Exact quantization methods for Multistage Stochastic Linear Problem

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Computational Management Science

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Multistage stochastic linear programming (MSLP)

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\begin{aligned}
\min_{(\boldsymbol{x}_t)_{t\in[T]}} \quad & \mathbb{E}\Big[\sum_{t=1}^{T} \boldsymbol{c}_t^\top \boldsymbol{x}_t\Big] & \\
\text{s.t.} \quad & \boldsymbol{A}_t \boldsymbol{x}_t + \boldsymbol{B}_t \boldsymbol{x}_{t-1} \leqslant \boldsymbol{b}_t & \forall t \in [T] \\
& \sigma(\boldsymbol{x}_t) \subset \sigma(\boldsymbol{c}_\tau, \boldsymbol{A}_\tau, \boldsymbol{B}_\tau, \boldsymbol{b}_\tau)_{\tau \leqslant t} & \forall t \in [T] \\
& \boldsymbol{x}_0 \equiv x_0 \text{ given} &
\end{aligned}
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$\boldsymbol{\xi}_t = (\boldsymbol{c}_t, \boldsymbol{A}_t, \boldsymbol{B}_t, \boldsymbol{b}_t)_{t\in[T]}$ is assumed to be stagewise independent.

We set $V_{T+1} \equiv 0$ and:

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V_t(x_{t-1}) := \mathbb{E}\big[\hat{V}_t(x_{t-1}, \boldsymbol{\xi}_t)\big] := \mathbb{E}\begin{bmatrix} \min\limits_{x_t \in \mathbb{R}^{n_t}} & \boldsymbol{c}_t^\top x_t + V_{t+1}(x_t) \\ \text{s.t.} & \boldsymbol{A}_t x_t + \boldsymbol{B}_t x_{t-1} \leqslant \boldsymbol{b}_t \end{bmatrix}
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➡ How to deal with continuous distributions ?

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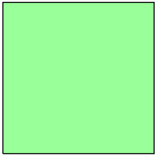
Quantization of a MSLP

Real problem

$$V_t(x) = \mathbb{E}\big[\hat{V}_t\big(x, \boldsymbol{\xi}_t\big)\big] = \mathbb{E}\begin{bmatrix} \min\limits_{y \in \mathbb{R}^{n_t}} & \boldsymbol{c}_t^\top y + V_{t+1}(y) \\ \text{s.t.} & \boldsymbol{A}_t y + \boldsymbol{B}_t x \leqslant \boldsymbol{b}_t \end{bmatrix}$$



$\boldsymbol{\xi}_t$ continuous

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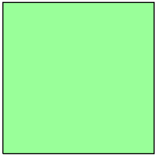
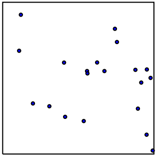


ξ_t continuous

Sample Average Approximation (SAA)

$$V_{t,N}^{SAA}(x) := \frac{1}{N}\sum_{k=1}^{N} \hat{V}_t\left(x, \xi^k\right)$$

$\xi^1, \cdots, \xi^N$ drawn by Monte Carlo



SAA $N = 20$

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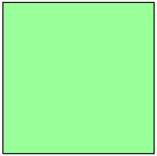

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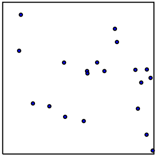

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Partition-based

$$V_{t,\mathcal{P}}(x) := \sum_{P \in \mathcal{P}} \check{p}_{t,P} \hat{V}_t(x, \check{\xi}_{t,P})$$

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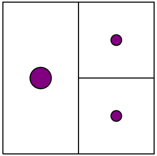

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Quantization of a MSLP

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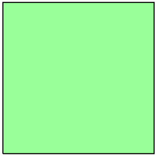

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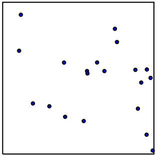

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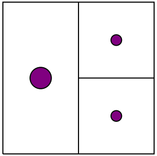
If $\xi \mapsto \hat{V}(x, \xi)$ is convex, $V_{t,\mathcal{P}}(x) \leqslant V_t(x)$.



Partition-based

Exact quantization

Definition

A MSP admits a local exact quantization at time t on x if there exists a finitely supported $(\check{\xi}_t)_{t\in[T]}$ *i.e.* such that

$$V_t(x) = \mathbb{E}\big[\hat{V}_t(x,\xi_t)\big] = \mathbb{E}\big[\hat{V}_t(x,\check{\xi}_t)\big].$$

We call an exact quantization

- uniform if it is locally exact at all $x \in \mathbb{R}^{n_t}$, and all $t \in [T]$.
- universal if there exists a partition $\mathcal{P}_{t,x}$ such that the induced quantization is exact at time t on x, for all distributions of $(\xi_\tau)_{\tau\in[T]}$.

Questions:

1. Under which condition does there exist an exact quantization ?
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A first counter example

Assume $V_{t+1} \equiv 0$ and denote $V := V_t$, $\hat{V} := \hat{V}_t$ and $\boldsymbol{\xi} := \boldsymbol{\xi}_t$ for now.

Let $\boldsymbol{A} = (-\boldsymbol{u})$, $\boldsymbol{B} \equiv (0)$, $\boldsymbol{b} \equiv (-1)$ where $\boldsymbol{u} \sim \mathcal{U}([1,2])$.

$$\hat{V}(x,\xi) = \begin{array}{ll} \min\limits_{y \in \mathbb{R}} & y \\ \text{s.t.} & uy \geqslant 1 \end{array} = \frac{1}{u}$$

By strict convexity, for all partition $\mathcal{P}$

$$\sum_{P \in \mathcal{P}} \check{p}_P \hat{V}\big(x, \check{\xi}_P\big) < V(x) = \mathbb{E}\Big[\frac{1}{\boldsymbol{u}}\Big]$$

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➥ There is no partition-based local, neither uniform or universal, exact quantization result for $\boldsymbol{A}$ non-finitely supported.

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Uniform exact quantization and polyhedrality

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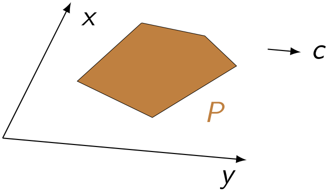

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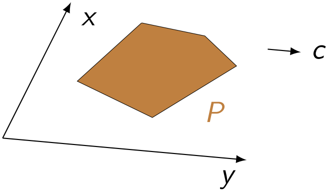

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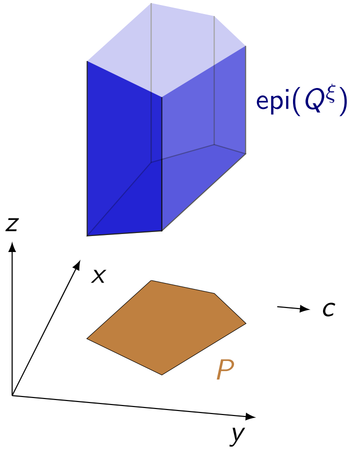
with $Q^\xi(x,y) := c^\top y + \mathbb{I}_{(x,y)\in P}$.



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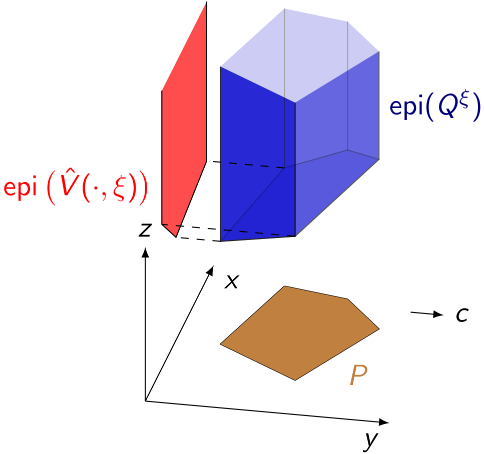

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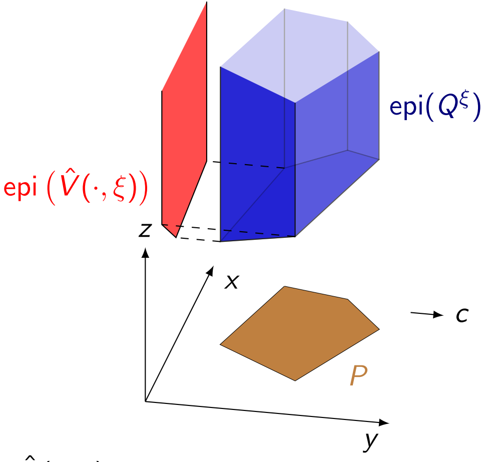

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$$V(x) = \mathbb{E}\left[\hat{V}(x,\boldsymbol{\xi})\right] = \sum_{\xi \in \mathrm{supp}(\check{\boldsymbol{\xi}})} p_\xi \hat{V}(x,\xi)$$

➥ If the noise is finitely supported, then V is polyhedral

Uniform exact quantization and polyhedrality

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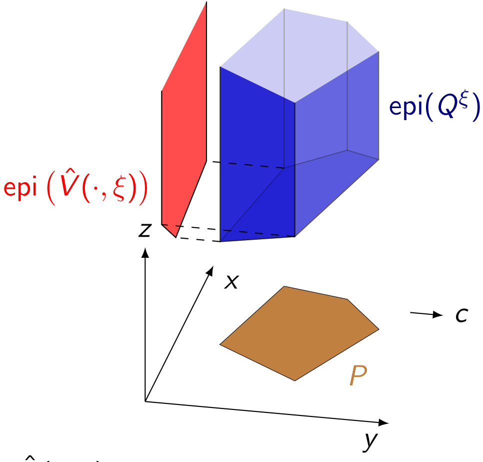

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- ➥ If the noise is finitely supported, then V is polyhedral
- ➥ Existence of uniform exact quantization implies polyhedrality of V.

Counter examples with stochastic constraints

Stochastic $\boldsymbol{B}$

$$
\begin{aligned}
V(x) &= \mathbb{E}\begin{bmatrix} \min\limits_{y\in\mathbb{R}^m} & y \\ \text{s.t.} & \boldsymbol{u}x - y \leqslant 0 \\ & y \geqslant 1 \end{bmatrix} \\
&= \mathbb{E}\big[\max(\boldsymbol{u}x, 1)\big] \\
&= \begin{cases} 1 & \text{if } x \leqslant 1 \\ \frac{x}{2} + \frac{1}{2x} & \text{if } x \geqslant 1 \end{cases}
\end{aligned}
$$

Stochastic $\boldsymbol{b}$

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➥ V is not polyhedral $\Rightarrow$ No uniform exact quantization for non-finitely supported $\boldsymbol{B}$ and $\boldsymbol{b}$.

$\boldsymbol{u}$ is uniform on $[0, 1]$

Counter examples with stochastic constraints

Stochastic $\boldsymbol{B}$

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Stochastic $\boldsymbol{b}$

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➥ V is not polyhedral $\Rightarrow$ No uniform exact quantization for non-finitely supported $\boldsymbol{B}$ and $\boldsymbol{b}$.

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Remaining cases

$$V(x) = \mathbb{E}\begin{bmatrix} \min\limits_{y\in\mathbb{R}^m} & \boldsymbol{c}^\top y \\ \text{s.t.} & \boldsymbol{A}y + \boldsymbol{B}x \leqslant \boldsymbol{b} \end{bmatrix}$$

	$\boldsymbol{A}$	$(\boldsymbol{B}, \boldsymbol{b})$	$\boldsymbol{c}$
Local	×	?	?
Uniform	×	×	?

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Local	×	✓	✓
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Theorem (GAPM, FL 2022)

If A is deterministic,
then there exists a universal and local exact quantization.

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Uniform	×	×	✓

Theorem (GAPM, FL 2022)

If A is deterministic,
then there exists a universal and local exact quantization.

Theorem (Exact quantization, FGL 2022)

If A, B and b are deterministic,
then there exists a universal and uniform exact quantization.

Contents

Reformulation of $V(x)$ highlighting the role of the fiber P_x

For a given x, (we still assume $V_{t+1} \equiv 0$)

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Illustrative running example:

$$P_x := \{y \in \mathbb{R}^m \mid \|y\|_1 \leqslant 1, \; y_1 \leqslant x, \; y_2 \leqslant x\}$$

Reformulation of $V(x)$ highlighting the role of the fiber P_x

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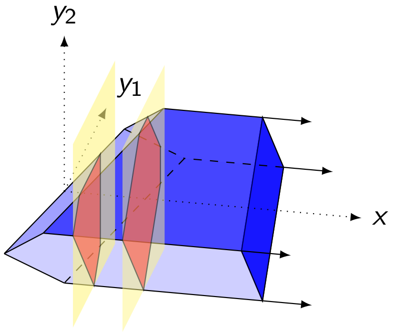

Normal fan $\mathcal{N}(P_x)$

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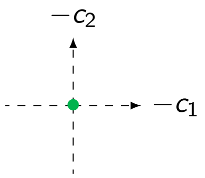
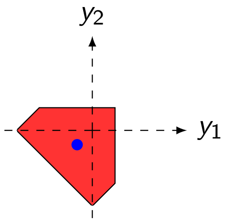


$N_{P_x}(y)$ for $x = 0.3$



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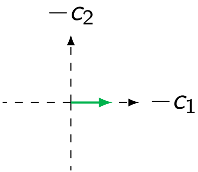

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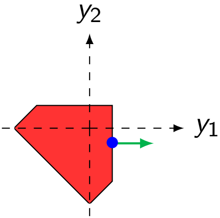

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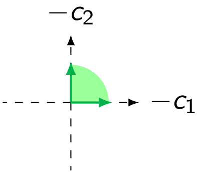

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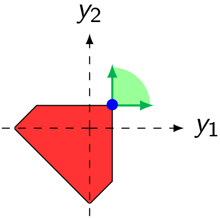

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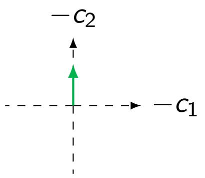

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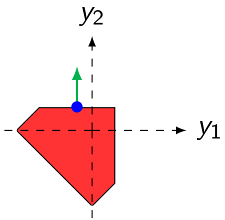

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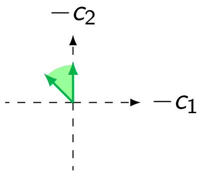

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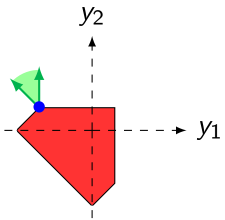

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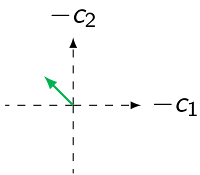

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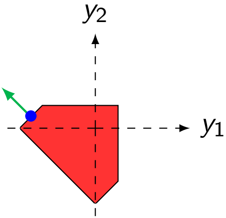

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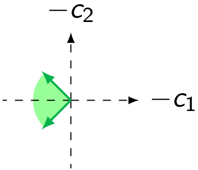

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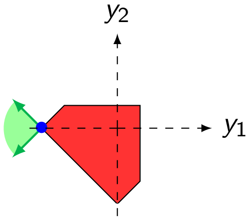

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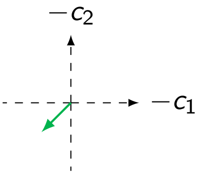

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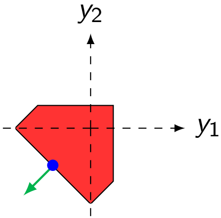

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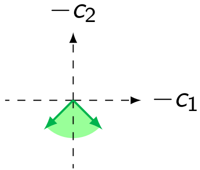

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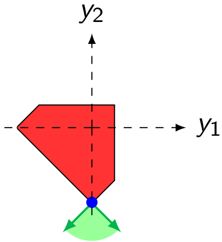

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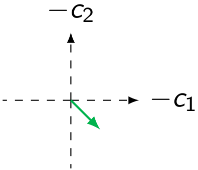

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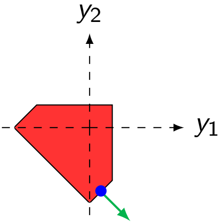

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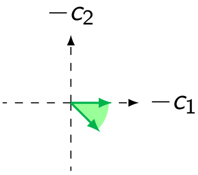

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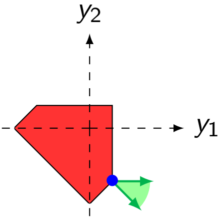

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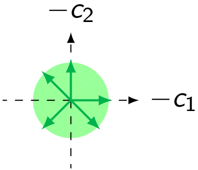
Proposition

If P_x *is bounded,* $\{\mathrm{ri}(N) \mid N \in \mathcal{N}(P_x)\}$ *is a partition of* $\mathbb{R}^m$*.*



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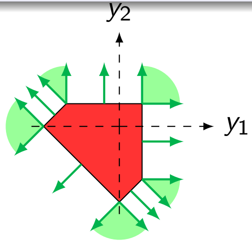

P_x and $\mathcal{N}(P_x)$ for $x = 0.3$

$\mathcal{N}(P_x)$: partition of cost coherent with the min

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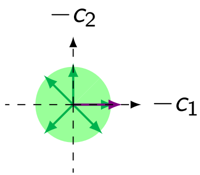
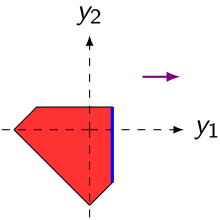
For any $N \in \mathcal{N}(P_x)$, $-c \mapsto \arg\min_{y \in P_x} c^\top y$ is constant for all $-c \in \mathrm{ri}(N)$.



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$



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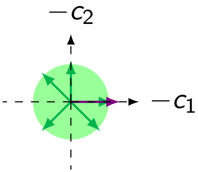

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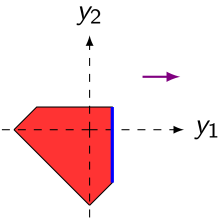

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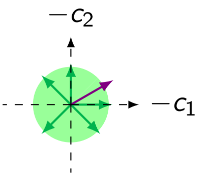

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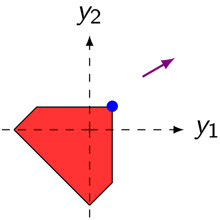

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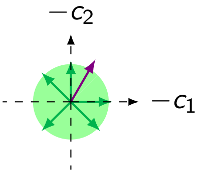

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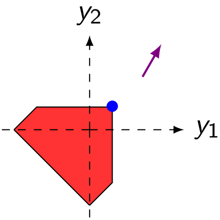

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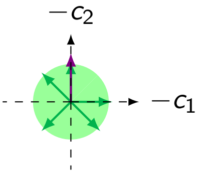

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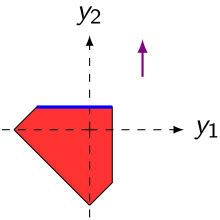

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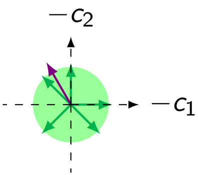

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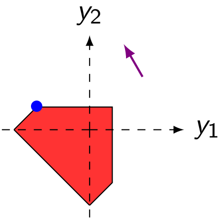

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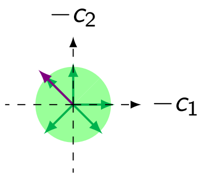

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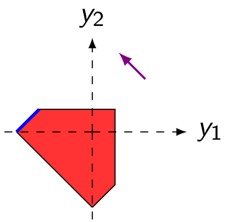

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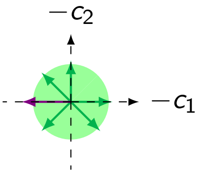

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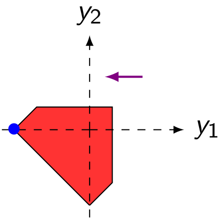

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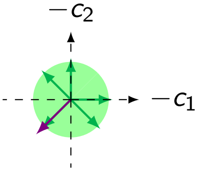

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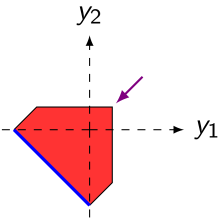

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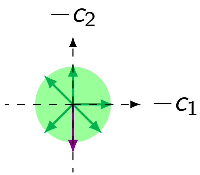
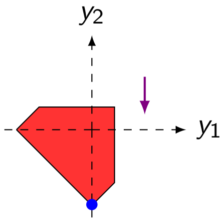
For any $N \in \mathcal{N}(P_x)$, $-c \mapsto \operatorname*{arg\,min}_{y \in P_x} c^\top y$ is constant for all $-c \in \mathrm{ri}(N)$.



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$



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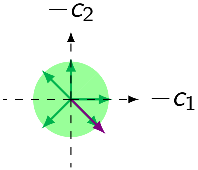

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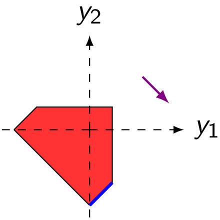

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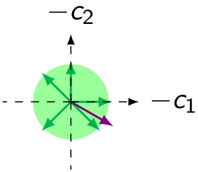

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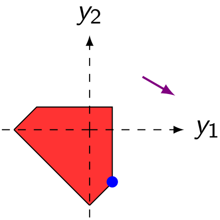

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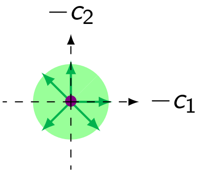

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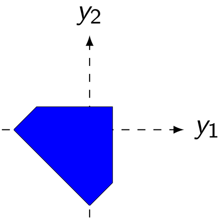

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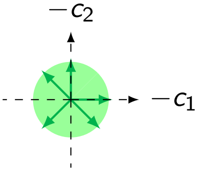

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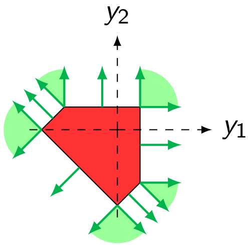
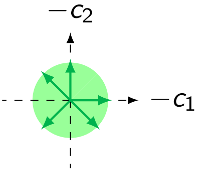


P_x for $x = 0.3$

Local and universal exact quantization for $\boldsymbol{c}$

$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \boldsymbol{c}^\top y\Big]$$
$$= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\Big[\mathbb{1}_{\boldsymbol{c} \in -\operatorname{ri} N} \min_{y \in P_x} \boldsymbol{c}^\top y\Big]$$



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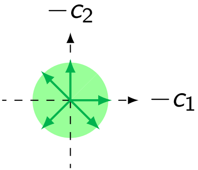
$$= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbb{1}_{\boldsymbol{c} \in -\operatorname{ri} N} \boldsymbol{c}^\top\big] y_N(x)$$

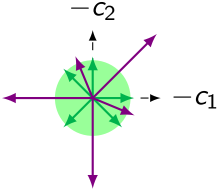


$\mathcal{N}(P_x)$ for $x = 0.3$

Local and universal exact quantization for $\boldsymbol{c}$

$$
\begin{aligned}
V(x) &= \mathbb{E}\Big[\min_{y\in P_x} \boldsymbol{c}^\top y\Big] \\
&= \sum_{N\in\mathcal{N}(P_x)} \mathbb{E}\Big[\mathbb{1}_{\boldsymbol{c}\in -\operatorname{ri} N} \min_{y\in P_x} \boldsymbol{c}^\top y\Big] \\
&= \sum_{N\in\mathcal{N}(P_x)} \mathbb{E}\Big[\mathbb{1}_{\boldsymbol{c}\in -\operatorname{ri} N} \boldsymbol{c}^\top\Big] y_N(x) \\
&= \sum_{N\in\mathcal{N}(P_x)} p_N \check{c}_N^\top y_N(x)
\end{aligned}
$$

where $y_N \in \arg\min_y \underbrace{c^\top}_{\in -\operatorname{ri} N} y$.



$\mathcal{N}(P_x)$ and $p_N \check{c}_N$ for $x = 0.3$

For $N \in \mathcal{N}(P_x)$,

$$
\begin{aligned}
p_N &:= \mathbb{P}\big[\boldsymbol{c} \in -\operatorname{ri} N\big] \\
\check{c}_N &:= \mathbb{E}\big[\boldsymbol{c} \mid \boldsymbol{c} \in -\operatorname{ri} N\big]
\end{aligned}
$$

We replace the continuous cost $\boldsymbol{c}$, by the discrete cost $\check{\boldsymbol{c}}$.

Local and universal exact quantization for $\boldsymbol{c}$

$$
\begin{aligned}
V(x) &= \mathbb{E}\Big[\min_{y \in P_x} \boldsymbol{c}^\top y\Big] \\
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&= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\Big[\mathbb{1}_{\boldsymbol{c} \in -\operatorname{ri} N} \boldsymbol{c}^\top\Big] y_N(x) \\
&= \sum_{N \in \mathcal{N}(P_x)} p_N \check{c}_N^\top y_N(x) \\
&= \sum_{N \in \mathcal{N}(P_x)} p_N \min_{y \in P_x} \check{c}_N^\top y
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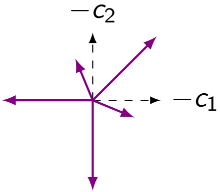
where $y_N \in \arg\min_y \underbrace{c^\top}_{\in -\operatorname{ri} N} y$.



$p_N \check{c}_N$ for $x = 0.3$

For $N \in \mathcal{N}(P_x)$,

$$
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\end{aligned}
$$

We replace the continuous cost $\boldsymbol{c}$, by the discrete cost $\boldsymbol{\check{c}}$.

Contents

$x \mapsto \mathcal{N}(P_x)$ is piecewise constant.

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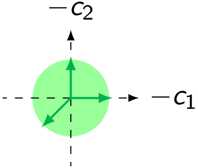
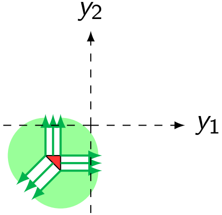
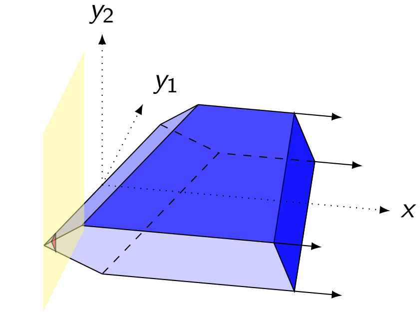


$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



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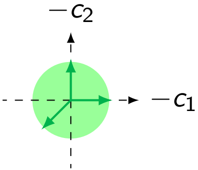

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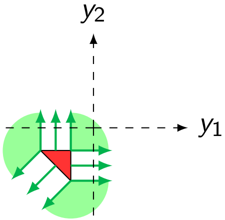

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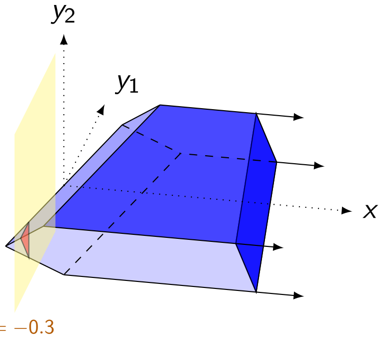

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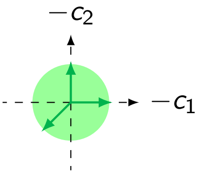

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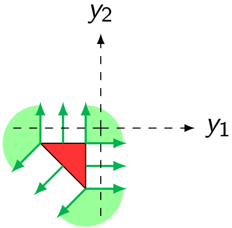

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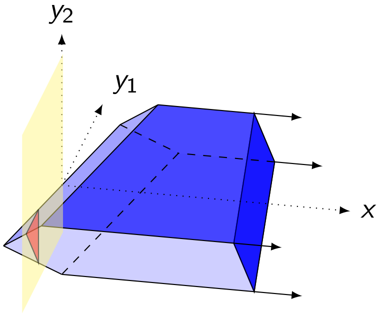

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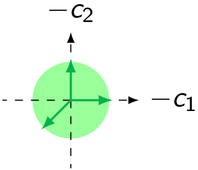

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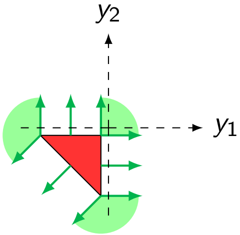

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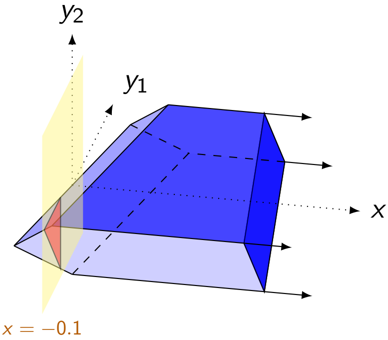

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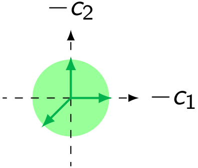

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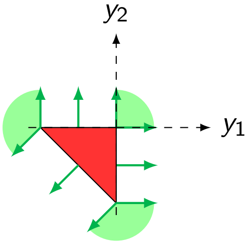

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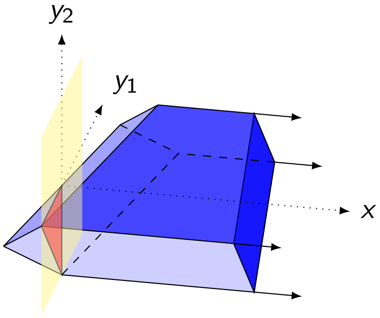

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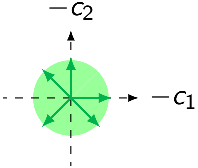

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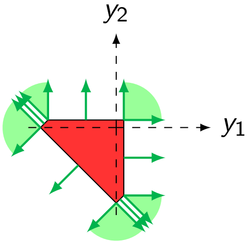

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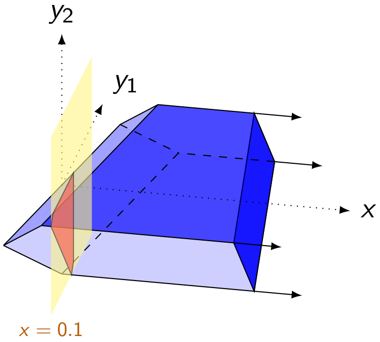

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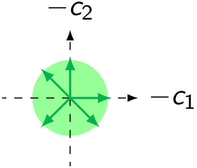

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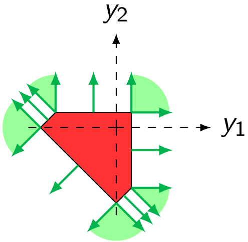

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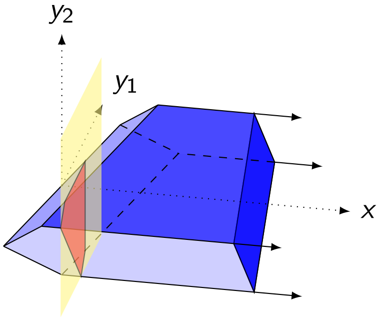

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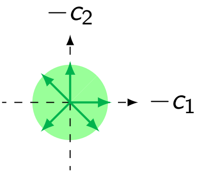

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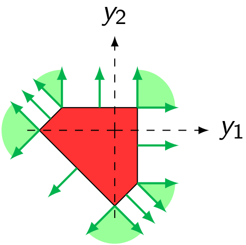

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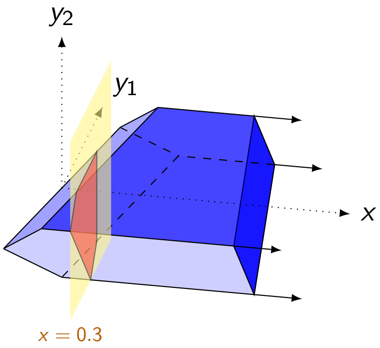

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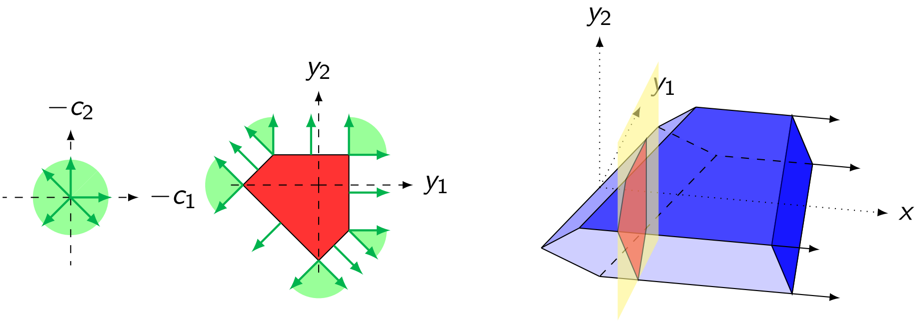

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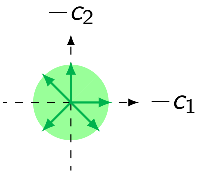

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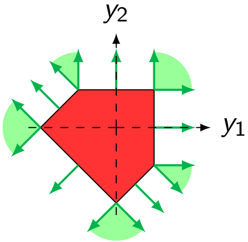

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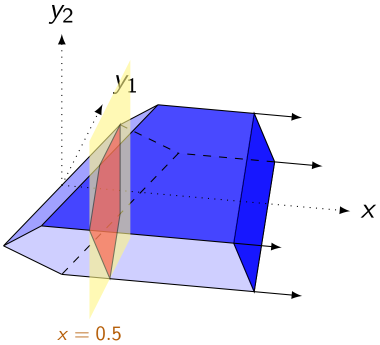

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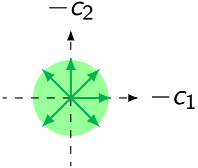

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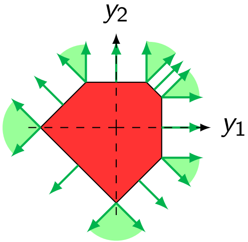

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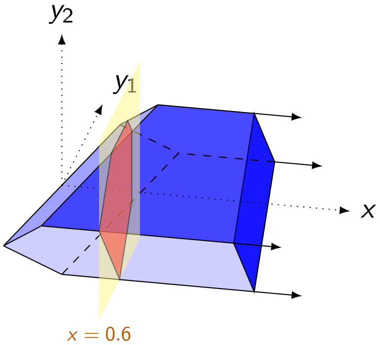

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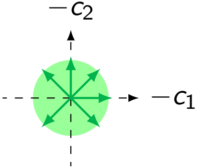

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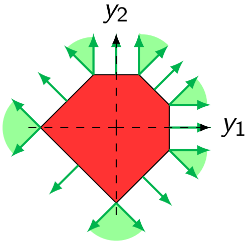

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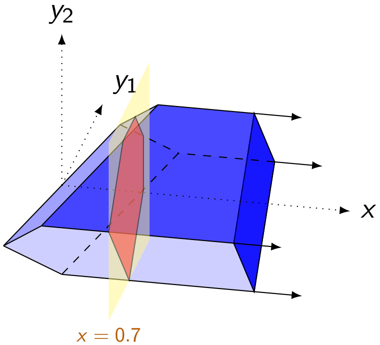

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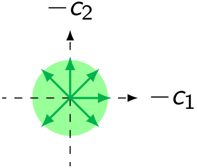

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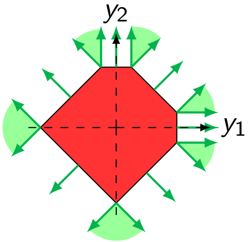

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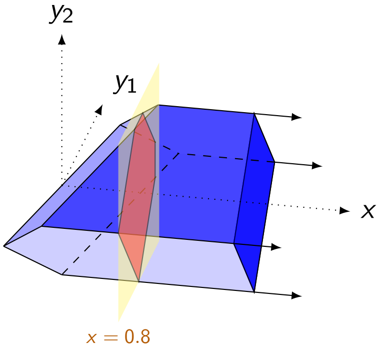

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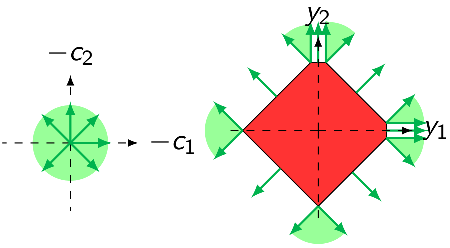
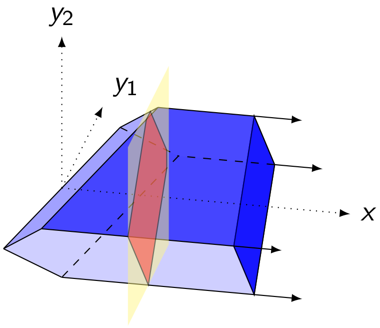


$\mathcal{N}(P_x)$ $\quad$ P_x and $\mathcal{N}(P_x)$



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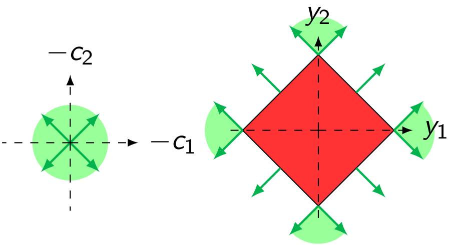

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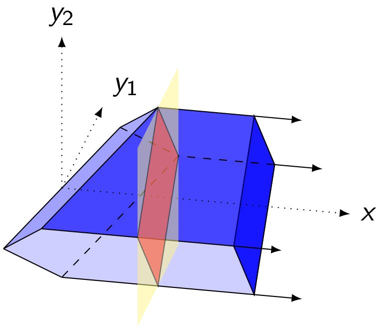

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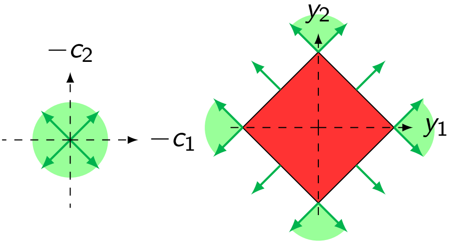

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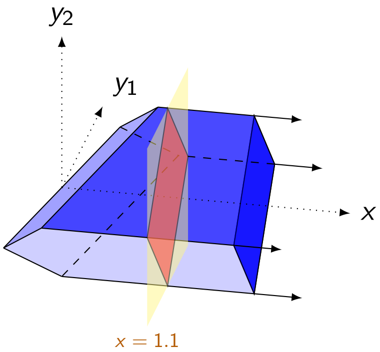

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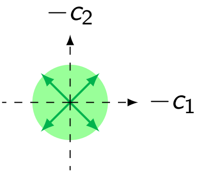

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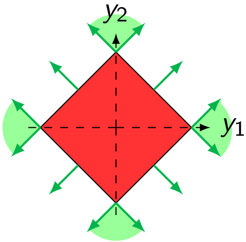

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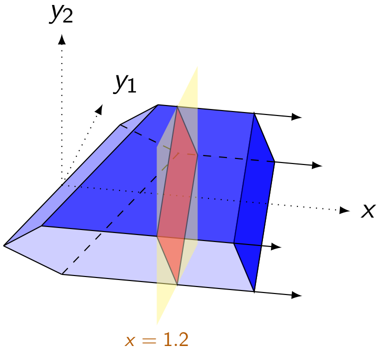

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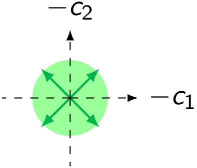

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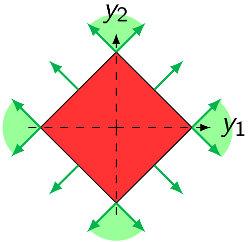

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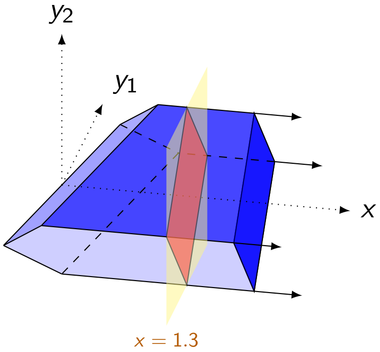

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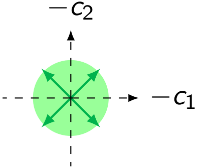

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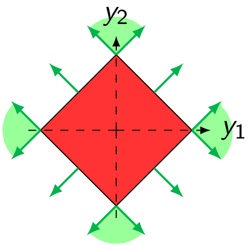

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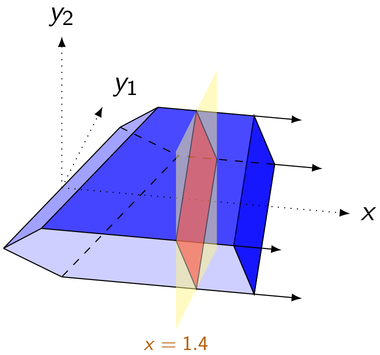

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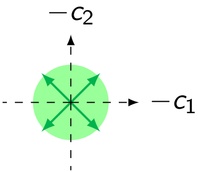

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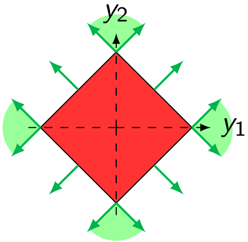

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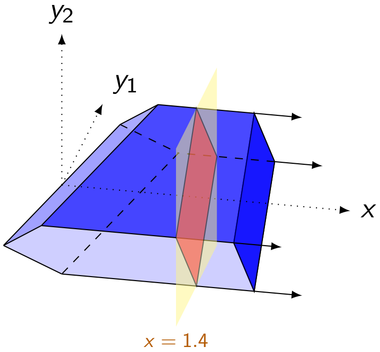

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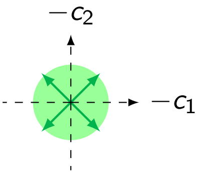

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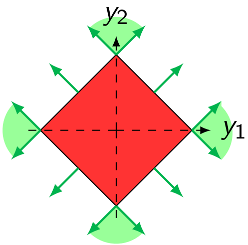

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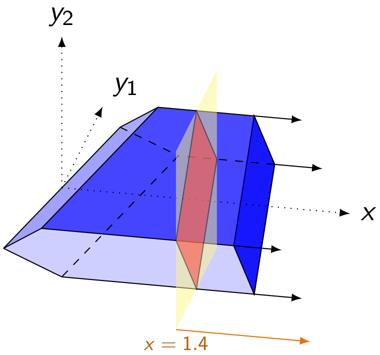

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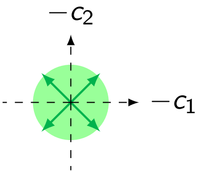

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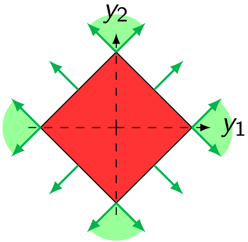

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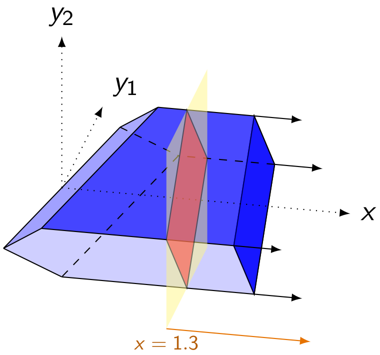

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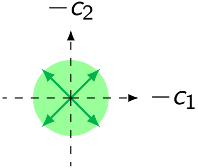

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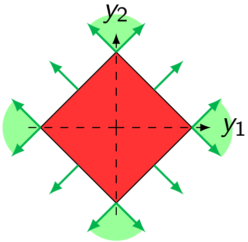

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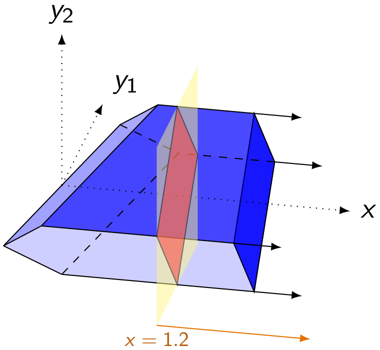

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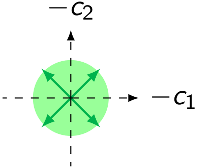

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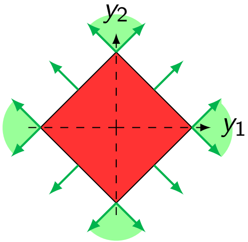

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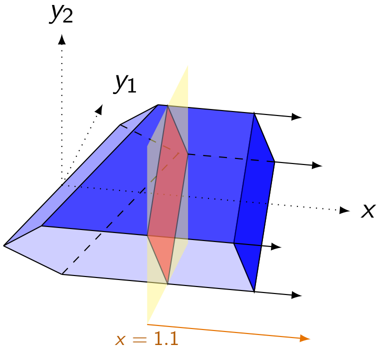

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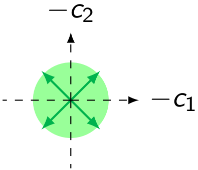

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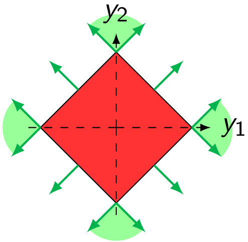

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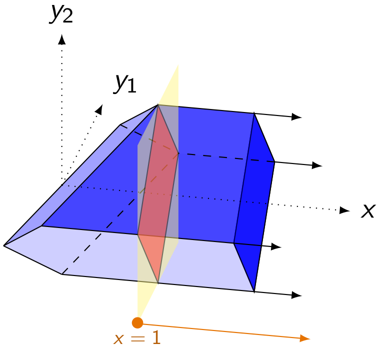

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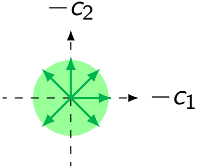

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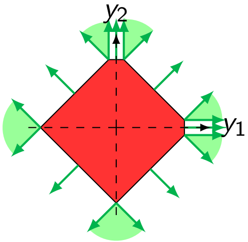

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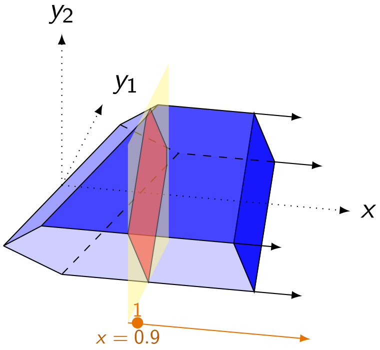

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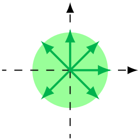

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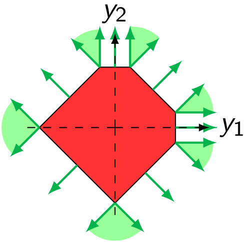

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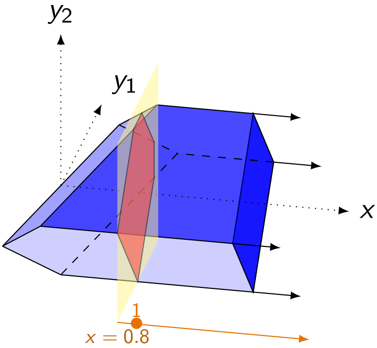

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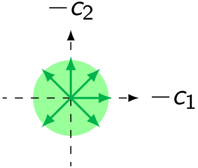

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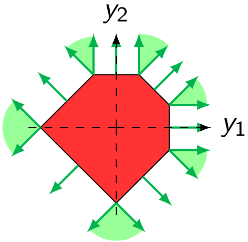

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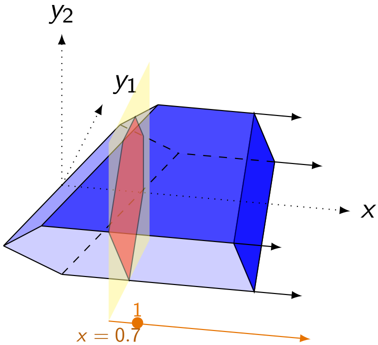

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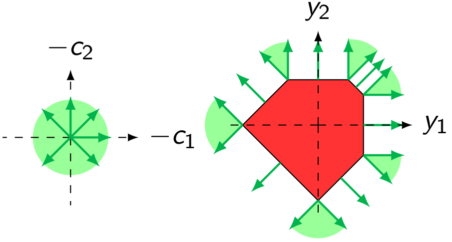

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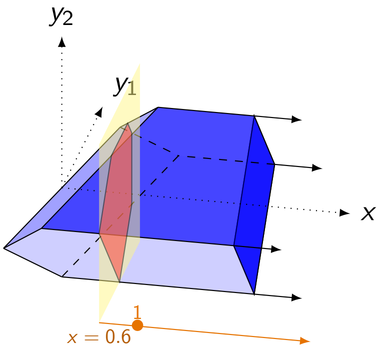

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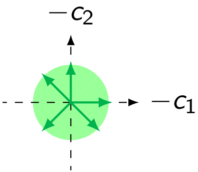

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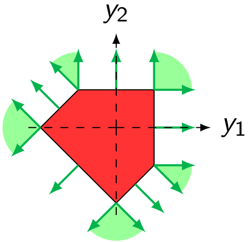

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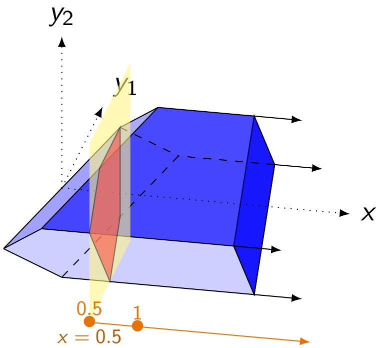

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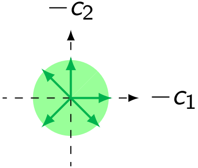

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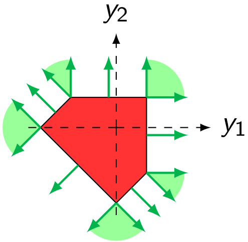

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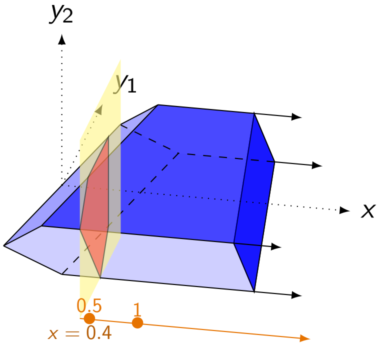

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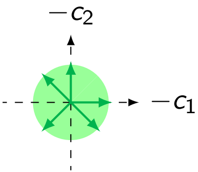

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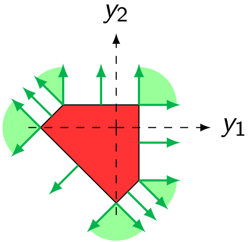

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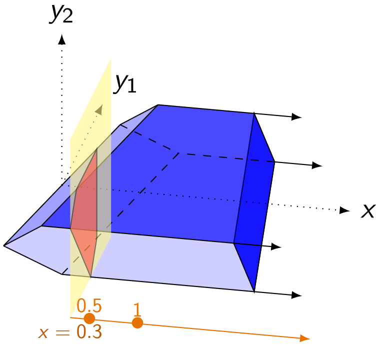

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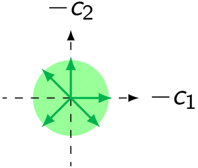

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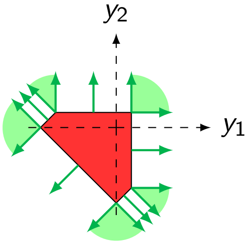

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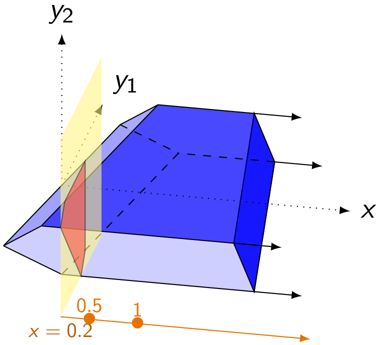
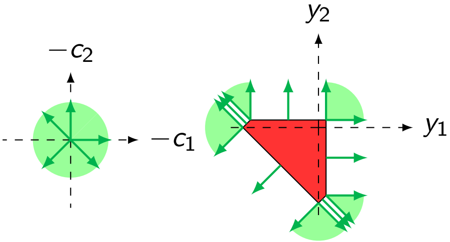


P and P_x

$x \mapsto \mathcal{N}(P_x)$ is piecewise constant.

$$P_x := \{y \mid Ay + Bx \leqslant b\} \quad \text{and} \quad P := \{(x,y) \mid Ay + Bx \leqslant b\}$$



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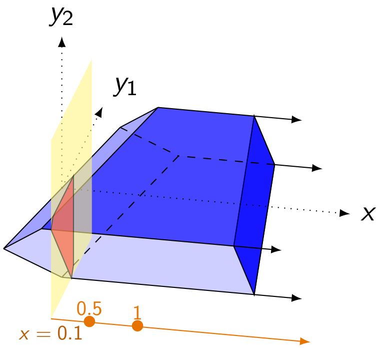

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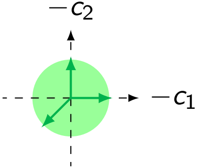

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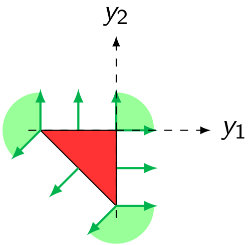

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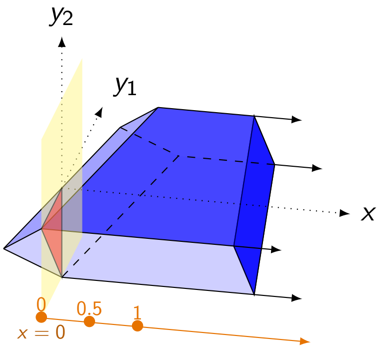

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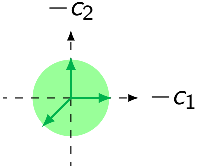

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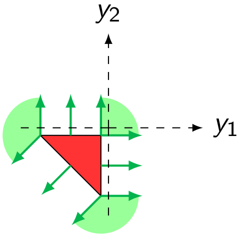

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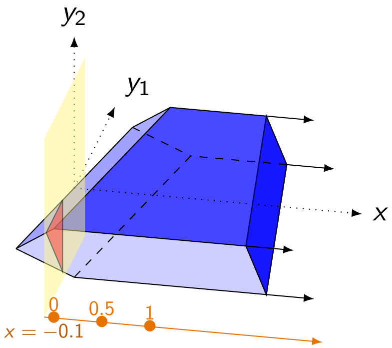

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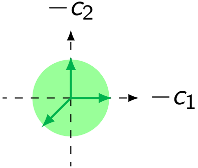

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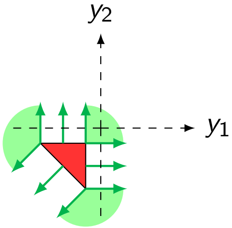

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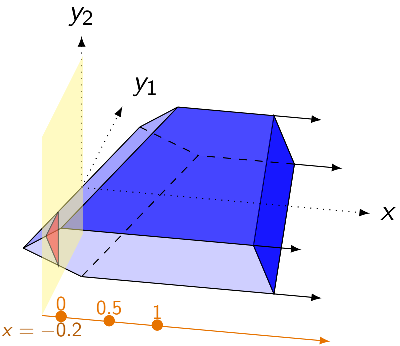

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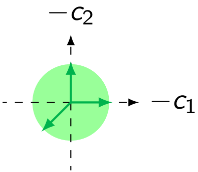

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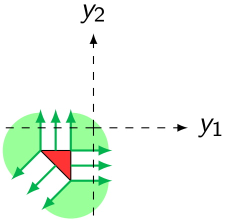

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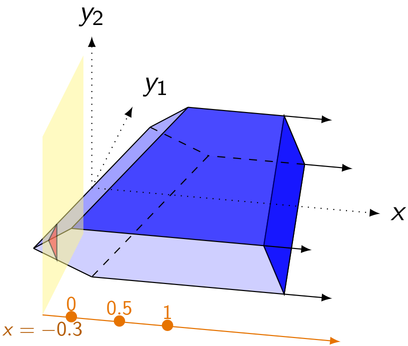

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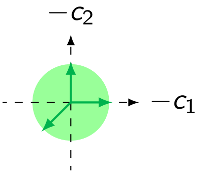

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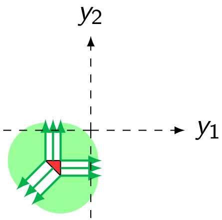

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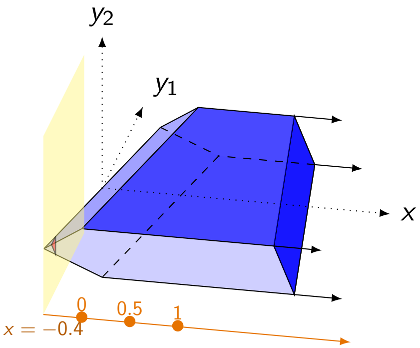


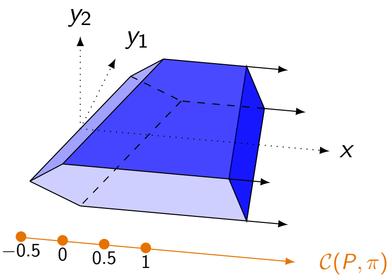
P and P_x

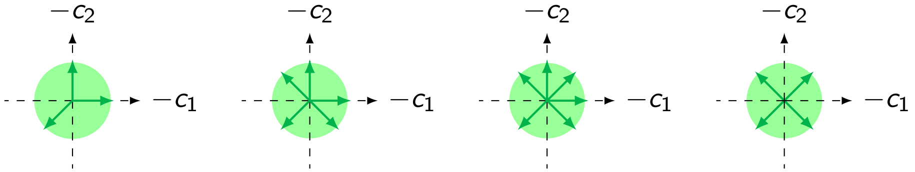
What are the constant regions of $x \mapsto \mathcal{N}(P_x)$?

Proposition

There exists a collection $\mathcal{C}(P,\pi)$ *called the* **chamber complex** *whose relative interior of cells are the constant regions of* $x \mapsto \mathcal{N}(P_x)$.

I.e, for $\sigma \in \mathcal{C}(P,\pi)$ *and* $x, x' \in \mathrm{ri}(\sigma)$, *we have* $\mathcal{N}(P_x) = \mathcal{N}(P_{x'}) =: \mathcal{N}_\sigma$





$\mathcal{N}_\sigma$ for $\sigma = [-0.5, 0]$ $\quad$ $\mathcal{N}_\sigma$ for $\sigma = [0, 0.5]$ $\quad$ $\mathcal{N}_\sigma$ for $\sigma = [0.5, 1]$ $\quad$ $\mathcal{N}_\sigma$ for $\sigma = [1, +\infty)$

Chamber complex

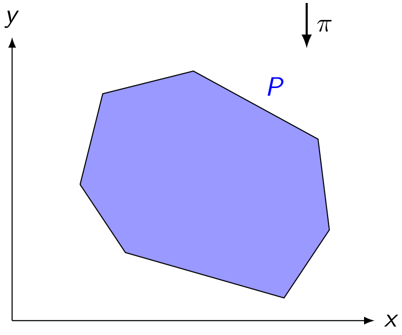
Definition

The *chamber complex* $\mathcal{C}(P, \pi)$ of P along π is

$$\mathcal{C}(P, \pi) := \{\sigma_{P,\pi}(x) \mid x \in \pi(P)\}$$

where

$$\sigma_{P,\pi}(x) := \bigcap_{F \in \mathcal{F}(P) \,|\, x \in \pi(F)} \pi(F)$$



where $\mathcal{F}(P)$ is the set of faces of P and π is the projection $(x, y) \mapsto x$.

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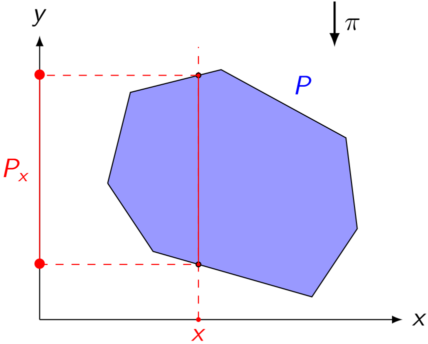

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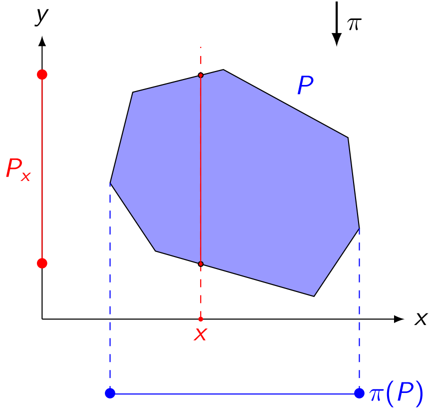

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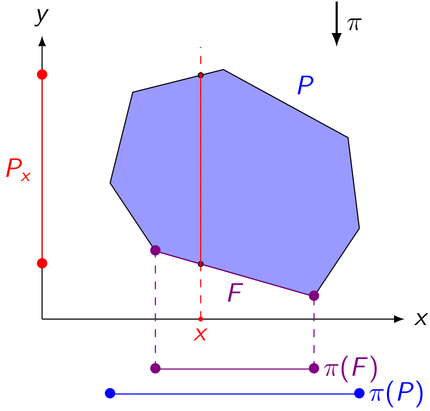

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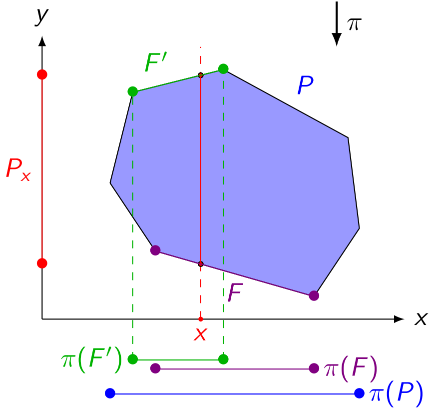

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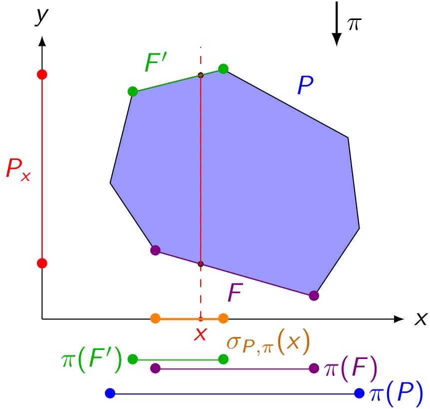

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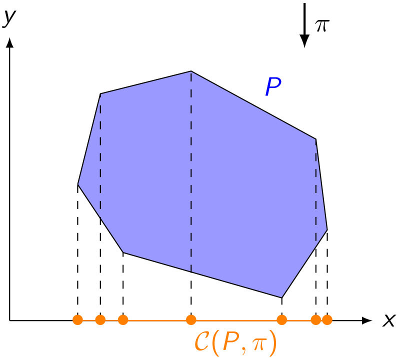

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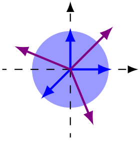
Common Refinement of Normal Fans

We can quantize $\boldsymbol{c}$ on each chamber.



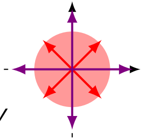
$\mathcal{N}_\sigma$ and $\check{\boldsymbol{c}}$

For all $x \in \mathrm{ri}(\sigma)$,

$$V(x) = \sum_{N \in \mathcal{N}_\sigma} p_N \min_{y \in P_x} \check{c}_N^\top y$$

For all $x' \in \mathrm{ri}(\tau)$,

$$V(x') = \sum_{N \in \mathcal{N}_\tau} p_N \min_{y \in P_x} \check{c}_N^\top y$$



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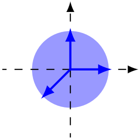

$\textcolor{blue}{\mathcal{N}_\sigma}$

For all $x \in \mathrm{ri}(\textcolor{blue}{\sigma})$,

$$V(x) = \sum_{N \in \textcolor{blue}{\mathcal{N}_\sigma}} p_N \min_{y \in P_x} \check{c}_N^\top y$$

For all $x' \in \mathrm{ri}(\textcolor{red}{\tau})$,

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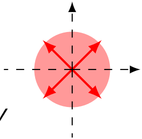

$\textcolor{red}{\mathcal{N}_\tau}$

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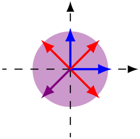
$$\mathcal{R} := \textcolor{blue}{\mathcal{N}_\sigma} \wedge \textcolor{red}{\mathcal{N}_\tau} = \{N \cap N' \mid N \in \mathcal{N}_\sigma, N' \in \mathcal{N}_\tau\}$$



$\textcolor{blue}{\mathcal{N}_\sigma} \wedge \textcolor{red}{\mathcal{N}_\tau}$

For all $x \in \mathrm{ri}(\textcolor{blue}{\sigma}) \cup \mathrm{ri}(\textcolor{red}{\tau})$,

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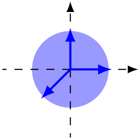

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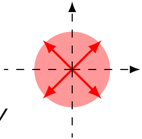

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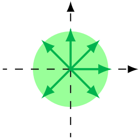

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Uniform exact quantization for $\boldsymbol{c}$

Let's sum up:

- local exact quantization at x induced by $\mathcal{N}(P_x)$;
- local exact quantization at x and x' by taking the refinement,
- $x \mapsto \mathcal{N}(P_x)$ is constant on each $\sigma \in \mathcal{C}(P, \pi)$

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Theorem (Uniform and universal quantization of the cost distribution)

Let $\mathcal{R} = \bigwedge_{\sigma \in \mathcal{C}(P,\pi)} -\mathcal{N}_\sigma$, *then* **for all** $x \in \mathbb{R}^n$

$$V(x) = \sum_{R \in \mathcal{R}} \check{p}_R \min_{y \in P_x} \check{c}_R^\top y$$

where $\check{p}_R := \mathbb{P}\big[\boldsymbol{c} \in \mathrm{ri}(R)\big]$ *and* $\check{c}_R := \mathbb{E}\big[\boldsymbol{c} \mid \boldsymbol{c} \in \mathrm{ri}(R)\big]$

Polyhedral characterization of V

Theorem

For all distributions of $\boldsymbol{c}$, V is affine on each cell of $\mathcal{C}(P,\pi)$.

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Under an affine change of variable, V is the support function of a polyhedron E

$$V(x) = \sigma_E\big(b - Bx\big) = \sup_{\lambda \in E} (b - Bx)^\top \lambda$$

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E is the *weighted fiber polyhedron*

$$E := \mathbb{E}\big[D_{\boldsymbol{c}}\big] = \int D_c \mathbb{P}(dc)$$

with D_c the dual admissible set

$$D_c := \big\{\lambda \,|\, A^\top \lambda + c = 0\big\}$$

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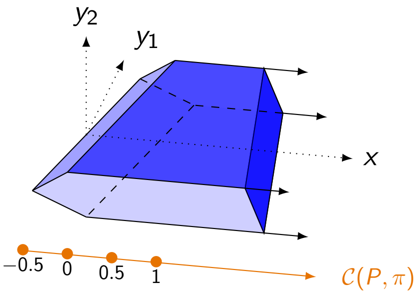
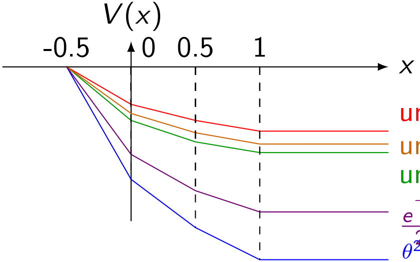
Extension of fiber polytope of

L. Billera, B. Sturmfels, Fiber polytopes, *Annals of Mathematics*, p527–549, 1992.

Explicit computation of the example

$$V(x) = \mathbb{E}\begin{bmatrix} \min\limits_{y\in\mathbb{R}^2} & \boldsymbol{c}^\top y \\ \text{s.t.} & \|y\|_1 \leqslant 1 \\ & y_1 \leqslant x \\ & y_2 \leqslant x \end{bmatrix}$$





Different distributions of $\boldsymbol{c}$:

uniform on norm 1 ball

uniform on norm 2 ball

uniform on norm ∞ ball

$\dfrac{e^{-\frac{\|c\|_2^2}{2\gamma^2}}}{2\pi\gamma^2}dc$

$\dfrac{\theta^2 e^{-\theta\|c\|_1}}{4}dc$

Contents

Extension to multistage and stochastic constraints

Theorem

All results generalizes to multistage problem with finitely supported stochastic constraints.

- $(V_t)_t$ *are affine on universal chamber complexes, i.e. independent of the law of* $(\boldsymbol{c}_t)_t$
- *We have an uniform and universal exact quantization.*

Core idea of the proof :
Iterated chamber complexes

$$\mathcal{P}_{t,\xi} := \mathcal{C}\Big((\mathbb{R}^{n_t} \times \mathcal{P}_{t+1}) \wedge \mathcal{F}\big(P_t(\xi)\big), \pi_{x_{t-1}}^{x_{t-1},x_t}\Big)$$

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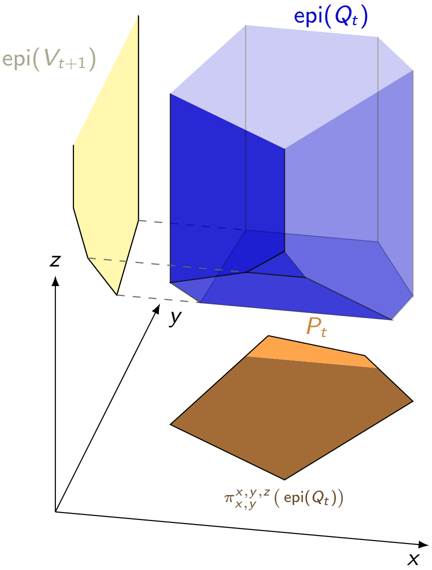
Multistage uniform and universal exact quantization

$$V_t(x) = \mathbb{E}\left[\begin{array}{ll} \min\limits_{y\in\mathbb{R}^{n_t}} & \boldsymbol{c}_t^\top y + V_{t+1}(y) \\ \\ \text{s.t.} & (x,y) \in P_t \end{array}\right]$$

with $Q_t(x,y) := V_{t+1}(y) + \mathbb{I}_{(x,y)\in P_t}$.



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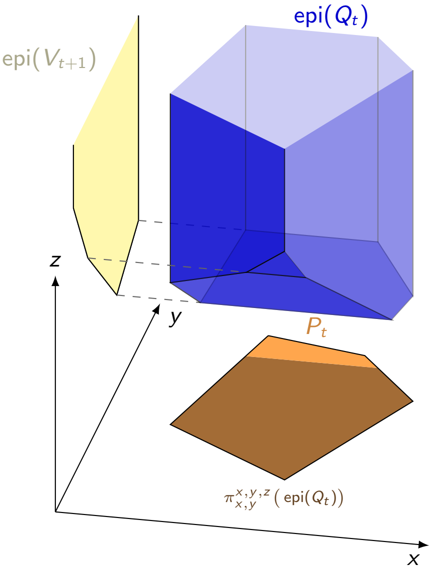

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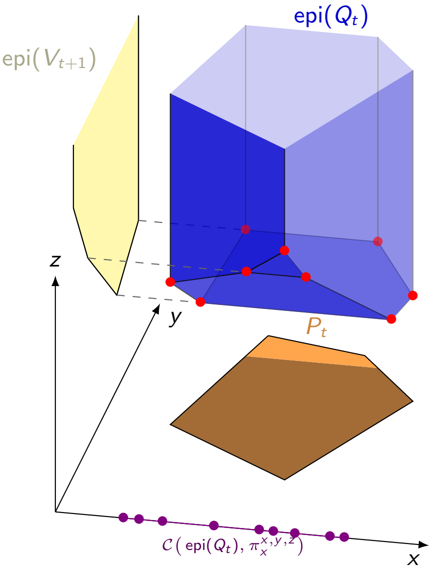

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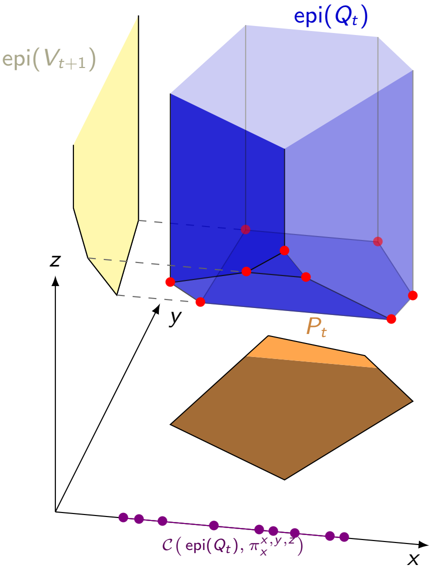

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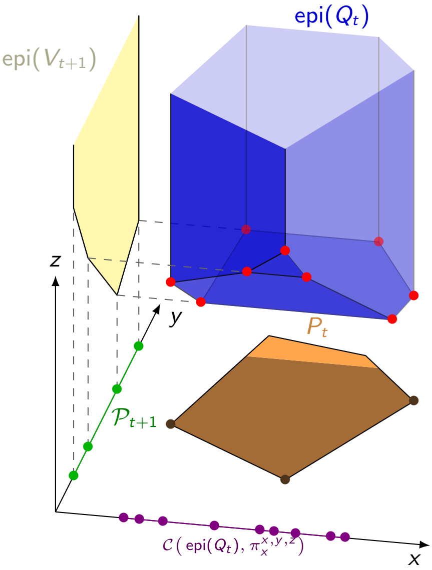

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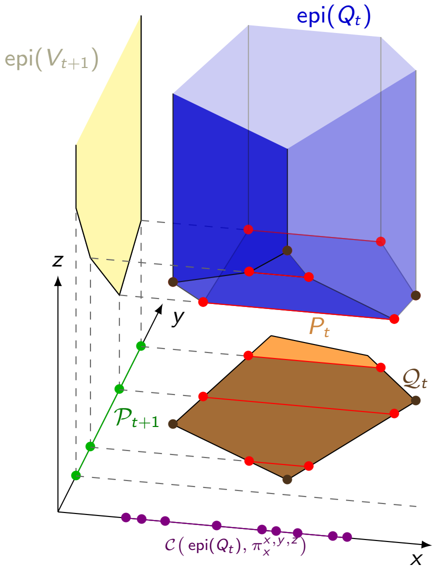

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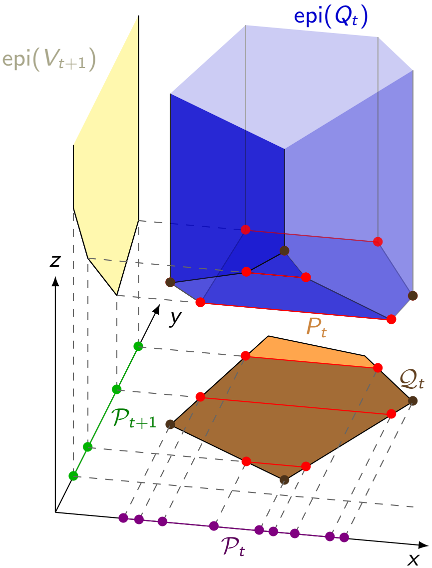

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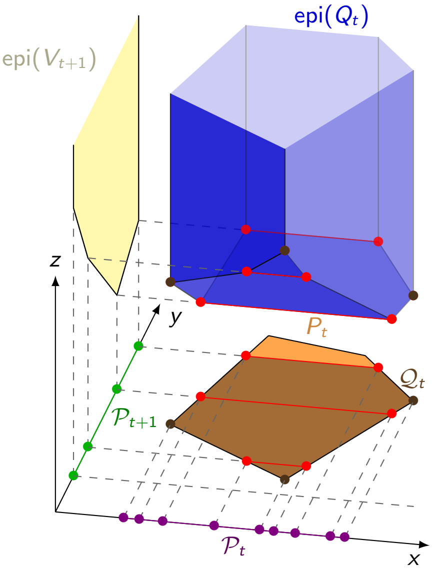
[FGL21, Lem. 4.1]: $\mathcal{P}_t \preccurlyeq \mathcal{C}(\mathrm{epi}(Q_t), \pi_x^{x,y,z})$

➥ V_t affine on $\mathcal{P}_t$, $\mathcal{N}(P_x)$ constant on $\mathcal{P}_t$



Contents

Earlier and new complexity results

Volume of a polytope

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Shapiro and Nemirovski (2005):
By SAA, we can solve MSLP, up to precision ε, in pseudo-polynomial time, i.e. polynomial in $\frac{1}{\varepsilon}$, with probability $1-\alpha$, when T is fixed.

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Assume that T, n_t, and $|\mathrm{supp}(\boldsymbol{A}_t, \boldsymbol{B}_t, \boldsymbol{b}_t)|$, for $t = 2, \ldots, T$, are fixed integers.[1]
Assume that $\boldsymbol{c}$ admits a density function with a bounded total variation.

Then, there exists an algorithm that either asserts that MSLP is unfeasible or finds an ε-solution in polynomial time in $\log(\frac{1}{\varepsilon})$ with probability 1.

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Then, there exists an algorithm that either asserts that MSLP is unfeasible or finds an ε-solution in polynomial time in $\log(\frac{1}{\varepsilon})$ with probability 1.

➥ Can be adapted to exact complexity when we can compute exactly $\mathbb{E}\big[\boldsymbol{c} \in C | (\boldsymbol{A}_t, \boldsymbol{B}_t, \boldsymbol{b}_t) = (A, B, b)\big]$ and $\mathbb{P}\big[\boldsymbol{c} \in C | (\boldsymbol{A}_t, \boldsymbol{B}_t, \boldsymbol{b}_t) = (A, B, b)\big]$.

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Conclusion and applications

- *Uniform and universal* exact quantization for an MSLP
  - ➥ New complexity results.

  Unfortunately this quantization might be very large.

- *Local* exact quantization for $\boldsymbol{c}$
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Thank you for listening ! Any question ?

M. Forcier, S. Gaubert, V. Leclère
Exact quantization of multistage stochastic linear problems.
arXiv preprint arXiv:2107.09566 (2021).

M. Forcier, V. Leclère
Generalized adaptive partition-based method for two-stage stochastic linear programs: convergence and generalization.
Operation Research Letters, to appear (2022).

M. Forcier, V. Leclère
Convergence of Trajectory Following Dynamic Programming algorithms for multistage stochastic problems without finite support assumptions
HAL Id : hal-03683697 (2022).

Local exact quantization and adapted partition

Local exact quantization

random cost

Recall that for a fixed x,

$$\mathbb{E}\big[\min_{y \in P_x} \boldsymbol{c}^\top y\big] = \sum_{N \in \mathcal{N}(P_x)} p_N \min_{y \in P_x} \check{c}_N{}^\top y$$

where,

$$p_N := \mathbb{P}\big[\boldsymbol{c} \in -\operatorname{ri} N\big]$$
$$\check{c}_N := \mathbb{E}\big[\boldsymbol{c} \mid \boldsymbol{c} \in -\operatorname{ri} N\big]$$

$$P_x := \{y \in \mathbb{R}^m \mid Ay + Bx \leqslant b\}$$

GAPM

random constraints

Similarly, for a given q, and all x,

$$\begin{aligned} V(x) &:= \mathbb{E}\big[Q(x, \boldsymbol{\xi})\big] \\ &= \mathbb{E}\big[\max_{\lambda \in D_q} (\boldsymbol{h} - \boldsymbol{T}x)^\top \lambda\big] \\ &= \sum_{N \in \mathcal{N}(D_q)} p_N \max_{\lambda \in D_q} \psi_{N,x}{}^\top \lambda \end{aligned}$$

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$$D_q := \{\lambda \in \mathbb{R}^l \mid W^\top \lambda \leqslant q\}$$

An explicit adapted partition

Consider $x \in \mathbb{R}^n$ and $N \in \mathcal{N}(D_q)$ a normal cone of D_q. We define

$$E_{N,x} := \{\xi \in \Xi \mid h - Tx \in \operatorname{ri} N\}$$

Theorem (FL 2021)

$\mathcal{R}_x := \left\{E_{N,x} \mid N \in \mathcal{N}(D_q)\right\}$ *is an adapted partition to* x
i.e. $V_{\mathcal{R}_x}(x) = V(x)$

Proof:

$$\begin{aligned} V(x) &:= \mathbb{E}\big[Q(x,\boldsymbol{\xi})\big] \\ &= \sum_{N \in \mathcal{N}(D)} \mathbb{P}\big[\boldsymbol{h} - \boldsymbol{T}x \in \operatorname{ri} N\big] \min_{\lambda \in D} \mathbb{E}\big[\boldsymbol{h} - \boldsymbol{T}x \,|\, \boldsymbol{h} - \boldsymbol{T}x \in \operatorname{ri} N\big]^\top \lambda \\ &= \sum_{N \in \mathcal{N}(D)} \mathbb{P}\big[\boldsymbol{\xi} \in E_{N,x}\big] Q\Big(\mathbb{E}\big[\boldsymbol{\xi} \,|\, \boldsymbol{\xi} \in E_{N,x}\big], x\Big) = V_{\mathcal{R}_x}(x) \end{aligned}$$

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Numerical Results - ProdMix

k	z_L^k	z_U^k	$z_U^k - z_L^k$	Total time	$\lvert\mathcal{P}^k\rvert$
1	-18666.67	-16939.71	1726.96	0.57 s	4
2	-17873.01	-17383.73	489.28	2.1 s	9
4	-17744.67	-17709.00	35.67	9.1 s	25
6	-17713.74	-17711.37	2.37	23.7 s	49
8	-17711.71	-17711.56	0.15	50.0 s	81
10	-17711.57	-17711.56	0.01	88.0 s	121

Table: Results for problem Prod-Mix

Comparison with SAA : we solved the same problem 100 times, each with 10 000 scenarios randomly drawn

$\rightsquigarrow$ 95% confidence interval centered in -17711, with radius 2.2.

$\rightsquigarrow$ required 2058s of computation.