Multistage stochastic optimization and polyhedral geometry

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PhD Defense, under the supervision of

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Modeling hydroelectric energy storage management

Modeling hydroelectric energy storage management

- $\color{red}{u}$ water hustled
- d demand
- c cost of unmet demand

$$\min_{\color{red}{u}} \; c(d - \color{red}{u})$$
$$s.c. \;\; 0 \leqslant \color{red}{u} \leqslant d$$

Modeling hydroelectric energy storage management

- u water hustled
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- c cost of unmet demand
- x_0/x_1 water in the reservoir
- $\overline{x}$ capacity of the reservoir

$$
\begin{aligned}
\min_{u} \quad & c(d - u) \\
s.c. \quad & 0 \leqslant u \leqslant d \\
& x_1 = x_0 - u \\
& 0 \leqslant x_0 \leqslant \overline{x},\ 0 \leqslant x_1 \leqslant \overline{x}
\end{aligned}
$$

Modeling hydroelectric energy storage management

- u water hustled
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- c cost of unmet demand
- x_0/x_1 water in the reservoir
- $\overline{x}$ capacity of the reservoir
- w rain and runoff

$$\begin{aligned} \min_{u} \quad & c(d - u) \\ s.c. \quad & 0 \leqslant u \leqslant d \\ & x_1 = x_0 - u + w \\ & 0 \leqslant x_0 \leqslant \overline{x}, \ 0 \leqslant x_1 \leqslant \overline{x} \end{aligned}$$

Modeling hydroelectric energy storage management

- u water hustled
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- x_0/x_1 water in the reservoir
- $\overline{x}$ capacity of the reservoir
- w rain and runoff
- v water evacuated by the valve

$$\begin{aligned} \min_{u,v} \quad & c(d-u) \\ s.c. \quad & 0 \leqslant u \leqslant d \\ & x_1 = x_0 - u + w - v \\ & 0 \leqslant x_0 \leqslant \overline{x},\ 0 \leqslant x_1 \leqslant \overline{x} \\ & 0 \leqslant v \end{aligned}$$

Modeling hydroelectric energy storage management

At step t

- u_t water hustled
- d_t demand
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$$
\begin{aligned}
\min_{u_t, v_t} \quad & \sum_{t=1}^{T} c_t(d_t - u_t) \\
s.c. \ & \forall t \in [T], \ 0 \leqslant u_t \leqslant d_t \\
& \forall t \in [T], \ x_{t+1} = x_t - u_t + w_t - v_t \\
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Modeling hydroelectric energy storage management

At step t

- $\boldsymbol{u}_t$ water hustled
- $\boldsymbol{d}_t$ demand
- $\boldsymbol{c}_t$ cost of unmet demand
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\end{aligned}
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Multistage stochastic linear programming (MSLP)

$$
\begin{aligned}
\min_{(\boldsymbol{x}_t)_{t\in[T]}} \quad & \mathbb{E}\Big[\sum_{t=1}^{T} \boldsymbol{c}_t^\top \boldsymbol{x}_t\Big] & \\
\text{s.t.} \quad & \boldsymbol{A}_t \boldsymbol{x}_t + \boldsymbol{B}_t \boldsymbol{x}_{t-1} \leqslant \boldsymbol{b}_t & \forall t \in [T] \\
& \sigma(\boldsymbol{x}_t) \subset \sigma(\boldsymbol{c}_\tau, \boldsymbol{A}_\tau, \boldsymbol{B}_\tau, \boldsymbol{b}_\tau)_{\tau \leqslant t} & \forall t \in [T] \\
& \boldsymbol{x}_0 \equiv x_0 \text{ given} &
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$\boldsymbol{\xi}_t = (\boldsymbol{c}_t, \boldsymbol{A}_t, \boldsymbol{B}_t, \boldsymbol{b}_t)_{t\in[T]}$ is assumed to be stagewise independent.

We set $V_{T+1} \equiv 0$ and:

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V_t(x_{t-1}) := \mathbb{E}\big[\hat{V}_t(x_{t-1}, \boldsymbol{\xi}_t)\big] := \mathbb{E}\begin{bmatrix} \min\limits_{x_t \in \mathbb{R}^{n_t}} & \boldsymbol{c}_t^\top x_t + V_{t+1}(x_t) \\ \text{s.t.} & \boldsymbol{A}_t x_t + \boldsymbol{B}_t x_{t-1} \leqslant \boldsymbol{b}_t \end{bmatrix}
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➥ How to deal with continuous distributions ?

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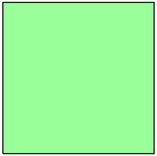
Quantization of a MSLP

Real problem

$$V_t(x) = \mathbb{E}\left[\hat{V}_t\left(x, \xi_t\right)\right] = \mathbb{E}\left[\begin{array}{ll} \min\limits_{y \in \mathbb{R}^{n_t}} & \boldsymbol{c}_t^\top y + V_{t+1}(y) \\ \text{s.t.} & \boldsymbol{A}_t y + \boldsymbol{B}_t x \leqslant \boldsymbol{b}_t \end{array}\right]$$



ξ_t continuous

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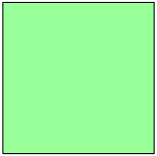
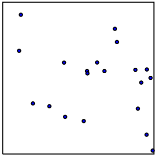


ξ_t continuous

Sample Average Approximation (SAA)

$$V_{t,N}^{SAA}(x) := \frac{1}{N}\sum_{k=1}^{N} \hat{V}_t\big(x, \xi^k\big)$$

$\xi^1, \cdots, \xi^N$ drawn by Monte Carlo



SAA $N = 20$

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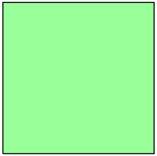

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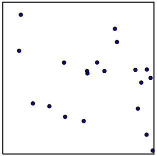
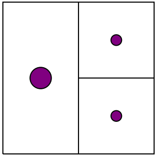


SAA $N = 20$

Partition-based

$$V_{t,\mathcal{P}}(x) := \sum_{P \in \mathcal{P}} \check{p}_{t,P} \hat{V}_t(x, \check{\xi}_{t,P})$$

with $\check{p}_{t,P} := \mathbb{P}\left[\boldsymbol{\xi}_t \in P\right]$ and $\check{\xi}_{t,P} := \mathbb{E}\left[\boldsymbol{\xi}_t \,\middle|\, \boldsymbol{\xi}_t \in P\right]$



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Quantization of a MSLP

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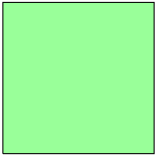

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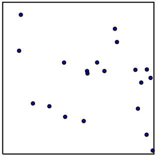

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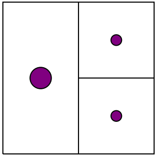
If $\xi \mapsto \hat{V}(x, \xi)$ is convex, $V_{t,\mathcal{P}}(x) \leqslant V_t(x)$.



Partition-based

Exact quantization

Definition

A MSLP admits a local exact quantization at time t on x if there exists a finitely supported $(\check{\xi}_t)_{t\in[T]}$ *i.e.* such that

$$V_t(x) = \mathbb{E}\big[\hat{V}_t(x, \xi_t)\big] = \mathbb{E}\big[\hat{V}_t(x, \check{\xi}_t)\big].$$

We call an exact quantization

- uniform if it is locally exact at all $x \in \mathbb{R}^{n_t}$, and all $t \in [T]$.
- universal if there exists a partition $\mathcal{P}_{t,x}$ such that the induced quantization is exact at time t on x, for all distributions of $(\xi_\tau)_{\tau\in[T]}$.

Questions:

1. Under which condition does there exist an exact quantization ?
2. Can we construct a uniform and universal exact quantization ?

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A first counter example

Assume $V_{t+1} \equiv 0$ and denote $V := V_t$, $\hat{V} := \hat{V}_t$ and $\boldsymbol{\xi} := \boldsymbol{\xi}_t$ for now.

Let $\boldsymbol{A} = (-\boldsymbol{u})$, $\boldsymbol{B} \equiv (0)$, $\boldsymbol{b} \equiv (-1)$ where $\boldsymbol{u} \sim \mathcal{U}([1,2])$.

$$\hat{V}(x,\xi) = \begin{array}{ll} \min\limits_{y \in \mathbb{R}} & y \\ \text{s.t.} & uy \geqslant 1 \end{array} = \frac{1}{u}$$

By strict convexity, for all partition $\mathcal{P}$

$$\sum_{P \in \mathcal{P}} \check{p}_P \hat{V}\big(x, \check{\xi}_P\big) < V(x) = \mathbb{E}\left[\frac{1}{\boldsymbol{u}}\right]$$

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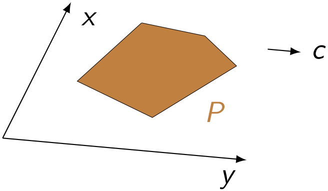
Uniform exact quantization and polyhedrality

$$\hat{V}(x,\xi) = \min_{y \in \mathbb{R}^m} \; c^\top y$$
$$\text{s.t.} \quad Ay + Bx \leqslant h$$



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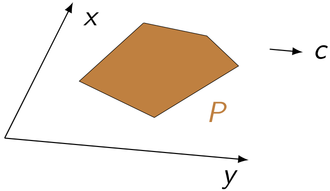

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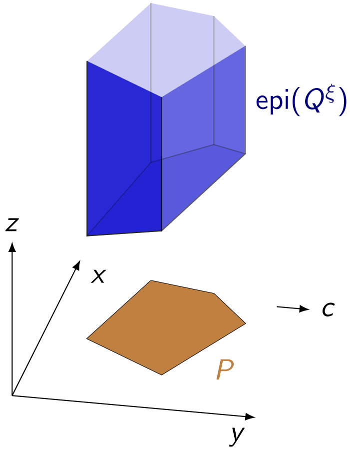
with $Q^\xi(x,y) := c^\top y + \mathbb{I}_{(x,y)\in P}$.



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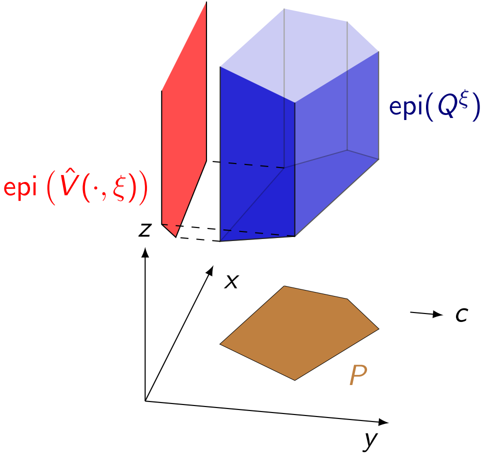

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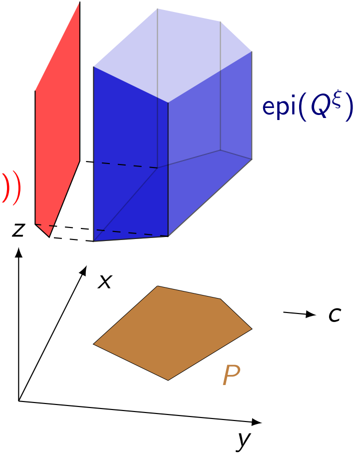

$$V(x) = \mathbb{E}\left[\hat{V}(x,\boldsymbol{\xi})\right] = \sum_{\xi\in\mathrm{supp}(\check{\boldsymbol{\xi}})} p_\xi \hat{V}(x,\xi)$$

➥ If the noise is finitely supported, then V is polyhedral

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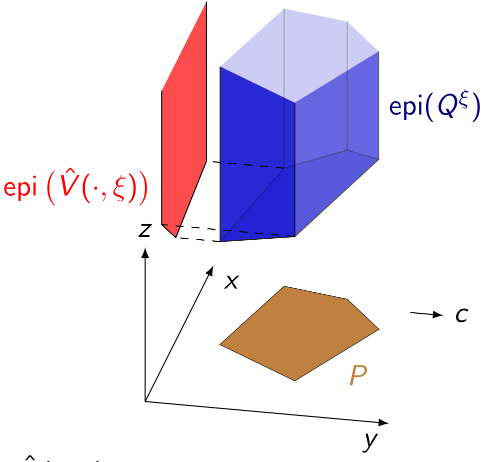

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- ➥ If the noise is finitely supported, then V is polyhedral
- ➥ Existence of uniform exact quantization implies polyhedrality of V.

Counter examples with stochastic constraints

Stochastic $\boldsymbol{B}$

$$V(x) = \mathbb{E}\left[\begin{array}{ll} \min\limits_{y \in \mathbb{R}^m} & y \\ \text{s.t.} & \boldsymbol{u}x - y \leqslant 0 \\ & y \geqslant 1 \end{array}\right]$$
$$= \mathbb{E}\left[\max(\boldsymbol{u}x, 1)\right]$$
$$= \begin{cases} 1 & \text{if } x \leqslant 1 \\ \frac{x}{2} + \frac{1}{2x} & \text{if } x \geqslant 1 \end{cases}$$

Stochastic $\boldsymbol{b}$

$$V(x) = \mathbb{E}\left[\begin{array}{ll} \min\limits_{y \in \mathbb{R}^m} & y \\ \text{s.t.} & y \geqslant \boldsymbol{u} \\ & x - y \leqslant 0 \end{array}\right]$$
$$= \mathbb{E}\left[\max(x, \boldsymbol{u})\right]$$
$$= \begin{cases} \frac{1}{2} & \text{if } x \leqslant 0 \\ \frac{x^2+1}{2} & \text{if } x \in [0,1] \\ x & \text{if } x \geqslant 1 \end{cases}$$

➥ V is not polyhedral $\Rightarrow$ No uniform exact quantization for non-finitely supported $\boldsymbol{B}$ and $\boldsymbol{b}$.

$\boldsymbol{u}$ is uniform on $[0, 1]$

Counter examples with stochastic constraints

Stochastic $\boldsymbol{B}$

$$V(x) = \mathbb{E}\begin{bmatrix} \min\limits_{y\in\mathbb{R}^m} & y \\ \text{s.t.} & \boldsymbol{u}x - y \leqslant 0 \\ & y \geqslant 1 \end{bmatrix}$$

$$= \mathbb{E}\big[\max(\boldsymbol{u}x, 1)\big]$$

$$= \begin{cases} 1 & \text{if } x \leqslant 1 \\ \frac{x}{2} + \frac{1}{2x} & \text{if } x \geqslant 1 \end{cases}$$

Stochastic $\boldsymbol{b}$

$$V(x) = \mathbb{E}\begin{bmatrix} \min\limits_{y\in\mathbb{R}^m} & y \\ \text{s.t.} & y \geqslant \boldsymbol{u} \\ & x - y \leqslant 0 \end{bmatrix}$$

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Remaining cases

$$V(x) = \mathbb{E}\left[\begin{array}{ll} \min\limits_{y\in\mathbb{R}^m} & \boldsymbol{c}^\top y \\ \text{s.t.} & \boldsymbol{A}y + \boldsymbol{B}x \leqslant \boldsymbol{b} \end{array}\right]$$

	$\boldsymbol{A}$	$(\boldsymbol{B}, \boldsymbol{b})$	$\boldsymbol{c}$
Local	$\times$	?	?
Uniform	$\times$	$\times$	?

Remaining cases

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	$\boldsymbol{A}$	$(\boldsymbol{B}, \boldsymbol{b})$	$\boldsymbol{c}$
Local	×	✓	✓
Uniform	×	×	?

Theorem (GAPM, FL 2022)

If A is deterministic,
then there exists a universal and local exact quantization.

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	$\boldsymbol{A}$	$(\boldsymbol{B}, \boldsymbol{b})$	$\boldsymbol{c}$
Local	$\times$	$\checkmark$	$\checkmark$
Uniform	$\times$	$\times$	$\checkmark$

Theorem (GAPM, FL 2022)

If A is deterministic,
then there exists a universal and local exact quantization.

Theorem (Exact quantization, FGL 2021)

If A, B and b are deterministic,
then there exists a universal and uniform exact quantization.

Contents

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Illustrative running example:

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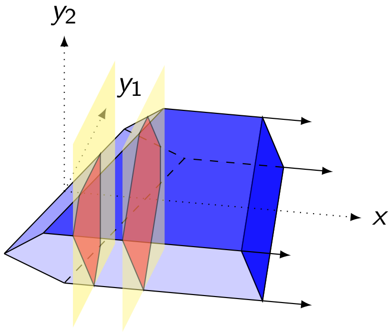

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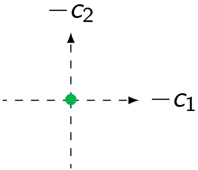

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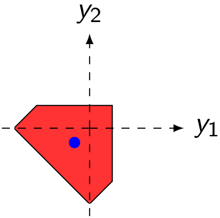

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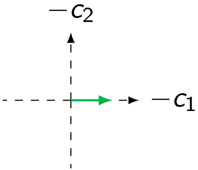

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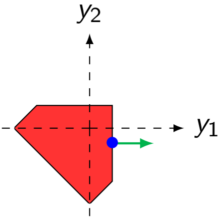

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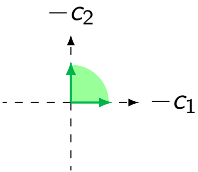

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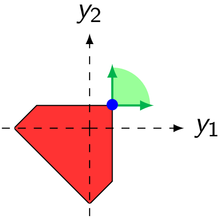

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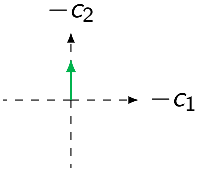

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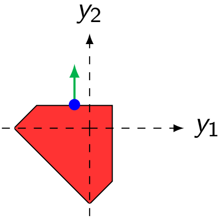

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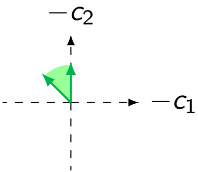

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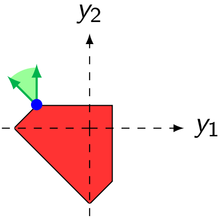

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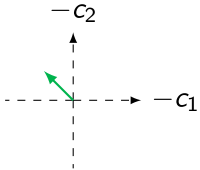

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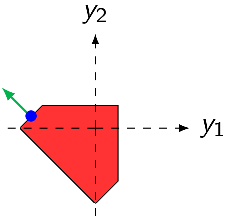

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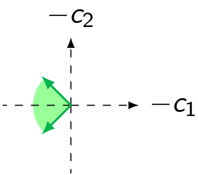

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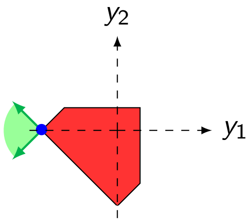

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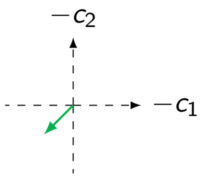

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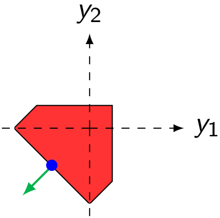

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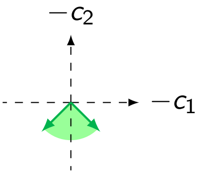

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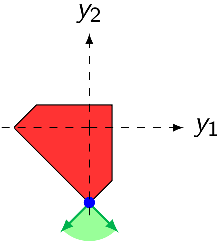

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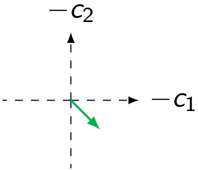

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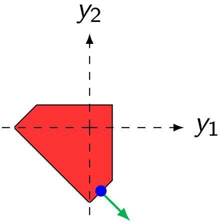

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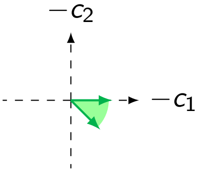

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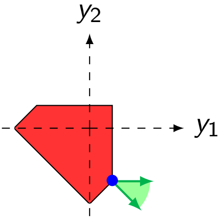

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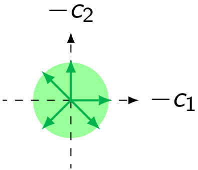
Proposition

If P_x is bounded, $\{\mathrm{ri}(N) \mid N \in \mathcal{N}(P_x)\}$ *is a partition of* $\mathbb{R}^m$.



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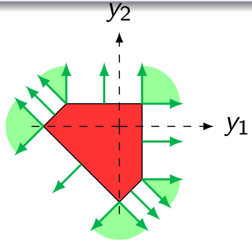

P_x and $\mathcal{N}(P_x)$ for $x = 0.3$

$\mathcal{N}(P_x)$: partition of cost coherent with the min

$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \boldsymbol{c}^\top y\big]$$

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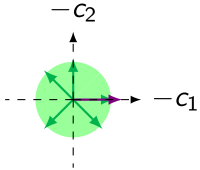
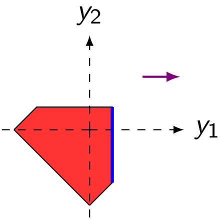


Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$



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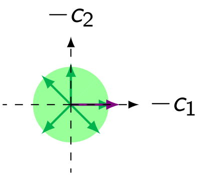

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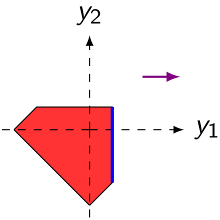

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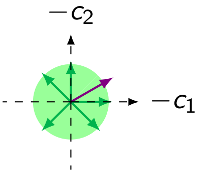

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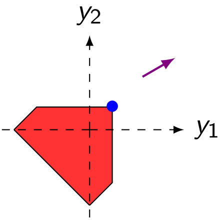

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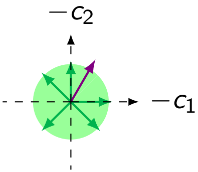

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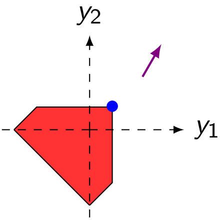

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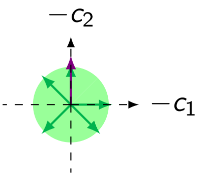

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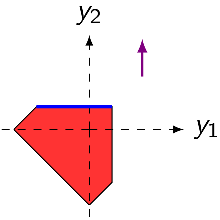

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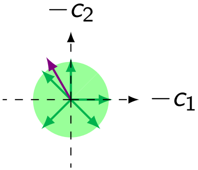

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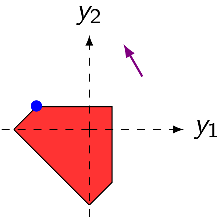

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$\mathcal{N}(P_x)$: partition of cost coherent with the min

$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \boldsymbol{c}^\top y\big]$$

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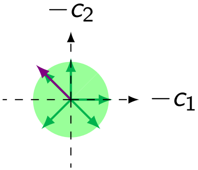

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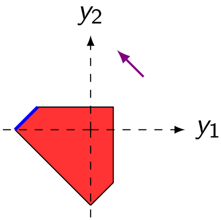

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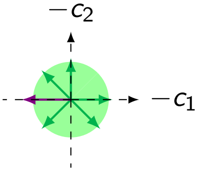

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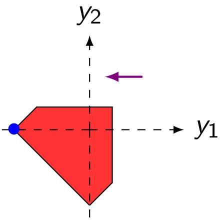

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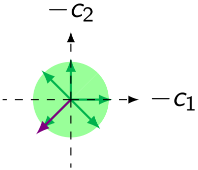

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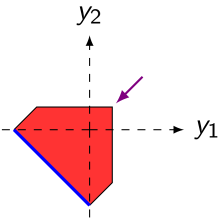

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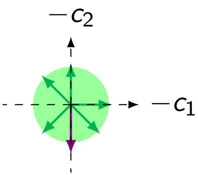

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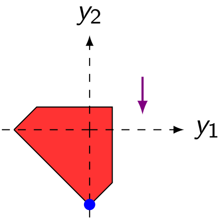

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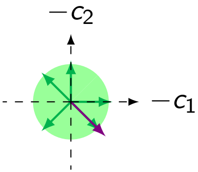

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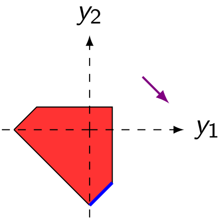

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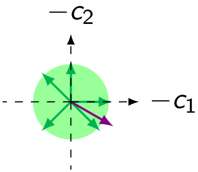

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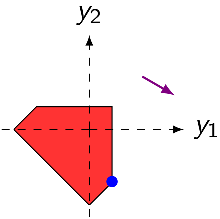

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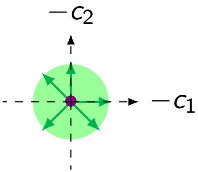

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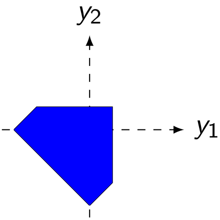

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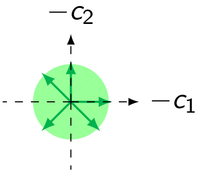

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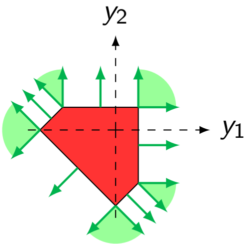
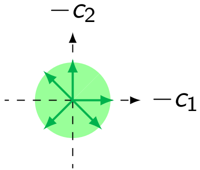


P_x for $x = 0.3$

Local and universal exact quantization for $\boldsymbol{c}$

$$
\begin{aligned}
V(x) &= \mathbb{E}\Big[\min_{y \in P_x} \boldsymbol{c}^\top y\Big] \\
&= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\Big[\mathbb{1}_{\boldsymbol{c} \in -\operatorname{ri} N} \min_{y \in P_x} \boldsymbol{c}^\top y\Big]
\end{aligned}
$$



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$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \boldsymbol{c}^\top y\Big]$$

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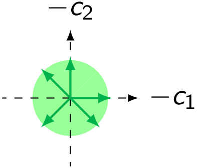
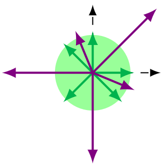
$$= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\Big[\mathbb{1}_{\boldsymbol{c} \in -\operatorname{ri} N} \boldsymbol{c}^\top\Big] y_N(x)$$



$\mathcal{N}(P_x)$ for $x = 0.3$

Local and universal exact quantization for $\boldsymbol{c}$

$$\begin{aligned} V(x) &= \mathbb{E}\big[\min_{y \in P_x} \boldsymbol{c}^\top y\big] \\ &= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbb{1}_{\boldsymbol{c} \in -\operatorname{ri} N} \min_{y \in P_x} \boldsymbol{c}^\top y\big] \quad \text{where } y_N(x) \in \arg\min_{y \in P_x} \underbrace{\boldsymbol{c}^\top}_{\in -\operatorname{ri} N} y. \\ &= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbb{1}_{\boldsymbol{c} \in -\operatorname{ri} N} \boldsymbol{c}^\top\big] y_N(x) \\ &= \sum_{N \in \mathcal{N}(P_x)} p_N \check{c}_N^\top y_N(x) \end{aligned}$$



$\mathcal{N}(P_x)$ and $p_N \check{c}_N$ for $x = 0.3$

For $N \in \mathcal{N}(P_x)$,

$$\begin{aligned} p_N &:= \mathbb{P}\big[\boldsymbol{c} \in -\operatorname{ri} N\big] \\ \check{c}_N &:= \mathbb{E}\big[\boldsymbol{c} \mid \boldsymbol{c} \in -\operatorname{ri} N\big] \end{aligned}$$

We replace the continuous cost $\boldsymbol{c}$, by the discrete cost $\check{\boldsymbol{c}}$.

Local and universal exact quantization for $\boldsymbol{c}$

$$
\begin{aligned}
V(x) &= \mathbb{E}\Big[\min_{y \in P_x} \boldsymbol{c}^\top y\Big] \\
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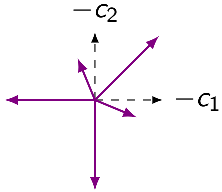
&= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\Big[\mathbb{1}_{\boldsymbol{c} \in -\operatorname{ri} N} \boldsymbol{c}^\top\Big] y_N(x) \\
&= \sum_{N \in \mathcal{N}(P_x)} p_N \check{c}_N{}^\top y_N(x) \\
&= \sum_{N \in \mathcal{N}(P_x)} p_N \min_{y \in P_x} \check{c}_N{}^\top y
\end{aligned}
$$



$p_N \check{c}_N$ for $x = 0.3$

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$$
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Contents

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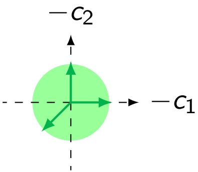
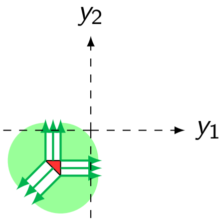


$\mathcal{N}(P_x)$



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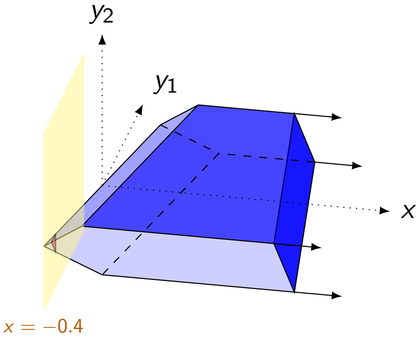

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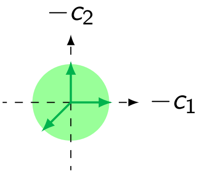

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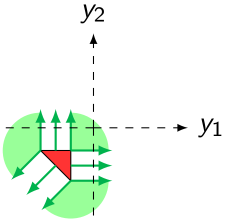

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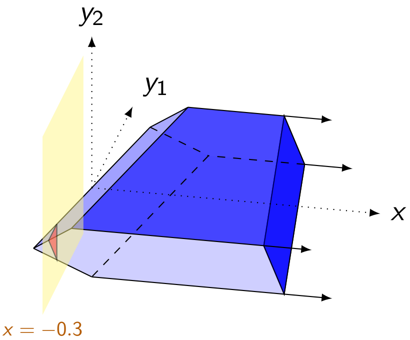

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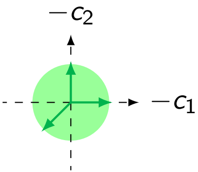

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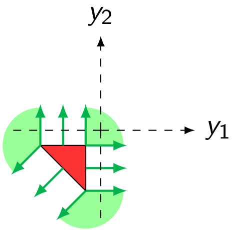

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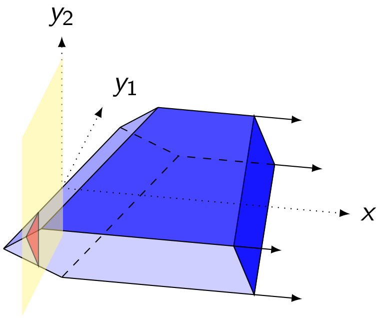

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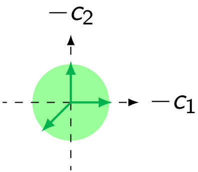

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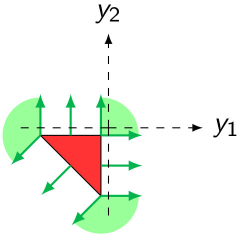

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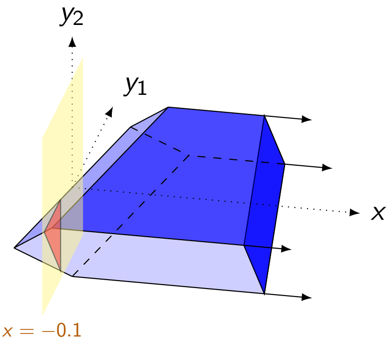

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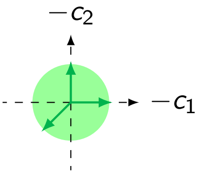

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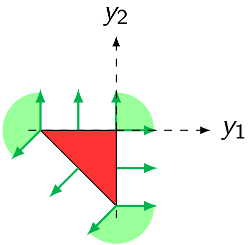

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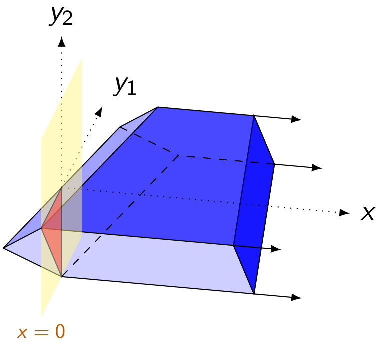

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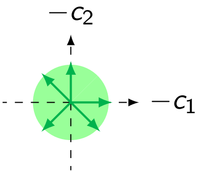

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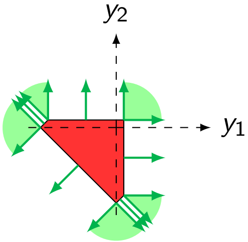

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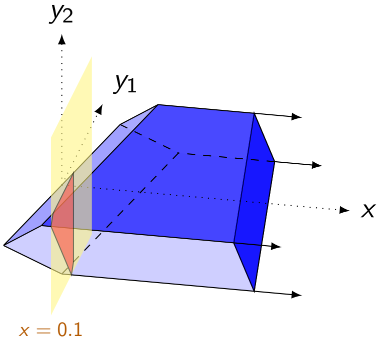

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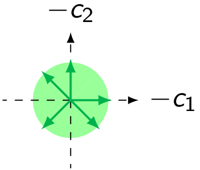

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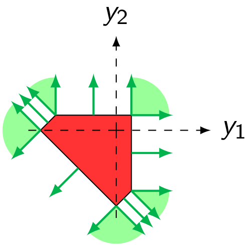

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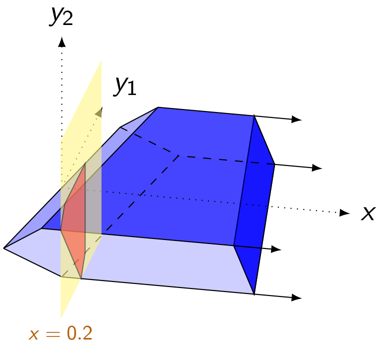

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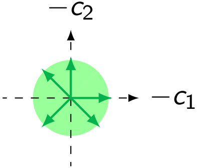

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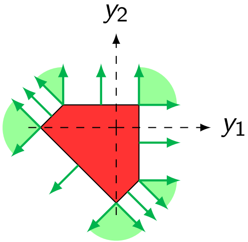

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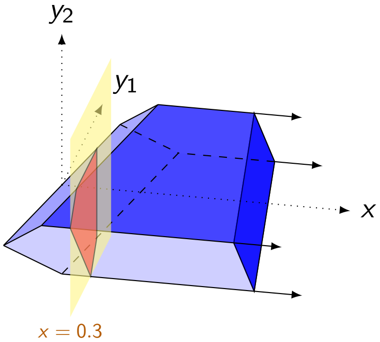

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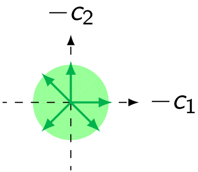

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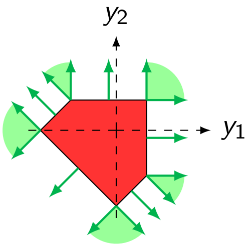

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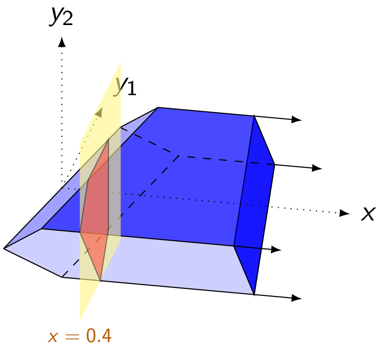

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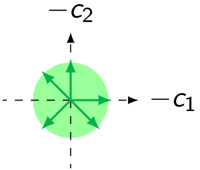

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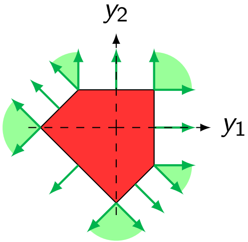

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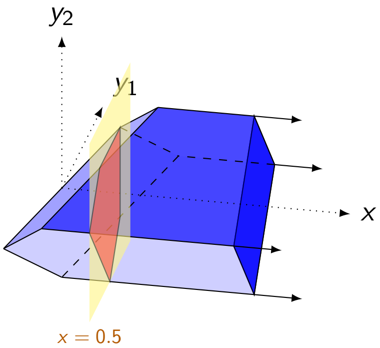

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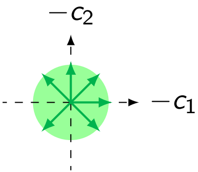

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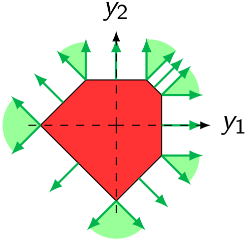

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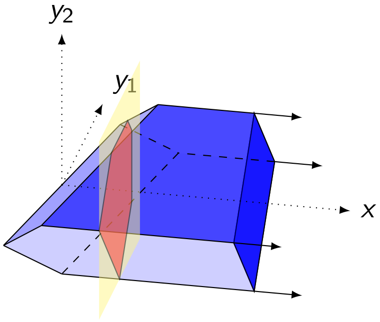

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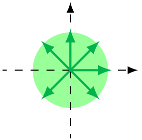

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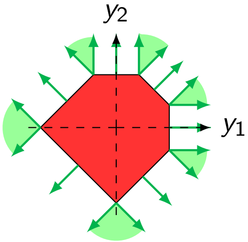

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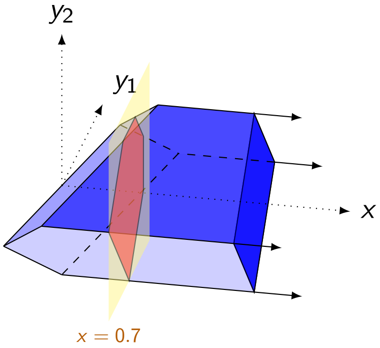

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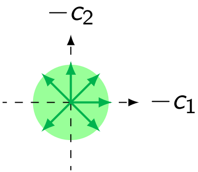

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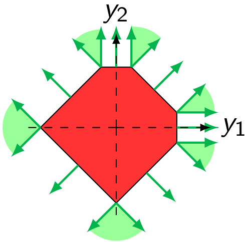

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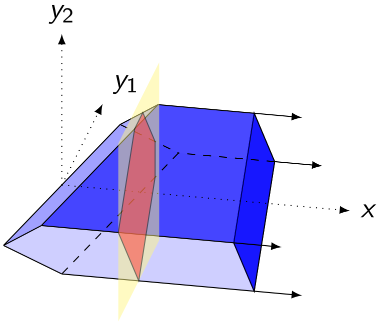

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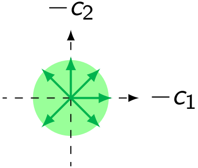

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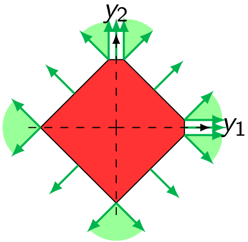

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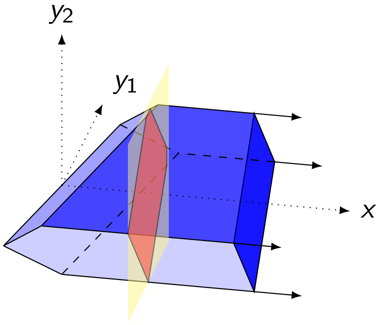

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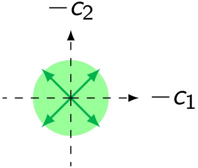

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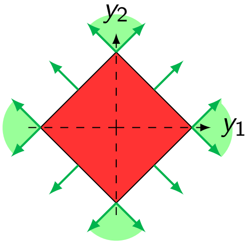

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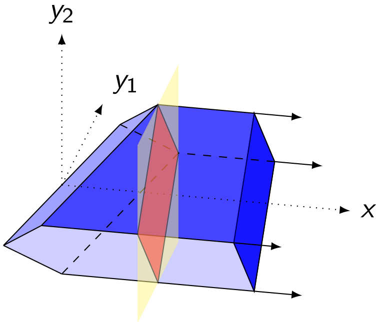

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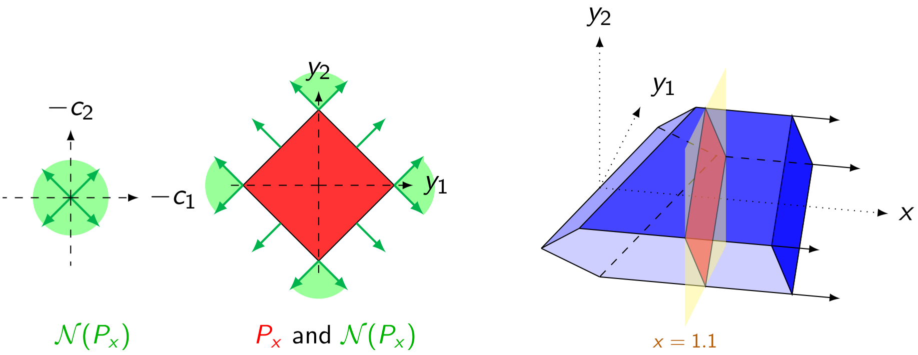

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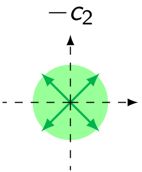

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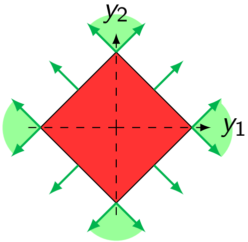

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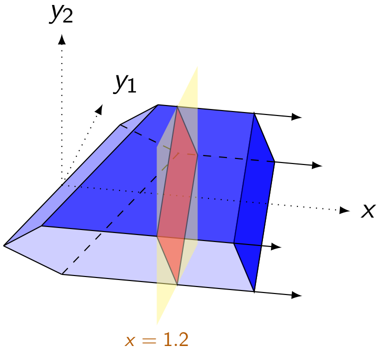

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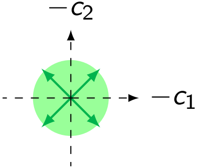

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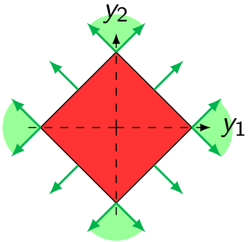

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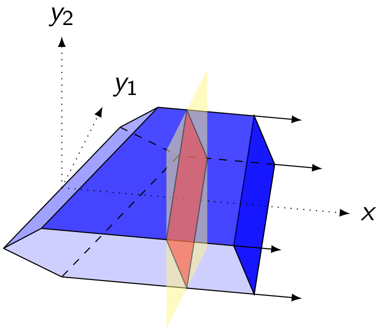

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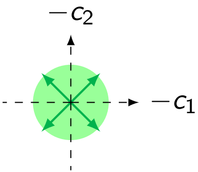

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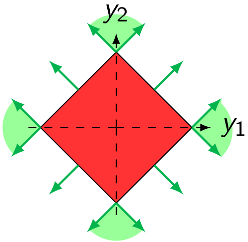

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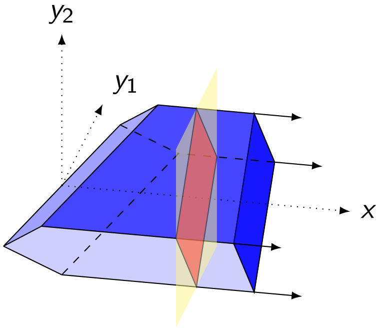

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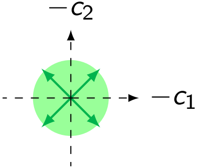

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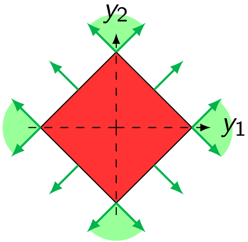

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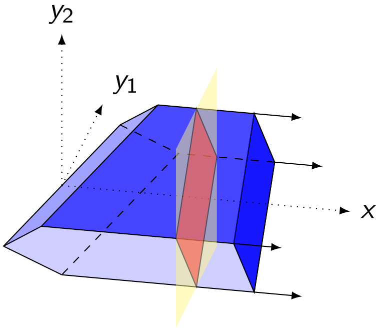

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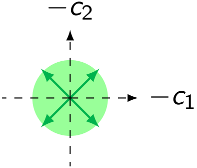

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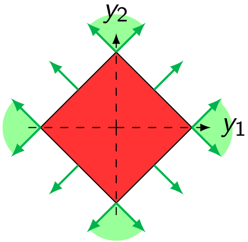

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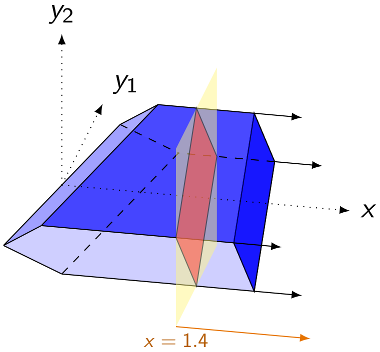

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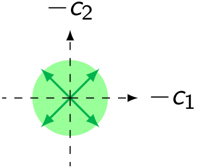

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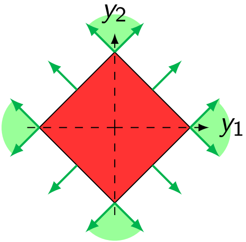

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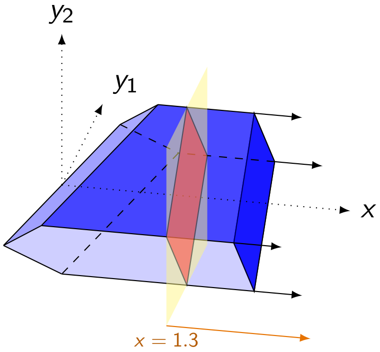

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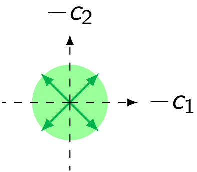

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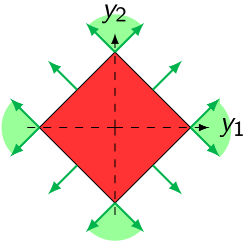

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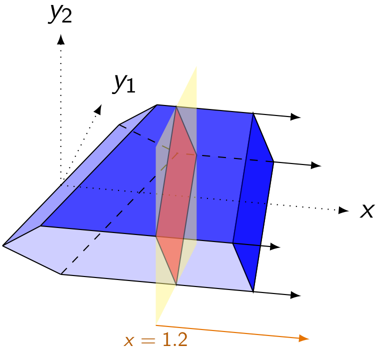

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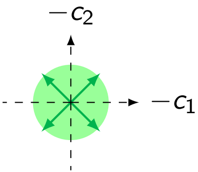

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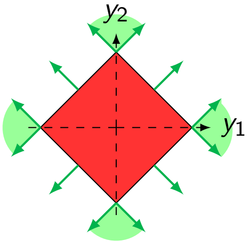

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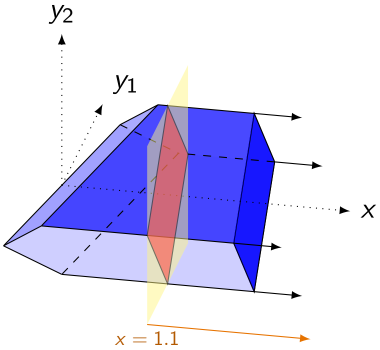

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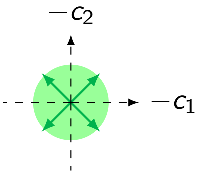

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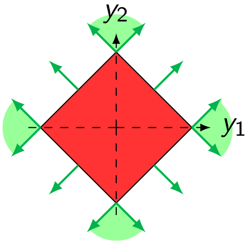

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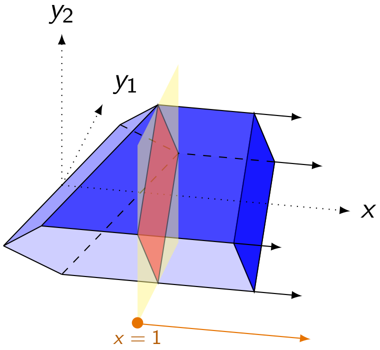

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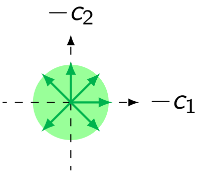

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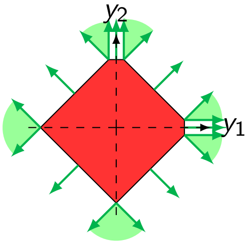

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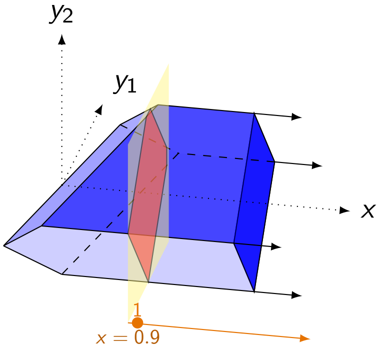

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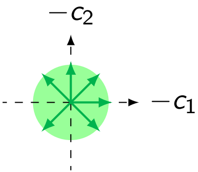

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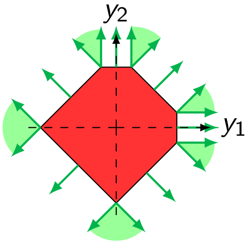

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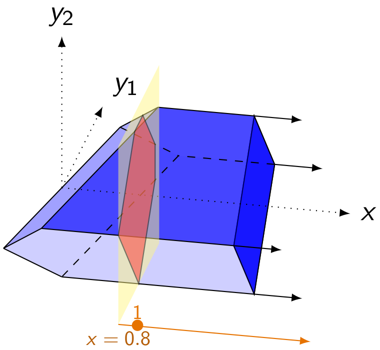

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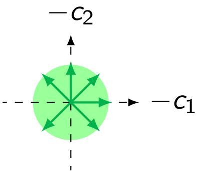

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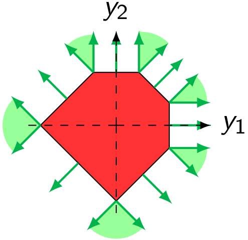

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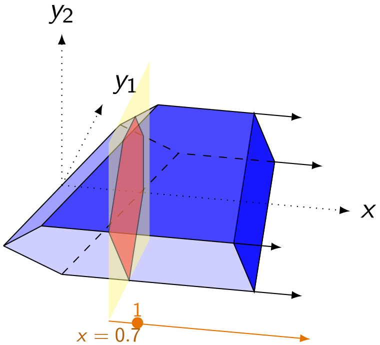

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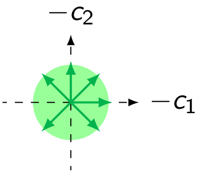

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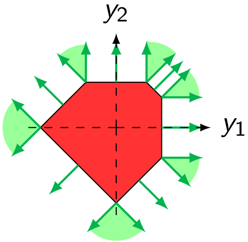

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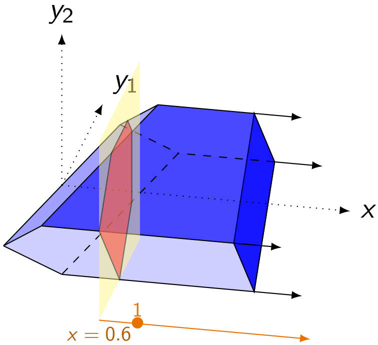

P and P_x

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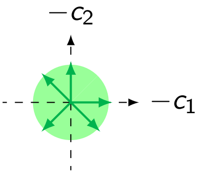
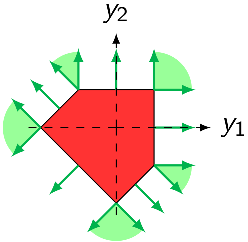


$\mathcal{N}(P_x)$



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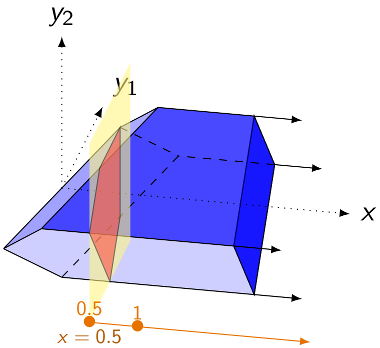

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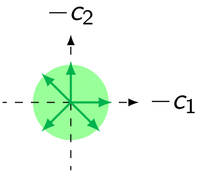

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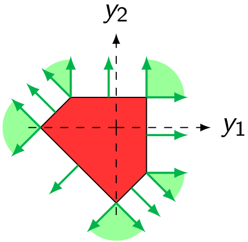

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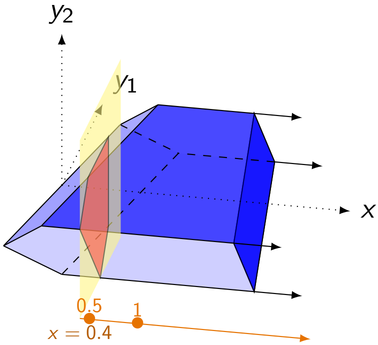

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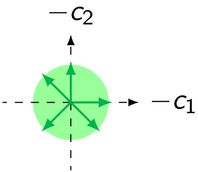

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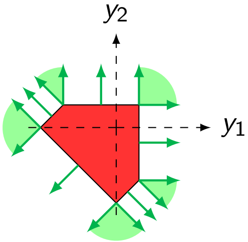

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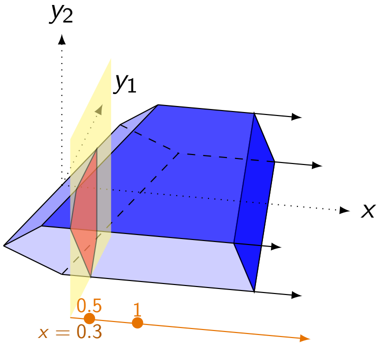

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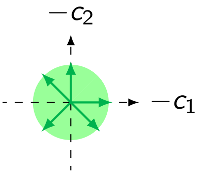

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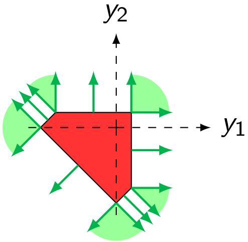

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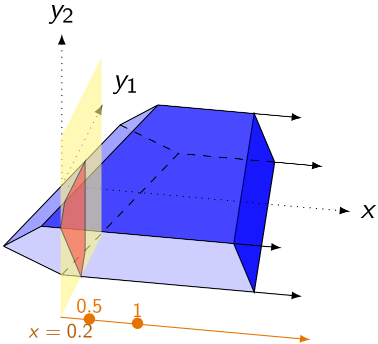

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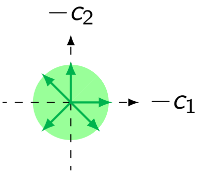

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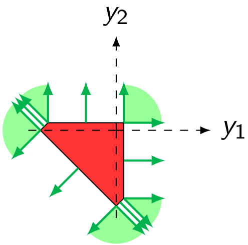

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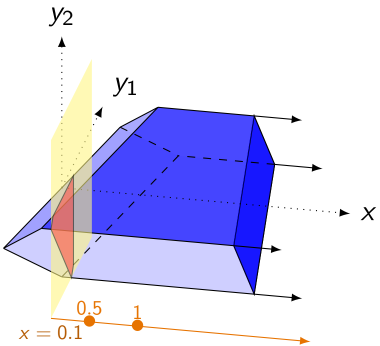

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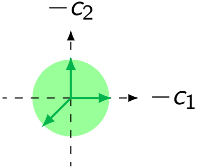

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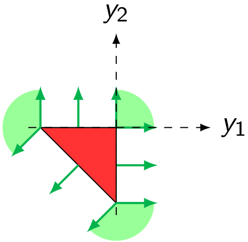

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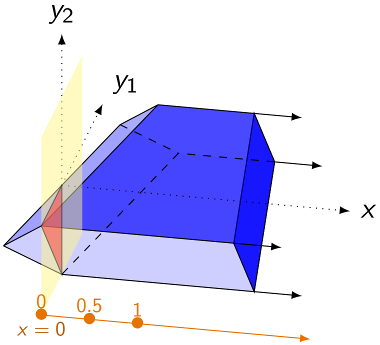

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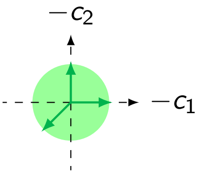

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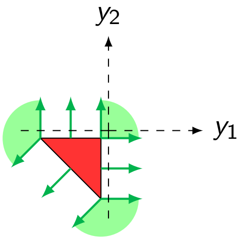

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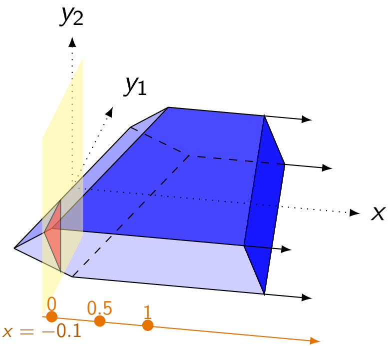

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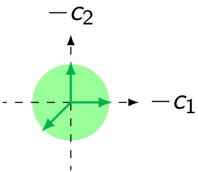

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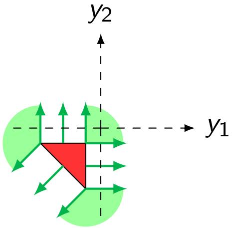

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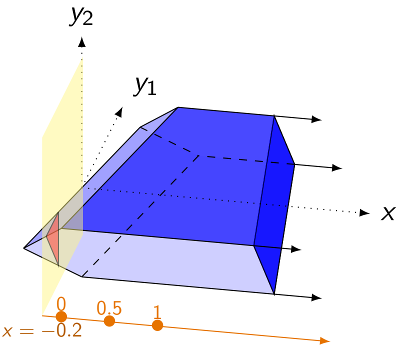

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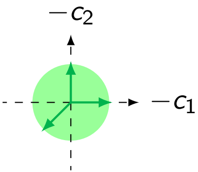

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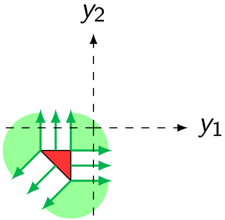

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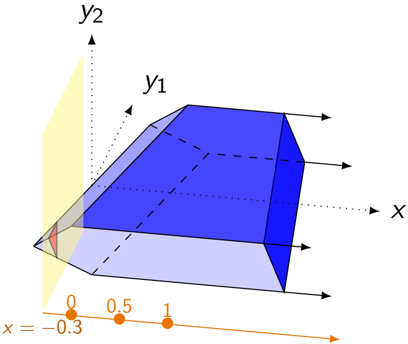

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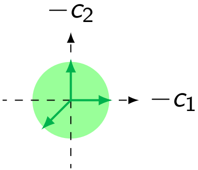

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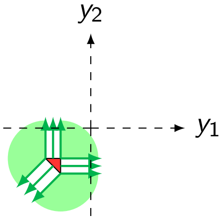

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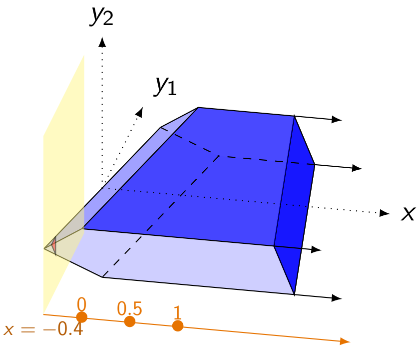


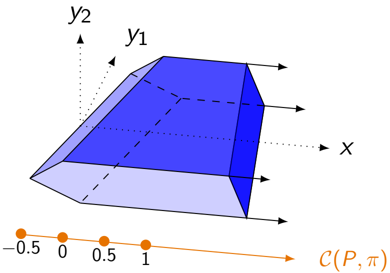
P and P_x

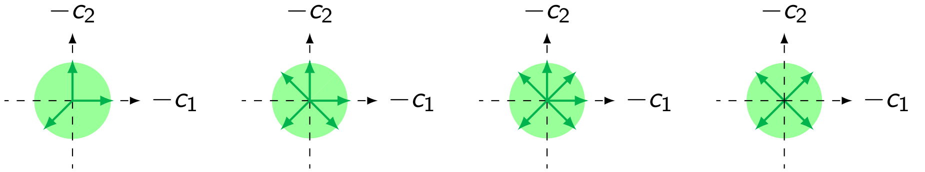
What are the constant regions of $x \mapsto \mathcal{N}(P_x)$?

Proposition

There exists a collection $\mathcal{C}(P,\pi)$ called the **chamber complex** *whose relative interior of cells are the constant regions of $x \mapsto \mathcal{N}(P_x)$.*

I.e, for $\sigma \in \mathcal{C}(P,\pi)$ and $x, x' \in \mathrm{ri}(\sigma)$, we have $\mathcal{N}(P_x) = \mathcal{N}(P_{x'}) =: \mathcal{N}_\sigma$





$\mathcal{N}_\sigma$ for $\sigma = [-0.5, 0]$ $\mathcal{N}_\sigma$ for $\sigma = [0, 0.5]$ $\mathcal{N}_\sigma$ for $\sigma = [0.5, 1]$ $\mathcal{N}_\sigma$ for $\sigma = [1, +\infty)$

Chamber complex

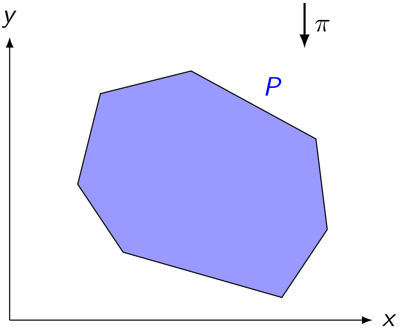
Definition

The *chamber complex* $\mathcal{C}(P,\pi)$ of P along π is

$$\mathcal{C}(P,\pi) := \{\sigma_{P,\pi}(x) \mid x \in \pi(P)\}$$

where

$$\sigma_{P,\pi}(x) := \bigcap_{F \in \mathcal{F}(P) \mid x \in \pi(F)} \pi(F)$$



where $\mathcal{F}(P)$ is the set of faces of P and π is the projection $(x,y) \mapsto x$.

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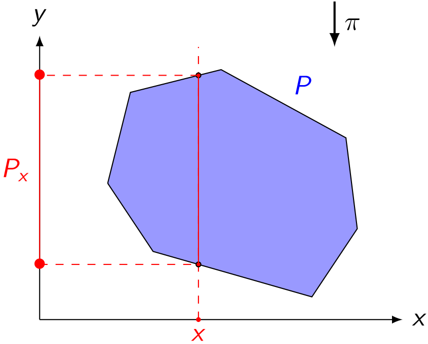

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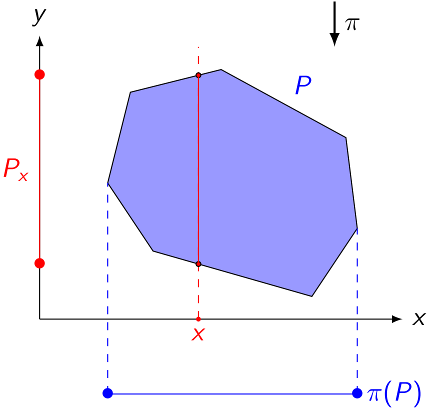

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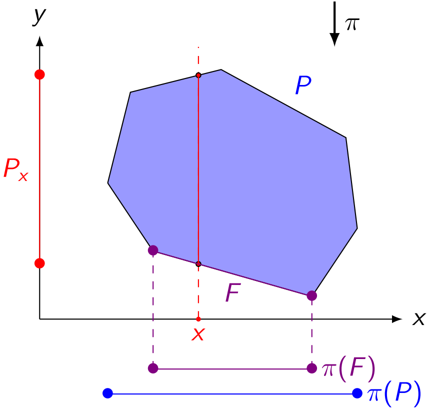

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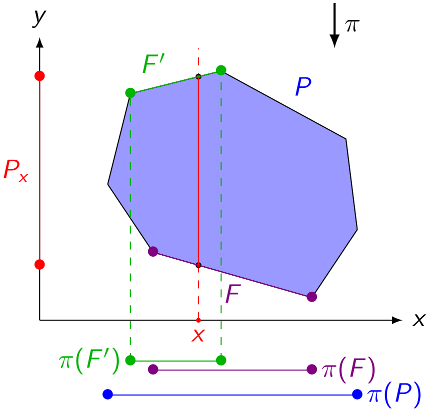

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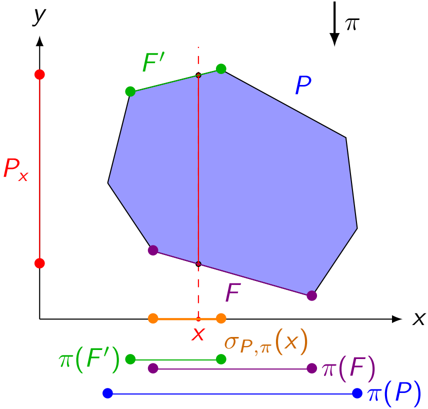

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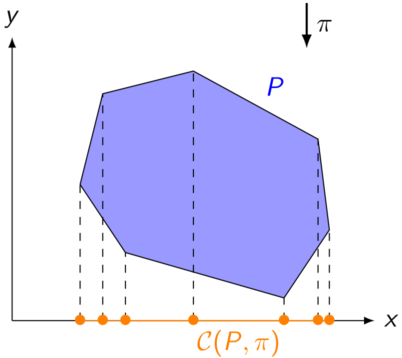
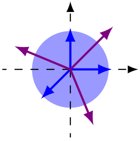


where $\mathcal{F}(P)$ is the set of faces of P and π is the projection $(x,y) \mapsto x$.

Common Refinement of Normal Fans

We can quantize $\boldsymbol{c}$ on each chamber.



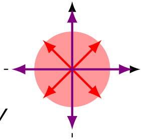
$\mathcal{N}_\sigma$ and $\check{\boldsymbol{c}}$

For all $x \in \mathrm{ri}(\sigma)$,

$$V(x) = \sum_{N \in \mathcal{N}_\sigma} p_N \min_{y \in P_x} \check{c}_N^\top y$$

For all $x' \in \mathrm{ri}(\tau)$,

$$V(x') = \sum_{N \in \mathcal{N}_\tau} p_N \min_{y \in P_x} \check{c}_N^\top y$$



$\mathcal{N}_\tau$ and $\check{\boldsymbol{c}}$

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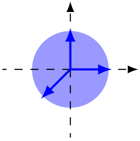

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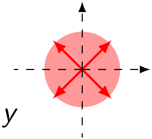
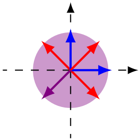


$\mathcal{N}_\tau$

We take the *common refinement*:

$$\mathcal{R} := \mathcal{N}_\sigma \wedge \mathcal{N}_\tau = \{N \cap N' \mid N \in \mathcal{N}_\sigma, N' \in \mathcal{N}_\tau\}$$



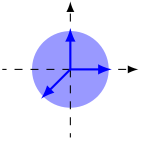
$\mathcal{N}_\sigma \wedge \mathcal{N}_\tau$

For all $x \in \mathrm{ri}(\sigma) \cup \mathrm{ri}(\tau)$,

$$V(x) = \sum_{N \in \mathcal{N}_\sigma \wedge \mathcal{N}_\tau} p_N \min_{y \in P_x} \check{c}_N^\top y$$

Common Refinement of Normal Fans

We can quantize $\boldsymbol{c}$ on each chamber.



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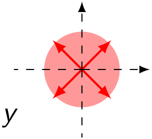

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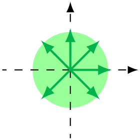

$\mathcal{R}$

For all $x \in \mathrm{ri}(\sigma) \cup \mathrm{ri}(\tau)$,

$$V(x) = \sum_{N \in \mathcal{R}} p_N \min_{y \in P_x} \check{c}_N^\top y$$

Uniform exact quantization for c

Let's sum up:

- local exact quantization at x induced by $\mathcal{N}(P_x)$,
- $x \mapsto \mathcal{N}(P_x)$ is constant on each $\sigma \in \mathcal{C}(P, \pi)$,
- local exact quantization at $\mathrm{ri}(\sigma)$ induced by $\mathcal{N}_\sigma$,
- local exact quantization at $\mathrm{ri}(\sigma) \cup \mathrm{ri}(\tau)$ induced by $\mathcal{N}_\sigma \wedge \mathcal{N}_\tau$.

Uniform exact quantization for $\boldsymbol{c}$

Let's sum up:

- local exact quantization at x induced by $\mathcal{N}(P_x)$,
- $x \mapsto \mathcal{N}(P_x)$ is constant on each $\sigma \in \mathcal{C}(P, \pi)$,
- local exact quantization at $\mathrm{ri}(\sigma)$ induced by $\mathcal{N}_\sigma$,
- local exact quantization at $\mathrm{ri}(\sigma) \cup \mathrm{ri}(\tau)$ induced by $\mathcal{N}_\sigma \wedge \mathcal{N}_\tau$.

Theorem (FGL21, Uniform and universal quantization of the cost)

Let $\mathcal{R} = \bigwedge_{\sigma \in \mathcal{C}(P,\pi)} -\mathcal{N}_\sigma$, *then* **for all** $x \in \mathbb{R}^n$

$$V(x) = \sum_{R \in \mathcal{R}} \check{p}_R \min_{y \in P_x} \check{c}_R^\top y$$

where $\check{p}_R := \mathbb{P}\big[\boldsymbol{c} \in \mathrm{ri}(R)\big]$ *and* $\check{c}_R := \mathbb{E}\big[\boldsymbol{c} \mid \boldsymbol{c} \in \mathrm{ri}(R)\big]$

Polyhedral characterization of V

Theorem (FGL21)

For all distributions of $\boldsymbol{c}$, V is affine on each cell of $\mathcal{C}(P, \pi)$.

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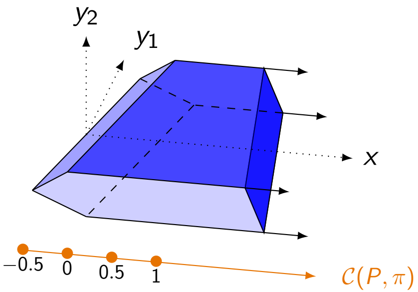
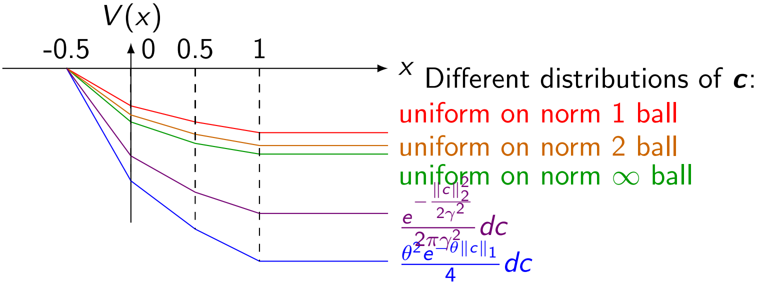
Extension of fiber polytope of

L. Billera, B. Sturmfels, Fiber polytopes, *Annals of Mathematics*, p527–549, 1992.

Explicit computation of the example

$$V(x) = \mathbb{E}\begin{bmatrix} \min\limits_{y\in\mathbb{R}^2} & \boldsymbol{c}^\top y \\ \text{s.t.} & \|y\|_1 \leqslant 1 \\ & y_1 \leqslant x \\ & y_2 \leqslant x \end{bmatrix}$$





Contents

Multistage uniform and universal exact quantization

$$V_t(x) = \mathbb{E}\left[\begin{array}{ll} \min\limits_{y \in \mathbb{R}^{n_t}} & \boldsymbol{c}_t^\top y + V_{t+1}(y) \\ \text{s.t.} & (x, y) \in P_t \end{array}\right]$$

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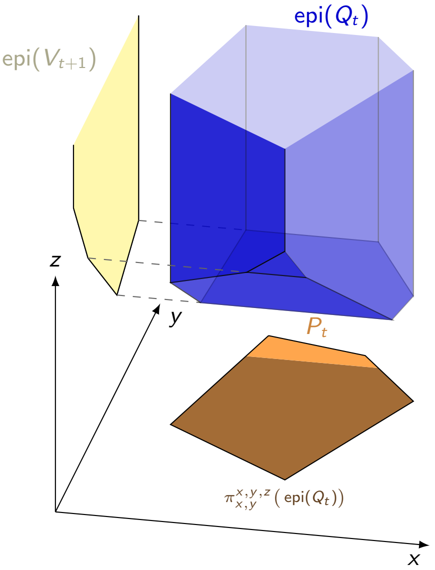

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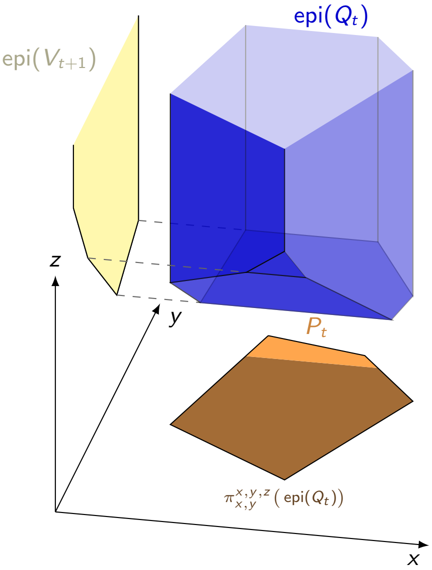

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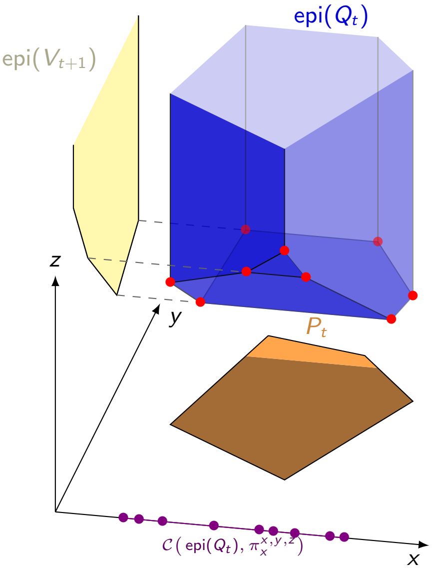

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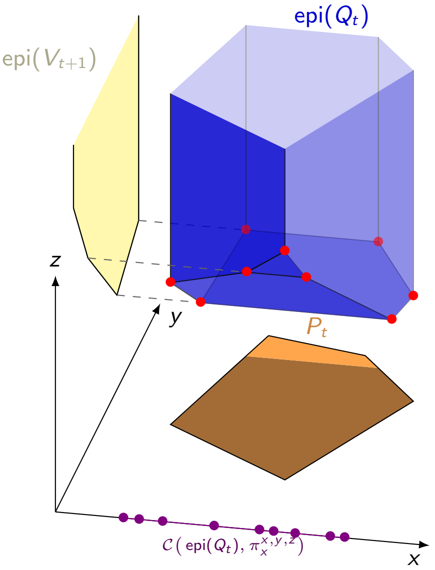

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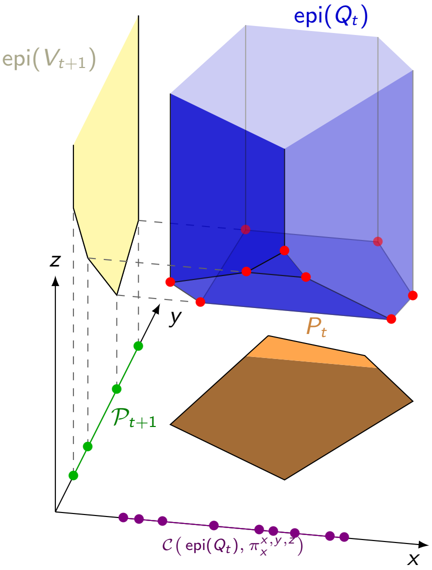

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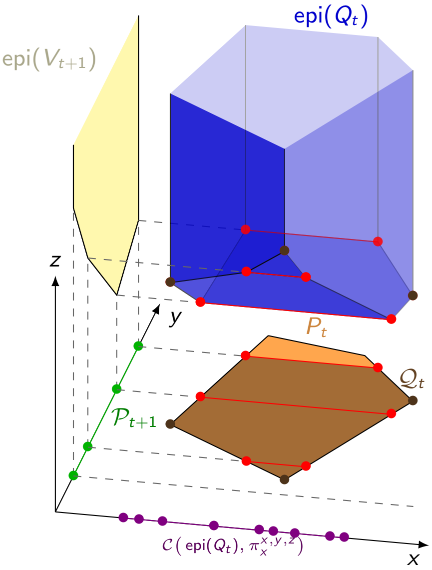

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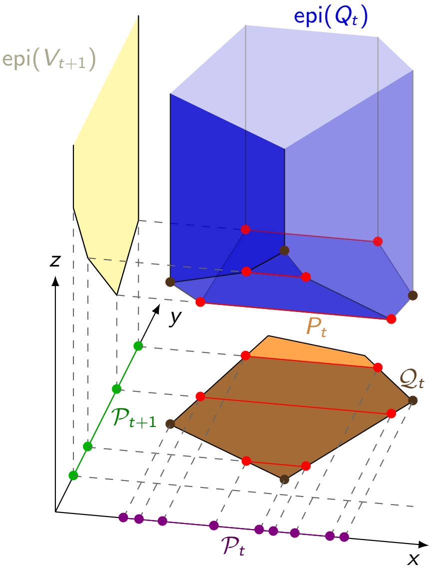

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[FGL21, Lem. 4.1]: $\mathcal{P}_t \preccurlyeq \mathcal{C}(\mathrm{epi}(Q_t), \pi_x^{x,y,z})$

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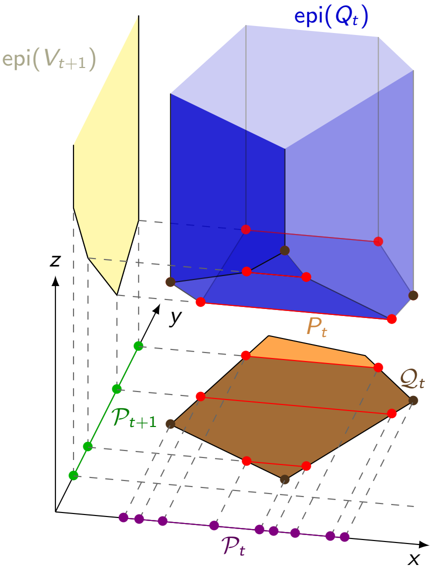

Extension to multistage and stochastic constraints

Iterated chamber complexes by backward induction

$$\mathcal{P}_{t,\xi} := \mathcal{C}\Big((\mathbb{R}^{n_t} \times \mathcal{P}_{t+1}) \wedge \mathcal{F}\big(P_t(\xi)\big), \pi^{x_{t-1},x_t}_{x_{t-1}} \Big)$$

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Theorem (FGL21)

All results generalizes to MSLP with finitely supported stochastic constraints.

- ➥ *$(V_t)_t$ are affine on universal chamber complexes, i.e. independent of the law of $(\boldsymbol{c}_t)_t$*
- ➥ *We have an uniform and universal exact quantization.*

Contents

Earlier and new complexity results

Volume of a polytope

$$\mathrm{Vol}\left(\{z \in \mathbb{R}^d \mid Az \leqslant b\}\right) \text{ or}$$
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Complexity result multistage

Theorem (FGL21: MSLP is polynomial for fixed dimensions)

Assume that T, $n_2, \cdots, n_T$, are fixed.[1]
Assume that $\boldsymbol{c}$ admits a density function with a bounded total variation.

Then, there exists an algorithm that either asserts that MSLP is unfeasible or finds an ε-solution in polynomial time in $\log(\frac{1}{\varepsilon})$ with probability 1.

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Same with SDDP: [Lan 2020][Zhang and Sun 2020]

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Explicit formulas for usual distributions

We need to compute the quantized probalit $\check{p}_S = \mathbb{P}\big[\boldsymbol{c} \in S\big]$ and the quantized cost $\check{c}_S = \mathbb{E}\big[\boldsymbol{\xi} \mid \boldsymbol{c} \in S\big]$ when S is a polyhedron.

Explicit formulas, valid for S full dimensional simplex or simplicial cone, which can be used through triangulation.

Distribution	Uniform on polytope	Exponential	Gaussian
$d\mathbb{P}(\xi)$	$\frac{\mathbb{1}_{\xi\in Q}}{\mathrm{Vol}_d(Q)}\mathcal{L}_{\mathrm{Aff}(Q)}(d\xi)$	$\frac{e^{\theta^\top \xi}\mathbb{1}_{\xi\in K}}{\Phi_K(\theta)}\mathcal{L}_{\mathrm{Aff}(K)}(d\xi)$	$\frac{e^{-\frac{1}{2}\xi^\top M^{-2}\xi}}{(2\pi)^{\frac{m}{2}}\det M}d\xi$
Support	Polytope : Q	Cone : K	$\mathbb{R}^m$
$\check{p}_S$	$\dfrac{\mathrm{Vol}_d(S)}{\mathrm{Vol}_d(Q)}$	$\dfrac{\lvert\det(\mathrm{Ray}(S))\rvert}{\Phi_K(\theta)}\prod_{r\in\mathrm{Ray}(S)}\dfrac{1}{-r^\top\theta}$	$\mathrm{Ang}\left(M^{-1}S\right)$
$\check{c}_S$	$\frac{1}{d}\sum_{v\in\mathrm{Vert}(S)} v$	$\left(\sum_{r\in\mathrm{Ray}(S)}\frac{-r_i}{r^\top\theta}\right)_{i\in[m]}$	$\frac{\sqrt{2}\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})}M\,\mathrm{Ctr}\left(S\cap\mathbb{S}_{m-1}\right)$

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$\check{c}_S$	$\frac{1}{d} \sum_{v \in \mathrm{Vert}(S)} v$	$\left(\sum_{r \in \mathrm{Ray}(S)} \frac{-r_i}{r^\top \theta} \right)_{i \in [m]}$	$\frac{\sqrt{2}\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})} M \,\mathrm{Ctr}\left(S \cap \mathbb{S}_{m-1}\right)$

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2 stage stochastic linear programming (2SLP)

$$\begin{aligned} \min_{x \in \mathbb{R}^n_+} \quad & c^\top x + \mathbb{E}\big[Q(x, \boldsymbol{\xi})\big] \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

where $\boldsymbol{\xi} = (\boldsymbol{T}, \boldsymbol{h})$ is random whereas q and W are deterministic[1]

$$Q(x, \xi) := \min_{y \in \mathbb{R}^m_+} q^\top y \quad = \max_{\lambda \in \mathbb{R}^n} (h - Tx)^\top \lambda$$

$$\text{s.t.} \quad Tx + Wy = h \qquad \text{s.t.} \quad W^\top \lambda \leqslant q$$

We define

$$X := \{x \in \mathbb{R}^n_+ \mid Ax = b\} \qquad D := \{\lambda \in \mathbb{R}^\ell \mid W^\top \lambda \leqslant q\}$$

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$$X := \{x \in \mathbb{R}^n_+ \mid Ax = b\} \qquad D := \{\lambda \in \mathbb{R}^\ell \mid W^\top \lambda \leqslant q\}$$

[1] Can be extended to generic random $\boldsymbol{q}$, and finitely supported $\boldsymbol{W}$

2 stage stochastic linear programming (2SLP)

$$\min_{x \in X} \qquad c^\top x + \mathbb{E}\big[Q(x, \boldsymbol{\xi})\big]$$

where $\boldsymbol{\xi} = (\boldsymbol{T}, \boldsymbol{h})$ is random whereas q and W are deterministic[1]

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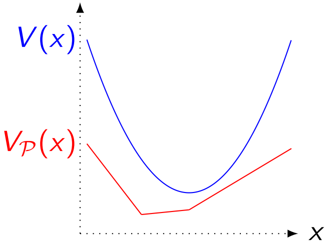
Properties of partitioned cost-to-go

Recall that

$$V(x) = \mathbb{E}\Big[Q(x,\boldsymbol{\xi})\Big]$$

$$V_{\mathcal{P}}(x) = \sum_{P\in\mathcal{P}} \mathbb{P}\big[P\big]\, Q\Big(x, \mathbb{E}\big[\boldsymbol{\xi}|P\big]\Big)$$

- $Q(x,\cdot)$ is convex $\rightsquigarrow V_{\mathcal{P}} \leqslant V$.
- $Q\big(\cdot, \mathbb{E}\big[\boldsymbol{\xi}|P\big]\big)$ is polyhedral $\rightsquigarrow V_{\mathcal{P}}$ is polyhedral.



Finally,

$$\min_{x\in X} c^\top x + V_{\mathcal{P}}(x) \qquad (2SLP_{\mathcal{P}})$$

is equivalent to

$$\min_{x\in X,(y_P)_{P\in\mathcal{P}}} \quad c^\top x + \sum_{P\in\mathcal{P}} \mathbb{P}\big[P\big] q^\top y_P$$

$$\mathbb{E}\big[\boldsymbol{T}|P\big]x + Wy_P \leqslant \mathbb{E}\big[\boldsymbol{h}|P\big] \qquad \forall P \in \mathcal{P}$$

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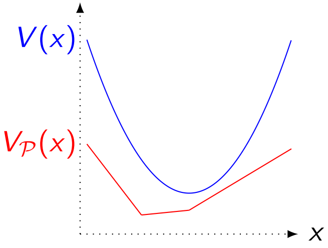

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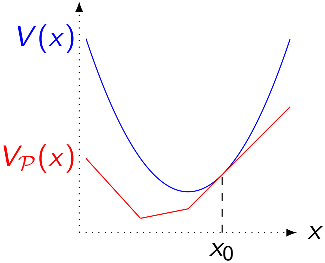
Adapted partition

Definition

We say that a partition $\mathcal{P}$ is adapted to x_0 if

$$V_{\mathcal{P}}(x_0) = V(x_0) := \mathbb{E}\left[Q(x_0, \xi)\right]$$



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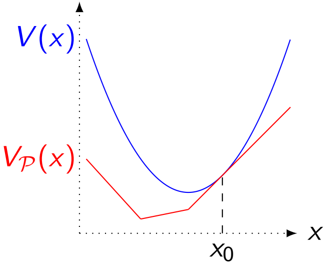

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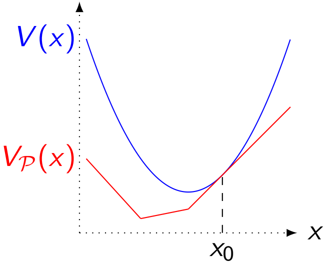

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Refinement

$\mathcal{R}$ refines $\mathcal{P}$ ($\mathcal{R} \preccurlyeq \mathcal{P}$) if

$$\forall R \in \mathcal{R}, \exists P \in P, R \subset P$$

[$\mathcal{R} \preccurlyeq_{\mathbb{P}} \mathcal{P}$ if $\mathcal{R}$ refines $\mathcal{P}$ up to $\mathbb{P}$-null sets.]

$\mathcal{P}$ $\mathcal{R}$

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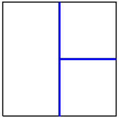
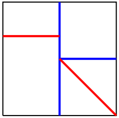


$\mathcal{P}$



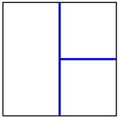
$\mathcal{R}$

The common refinement of $\mathcal{P}$ and $\mathcal{P}'$ is

$$\mathcal{P} \wedge \mathcal{P}' := \{P \cap P' \mid P \in \mathcal{P}, P' \in \mathcal{P}'\}$$

Since $\mathcal{P} \wedge \mathcal{P}'$ refines $\mathcal{P}$ and $\mathcal{P}'$

$$\max(V_{\mathcal{P}}, V_{\mathcal{P}'}) \leqslant V_{\mathcal{P} \wedge \mathcal{P}'}$$



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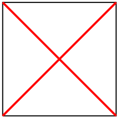
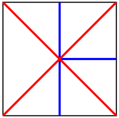


$\mathcal{P}'$



$\mathcal{P} \wedge \mathcal{P}'$

General framework for APM

$k \leftarrow 0$, $z_U^0 \leftarrow +\infty$, $z_L^0 \leftarrow -\infty$, $\mathcal{P}^0 \leftarrow \{\Xi\}$;

while $z_U^k - z_L^k > \varepsilon$ **do**

$k \leftarrow k+1$;

Solve (for x^k) $\quad z_L^k \leftarrow \min_{x \in X} c^\top x + V_{\mathcal{P}^{k-1}}(x)$;

$\mathcal{P}_{x^k} \leftarrow \mathrm{Oracle}(x^k)$;

$\mathcal{P}^k \leftarrow \mathcal{P}^{k-1} \wedge \mathcal{P}_{x^k}$;

$z_U^k \leftarrow \min\left(z_U^{k-1}, c^\top x^k + V_{\mathcal{P}^k}(x^k)\right)$;

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Algorithm 1: Generic framework for APM.

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Algorithm 1: Generic framework for APM.

Theorem (FL2021)

If the oracle is adapted, then x^k is an ε-solution of problem (2SLP) for $k \geqslant \left(\frac{L diam(X)}{\varepsilon} + 1\right)^n$.

Previous APM methods

Lemma (Song & Luedtke)

Let $\mathcal{P}$ a partition of Ξ. $\mathcal{P}$ is adapted at x iff for all set of scenarios $P \in \mathcal{P}$, there exists a common optimal multiplier λ_P, i.e.

$$\forall P \in \mathcal{P}, \quad \exists \lambda_P \in D, \quad \forall \xi_k \in P, \qquad \lambda_P \in \underset{\lambda \in D}{\mathrm{argmax}} (h^k - T^k x)^\top \lambda$$

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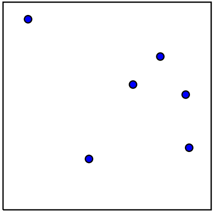
Idea

- Sample a large number of scenario
- without loss of precision aggregate scenarios



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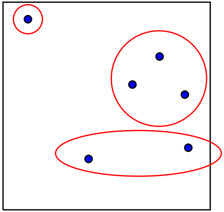

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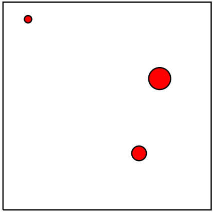

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Lemma (Ramirez-Pico & Moreno)

Let $\mathcal{P}$ a partition of Ξ. If there exists $\lambda(\boldsymbol{\xi})$ such that, for all $P \in \mathcal{P}$,

$$\mathbb{E}\left[\boldsymbol{h}|P\right]^\top \mathbb{E}\left[\lambda(\boldsymbol{\xi})|P\right] = \mathbb{E}\left[\boldsymbol{h}^\top \lambda(\boldsymbol{\xi})|P\right]$$
$$x^\top \mathbb{E}\left[\boldsymbol{T}|P\right]^\top \mathbb{E}\left[\lambda(\boldsymbol{\xi})|P\right] = x^\top \mathbb{E}\left[\boldsymbol{T}^\top \lambda(\boldsymbol{\xi})|P\right]$$

then $\mathcal{P}$ is an adapted partition.

A (partial) comparison between partition based results

Paper	Song, Luedtke (2015)	Ramirez-Pico, Moreno (2020)	F., Leclère (2021)
Non-finite supp(ξ)	$\times$	$\checkmark$	$\checkmark$
Explicit oracle	$\checkmark$	$\times$	$\checkmark$
Proof of convergence	$\checkmark$	$\times$	$\checkmark$
Complexity result	$\times$	$\times$	$\checkmark$
Fast iteration	$\checkmark$	$\times$	$\times$

Local exact quantization and adapted partition

Local exact quantization

random cost

Recall that for a fixed x,

$$\mathbb{E}\Big[\min_{y \in P_x} \boldsymbol{c}^\top y\Big] = \sum_{N \in \mathcal{N}(P_x)} p_N \min_{y \in P_x} \check{c}_N{}^\top y$$

where,

$$p_N := \mathbb{P}\big[\boldsymbol{c} \in -\operatorname{ri} N\big]$$
$$\check{c}_N := \mathbb{E}\big[\boldsymbol{c} \mid \boldsymbol{c} \in -\operatorname{ri} N\big]$$

$$P_x := \{y \in \mathbb{R}^m \mid Ay + Bx \leqslant b\}$$

GAPM

random constraints

Similarly, for a given q, and all x,

$$\begin{aligned} V(x) &:= \mathbb{E}\big[Q(x, \boldsymbol{\xi})\big] \\ &= \mathbb{E}\Big[\max_{\lambda \in D_q} (\boldsymbol{h} - \boldsymbol{T}x)^\top \lambda\Big] \\ &= \sum_{N \in \mathcal{N}(D_q)} p_N \max_{\lambda \in D_q} \psi_{N,x}{}^\top \lambda \end{aligned}$$

where,

$$p_N := \mathbb{P}\big[\boldsymbol{h} - \boldsymbol{T}x \in \operatorname{ri} N\big]$$
$$\psi_{N,x} := \mathbb{E}\big[\boldsymbol{h} - \boldsymbol{T}x \mid \boldsymbol{h} - \boldsymbol{T}x \in \operatorname{ri} N\big]$$

$$D_q := \{\lambda \in \mathbb{R}^l \mid W^\top \lambda \leqslant q\}$$

An explicit adapted partition

Consider $x \in \mathbb{R}^n$ and $N \in \mathcal{N}(D_q)$ a normal cone of D_q. We define

$$E_{N,x} := \{\xi \in \Xi \mid h - Tx \in \operatorname{ri} N\}$$

Theorem (FL 2021)

$\mathcal{R}_x := \left\{ E_{N,x} \mid N \in \mathcal{N}(D_q) \right\}$ *is an adapted partition to* x
i.e. $V_{\mathcal{R}_x}(x) = V(x)$

Proof:

$$\begin{aligned} V(x) &:= \mathbb{E}\left[Q(x,\boldsymbol{\xi})\right] \\ &= \sum_{N \in \mathcal{N}(D)} \mathbb{P}\left[\boldsymbol{h} - \boldsymbol{T}x \in \operatorname{ri} N\right] \min_{\lambda \in D} \mathbb{E}\left[\boldsymbol{h} - \boldsymbol{T}x \,|\, \boldsymbol{h} - \boldsymbol{T}x \in \operatorname{ri} N\right]^\top \lambda \\ &= \sum_{N \in \mathcal{N}(D)} \mathbb{P}\left[\boldsymbol{\xi} \in E_{N,x}\right] Q\Big(\mathbb{E}\left[\boldsymbol{\xi} \,|\, \boldsymbol{\xi} \in E_{N,x}\right], x\Big) = V_{\mathcal{R}_x}(x) \end{aligned}$$

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➥ Is it the coarsest one ?

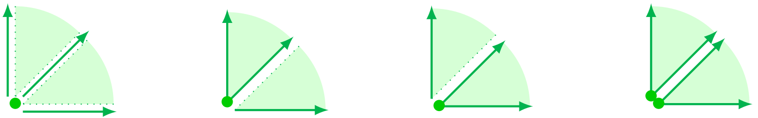
Conditions for a partition to be adapted

Theorem (FL 2021)

For $x \in \mathbb{R}^n$ and $\mathcal{P}$ a partition of Ξ, there exists $\overline{\mathcal{R}}_x \succcurlyeq_{\mathbb{P}} \mathcal{R}_x$ such that

$$\mathcal{P} \preccurlyeq_{\mathbb{P}} \overline{\mathcal{R}}_x \iff V_{\mathcal{P}}(x) = V(x).$$

- If ξ admits a density, $\mathcal{R}_x =_{\mathbb{P}} \overline{\mathcal{R}}_x$.
- An oracle is adapted if and only if it returns a partition $\mathcal{P}$ refining $\overline{\mathcal{R}}_x$.



$\mathcal{R}_x$ $\quad$ $\mathcal{P}$ $\quad$ $\mathcal{P}'$ $\quad$ $\overline{\mathcal{R}}_x$

$$E_{N,x} := \{\xi \in \Xi \mid h - Tx \in \mathrm{ri}(N)\}$$
$$\mathcal{R}_x := \left\{E_{N,x} \mid N \in \mathcal{N}(D_q)\right\}$$

$$\overline{E}_{N,x} := \{\xi \in \Xi \mid h - Tx \in N\}$$
$$\overline{\mathcal{R}}_x := \left\{E_{N,x} \mid N \in \mathcal{N}(D_q)^{\max}\right\}.$$

Subgradient of partition function

Recall that if $\mathcal{P} \preccurlyeq_{\mathbb{P}} \mathcal{R}_x$ then

$$V_{\mathcal{R}_x}(x) = V_{\mathcal{P}}(x) = V(x)$$

$$V_{\mathcal{R}_x}(\cdot) \leqslant V_{\mathcal{P}}(\cdot) \leqslant V(\cdot)$$

Lemma

Let $x \in \mathrm{dom}(V)$ and $\mathcal{P}$ be a refinement of $\mathcal{R}_x$, i.e. $\mathcal{P} \preccurlyeq \mathcal{R}_x$, then

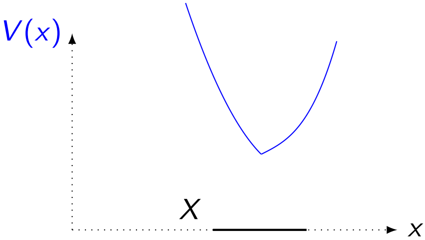
$$\partial V_{\mathcal{R}_x}(x) \subset \partial V_{\mathcal{P}}(x) \subset \partial V(x)$$

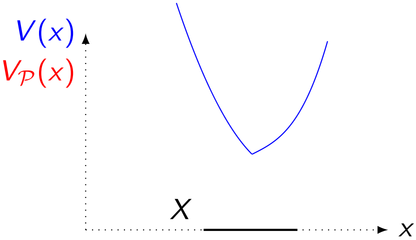
Furthermore, if $x \in \mathrm{ri}\,\mathrm{dom}(V)$,

$$\partial V_{\mathcal{R}_x}(x) = \partial V_{\mathcal{P}}(x) = \partial V(x)$$

Link with Benders decomposition and L-shaped

Partition based method can be seen as a tangent cone method: a cutting plane method where we add all active cuts instead of a single one.





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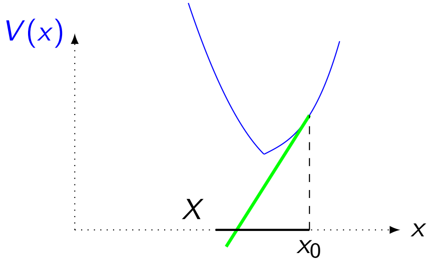

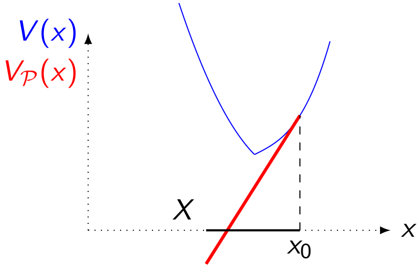

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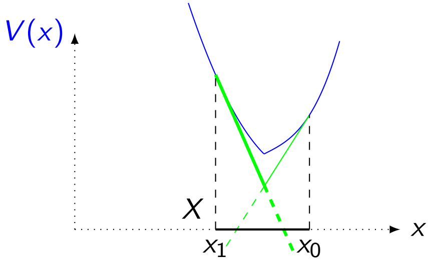

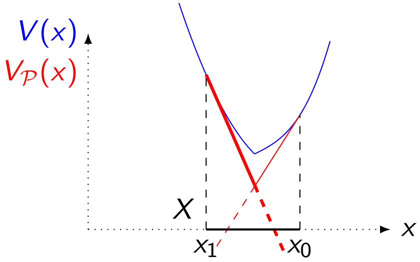

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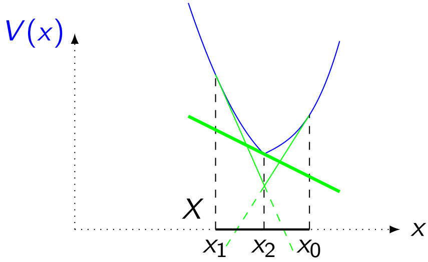

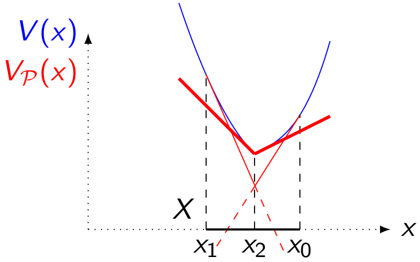

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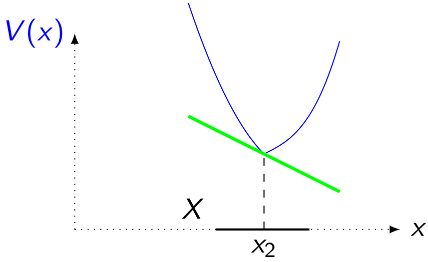

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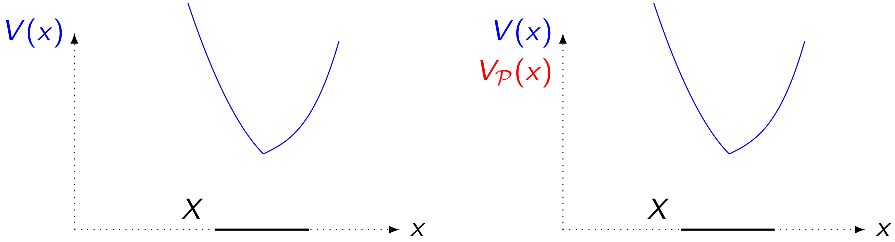
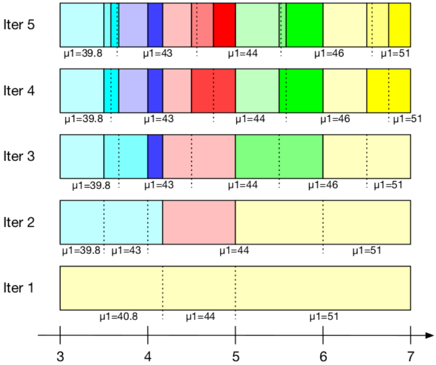


Theorem (Convergence and complexity results)

If $X \cap \mathrm{dom}(V) \subset \mathbb{R}^+$ is contained in a ball of diameter $M \in \mathbb{R}^+$ and $x \to c^\top x + V(x)$ is Lipschitz with constant L then the partition based method finds an ε-solution in at most $\left(\frac{LM}{\varepsilon} + 1\right)^n$ iterations.

Numerical Results - LandS



Iter	x_1	x_2	x_3	x_4
1	0.833	3.000	4.167	4.000
2	2.500	3.000	3.500	3.000
3	1.833	4.000	3.667	2.500
4	2.000	4.167	3.583	2.250
5	1.917	4.083	3.625	2.375
6	1.875	4.042	3.646	2.438

Iter	LB	UB	Gap
1	378.667	382.711	1.0567%
2	380.122	381.100	0.2567%
3	380.601	380.844	0.0640%
4	380.842	380.893	0.0007%
5	380.843	380.856	0.0004%
6	380.844	380.847	0.0002%

Results given by GAPM for LandS problem[2]

[2]illustration from Ramirez-Pico and Moreno

Numerical Results - ProdMix

k	x_k	z_L^k	z_U^k	Gap	$\lvert\mathcal{P}_k^{\max}\rvert$
1	$(1333.33, 66.67)$	-18666.67	-16939.71	9.3%	4
2	$(1441.41, 59.57)$	-17873.01	-17383.73	2.7%	9
3	$(1399.05, 57.91)$	-17789.88	-17659.19	0.74%	16
4	$(1379.98, 56.64)$	-17744.67	-17708.00	0.20%	25
5	$(1371.36, 55.71)$	-17718.96	-17709.05	0.056%	36
6	$(1375.55, 56.21)$	-17713.74	-17711.37	0.013%	49

Table: Results for problem Prod-Mix

To compare our approach with SAA, we solved the same problem 100 times, each with 10 000 scenarios randomly drawn, yielding a 95% confidence interval centered in -17711, with radius 2.2.

Conclusion

	$\boldsymbol{A}$	$(\boldsymbol{B}, \boldsymbol{b})$	$\boldsymbol{c}$
Local	×	✓	✓
Uniform	×	×	✓

- Links with fundamental polyhedral geometry, regular subdivisions and fiber polytope (Chap. 3 and 4).
- *Uniform and universal* exact quantization for $\boldsymbol{c}$ in MSLP (Chap.4).
  - ➥ New complexity results.
- *Local* exact quantization for $\boldsymbol{B}$ and $\boldsymbol{b}$.
  - ➥ Adaptive Partition-based Methods (APM) for general distribution: solves small 2SLP with high precision (Chap. 5).
- Extension of Stochastic Dual Dynamic Programming algorithms and more generally all Trajectory Following Dynamic Programming algorithm for non finitely supported distribution (Chap. 6).

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- Higher order simplex algorithm on the chamber complex for 2SLP.
- 2-time scale MSLP, nested fiber polyhedra and convex bodies.
- Reintroduce approximation or sampling.
- Exact quantization for stochastic integer linear problems
- Understanding the complexity of MSLP.

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Thank you for listening ! Any question ?

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