

Exact quantization methods for Multistage Stochastic Linear Problem

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Multistage stochastic linear programming (MSLP)

$$\begin{aligned} \min_{(\mathbf{x}_t)_{t \in [T]}} \quad & \mathbb{E} \left[\sum_{t=1}^T \mathbf{c}_t^\top \mathbf{x}_t \right] \\ \text{s.t.} \quad & \mathbf{A}_t \mathbf{x}_t + \mathbf{B}_t \mathbf{x}_{t-1} \leq \mathbf{b}_t \quad \forall t \in [T] \\ & \sigma(\mathbf{x}_t) \subset \sigma(\mathbf{c}_\tau, \mathbf{A}_\tau, \mathbf{B}_\tau, \mathbf{b}_\tau)_{\tau \leq t} \quad \forall t \in [T] \\ & \mathbf{x}_0 \equiv \mathbf{x}_0 \text{ given} \end{aligned}$$

$\xi_t = (\mathbf{c}_t, \mathbf{A}_t, \mathbf{B}_t, \mathbf{b}_t)_{t \in [T]}$ is assumed to be **stagewise independent**.

We set $V_{T+1} \equiv 0$ and:

$$V_t(\mathbf{x}_{t-1}) := \mathbb{E} [\hat{V}_t(\mathbf{x}_{t-1}, \xi_t)] := \mathbb{E} \left[\begin{array}{ll} \min_{\mathbf{x}_t \in \mathbb{R}^{n_t}} & \mathbf{c}_t^\top \mathbf{x}_t + V_{t+1}(\mathbf{x}_t) \\ \text{s.t.} & \mathbf{A}_t \mathbf{x}_t + \mathbf{B}_t \mathbf{x}_{t-1} \leq \mathbf{b}_t \end{array} \right]$$

➡ How to deal with continuous distributions ?

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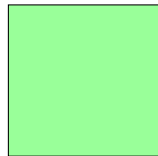
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Quantization of a MSLP

Real problem

$$V_t(x) = \mathbb{E}[\hat{V}_t(x, \xi_t)] = \mathbb{E} \left[\begin{array}{ll} \min_{y \in \mathbb{R}^{n_t}} & \mathbf{c}_t^\top y + V_{t+1}(y) \\ \text{s.t.} & \mathbf{A}_t y + \mathbf{B}_t x \leq \mathbf{b}_t \end{array} \right]$$

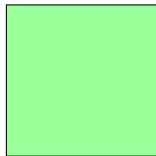


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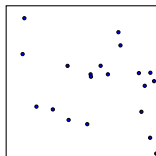


ξ_t continuous

Sample Average Approximation (SAA)

$$V_{t,N}^{SAA}(x) := \frac{1}{N} \sum_{k=1}^N \hat{V}_t(x, \xi^k)$$

ξ^1, \dots, ξ^N drawn by Monte Carlo

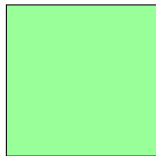


SAA $N = 20$

Quantization of a MSLP

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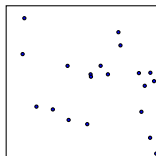


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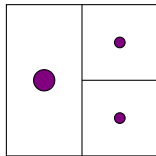


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Partition-based

$$V_{t,P}(x) := \sum_{P \in \mathcal{P}} \check{p}_{t,P} \hat{V}_t(x, \check{\xi}_{t,P})$$

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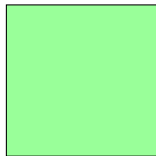


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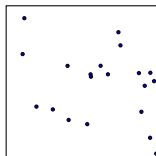


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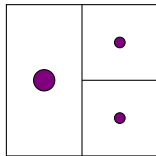


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$$V_{t,\mathcal{P}}(x) := \sum_{P \in \mathcal{P}} \check{p}_{t,P} \hat{V}_t(x, \check{\xi}_{t,P})$$

with $\check{p}_{t,P} := \mathbb{P}[\xi_t \in P]$ and $\check{\xi}_{t,P} := \mathbb{E}[\xi_t | \xi_t \in P]$
If $\xi \mapsto \hat{V}(x, \xi)$ is convex, $V_{t,\mathcal{P}}(x) \leq V_t(x)$.



Partition-based

Exact quantization

Definition

A MSLP admits a **local exact quantization** at time t on x if there exists a finitely supported $(\check{\xi}_t)_{t \in [T]}$ i.e. such that

$$V_t(x) = \mathbb{E}[\hat{V}_t(x, \xi_t)] = \mathbb{E}[\hat{V}_t(x, \check{\xi}_t)].$$

We call an exact quantization

- **uniform** if it is locally exact at all $x \in \mathbb{R}^{n_t}$, and all $t \in [T]$.
- **universal** if there exists a partition $\mathcal{P}_{t,x}$ such that the induced quantization is exact at time t on x , for all distributions of $(\xi_\tau)_{\tau \in [T]}$.

Questions:

- 1 Under which condition does there exist an exact quantization ?
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A first counter example

Assume $V_{t+1} \equiv 0$ and denote $V := V_t$, $\hat{V} := \hat{V}_t$ and $\xi := \xi_t$ for now.

Let $\mathbf{A} = (-\mathbf{u})$, $\mathbf{B} \equiv (0)$, $\mathbf{b} \equiv (-1)$ where $\mathbf{u} \sim \mathcal{U}([1, 2])$.

$$\hat{V}(x, \xi) = \min_{y \in \mathbb{R}} y \quad \text{s.t.} \quad \mathbf{u}y \geq 1 = \frac{1}{\mathbf{u}}$$

By strict convexity, for all partition \mathcal{P}

$$\sum_{P \in \mathcal{P}} \check{p}_P \hat{V}(x, \check{\xi}_P) < V(x) = \mathbb{E} \left[\frac{1}{\mathbf{u}} \right]$$

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➡ There is no partition-based local, neither uniform or universal, exact quantization result for \mathbf{A} non-finitely supported.

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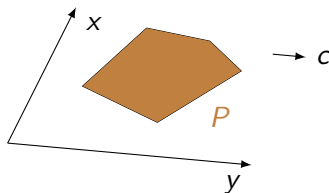
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Uniform exact quantization and polyhedrality

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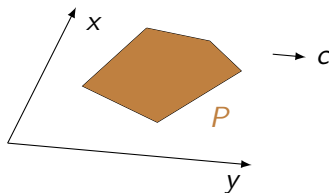
s.t. $Ay + Bx \leq h$



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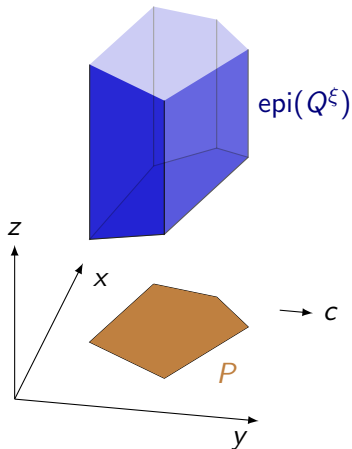
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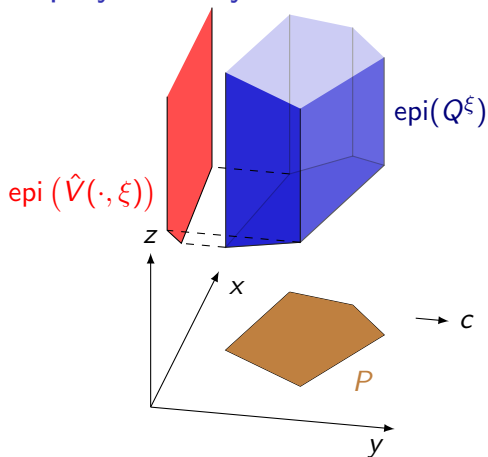


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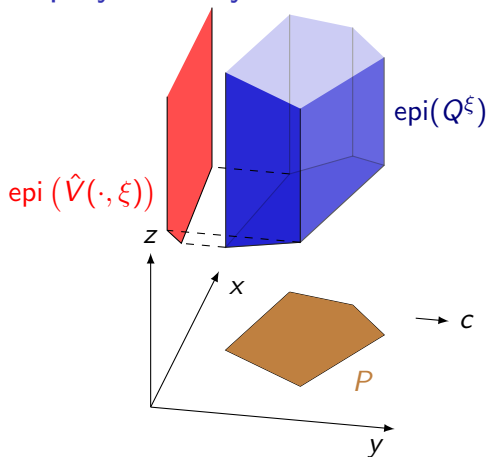


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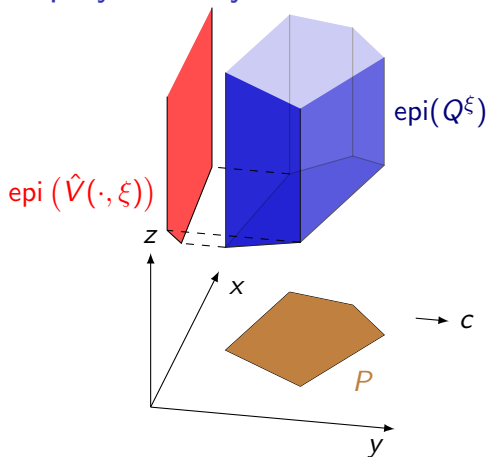
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- ➡ If the noise is finitely supported, then V is polyhedral
- ➡ Existence of uniform exact quantization implies polyhedrality of V .

Counter examples with stochastic constraints

Stochastic \mathbf{B}

$$\begin{aligned} V(x) &= \mathbb{E} \left[\begin{array}{l} \min_{y \in \mathbb{R}^m} \quad y \\ \text{s.t.} \quad \mathbf{u}x - y \leq 0 \\ \quad \quad y \geq 1 \end{array} \right] \\ &= \mathbb{E} [\max(\mathbf{u}x, 1)] \\ &= \begin{cases} 1 & \text{if } x \leq 1 \\ \frac{x}{2} + \frac{1}{2x} & \text{if } x \geq 1 \end{cases} \end{aligned}$$

Stochastic \mathbf{b}

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➡ V is not polyhedral \Rightarrow No uniform exact quantization for non-finitely supported \mathbf{B} and \mathbf{b} .

\mathbf{u} is uniform on $[0, 1]$

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Remaining cases

$$V(x) = \mathbb{E} \left[\begin{array}{ll} \min_{y \in \mathbb{R}^m} & \mathbf{c}^\top y \\ \text{s.t.} & \mathbf{A}y + \mathbf{B}x \leq \mathbf{b} \end{array} \right]$$

	\mathbf{A}	(\mathbf{B}, \mathbf{b})	\mathbf{c}
Local	×	?	?
Uniform	×	×	?

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Theorem (GAPM, FL 2022)

If \mathbf{A} is deterministic,
then there exists a *universal and local* exact quantization.

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Theorem (Exact quantization, FGL 2022)

If \mathbf{A} , \mathbf{B} and \mathbf{b} are deterministic,
then there exists a *universal and uniform* exact quantization.

Contents

- 1 Local and Universal Exact Quantization for cost in 2-stage
- 2 Uniform and Universal Exact Quantization for cost in 2-stage
- 3 Uniform and Universal Exact Quantization for cost in multistage
- 4 Complexity results

Reformulation of $V(x)$ highlighting the role of the fiber P_x

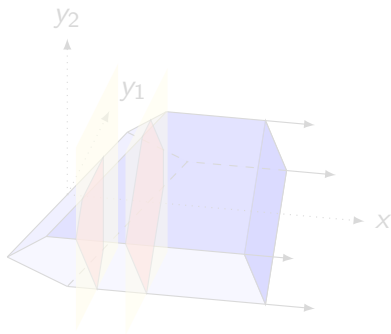
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Illustrative running example:

$$P_x := \{y \in \mathbb{R}^m \mid \|y\|_1 \leq 1, \\ y_1 \leq x, y_2 \leq x\}$$



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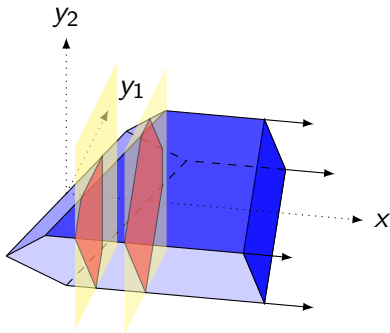
For a given x , (we still assume $V_{t+1} \equiv 0$)

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$$V(x) = \mathbb{E} \left[\min_{y \in P_x} \mathbf{c}^\top y \right] \quad \text{where} \quad P_x := \{y \in \mathbb{R}^m \mid Ay + Bx \leq b\}$$

Illustrative running example:

$$P_x := \{y \in \mathbb{R}^m \mid \|y\|_1 \leq 1, \\ y_1 \leq x, y_2 \leq x\}$$



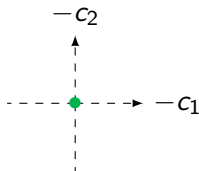
Normal fan $\mathcal{N}(P_x)$

Definition

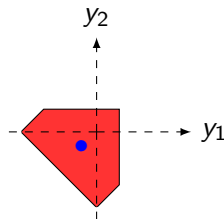
The normal fan of the fiber P_x is

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with $N_{P_x}(y) = \{c \mid \forall y' \in P_x, c^\top(y' - y) \leq 0\}$ the normal cone of P_x at y .



$N_{P_x}(y)$ for $x = 0.3$



P_x, y and $N_{P_x}(y)$ for $x = 0.3$

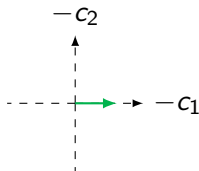
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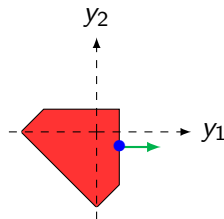
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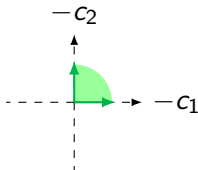
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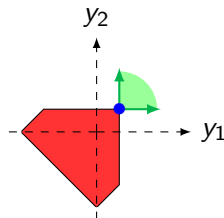
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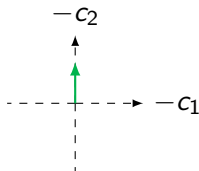
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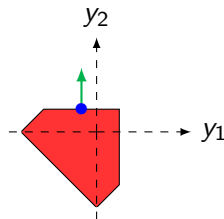
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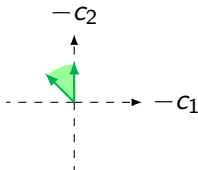
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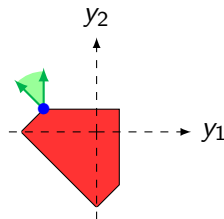
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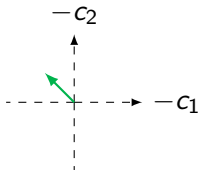
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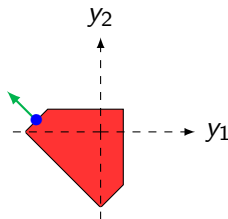
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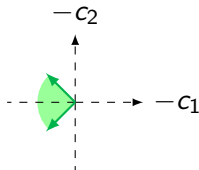
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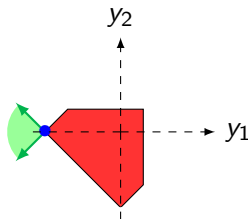
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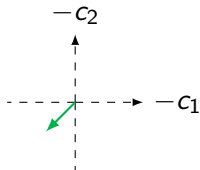
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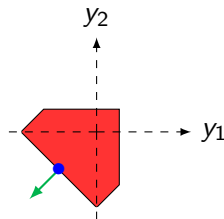
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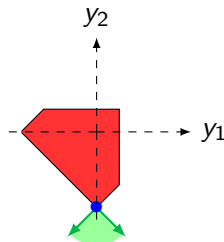
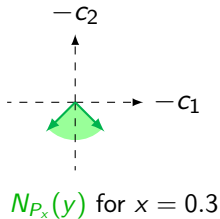
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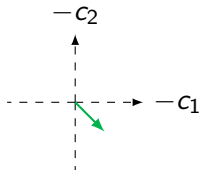
Normal fan $\mathcal{N}(P_x)$

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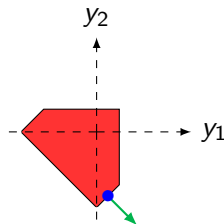
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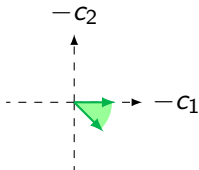
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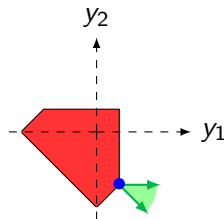
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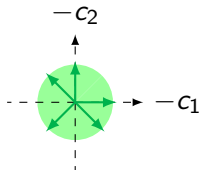
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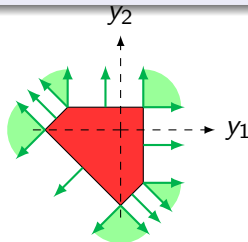
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Proposition

If P_x is bounded, $\{\text{ri}(N) \mid N \in \mathcal{N}(P_x)\}$ is a partition of \mathbb{R}^m .



$\mathcal{N}(P_x)$ for $x = 0.3$

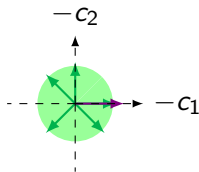


P_x and $\mathcal{N}(P_x)$ for $x = 0.3$

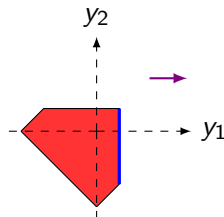
$\mathcal{N}(P_x)$: partition of cost coherent with the min

$$V(x) = \mathbb{E} \left[\min_{y \in P_x} c^\top y \right]$$

For any $N \in \mathcal{N}(P_x)$, $-c \mapsto \arg \min_{y \in P_x} c^\top y$ is constant for all $-c \in \text{ri}(N)$.



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$

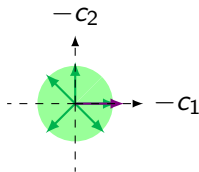


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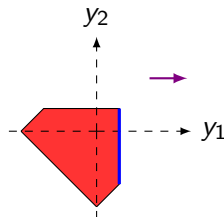
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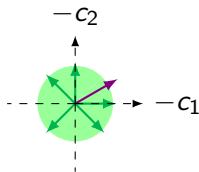


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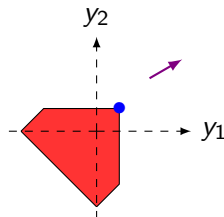
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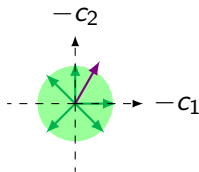


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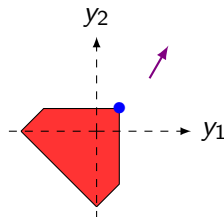
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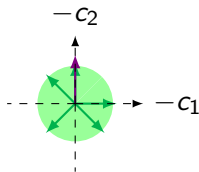


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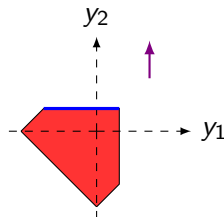
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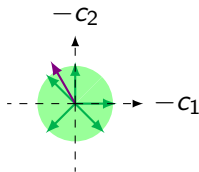


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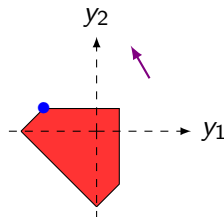
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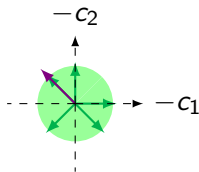


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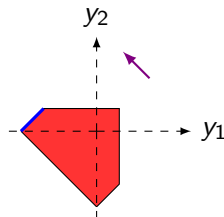
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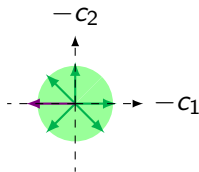


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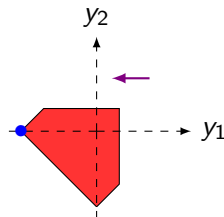
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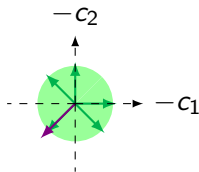


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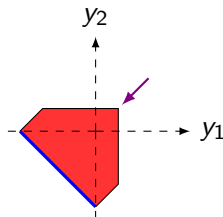
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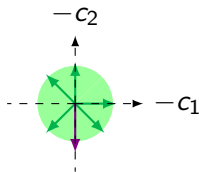


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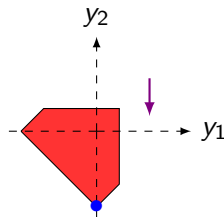
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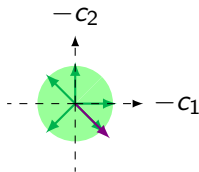


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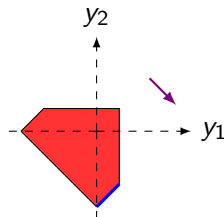
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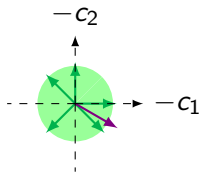


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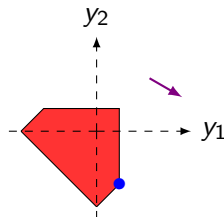
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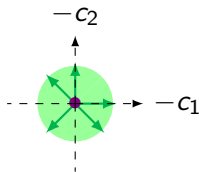


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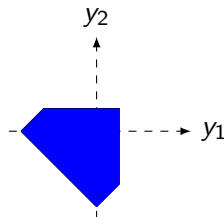
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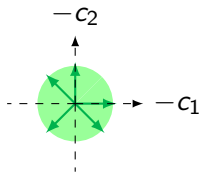


P_x for $x = 0.3$

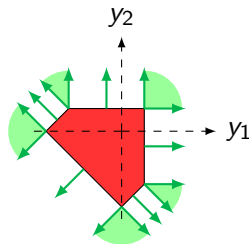
$\mathcal{N}(P_x)$: partition of cost coherent with the min

$$V(x) = \mathbb{E} \left[\min_{y \in P_x} c^\top y \right]$$

For any $N \in \mathcal{N}(P_x)$, $-c \mapsto \arg \min_{y \in P_x} c^\top y$ is constant for all $-c \in \text{ri}(N)$.



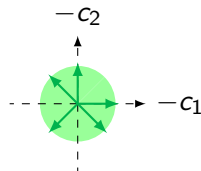
Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$



P_x for $x = 0.3$

Local and universal exact quantization for c

$$\begin{aligned} V(x) &= \mathbb{E} \left[\min_{y \in P_x} c^\top y \right] \\ &= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E} \left[\mathbb{1}_{c \in -\text{ri } N} \min_{y \in P_x} c^\top y \right] \end{aligned}$$

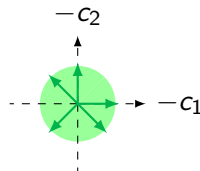


$\mathcal{N}(P_x)$

for $x = 0.3$

Local and universal exact quantization for c

$$\begin{aligned}
 V(x) &= \mathbb{E} \left[\min_{y \in P_x} c^\top y \right] \\
 &= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E} \left[\mathbb{1}_{c \in -\text{ri } N} \min_{y \in P_x} c^\top y \right] \quad \text{where } y_N(x) \in \arg \min_{y \in P_x} \underbrace{c^\top}_{\in -\text{ri } N} y. \\
 &= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E} \left[\mathbb{1}_{c \in -\text{ri } N} c^\top \right] y_N(x)
 \end{aligned}$$

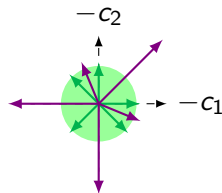


$\mathcal{N}(P_x)$

for $x = 0.3$

Local and universal exact quantization for \mathbf{c}

$$\begin{aligned}
 V(x) &= \mathbb{E} \left[\min_{y \in P_x} \mathbf{c}^\top y \right] \\
 &= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E} \left[\mathbb{1}_{\mathbf{c} \in -\text{ri } N} \min_{y \in P_x} \mathbf{c}^\top y \right] \quad \text{where } y_N(x) \in \arg \min_{y \in P_x} \underbrace{\mathbf{c}^\top}_{\in -\text{ri } N} y. \\
 &= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E} \left[\mathbb{1}_{\mathbf{c} \in -\text{ri } N} \mathbf{c}^\top \right] y_N(x) \\
 &= \sum_{N \in \mathcal{N}(P_x)} p_N \check{\mathbf{c}}_N^\top y_N(x)
 \end{aligned}$$



$\mathcal{N}(P_x)$ and $p_N \check{\mathbf{c}}_N$ for $x = 0.3$

For $N \in \mathcal{N}(P_x)$,

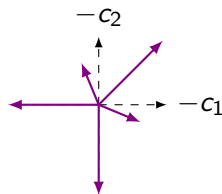
$$p_N := \mathbb{P}[\mathbf{c} \in -\text{ri } N]$$

$$\check{\mathbf{c}}_N := \mathbb{E}[\mathbf{c} \mid \mathbf{c} \in -\text{ri } N]$$

We replace the continuous cost \mathbf{c} ,
by the discrete cost $\check{\mathbf{c}}$.

Local and universal exact quantization for \mathbf{c}

$$\begin{aligned}
 V(x) &= \mathbb{E} \left[\min_{y \in P_x} \mathbf{c}^\top y \right] \\
 &= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E} \left[\mathbb{1}_{\mathbf{c} \in -\text{ri } N} \min_{y \in P_x} \mathbf{c}^\top y \right] \quad \text{where } y_N(x) \in \arg \min_{y \in P_x} \underbrace{\mathbf{c}^\top}_{\in -\text{ri } N} y. \\
 &= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E} \left[\mathbb{1}_{\mathbf{c} \in -\text{ri } N} \mathbf{c}^\top \right] y_N(x) \\
 &= \sum_{N \in \mathcal{N}(P_x)} p_N \check{\mathbf{c}}_N^\top y_N(x) \\
 &= \sum_{N \in \mathcal{N}(P_x)} p_N \min_{y \in P_x} \check{\mathbf{c}}_N^\top y
 \end{aligned}$$



$p_N \check{\mathbf{c}}_N$ for $x = 0.3$

For $N \in \mathcal{N}(P_x)$,

$$p_N := \mathbb{P}[\mathbf{c} \in -\text{ri } N]$$

$$\check{\mathbf{c}}_N := \mathbb{E}[\mathbf{c} \mid \mathbf{c} \in -\text{ri } N]$$

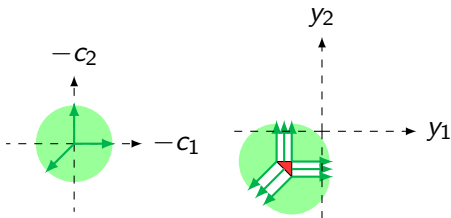
We replace the continuous cost \mathbf{c} ,
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Contents

- 1 Local and Universal Exact Quantization for cost in 2-stage
- 2 Uniform and Universal Exact Quantization for cost in 2-stage**
- 3 Uniform and Universal Exact Quantization for cost in multistage
- 4 Complexity results

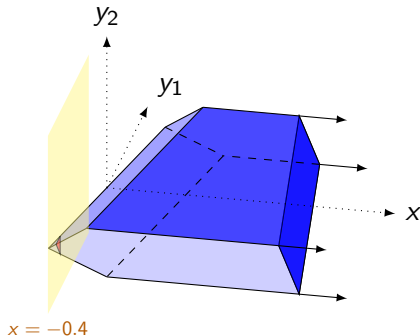
$x \mapsto \mathcal{N}(P_x)$ is piecewise constant.

$$P_x := \{y \mid Ay + Bx \leq b\} \quad \text{and} \quad P := \{(x, y) \mid Ay + Bx \leq b\}$$



$\mathcal{N}(P_x)$

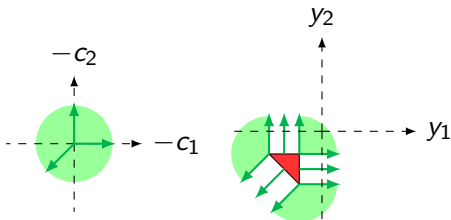
P_x and $\mathcal{N}(P_x)$



P and P_x

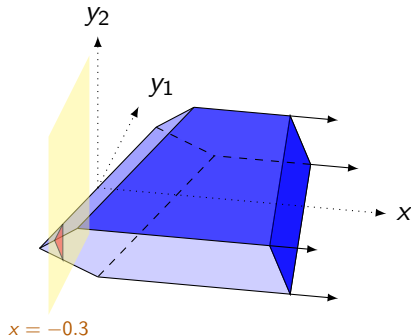
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$\mathcal{N}(P_x)$

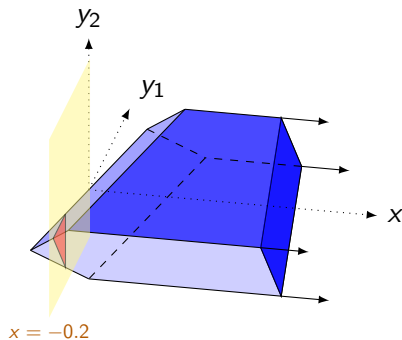
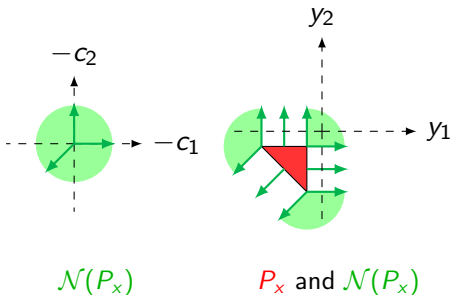
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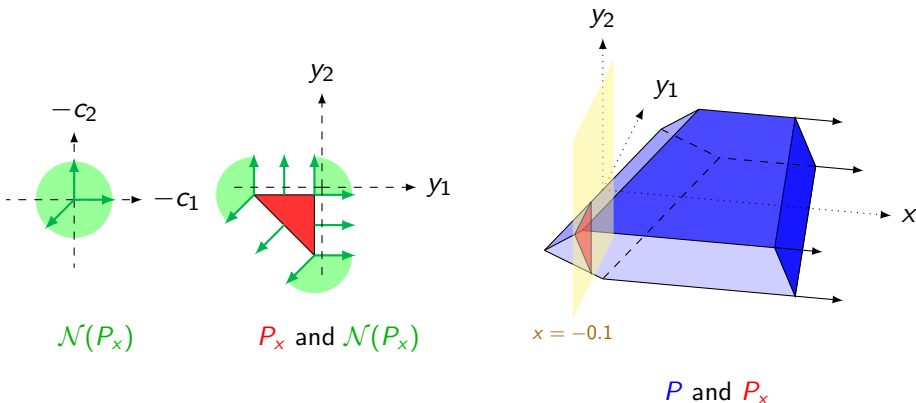
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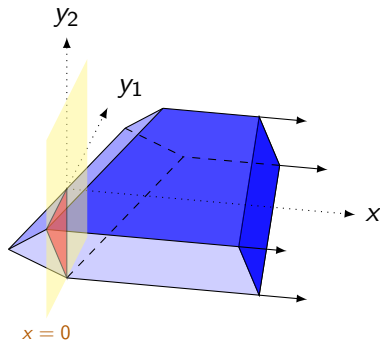
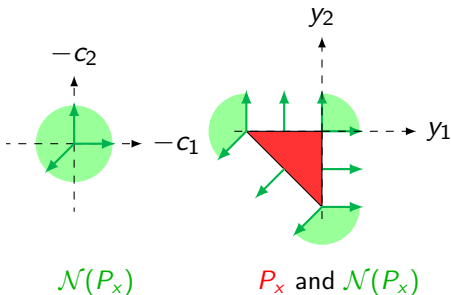
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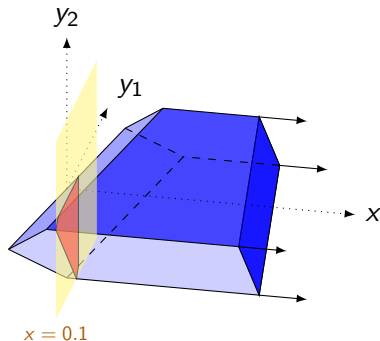
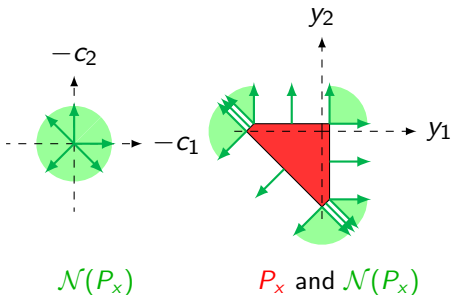
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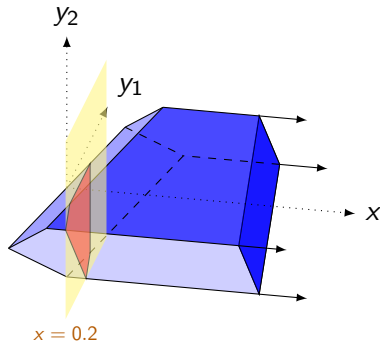
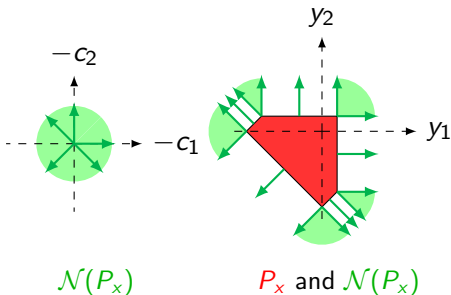
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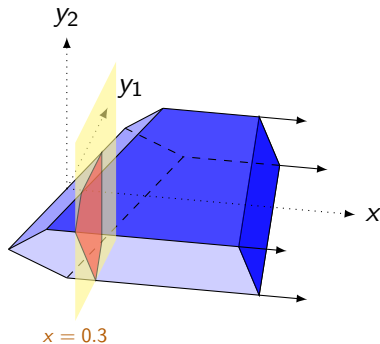
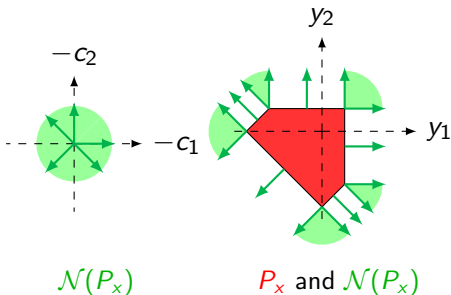
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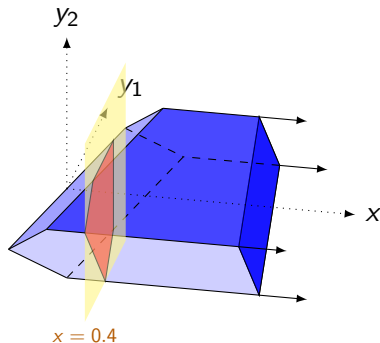
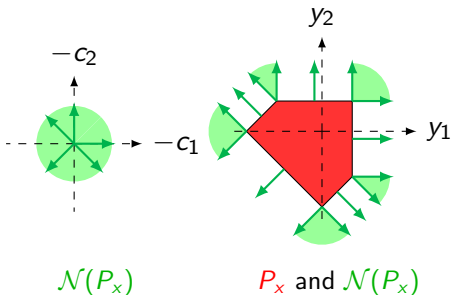
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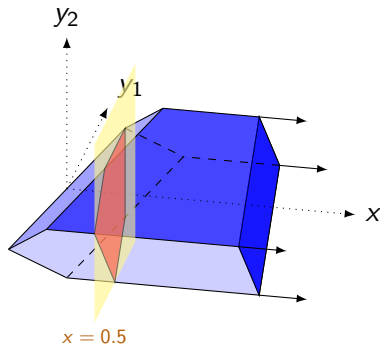
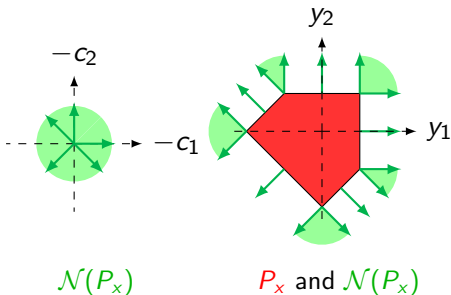
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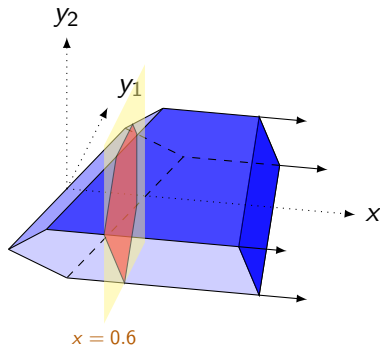
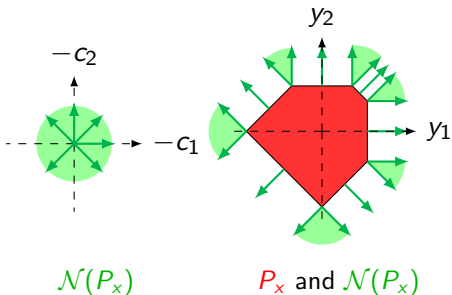
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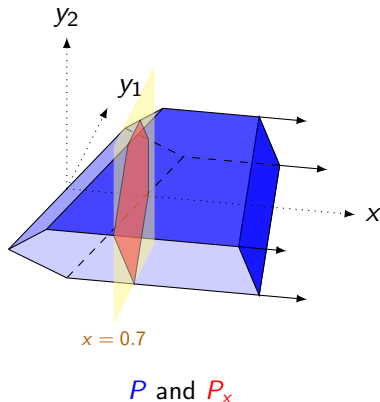
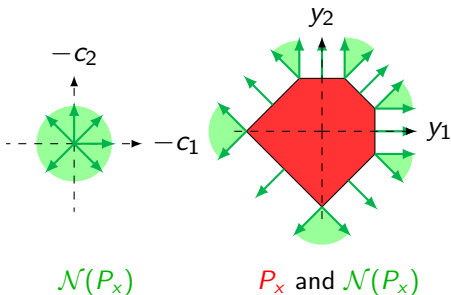
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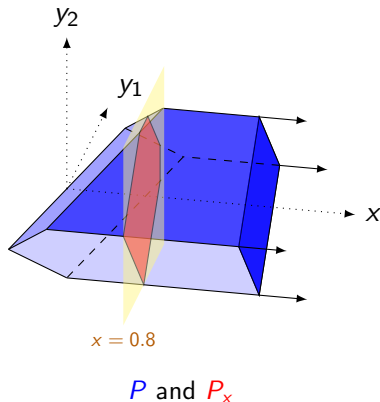
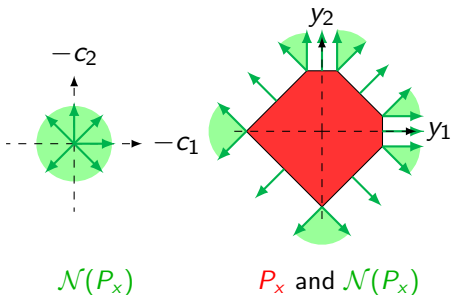
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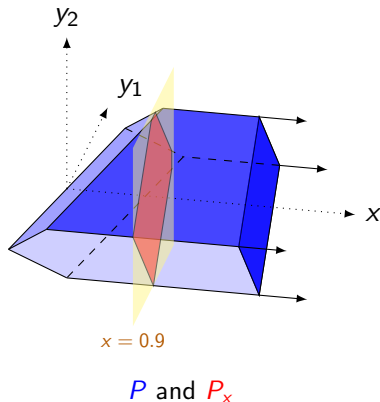
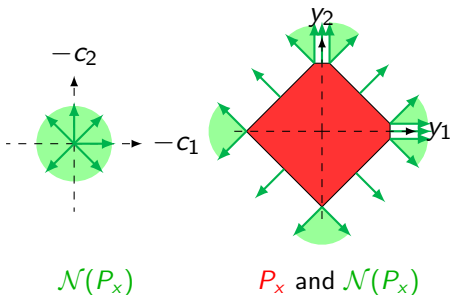
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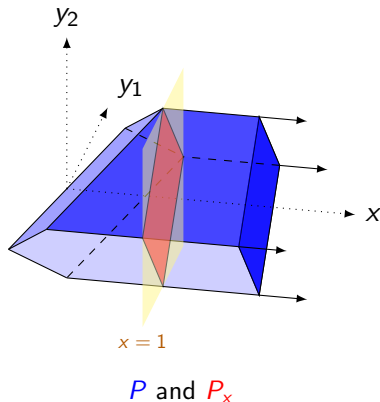
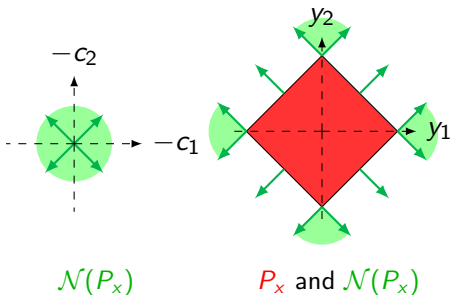
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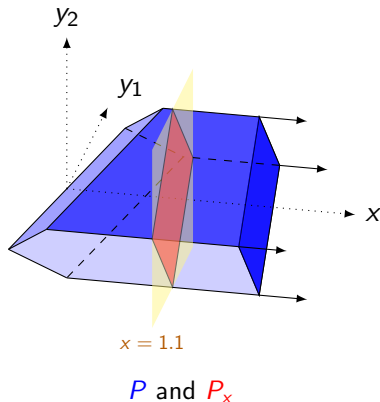
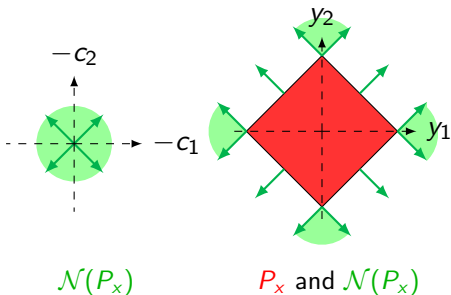
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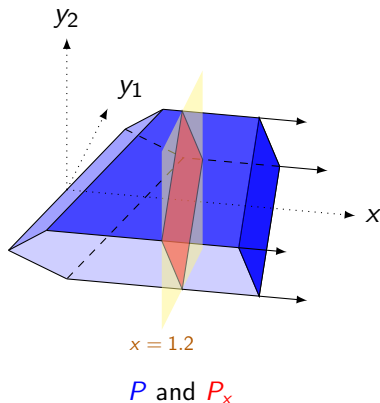
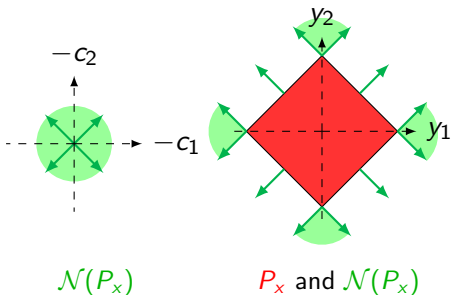
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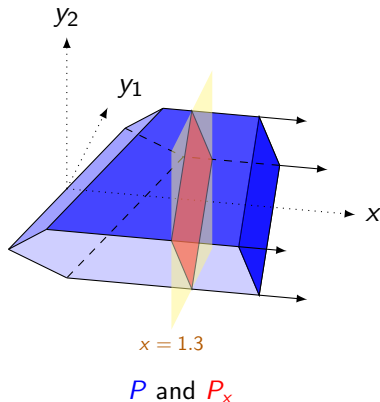
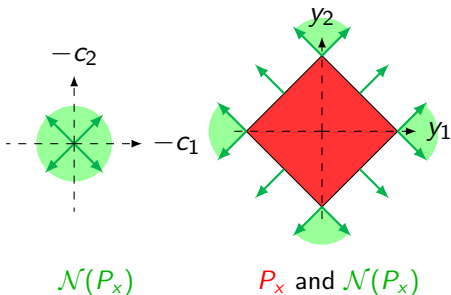
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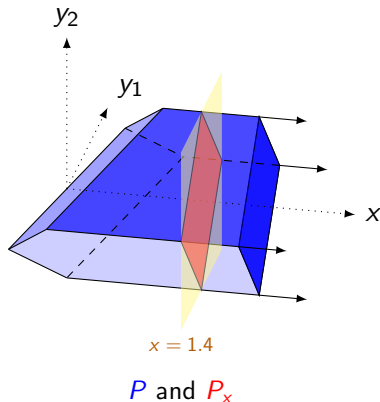
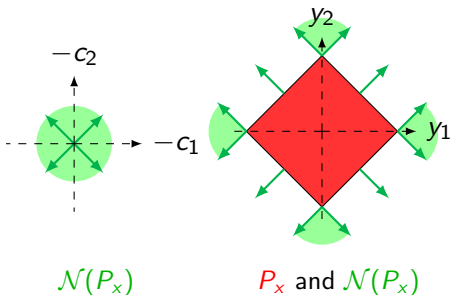
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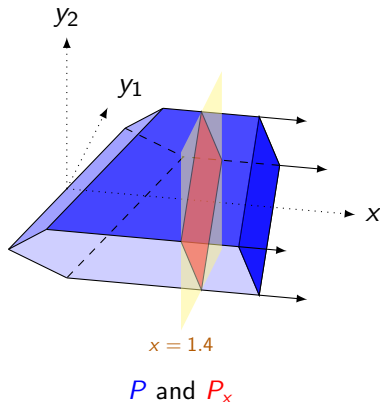
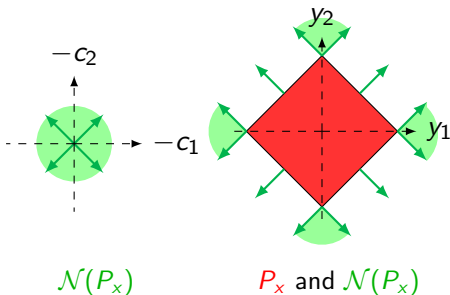
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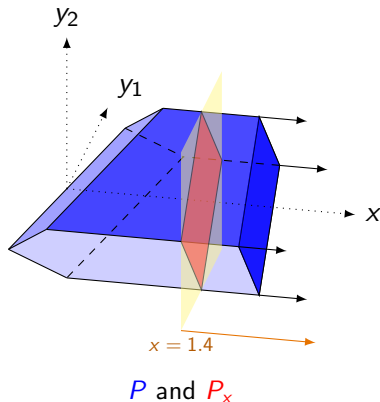
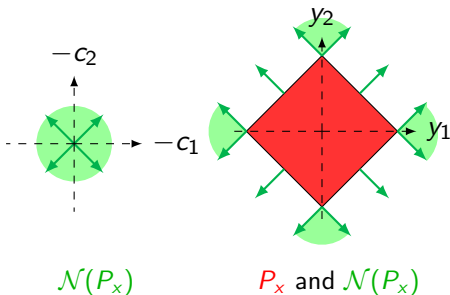
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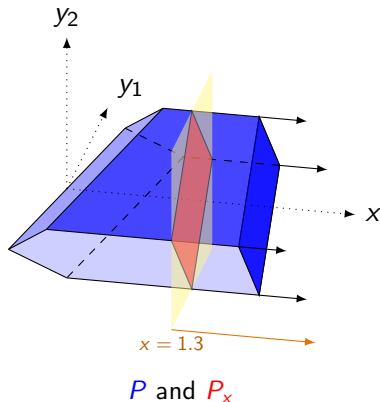
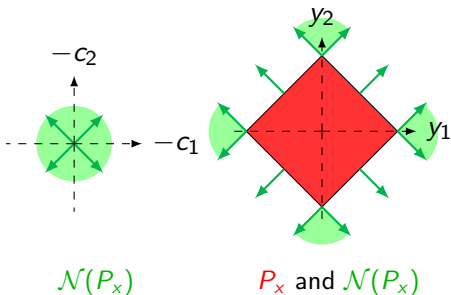
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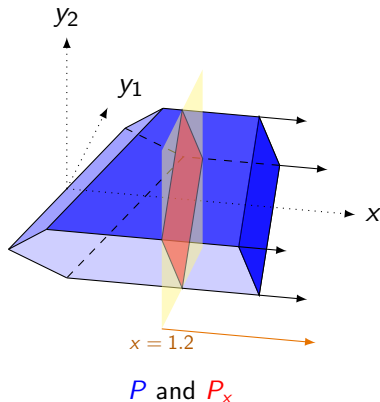
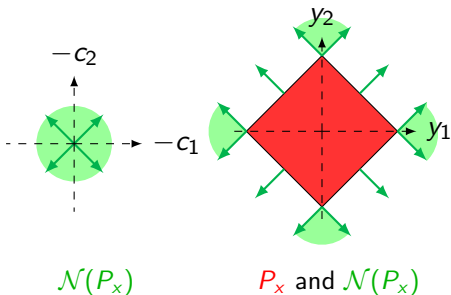
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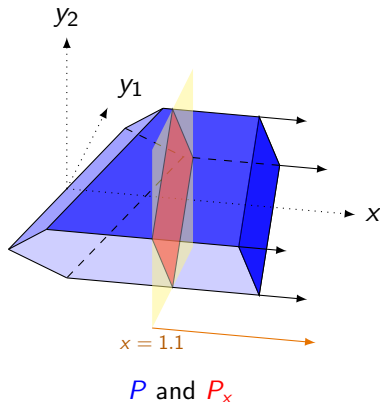
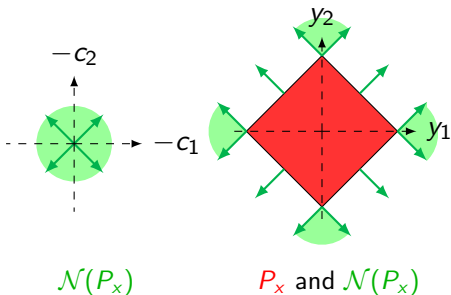
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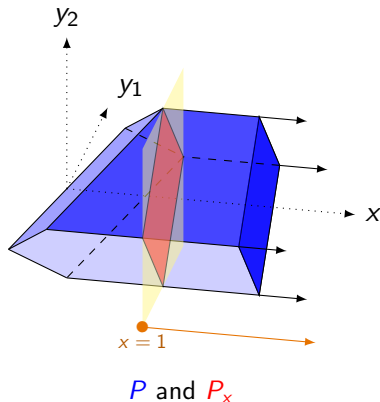
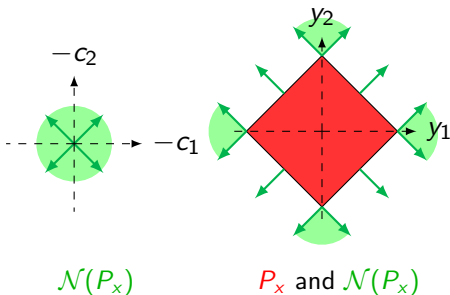
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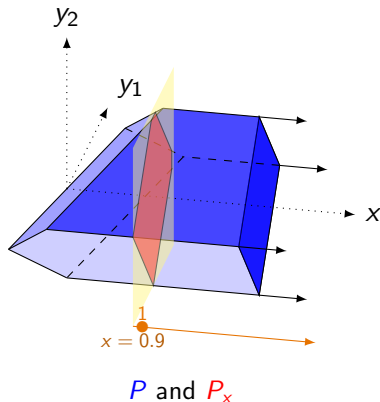
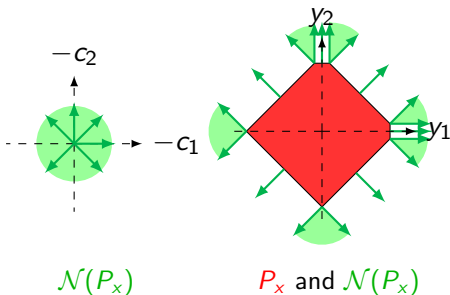
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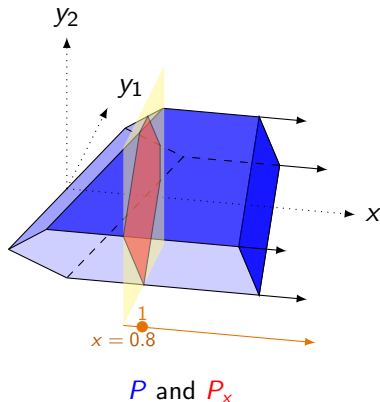
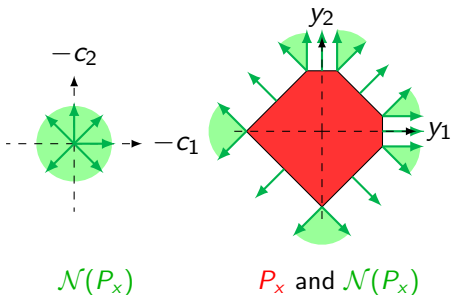
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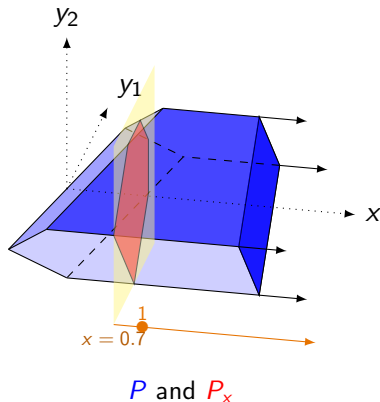
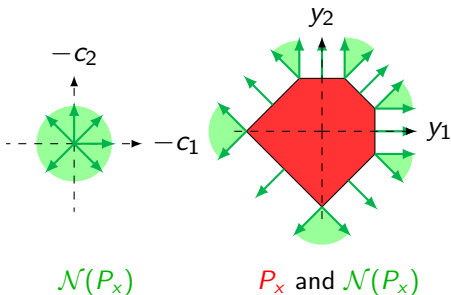
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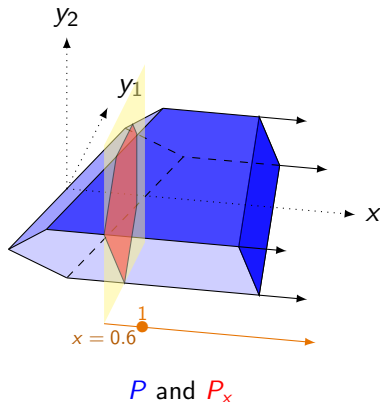
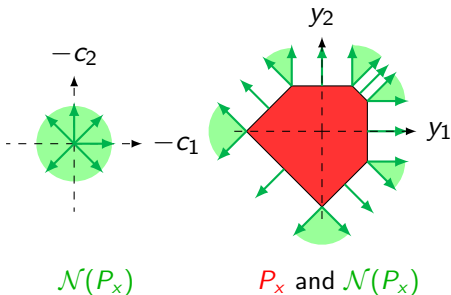
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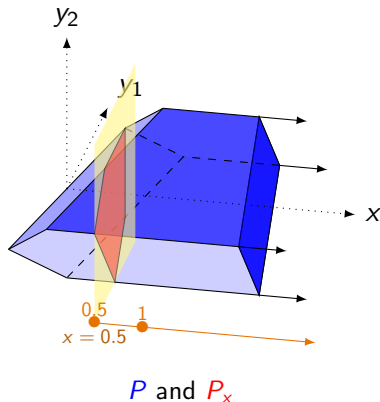
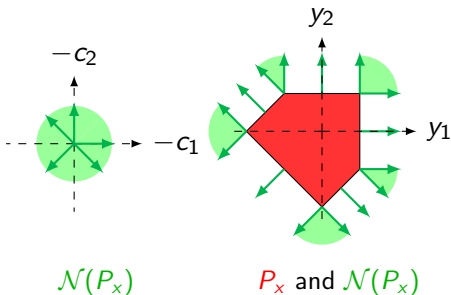
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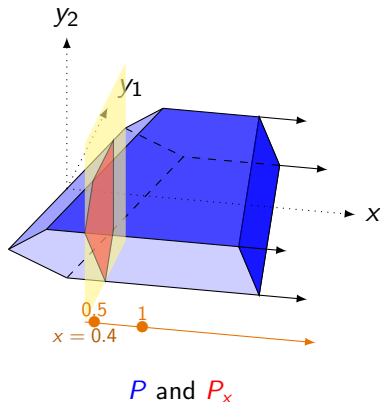
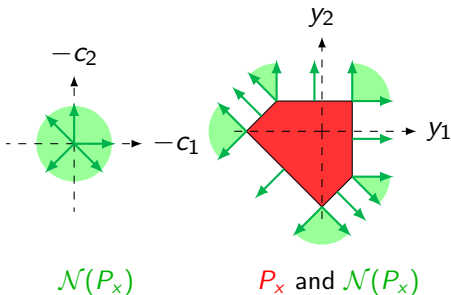
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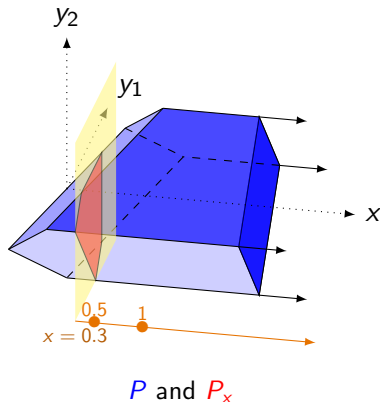
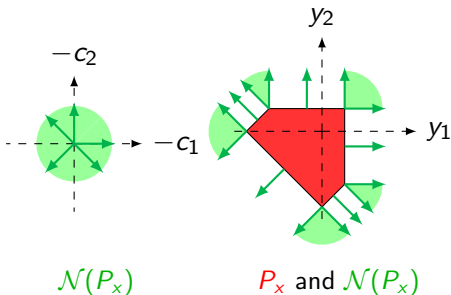
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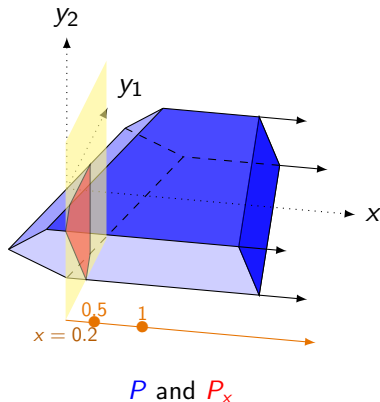
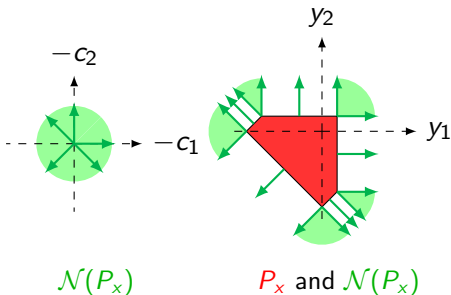
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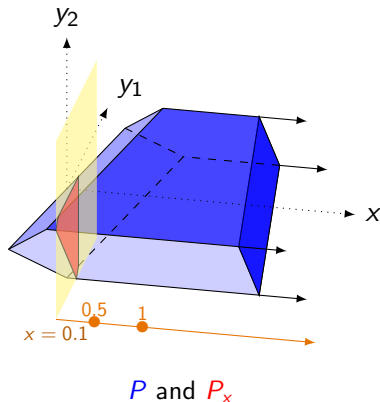
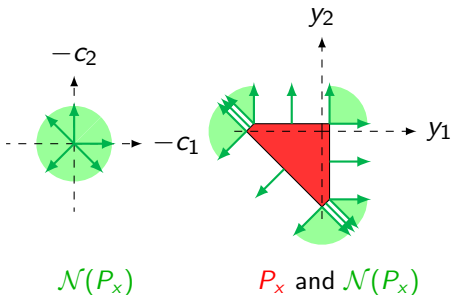
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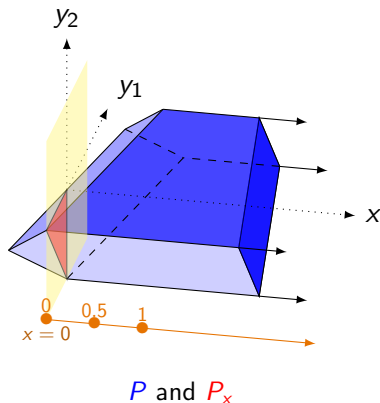
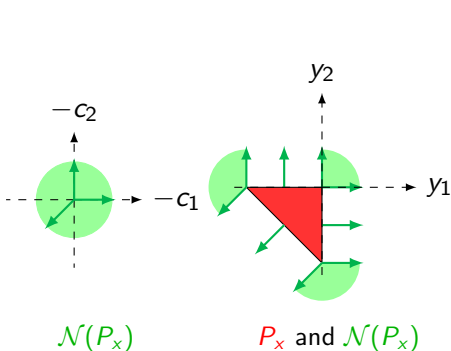
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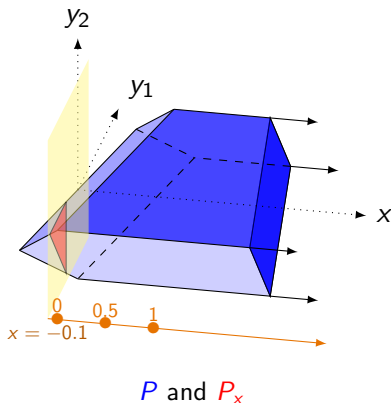
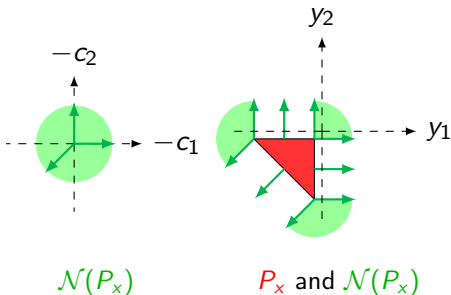
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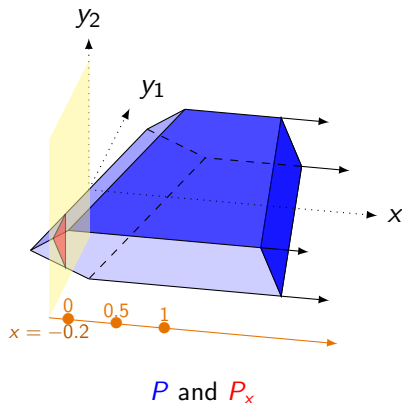
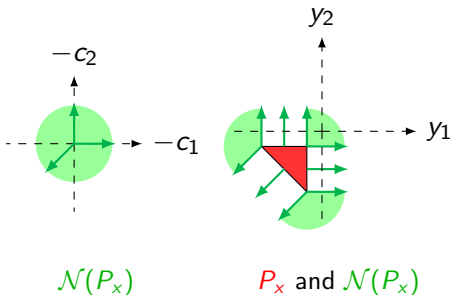
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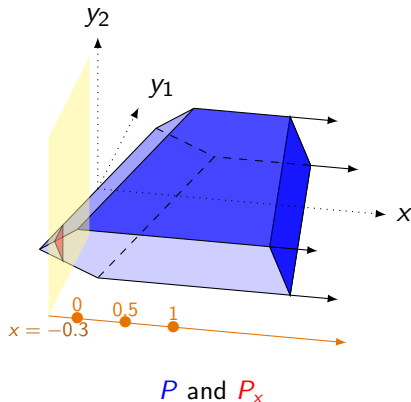
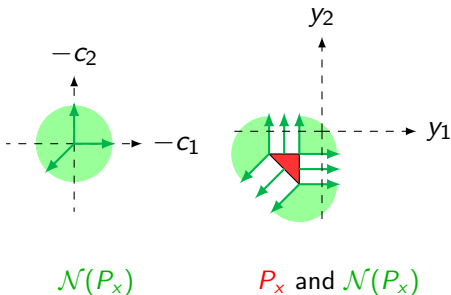
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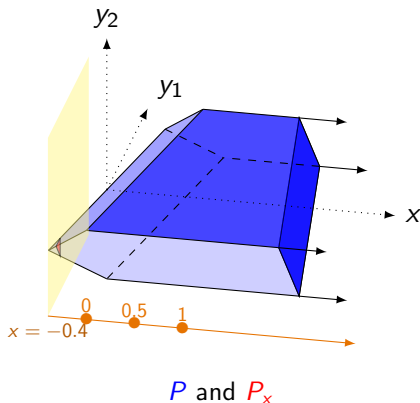
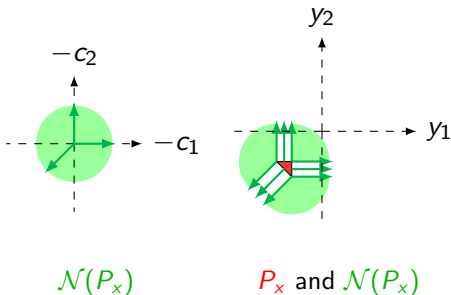
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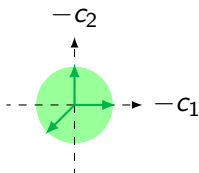
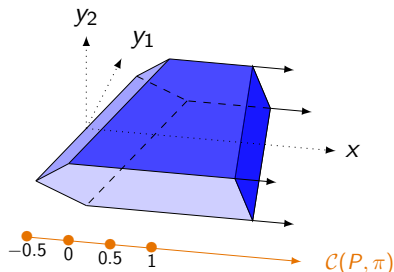


What are the constant regions of $x \mapsto \mathcal{N}(P_x)$?

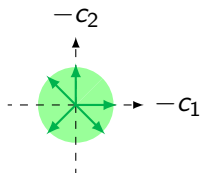
Proposition

There exists a collection $\mathcal{C}(P, \pi)$ called the **chamber complex** whose relative interior of cells are the constant regions of $x \mapsto \mathcal{N}(P_x)$.

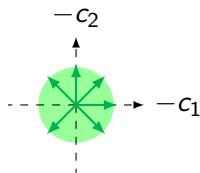
I.e, for $\sigma \in \mathcal{C}(P, \pi)$ and $x, x' \in \text{ri}(\sigma)$, we have $\mathcal{N}(P_x) = \mathcal{N}(P_{x'}) =: \mathcal{N}_\sigma$



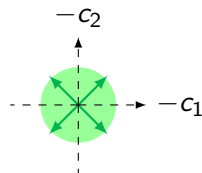
\mathcal{N}_σ for $\sigma = [-0.5, 0]$



\mathcal{N}_σ for $\sigma = [0, 0.5]$



\mathcal{N}_σ for $\sigma = [0.5, 1]$



\mathcal{N}_σ for $\sigma = [1, +\infty)$

Chamber complex

Definition

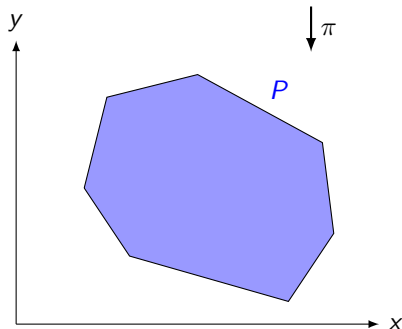
The *chamber complex* $\mathcal{C}(P, \pi)$ of P along π is

$$\mathcal{C}(P, \pi) := \{\sigma_{P, \pi}(x) \mid x \in \pi(P)\}$$

where

$$\sigma_{P, \pi}(x) := \bigcap_{F \in \mathcal{F}(P) \mid x \in \pi(F)} \pi(F)$$

where $\mathcal{F}(P)$ is the set of faces of P and π is the projection $(x, y) \mapsto x$.



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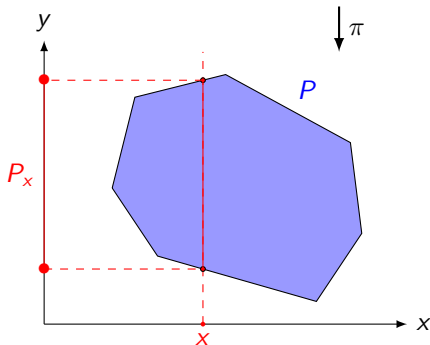
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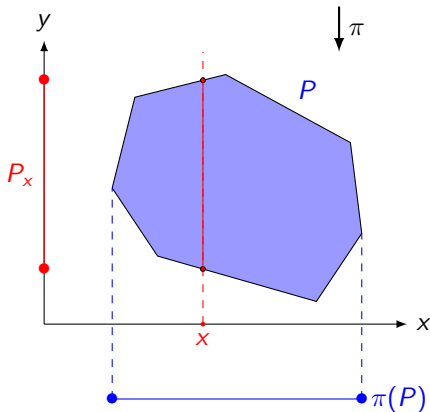
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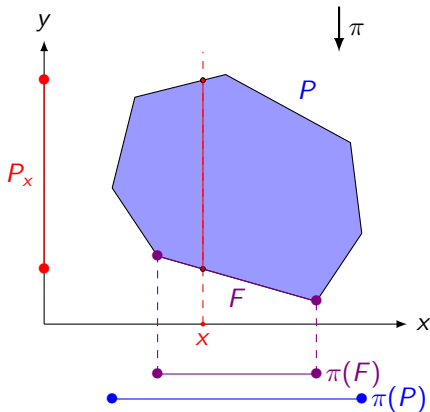
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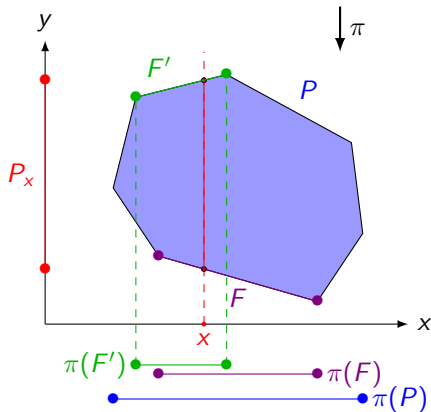
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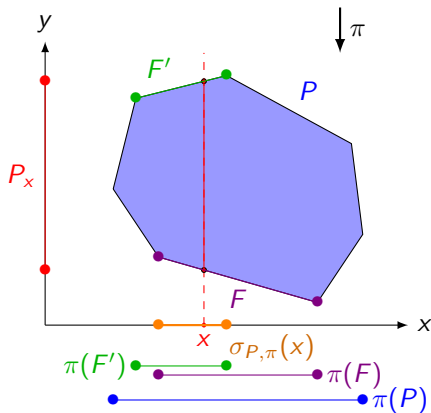
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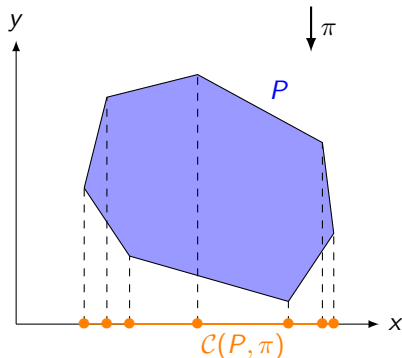
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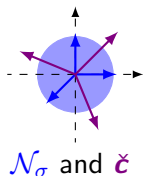
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Common Refinement of Normal Fans

We can quantize \mathbf{c} on each chamber.

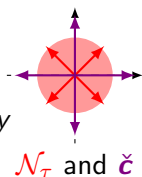


For all $x \in \text{ri}(\sigma)$,

$$V(x) = \sum_{N \in \mathcal{N}_\sigma} p_N \min_{y \in P_x} \check{\mathbf{c}}_N^\top y$$

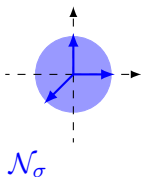
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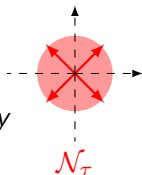


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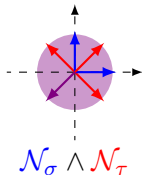
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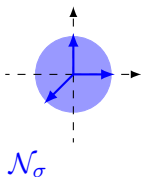


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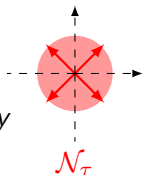


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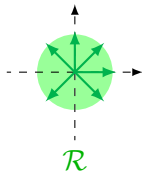
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Uniform exact quantization for \mathcal{C}

Let's sum up:

- local exact quantization at x induced by $\mathcal{N}(P_x)$;
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Theorem (Uniform and universal quantization of the cost distribution)

Let $\mathcal{R} = \bigwedge_{\sigma \in \mathcal{C}(P, \pi)} -\mathcal{N}_\sigma$, then **for all** $x \in \mathbb{R}^n$

$$V(x) = \sum_{R \in \mathcal{R}} \check{p}_R \min_{y \in P_x} \check{c}_R^\top y$$

where $\check{p}_R := \mathbb{P}[\mathbf{c} \in \text{ri}(R)]$ and $\check{c}_R := \mathbb{E}[\mathbf{c} \mid \mathbf{c} \in \text{ri}(R)]$

Polyhedral characterization of V

Theorem

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E is the *weighted fiber polyhedron*

$$E := \mathbb{E}[D_{\mathbf{c}}] = \int D_{\mathbf{c}} \mathbb{P}(d\mathbf{c})$$

with $D_{\mathbf{c}}$ the dual admissible set

$$D_{\mathbf{c}} := \{\lambda \mid A^\top \lambda + \mathbf{c} = 0\}$$

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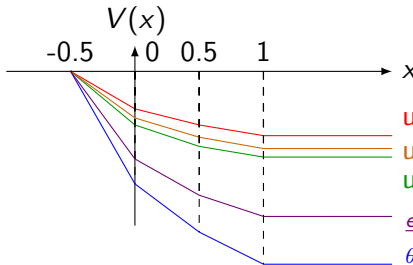
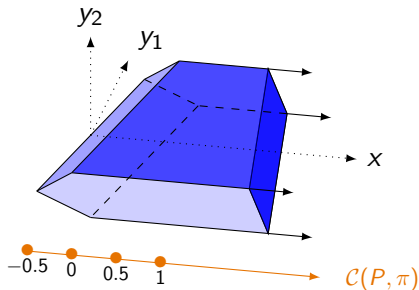
Extension of fiber polytope of



L. Billera, B. Sturmfels, Fiber polytopes, *Annals of Mathematics*, p527–549, 1992.

Explicit computation of the example

$$V(x) = \mathbb{E} \left[\begin{array}{ll} \min_{y \in \mathbb{R}^2} & \mathbf{c}^\top y \\ \text{s.t.} & \|y\|_1 \leq 1 \\ & y_1 \leq x \\ & y_2 \leq x \end{array} \right]$$



Different distributions of \mathbf{c} :

uniform on norm 1 ball

uniform on norm 2 ball

uniform on norm ∞ ball

$$\frac{e^{-\frac{\|\mathbf{c}\|_2^2}{2\gamma^2}}}{\theta^2 e^{-\frac{\|\mathbf{c}\|_1}{\theta}}} d\mathbf{c}$$

Contents

- 1 Local and Universal Exact Quantization for cost in 2-stage
- 2 Uniform and Universal Exact Quantization for cost in 2-stage
- 3 Uniform and Universal Exact Quantization for cost in multistage
- 4 Complexity results

Extension to multistage and stochastic constraints

Theorem

All results generalizes to multistage problem with finitely supported stochastic constraints.

- ➡ $(V_t)_t$ are affine on *universal* chamber complexes, i.e. independent of the law of $(c_t)_t$
- ➡ We have an *uniform and universal* exact quantization.

Core idea of the proof :

Iterated chamber complexes

$$\mathcal{P}_{t,\xi} := \mathcal{C}\left((\mathbb{R}^{n_t} \times \mathcal{P}_{t+1}) \wedge \mathcal{F}(P_t(\xi)), \pi_{x_{t-1}}^{x_{t-1}, x_t}\right)$$
$$\mathcal{P}_t := \bigwedge_{\xi_t \in \text{supp } \xi_t} \mathcal{P}_{t,\xi}$$

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- ➡ $(V_t)_t$ are affine on *universal* chamber complexes, i.e. independent of the law of $(\mathbf{c}_t)_t$
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Core idea of the proof :

Iterated chamber complexes

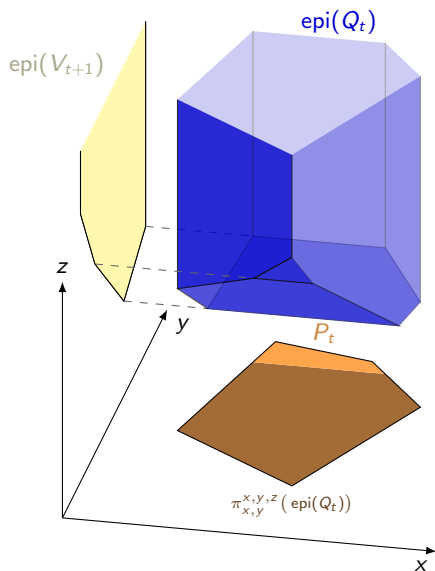
$$\mathcal{P}_{t,\xi} := \mathcal{C}\left((\mathbb{R}^{n_t} \times \mathcal{P}_{t+1}) \wedge \mathcal{F}(P_t(\xi)), \pi_{x_{t-1}}^{x_{t-1}, x_t}\right)$$
$$\mathcal{P}_t := \bigwedge_{\xi_t \in \text{supp } \xi_t} \mathcal{P}_{t,\xi}$$

Multistage uniform and universal exact quantization

$$V_t(x) = \mathbb{E} \left[\min_{y \in \mathbb{R}^{n_t}} \mathbf{c}_t^\top y + V_{t+1}(y) \right]$$

$$\text{s.t. } (x, y) \in P_t$$

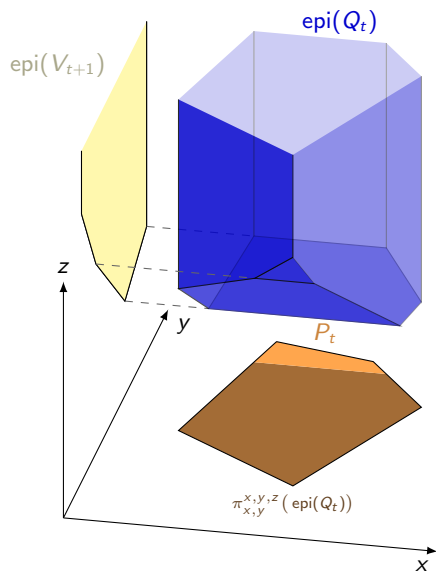
with $Q_t(x, y) := V_{t+1}(y) + \mathbb{I}_{(x, y) \in P_t}$.



Multistage uniform and universal exact quantization

$$V_t(x) = \mathbb{E} \left[\min_{\substack{y \in \mathbb{R}^{n_t} \\ z \in \mathbb{R}}} \mathbf{c}_t^\top y + z \right. \\ \left. \text{s.t. } (x, y, z) \in \text{epi}(Q_t) \right]$$

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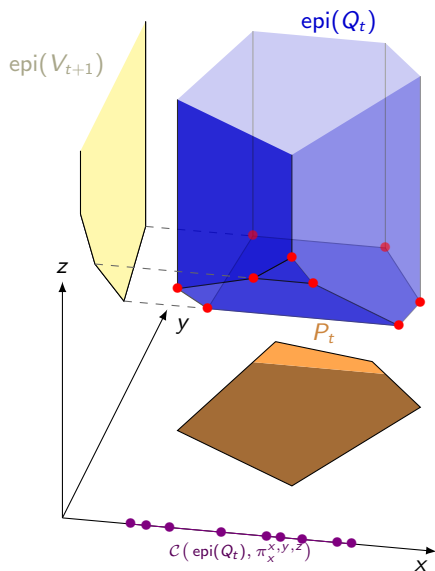


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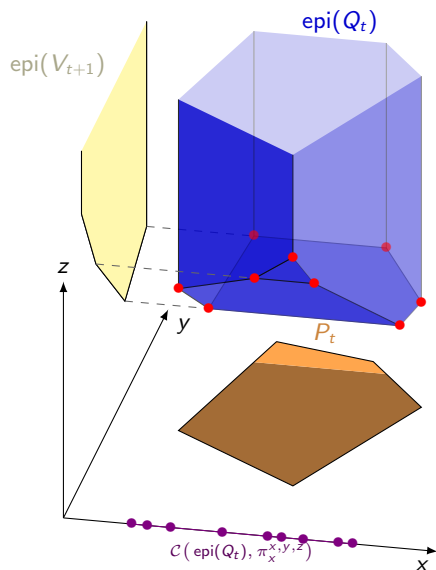
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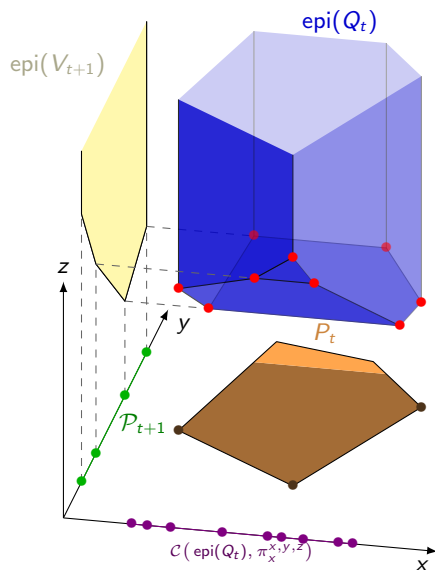
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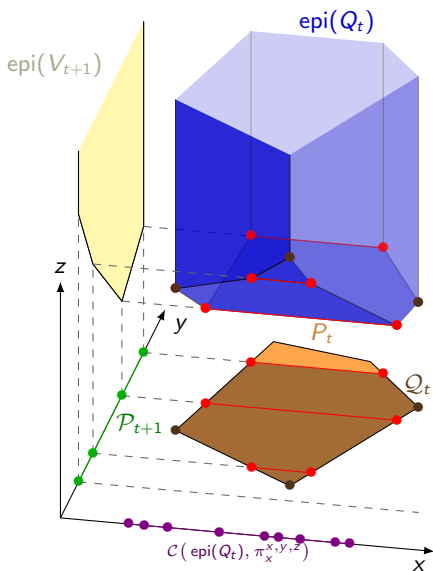
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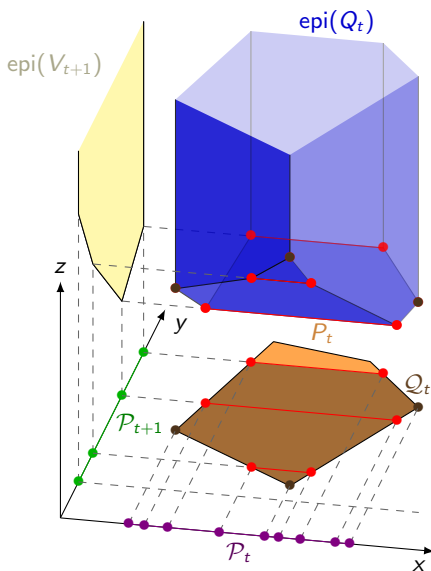
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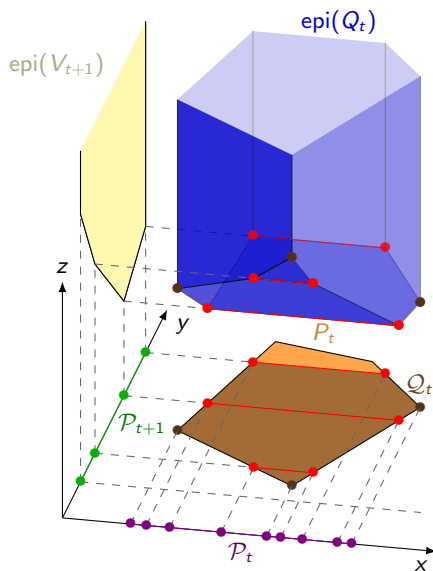
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[FGL21, Lem. 4.1]: $P_t \preceq \mathcal{C}(\text{epi}(Q_t), \pi_x^{x,y,z})$

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Contents

- 1 Local and Universal Exact Quantization for cost in 2-stage
- 2 Uniform and Universal Exact Quantization for cost in 2-stage
- 3 Uniform and Universal Exact Quantization for cost in multistage
- 4 Complexity results

Earlier and new complexity results

Volume of a polytope

$$\text{Vol}(\{z \in \mathbb{R}^d \mid Az \leq b\}) \text{ or}$$
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- $\#P$ -complete:
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 - \rightsquigarrow Exact case
 - \rightsquigarrow Approximated case

Complexity result multistage

Theorem (FGL: MSLP is polynomial for fixed dimensions)

Assume that T , n_t , and $|\text{supp}(\mathbf{A}_t, \mathbf{B}_t, \mathbf{b}_t)|$, for $t = 2, \dots, T$, are fixed integers.¹

Assume that \mathbf{c} admits a density function with a bounded total variation.

*Then, there exists an algorithm that either asserts that MSLP is unfeasible or finds an ε -solution in **polynomial** time in $\log(\frac{1}{\varepsilon})$ with **probability 1**.*

¹No requirement for the first decision.

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By SAA, we can solve MSLP, up to precision ε , in **pseudo-polynomial** time, i.e. polynomial in $\frac{1}{\varepsilon}$, with **probability $1 - \alpha$** , when T, n_1, \dots, n_T are fixed.

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Conclusion and applications

- *Uniform and universal* exact quantization for an MSLP

➡ New complexity results.

Unfortunately this quantization might be very large.

- *Local* exact quantization for \mathbf{c}

➡ Higher order simplex algorithm on the chamber complex solves 2SLP of dimension $100 + 10$.

- *Local* exact quantization for \mathbf{B} and \mathbf{b} .

➡ Adaptive Partition-based Methods (APM) for general distribution: solves small 2SLP with high precision.

- Extension of Stochastic Dual Dynamic Programming algorithms for non finitely supported distribution.

- Links with fundamental polyhedral geometry, regular subdivisions and fiber polytope.

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Thank you for listening ! Any question ?



M. Forcier, S. Gaubert, V. Leclère

Exact quantization of multistage stochastic linear problems.

arXiv preprint arXiv:2107.09566 (2021).



M. Forcier, V. Leclère

Generalized adaptive partition-based method for two-stage stochastic linear programs: convergence and generalization.

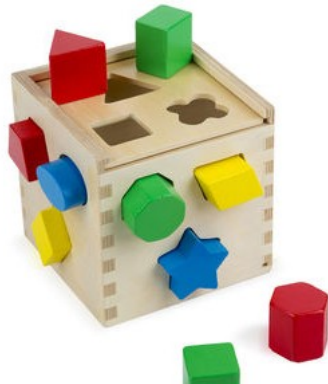
Operation Research Letters, to appear (2022).



M. Forcier, V. Leclère

Convergence of Trajectory Following Dynamic Programming algorithms for multistage stochastic problems without finite support assumptions

HAL Id : hal-03683697 (2022).



Local exact quantization and adapted partition

Local exact quantization

random cost

Recall that for a fixed x ,

$$\begin{aligned}\mathbb{E} \left[\min_{y \in P_x} \mathbf{c}^\top y \right] \\ = \sum_{N \in \mathcal{N}(P_x)} p_N \min_{y \in P_x} \check{\mathbf{c}}_N^\top y\end{aligned}$$

where,

$$\begin{aligned}p_N &:= \mathbb{P}[\mathbf{c} \in -\text{ri } N] \\ \check{\mathbf{c}}_N &:= \mathbb{E}[\mathbf{c} \mid \mathbf{c} \in -\text{ri } N]\end{aligned}$$

$$P_x := \{y \in \mathbb{R}^m \mid Ay + Bx \leq b\}$$

GAPM

random constraints

Similarly, for a given q , and all x ,

$$\begin{aligned}V(x) &:= \mathbb{E}[Q(x, \boldsymbol{\xi})] \\ &= \mathbb{E} \left[\max_{\lambda \in D_q} (\mathbf{h} - \mathbf{T}x)^\top \lambda \right] \\ &= \sum_{N \in \mathcal{N}(D_q)} p_N \max_{\lambda \in D_q} \psi_{N,x}^\top \lambda\end{aligned}$$

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An explicit adapted partition

Consider $x \in \mathbb{R}^n$ and $N \in \mathcal{N}(D_q)$ a normal cone of D_q . We define

$$E_{N,x} := \{\xi \in \Xi \mid h - Tx \in \text{ri } N\}$$

Theorem (FL 2021)

$\mathcal{R}_x := \{E_{N,x} \mid N \in \mathcal{N}(D_q)\}$ is an adapted partition to x
i.e. $V_{\mathcal{R}_x}(x) = V(x)$

Proof:

$$\begin{aligned} V(x) &:= \mathbb{E}[Q(x, \xi)] \\ &= \sum_{N \in \mathcal{N}(D)} \mathbb{P}[h - Tx \in \text{ri } N] \min_{\lambda \in D} \mathbb{E}[h - Tx \mid h - Tx \in \text{ri } N]^\top \lambda \\ &= \sum_{N \in \mathcal{N}(D)} \mathbb{P}[\xi \in E_{N,x}] Q\left(\mathbb{E}[\xi \mid \xi \in E_{N,x}], x\right) = V_{\mathcal{R}_x}(x) \end{aligned}$$

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Numerical Results - ProdMix

k	z_L^k	z_U^k	$z_U^k - z_L^k$	Total time	$ \mathcal{P}^k $
1	-18666.67	-16939.71	1726.96	0.57 s	4
2	-17873.01	-17383.73	489.28	2.1 s	9
4	-17744.67	-17709.00	35.67	9.1 s	25
6	-17713.74	-17711.37	2.37	23.7 s	49
8	-17711.71	-17711.56	0.15	50.0 s	81
10	-17711.57	-17711.56	0.01	88.0 s	121

Table: Results for problem Prod-Mix

Comparison with SAA : we solved the same problem 100 times, each with 10 000 scenarios randomly drawn

↪ 95% confidence interval centered in -17711 , with radius 2.2.

↪ required 2058s of computation.