

# Exact quantization methods for Multistage Stochastic Linear Problem

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# Multistage stochastic linear programming (MSLP)

$$\begin{aligned} \min_{(\mathbf{x}_t)_{t \in [T]}} \quad & \mathbb{E} \left[ \sum_{t=1}^T \mathbf{c}_t^\top \mathbf{x}_t \right] \\ \text{s.t.} \quad & \mathbf{A}_t \mathbf{x}_t + \mathbf{B}_t \mathbf{x}_{t-1} \leq \mathbf{b}_t \quad \forall t \in [T] \\ & \sigma(\mathbf{x}_t) \subset \sigma(\mathbf{c}_\tau, \mathbf{A}_\tau, \mathbf{B}_\tau, \mathbf{b}_\tau)_{\tau \leq t} \quad \forall t \in [T] \\ & \mathbf{x}_0 \equiv \mathbf{x}_0 \text{ given} \end{aligned}$$

$\xi_t = (\mathbf{c}_t, \mathbf{A}_t, \mathbf{B}_t, \mathbf{b}_t)_{t \in [T]}$  is assumed to be **stagewise independent**.

We set  $V_{T+1} \equiv 0$  and:

$$V_t(\mathbf{x}_{t-1}) := \mathbb{E} [\hat{V}_t(\mathbf{x}_{t-1}, \xi_t)] := \mathbb{E} \left[ \begin{array}{ll} \min_{\mathbf{x}_t \in \mathbb{R}^{n_t}} & \mathbf{c}_t^\top \mathbf{x}_t + V_{t+1}(\mathbf{x}_t) \\ \text{s.t.} & \mathbf{A}_t \mathbf{x}_t + \mathbf{B}_t \mathbf{x}_{t-1} \leq \mathbf{b}_t \end{array} \right]$$

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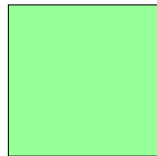
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# Quantization of a MSLP

Real problem

$$V_t(x) = \mathbb{E}[\hat{V}_t(x, \xi_t)] = \mathbb{E} \left[ \begin{array}{ll} \min_{y \in \mathbb{R}^{n_t}} & \mathbf{c}_t^\top y + V_{t+1}(y) \\ \text{s.t.} & \mathbf{A}_t y + \mathbf{B}_t x \leq \mathbf{b}_t \end{array} \right]$$

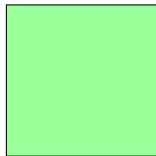


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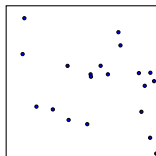


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Sample Average Approximation (SAA)

$$V_{t,N}^{SAA}(x) := \frac{1}{N} \sum_{k=1}^N \hat{V}_t(x, \xi^k)$$

$\xi^1, \dots, \xi^N$  drawn by Monte Carlo

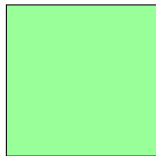


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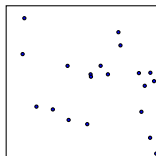


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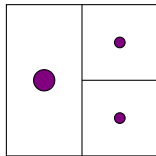


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$$V_{t,P}(x) := \sum_{P \in \mathcal{P}} \check{p}_{t,P} \hat{V}_t(x, \check{\xi}_{t,P})$$

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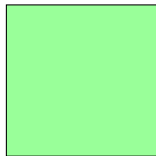


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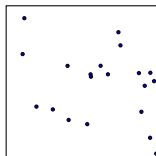


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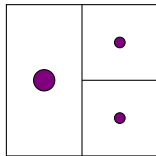


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$$V_{t,\mathcal{P}}(x) := \sum_{P \in \mathcal{P}} \check{p}_{t,P} \hat{V}_t(x, \check{\xi}_{t,P})$$

with  $\check{p}_{t,P} := \mathbb{P}[\xi_t \in P]$  and  $\check{\xi}_{t,P} := \mathbb{E}[\xi_t | \xi_t \in P]$   
If  $\xi \mapsto \hat{V}(x, \xi)$  is convex,  $V_{t,\mathcal{P}}(x) \leq V_t(x)$ .



Partition-based



# Exact quantization

## Definition

A MSP admits a **local exact quantization** at time  $t$  on  $x$  if there exists a finitely supported  $(\check{\xi}_t)_{t \in [T]}$  i.e. such that

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We call an exact quantization

- **uniform** if it is locally exact at all  $x \in \mathbb{R}^{n_t}$ , and all  $t \in [T]$ .
- **universal** if there exists a partition  $\mathcal{P}_{t,x}$  such that the induced quantization is exact at time  $t$  on  $x$ , for all distributions of  $(\xi_\tau)_{\tau \in [T]}$ .

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## A first counter example

Assume  $V_{t+1} \equiv 0$  for now and denote  $V := V_t$ ,  $\hat{V} := \hat{V}_t$  and  $\xi := \xi_t$ .

Let  $\mathbf{A} = (-\mathbf{u})$ ,  $\mathbf{B} \equiv (0)$ ,  $\mathbf{b} \equiv (-1)$  where  $\mathbf{u} \sim \mathcal{U}([1, 2])$ . Then,

$$\hat{V}(x, \xi) = \min_{y \in \mathbb{R}} y \quad \text{s.t.} \quad \mathbf{u}y \geq 1 = \frac{1}{\mathbf{u}}$$

By strict convexity, for all partition  $\mathcal{P}$

$$\sum_{P \in \mathcal{P}} \check{p}_P \hat{V}(x, \check{\xi}_P) < V(x) = \mathbb{E} \left[ \frac{1}{\mathbf{u}} \right]$$

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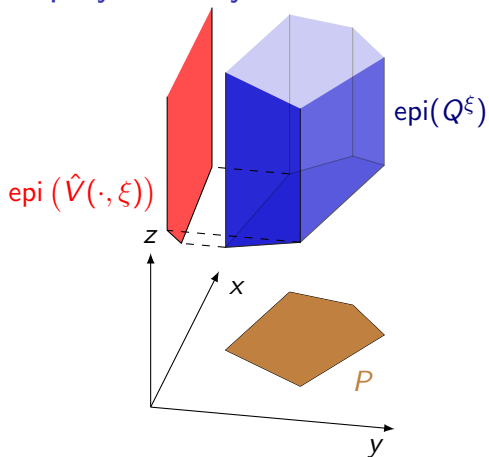
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$$\hat{V}(x, \xi) = \min_{y \in \mathbb{R}^m} c^\top y$$

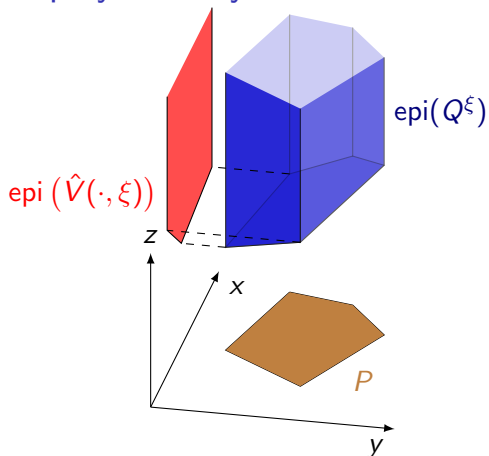
s.t.  $Ay + Bx \leq h$



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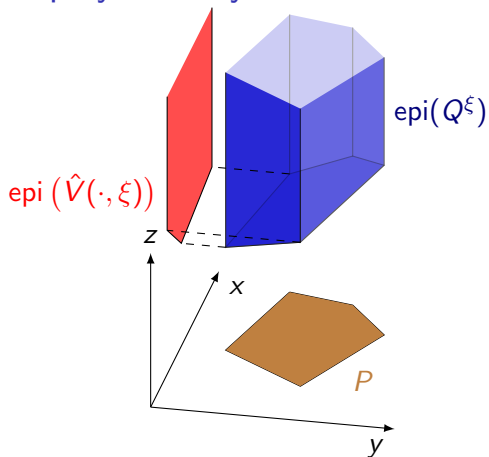




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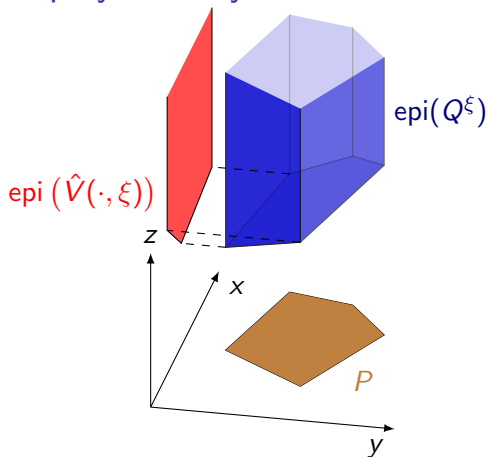


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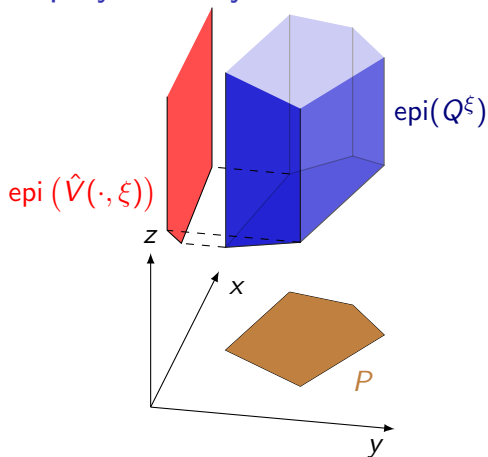


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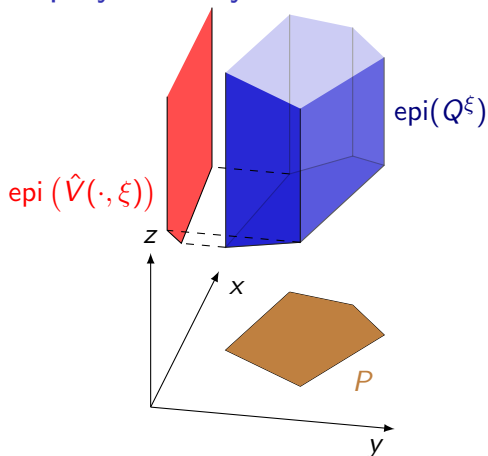
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- ➡ If the noise is finitely supported, then  $V$  is polyhedral
- ➡ Existence of uniform exact quantization implies polyhedrality of  $V$ .

# Counter examples with stochastic constraints

Stochastic  $\mathbf{B}$

$$\begin{aligned} V(x) &= \mathbb{E} \left[ \begin{array}{l} \min_{y \in \mathbb{R}^m} \quad y \\ \text{s.t.} \quad \mathbf{u}x - y \leq 0 \\ \quad \quad y \geq 1 \end{array} \right] \\ &= \mathbb{E} [\max(\mathbf{u}x, 1)] \\ &= \begin{cases} 1 & \text{if } x \leq 1 \\ \frac{x}{2} + \frac{1}{2x} & \text{if } x \geq 1 \end{cases} \end{aligned}$$

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Remaining case: only  $c$  stochastic

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### Theorem (FGL 2021)

If  $A$ ,  $B$  and  $b$  are deterministic,  
then there exists a *universal and uniform* exact quantization.

➡ This extends to finitely supported random  $A$ ,  $B$  and  $b$ .

Let's dive in !

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# Contents

- 1 Local and Universal Exact Quantization for cost in 2-stage
- 2 Uniform and Universal Exact Quantization for cost in 2-stage
- 3 Uniform and Universal Exact Quantization for cost in multistage
- 4 Complexity results

# Reformulation of $V(x)$ highlighting the role of the fiber $P_x$

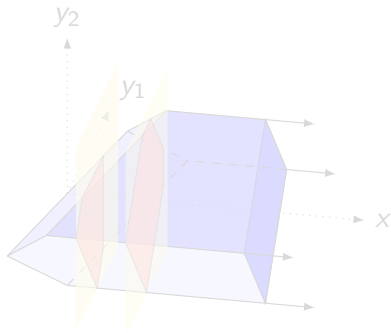
For a given  $x$ , (we still assume  $V_{t+1} \equiv 0$ )

$$V(x) := \mathbb{E} \left[ \min_{y \in \mathbb{R}^m} \mathbf{c}^\top y \right. \\ \left. \text{s.t. } Ay + Bx \leq b \right]$$

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Illustrative running example:

$$P_x := \{y \in \mathbb{R}^m \mid \|y\|_1 \leq 1, \\ y_1 \leq x, y_2 \leq x\}$$



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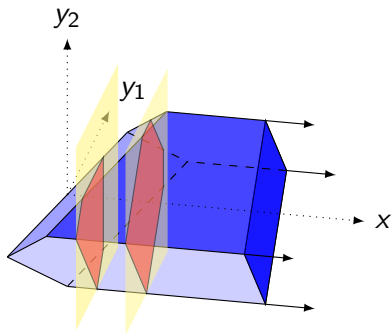
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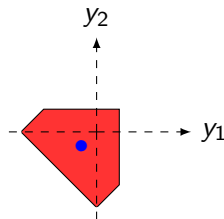
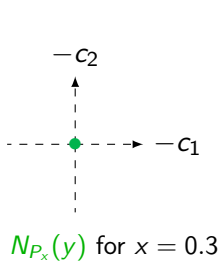
# Normal fan $\mathcal{N}(P_x)$

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$P_x$ ,  $y$  and  $N_{P_x}(y)$  for  $x = 0.3$

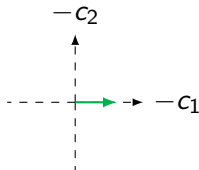
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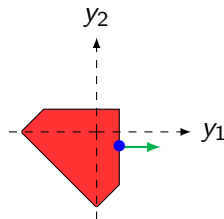
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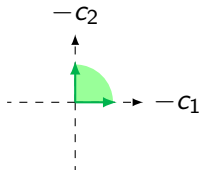
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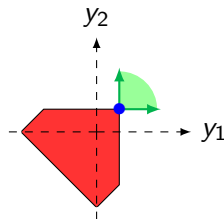
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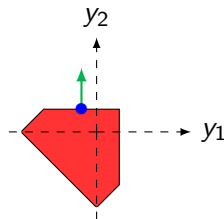
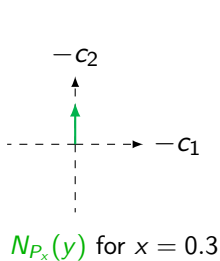
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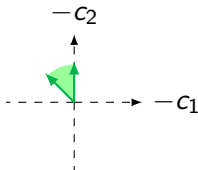
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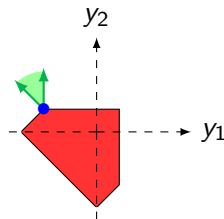
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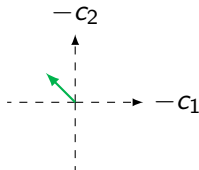
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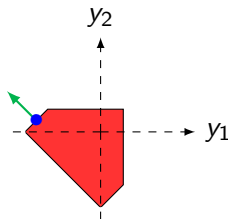
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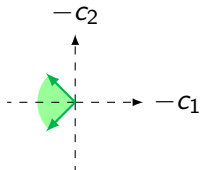
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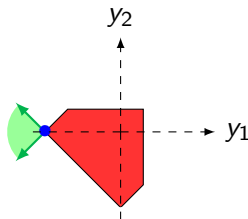
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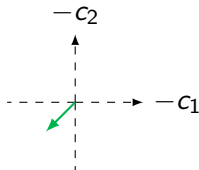
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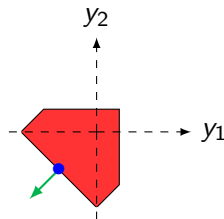
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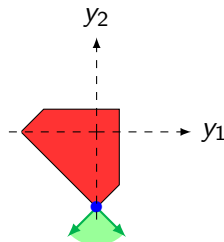
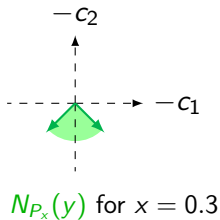
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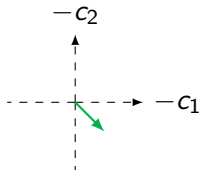
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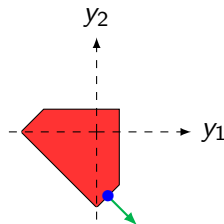
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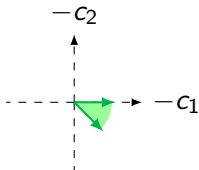
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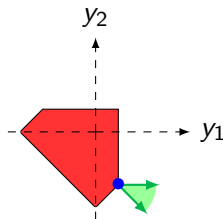
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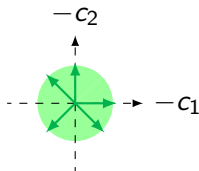
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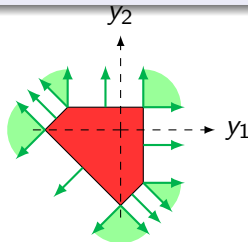
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## Proposition

If  $P_x$  is bounded,  $\{\text{ri}(N) \mid N \in \mathcal{N}(P_x)\}$  is a partition of  $\mathbb{R}^m$ .



$\mathcal{N}(P_x)$  for  $x = 0.3$



$P_x$  and  $\mathcal{N}(P_x)$  for  $x = 0.3$

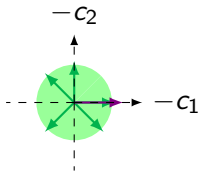
$\mathcal{N}(P_x)$ : partition of cost coherent with the min

For a given  $x$ , we have

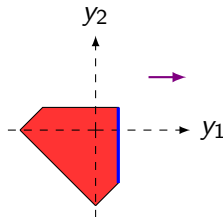
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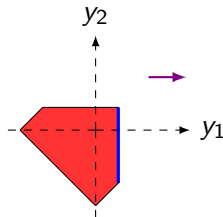
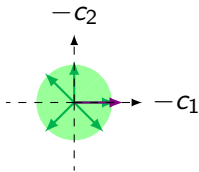
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Cost  $-c$  and  $\mathcal{N}(P_x)$  for  $x = 0.3$

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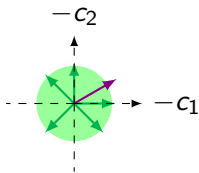
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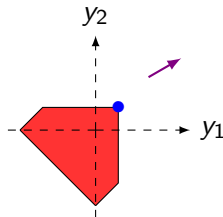
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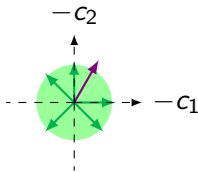
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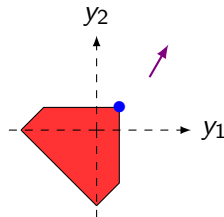
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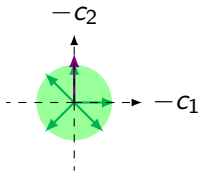
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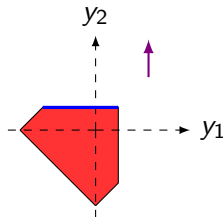
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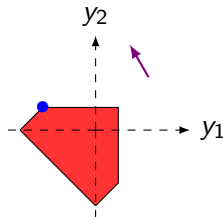
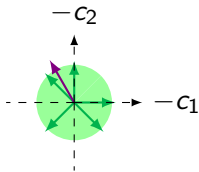
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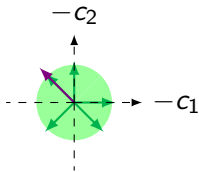
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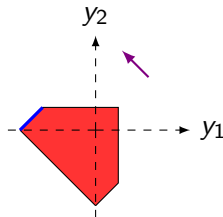
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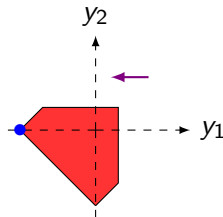
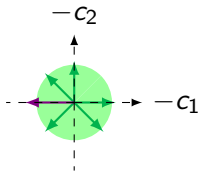
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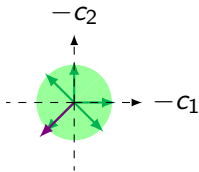
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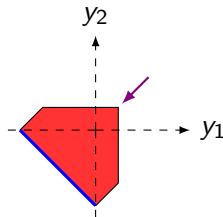
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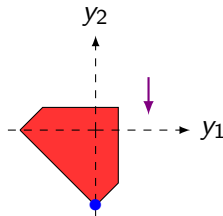
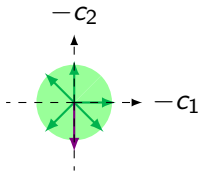
$\mathcal{N}(P_x)$ : partition of cost coherent with the min

For a given  $x$ , we have

$$V(x) = \mathbb{E} \left[ \min_{y \in P_x} c^\top y \right]$$

For any  $N \in \mathcal{N}(P_x)$ ,  $-c \mapsto \arg \min_{y \in P_x} c^\top y$  is constant for all  $-c \in \text{ri}(N)$ .

$\arg \min_{y \in P_x} c^\top y$  is a face of  $P_x$ .



Cost  $-c$  and  $\mathcal{N}(P_x)$  for  $x = 0.3$

$P_x$  for  $x = 0.3$

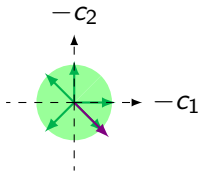
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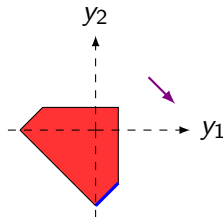
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Cost  $-c$  and  $\mathcal{N}(P_x)$  for  $x = 0.3$



$P_x$  for  $x = 0.3$

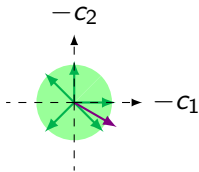
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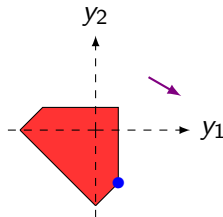
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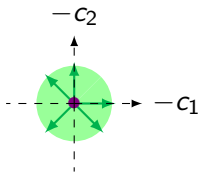
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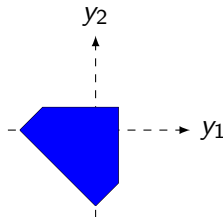
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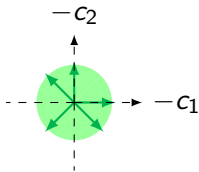
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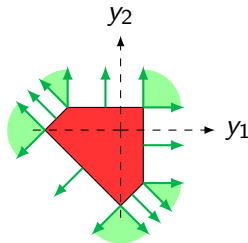
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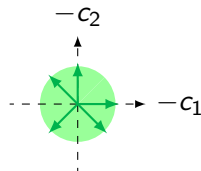
$\mathcal{N}(P_x)$  for  $x = 0.3$



$P_x$  for  $x = 0.3$

# Local and universal exact quantization for $c$

$$\begin{aligned} V(x) &= \mathbb{E} \left[ \min_{y \in P_x} c^\top y \right] \\ &= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E} \left[ \mathbb{1}_{c \in -\text{ri } N} \min_{y \in P_x} c^\top y \right] \end{aligned}$$



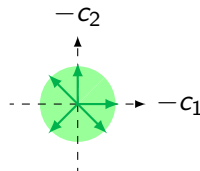
$\mathcal{N}(P_x)$

for  $x = 0.3$



# Local and universal exact quantization for $c$

$$\begin{aligned}
 V(x) &= \mathbb{E} \left[ \min_{y \in P_x} c^\top y \right] \\
 &= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E} \left[ \mathbb{1}_{c \in -ri N} \min_{y \in P_x} c^\top y \right] \quad \text{where } y_N \in \arg \min_{y \in -ri N} c^\top y. \\
 &= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E} \left[ \mathbb{1}_{c \in -ri N} c^\top \right] y_N(x)
 \end{aligned}$$

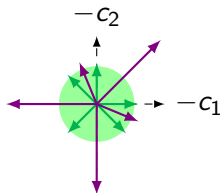


$\mathcal{N}(P_x)$

for  $x = 0.3$

# Local and universal exact quantization for $\mathbf{c}$

$$\begin{aligned}
 V(x) &= \mathbb{E} \left[ \min_{y \in P_x} \mathbf{c}^\top y \right] \\
 &= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E} \left[ \mathbb{1}_{\mathbf{c} \in -\text{ri } N} \min_{y \in P_x} \mathbf{c}^\top y \right] \quad \text{where } y_N \in \arg \min_{y \in -\text{ri } N} \mathbf{c}^\top y. \\
 &= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E} \left[ \mathbb{1}_{\mathbf{c} \in -\text{ri } N} \mathbf{c}^\top \right] y_N(x) \\
 &= \sum_{N \in \mathcal{N}(P_x)} p_N \check{\mathbf{c}}_N^\top y_N(x)
 \end{aligned}$$



$\mathcal{N}(P_x)$  and  $p_N \check{\mathbf{c}}_N$  for  $x = 0.3$

For  $N \in \mathcal{N}(P_x)$ ,

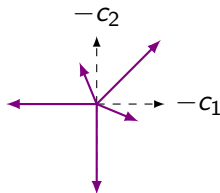
$$p_N := \mathbb{P}[\mathbf{c} \in -\text{ri } N]$$

$$\check{\mathbf{c}}_N := \mathbb{E}[\mathbf{c} \mid \mathbf{c} \in -\text{ri } N]$$

We replace the continuous cost  $\mathbf{c}$ ,  
by the discrete cost  $\check{\mathbf{c}}$ .

# Local and universal exact quantization for $\mathbf{c}$

$$\begin{aligned}
 V(x) &= \mathbb{E} \left[ \min_{y \in P_x} \mathbf{c}^\top y \right] \\
 &= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E} \left[ \mathbb{1}_{\mathbf{c} \in -\text{ri } N} \min_{y \in P_x} \mathbf{c}^\top y \right] \quad \text{where } y_N \in \arg \min_{y \in P_x} \underbrace{\mathbf{c}^\top}_\in -\text{ri } N y. \\
 &= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E} \left[ \mathbb{1}_{\mathbf{c} \in -\text{ri } N} \mathbf{c}^\top \right] y_N(x) \\
 &= \sum_{N \in \mathcal{N}(P_x)} p_N \check{\mathbf{c}}_N^\top y_N(x) \\
 &= \sum_{N \in \mathcal{N}(P_x)} p_N \min_{y \in P_x} \check{\mathbf{c}}_N^\top y
 \end{aligned}$$



$p_N \check{\mathbf{c}}_N$  for  $x = 0.3$

For  $N \in \mathcal{N}(P_x)$ ,

$$p_N := \mathbb{P}[\mathbf{c} \in -\text{ri } N]$$

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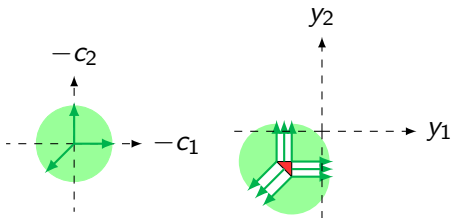
We replace the continuous cost  $\mathbf{c}$ ,  
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# Contents

- 1 Local and Universal Exact Quantization for cost in 2-stage
- 2 Uniform and Universal Exact Quantization for cost in 2-stage**
- 3 Uniform and Universal Exact Quantization for cost in multistage
- 4 Complexity results

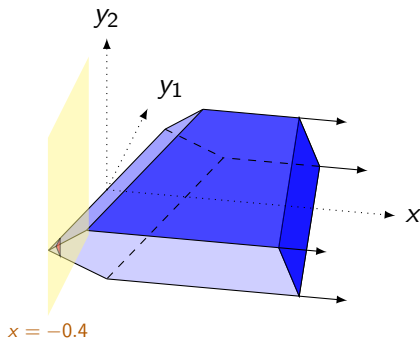
$x \mapsto \mathcal{N}(P_x)$  is piecewise constant.

$$P_x := \{y \mid Ay + Bx \leq b\} \quad \text{and} \quad P := \{(x, y) \mid Ay + Bx \leq b\}$$



$\mathcal{N}(P_x)$

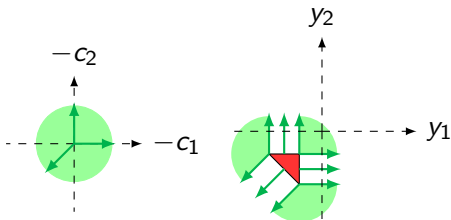
$P_x$  and  $\mathcal{N}(P_x)$



$P$  and  $P_x$

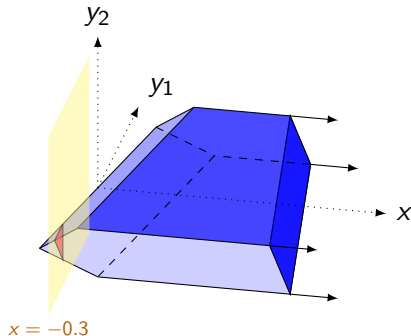
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$\mathcal{N}(P_x)$

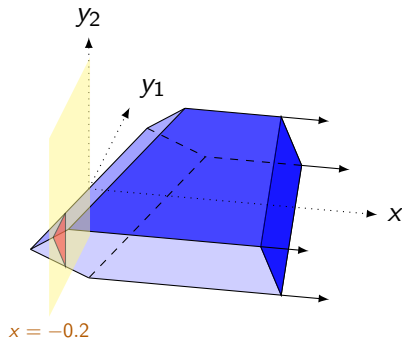
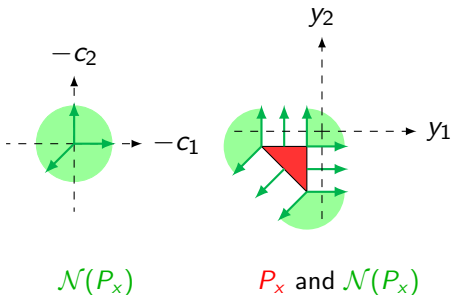
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$P$  and  $P_x$

$x \mapsto \mathcal{N}(P_x)$  is piecewise constant.

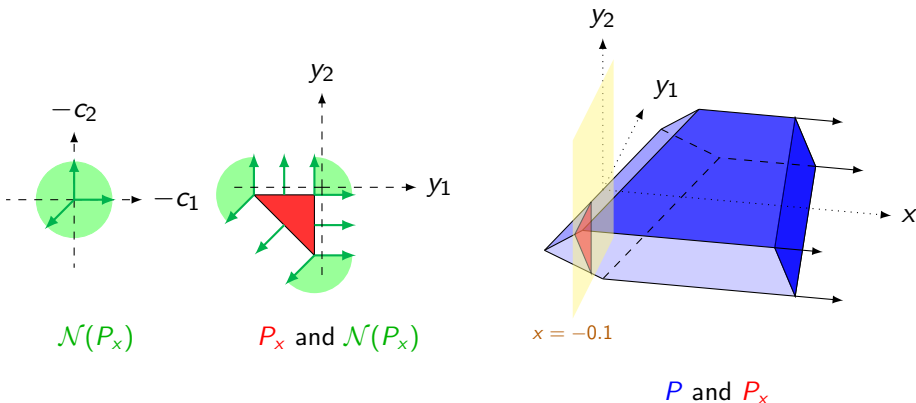
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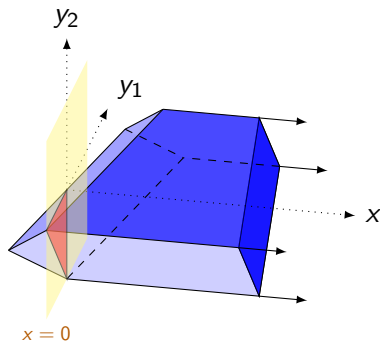
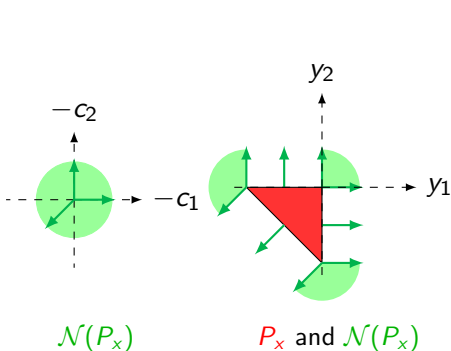
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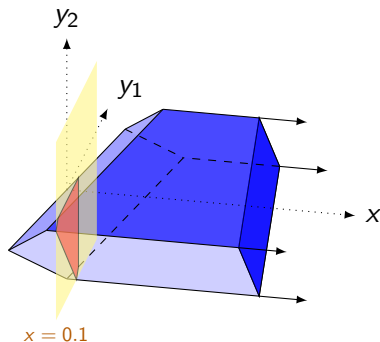
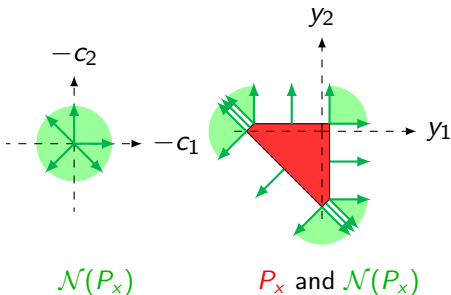
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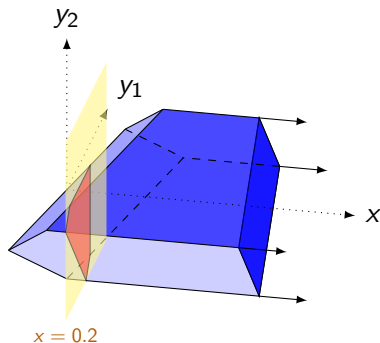
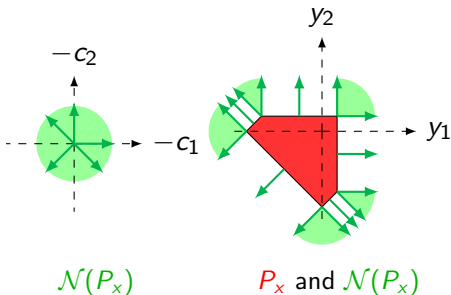
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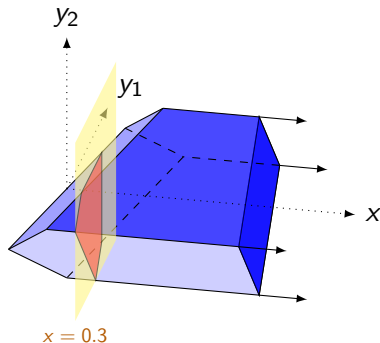
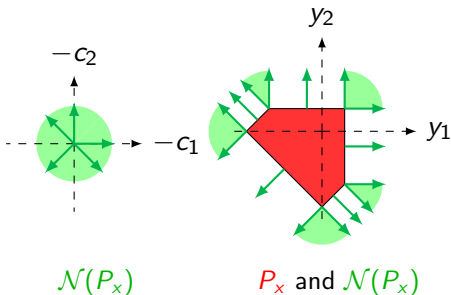
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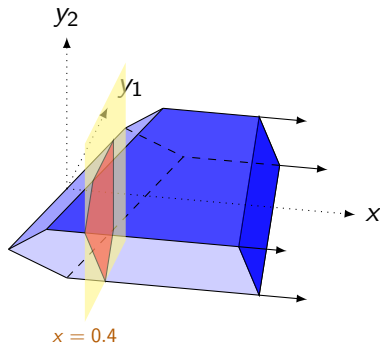
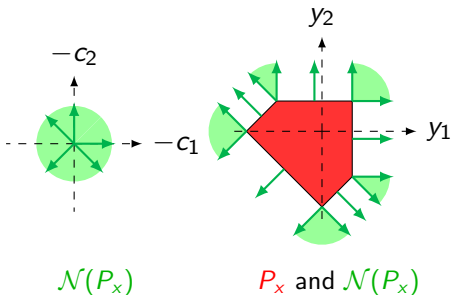
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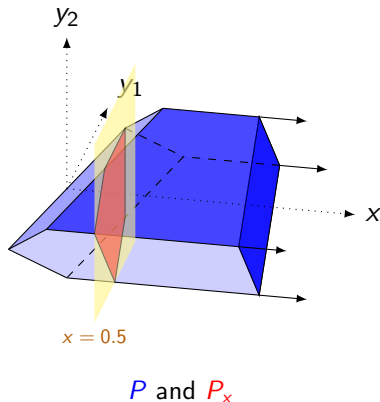
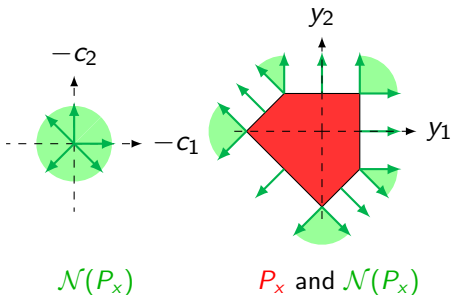
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$P$  and  $P_x$

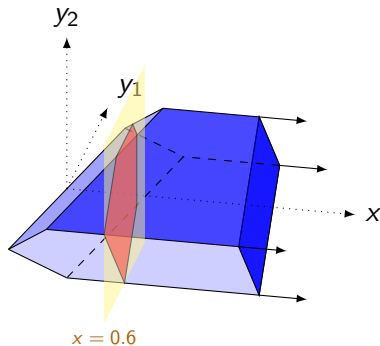
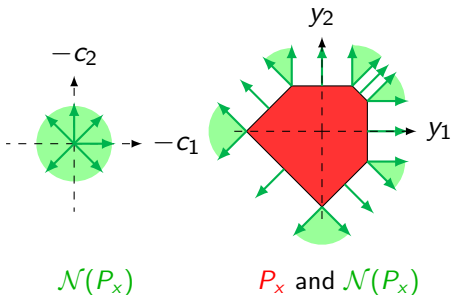
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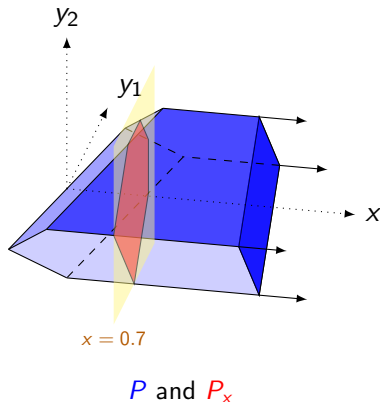
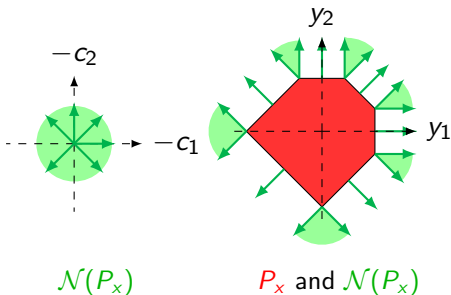
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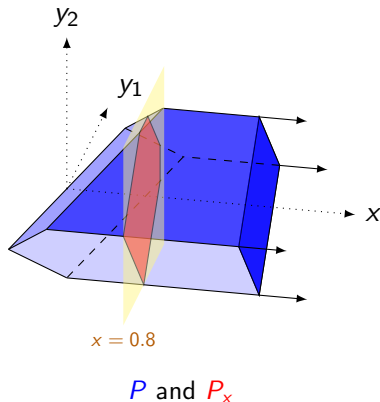
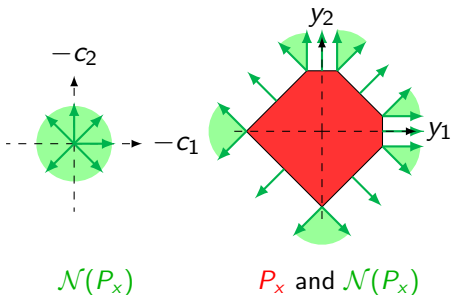
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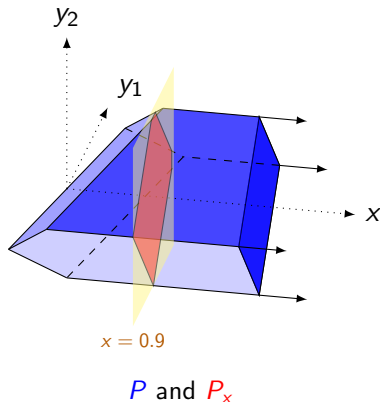
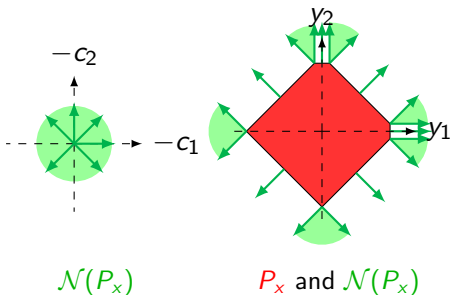
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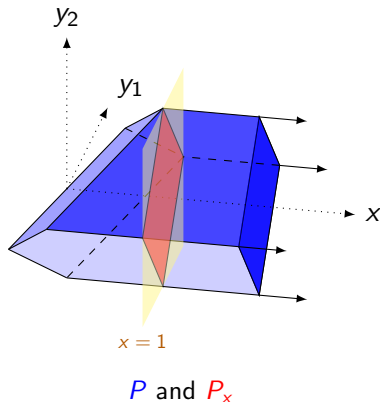
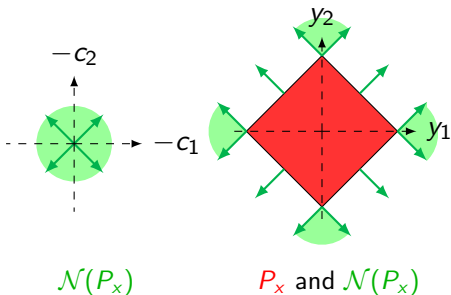
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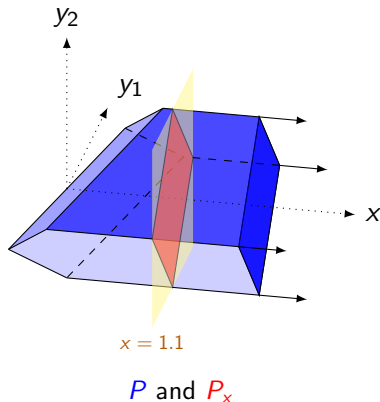
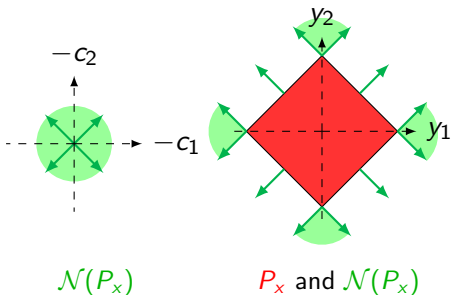
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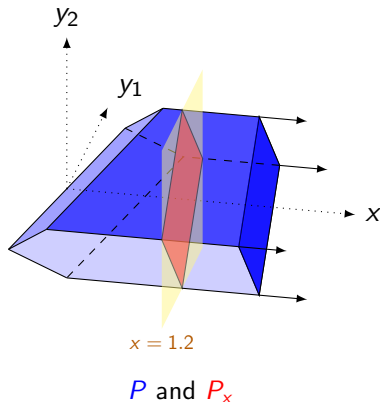
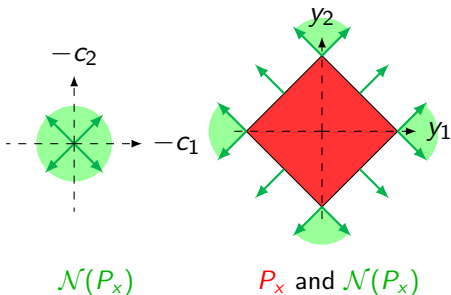
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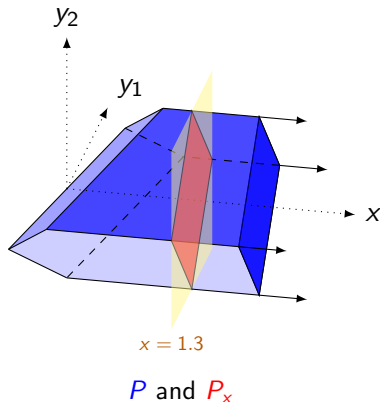
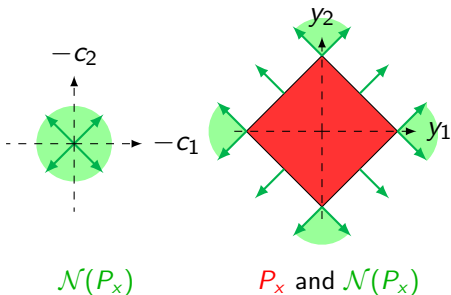
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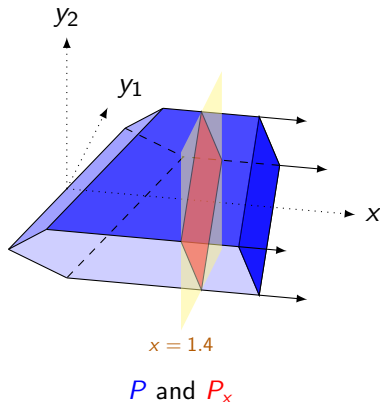
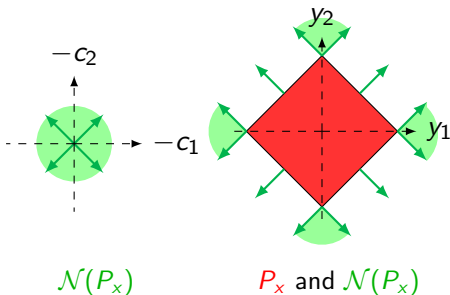
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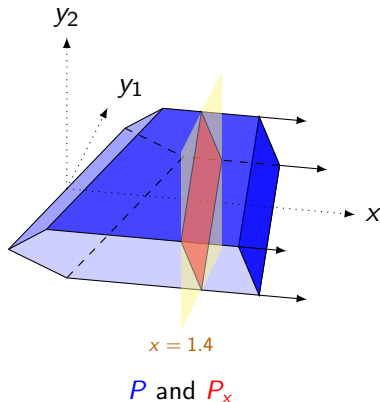
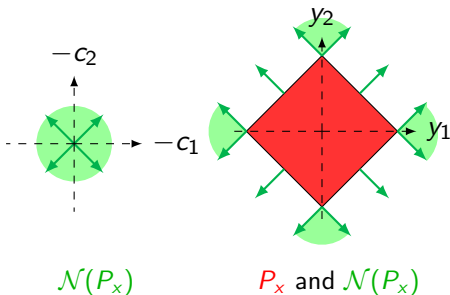
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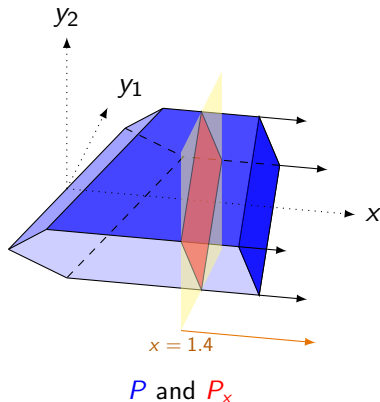
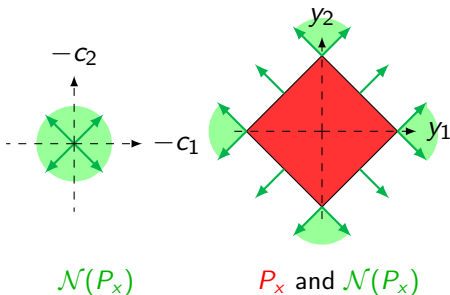
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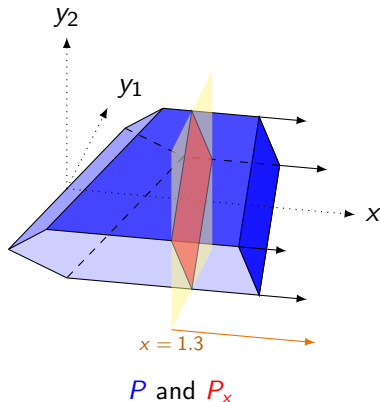
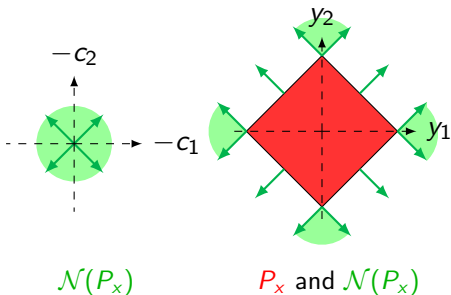
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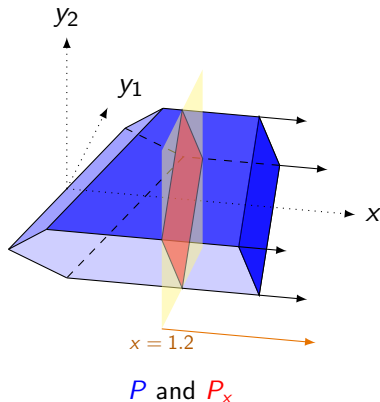
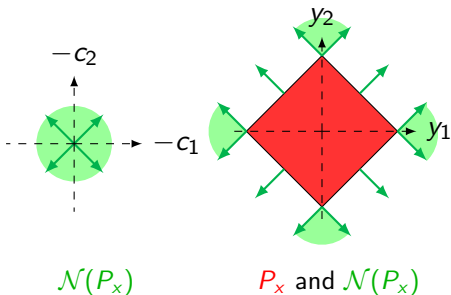
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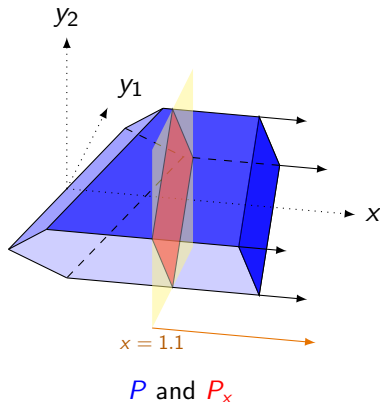
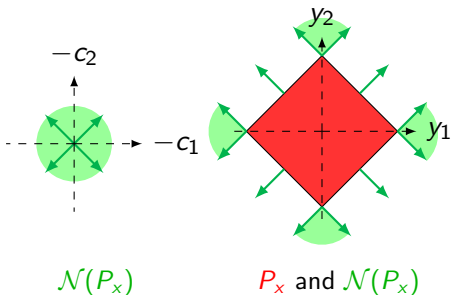
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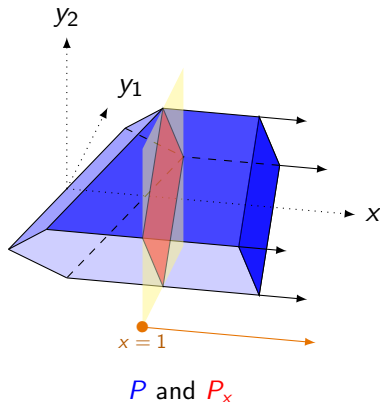
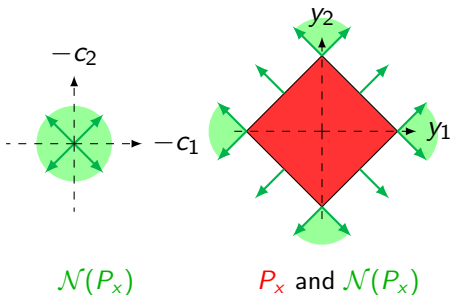
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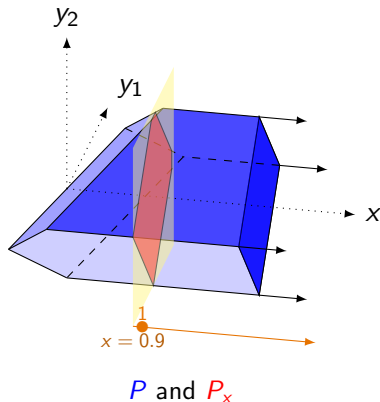
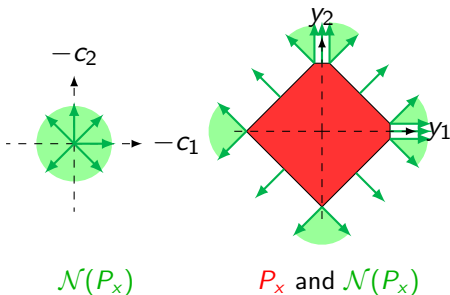
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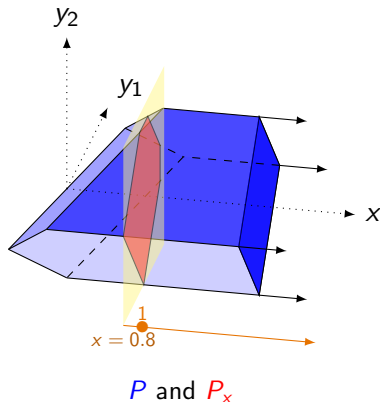
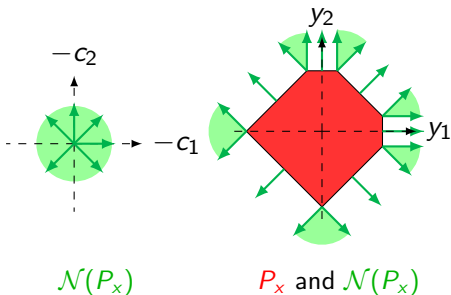
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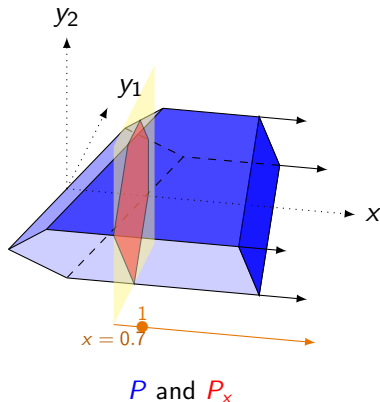
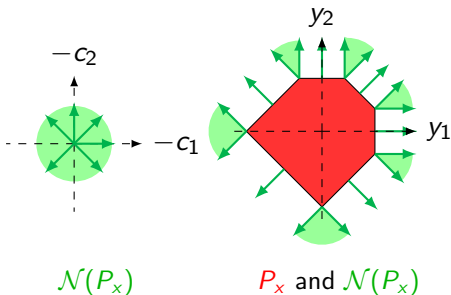
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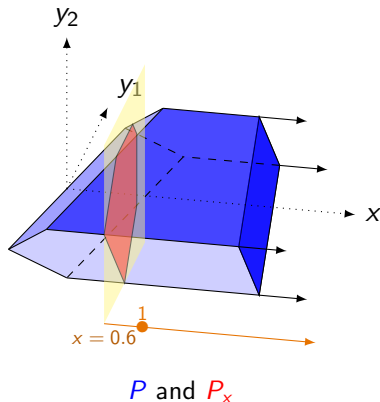
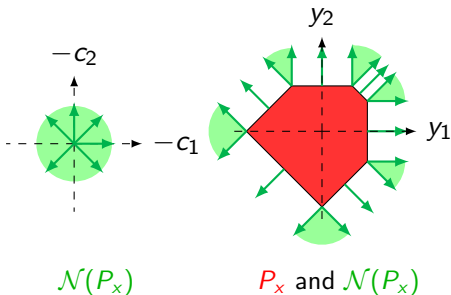
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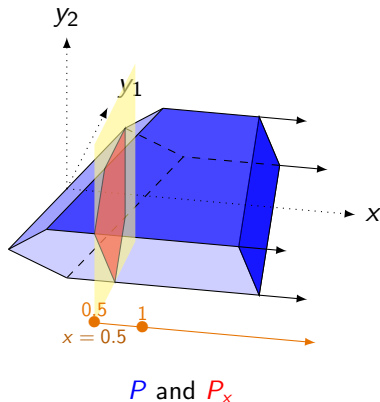
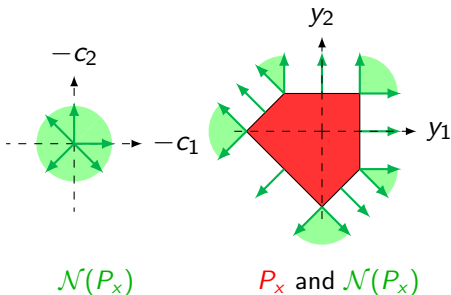
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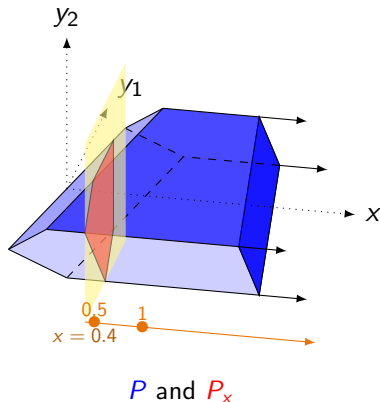
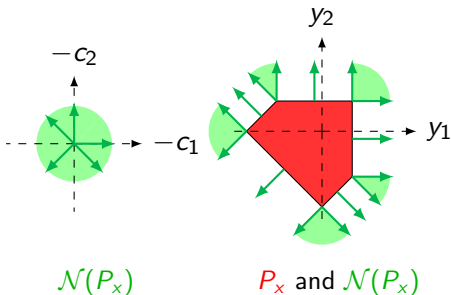
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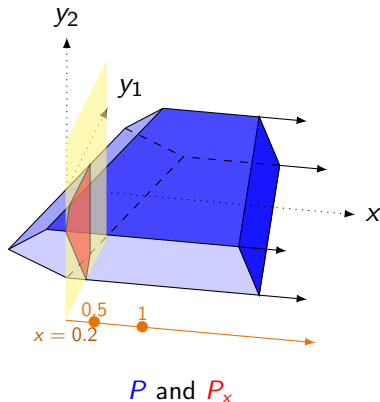
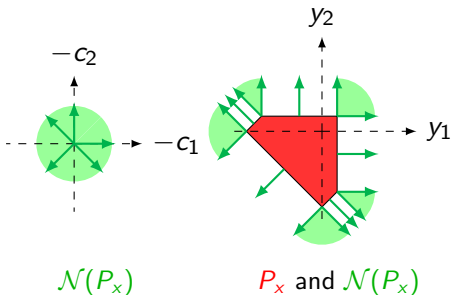
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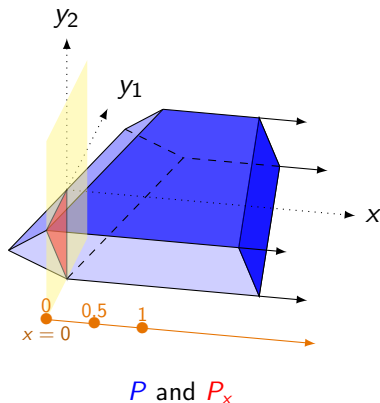
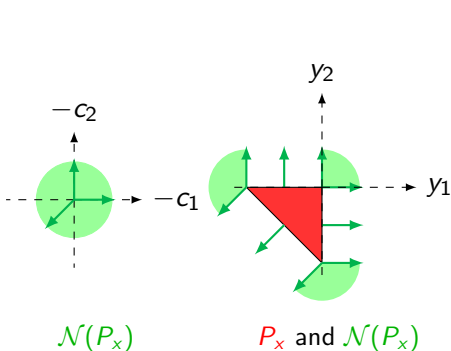
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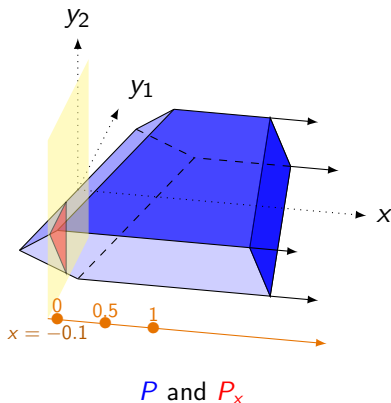
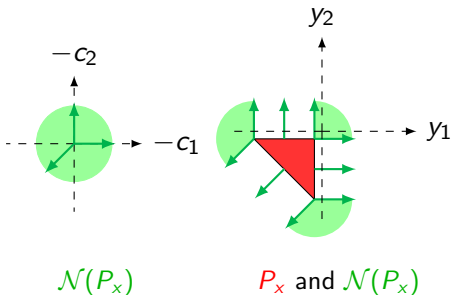
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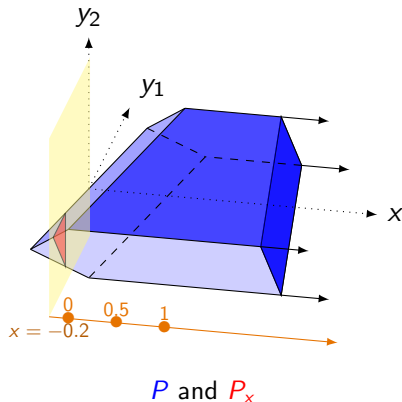
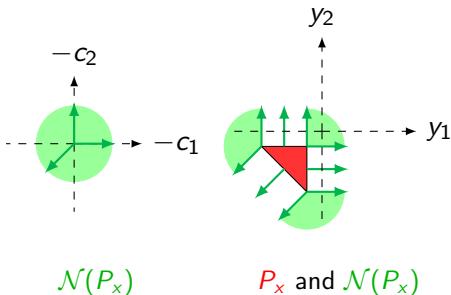
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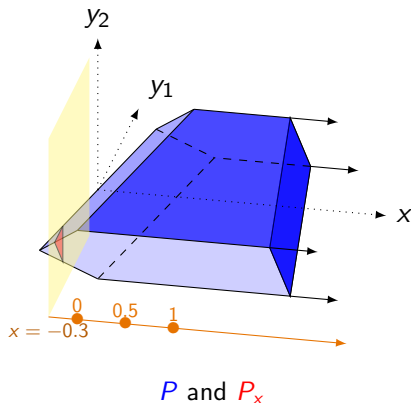
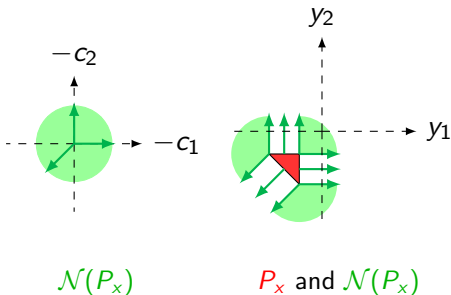
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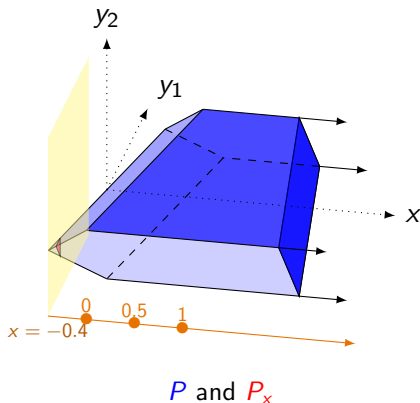
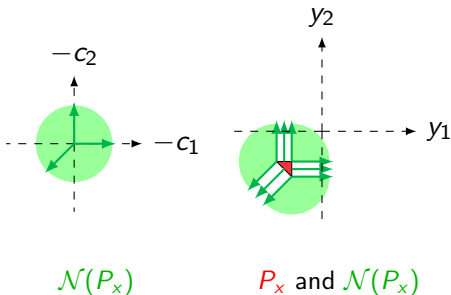
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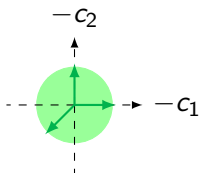
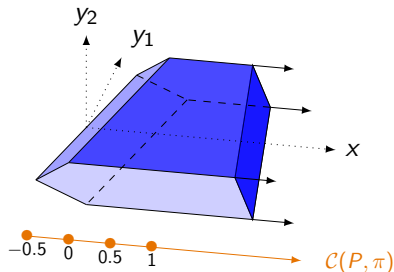


# What are the constant regions of $x \mapsto \mathcal{N}(P_x)$ ?

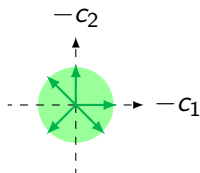
## Proposition

There exists a collection  $\mathcal{C}(P, \pi)$  called the **chamber complex** whose relative interior of cells are the constant regions of  $x \mapsto \mathcal{N}(P_x)$ .

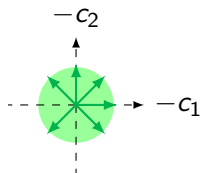
I.e, for  $\sigma \in \mathcal{C}(P, \pi)$  and  $x, x' \in \text{ri}(\sigma)$ , we have  $\mathcal{N}(P_x) = \mathcal{N}(P_{x'}) =: \mathcal{N}_\sigma$



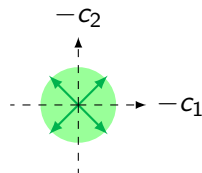
$\mathcal{N}_\sigma$  for  $\sigma = [-0.5, 0]$



$\mathcal{N}_\sigma$  for  $\sigma = [0, 0.5]$



$\mathcal{N}_\sigma$  for  $\sigma = [0.5, 1]$



$\mathcal{N}_\sigma$  for  $\sigma = [1, +\infty)$

# Chamber complex

## Definition

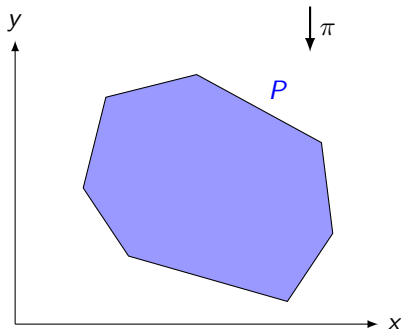
The *chamber complex*  $\mathcal{C}(P, \pi)$  of  $P$  along  $\pi$  is

$$\mathcal{C}(P, \pi) := \{\sigma_{P, \pi}(x) \mid x \in \pi(P)\}$$

where

$$\sigma_{P, \pi}(x) := \bigcap_{F \in \mathcal{F}(P) \mid x \in \pi(F)} \pi(F)$$

where  $\mathcal{F}(P)$  is the set of faces of  $P$  and  $\pi$  is the projection  $(x, y) \mapsto x$ .



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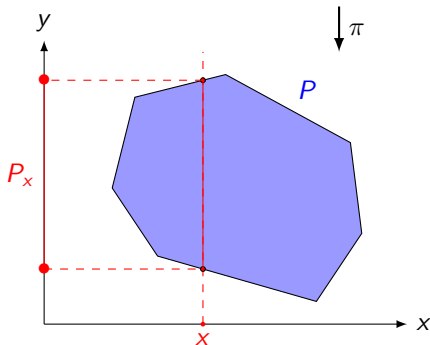
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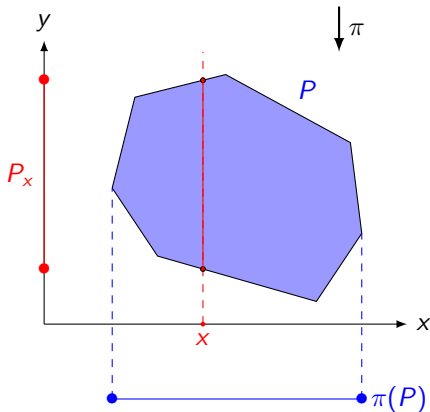
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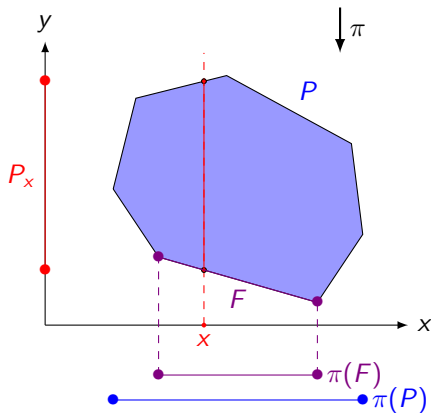
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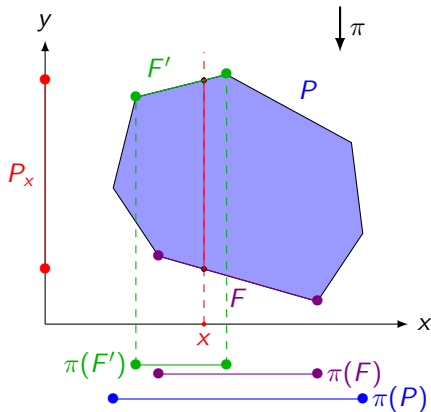
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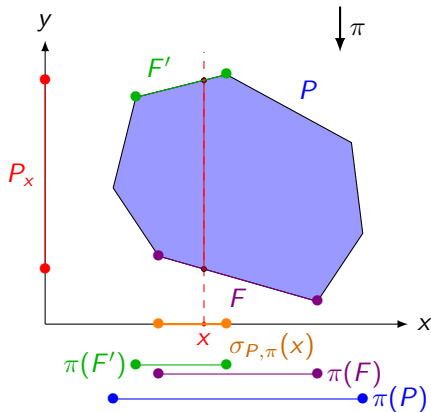
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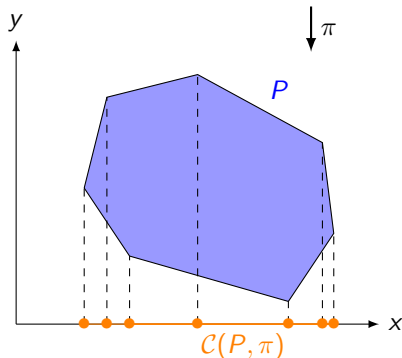
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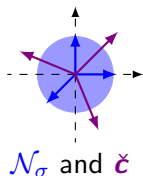
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# Common Refinement of Normal Fans

We can quantize  $\mathbf{c}$  on each chamber.

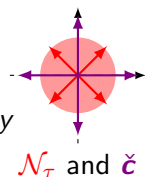


For all  $x \in \text{ri}(\sigma)$ ,

$$V(x) = \sum_{N \in \mathcal{N}_\sigma} p_N \min_{y \in P_x} \check{\mathbf{c}}_N^\top y$$

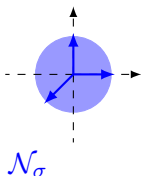
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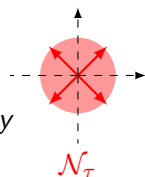


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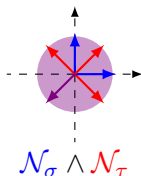
For all  $x' \in \text{ri}(\tau)$ ,

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We take the *common refinement*:

$$\mathcal{R} := \mathcal{N}_\sigma \wedge \mathcal{N}_\tau = \{N \cap N' \mid N \in \mathcal{N}_\sigma, N' \in \mathcal{N}_\tau\}$$

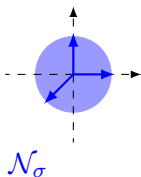


For all  $x \in \text{ri}(\sigma) \cup \text{ri}(\tau)$ ,

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# Common Refinement of Normal Fans

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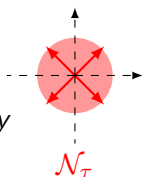


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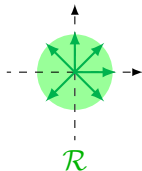
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# Uniform exact quantization for $\mathcal{C}$

Let's sum up:

- local exact quantization at  $x$  induced by  $\mathcal{N}(P_x)$ ;
- local exact quantization at  $x$  and  $x'$  by taking the refinement,
- $x \mapsto \mathcal{N}(P_x)$  is constant on each  $\sigma \in \mathcal{C}(P, \pi)$

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- local exact quantization at  $x$  induced by  $\mathcal{N}(P_x)$ ;
- local exact quantization at  $x$  and  $x'$  by taking the refinement,
- $x \mapsto \mathcal{N}(P_x)$  is constant on each  $\sigma \in \mathcal{C}(P, \pi)$

## Theorem (Uniform and universal quantization of the cost distribution)

Let  $\mathcal{R} = \bigwedge_{\sigma \in \mathcal{C}(P, \pi)} -\mathcal{N}_\sigma$ , then **for all**  $x \in \mathbb{R}^n$

$$V(x) = \sum_{R \in \mathcal{R}} \check{p}_R \min_{y \in P_x} \check{c}_R^\top y$$

where  $\check{p}_R := \mathbb{P}[\mathbf{c} \in \text{ri}(R)]$  and  $\check{c}_R := \mathbb{E}[\mathbf{c} \mid \mathbf{c} \in \text{ri}(R)]$



# Polyhedral dual characterization of $V$

Dual admissible set

$$D_c := \{\lambda \mid A^\top \lambda + c = 0\}$$

*Weighted fiber polyhedron*

$$E = \mathbb{E}[D_c] = \int D_c \mathbb{P}(dc)$$

Extension of fiber polytope of



L. Billera, B. Sturmfels, Fiber polytopes, *Annals of Mathematics*, p527–549, 1992.

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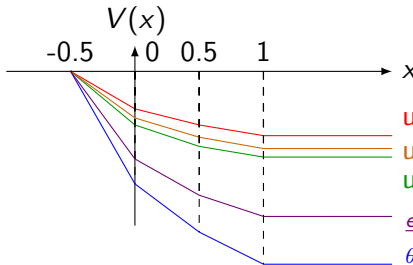
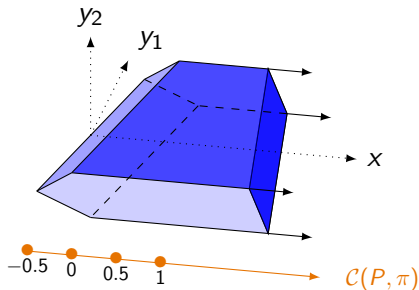
*Under an affine change of variable,  $V$  is the support function of  $E$*

$$V(x) = \sigma_E(b - Bx) = \sup_{\lambda \in E} (b - Bx)^\top \lambda$$

*Moreover, for all distributions of  $c$ ,  $V$  is affine on each cell of  $\mathcal{C}(P, \pi)$ .*

# Explicit computation of an example

$$V(x) = \mathbb{E} \left[ \begin{array}{ll} \min_{y \in \mathbb{R}^2} & \mathbf{c}^\top y \\ \text{s.t.} & \|y\|_1 \leq 1 \\ & y_1 \leq x \\ & y_2 \leq x \end{array} \right]$$



Different distributions of  $\mathbf{c}$ :

uniform on norm 1 ball

uniform on norm 2 ball

uniform on norm  $\infty$  ball

$$\frac{e^{-\frac{\|\mathbf{c}\|_2^2}{2\gamma^2}}}{\theta^2 e^{-\frac{\theta}{\|\mathbf{c}\|_1}}} d\mathbf{c}$$

$$\frac{1}{4} d\mathbf{c}$$

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# Extension to multistage and stochastic constraints

## Theorem

*All results generalizes to multistage problem with finitely supported stochastic constraints.*

- ➡  $(V_t)_t$  are affine on *universal* chamber complexes, i.e. independent of the law of  $(\mathbf{c}_t)_t$
- ➡ We have an *uniform and universal* exact quantization.

Core idea of the proof :

Iterated chamber complexes

$$\mathcal{P}_{t,\xi} := \mathcal{C}\left((\mathbb{R}^{n_t} \times \mathcal{P}_{t+1}) \wedge \mathcal{F}(P_t(\xi)), \pi_{x_{t-1}}^{x_{t-1}, x_t}\right)$$
$$\mathcal{P}_t := \bigwedge_{\xi_t \in \text{supp } \xi_t} \mathcal{P}_{t,\xi}$$

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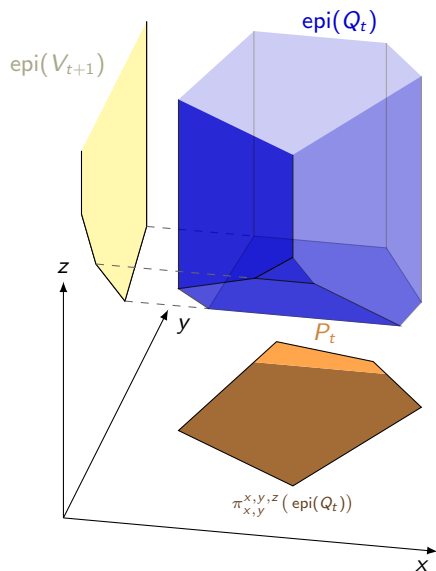
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# Multistage uniform and universal exact quantization

$$V_t(x) = \mathbb{E} \left[ \min_{y \in \mathbb{R}^{n_t}} \mathbf{c}_t^\top y + V_{t+1}(y) \right]$$

$$\text{s.t. } (x, y) \in P_t$$

with  $Q_t(x, y) := V_{t+1}(y) + \mathbb{I}_{(x, y) \in P_t}$ .

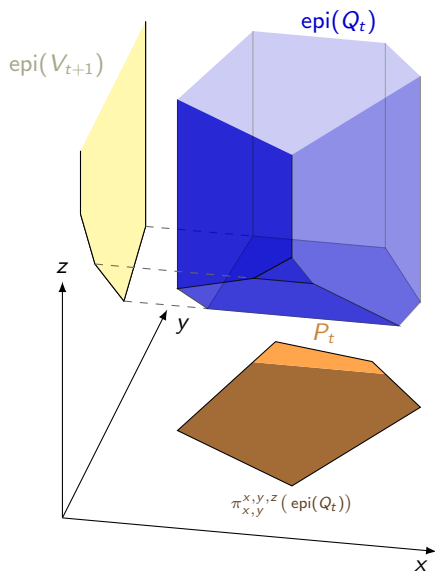




# Multistage uniform and universal exact quantization

$$V_t(x) = \mathbb{E} \left[ \min_{\substack{y \in \mathbb{R}^{n_t} \\ z \in \mathbb{R}}} \mathbf{c}_t^\top y + z \right. \\ \left. \text{s.t. } (x, y, z) \in \text{epi}(Q_t) \right]$$

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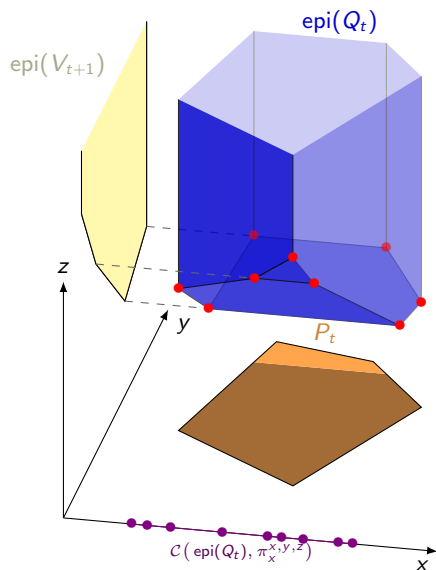


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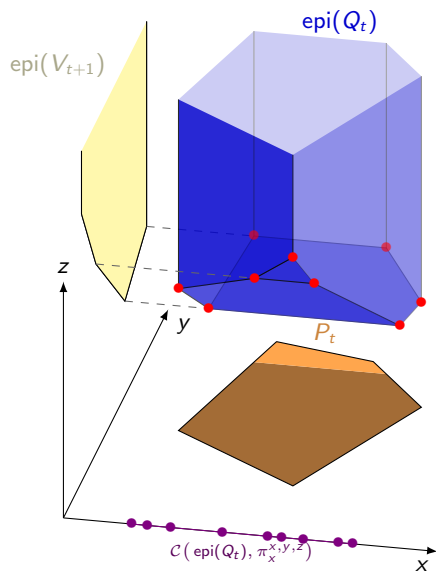
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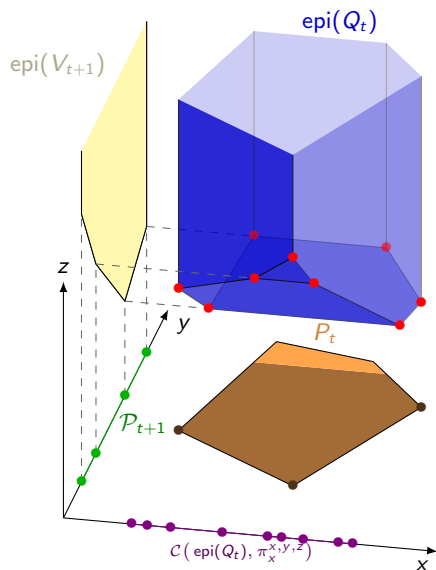
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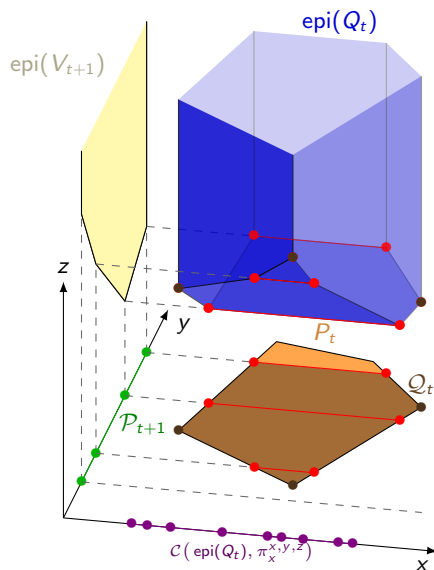
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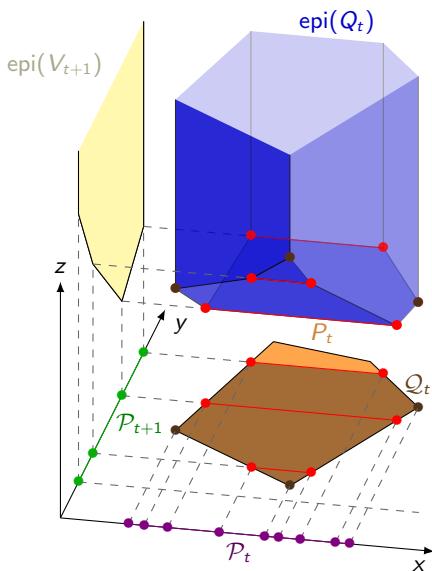
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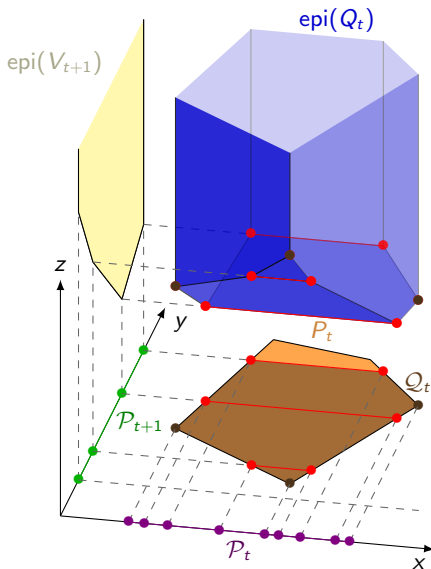
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[FGL21, Lem. 4.1]:  $P_t \preceq \mathcal{C}(\text{epi}(Q_t), \pi_x^{x,y,z})$

➡  $V_t$  affine on  $P_t$ ,  $\mathcal{N}(P_x)$  constant on  $P_t$



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# Earlier and new complexity results

## Volume of a polytope

$$\text{Vol}(\{z \in \mathbb{R}^d \mid Az \leq b\}) \text{ or}$$
$$\text{Vol}(\text{Conv}(v_1, \dots, v_n))$$

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  - $\rightsquigarrow$  Exact case
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## Complexity result multistage

Shapiro and Nemirovski (2005):

By SAA, we can solve MSLP, up to precision  $\varepsilon$ , in **pseudo-polynomial** time, i.e. polynomial in  $\frac{1}{\varepsilon}$ , with **probability**  $1 - \alpha$ , when  $T$  is fixed.

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*Assume that  $T$ ,  $n_t$ , and  $|\text{supp}(\mathbf{A}_t, \mathbf{B}_t, \mathbf{b}_t)|$ , for  $t = 2, \dots, T$ , are fixed integers.<sup>1</sup>*

*Assume that  $\mathbf{c}$  admits a density function with a bounded total variation.*

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➡ Can be adapted to exact complexity when we can compute exactly  $\mathbb{E}[\mathbf{c} \in C | (\mathbf{A}_t, \mathbf{B}_t, \mathbf{b}_t) = (A, B, b)]$  and  $\mathbb{P}[\mathbf{c} \in C | (\mathbf{A}_t, \mathbf{B}_t, \mathbf{b}_t) = (A, B, b)]$ .

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# Conclusion and applications

- *Uniform and universal* exact quantization for an MSLP

➡ New complexity results.

Unfortunately this quantization might be very large.

- *Local* exact quantization on  $\mathcal{C}$

➡ Higher order simplex algorithm on the chamber complex solves 2SLP of dimension  $100 + 10$ .

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# Thank you for listening ! Any question ?



M. Forcier, S. Gaubert, V. Leclère

Exact quantization of multistage stochastic linear problems.

*arXiv preprint arXiv:2107.09566 (2021).*



M. Forcier, V. Leclère

Generalized adaptive partition-based method for two-stage stochastic linear programs: convergence and generalization.

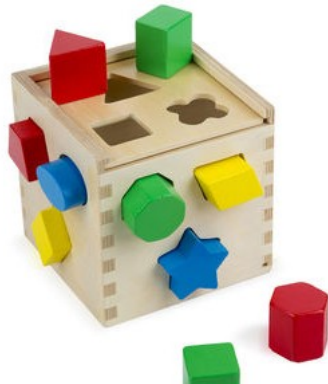
*Operation Research Letters, to appear (2022).*



M. Forcier, V. Leclère

Convergence of Trajectory Following Dynamic Programming algorithms for multistage stochastic problems without finite support assumptions

*HAL Id : hal-03683697 (2022).*



# Local exact quantization and adapted partition

## Local exact quantization

random cost

Recall that for a fixed  $x$ ,

$$\begin{aligned}\mathbb{E} \left[ \min_{y \in P_x} \mathbf{c}^\top y \right] \\ = \sum_{N \in \mathcal{N}(P_x)} p_N \min_{y \in P_x} \check{\mathbf{c}}_N^\top y\end{aligned}$$

where,

$$\begin{aligned}p_N &:= \mathbb{P}[\mathbf{c} \in -\text{ri } N] \\ \check{\mathbf{c}}_N &:= \mathbb{E}[\mathbf{c} \mid \mathbf{c} \in -\text{ri } N]\end{aligned}$$

$$P_x := \{y \in \mathbb{R}^m \mid Ay + Bx \leq b\}$$

## GAPM

random constraints

Similarly, for a given  $q$ , and all  $x$ ,

$$\begin{aligned}V(x) &:= \mathbb{E}[Q(x, \boldsymbol{\xi})] \\ &= \mathbb{E} \left[ \max_{\lambda \in D_q} (\mathbf{h} - \mathbf{T}x)^\top \lambda \right] \\ &= \sum_{N \in \mathcal{N}(D_q)} p_N \max_{\lambda \in D_q} \psi_{N,x}^\top \lambda\end{aligned}$$

where,

$$\begin{aligned}p_N &:= \mathbb{P}[\mathbf{h} - \mathbf{T}x \in \text{ri } N] \\ \psi_{N,x} &:= \mathbb{E}[\mathbf{h} - \mathbf{T}x \mid \mathbf{h} - \mathbf{T}x \in \text{ri } N]\end{aligned}$$

$$D_q := \{\lambda \in \mathbb{R}^l \mid W^\top \lambda \leq q\}$$

# An explicit adapted partition

Consider  $x \in \mathbb{R}^n$  and  $N \in \mathcal{N}(D_q)$  a normal cone of  $D_q$ . We define

$$E_{N,x} := \{\xi \in \Xi \mid h - Tx \in \text{ri } N\}$$

## Theorem (FL 2021)

$\mathcal{R}_x := \{E_{N,x} \mid N \in \mathcal{N}(D_q)\}$  is an adapted partition to  $x$   
i.e.  $V_{\mathcal{R}_x}(x) = V(x)$

Proof:

$$\begin{aligned} V(x) &:= \mathbb{E}[Q(x, \xi)] \\ &= \sum_{N \in \mathcal{N}(D)} \mathbb{P}[h - Tx \in \text{ri } N] \min_{\lambda \in D} \mathbb{E}[h - Tx \mid h - Tx \in \text{ri } N]^\top \lambda \\ &= \sum_{N \in \mathcal{N}(D)} \mathbb{P}[\xi \in E_{N,x}] Q\left(\mathbb{E}[\xi \mid \xi \in E_{N,x}], x\right) = V_{\mathcal{R}_x}(x) \end{aligned}$$

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i.e.  $V_{\mathcal{R}_x}(x) = V(x)$

Proof:

$$\begin{aligned} V(x) &:= \mathbb{E}[Q(x, \xi)] \\ &= \sum_{N \in \mathcal{N}(D)} \mathbb{P}[h - Tx \in \text{ri } N] \min_{\lambda \in D} \mathbb{E}[h - Tx \mid h - Tx \in \text{ri } N]^\top \lambda \\ &= \sum_{N \in \mathcal{N}(D)} \mathbb{P}[\xi \in E_{N,x}] Q\left(\mathbb{E}[\xi \mid \xi \in E_{N,x}], x\right) = V_{\mathcal{R}_x}(x) \end{aligned}$$



# Synthesis of local and uniform quantization results

	$\mathbf{W}$	$(\mathbf{T}, \mathbf{h})$	$\mathbf{q}$
Local	$\emptyset$	$\mathcal{R}_x$	$\mathcal{N}(P_x)$
Uniform	$\emptyset$	$\emptyset$	$\bigwedge_{\sigma \in \mathcal{C}(P, \pi)} \mathcal{N}_\sigma$

## Numerical Results - ProdMix

$k$	$z_L^k$	$z_U^k$	$z_U^k - z_L^k$	Total time	$ \mathcal{P}^k $
1	-18666.67	-16939.71	1726.96	0.57 s	4
2	-17873.01	-17383.73	489.28	2.1 s	9
4	-17744.67	-17709.00	35.67	9.1 s	25
6	-17713.74	-17711.37	2.37	23.7 s	49
8	-17711.71	-17711.56	0.15	50.0 s	81
10	-17711.57	-17711.56	0.01	88.0 s	121

Table: Results for problem Prod-Mix

Comparison with SAA : we solved the same problem 100 times, each with 10 000 scenarios randomly drawn

↪ 95% confidence interval centered in  $-17711$ , with radius 2.2.

↪ required 2058s of computation.