MODELLING OF EPIDEMICS ON GRAPHS.

By Manuel Ehrlich

Why graphs? For the contact structure.

What is a graph? A set of <u>nodes</u> V and a set of <u>edges</u> E connecting various nodes. Each node v has a <u>node degree</u>: the number of edges connecting node v to other edges in the graph. If $u \rightarrow v_1 \rightarrow \ldots \rightarrow v_k$ exists and is the shortest path between nodes u&v_k, then we say that the <u>(minimal) distance</u> between u and v_k is k.

Examples of graphs?

- <u>Random graphs</u>. Given N nodes, we assign edges between any two nodes at random.
- <u>Lattice:</u> a 2D grid.

How do we want our graph to be? What properties do “real graphs” possess?

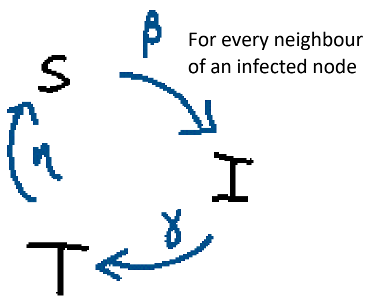
- <u>Geometry</u>
- <u>Sparsity</u>.
- <u>Clustering</u>. The friend of my friend is likely my friend.
- <u>Giant component</u>. Everyone is connected to everyone.
- Node <u>degree distribution</u>. Node degrees are not all equal, they follow a power law distribution.
  $p(k) = C \cdot k^{-(\tau-1)}$
- <u>Small world</u> phenomena. All k-th moments of node degrees are infinite, for $k > \tau - 1$. Average distances in the graph are small, less than log(N), where N is the graph size. If $2 < \tau < 3$, the variance of the node degree is infinite, and the average distance is less than log(log(N)). Such case is called <u>ultra-small world.</u> For social network that value is less than 10. That means you can “reach” any person in the world with less than 10 acquaintances of acquaintances.

THE MODEL

The epidemic is modelled as an ODE model for comparison and on three different types of graphs.

- A lattice (2D grid on a torus, so that there’s no boundary behaviour). It is geometric, but all nodes have the same degree.
- The <u>configuration model</u> (a random graph, which is constructed such that the nodes have a power law degree distribution). It has clusters, the small world (or ultra-small world) property, but it has no geometry.
- The <u>Geometric inhomogeneous random graph</u> (GIRG), a mixture between the previous two. It has all the properties we require. Construction: given N nodes, each node u is assigned a weight w_u according to a power-law distribution, and a uniformly random position x_u. Given nodes u, v, values w_u, x_u, x_v, w_v and parameter, the likelihood that an edge is assigned between u and v is: $Prob(edge\ between\ u\ and\ v) = \left(\frac{w_u \cdot w_v}{dist(x_u,\ x_v)^2}\right)^{\alpha}$. Increasing α we decrease the probability of having long edges.

SITS model: $\begin{cases} s' = -\beta \cdot s\, i + \eta \cdot t \\ i' = \beta \cdot s\, i - \gamma \cdot i \\ t' = \gamma \cdot i - \eta \cdot t \end{cases}$



On graphs, each node can be in one of three states: susceptible (S), infective (I) and temporary immune (T). Each (discrete) time step each infective node can infect each susceptible neighbour with probability β, it can itself heal with probability γ; temporary immune nodes can lose their immunity with probability η.

As already seen, in the ode model, the disease either dies out if $R_0<1$, or it has an initial exponential increase, followed by damped oscillations until it reaches stability.

The graphs, having a stochastic nature, allow the disease to die out with non-zero probability also for $R_0>1$. In addition, all three graphs have an additional possible outcome: if η (loss-of-immunity parameter) is big enough there's one peak and then the disease dies out (SIR situation). There is a sharp transition of parameters between the three possible phases.

- Note: ODE, Configuration model & GIRG (with many hubs and long connections) have all initial exponential increase, when $R_0>1$. In the lattice & GIRG (without many hubs and long connections) it's linear.

RESTRICTION MODELLING

- Travel restrictions. Decrease long-range connections by:
  1. Increasing parameter α.
  2. For each edge between nodes u, v, set a cut-off value L and if $L < d(x_u, x_v)$ then we delete the edge.
- Set a maximum number of contacts per individual. → Set a maximal node degree M for all nodes v that have a higher node degree we randomly delete edges connected to v until its degree is within M.
- Keeping physical distance → randomly remove edges from the graph.

RESULTS

All interventions decrease the height of the first peak. On the other hand, they extend the time period of the peak. The critical η is shortened, so it is possible that intervention cause the disease to survive instead of dying out after the first peak. Travel restrictions (2) are the most effective in lowering the first peak.

REFERENCES

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  [The paper I followed for the modelling of the epidemic and the interventions]
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  [Theoretical approach on GIRGs]