Non-Newtonian Fluid Mechanics

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Abstract

Non-Newtonian fluids are hugely important in our lives; from the blood in our veins to the ketchup in our kitchen cupboards, we come into contact with them every day. They are essential in a wide range of subjects, such as chemical engineering and pharmaceutics, so it is useful to develop and understand the mathematics behind these fluids. In this project, we aim to compare properties of Newtonian and non-Newtonian fluids both analytically and numerically. Starting with Newtonian fluids, we explore problems such as the Couette Plane Problem and the Stokes Second Problem. We introduce the idea of modelling non-Newtonian fluids, in particular we explore the Oldroyd-B model in order to differentiate between viscoelastic fluids and Newtonian fluids in similar problems. We go on to numerically investigate solutions for non-Newtonian fluids based on more complex models, such as the power-law and Carreau-Yasuda models.

Contents

1 Introduction

There are many examples of how fluid mechanics can be used. One example is the use of fluid bearings. In machinery with moving metallic parts that slide along each other there will be a lot of friction, but when you have a thin layer of fluid in between the parts the friction will be lower. This setting is analogous to the plane Couette flow which we will explore in more detail later.

Throughout this paper we are interested in fluid flow between two infinitely wide parallel plates. We begin by introducing the concept of Non-Newtonian fluids and how we can describe their flow, then we lead on to how the different surroundings of the fluid affects the solution to working out its flow (such as the fluid being between two oscillating plates). We will also go into detail about the Oldroyd-B model and how to use it for viscoelastic fluid flows, then we move on to numerically calculating solutions for both Newtonian and Non-Newtonian fluid flow problems using Matlab and comparing the numerical and analytical solutions.

1.1 Definitions

- Stress, σ, is a measure of the forces between particles in a substance
- Shear stress is the component of stress which acts in the same plane as the cross-section of the substance
- Fluids are substances which deform when a shear stress is applied to them
- Strain, γ, is a measure of the deformation of a substance
- Strain rate, $\dot{\gamma}$, is a measure of the rate of change of the distance between particles in the substance
- Viscosity, η or μ, is the measure of a fluid's resistance to a shear stress

1.2 Differences between Newtonian and Non-Newtonian Fluids

All fluids can be separated in to two groups: Newtonian and non-Newtonian.

Newtonian fluids follow Newton's law of viscosity.

$$\sigma = \mu\dot{\gamma}$$

σ is the shear stress
μ is the coefficient of viscosity
$\dot{\gamma}$ is the strain rate

In other words, for a Newtonian fluid, the shear stress is directly proportional to the strain rate.

Non-Newtonian fluids do not follow Newton's law of viscosity. That is, their shear stress is not directly proportional to the strain rate.

Common examples of Newtonian fluids include: water, oil and alcohol.

Common examples of non-Newtonian fluids include: ketchup, toothpaste, paint and whipped cream.

1.2.1 Types of Non-Newtonian Fluids

Shear thinning fluids have the property that viscosity decreases as shear rate increases (such as ketchup)

Shear thickening fluids have the property that viscosity increases as shear rate increases (such as cornstarch in water)

A Bingham plastic behaves as a solid at low shear stresses, but once the yield stress is reached it will start to flow (such as toothpaste) [1]

Rheopectic fluids are similar to shear thickening fluids as their strain rate increases with time (printer ink)

Thixotropic fluids are similar to shear thinning fluids as their strain rate decreases with time (such as many paints)

Viscoelastic fluids exhibit both viscosity and elasticity, so in some conditions it flows under stress but returns to the original state after a stress is removed (such as whipped cream) [2]

1.3 Modelling Flows

- Newtonian

$$\eta(\dot{\gamma}) = \mu$$

- Power-Law

$$\eta(\dot{\gamma}) = \mu|\dot{\gamma}|^{n-1}$$

$n = 1$ is Newtonian
$n > 1$ is shear thickening
$0 < n < 1$ is shear thinning

In the Power-Law model [3], the viscosity can be bounded by an upper and lower bound, outside of which there is significant variation from the model. It is therefore not valid over all shear rates but it is the most simple model for Non-Newtonian fluids.

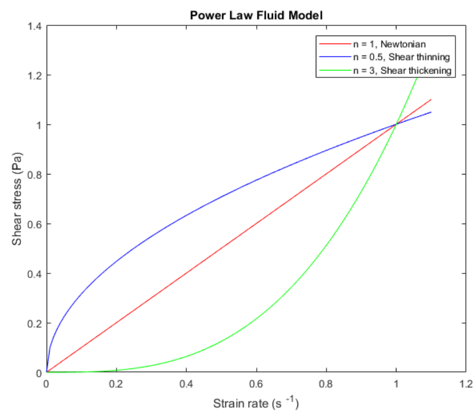


Figure 1: This graph, plotted in Matlab, displays the different behaviour of Newtonian fluids (in red), shear thinning fluids (in blue) and shear thickening fluids (in green) according to the Power-Law model

- Carreau

$$\eta(\dot{\gamma}) = \eta_\infty + (\eta_0 - \eta_\infty)(1 + (\dot{\gamma}\lambda)^2)^{\frac{n-1}{2}}$$

η_∞ is the viscosity at infinite shear rate
η_0 is the viscosity at zero shear rate
λ is the relaxation time

The Carreau model [3] is valid over all shear rates. At a low shear rate, a Carreau fluid behaves as a Newtonian fluid and at a high shear rate a Carreau fluid behaves as a power-law fluid. This model is used in blood flow applications as well as other areas.

2 Fluid Flow Problems

2.1 Navier-Stokes Equation

In mechanics we solve the following familiar equation for $\mathbf{x}$

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = \frac{\mathbf{F}}{m} \tag{1}$$

Whilst, in fluid mechanics we are more interested in momentum and velocities so we solve something similar to equation (1) (the momentum equation) for $\mathbf{u}$

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{\sum \boldsymbol{f}}{\rho} \tag{2}$$

where ρ is the fluid density at that point and $\boldsymbol{f}$ is the force per unit volume

As $\mathbf{u} = \mathbf{u}(x,\, y,\, z,\, t)$

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} \tag{3}$$

Force of gravity, $\boldsymbol{f}_g = -\rho g\hat{\mathbf{k}}$, is not taken in to account here because we will only be considering fluids in the $x - y$ plane.

Force of pressure, $\boldsymbol{f}_p = -\nabla p$, is negative because as the flow increases in the x-direction it moves from an area of high pressure to an area of low pressure (i.e. pressure decreases in the same direction as flow increases).

Force from stress, $\boldsymbol{f}_s = \nabla\cdot\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is the symmetric matrix of stresses [4] such that

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix}$$

So equation (2) with equation (3) becomes:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla \boldsymbol{p} + \nabla\cdot\boldsymbol{\sigma} \tag{4}$$

Equation (4) can be written in the following form which is called the Navier-Stokes Equation [5]

$$\rho\left(\frac{\partial u_j}{\partial t} + u_i\frac{\partial u_j}{\partial x_i}\right) = -\frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_i}\left(\mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) + \left(\mu_v - \frac{2}{3}\mu\right)\frac{\partial u_m}{\partial x_m}\delta_{ij}\right) \tag{5}$$

We start with the differential form of the continuity equation and the material derivative of the density, with the fluid being incompressible (the derivative of the density with respect to time and space is zero) [5]

$$\begin{cases} \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \\ \frac{\partial \rho}{\partial t} + (\mathbf{u}\cdot\nabla)\rho = 0 \end{cases}$$

$$\begin{cases} \quad \frac{\partial \rho}{\partial t} + \rho(\nabla\cdot\mathbf{u}) + \mathbf{u}(\nabla\cdot\rho) = 0 \\ \quad \frac{\partial \rho}{\partial t} + (\mathbf{u}\cdot\nabla)\rho = 0 \end{cases}$$

$$\begin{cases} \frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{u}) = 0 \\ \frac{\partial \rho}{\partial t} = 0 \end{cases}$$

From this, we get the continuity equation for incompressible fluids,

$$\nabla \cdot \mathbf{u} = 0 \tag{6}$$

As we are working in 2 dimensions, (6) becomes,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

We take $u(y, t)$ for shear flows, so that,

$$\frac{\partial u}{\partial x} = 0$$

and so from (6)

$$\frac{\partial v}{\partial y} = 0$$

So we can simplify the Navier-Stokes equation for a shear flow, i.e. equation (5) becomes:

$$\rho\left(\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i}\right) = -\frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_i}\left(\mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\right)$$

which can be written as two separate equations

$$\begin{cases} \rho\left(\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} \\ \\ \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x}\right) = -\frac{\partial p}{\partial y} + \mu\frac{\partial^2 v}{\partial x^2} \end{cases}$$

2.2 Plane Couette Flow

In order to study the behavior of fluids, one needs to start by considering the simple cases first and adding complexity afterwards. We firstly consider a Newtonian fluid, a particular type of fluid whose stress is proportional to the shear ('parallel displacement') everywhere. The equations that govern this kind of flow are called the 'Navier-Stokes' equations and they are derived from conservation of mass and momentum. There are non-linear terms in the equations, which means that only in few cases an explicit solution can be found. One must, hence, look for suitable physical approximations that lead to cancellation of some terms that permit us to solve the equations. The equations in two dimensions are:

$$\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\ \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\ \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \end{cases} \tag{7}$$

We start our study by considering the Plane Couette flow [5], where we have two parallel plates of distance H apart and of infinite width, moving with speed U_1 and U_2 in the tangent direction, respectively. The pressure gradient is considered to be zero for simplicity. The flow between the two plates is steady and since the plates are infinite and moving in the tangent direction we have that for fixed height y, the velocity must be the same for all x. Thus $\mathbf{u} = (u, v)$ is independent on x and t. Equations (7.1), (7.2), (7.3) get simplified to:

$$\begin{cases} \frac{\partial v}{\partial y} = 0 \\ v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \\ v\frac{\partial v}{\partial y} = \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\frac{\partial^2 v}{\partial y^2} \end{cases} \tag{8}$$

On top of these constraints one must consider the boundary condition, also derived by physical constraints which characterize each problem and lead to fundamental differences in the solutions to problems. For this problem we use the impermeability condition and the no-slip condition. The impermeability condition implies that there is no flow coming out from the plates, so we have that at $y = 0$ and $y = H$, $v = 0$; i.e., $v(0) = v(H) = 0$. The no-slip condition states that the velocity of the fluid at the plates is the same as the velocity of the plates due to the effect of viscosity; i.e. $u(0) = U_1$, $u(H) = U_2$.
Thus from (8.1), we have that v is a constant and since $v(0) = 0$ by boundary condition we have that $v = 0$.

Equation (8.3) becomes $\frac{\partial p}{\partial y} = 0$, which implies that $p = p(x)$. As the pressure gradient is 0 we have that pressure is constant. (7.2) simplifies to $\frac{\partial^2 u}{\partial y^2} = 0$, which is easily solvable to $u(y) = ay + b$
Applying the Boundary conditions we get the integrating constants: $b = U_1$, $a = \frac{U_2 - U_1}{h}$.

Finally, the solution is

$$\mathbf{u}(y) = \left(\frac{U_2 - U_1}{h}y + U_1, 0\right)$$

This is a shear flow as shear stress is non-zero. In fact we have that shear stress for $U_1 \neq U_2$ is:

$$S(y) = \mu\frac{\partial u}{\partial y} = \rho\frac{U_2 - U_1}{h} \neq 0$$

2.3 Oscillating Case (semi-infinite case)

We would like to see the behavior of a fluid inside two plates oscillating in the tangent direction (with the same frequency ω and same amplitude U, say).

Let's first study the influence of only one oscillating plate on fluid particles. So assume that one plate is stationary and infinitely far away from the other one. This means that any fluid particle will be affected uniquely by the first. We use again the impermeability and non-slip conditions. Then the boundary conditions are:

$$u(0, t) = U\cos(wt)$$

and

$$\lim_{y\to\infty} u(y, t) = 0$$

$v(0, t) = 0$ and

$$\lim_{y\to\infty} v(y, t) = 0$$

As in the previous problem, the pressure gradient is zero and as the velocity is tangential, we have that the velocity $\mathbf{u} = (u, v)$ is independent of x.

The Navier-Stokes equations (7) can be simplified to:

$$\begin{cases} \frac{\partial v}{\partial y} = 0 \\ \frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \end{cases} \tag{9}$$

(9.1) implies that $v = v(t)$, but since $v(t) = 0$ on the boundaries for all t it means that $v = 0$.
(9.3) implies that $p =$ constant as the pressure gradient is 0.
Finally, we are left with

$$\frac{\partial u}{\partial t} = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \tag{10}$$

In the problem we have an oscillating term, with constant frequency ω, which depends only on time. Hence it can be solved by separation of variables by setting $u(y,t) = f(y)e^{iwt}$, taking only the real part of the expression.

(10) simplifies to $iwf = \frac{\mu}{\rho}f^{''}$ so we can find

$$f(y) = Ae^{cy} + Be^{-cy}, \tag{11}$$

where A, B are constants that can be found using the boundary conditions and $c = (1+i)\sqrt{\frac{w\rho}{2\mu}}$. As y goes to infinity, U goes to 0 but the first term of f gets exponentially large, so $A = 0$.

Also, $u(0,0) = f(0) = B = U$.

Hence, we have that $f(y) = Ue^{-cy}$ and $u(y,t) = Ue^{-cy+iwt}$.

2.4 Oscillating Plates (finite case)

Now we see how the fluid is affected by two oscillating plates, rather than one. For simplicity, we assume that the frequency and amplitude for both plates are the same. We can write these constraints as $u(0,t) = u(H,t) = Ucos(wt)$, where H is the distance between the plates. The problem conditions are the same as before, so we get the same general solution by separation of variables: $u(y,t) = f(y)e^{iwt}$ Applying the boundary conditions we get:

$$f(y) = A\cosh cy + B\sinh cy,$$

This time we use *cosh* and *sinh* because we have a bounded domain.
This time $u(0,0) = f(0) = A = U$. Using the second boundary condition we get:

$$u(H,0) = f(H) = U\cosh cH + B\sinh cH = U$$

from which we get that $B = \dfrac{U\,(1-\cosh cH)}{\sinh cH}$. Let $k = \sqrt{\dfrac{w\rho}{2\mu}}$. The solution is:

$$u(y,t) = \left(U\cosh cy + \frac{U\,(1-\cosh cH)}{\sinh cH}\sinh cy\right)e^{iwt} \tag{12}$$

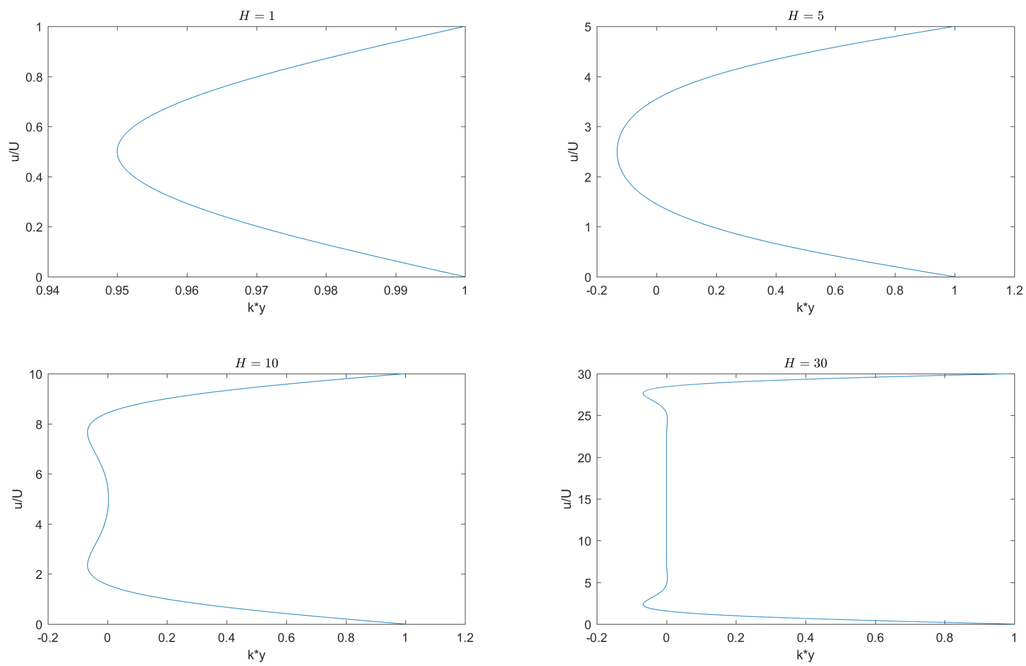


Figure 2: These are the velocity profiles for different distances between the plates for some fixed time t=0. As one can see, if the plates are far apart, fluid particles in the middle are not influenced by the plates.

2.5 Non-Dimensionalisation

Another way to solve systems of equations in fluid dynamics is through non-dimensionalisation. Instead of writing the problem depending on units, we express it in terms of some characteristic scales of the problem [9]. This allows us not to worry about many parameters as we bring them together in some other minimal parameters. Let's consider the following non-dimensionalisation, where k^* represents dimensional quantities, and the characteristic scales are taken from the plane Couette flow problem.
Let

$$\begin{cases} u^* = U_1 u \\ x^* = Hx \\ t^* = \frac{H}{U_1} t \\ v^* = U_1 v \\ y^* = Hy \\ p^* = \frac{\mu U_1}{H} p \end{cases} \tag{13}$$

The Navier-Stokes equations (7*) become

$$\begin{cases} \frac{U_1}{H}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0 \\ \\ \frac{U_1^2}{H}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\frac{\mu U_1}{\rho H^2}\left(-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \\ \\ \frac{U_1^2}{H}\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=\frac{\mu U_1}{\rho H^2}\left(-\frac{\partial p}{\partial y}+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \end{cases} \tag{14}$$

So by cancelling out the scale parameters and defining the Reynolds number as $R=\frac{\rho UH}{\mu}$ we simplify (14) and get:

$$\begin{cases} \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \\ \\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{R}\left(-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \\ \\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\frac{1}{R}\left(-\frac{\partial p}{\partial y}+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \end{cases} \tag{15}$$

The only parameter left is the Reynold's number: $\frac{\rho U_1 H}{\mu}$, this non-dimensional value is important in understanding what the relevant forces in some system are. That's because the left hand side of (15.2) and (15.3) represents the inertial forces, while the right hand side represent the viscous forces. So by knowing the value of the Reynold's number for some problem, one might be able to do useful approximations in order to model the solution. In fact, if the Reynold's number is really small it means that viscous forces are ruling the system while inertial forces are irrelevant, while if it is really big then the opposite happens. For some transition value both forces are relevant so no such approximation can be done. For this reason, by comparing the Reynold's number of different fluids one can predict whether these fluids will behave the same. Similar considerations can be done for systems of more equations which will give more non-dimensional parameters.

Taking the case for the plane Couette flow we have that the non-dimensional boundary conditions are: $u(0)=1$ and $u(1)=U_2/U_1$. Cancelling out terms as previously done we get that the solution is of the form $u(y)=ay+b$ and using the boundary conditions we get $u(y)=(U_2/U_1-1)y+1$. To switch back to the dimensional solution again one needs to do the inverse substitution:

$$\frac{u^*}{U_1}=\frac{(U_2-U_1)}{U_1}\frac{y^*}{H}+1 \qquad \Rightarrow \qquad u^*(y^*)=\frac{U_2-U_1}{H}y^*+U_1$$

which is consistent with what was previously found.

3 Oldroyd-B Model and its applications

3.1 Generalised Oldroyd-B model

$$\begin{cases} \nabla \cdot \mathbf{u} = 0 \\ \boldsymbol{\rho}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u}\right) = \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} \\ \boldsymbol{\sigma} = -p\mathbf{I} + \eta\left(\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^T\right) + G\mathbf{A} \\ \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{A} - \mathbf{A} \cdot \boldsymbol{\nabla}\mathbf{u} - (\boldsymbol{\nabla}\mathbf{u})^T \cdot \mathbf{A} = -\frac{1}{\tau}(\mathbf{A} - \mathbf{I}) \end{cases} \tag{16}$$

The $G\mathbf{A}$ term is called the polymeric stress and it is an added component to the Newtonian case [1]. τ is the relaxation time for the fluid; this is the time required by a viscous fluid to recover to initial condition after a force has been applied to it. G is the shear modulus. In 2D we have:

$$\mathbf{u} = (u, v)$$

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} A_{xx} & A_{xy} \\ A_{xy} & A_{yy} \end{pmatrix}$$

$$\boldsymbol{\nabla}\mathbf{u} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{pmatrix}$$

$$\mathbf{A} \cdot \boldsymbol{\nabla}\mathbf{u} = \begin{pmatrix} A_{xx} & A_{xy} \\ A_{xy} & A_{yy} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} A_{xx}\frac{\partial u}{\partial x} + A_{xy}\frac{\partial u}{\partial y} & A_{xx}\frac{\partial v}{\partial x} + A_{xy}\frac{\partial v}{\partial y} \\ A_{xy}\frac{\partial u}{\partial x} + A_{yy}\frac{\partial u}{\partial y} & A_{xy}\frac{\partial v}{\partial x} + A_{yy}\frac{\partial v}{\partial y} \end{pmatrix}$$

$$(\boldsymbol{\nabla}\mathbf{u})^T \cdot \mathbf{A} = (\boldsymbol{\nabla}\mathbf{u})^T \cdot \mathbf{A}^T = (\mathbf{A} \cdot \boldsymbol{\nabla}\mathbf{u})^T$$

(16.1) can be expanded as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{17}$$

(16.2) can be expanded as

$$\begin{cases} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y}\right) \\ \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{1}{\rho}\left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y}\right) \end{cases} \tag{18}$$

(16.3) can be expanded as

$$\begin{cases} \sigma_{xx} = -p + 2\eta\frac{\partial u}{\partial x} + GA_{xx} \\ \\ \sigma_{xy} = \eta\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + GA_{xy} \\ \\ \sigma_{yy} = -p + 2\eta\frac{\partial v}{\partial y} + GA_{yy} \end{cases} \tag{19}$$

(16.4) can be expanded as

$$\begin{cases} \frac{\partial A_{xx}}{\partial t} + u\frac{\partial A_{xx}}{\partial x} + v\frac{\partial A_{xx}}{\partial y} - 2\left(A_{xx}\frac{\partial u}{\partial x} + A_{xy}\frac{\partial u}{\partial y}\right) = -\frac{1}{\tau}\left(A_{xx} - 1\right) \\ \frac{\partial A_{xy}}{\partial t} + u\frac{\partial A_{xy}}{\partial x} + v\frac{\partial A_{xy}}{\partial y} - A_{xx}\frac{\partial v}{\partial x} - A_{yy}\frac{\partial u}{\partial y} = -\frac{1}{\tau}A_{xy} \\ \frac{\partial A_{yy}}{\partial t} + u\frac{\partial A_{yy}}{\partial x} + v\frac{\partial A_{yy}}{\partial y} - 2\left(A_{xy}\frac{\partial v}{\partial x} + A_{yy}\frac{\partial v}{\partial y}\right) = -\frac{1}{\tau}\left(A_{yy} - 1\right) \end{cases} \tag{20}$$

3.2 Applications of the Oldroyd-B Model

The Oldroyd-B Model for 2-dimensional shear flows:

$$\rho\frac{\partial u}{\partial t} = G\frac{\partial A_{xy}}{\partial y} + \eta\frac{\partial^2 u}{\partial y^2} \tag{21}$$

$$0 = -\frac{\partial p}{\partial y} + G\frac{\partial A_{yy}}{\partial y} \tag{22}$$

Where the components of the conformal tensor are defined by

$$\frac{\partial A_{xx}}{\partial t} - 2A_{xy}\frac{\partial u}{\partial y} = -\frac{1}{\tau}(A_{xx} - 1) \tag{23}$$

$$\frac{\partial A_{yy}}{\partial t} = -\frac{1}{\tau}(A_{yy} - 1) \tag{24}$$

$$\frac{\partial A_{xy}}{\partial t} - A_{yy}\frac{\partial u}{\partial y} = -\frac{1}{\tau}A_{xy} \tag{25}$$

Note that, as this is a shear flow, $\boldsymbol{u} = (u(t, y), 0)$, so $v = 0$ and furthermore, this means other physical quantities such as the conformal tensor components can only depend on y and t, hence the results above [6].

3.3 Viscoelastic plane Couette flow

In this example, we can think of a bounded flow, with plates of infinite width at $y = 0$ and $y = H$ which are moving with speeds U_1, U_2 respectively. In a Couette flow, we have a zero pressure gradient [5], i.e $\nabla p = 0$ and due to the boundary conditions, the velocity of the flow is a function of y only, which means that all other physical quantities are also only a function of y.

We wish to solve this problem using the Oldroyd-B model, and by equation (24), the left hand side becomes zero, meaning $A_{yy} = 1$. This result holds for equation (22) as A_{yy} is constant. Now use A_{yy} to solve for A_{xy} and write $u = f(y)$, then from (23) and (25),

$$\begin{aligned} A_{xy} &= \tau f'(y) \\ 2A_{xy} f'(y) &= \frac{1}{\tau}(A_{xx} - 1) \\ \Rightarrow A_{xx} &= 1 + 2\left(\tau f'(y)\right)^2 \end{aligned}$$

Now, solving (21) we get

$$0 = G\tau f''(y) + \eta f''(y) \quad \Rightarrow \quad f''(y) = 0 \quad \Rightarrow \quad f(y) = \alpha + \beta y$$

Where the constants α and β depend on the boundary conditions $u(0) = U_1$ and $u(H) = U_2$. From these, the final answer is $u(y) = f(y) = U_1 + \left(\frac{U_2 - U_1}{H}\right) y$.

Hence, the conformal tensor components are

$$A_{yy} = 1, \quad A_{xy} = \frac{\tau(U_2 - U_1)}{H}, \quad A_{xx} = 1 + \frac{2\tau^2(U_2 - U_1)^2}{H^2}$$

Now, we can define the effective viscosity ν^e to be the ratio of the shear stress and stress, $\sigma_{xy}/\frac{\mathrm{d}u}{\mathrm{d}y}$, where $\boldsymbol{\sigma}$ is defined (in the 2-dimensional case here) in [6] as

$$\boldsymbol{\sigma} = -p\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \eta\left(\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} + \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\right) + G\begin{pmatrix} A_{xx} & A_{xy} \\ A_{xy} & A_{yy} \end{pmatrix} \tag{$\star$}$$

Using ($\star$), we get that

$$\begin{aligned} \sigma_{xy} &= 0 + \eta\frac{\mathrm{d}u}{\mathrm{d}y} + GA_{xy} \\ &= (\eta + G\tau)\frac{\mathrm{d}u}{\mathrm{d}y} \\ \Rightarrow \quad \nu^e &\equiv \sigma_{xy}\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^{-1} \\ &= \eta + G\tau \end{aligned}$$

So, the viscoelastic plane Couette flow has constant effective viscosity $\eta + G\tau$, which is bigger than the Newtonian viscosity, η.

[1] The power-law model has viscosity that is given to be $\nu^p := \eta\left(f'(y)\right) = \eta|f'(y)|^{n-1}$ where η is the usual Newtonian viscosity and $n > 0$ is picked in order to model the viscosity for a non-Newtonian fluid depending on the shear. For $n = 1$, we get Newtonian viscosity, $n > 1$ is for shear thickening and $0 < n < 1$

is for shear-thinning. Let us now suppose that the values of U_1 and U_2 are chosen so that they're not equal, thus implying that $f'(y) \equiv (U_2 - U_1)/H \neq 0$.

For $0 < n < 1$, then $\nu^p < \eta < \eta + G\tau \equiv \nu^e$. Similarly, for $n > 1$, then $\nu^p > \eta$ and in particular, for some value of $N > 1$, then $\forall\ n > N, \quad \nu^p > \nu^e$. We can easily draw a conclusion that a possible limitation of using the Oldroyd-B model is that as its viscosity is constant for any type of shear (shear-thickening/thinning), then in particular it does not model well situations when we have shear-thickening in comparison to the power-law model. This follows from the idea that models are not suited for every sort of situation in non-Newtonian flows and care should be taken to ensure that when we pick models such as Oldroyd-B, we should clarify whether it models our current problem accurately.

Let's now consider the normal stress, σ_{xx}, for the Couette flow in the Oldroyd-B model. Using ($\star$),

$$\begin{aligned}\sigma_{xx} &= -p + GA_{xx} \\ &= -p + G\left(1 + \frac{2\tau^2(U_2 - U_1)^2}{H^2}\right)\end{aligned}$$

The only parameters that can vary here are U_1, U_2 and H. Varying these parameters so that $U_2 \gg U_1$ or $U_2 \ll U_1$ with H being relatively small, gives a large result for σ_{xx}. This means that our value for $\frac{du}{dy}$ is large, so our flow is stretching in the horizontal direction, due to the shear acting in the horizontal direction. This follows from the fact that σ_{xx} points in the horizontal direction in the x plane, meaning if σ_{xx} is large, then the shear in the horizontal direction is large.

3.4 Viscoelastic flow between two oscillating plates

Again, let us consider the pressure gradient to be zero, and due to the oscillating nature of our new boundary conditions, we have that the horizontal velocity of the fluid is oscillating in time. As in the Newtonian case, $u(t, y) = e^{iwt}f(y)$, where $f(y)$ is to be found due to the nature of the problem. Let $A_{yy}(t = 0) = 1$ and $u(t, 0) = u(t, H) = U\cos(wt)$ be our initial condition and boundary conditions.

As in the Couette flow problem, we start by solving (24), giving us $A_{yy} = 1 + C(y)\exp\left(-\frac{t}{\tau}\right)$ by the integration factor and due to our initial condition, then $C = 0$, hence $A_{yy} = 1$. Now, solving (25) we get

$$\begin{aligned}\frac{\partial A_{xy}}{\partial t} - \frac{\partial u}{\partial y} &= -\frac{1}{\tau}A_{xy} \\ \text{where,} \quad \frac{\partial u}{\partial y} &= \exp(iwt)f'(y) \\ \Rightarrow \frac{\partial A_{xy}}{\partial t} + \frac{1}{\tau}A_{xy} &= \exp(iwt)f'(y) \\ \Rightarrow \frac{\partial}{\partial t}\left(\exp\left(\frac{t}{\tau}\right)A_{xy}\right) &= \exp\left(t\left(iw + \frac{1}{\tau}\right)\right)f'(y) \\ \Rightarrow A_{xy} &= D\exp\left(-\frac{t}{\tau}\right) + \frac{\tau}{1 + iw\tau}e^{iwt}f'(y) \\ &= D\exp\left(-\frac{t}{\tau}\right) + \frac{\tau(1 - w\tau i)}{1 + w^2\tau^2}e^{iwt}f'(y)\end{aligned}$$

Where D is a arbitrary function of y. In a similar fashion, one can use the integration factor method to find the solution to A_{xx} using equation (23), giving

$$A_{xx} = E\exp\left(-\frac{t}{\tau}\right) + 1 + 2Dt\exp\left(-\frac{t}{\tau}\right) + \frac{2\tau^2(1 - w\tau i)^2}{(1 + w^2\tau^2)^2}e^{iwt}f'(y)$$

Where E and D are arbitrary functions of y which can be found from boundary conditions of A_{xx} and A_{xy}. Now that we have the components for the conformal tensor matrix A, we can find an expression using equation (21) for $f(y)$. (21) becomes,

$$iw\rho f(y) = f''(y)\left(\eta + \frac{G\tau(1-w\tau i)}{1+w^2\tau^2}\right)$$
$$\text{i.e.} \quad f''(y) - k^2 f(y) = 0$$

where $k^2 = iw\rho\left(\eta + \frac{G\tau(1-w\tau i)}{1+w^2\tau^2}\right)^{-1}$. The general form for $f(y)$ becomes

$$f(y) = \alpha\cosh(ky) + \beta\sinh(ky) \tag{\dagger}$$

The boundary conditions give us that $f(0) = f(H) = U$, hence $\alpha = U$, similarly, substituting $y = H$,

$$U = U\cosh(kH) + \beta\sinh(kH) \quad \Rightarrow \quad \beta = \frac{U(1-\cosh(kH))}{\sinh(kH)}$$

This almost has the exact same form as in the Newtonian case in section 2.4, the main difference being the definition for the value of k^2, where in the Newtonian case, $k^2 = iw\rho/\eta$. It's clear that both their values of k will be complex. When looking at our solution of $u(t, y)$ with the combination of (†), we can plot the real part of u at certain times, here is a velocity profile for the viscoelastic case,

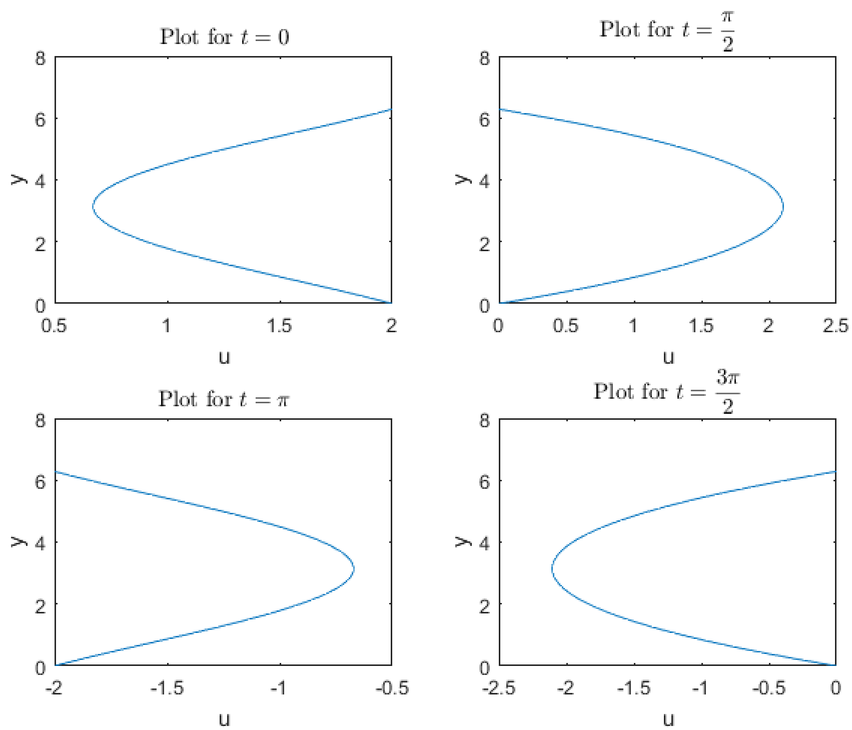


Figure 3: Velocity profile with: $U = 2,\ w = 1,\ \rho = 1,\ \eta = \pi,\ \tau = 2\pi,\ G = 2,\ H = 2\pi$

With changes to the parameters, different shapes emerge, but ultimately, the flow is still fairly similar.

Let's now analyse the behaviour of the effective viscosity for an oscillating boundary problem for viscoelastic fluid. As before, define $\nu^e = \sigma_{xy}/\left(\frac{\partial u}{\partial y}\right)$, giving

$$\nu^e = \eta + \frac{GDe^{\left(-\frac{t}{\tau}\right)}}{e^{iwt}f'(y)} + \frac{G\tau(1-w\tau i)}{1+w^2\tau^2} \tag{\dagger\dagger}$$

For simplicity, we can take $D(y) = 0$ and now consider the real and imaginary part of (††), i.e.

$$\Re(\nu^e) = \eta + \frac{G\tau}{1 + w^2\tau^2}$$
$$\Im(\nu^e) = -\frac{Gw\tau^2}{1 + w^2\tau^2}$$

(See Figure 5 below to see the plots of the imaginary and real parts of (††)) The behaviour of these two functions of frequency have interesting results in two special cases:

Case $w = 0$:

When $w = 0$, then the imaginary part is zero, leaving $\nu^e = \eta + G\tau$, this is exactly the result we obtained when we had the Viscoelastic plane Couette flow, since, if $w = 0$, this implies that the boundaries are not oscillating, and only move with a constant velocity U; this means that we lose the time dependence on the velocity of the flow.

Case $|w| \to \infty$:

As $|w| \to \infty$, then the imaginary part tends to 0, giving $\nu^e = \eta$, i.e. the effective viscosity tends to the Newtonian viscosity in the setting when the frequency becomes very large, i.e. Oscillations become infinitely fast. It's easy to see that as $|w| \to \infty$, $u \to 0$ (which can be verified with my numerical plot of the real part of u), the shear on the viscoelastic fluid loses its presence and since there are no other forces acting on the fluid, if the fluid starts at rest, then it will remain at rest. Hence the result of a viscoelastic fluid with Newtonian viscosity which stays stationary.

To see this result, we first consider what happens to $f(y)$ when $w \to \pm\infty$. Looking at the complex valued constant k defined in (†), it's easy to see that $k = k'(1 \pm i)$ where $k' \to \infty$. From here, we can show that $f(y) \to 0$ by numerical calculations or one can observe that $\beta \to -U$ giving $f(y) \to U(\cosh(ky) - \sinh(ky))$. Using the fact that $\sinh(ky) \approx \frac{1}{2}e^{ky}$ when k is large and similarly for $\cosh(ky)$, then it's clear that $f(y) \to 0$, and hence, for any time t, then $u \to 0$. These results can be seen by taking $w = 1000$ and plotting the velocity profiles with the same values for the other parameters as before:

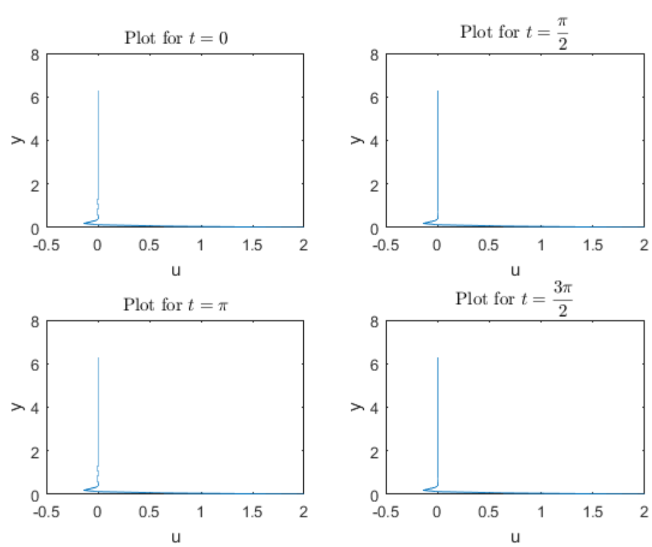


Figure 4: Velocity profiles with $w = 1000$, showing that the solution to $u \to 0$

Analysis for semi-infinite oscillating plate:

Rather than deriving separate solutions as in Section 2.3, one can take the pair of infinite oscillating plates and look at the behaviour of u as $H \to \infty$. Consider the function $f(y)$ as in (†) and look at points away from the two ends. Now when H is large then $\beta \approx -U$ as before, giving

$$f(y) \approx U(\cosh(ky) - \sinh(ky)) = Ue^{-ky}$$

This shows that for points away from the end points and for large H, then our solution is decaying exponentially the bigger y gets. We conclude that for large points away from the end points, $u \approx 0$. If H is large but finite, then the two end points have a nice convergence. Namely, for $y \approx 0$ then $\sinh(ky) \approx ky$ and similarly, $\cosh(ky) \approx 1$, hence for H large, $f(y) \approx U(1-ky) = U + O(y)$. Now, if $y \approx H$, with H large and finite then,

$$\beta \sinh(ky) = \frac{U \sinh(ky)(1-\cosh(kH))}{\sinh(kH)} \approx U(1-\cosh(kH))$$

Hence, for $y \approx H \Rightarrow f(y) \approx U\cosh(kH) + U(1-\cosh(kH)) = U$. So we conclude that, near the end points the velocity of the fluid is U, but away from the end points, u is converging exponentially to 0.

The analysis above can be seen numerically. Using the same parameter values as in the previous profiles, but increasing H to 100, and keeping $w = 1$, the following velocity profiles emerge:

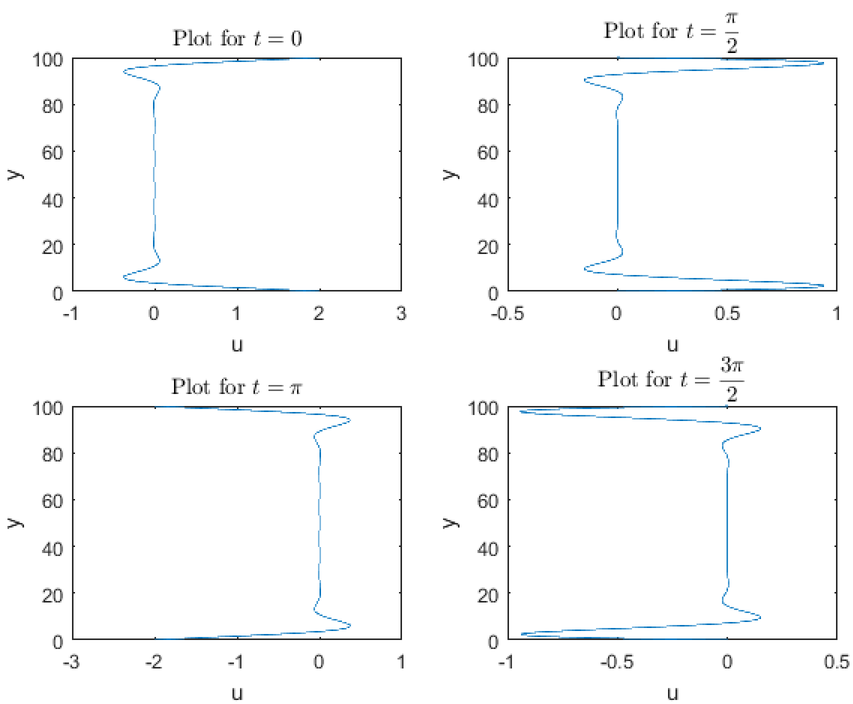


Figure 5: Velocity profiles for $H = 100$, $w = 1$, showing the convergence of $u \to 0$ as H gets large.

The behaviour of the effective viscosity's real and imaginary parts can be better seen from their plots, here are the plots for the real and imaginary axis against the frequency w,

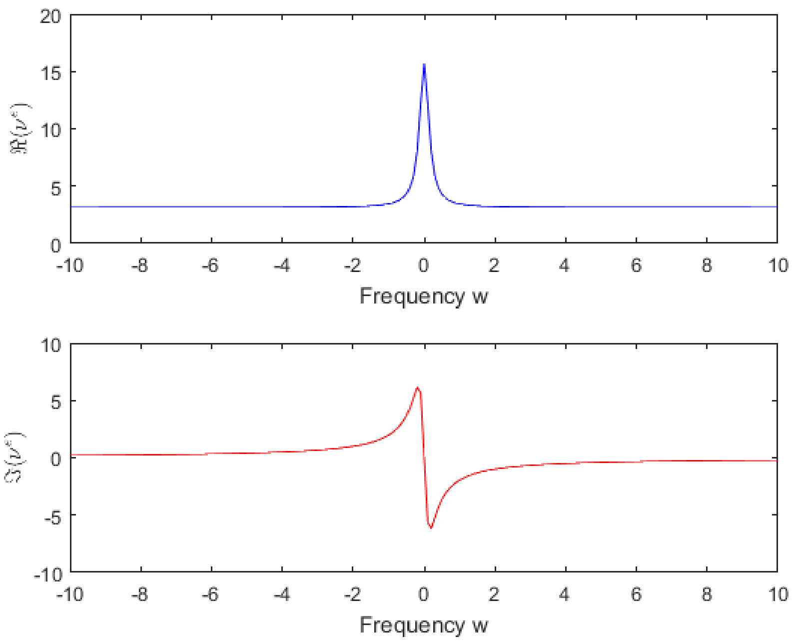


Figure 6: Plots of (††)'s real and imaginary part with, $w = 1,\ \eta = \pi\ , \tau = 2\pi,\ G = 2$

4 Numerical Investigation of Newtonian Fluids

So far we have considered various flows, and solving the governing equations using analytical methods. These solutions are exact and therefore preferable to numerical solutions, however for more complicated flows, it is not possible to find an exact solution. In this section we discuss methods for approximating derivatives and implementing these methods in Matlab to solve governing equations.

4.1 Finite Difference Approximations

The method of finite differences can be used to approximate derivatives of a function when the domain is split into a discrete set of points, ie $y_0, y_1, ..., y_N$. It is possible to derive formulas for finite differences from the Taylor expansions of the function at the point y_j:

$$f(y_{j+1}) = f(y_j) + (y_{j+1} - y_j)f'(y_j) + \frac{(y_{j+1} - y_j)^2}{2}\ f''(y_j) + ...$$

Setting $d = y_{j+1} - y_j$ and rearranging the above formula:

$$f'(y_j) = \frac{f(y_{j+1}) - f(y_j)}{d}\ - \frac{d}{2}\ f''(y_j) + ... \tag{26}$$

This gives us a finite difference approximation to the derivative, with the principal error term being of order d. This shows that if the distance between the points in the domain is reduced by a factor x, then the error will be reduced by the same factor x. This is useful for a computational approach, as we can control the error between the numerical and exact solutions.
In a similar way we can find a formula for the second derivative by using Taylor expansions:

$$f(y_{j+1}) = f(y_j) + df'((y_j) + \frac{d^2}{2}\ f''(y_j) + \frac{d^3}{3!}\ f'''(y_j) + ... \tag{27}$$

$$f(y_{j-1}) = f(y_j) - df'(y_j) + \frac{d^2}{2}\ f''(y_j) - \frac{d^3}{3!}\ f'''(y_j) + ... \tag{28}$$

Adding equations (27) and (28) together gives:

$$f''(y_j) = \frac{f(y_{j+1}) - 2f(y_j) + f(y_{j-1})}{d^2}\ + O(d^2) \tag{29}$$

Here the error term is of order d^2, i.e when d is decreased by a factor x, the error is reduced by a factor x^2. It is worth noting that it is possible to increase the accuracy of finite difference approximations by introducing higher order formulas. These are derived from Taylor expansions in a similar way to before, for example:

$$f'(y_j) = \frac{f(y_{j+1}) - f(y_{j-1})}{2d} + O(d^2) \tag{30}$$

While these are more accurate, they can prove problematic if the function is not defined outside of the discrete set of points in the domain. Nevertheless, we have occasionally used higher order formulas in our numerical solutions to increase the accuracy of the final result.
In the following examples, we show how Matlab can be used to generate approximations to first and second derivatives of the *sin* function. We also show how the error varies as we reduce the size of the difference between the points in the domain, d. [7]

Example 1 In the following example, we aim to find the finite difference approximation to the first derivative of $f = sin(x)$. We set $y_0 = 0$ and $y_N = 2\pi$ and to investigate how accurate the approximation is, we vary N, where

$$d = \frac{2\pi - 0}{N}$$

Below (Figure 7) are the plots of the finite difference approximations against the true value of the derivative, $cos(x)$, and one can clearly see that as N increases (as d decreases), the approximation becomes extremely accurate.

The final plot (Figure 8) is the error plotted against d, and it is as we would expect from our finite difference formula; as you decrease d by factor x, you decrease the error by factor x (roughly). The graph is not a perfectly straight line due to the additional error terms in the finite difference formula.

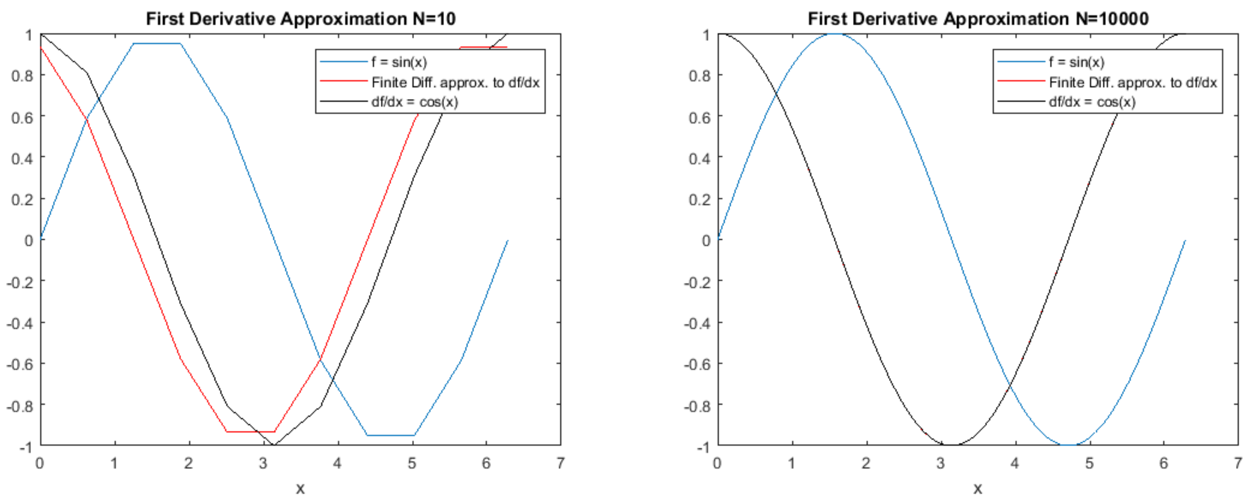


Figure 7: Graphs of first derivative approximations for different values of N

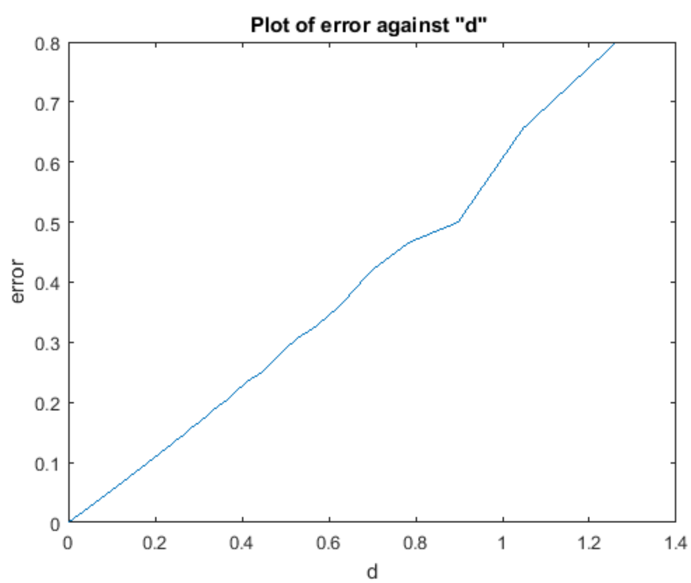


Figure 8: Plot of error for varying d

Example 2 This is a continuation of the previous example, and here we aim to approximate the second derivative of $f = sin(x)$. Using the same domain as Example 1, we use the second order finite difference formula, equation (29) as outlined in the previous section:
This formula was implemented into Matlab using a *for* loop.
Below we see the plots of the approximation and the exact solution (Figure 9), as well as the error against d (Figure 10). As is expected, the finite difference approximation becomes more accurate as N increases, and the error decreases in the predicted fashion. The finite difference formula outlined in the previous section shows the error is of order d^2, so as d is decreased by factor x, the error is decreased approximately by factor x^2; the roughly parabolic shape verifies this.

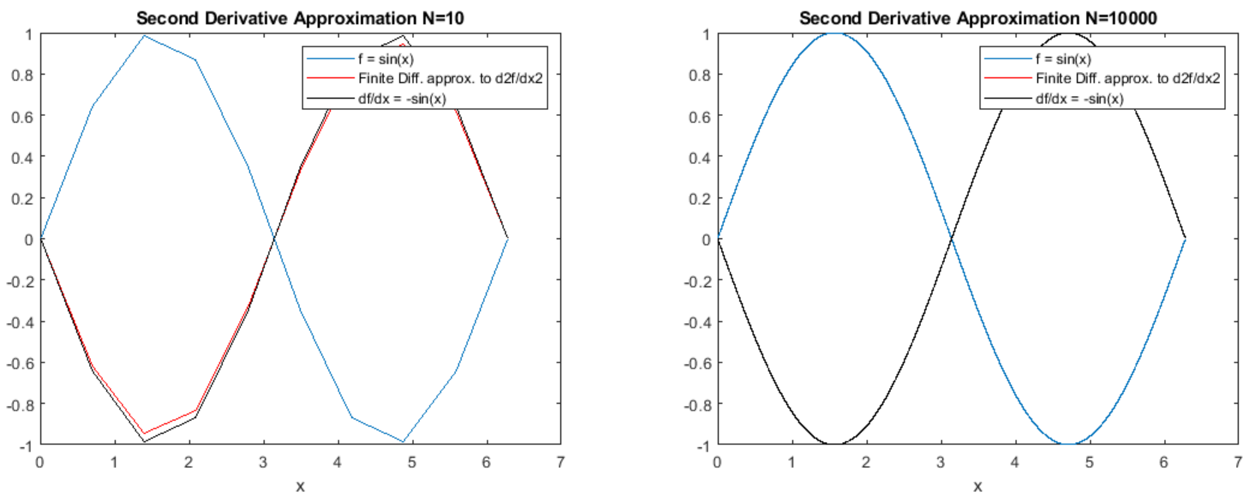


Figure 9: Graphs of second order derivatives for different values of N

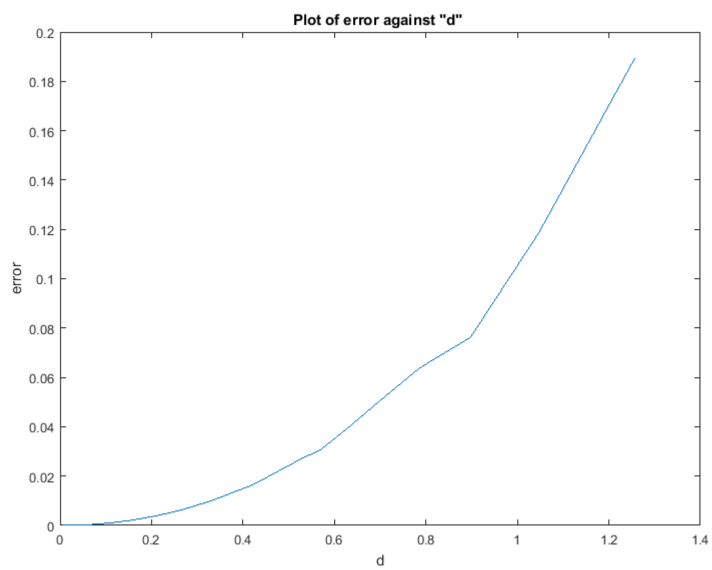


Figure 10: Plot of error for varying d

4.2 Oscillating Plate Problem: Newtonian Case

Earlier in the report we discussed the solution to a plane Couette flow between two oscillating plates of height H apart, and the velocity was found to satisfy the following diffusion equation [5]:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{31}$$

where

$$\nu = \frac{\mu}{\rho}$$

with boundary conditions:

$$u(y = 0) = u(y = H) = U_0 cos(\omega t)$$

We now aim to solve this equation numerically using Matlab, using the inbuilt differential equation solver *ode15s*. This will allow us to move onto a more complicated Non-Newtonian case in the next section, where we consider a power-law fluid. To solve this equation numerically we use finite difference approximations to split it up into a system of ODE's in the following way:

$$\frac{du(y_j)}{dt} = \nu \frac{u(y_{j+1}) - 2u(y_j) + u(y_{j-1})}{d^2} \tag{32}$$

$$d = y_j - y_{j-1} \qquad j = 1, ..., N \qquad y_0 = 0$$

Here we have split up y into the set of discrete points, $0, y_1, ..., y_{N-1}, y_N$ with $y_N = H$. We can vary N to increase the level of accuracy of the solution; in the previous section we showed how the error of the second derivative approximation varies as you increase N or equivalently decrease d.
We can then set up a Matlab to work out $\frac{du}{dt}$ in terms of $u(y_j)$, which can be done by using a *for* loop. To impose the boundary conditions, the first and last component of the $\frac{du}{dt}$ vector were set to $-U_0\omega sin(\omega t)$ (the derivative with respect to t of the boundary conditions).
To use *ode15s*, an initial condition, $u(t = 0)$ needs to be specified. We can use the analytical solution found in a previous chapter to find this, and it can be expressed in the form:

$$u(y, t = 0) = \Re\left(\frac{sin(ky)}{sin(kH)} + \frac{sin(k(H - y))}{sin(kH)}\right) \tag{33}$$

with

$$k = \frac{i + 1}{\sqrt{2}}\sqrt{\frac{\omega}{\nu}}$$

To find the initial condition, we simply evaluate the above equation at each point, y_j in the domain $[0, H]$.
The final step is to specify the range of time over which *ode15s* is to solve the equation, and input $\frac{du}{dt}$ and u_0.

4.2.1 Comparison of Numerical and Analytical Solutions, $N = 100$

Since the exact solution for this equation is known, it is possible to compare it to the *ode15s* solution and observe how the error varies between them. Figure 11 below shows the numerical solutions for $t = [0, 10]$; initially the parameters are set as follows:

$$\nu = 1, N = 100, \omega = 1, H = 10, U_0 = 1$$

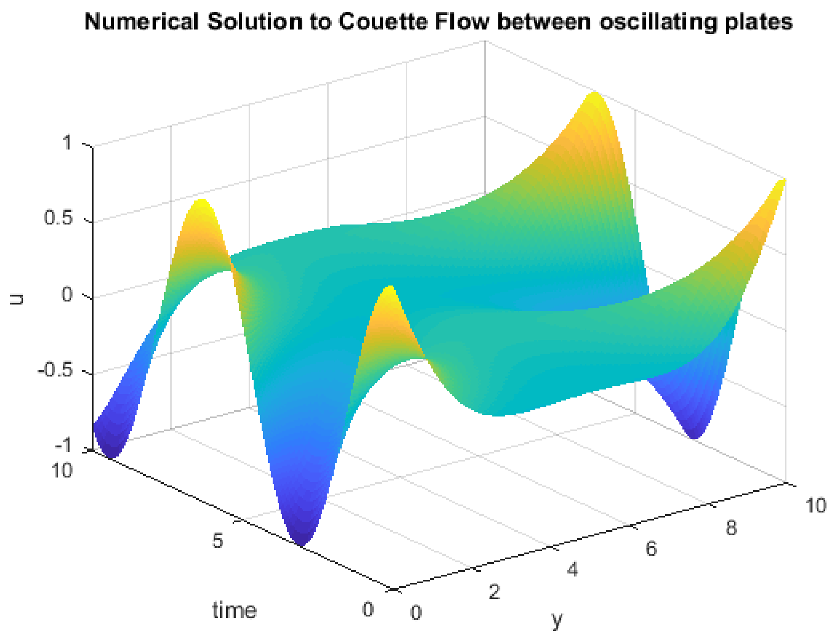


Figure 11

The surface plot below, Figure 12 is the exact solution for the same parameters and range of time. By inspection, the numerical and exact solutions appear the same, and the error between them is considered next.

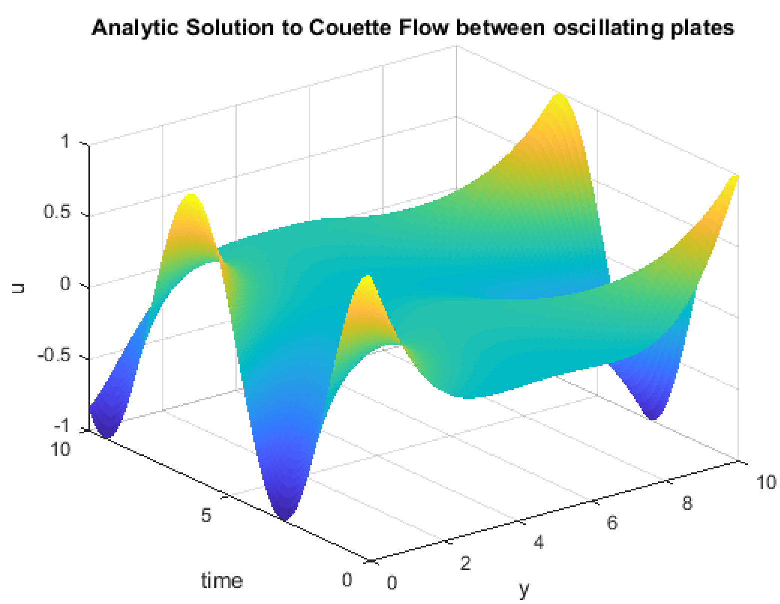


Figure 12

4.2.2 Error between Numerical and Analytical Solutions when $N = 100$

Figure 13 shows how the error varies at each point of the solution. The initial spike in error is due to the limitations of the *ode15s* solver; the first solution it calculates has a larger numerical error, which is dissipated out as time progresses. [8]

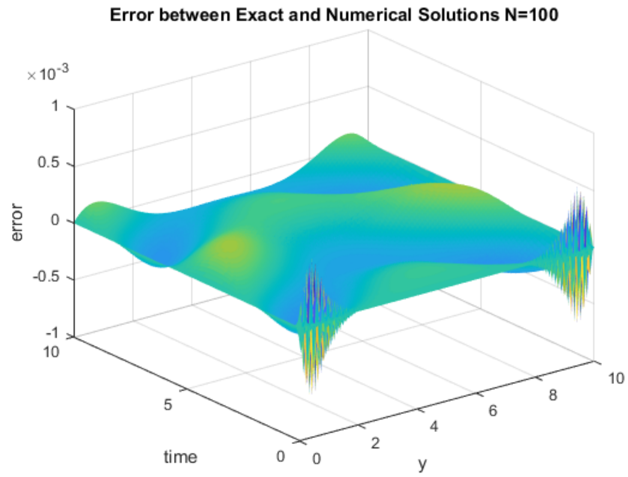


Figure 13

4.2.3 Comparison of Numerical and Analytical Solutions for increasing N

From our knowledge of the finite difference approximations, we can control the value of the error, up to the accuracy of *ode15s*. For this equation, the minimum error was found to be of the order 10^{-6}. Figure 14 shows how error decreases as we decrease d, and it is roughly parabolic, which is expected from the finite difference error, discussed in the previous subsection.

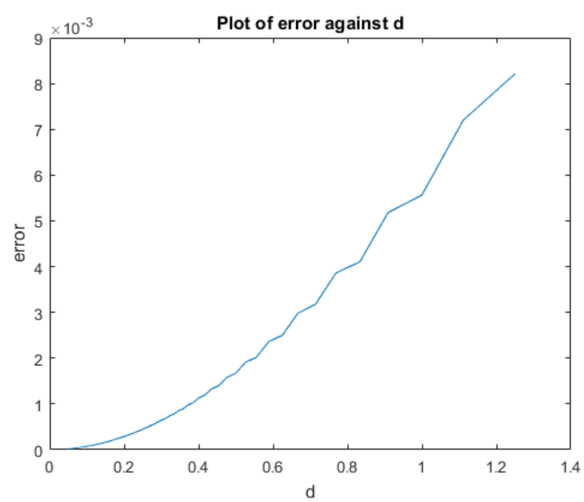


Figure 14

4.2.4 Varying Parameters

Here we investigate how varying the parameters affects the surface plot of $u(y,t)$. The parameter ω simply determines the frequency of oscillation of the boundary plates; U_0 varies the amplitude of these oscillations; H is the distance between the plates.
In this case, the fluid density ρ is constant, so varying ν is equivalent to varying the viscosity, μ. As μ is increased, the oscillations at the boundary have a greater effect on the the fluid in the centre of the channel, so the oscillations spread across a greater range of y. Likewise, when μ is decreased, the boundary oscillations have less of an effect on the fluid, so the fluid appears 'flatter' in the centre of the channel. The plots below (Figure 15) show four different values of ν, while keeping the other parameters as follows;

$$N = 100, \omega = 1, H = 10, U_0 = 1$$

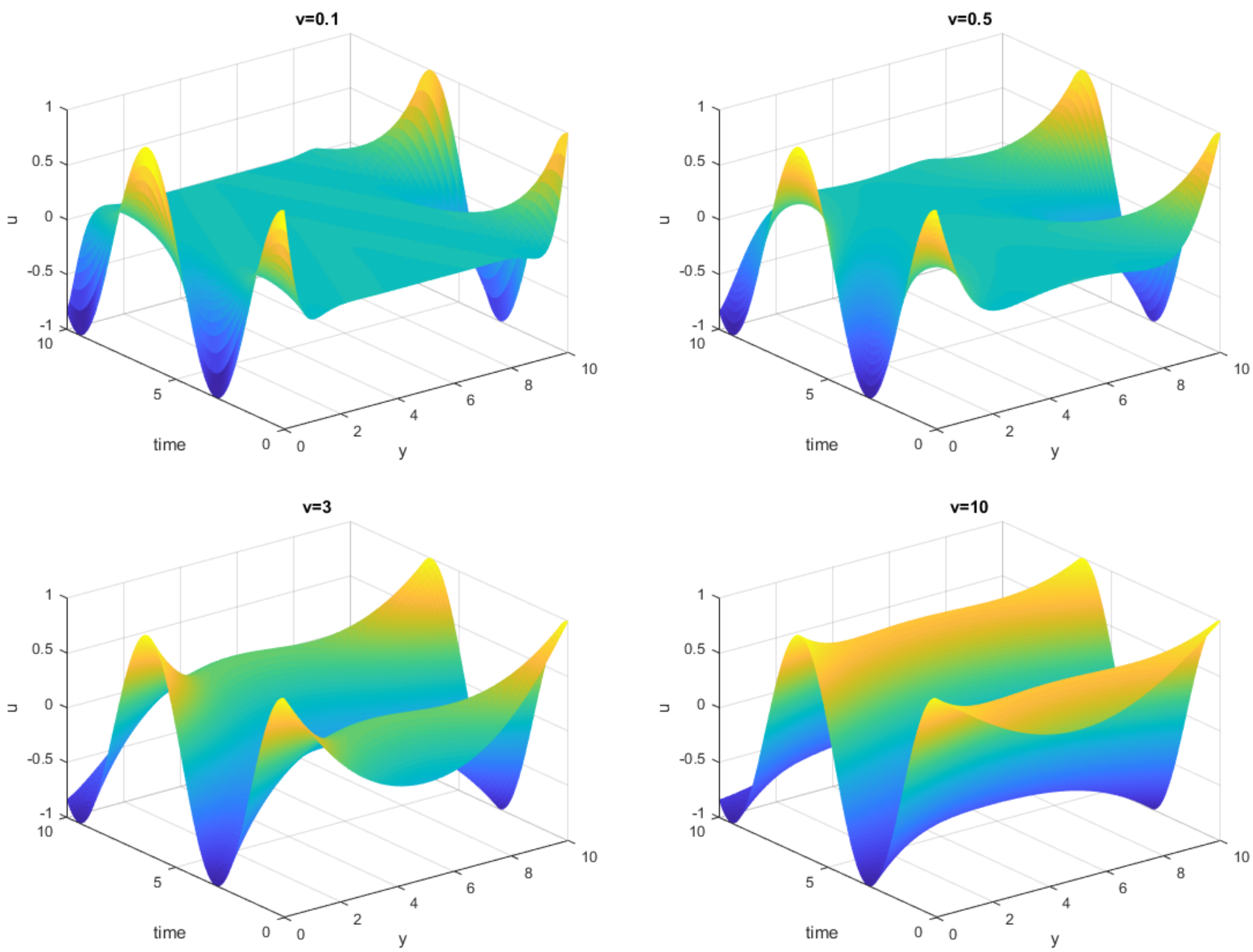


Figure 15: Surface plots for varying fluid viscosity

This concludes the Newtonian case; in the following section we discuss the equivalent situation for the Non-Newtonian case using the power-law model.

5 Numerical Investigation of Non-Newtonian Fluids

5.1 Governing Equation for the Power-law Model

Considering a 2-D fluid, with $\boldsymbol{u} = (u(y,t), 0)$, the Naiver-Stokes Equations can be simplified to be

$$\frac{\partial u}{\partial t} = \frac{\mu}{\rho}\frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right)$$

where ρ is the density of the fluid.
To solve this equation numerically, we introduce a new variable $v = \left|\frac{\partial u}{\partial y}\right|^{n-1}$, and the above equation could be rearranged to

$$\frac{\partial u}{\partial t} = \frac{\mu}{\rho}\left(v\frac{\partial^2 u}{\partial y^2} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\right)$$

Then we apply finite differences to the right-hand side to convert this PDE to a system of $N+1$ ODEs following a similar fashion as we do in the Newtonian case.

$$\frac{du(y_j)}{dt} = \frac{\mu}{\rho}\left(v_j\frac{u(y_{j+1}) - 2u(y_j) + u(y_{j-1})}{d^2} + \frac{v_{j+1} - v_{j-1}}{2d} \times \frac{u(y_{j+1}) - u(y_{j-1})}{2d}\right)$$

$$j = 1 \to N-1$$

where d is defined as in the Newtonian case and $v_j = \left|\frac{u(y_{j+1})-u(y_{j-1})}{2d}\right|^{n-1}$.

When $j = 0$ or N, we apply forward and backward difference respectively to calculate the first order derivative.

With the same initial and boundary conditions as the Newtonian case, fluids with different values of n ($n \geq 1$) are plotted. (Figure 16)

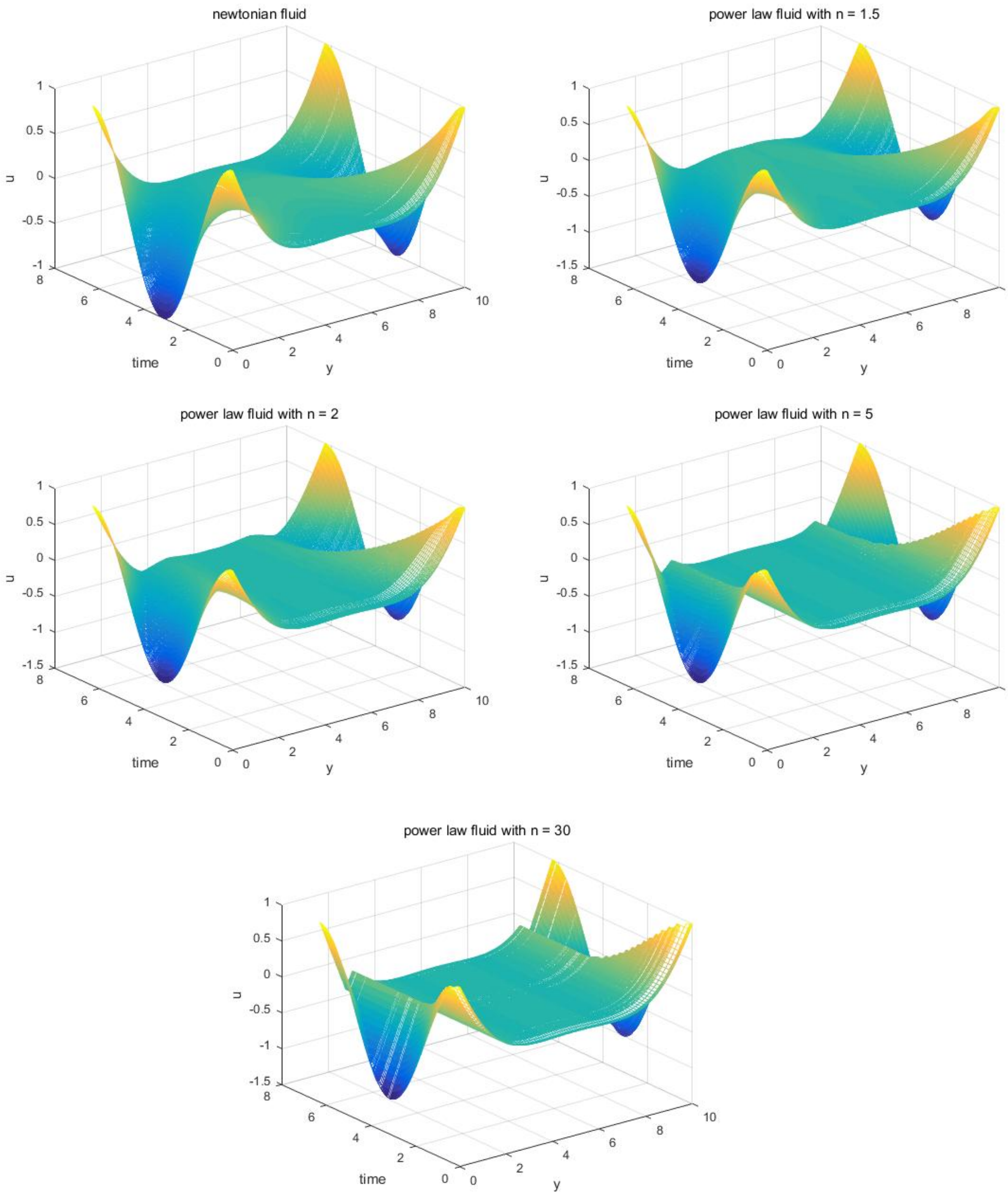


Figure 16: Surface plots for power-law model with $n \geq 1$

As we can see from the figures, near the boundaries of y, $\frac{\partial u}{\partial y}$ is large. Therefore, as n increases, fluid around those locations has very high viscosity and becomes more and more reluctant to follow the periodic movement of the boundaries. This is why we begin to have two ridges when n gets large.

5.2 Limitation of Power-law Model for Shear-thinning Fluids ($n < 1$)

When n was set to be smaller than one, Matlab indicated a math error. My first move was to check the code again, but I did not notice any mistakes. After careful tracing of the evolution of the values of v, it was certain that we needed to adjust the model.

In the power-law model,

$$v = \left|\frac{\partial u}{\partial y}\right|^{n-1} \to \infty, \text{ as } \left|\frac{\partial u}{\partial y}\right| \to 0$$

This means that a shear-thinning fluid acts perfectly like a solid when the shear rate is zero, since viscosity is positively related to v. Once it acts like a solid, it does not deform continuously, and the governing equation will fail to work. In addition, Matlab also fails to cope with very large value of v.

In reality, when the shear rate is small, the viscosity of a shear-thinning fluid does not become infinity, but a large constant instead. To overcome the limitation of the power-law model, an upper limit for v must be set. A robust way to do this is to set a lower limit for $\left|\frac{\partial u}{\partial y}\right|$, thus we adjust the model to be

$$\eta = \mu \left(\left|\frac{\partial u}{\partial y}\right| + a \right)^{n-1}$$

where a is a small positive constant depending on the nature of the fluid.

However, a cannot be too small, since that means the fluid has very high viscosity when the shear rate is close to zero. Thus, a is chosen to be 10^{-4}. Fluids with different values of n ($n < 1$) are then plotted. (Figure 17)

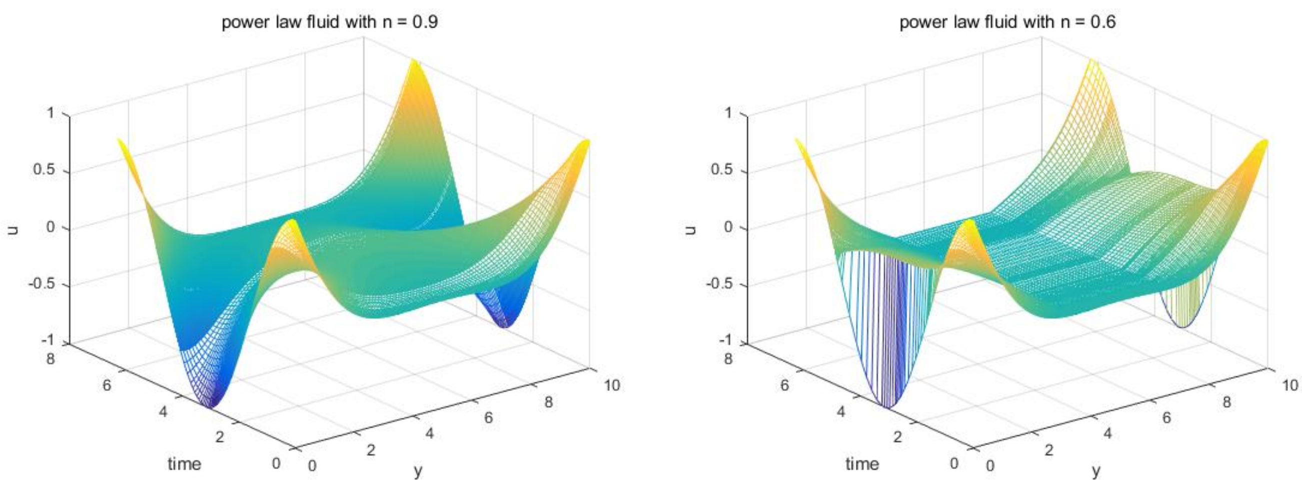


Figure 17: Surface plots for power-law model with $n < 1$

As we can see, the middle part of the above figure is almost flat. This is because those locations are where $\frac{\partial u}{\partial y}$ is very small, leading to more solid-like behaviour, so the fluid keeps the initial velocity throughout the entire period.

Near the boundaries, the fluid has an interesting property. At the beginning, $\frac{\partial u}{\partial y}$ is quite large, so the fluid follows the boundary conditions. When $t \approx \pi$, $\frac{\partial u}{\partial y}$ becomes very small. Thus, instead of following the boundary conditions to have a negative velocity like a Newtonian fluid, it maintains positive velocity. Finally, $\frac{\partial u}{\partial y}$ becomes large again, so the fluid begins to follow the boundary conditions again.

5.3 Alternative Model for Shear-thinning Fluids

An alternative model to deal with shear-thinning fluids is Carreau-Yasuda[6] :

$$\eta = \eta_\infty + (\eta_0 - \eta_\infty)\left[1 + \left(\tau \frac{\partial u}{\partial y}\right)^2\right]^{(n-1)/2}$$

where τ is the relaxation time. Thus $\eta \to \eta_\infty$ when $\frac{\partial u}{\partial y} \to \infty$, and $\eta \to \eta_0$ when $\frac{\partial u}{\partial y} \to 0$.

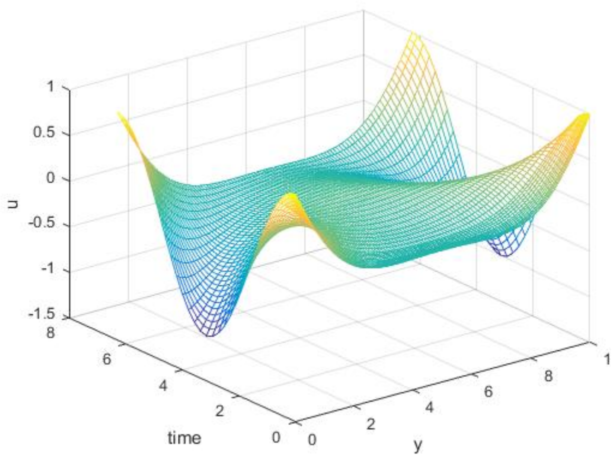
The governing equation now becomes

$$\frac{\partial u}{\partial t} = \frac{1}{\rho}\left(v \frac{\partial^2 u}{\partial y^2} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\right)$$

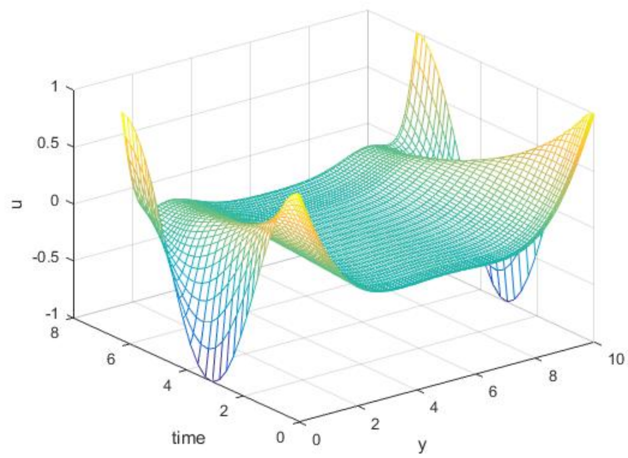
where $v = \eta_\infty + (\eta_0 - \eta_\infty)\left[1 + \left(\tau \frac{\partial u}{\partial y}\right)^2\right]^{(n-1)/2}$.

Using the same numerical method as above and choosing $n = 0.9$, different values of η_0 and $\eta_\infty$are chosen to plot the graph, under the same initial and boundary conditions. (Figure 18)

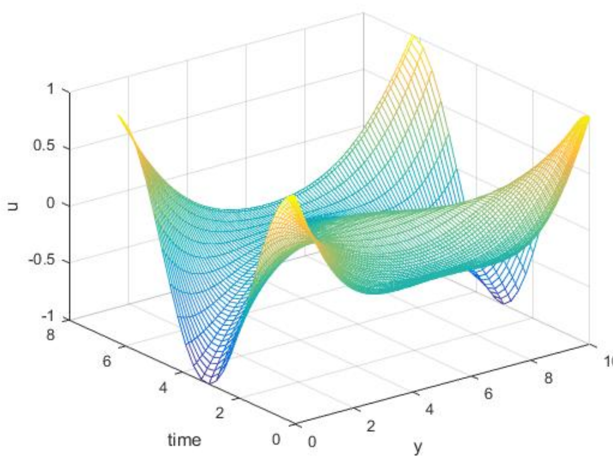
$\eta_0 = 1.1\,, \eta_\infty = 1$:



$\eta_0 = 0.1\,, \eta_\infty = 0$:



$\eta_0 = 2.1\,, \eta_\infty = 1$:



$\eta_0 = 25\,, \eta_\infty = 1$:

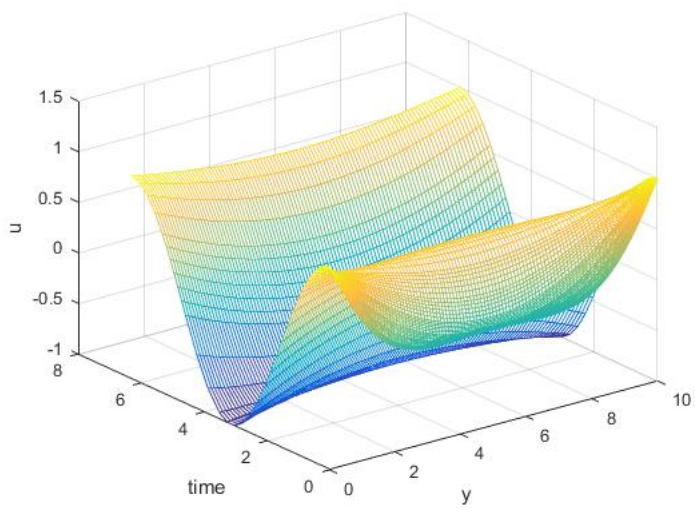


Figure 18: Surface plots for Carreau-Yasuda model

6 Conclusion

Non-Newtonian fluids are incredibly important in our lives and we come into contact with them all the time; from toothpaste, to the honey in our kitchens. They act very differently to Newtonian fluids and display interesting properties which contradict our intuition; it is possible to walk on custard! Further to this, they are useful for a wide range of tasks; a few examples of these fluids are glue, cosmetics, asphalt and paint. With these applications of Non-Newtonian fluids in mind, it was very interesting to try and understand some of the mathematics that governs their behaviour.

Having started with simple definitions and the Navier-Stokes equation for Newtonian fluids, we developed the existing equations to further our understanding of viscoelastic fluids. We have attempted to solve various flow models analytically in both the Newtonian and Non-Newtonian cases, considering a range of boundary conditions and how the solutions change with varying parameters.

We then went on to discuss solving these models from a numerical approach using Matlab. For the Newtonian case it was possible to compare these generated solutions to the exact solutions allowing us to consider the error between them. It was possible to see how our solutions changed by varying parameters and looking at the generated surface plots.

To develop this theory further, one could use different models such as the Bingham Yield fluid or Giesekus model. It would also be interesting to apply our methods to solve solutions for different flows, for example considering Poiseuille or Extensional flow.

Finally, we would like to thank Dr. Ray for his help and guidance in developing our understanding of this fascinating subject.

7 References

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