersson-Björck for the heart flow rates

- In this section $\tilde{\eta}(t)$ will denote the numerical solution of the heart flow rates which require root-finding.

- At k-th iteration, the root is between t_a and t_b. Andersson-Björck gives an estimate $\vec{t}_c$.

  We have two solutions $Q_a := \tilde{\eta}(t_a)$, $Q_b := \tilde{\eta}(t_b)$ with $t_a < t_b$.

- Initially $Q_a := \tilde{\eta}(t_{old})$ and $Q_b := \tilde{\eta}(t_{new}) = Q_c$.

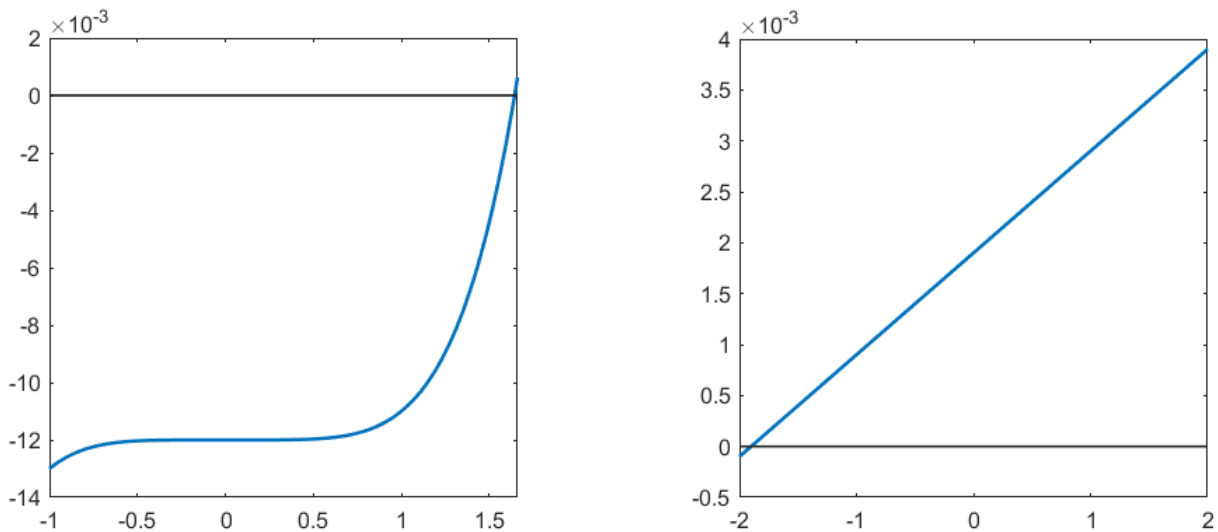


Figure: Example of cases where there is a change of sign but we do not apply Andersson-Björck.

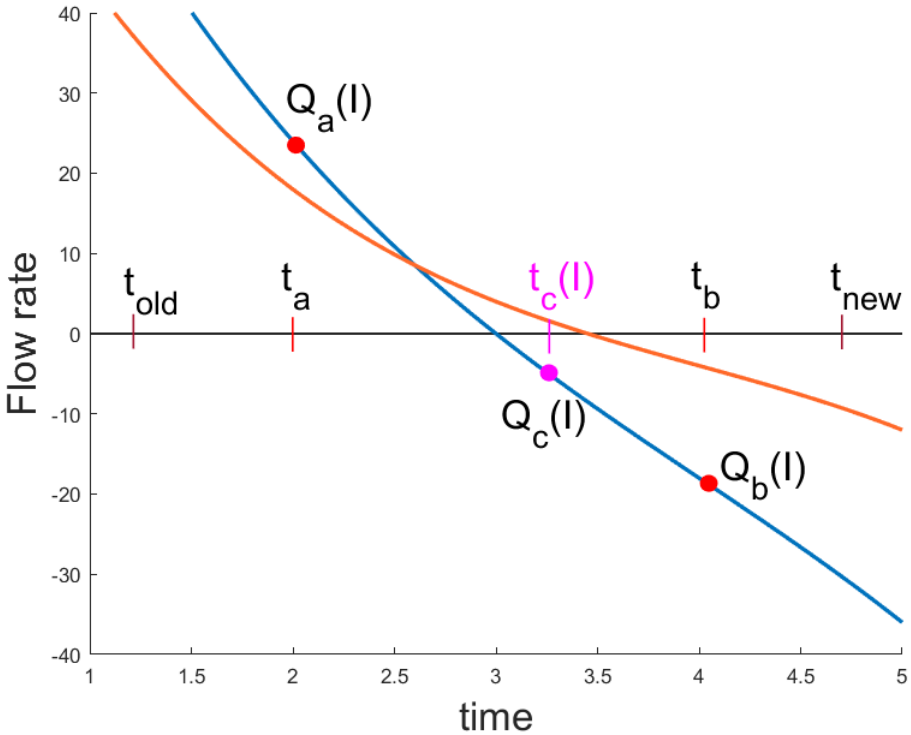
Steps of Andersson-Björck

1. Define $\vec{m}_i = \begin{cases} 1 - \dfrac{\vec{Q}_{c,i}}{\vec{Q}_{b,i}} & \text{, if } \vec{Q}_{c,i} < \vec{Q}_{b,i} \\ 0.5 & \text{, otherwise} \end{cases}$

2. Compute $\vec{t_c} = \dfrac{\vec{Q}_b t_a - \vec{m}\vec{Q}_a t_b}{\vec{Q}_b - \vec{m}\vec{Q}_a}$

3. Find $dt_c := \min(\vec{t_c} - t_{old})$,
   $I_c := argmin(\vec{t_c} - t_{old})$

4. Compute $\vec{Q}_c$ with RK from t_{old} and dt_c.



5.a If $|\vec{Q}_c(I)| < tol$, save solution and finish.

5.b Else if $sign(\vec{Q}_c(I)) = sign(\vec{Q}_a(I))$, repeat steps 1.-5. with $(\vec{Q}_a, dt_a) = (\vec{Q}_c, dt_c)$.

5.c Else, repeat steps 1.-5. with $(\vec{Q}_b, dt_b) = (\vec{Q}_c, dt_c)$.

Approximation functions

- Use approximation to absolute value $h_k(x) = \sqrt{x^2 + 10^{-k}} \in C^\infty(\mathbb{R}) \quad \forall k$
- Pointwise error $e(x,k) = -|x| + 10^{-k/2} + \dfrac{x^2}{20^{-k/2}} - \dfrac{x^4}{80^{-3k/2}} + O(x^6) \quad \forall k$
- Error L_2-norm decreases exponentially.

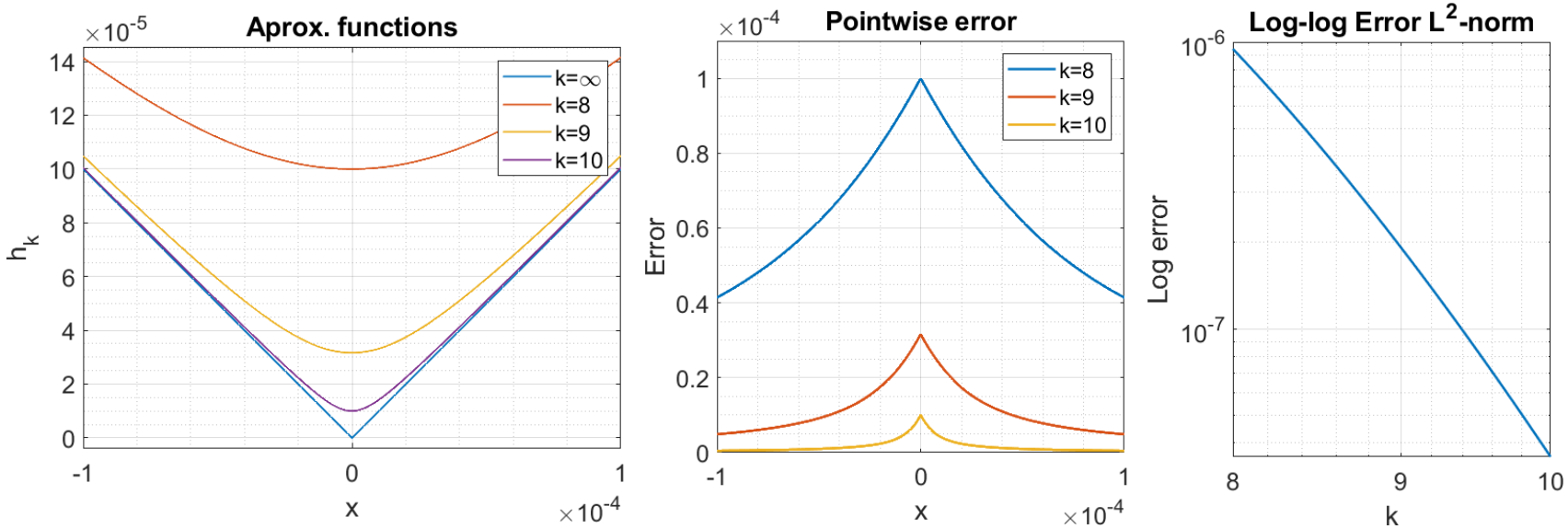


Figure: Left: $h_k(x)$. Center: Pointwise error. Right: Log-log plot of L_2-norm error, $x \in [-10^{-4}, 10^{-4}]$.

Two methods to address smoothness

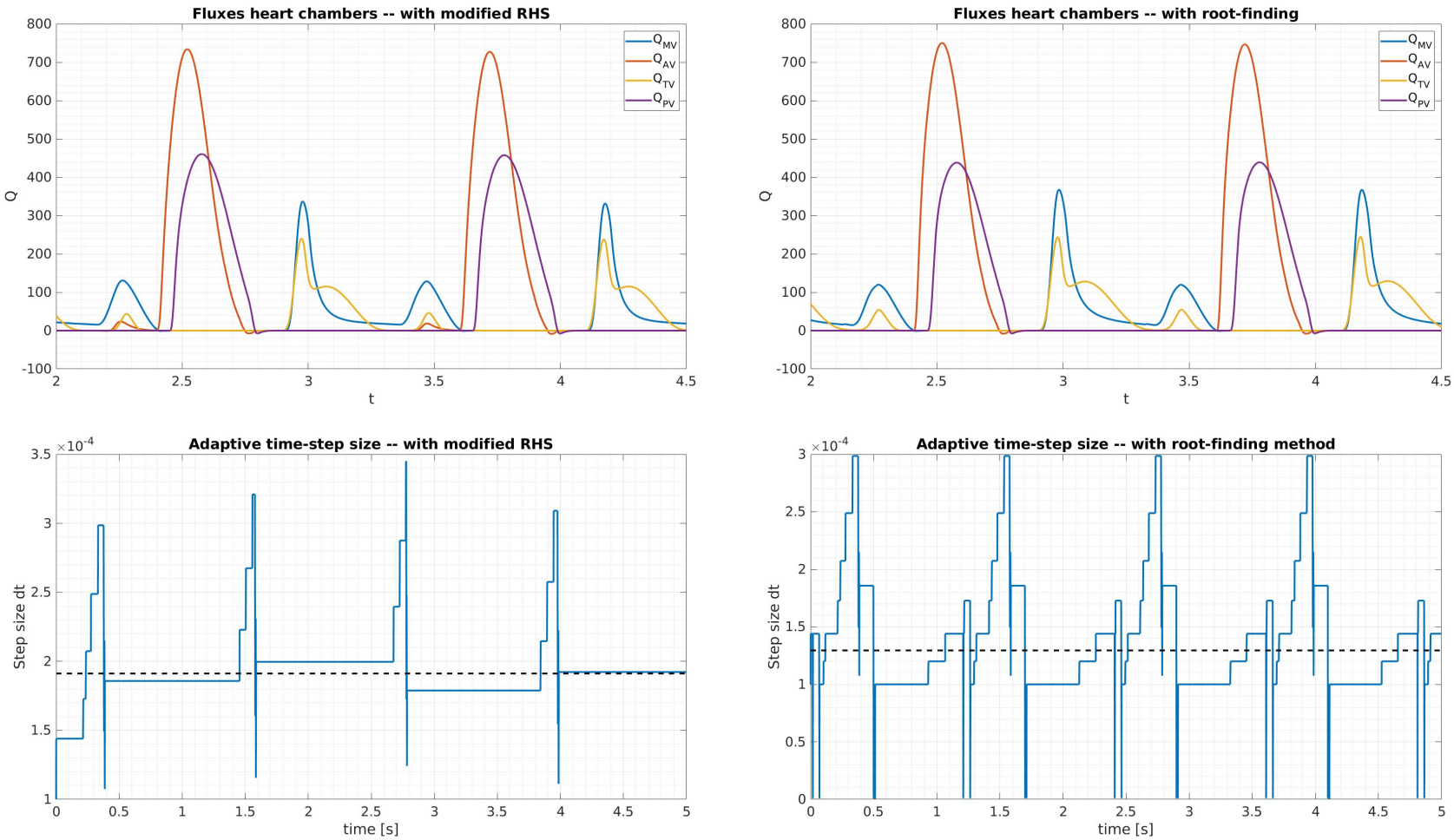


Figure: On the left we run the simulation modifying the right hand size of the ODE, while on the right we run the simulation with the root-finding method.

Adaptive Time-step

- Two methods of order p, $p+1$ and a single step size h.
- $\eta(t_0 + h_{new})$ is the solution from the method of order p
- $\hat{\eta}(t_0 + h_{new})$ is the solution from the method of order $p+1$
- The goal is to bound the local discretization error $\|\eta(t_0 + h_{new}) - \hat{\eta}(t_0 + h_{new})\| \leq tol$
- Two implementations.

Steps of first version of adaptive Time-step

We introduce two control parameters:

f_{max}: Maximum increase factor.

f_{min}: Minimum increase factor.

Initially $f_{max} = 1.5$, $f_{min} = 0.9$.

1. Compute $\eta(t_0 + h_{new})$ and $\hat{\eta}(t_0 + h_{new})$ with step size h_{old}.

2. The estimated error is $err = \|\eta(t_0 + h_{new}) - \hat{\eta}(t_0 + h_{new})\|$

3. Compute $h_{new} = h_{old} \min\left(f_{max}, \max\left(f_{min}, 0.9\left(\frac{tol}{err}\right)^{\frac{1}{p+1}}\right)\right)$

4.a. If $err > tol$, assign $h_{old} = h_{new}$. Repeat steps 1.- 4.

4.b. If $err \leq tol$, finish the loop.

Adaptive Time-step: second version

Implementing our code we have found that another approach to time-step evaluation worked better for us.
We estimate the error as previously explained.

- If the error estimate is above the error tolerance, we half the step-size and repeat the step.
- If the error estimate is close, but below, the error tolerance, we complete our step and we keep the same step size for the next step.
- If the error estimate is sufficiently below the error tolerance, we complete our step. The next step will be increased by 20%.

Speed comparisons

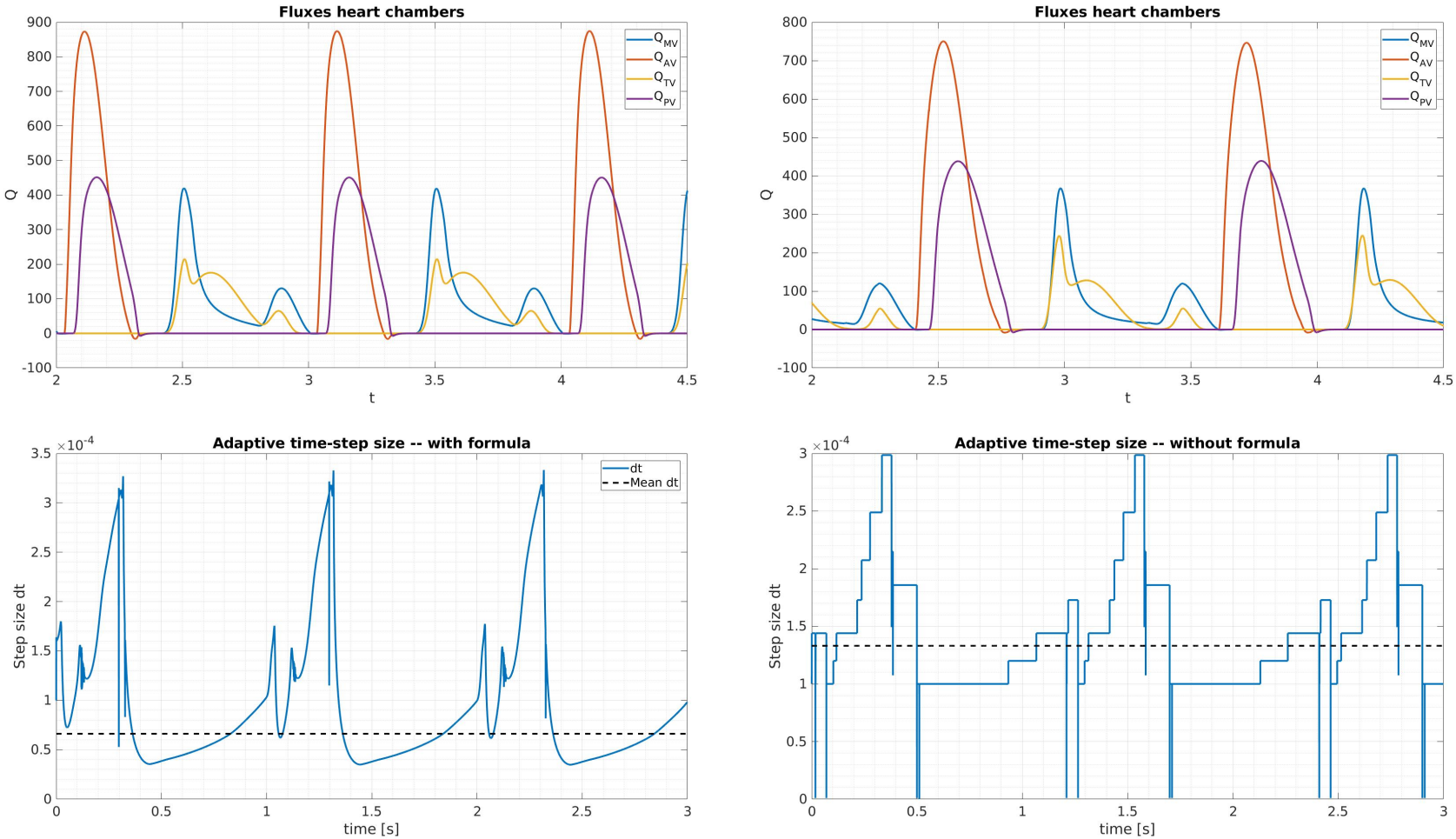


Figure: Different implementations of adaptive time step

Runge-Kutta Fehlberg method

Idea: Instead of calculating independently to approximate solutions of different orders, we extrapolate both approximate solutions using the same intermediate steps.

$$
\begin{array}{c|cccccc}
0 & & & & & & \\
\frac{1}{4} & \frac{1}{4} & & & & & \\
\frac{3}{8} & \frac{3}{32} & \frac{9}{32} & & & & \\
\frac{12}{32} & \frac{1932}{2197} & \frac{-7200}{2197} & \frac{7296}{2197} & & & \\
1 & \frac{439}{216} & -8 & \frac{3680}{513} & \frac{-845}{4104} & & \\
\frac{1}{2} & \frac{-8}{27} & 2 & \frac{3544}{2565} & \frac{1859}{4104} & \frac{-11}{40} & \\
\hline
 & \frac{16}{135} & 0 & \frac{6656}{12825} & \frac{28561}{56430} & \frac{-9}{50} & \frac{2}{55} \\
 & \frac{25}{135} & 0 & \frac{1408}{2565} & \frac{2197}{4104} & \frac{-1}{5} & 0
\end{array}
$$

We extrapolate both the fourth order approximate solution and the fifth order approximate solution using the same intermediate Runge Kutta steps.

Other Adaptive Runge-Kutta Methods

1. Bogacki–Shampine method (ode23 in Matlab): order 3(2).
2. Cash–Karp method.
3. Dormand–Prince (ode45 in Matlab) method.

Running our simulation using the:

- Anderson-Björck root finding-algorithm
- A local discretisation error tolerance of 10^{-4}

We find that the Fehlberg method and the Cash-Karp method require about 7000 grid points to perform one heart cycle, while the Dormand-Prince and the Bogacki-Shampine method are less efficient in our situation, requiring about 10000 grid points.

Comparison of 4 methods

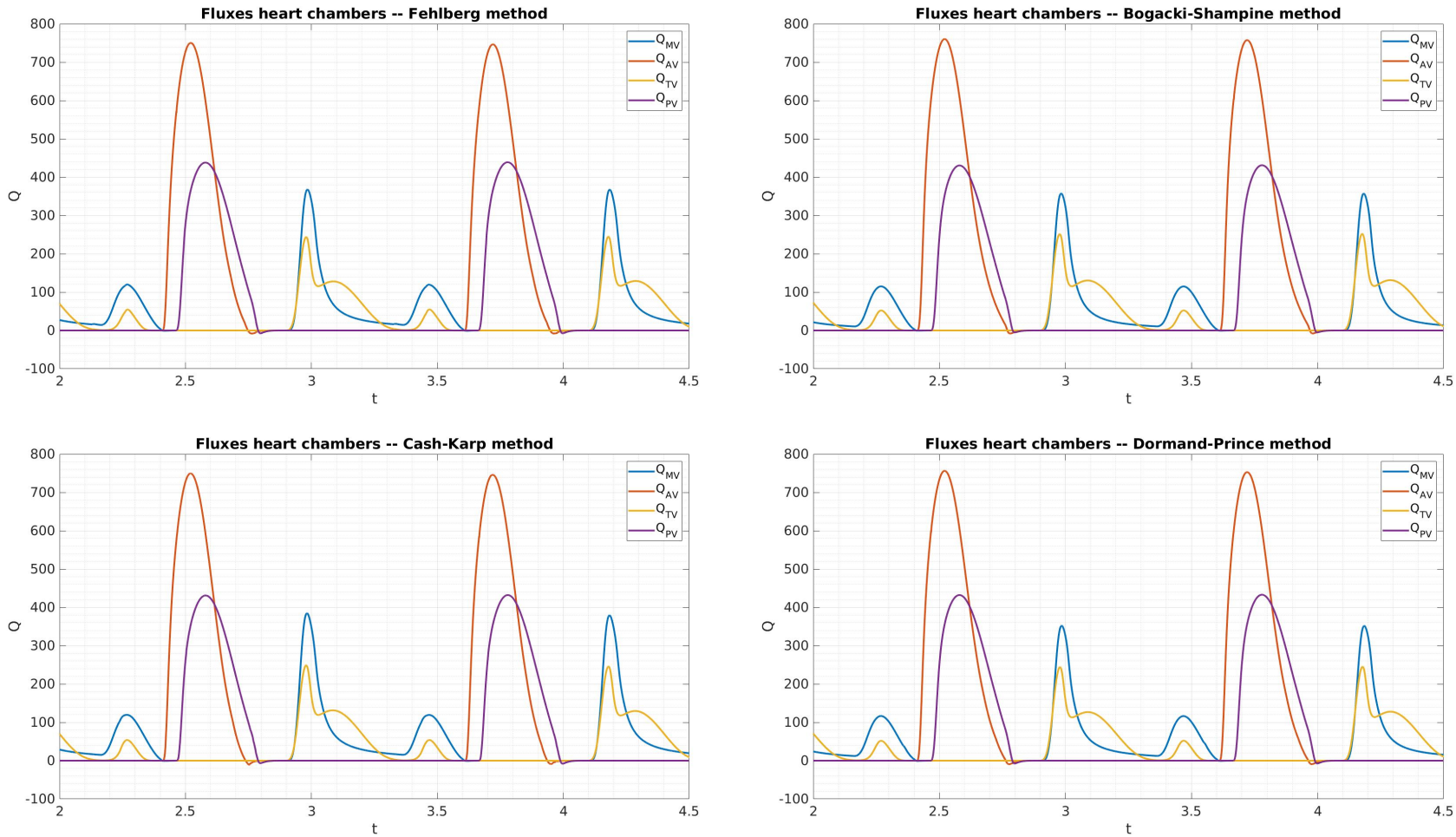


Figure: The solutions for all 4 methods is comparable.

Time-step size comparison

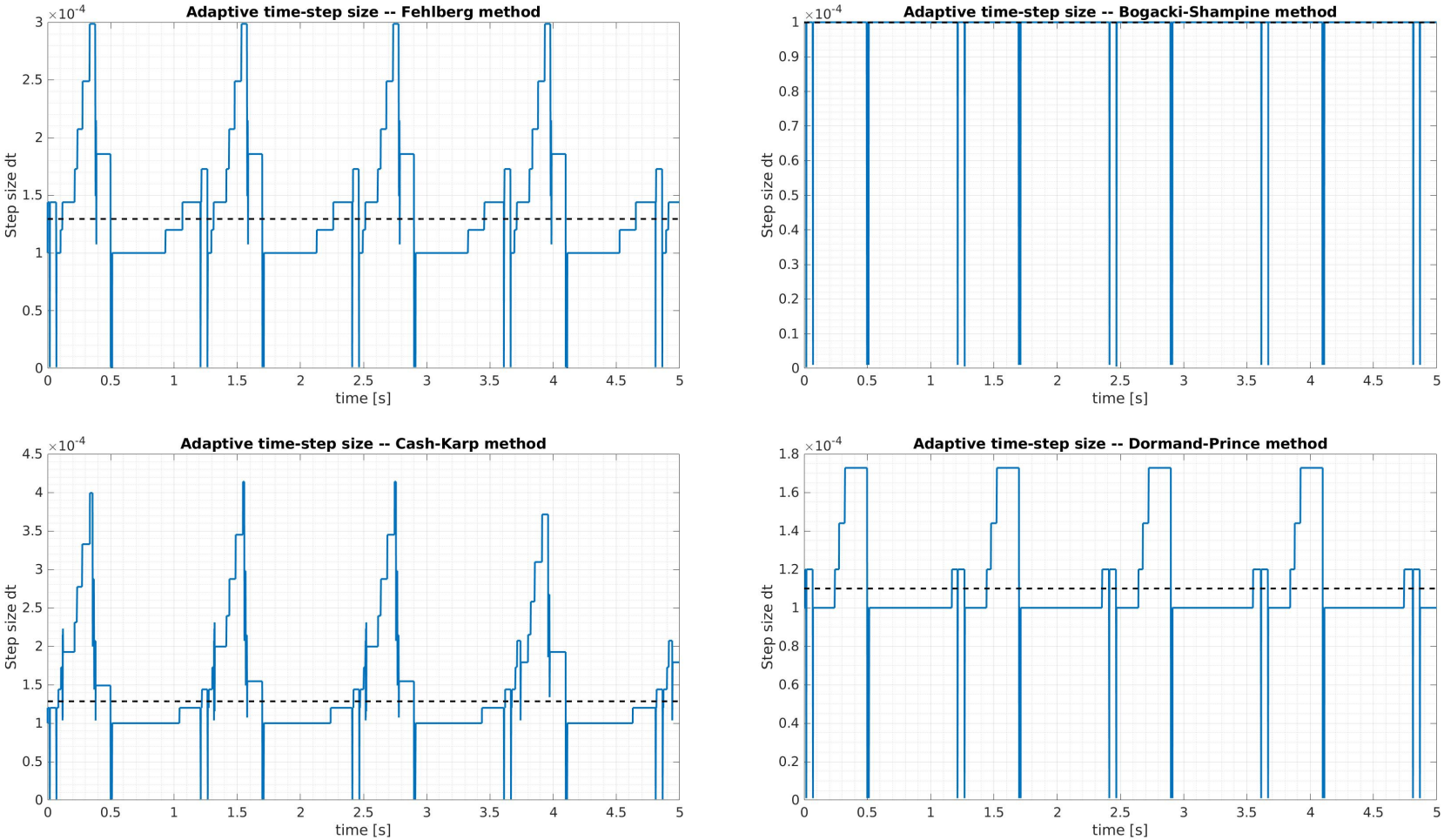
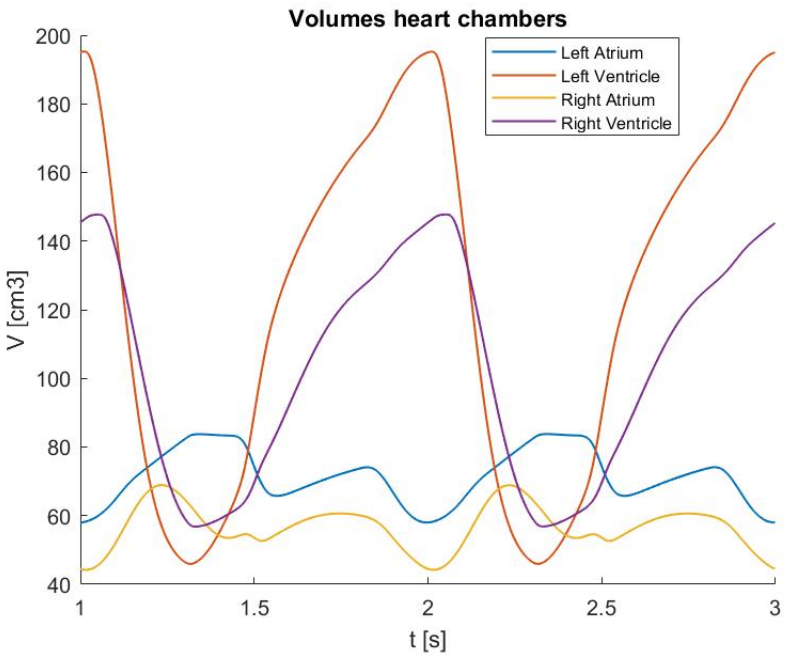


Figure: The step sizes for the 4 methods is not comparable.

Volumes in the heart chambers



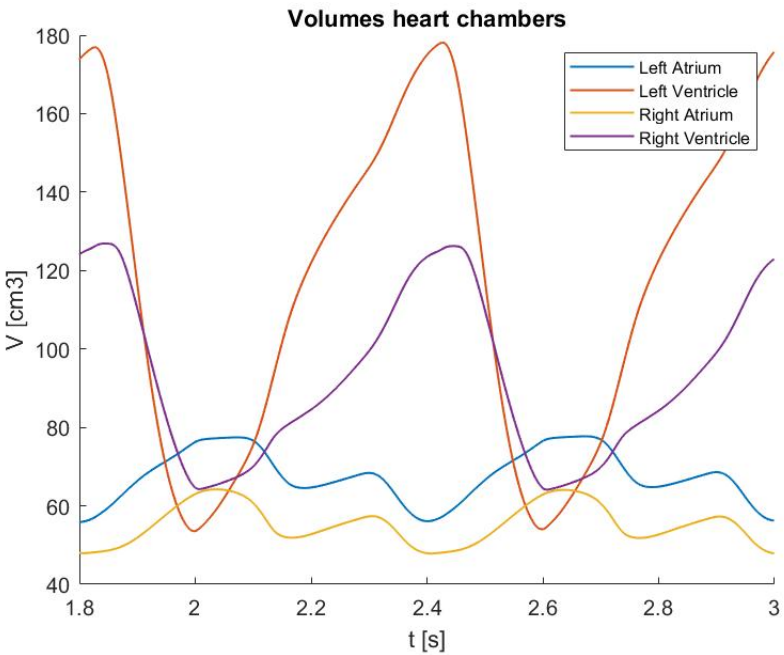


Figure: Left: 60bpm. Right: 100bpm. At lower heart rates, the heart chambers contain more blood volume.

Outflow rate at valves

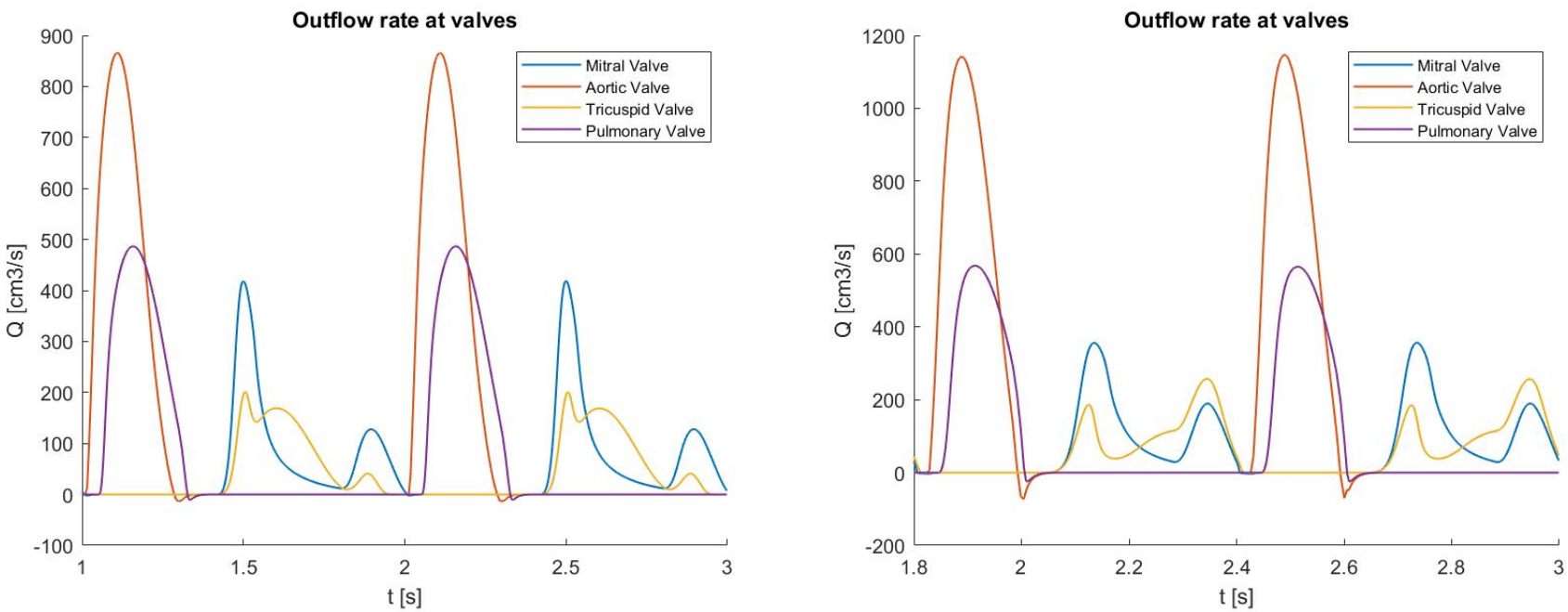
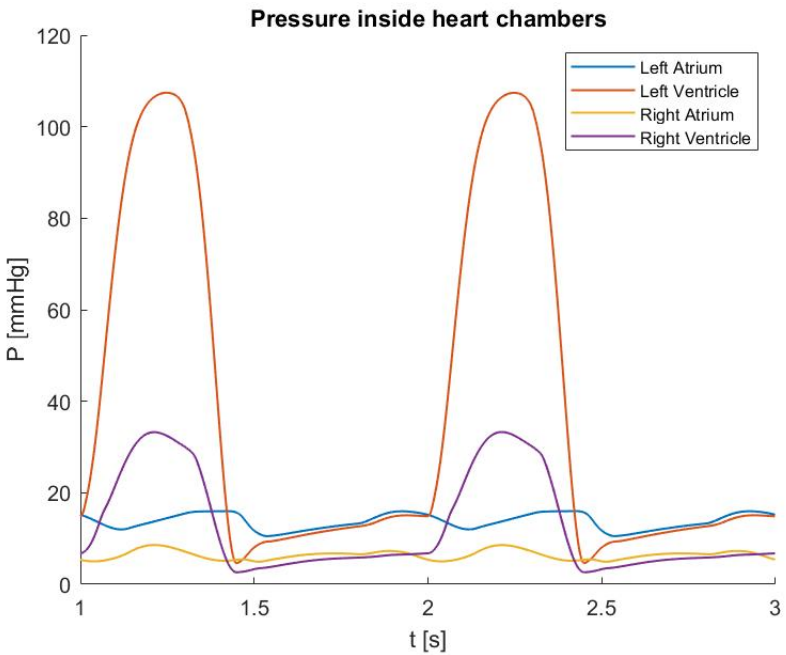


Figure: Left: 60bpm. Right: 100bpm. Higher heart rate results in more back-flow. The outflow rate at the Tricuspid Valve (Yellow) is changed.

Pressure in the heart chambers



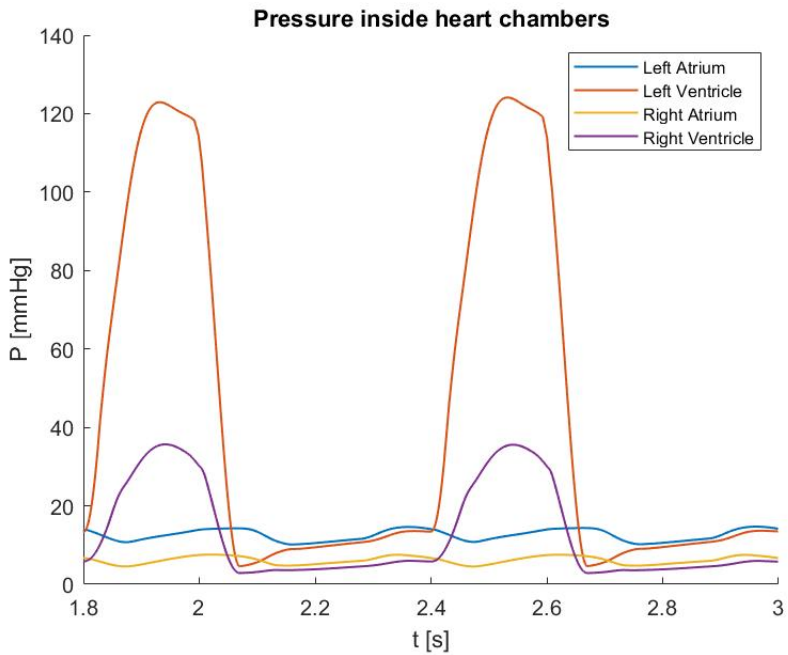
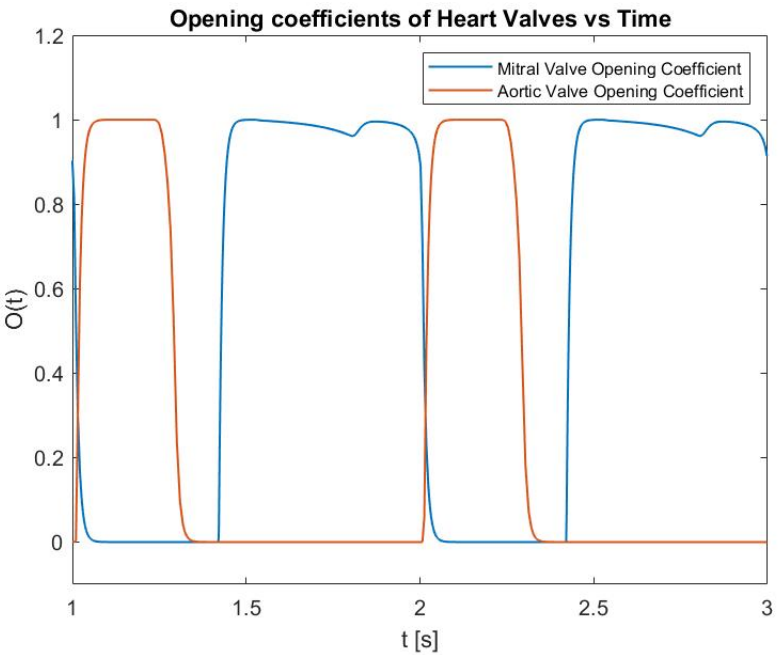


Figure: Left: 60bpm. Right: 100bpm. High pressure is required for the Left Ventricle, to pump oxygen-rich blood to tissues all over the body. Higher heat rate requires higher pressure in the heart chambers.

Valve behaviour



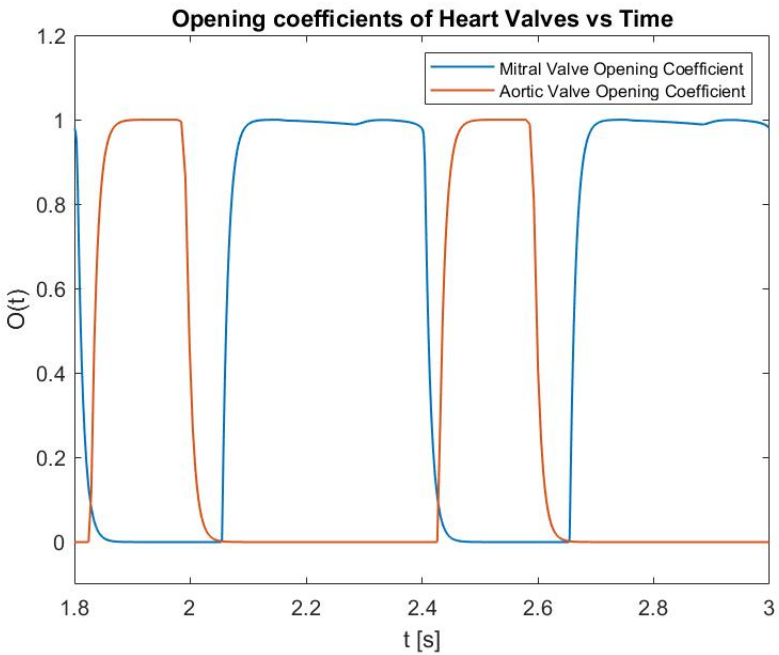


Figure: Left: 60bpm. Right: 100bpm. The valves switch very fast between 0 (fully closed) and 1 (fully open). Higher heart rate requires the Mitral valve to stay fully open, and the valves close faster (steeper descent).

Graphical User Interface

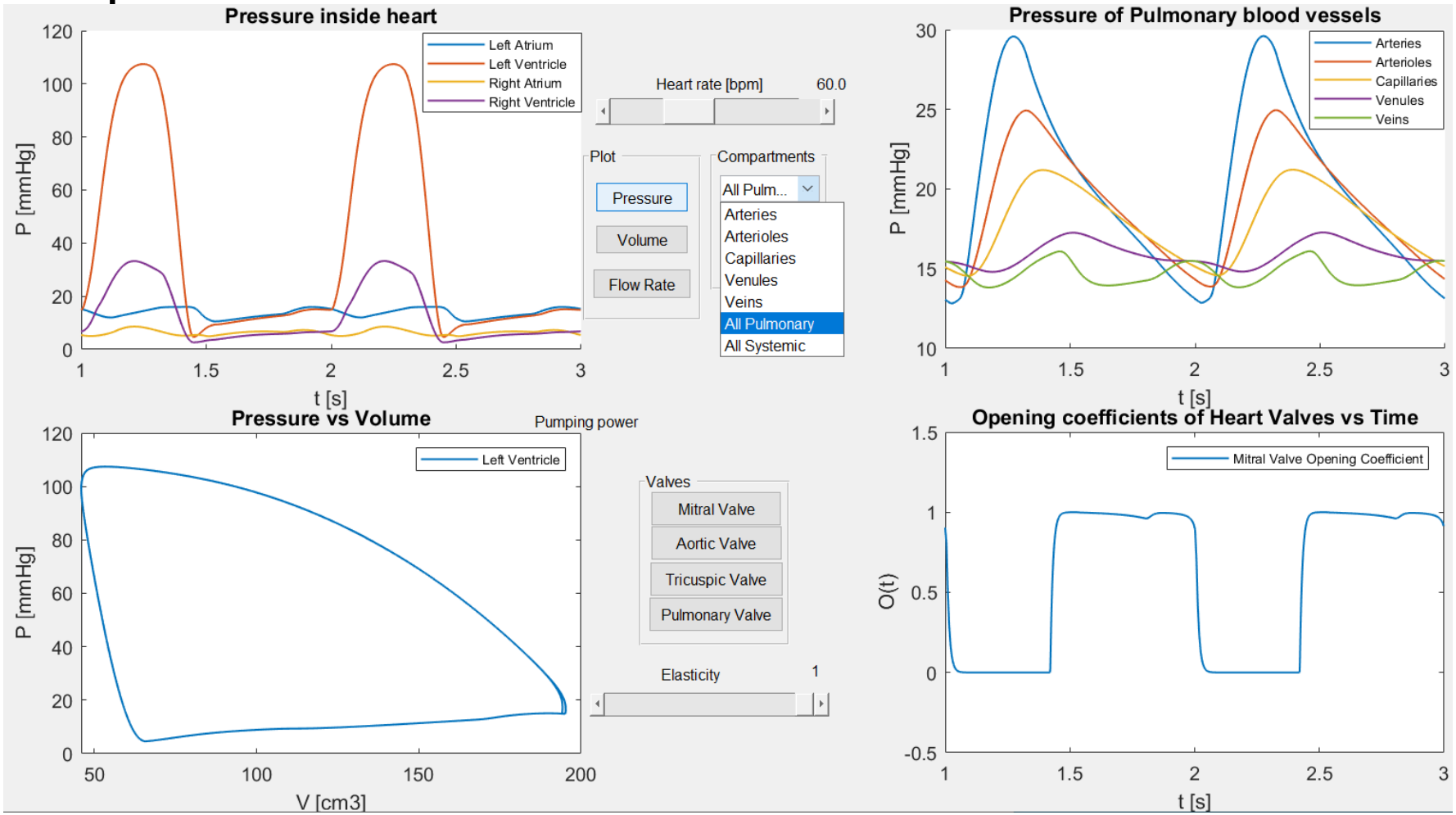


Figure: GUI for Model. Top two axes plot P, V, or Q. Lower two axes plot Pumping Power, and Opening Coefficients.

Graphical User Interface

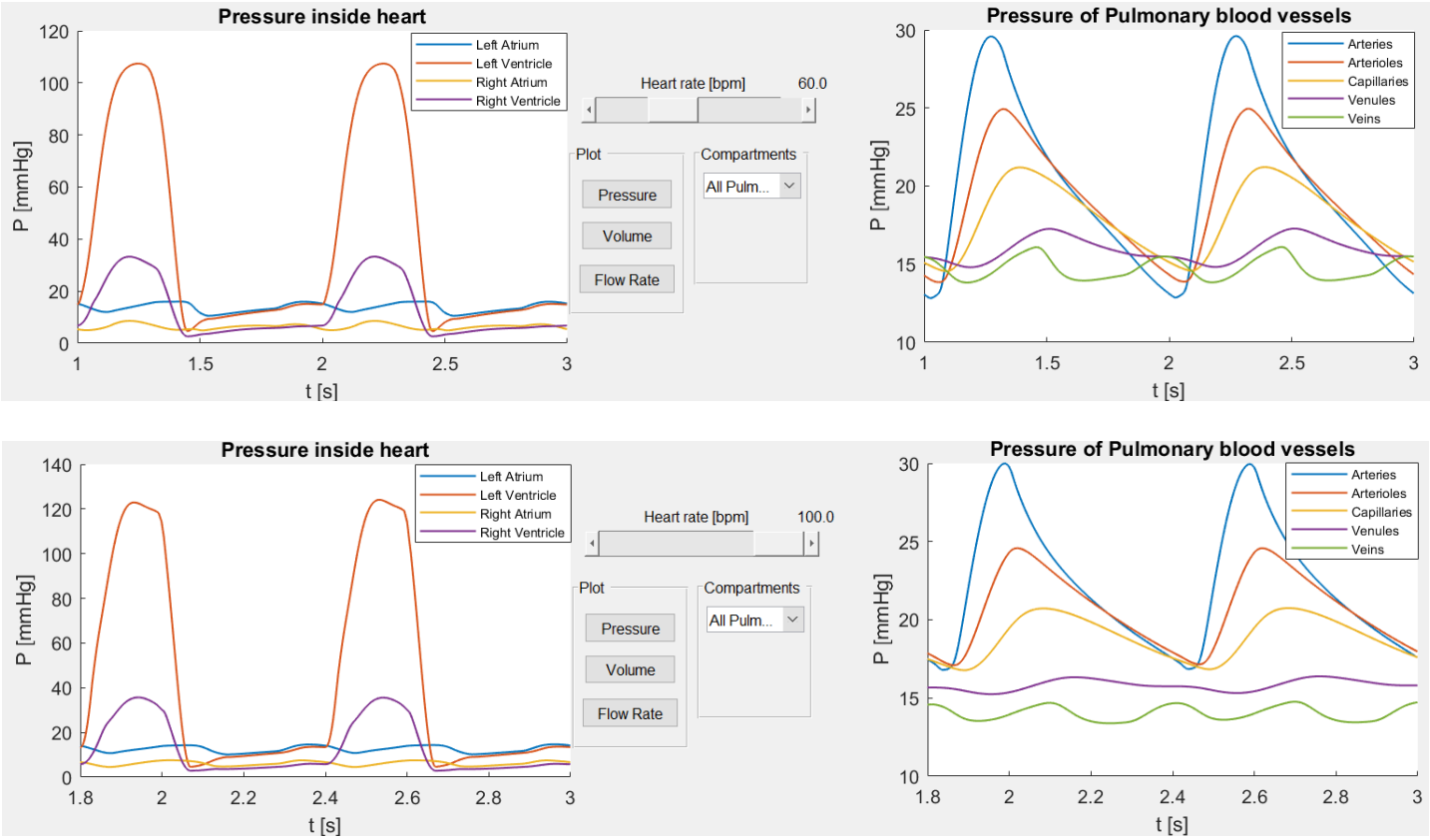
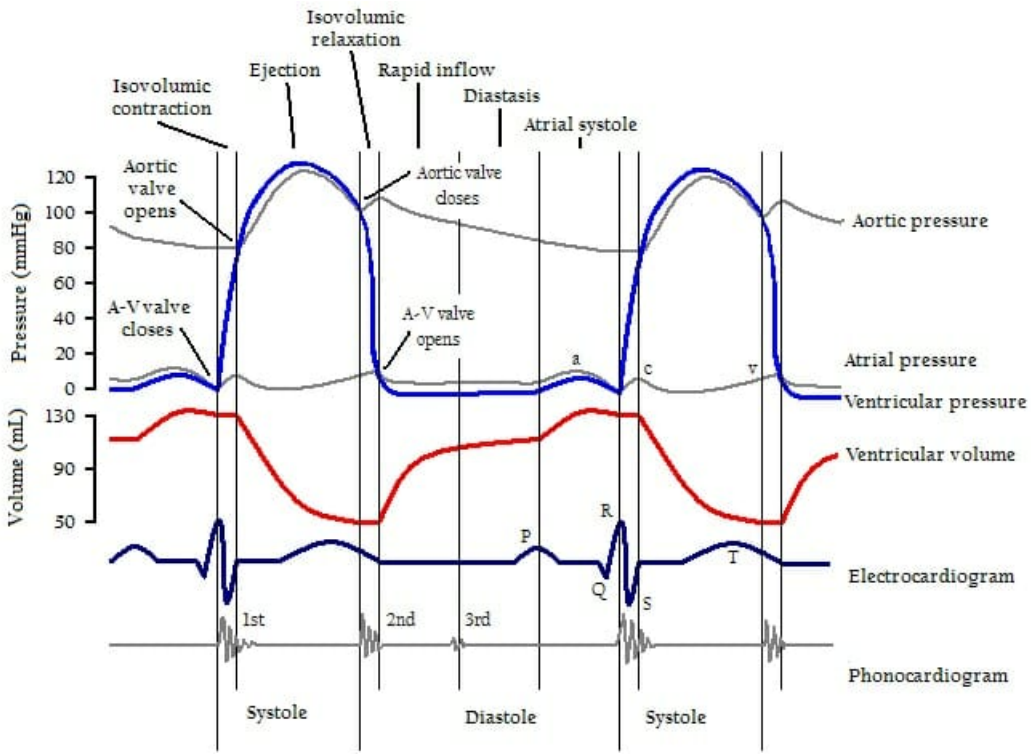


Figure: Top: 60bpm. Bottom: 100bpm. Pressure drops faster inside the heart. Peak behaves differently.

Comparison to medical data



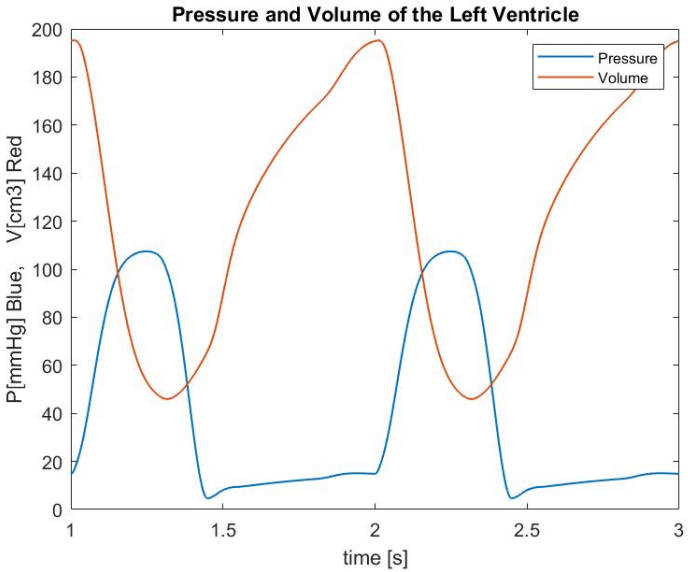


Figure: On the left; medical results for Ventricular Pressure (Blue) and Volume (Red), in contrast to analytical results from the model (Right).

Conclusions and Outlook

- Our implementation is much faster (7 times faster) than Matlab's ode45 solver.
- We found two ways to solve the issue with the non-smoothness points: by using Andersson-Björck method and with the approximation function.
- We implemented a version of the automatic step control which improves the performance considerably.
- Interactive GUI to view multiple results for different compartments. Evaluate the effects of varying heart rates.
- Our plots are coherent with measured data in medical literature.

Acknowledgments

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Thank you for your attention!

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