Multistage stochastic optimization and polyhedral geometry

Maël Forcier

PhD Defense, under the supervision of

Jean-Philippe Chancelier, Stéphane Gaubert and Vincent Leclère

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Modeling hydroelectric energy storage management

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- u water hustled
- d demand
- c cost of unmet demand

$$\min_{u} \; c(d - u)$$
$$s.c. \; 0 \leqslant u \leqslant d$$

Modeling hydroelectric energy storage management

- u water hustled
- d demand
- c cost of unmet demand
- x_0/x_1 water in the reservoir
- $\overline{x}$ capacity of the reservoir

$$
\begin{aligned}
\min_{u} \quad & c(d - u) \\
s.c. \quad & 0 \leqslant u \leqslant d \\
& x_1 = x_0 - u \\
& 0 \leqslant x_0 \leqslant \overline{x}, \; 0 \leqslant x_1 \leqslant \overline{x}
\end{aligned}
$$

Modeling hydroelectric energy storage management

- u water hustled
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- x_0/x_1 water in the reservoir
- $\overline{x}$ capacity of the reservoir
- w rain and runoff

$$\begin{aligned} \min_{u} \quad & c(d - u) \\ s.c. \quad & 0 \leqslant u \leqslant d \\ & x_1 = x_0 - u + w \\ & 0 \leqslant x_0 \leqslant \overline{x},\ 0 \leqslant x_1 \leqslant \overline{x} \end{aligned}$$

Modeling hydroelectric energy storage management

- u water hustled
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- c cost of unmet demand
- x_0 / x_1 water in the reservoir
- $\overline{x}$ capacity of the reservoir
- w rain and runoff
- v water evacuated by the valve

$$
\begin{aligned}
\min_{u,v} \quad & c(d-u) \\
s.c. \quad & 0 \leqslant u \leqslant d \\
& x_1 = x_0 - u + w - v \\
& 0 \leqslant x_0 \leqslant \overline{x}, \; 0 \leqslant x_1 \leqslant \overline{x} \\
& 0 \leqslant v
\end{aligned}
$$

Modeling hydroelectric energy storage management

At step t

- u_t water hustled
- d_t demand
- c_t cost of unmet demand
- x_t water in the reservoir
- $\overline{x}$ capacity of the reservoir
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- v_t water evacuated by the valve

$$
\begin{aligned}
\min_{u_t, v_t} \quad & \sum_{t=1}^{T} c_t(d_t - u_t) \\
s.c.\ & \forall t \in [T],\ 0 \leqslant u_t \leqslant d_t \\
& \forall t \in [T],\ x_{t+1} = x_t - u_t + w_t - v_t \\
& \forall t \in [T],\ 0 \leqslant x_t \leqslant \overline{x} \\
& \forall t \in [T],\ 0 \leqslant v_t
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$$

Modeling hydroelectric energy storage management

At step t

- $\boldsymbol{u}_t$ water hustled
- $\boldsymbol{d}_t$ demand
- $\boldsymbol{c}_t$ cost of unmet demand
- $\boldsymbol{x}_t$ water in the reservoir
- $\overline{x}$ capacity of the reservoir
- $\boldsymbol{w}_t$ rain and runoff
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$$
\begin{aligned}
&\min_{\boldsymbol{u}_t, \boldsymbol{v}_t} \mathbb{E}\left[\sum_{t=1}^{T} \boldsymbol{c}_t(\boldsymbol{d}_t - \boldsymbol{u}_t)\right] \\
s.c.\ &\forall t \in [T],\ 0 \leqslant \boldsymbol{u}_t \leqslant \boldsymbol{d}_t \\
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&\forall t \in [T],\ 0 \leqslant \boldsymbol{x}_t \leqslant \overline{x} \\
&\forall t \in [T],\ 0 \leqslant \boldsymbol{v}_t
\end{aligned}
$$

Multistage stochastic linear programming (MSLP)

$$\begin{aligned} \min_{(\boldsymbol{x}_t)_{t\in[T]}} \quad & \mathbb{E}\Big[\sum_{t=1}^{T} \boldsymbol{c}_t^\top \boldsymbol{x}_t\Big] & \\ \text{s.t.} \quad & \boldsymbol{A}_t\boldsymbol{x}_t + \boldsymbol{B}_t\boldsymbol{x}_{t-1} \leqslant \boldsymbol{b}_t & \forall t \in [T] \\ & \sigma(\boldsymbol{x}_t) \subset \sigma(\boldsymbol{c}_\tau, \boldsymbol{A}_\tau, \boldsymbol{B}_\tau, \boldsymbol{b}_\tau)_{\tau \leqslant t} & \forall t \in [T] \\ & \boldsymbol{x}_0 \equiv x_0 \text{ given} & \end{aligned}$$

$\boldsymbol{\xi}_t = (\boldsymbol{c}_t, \boldsymbol{A}_t, \boldsymbol{B}_t, \boldsymbol{b}_t)_{t\in[T]}$ is assumed to be stagewise independent.

We set $V_{T+1} \equiv 0$ and:

$$V_t(x_{t-1}) := \mathbb{E}\big[\hat{V}_t(x_{t-1}, \boldsymbol{\xi}_t)\big] := \mathbb{E}\begin{bmatrix} \min\limits_{x_t \in \mathbb{R}^{n_t}} & \boldsymbol{c}_t^\top x_t + V_{t+1}(x_t) \\ \text{s.t.} & \boldsymbol{A}_t x_t + \boldsymbol{B}_t x_{t-1} \leqslant \boldsymbol{b}_t \end{bmatrix}$$

➥ How to deal with continuous distributions ?

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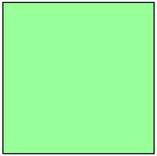
➥ How to deal with continuous distributions ?

Quantization of a MSLP

Real problem

$$V_t(x) = \mathbb{E}\big[\hat{V}_t\big(x, \xi_t\big)\big] = \mathbb{E}\begin{bmatrix} \min\limits_{y \in \mathbb{R}^{n_t}} & \boldsymbol{c}_t^\top y + V_{t+1}(y) \\ \text{s.t.} & \boldsymbol{A}_t y + \boldsymbol{B}_t x \leqslant \boldsymbol{b}_t \end{bmatrix}$$



ξ_t continuous

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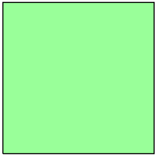
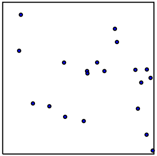


ξ_t continuous

Sample Average Approximation (SAA)

$$V_{t,N}^{SAA}(x) := \frac{1}{N}\sum_{k=1}^{N} \hat{V}_t\left(x, \xi^k\right)$$

$\xi^1, \cdots, \xi^N$ drawn by Monte Carlo



SAA $N = 20$

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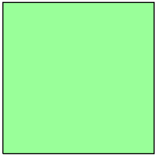

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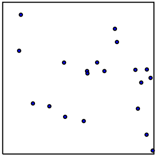
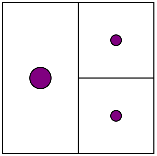


SAA $N = 20$

Partition-based

$$V_{t,\mathcal{P}}(x) := \sum_{P \in \mathcal{P}} \check{p}_{t,P} \hat{V}_t(x, \check{\xi}_{t,P})$$

with $\check{p}_{t,P} := \mathbb{P}\big[\boldsymbol{\xi}_t \in P\big]$ and $\check{\xi}_{t,P} := \mathbb{E}\big[\boldsymbol{\xi}_t \,|\, \boldsymbol{\xi}_t \in P\big]$



Partition-based

Quantization of a MSLP

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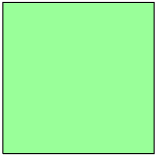

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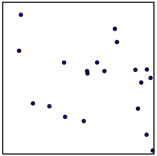

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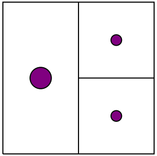

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If $\xi \mapsto \hat{V}(x, \xi)$ is convex, $V_{t,\mathcal{P}}(x) \leqslant V_t(x)$.

Exact quantization

Definition

A MSLP admits a local exact quantization at time t on x if there exists a finitely supported $(\check{\xi}_t)_{t\in[T]}$ *i.e.* such that

$$V_t(x) = \mathbb{E}\big[\hat{V}_t(x,\xi_t)\big] = \mathbb{E}\big[\hat{V}_t(x,\check{\xi}_t)\big].$$

We call an exact quantization

- uniform if it is locally exact at all $x \in \mathbb{R}^{n_t}$, and all $t \in [T]$.
- universal if there exists a partition $\mathcal{P}_{t,x}$ such that the induced quantization is exact at time t on x, for all distributions of $(\xi_\tau)_{\tau\in[T]}$.

Questions:

1. Under which condition does there exist an exact quantization ?
2. Can we construct a uniform and universal exact quantization ?

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A first counter example

Assume $V_{t+1} \equiv 0$ and denote $V := V_t$, $\hat{V} := \hat{V}_t$ and $\boldsymbol{\xi} := \boldsymbol{\xi}_t$ for now.

Let $\boldsymbol{A} = (-\boldsymbol{u})$, $\boldsymbol{B} \equiv (0)$, $\boldsymbol{b} \equiv (-1)$ where $\boldsymbol{u} \sim \mathcal{U}([1, 2])$.

$$\hat{V}(x, \xi) = \begin{array}{cc} \min\limits_{y \in \mathbb{R}} & y \\ \text{s.t.} & uy \geqslant 1 \end{array} = \frac{1}{u}$$

By strict convexity, for all partition $\mathcal{P}$

$$\sum_{P \in \mathcal{P}} \check{p}_P \hat{V}(x, \check{\xi}_P) < V(x) = \mathbb{E}\left[\frac{1}{\boldsymbol{u}}\right]$$

with $\check{p}_P = \mathbb{P}[\boldsymbol{\xi} \in P]$, $\check{\xi}_P = \mathbb{E}[\boldsymbol{\xi} \mid \boldsymbol{\xi} \in P]$.

➥ There is no partition-based local, neither uniform nor universal, exact quantization result for $\boldsymbol{A}$ non-finitely supported.

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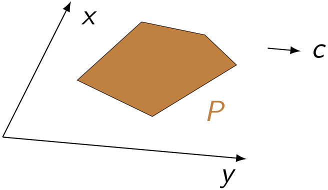
Uniform exact quantization and polyhedrality

$$
\begin{aligned}
\hat{V}(x,\xi) = \min_{y \in \mathbb{R}^m} \ & c^\top y \\
\text{s.t.} \ & Ay + Bx \leqslant h
\end{aligned}
$$



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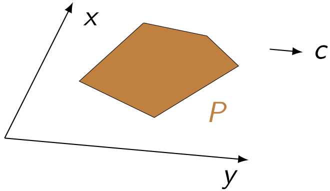

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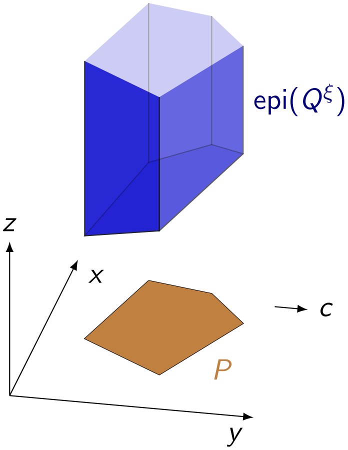
$$\text{s.t.} \quad (x,y) \in P$$
$$= \min_{y \in \mathbb{R}^m} Q^\xi(x,y)$$

with $Q^\xi(x,y) := c^\top y + \mathbb{I}_{(x,y)\in P}$.



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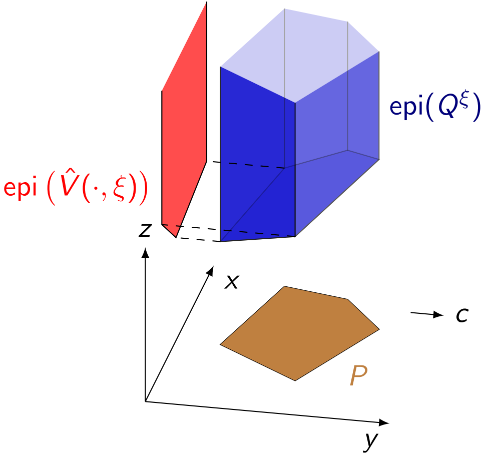
$\hat{V}(\cdot,\xi)$ is polyhedral because epi $\big(\hat{V}(\cdot,\xi)\big)$ is the projection of epi(Q^ξ).



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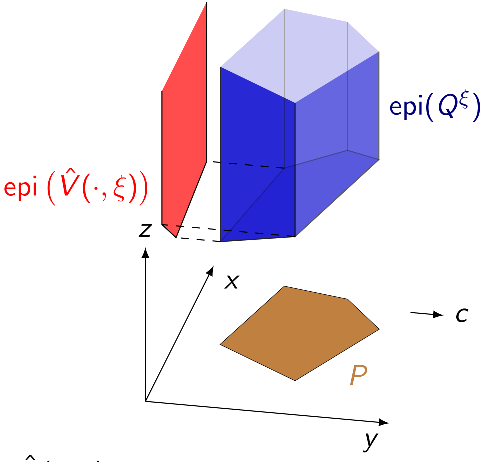

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$$V(x) = \mathbb{E}\left[\hat{V}(x,\boldsymbol{\xi})\right] = \sum_{\xi\in\text{supp}(\check{\boldsymbol{\xi}})} p_\xi \hat{V}(x,\xi)$$

➥ If the noise is finitely supported, then V is polyhedral

Uniform exact quantization and polyhedrality

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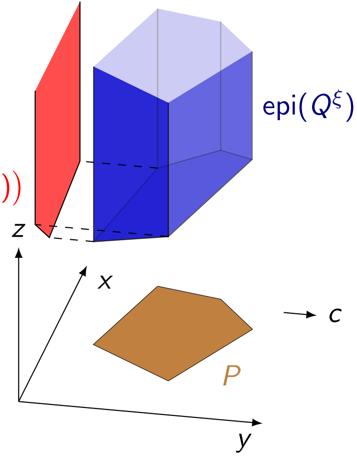

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- ➥ If the noise is finitely supported, then V is polyhedral
- ➥ Existence of uniform exact quantization implies polyhedrality of V.

Counter examples with stochastic constraints

Stochastic $\boldsymbol{B}$

$$V(x) = \mathbb{E}\left[\begin{array}{ll} \min\limits_{y\in\mathbb{R}^m} & y \\ \text{s.t.} & \boldsymbol{u}x - y \leqslant 0 \\ & y \geqslant 1 \end{array}\right]$$

$$= \mathbb{E}\Big[\max(\boldsymbol{u}x, 1)\Big]$$

$$= \begin{cases} 1 & \text{if } x \leqslant 1 \\ \frac{x}{2} + \frac{1}{2x} & \text{if } x \geqslant 1 \end{cases}$$

Stochastic $\boldsymbol{b}$

$$V(x) = \mathbb{E}\left[\begin{array}{ll} \min\limits_{y\in\mathbb{R}^m} & y \\ \text{s.t.} & y \geqslant \boldsymbol{u} \\ & x - y \leqslant 0 \end{array}\right]$$

$$= \mathbb{E}\Big[\max(x, \boldsymbol{u})\Big]$$

$$= \begin{cases} \frac{1}{2} & \text{if } x \leqslant 0 \\ \frac{x^2+1}{2} & \text{if } x \in [0,1] \\ x & \text{if } x \geqslant 1 \end{cases}$$

➥ V is not polyhedral $\Rightarrow$ No uniform exact quantization for non-finitely supported $\boldsymbol{B}$ and $\boldsymbol{b}$.

$\boldsymbol{u}$ is uniform on $[0, 1]$

Counter examples with stochastic constraints

Stochastic $\boldsymbol{B}$

$$
\begin{aligned}
V(x) &= \mathbb{E}\left[\begin{array}{ll} \min\limits_{y\in\mathbb{R}^m} & y \\ \text{s.t.} & \boldsymbol{u}x - y \leqslant 0 \\ & y \geqslant 1 \end{array}\right] \\
&= \mathbb{E}\big[\max(\boldsymbol{u}x, 1)\big] \\
&= \begin{cases} 1 & \text{if } x \leqslant 1 \\ \frac{x}{2} + \frac{1}{2x} & \text{if } x \geqslant 1 \end{cases}
\end{aligned}
$$

Stochastic $\boldsymbol{b}$

$$
\begin{aligned}
V(x) &= \mathbb{E}\left[\begin{array}{ll} \min\limits_{y\in\mathbb{R}^m} & y \\ \text{s.t.} & y \geqslant \boldsymbol{u} \\ & x - y \leqslant 0 \end{array}\right] \\
&= \mathbb{E}\big[\max(x, \boldsymbol{u})\big] \\
&= \begin{cases} \frac{1}{2} & \text{if } x \leqslant 0 \\ \frac{x^2+1}{2} & \text{if } x \in [0,1] \\ x & \text{if } x \geqslant 1 \end{cases}
\end{aligned}
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➥ V is not polyhedral $\Rightarrow$ No uniform exact quantization for non-finitely supported $\boldsymbol{B}$ and $\boldsymbol{b}$.

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Remaining cases

$$V(x) = \mathbb{E}\left[\begin{array}{ll} \min\limits_{y\in\mathbb{R}^m} & \boldsymbol{c}^\top y \\ \text{s.t.} & \boldsymbol{A}y + \boldsymbol{B}x \leqslant \boldsymbol{b} \end{array}\right]$$

	$\boldsymbol{A}$	$(\boldsymbol{B}, \boldsymbol{b})$	$\boldsymbol{c}$
Local	×	?	?
Uniform	×	×	?

Remaining cases

$$V(x) = \mathbb{E}\left[\begin{array}{ll} \min\limits_{y\in\mathbb{R}^m} & \boldsymbol{c}^\top y \\ \text{s.t.} & \boldsymbol{A}y + \boldsymbol{B}x \leqslant \boldsymbol{b} \end{array}\right]$$

	$\boldsymbol{A}$	$(\boldsymbol{B}, \boldsymbol{b})$	$\boldsymbol{c}$
Local	$\times$	$\checkmark$	$\checkmark$
Uniform	$\times$	$\times$	?

Theorem (GAPM, FL 2022)

If A is deterministic,
then there exists a universal and local exact quantization.

Remaining cases

$$V(x) = \mathbb{E}\begin{bmatrix} \min\limits_{y \in \mathbb{R}^m} & \boldsymbol{c}^\top y \\ \text{s.t.} & \boldsymbol{A}y + \boldsymbol{B}x \leqslant \boldsymbol{b} \end{bmatrix}$$

	$\boldsymbol{A}$	$(\boldsymbol{B}, \boldsymbol{b})$	$\boldsymbol{c}$
Local	×	✓	✓
Uniform	×	×	✓

Theorem (GAPM, FL 2022)

If A is deterministic,
then there exists a universal and local exact quantization.

Theorem (Exact quantization, FGL 2021)

If A, B and b are deterministic,
then there exists a universal and uniform exact quantization.

Contents

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Illustrative running example:

$$P_x := \{y \in \mathbb{R}^m \mid \|y\|_1 \leqslant 1, \\ y_1 \leqslant x,\ y_2 \leqslant x\}$$

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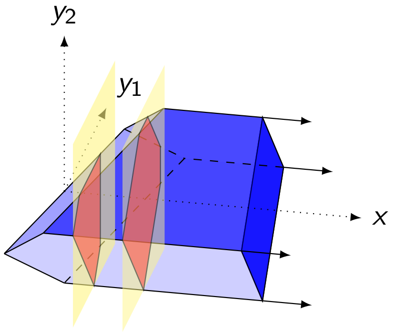

Normal fan $\mathcal{N}(P_x)$

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$$\mathcal{N}(P_x) := \{N_{P_x}(y) \,|\, y \in P_x\}$$

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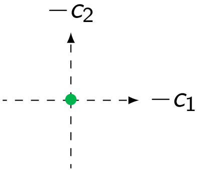

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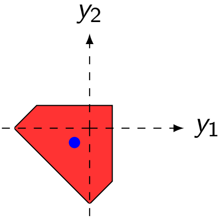

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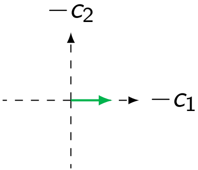

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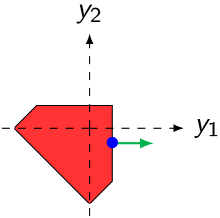

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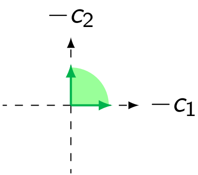

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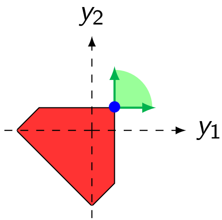

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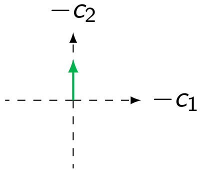

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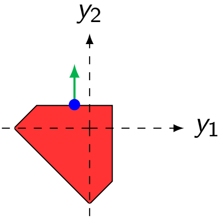

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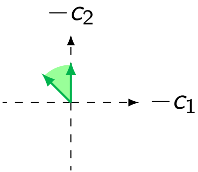

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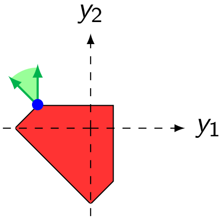

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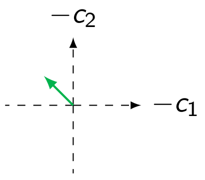

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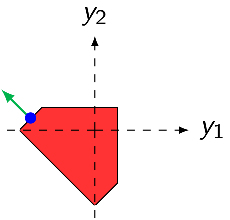

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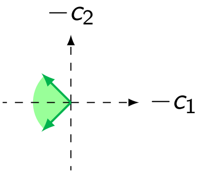

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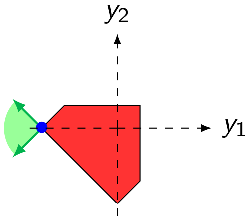

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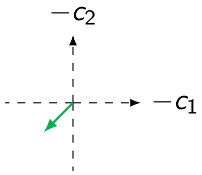

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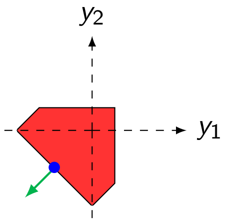

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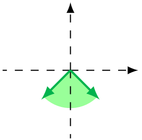

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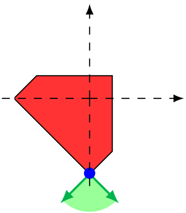

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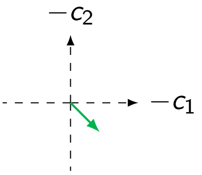

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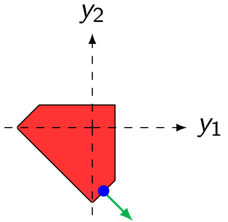

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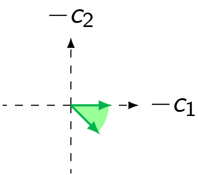

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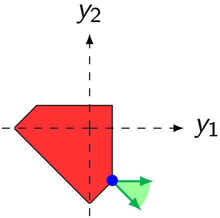

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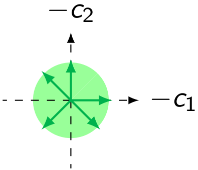
Proposition

If P_x is bounded, $\{\mathrm{ri}(N) \mid N \in \mathcal{N}(P_x)\}$ *is a partition of $\mathbb{R}^m$.*



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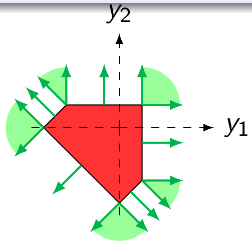

P_x and $\mathcal{N}(P_x)$ for $x = 0.3$

$\mathcal{N}(P_x)$: partition of cost coherent with the min

$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \boldsymbol{c}^\top y\big]$$

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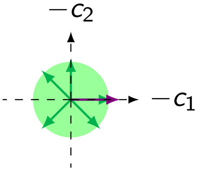
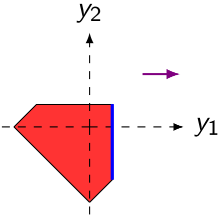


Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$



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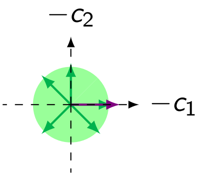

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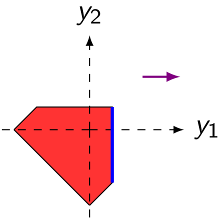

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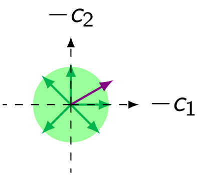

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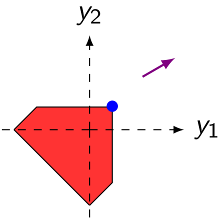

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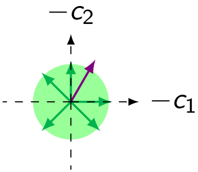

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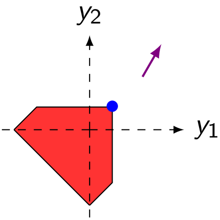

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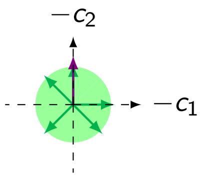

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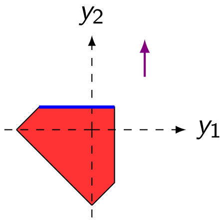

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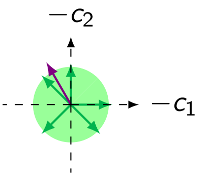

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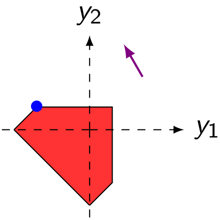

P_x for $x = 0.3$

$\mathcal{N}(P_x)$: partition of cost coherent with the min

$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \boldsymbol{c}^\top y\big]$$

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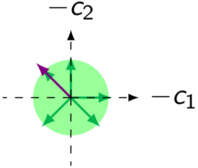

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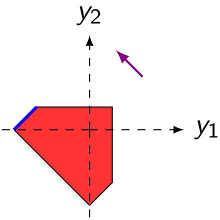

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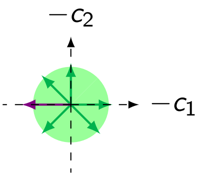

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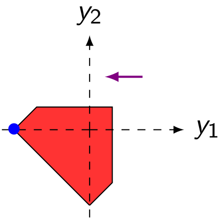

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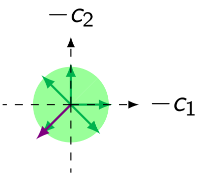

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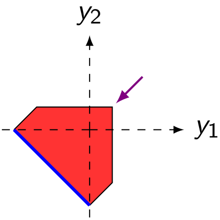

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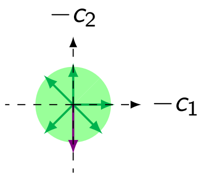

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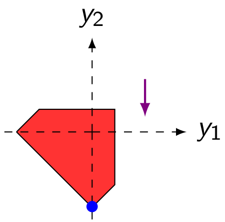

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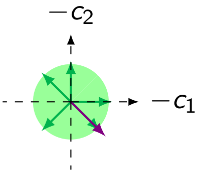

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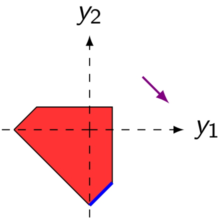

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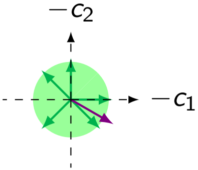

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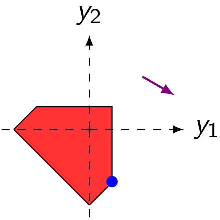

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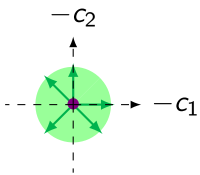

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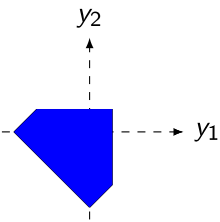

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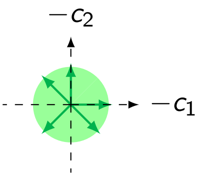

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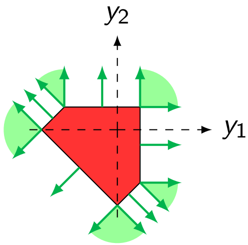
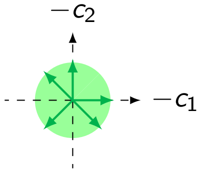


P_x for $x = 0.3$

Local and universal exact quantization for $\boldsymbol{c}$

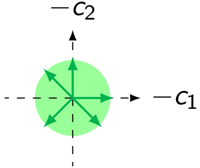
$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \boldsymbol{c}^\top y\Big]$$
$$= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\Big[\mathbb{1}_{\boldsymbol{c} \in -\operatorname{ri} N} \min_{y \in P_x} \boldsymbol{c}^\top y\Big]$$



$\mathcal{N}(P_x)$ for $x = 0.3$

Local and universal exact quantization for $\boldsymbol{c}$

$$
\begin{aligned}
V(x) &= \mathbb{E}\Big[\min_{y \in P_x} \boldsymbol{c}^\top y\Big] \\
&= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\Big[\mathbb{1}_{\boldsymbol{c} \in -\operatorname{ri} N} \min_{y \in P_x} \boldsymbol{c}^\top y\Big] \quad \text{where } y_N(x) \in \arg\min_{y \in P_x} \underbrace{c^\top}_{\in -\operatorname{ri} N} y. \\
&= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\Big[\mathbb{1}_{\boldsymbol{c} \in -\operatorname{ri} N} \boldsymbol{c}^\top\Big] y_N(x)
\end{aligned}
$$



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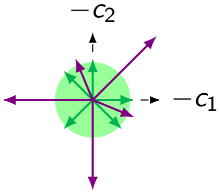
\end{aligned}
$$



$\mathcal{N}(P_x)$ and $p_N \check{c}_N$ for $x = 0.3$

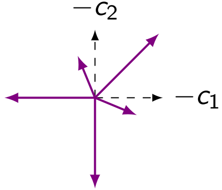
For $N \in \mathcal{N}(P_x)$,

$$
\begin{aligned}
p_N &:= \mathbb{P}\big[\boldsymbol{c} \in -\operatorname{ri} N\big] \\
\check{c}_N &:= \mathbb{E}\big[\boldsymbol{c} \mid \boldsymbol{c} \in -\operatorname{ri} N\big]
\end{aligned}
$$

We replace the continuous cost $\boldsymbol{c}$, by the discrete cost $\check{\boldsymbol{c}}$.

Local and universal exact quantization for $\boldsymbol{c}$

$$
\begin{aligned}
V(x) &= \mathbb{E}\big[\min_{y \in P_x} \boldsymbol{c}^\top y\big] \\
&= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbb{1}_{\boldsymbol{c} \in -\operatorname{ri} N} \min_{y \in P_x} \boldsymbol{c}^\top y\big] \quad \text{where } y_N(x) \in \arg\min_{y \in P_x} \underbrace{c^\top}_{\in -\operatorname{ri} N} y. \\
&= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbb{1}_{\boldsymbol{c} \in -\operatorname{ri} N} \boldsymbol{c}^\top\big] y_N(x) \\
&= \sum_{N \in \mathcal{N}(P_x)} p_N \check{c}_N{}^\top y_N(x) \\
&= \sum_{N \in \mathcal{N}(P_x)} p_N \min_{y \in P_x} \check{c}_N{}^\top y
\end{aligned}
$$



$p_N \check{c}_N$ for $x = 0.3$

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Contents

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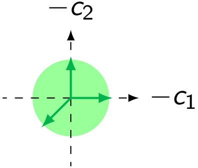
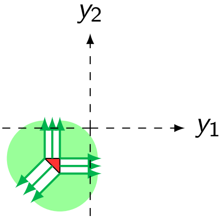


$\mathcal{N}(P_x)$



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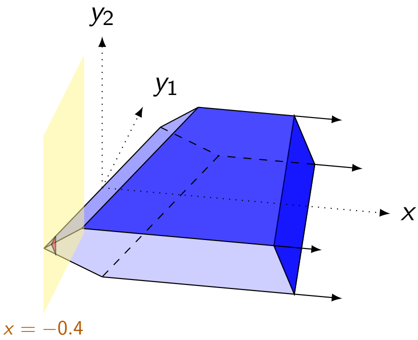

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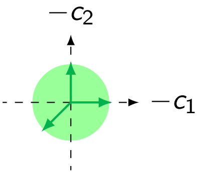

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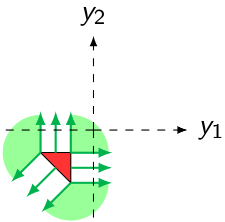

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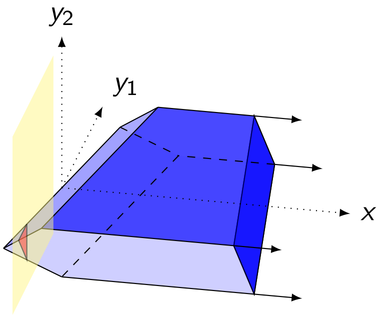

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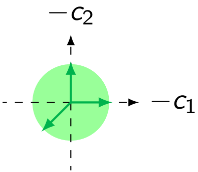

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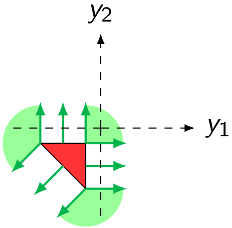

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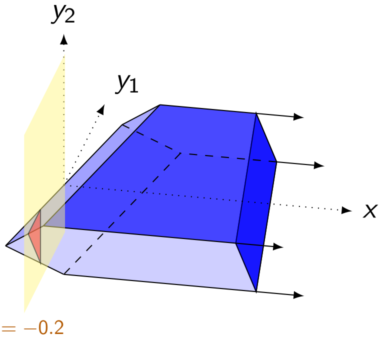

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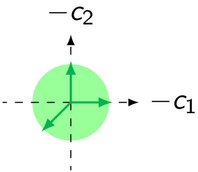

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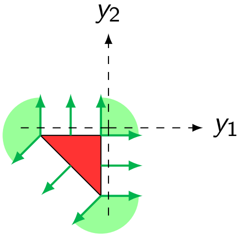

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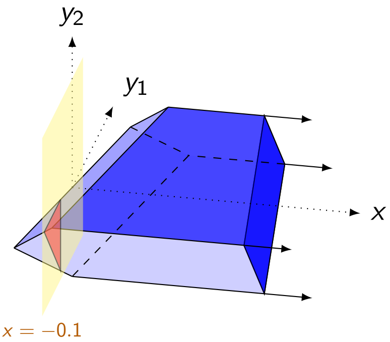

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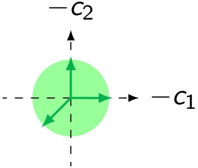

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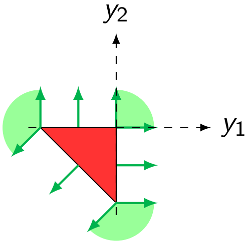

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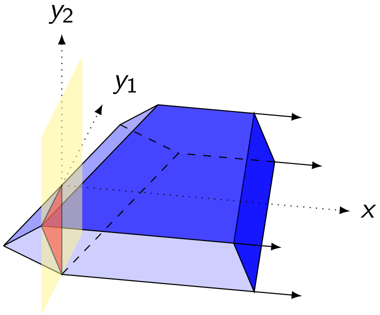

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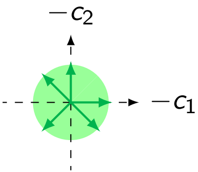

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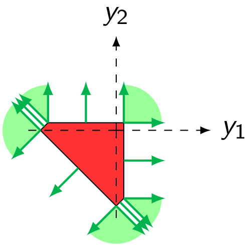

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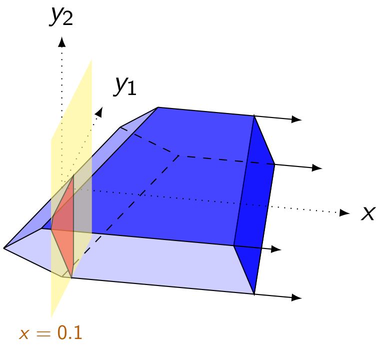

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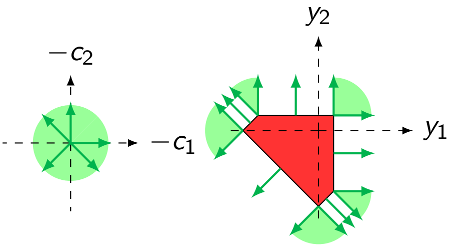

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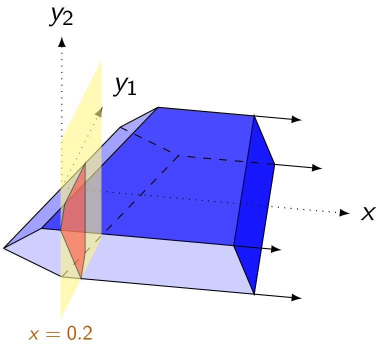

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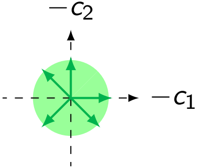

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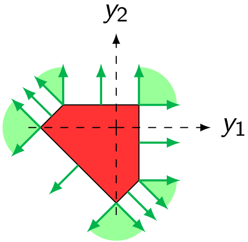

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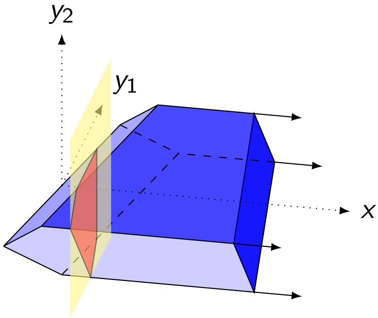

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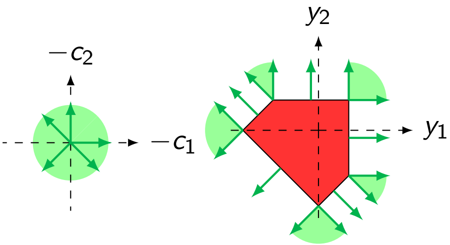
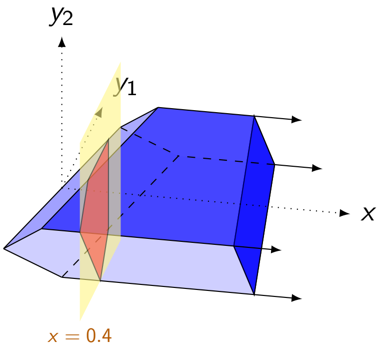


$\mathcal{N}(P_x)$ $\qquad$ P_x and $\mathcal{N}(P_x)$



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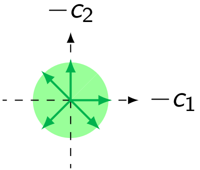

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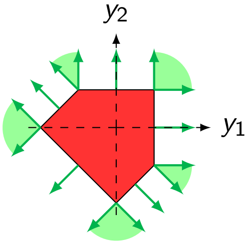

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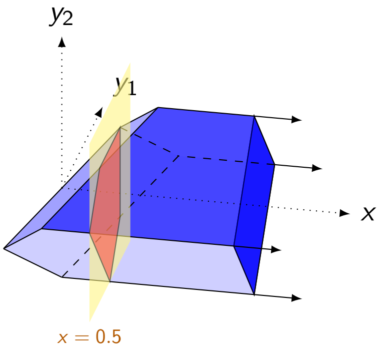

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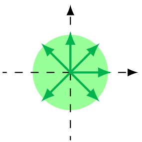

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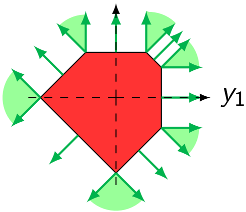

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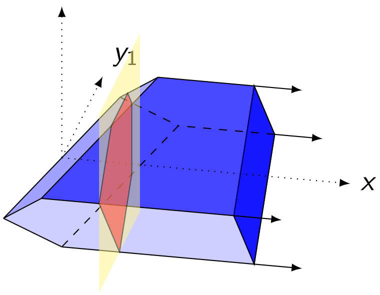

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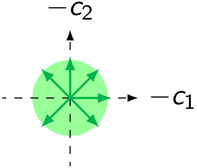

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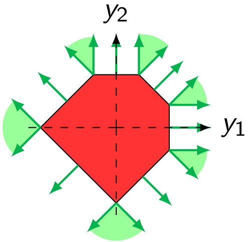

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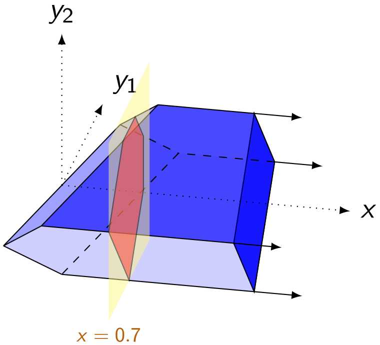

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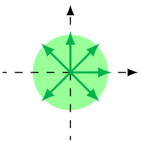

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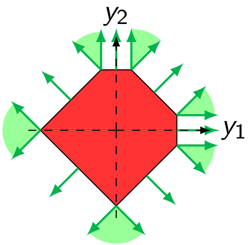

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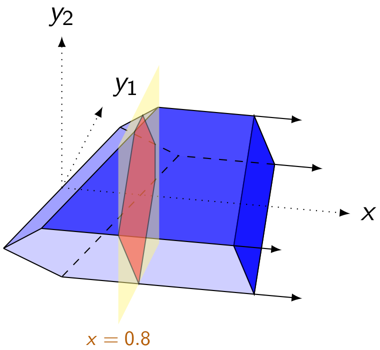

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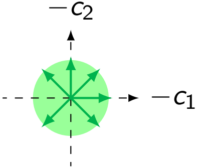

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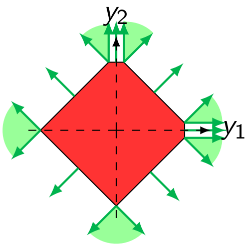

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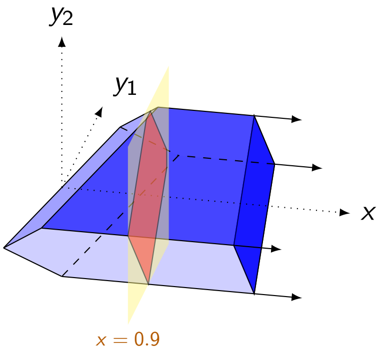

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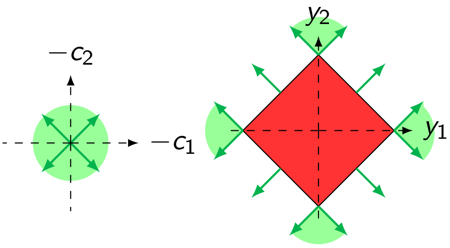

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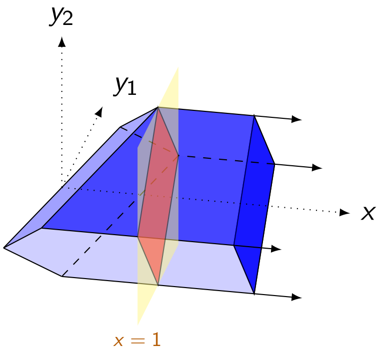

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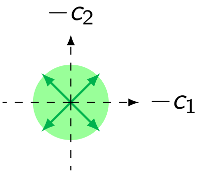

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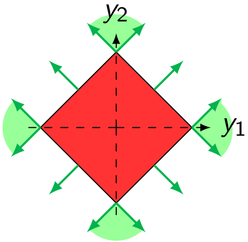

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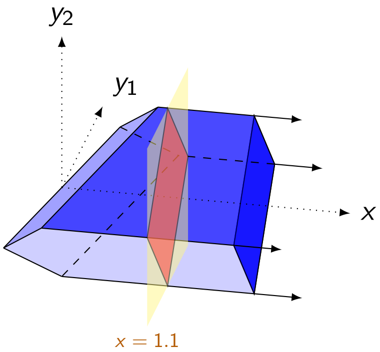

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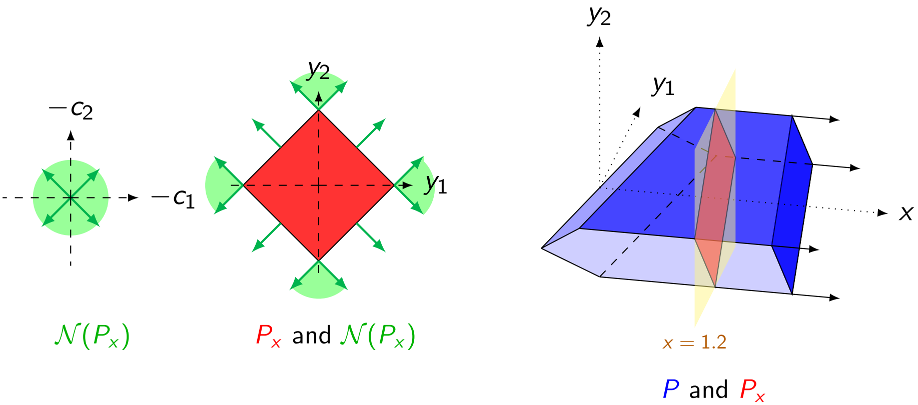

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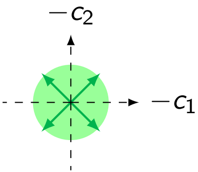

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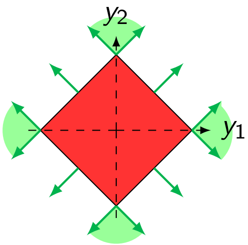

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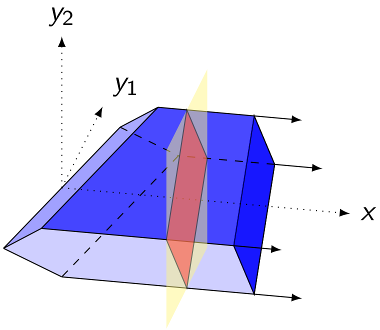

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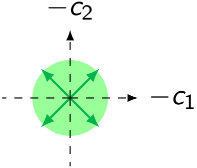

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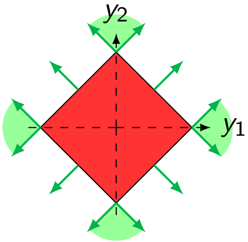

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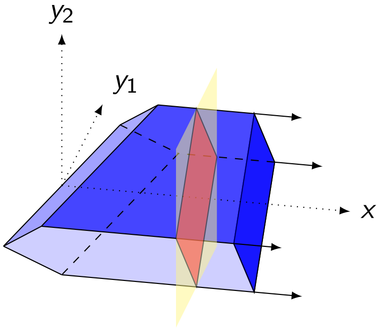

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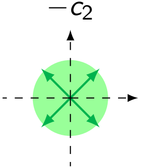

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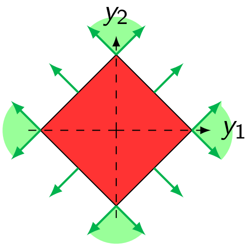

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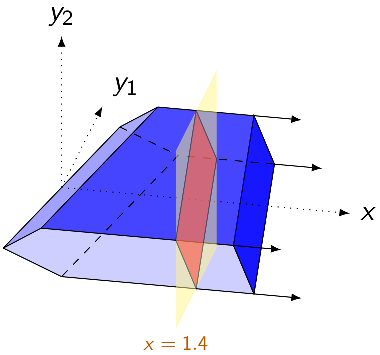

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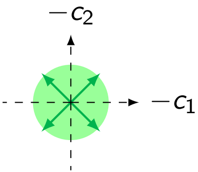

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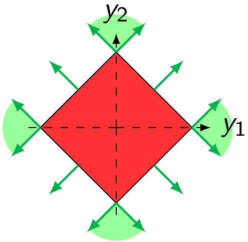

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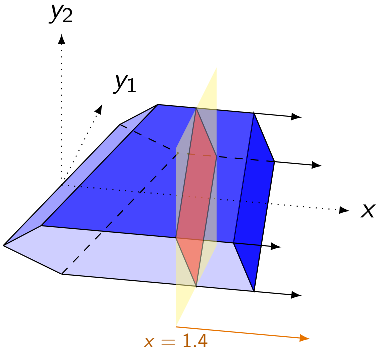

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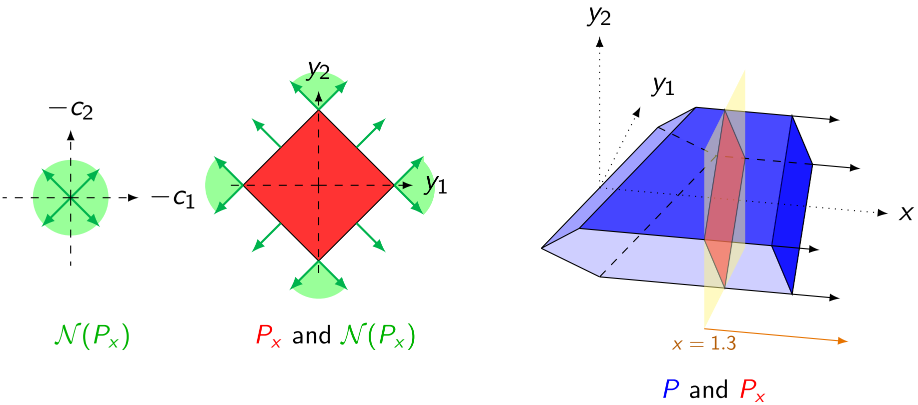

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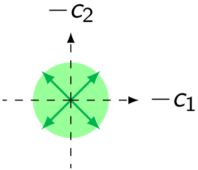

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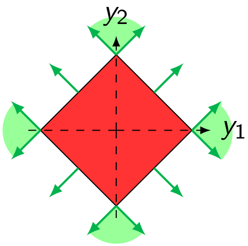

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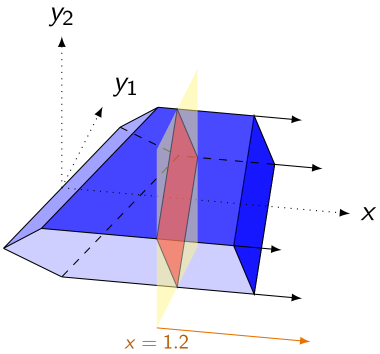

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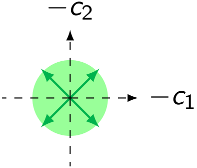

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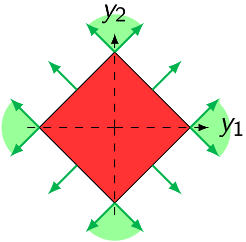

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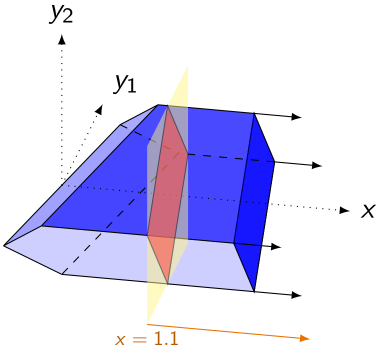

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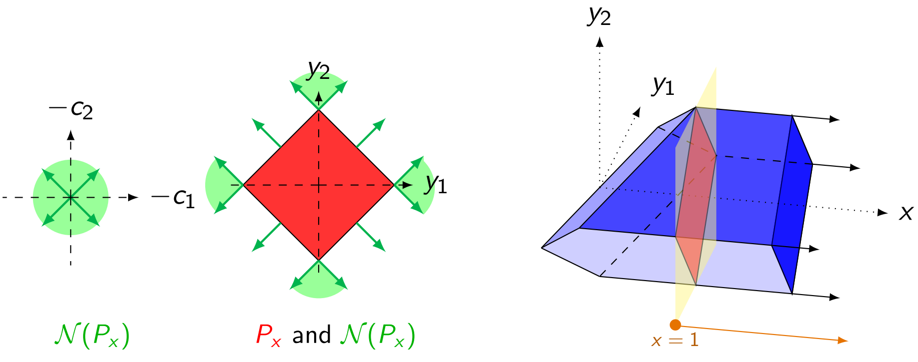

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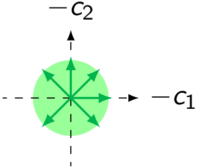

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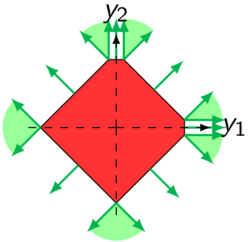

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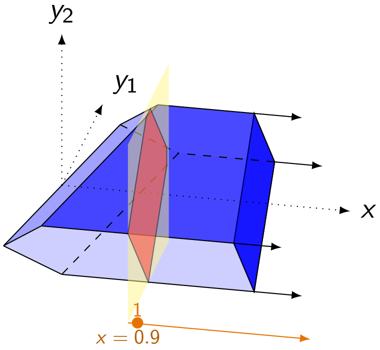

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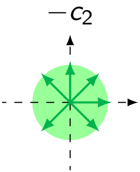

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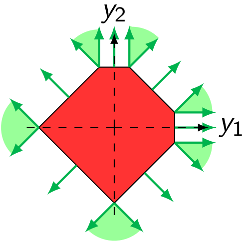

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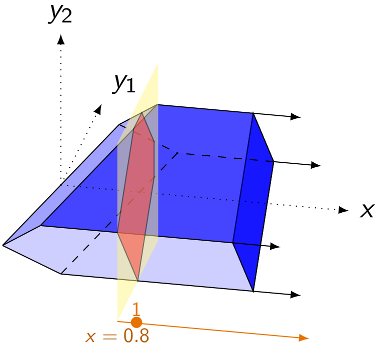

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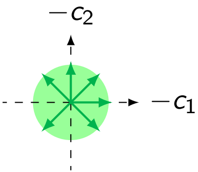

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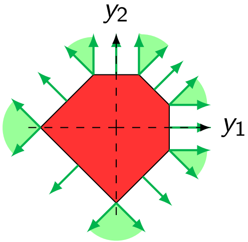

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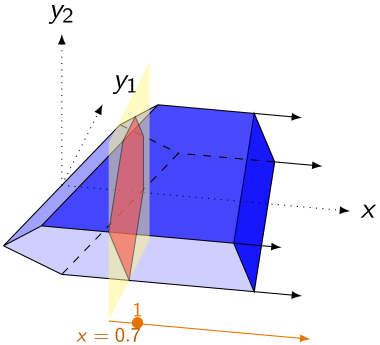

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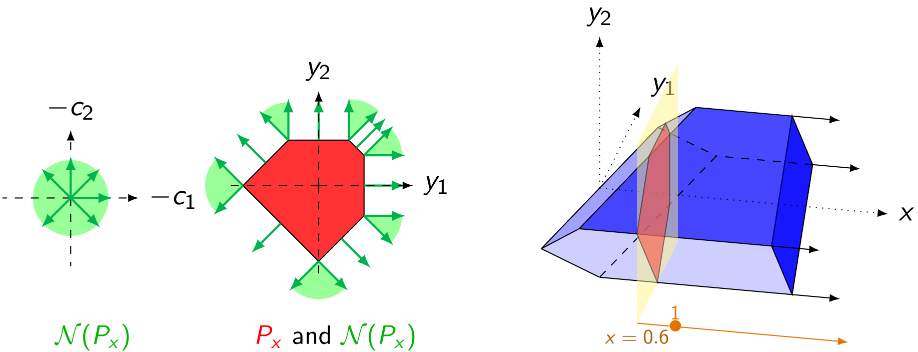

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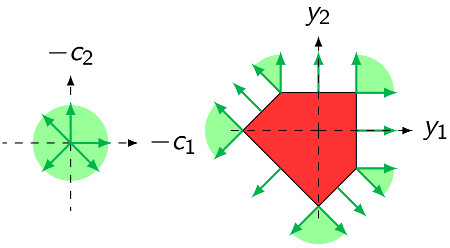

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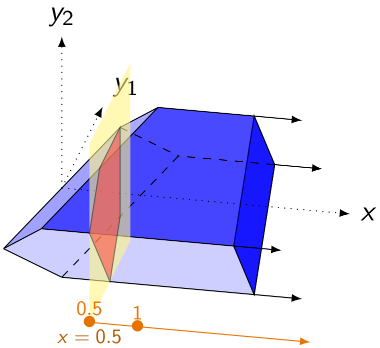

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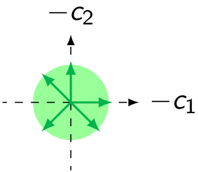

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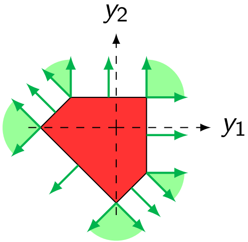

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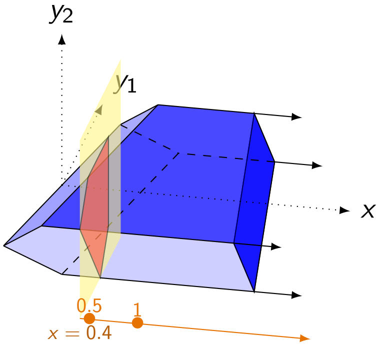

P and P_x

$x \mapsto \mathcal{N}(P_x)$ is piecewise constant.

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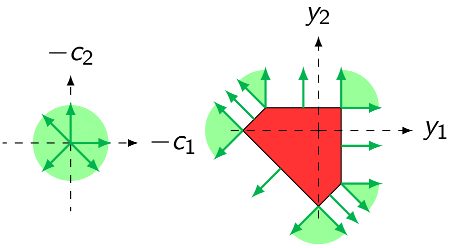
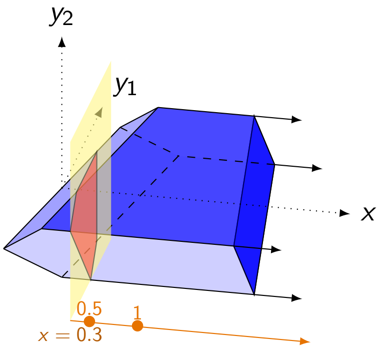


$\mathcal{N}(P_x)$ $\quad$ P_x and $\mathcal{N}(P_x)$



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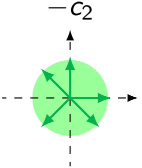

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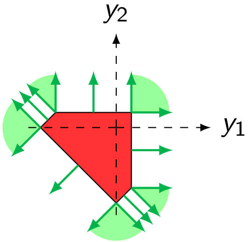

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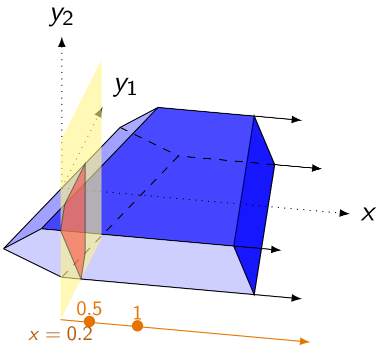

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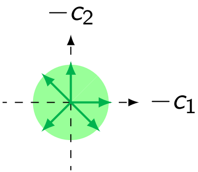

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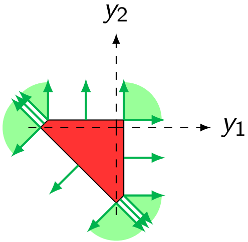

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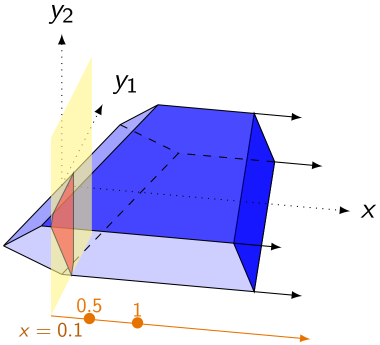

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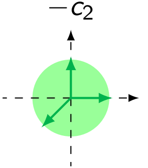

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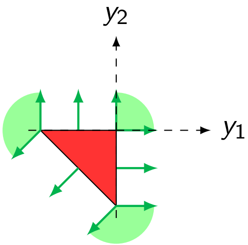

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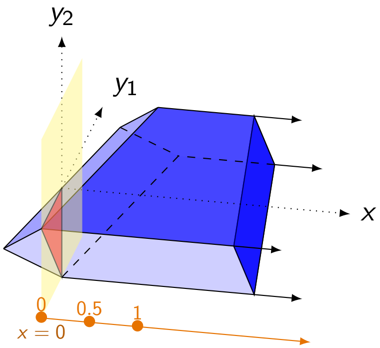

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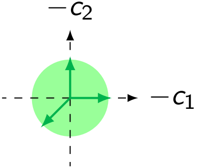

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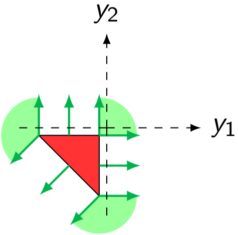

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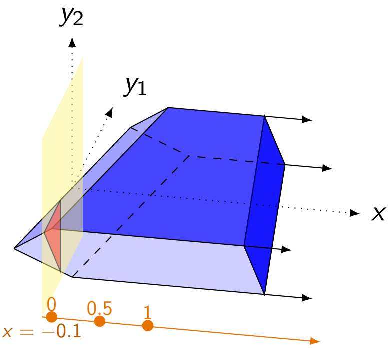

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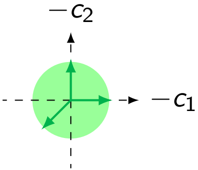

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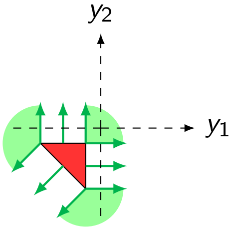

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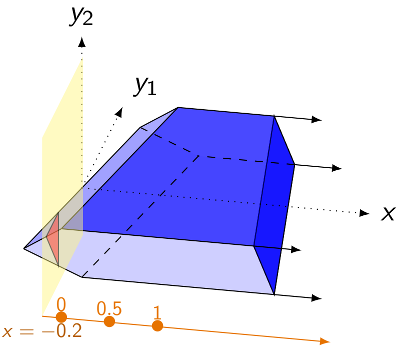

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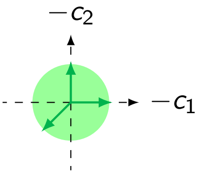

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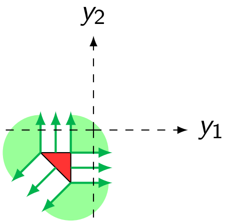

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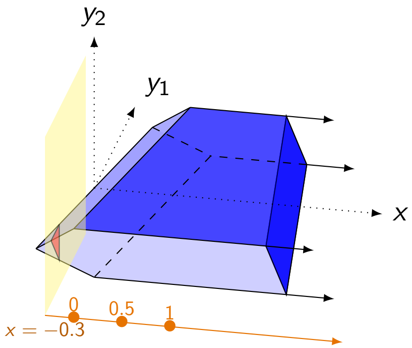

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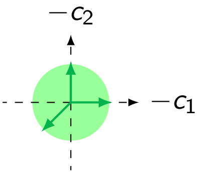

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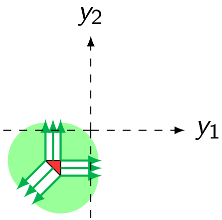

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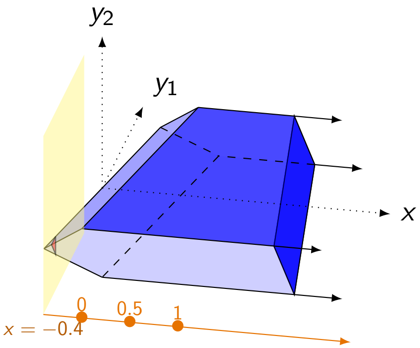


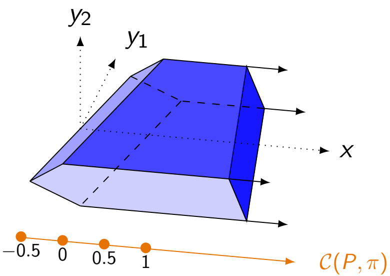
P and P_x

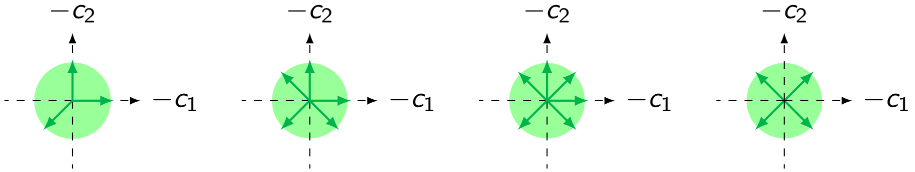
What are the constant regions of $x \mapsto \mathcal{N}(P_x)$?

Proposition

There exists a collection $\mathcal{C}(P,\pi)$ called the **chamber complex** *whose relative interior of cells are the constant regions of $x \mapsto \mathcal{N}(P_x)$.*

I.e, for $\sigma \in \mathcal{C}(P,\pi)$ and $x, x' \in \mathrm{ri}(\sigma)$, we have $\mathcal{N}(P_x) = \mathcal{N}(P_{x'}) =: \mathcal{N}_\sigma$





$\mathcal{N}_\sigma$ for $\sigma = [-0.5, 0]$ $\mathcal{N}_\sigma$ for $\sigma = [0, 0.5]$ $\mathcal{N}_\sigma$ for $\sigma = [0.5, 1]$ $\mathcal{N}_\sigma$ for $\sigma = [1, +\infty)$

Chamber complex

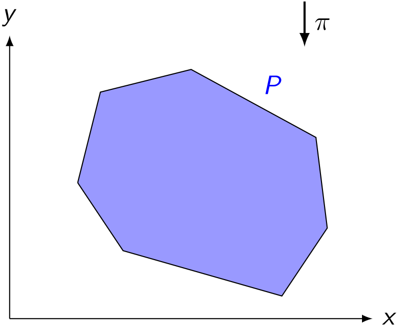
Definition

The *chamber complex* $\mathcal{C}(P, \pi)$ of P along π is

$$\mathcal{C}(P, \pi) := \{\sigma_{P,\pi}(x) \mid x \in \pi(P)\}$$

where

$$\sigma_{P,\pi}(x) := \bigcap_{F \in \mathcal{F}(P) \,|\, x \in \pi(F)} \pi(F)$$



where $\mathcal{F}(P)$ is the set of faces of P and π is the projection $(x, y) \mapsto x$.

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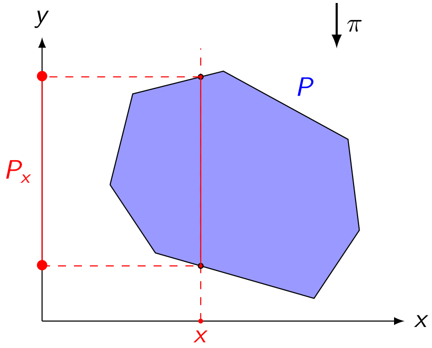

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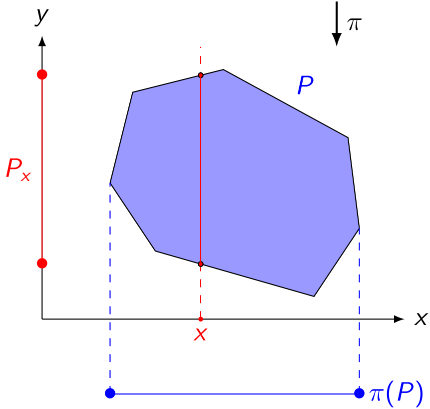

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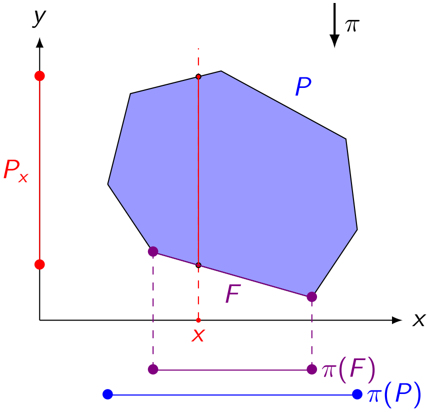

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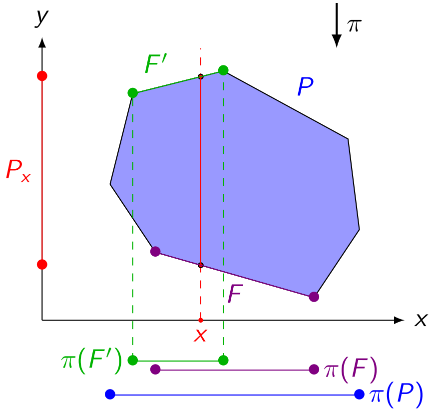

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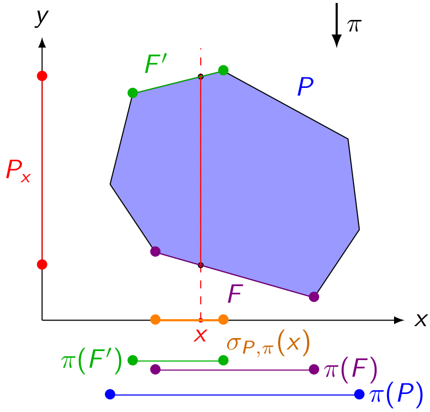

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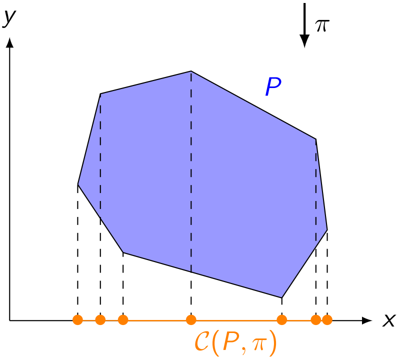

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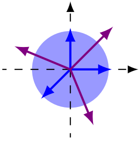
Common Refinement of Normal Fans

We can quantize $\boldsymbol{c}$ on each chamber.



$\mathcal{N}_\sigma$ and $\check{\boldsymbol{c}}$

For all $x \in \mathrm{ri}(\sigma)$,

$$V(x) = \sum_{N \in \mathcal{N}_\sigma} p_N \min_{y \in P_x} \check{c}_N^\top y$$

For all $x' \in \mathrm{ri}(\tau)$,

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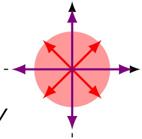

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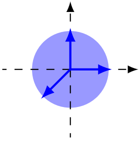

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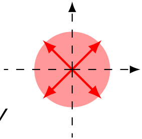

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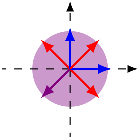
$$\mathcal{R} := \mathcal{N}_\sigma \wedge \mathcal{N}_\tau = \{N \cap N' \mid N \in \mathcal{N}_\sigma, N' \in \mathcal{N}_\tau\}$$



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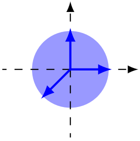

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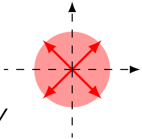

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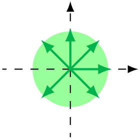

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Uniform exact quantization for c

Let's sum up:

- local exact quantization at x induced by $\mathcal{N}(P_x)$,
- $x \mapsto \mathcal{N}(P_x)$ is constant on each $\sigma \in \mathcal{C}(P, \pi)$,
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Theorem (FGL21, Uniform and universal quantization of the cost)

Let $\mathcal{R} = \bigwedge_{\sigma \in \mathcal{C}(P,\pi)} -\mathcal{N}_\sigma$, *then* **for all** $x \in \mathbb{R}^n$

$$V(x) = \sum_{R \in \mathcal{R}} \check{p}_R \min_{y \in P_x} \check{c}_R^\top y$$

where $\check{p}_R := \mathbb{P}\big[\boldsymbol{c} \in \mathrm{ri}(R)\big]$ *and* $\check{c}_R := \mathbb{E}\big[\boldsymbol{c} \,|\, \boldsymbol{c} \in \mathrm{ri}(R)\big]$

Polyhedral characterization of V

Theorem (FGL21)

For all distributions of $\boldsymbol{c}$, V *is affine on each cell of* $\mathcal{C}(P, \pi)$.

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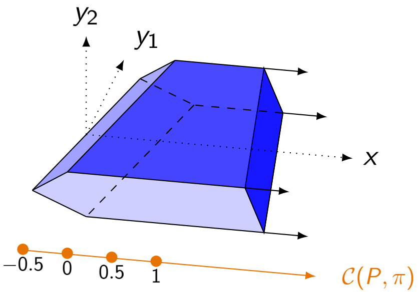
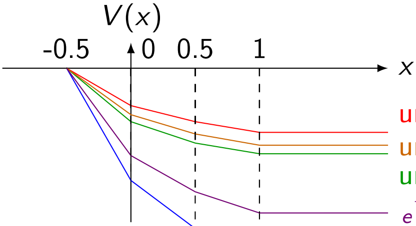
Extension of fiber polytope of

L. Billera, B. Sturmfels, Fiber polytopes, *Annals of Mathematics*, p527–549, 1992.

Explicit computation of the example

$$V(x) = \mathbb{E}\left[\begin{array}{ll} \min\limits_{y\in\mathbb{R}^2} & \boldsymbol{c}^\top y \\ \text{s.t.} & \|y\|_1 \leqslant 1 \\ & y_1 \leqslant x \\ & y_2 \leqslant x \end{array}\right]$$





Different distributions of $\boldsymbol{c}$:

uniform on norm 1 ball

uniform on norm 2 ball

uniform on norm ∞ ball

$\dfrac{e^{-\frac{\|c\|_2^2}{2\gamma^2}}}{2\pi\gamma^2}dc$

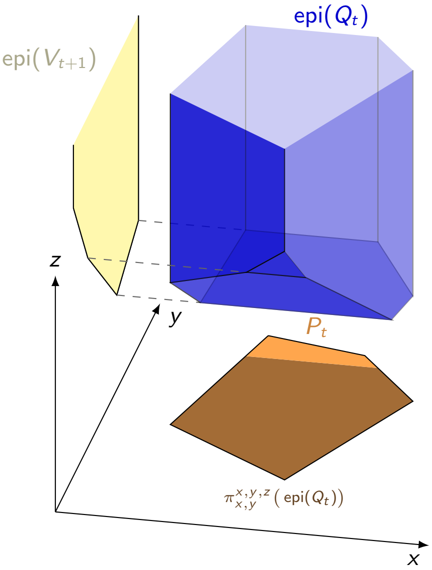
$\dfrac{\theta^2 e^{-\theta\|c\|_1}}{4}dc$

Contents

Multistage uniform and universal exact quantization

$$V_t(x) = \mathbb{E}\left[\begin{array}{ll}\min\limits_{y \in \mathbb{R}^{n_t}} & \boldsymbol{c}_t^\top y + V_{t+1}(y) \\ \\ \text{s.t.} & (x, y) \in P_t\end{array}\right]$$

with $Q_t(x, y) := V_{t+1}(y) + \mathbb{I}_{(x,y) \in P_t}$.



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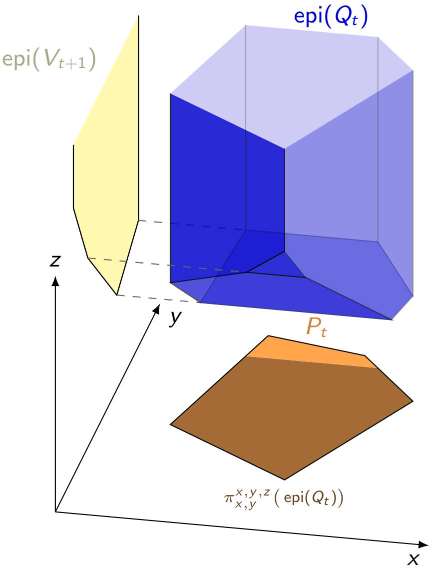

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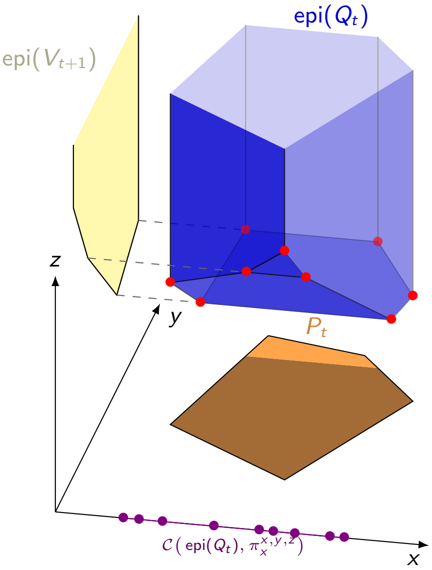

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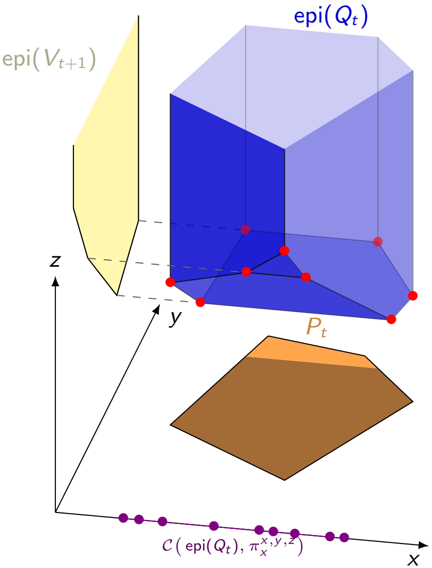

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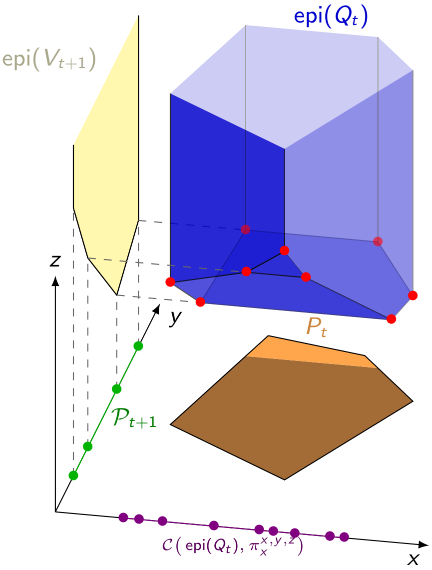

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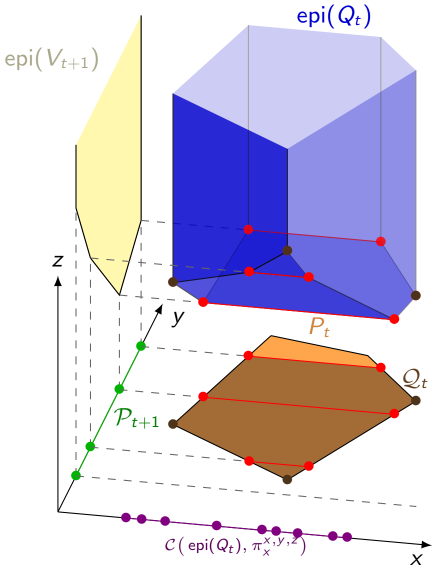

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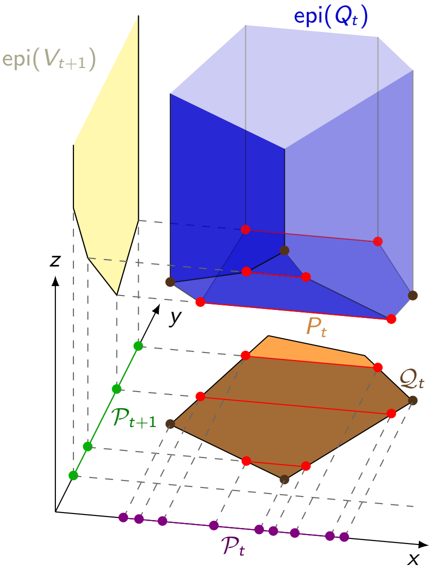

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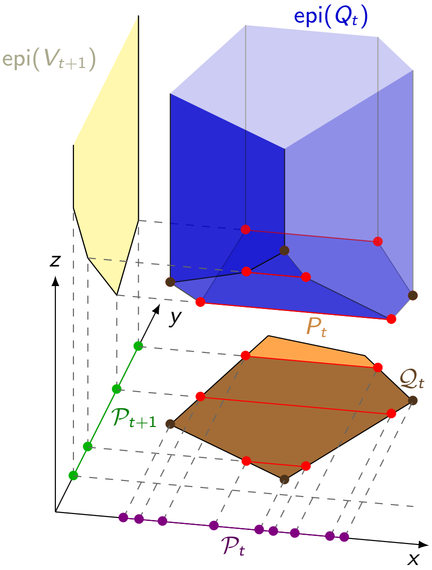
[FGL21, Lem. 4.1]: $\mathcal{P}_t \preccurlyeq \mathcal{C}(\operatorname{epi}(Q_t), \pi_x^{x,y,z})$

➡ V_t affine on $\mathcal{P}_t$, $\mathcal{N}(P_x)$ constant on $\mathcal{P}_t$



Extension to multistage and stochastic constraints

Iterated chamber complexes by backward induction

$$\mathcal{P}_{t,\xi} := \mathcal{C}\Big((\mathbb{R}^{n_t} \times \mathcal{P}_{t+1}) \wedge \mathcal{F}\big(P_t(\xi)\big), \pi_{x_{t-1}}^{x_{t-1},x_t} \Big)$$

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Theorem (FGL21)

All results generalizes to MSLP with finitely supported stochastic constraints.

- $(V_t)_t$ *are affine on universal chamber complexes, i.e. independent of the law of* $(\boldsymbol{c}_t)_t$
- *We have an uniform and universal exact quantization.*

Contents

Earlier and new complexity results

Volume of a polytope

$$\mathrm{Vol}\left(\{z \in \mathbb{R}^d \mid Az \leqslant b\}\right) \text{ or}$$
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- $\sharp P$-complete: Dyer and Frieze (1988)
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$$\min_{x \in \mathbb{R}^n} c^\top x + \mathbb{E}\begin{bmatrix} \min\limits_{y \in \mathbb{R}^m} \boldsymbol{q}^\top y \\ \text{s.t. } \; Tx + Wy \leqslant h \end{bmatrix}$$
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Complexity result multistage

Theorem (FGL21: MSLP is polynomial for fixed dimensions)

Assume that T, $n_2, \cdots, n_T$, are fixed.[1]
Assume that $\boldsymbol{c}$ admits a density function with a bounded total variation.

Then, there exists an algorithm that either asserts that MSLP is unfeasible or finds an ε-solution in polynomial time in $\log(\frac{1}{\varepsilon})$ with probability 1.

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Same with SDDP: [Lan 2020][Zhang and Sun 2020]

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We need to compute the quantized probalit $\check{p}_S = \mathbb{P}\big[\boldsymbol{c} \in S\big]$ and the quantized cost $\check{c}_S = \mathbb{E}\big[\boldsymbol{\xi} \,|\, \boldsymbol{c} \in S\big]$ when S is a polyhedron.

Explicit formulas, valid for S full dimensional simplex or simplicial cone, which can be used through triangulation.

Distribution	Uniform on polytope	Exponential	Gaussian
$d\mathbb{P}(\xi)$	$\frac{\mathbb{1}_{\xi \in Q}}{\mathrm{Vol}_d(Q)} \mathcal{L}_{\mathrm{Aff}(Q)}(d\xi)$	$\frac{e^{\theta^\top \xi} \mathbb{1}_{\xi \in K}}{\Phi_K(\theta)} \mathcal{L}_{\mathrm{Aff}(K)}(d\xi)$	$\frac{e^{-\frac{1}{2}\xi^\top M^{-2} \xi}}{(2\pi)^{\frac{m}{2}} \det M} d\xi$
Support	Polytope : Q	Cone : K	$\mathbb{R}^m$
$\check{p}_S$	$\dfrac{\mathrm{Vol}_d(S)}{\mathrm{Vol}_d(Q)}$	$\dfrac{\lvert \det(\mathrm{Ray}(S)) \rvert}{\Phi_K(\theta)} \prod_{r \in \mathrm{Ray}(S)} \dfrac{1}{-r^\top \theta}$	$\mathrm{Ang}\left(M^{-1} S\right)$
$\check{c}_S$	$\frac{1}{d} \sum_{v \in \mathrm{Vert}(S)} v$	$\left(\sum_{r \in \mathrm{Ray}(S)} \frac{-r_i}{r^\top \theta}\right)_{i \in [m]}$	$\frac{\sqrt{2}\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})} M \,\mathrm{Ctr}\left(S \cap \mathbb{S}_{m-1}\right)$

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2 stage stochastic linear programming (2SLP)

$$\begin{aligned} \min_{x \in \mathbb{R}^n_+} \quad & c^\top x + \mathbb{E}\big[Q(x, \boldsymbol{\xi})\big] \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

where $\boldsymbol{\xi} = (\boldsymbol{T}, \boldsymbol{h})$ is random whereas q and W are deterministic[1]

$$Q(x, \xi) := \min_{y \in \mathbb{R}^m_+} q^\top y \quad = \max_{\lambda \in \mathbb{R}^n} (h - Tx)^\top \lambda$$

$$\text{s.t.} \quad Tx + Wy = h \qquad \text{s.t.} \quad W^\top \lambda \leqslant q$$

We define

$$X := \{x \in \mathbb{R}^n_+ \mid Ax = b\} \qquad D := \{\lambda \in \mathbb{R}^\ell \mid W^\top \lambda \leqslant q\}$$

[1] Can be extended to generic random $\boldsymbol{q}$, and finitely supported $\boldsymbol{W}$

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$\rightsquigarrow$ need to discretize $\boldsymbol{\xi}$

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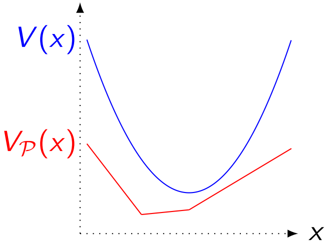
Properties of partitioned cost-to-go

Recall that

$$V(x) = \mathbb{E}\Big[Q(x,\xi)\Big]$$
$$V_{\mathcal{P}}(x) = \sum_{P\in\mathcal{P}} \mathbb{P}\big[P\big]\, Q\big(x, \mathbb{E}\big[\xi|P\big]\big)$$

- $Q(x,\cdot)$ is convex $\rightsquigarrow V_{\mathcal{P}} \leqslant V$.
- $Q\big(\cdot,\mathbb{E}\big[\xi|P\big]\big)$ is polyhedral $\rightsquigarrow V_{\mathcal{P}}$ is polyhedral.



Finally,

$$\min_{x\in X} c^\top x + V_{\mathcal{P}}(x) \qquad (2SLP_{\mathcal{P}})$$

is equivalent to

$$\min_{x\in X,(y_P)_{P\in\mathcal{P}}} \quad c^\top x + \sum_{P\in\mathcal{P}} \mathbb{P}\big[P\big] q^\top y_P$$
$$\mathbb{E}\big[\boldsymbol{T}|P\big]x + Wy_P \leqslant \mathbb{E}\big[\boldsymbol{h}|P\big] \qquad \forall P\in\mathcal{P}$$

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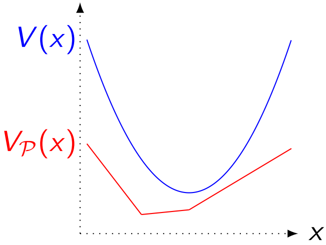

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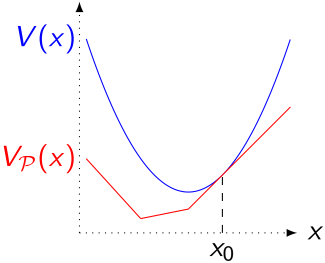
Adapted partition

Definition

We say that a partition $\mathcal{P}$ is adapted to x_0 if

$$V_{\mathcal{P}}(x_0) = V(x_0) := \mathbb{E}\big[Q(x_0, \xi)\big]$$



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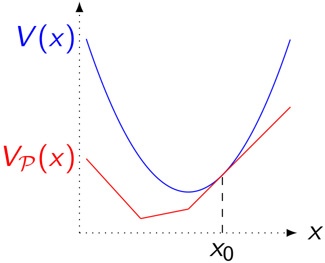

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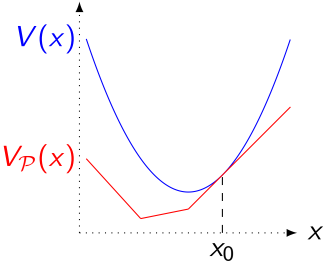

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An adapted partition oracle is a function taking a first stage decision x^k as argument and returning an adapted to x^k partition of Ξ.

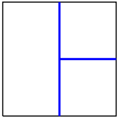
Refinement

$\mathcal{R}$ refines $\mathcal{P}$ ($\mathcal{R} \preccurlyeq \mathcal{P}$) if

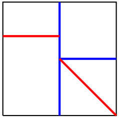
$$\forall R \in \mathcal{R}, \exists P \in P, R \subset P$$

[$\mathcal{R} \preccurlyeq_{\mathbb{P}} \mathcal{P}$ if $\mathcal{R}$ refines $\mathcal{P}$ up to $\mathbb{P}$-null sets.]

Then, $\quad \mathcal{R} \preccurlyeq_{\mathbb{P}} \mathcal{P} \Rightarrow V_{\mathcal{R}} \geqslant V_{\mathcal{P}}$



$\mathcal{P}$



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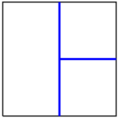

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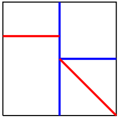


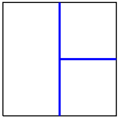
$\mathcal{R}$

The common refinement of $\mathcal{P}$ and $\mathcal{P}'$ is

$$\mathcal{P} \wedge \mathcal{P}' := \{P \cap P' \mid P \in \mathcal{P}, P' \in \mathcal{P}'\}$$

Since $\mathcal{P} \wedge \mathcal{P}'$ refines $\mathcal{P}$ and $\mathcal{P}'$

$$\max(V_{\mathcal{P}}, V_{\mathcal{P}'}) \leqslant V_{\mathcal{P} \wedge \mathcal{P}'}$$



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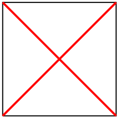
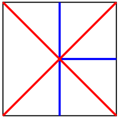


$\mathcal{P}'$



$\mathcal{P} \wedge \mathcal{P}'$

General framework for APM

$k \leftarrow 0$, $z_U^0 \leftarrow +\infty$, $z_L^0 \leftarrow -\infty$, $\mathcal{P}^0 \leftarrow \{\Xi\}$;

while $z_U^k - z_L^k > \varepsilon$ **do**

  $k \leftarrow k+1$;

  Solve (for x^k) $\quad z_L^k \leftarrow \min_{x \in X} c^\top x + V_{\mathcal{P}^{k-1}}(x)$;

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Algorithm 1: Generic framework for APM.

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Algorithm 1: Generic framework for APM.

Theorem (FL2021)

If the oracle is adapted, then x^k is an ε-solution of problem (2SLP) for
$$k \geqslant \left(\frac{L diam(X)}{\varepsilon} + 1\right)^n.$$

Previous APM methods

Lemma (Song & Luedtke)

Let $\mathcal{P}$ a partition of Ξ. $\mathcal{P}$ is adapted at x iff for all set of scenarios $P \in \mathcal{P}$, there exists a common optimal multiplier λ_P, i.e.

$$\forall P \in \mathcal{P}, \quad \exists \lambda_P \in D, \quad \forall \xi_k \in P, \qquad \lambda_P \in \operatorname*{argmax}_{\lambda \in D} (h^k - T^k x)^\top \lambda$$

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Idea

- Sample a large number of scenario
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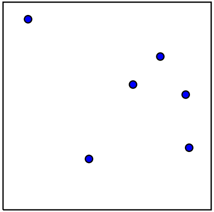

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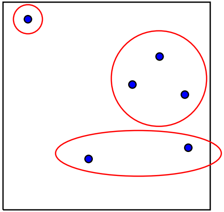

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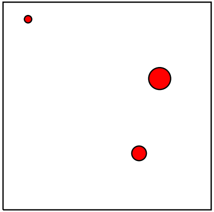

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Lemma (Ramirez-Pico & Moreno)

Let $\mathcal{P}$ a partition of Ξ. If there exists $\lambda(\boldsymbol{\xi})$ such that, for all $P \in \mathcal{P}$,

$$\mathbb{E}\big[\boldsymbol{h}|P\big]^\top \mathbb{E}\big[\lambda(\boldsymbol{\xi})|P\big] = \mathbb{E}\big[\boldsymbol{h}^\top \lambda(\boldsymbol{\xi})|P\big]$$

$$x^\top \mathbb{E}\big[\boldsymbol{T}|P\big]^\top \mathbb{E}\big[\lambda(\boldsymbol{\xi})|P\big] = x^\top \mathbb{E}\big[\boldsymbol{T}^\top \lambda(\boldsymbol{\xi})|P\big]$$

then $\mathcal{P}$ is an adapted partition.

A (partial) comparison between partition based results

Paper	Song, Luedtke (2015)	Ramirez-Pico, Moreno (2020)	F., Leclère (2021)
Non-finite supp(ξ)	×	✓	✓
Explicit oracle	✓	×	✓
Proof of convergence	✓	×	✓
Complexity result	×	×	✓
Fast iteration	✓	×	×

Local exact quantization and adapted partition

Local exact quantization

random cost

Recall that for a fixed x,

$$\mathbb{E}\Big[\min_{y \in P_x} \boldsymbol{c}^\top y\Big] = \sum_{N \in \mathcal{N}(P_x)} p_N \min_{y \in P_x} \check{c}_N{}^\top y$$

where,

$$p_N := \mathbb{P}\big[\boldsymbol{c} \in -\operatorname{ri} N\big]$$
$$\check{c}_N := \mathbb{E}\big[\boldsymbol{c} \mid \boldsymbol{c} \in -\operatorname{ri} N\big]$$

$$P_x := \{y \in \mathbb{R}^m \mid Ay + Bx \leqslant b\}$$

GAPM

random constraints

Similarly, for a given q, and all x,

$$\begin{aligned} V(x) &:= \mathbb{E}\big[Q(x, \boldsymbol{\xi})\big] \\ &= \mathbb{E}\Big[\max_{\lambda \in D_q} (\boldsymbol{h} - \boldsymbol{T}x)^\top \lambda\Big] \\ &= \sum_{N \in \mathcal{N}(D_q)} p_N \max_{\lambda \in D_q} \psi_{N,x}{}^\top \lambda \end{aligned}$$

where,

$$p_N := \mathbb{P}\big[\boldsymbol{h} - \boldsymbol{T}x \in \operatorname{ri} N\big]$$
$$\psi_{N,x} := \mathbb{E}\big[\boldsymbol{h} - \boldsymbol{T}x \mid \boldsymbol{h} - \boldsymbol{T}x \in \operatorname{ri} N\big]$$

$$D_q := \{\lambda \in \mathbb{R}^l \mid W^\top \lambda \leqslant q\}$$

An explicit adapted partition

Consider $x \in \mathbb{R}^n$ and $N \in \mathcal{N}(D_q)$ a normal cone of D_q. We define

$$E_{N,x} := \{\xi \in \Xi \mid h - Tx \in \operatorname{ri} N\}$$

Theorem (FL 2021)

$\mathcal{R}_x := \left\{E_{N,x} \mid N \in \mathcal{N}(D_q)\right\}$ *is an adapted partition to* x
i.e. $V_{\mathcal{R}_x}(x) = V(x)$

Proof:

$$\begin{aligned} V(x) &:= \mathbb{E}\left[Q(x,\xi)\right] \\ &= \sum_{N \in \mathcal{N}(D)} \mathbb{P}\left[\boldsymbol{h} - \boldsymbol{T}x \in \operatorname{ri} N\right] \min_{\lambda \in D} \mathbb{E}\left[\boldsymbol{h} - \boldsymbol{T}x \,|\, \boldsymbol{h} - \boldsymbol{T}x \in \operatorname{ri} N\right]^\top \lambda \\ &= \sum_{N \in \mathcal{N}(D)} \mathbb{P}\left[\boldsymbol{\xi} \in E_{N,x}\right] Q\Big(\mathbb{E}\left[\boldsymbol{\xi} \,|\, \boldsymbol{\xi} \in E_{N,x}\right], x\Big) = V_{\mathcal{R}_x}(x) \end{aligned}$$

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➥ Is it the coarsest one ?

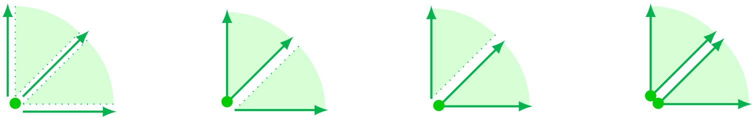
Conditions for a partition to be adapted

Theorem (FL 2021)

For $x \in \mathbb{R}^n$ and $\mathcal{P}$ a partition of Ξ, there exists $\overline{\mathcal{R}}_x \succcurlyeq_{\mathbb{P}} \mathcal{R}_x$ such that

$$\mathcal{P} \preccurlyeq_{\mathbb{P}} \overline{\mathcal{R}}_x \iff V_{\mathcal{P}}(x) = V(x).$$

- If ξ admits a density, $\mathcal{R}_x =_{\mathbb{P}} \overline{\mathcal{R}}_x$.
- An oracle is adapted if and only if it returns a partition $\mathcal{P}$ refining $\overline{\mathcal{R}}_x$.



$\mathcal{R}_x$ $\qquad$ $\mathcal{P}$ $\qquad$ $\mathcal{P}'$ $\qquad$ $\overline{\mathcal{R}}_x$

$$E_{N,x} := \{\xi \in \Xi \mid h - Tx \in \mathrm{ri}(N)\}$$
$$\mathcal{R}_x := \left\{E_{N,x} \mid N \in \mathcal{N}(D_q)\right\}$$

$$\overline{E}_{N,x} := \{\xi \in \Xi \mid h - Tx \in N\}$$
$$\overline{\mathcal{R}}_x := \left\{E_{N,x} \mid N \in \mathcal{N}(D_q)^{\max}\right\}.$$

Subgradient of partition function

Recall that if $\mathcal{P} \preccurlyeq_{\mathbb{P}} \mathcal{R}_x$ then

$$V_{\mathcal{R}_x}(x) = V_{\mathcal{P}}(x) = V(x)$$

$$V_{\mathcal{R}_x}(\cdot) \leqslant V_{\mathcal{P}}(\cdot) \leqslant V(\cdot)$$

Lemma

Let $x \in \mathrm{dom}(V)$ *and* $\mathcal{P}$ *be a refinement of* $\mathcal{R}_x$*, i.e.* $\mathcal{P} \preccurlyeq \mathcal{R}_x$*, then*

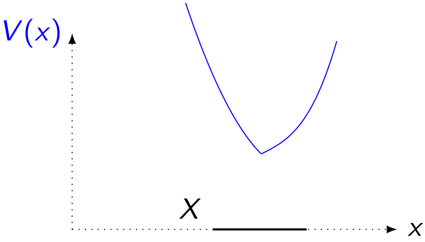
$$\partial V_{\mathcal{R}_x}(x) \subset \partial V_{\mathcal{P}}(x) \subset \partial V(x)$$

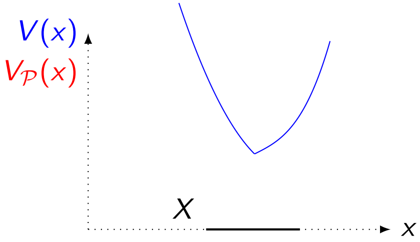
Furthermore, if $x \in \mathrm{ri}\,\mathrm{dom}(V)$,

$$\partial V_{\mathcal{R}_x}(x) = \partial V_{\mathcal{P}}(x) = \partial V(x)$$

Link with Benders decomposition and L-shaped

Partition based method can be seen as a tangent cone method: a cutting plane method where we add all active cuts instead of a single one.





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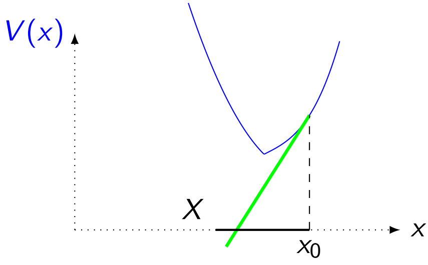

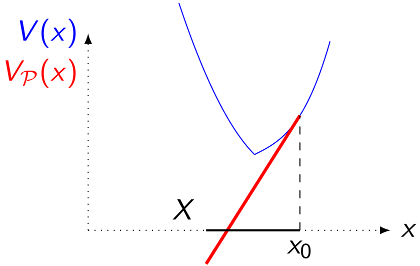

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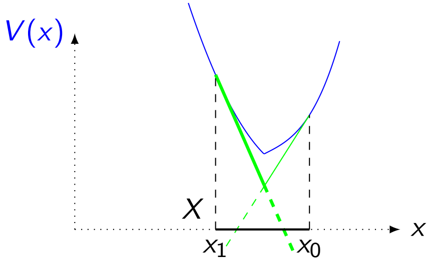

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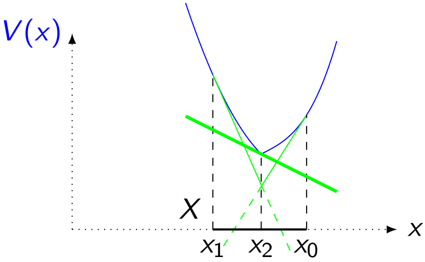

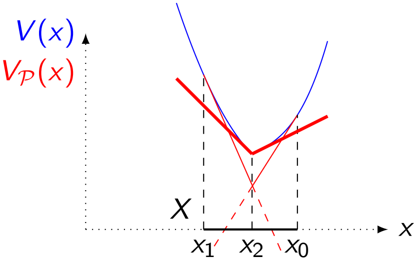

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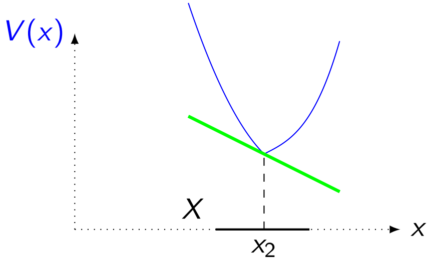

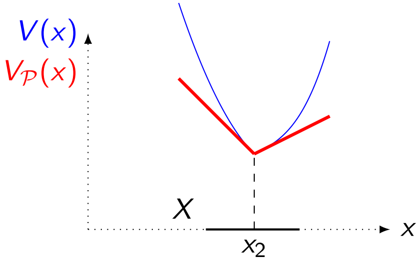

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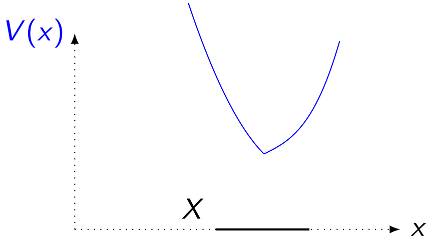
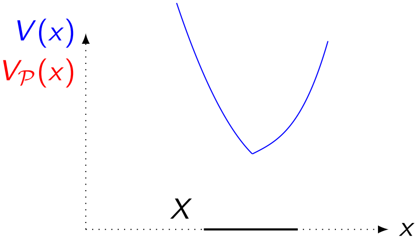
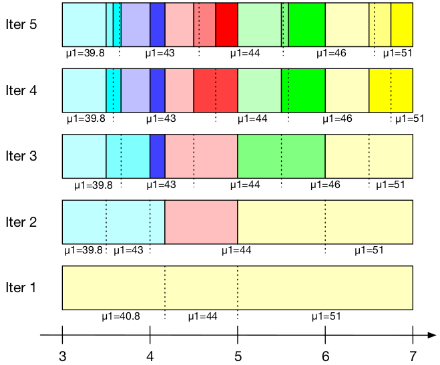




Theorem (Convergence and complexity results)

If $X \cap \mathrm{dom}(V) \subset \mathbb{R}^+$ is contained in a ball of diameter $M \in \mathbb{R}^+$ and $x \to c^\top x + V(x)$ is Lipschitz with constant L then the partition based method finds an ε-solution in at most $\left(\frac{LM}{\varepsilon} + 1\right)^n$ iterations.

Numerical Results - LandS



Iter	x_1	x_2	x_3	x_4
1	0.833	3.000	4.167	4.000
2	2.500	3.000	3.500	3.000
3	1.833	4.000	3.667	2.500
4	2.000	4.167	3.583	2.250
5	1.917	4.083	3.625	2.375
6	1.875	4.042	3.646	2.438

Iter	LB	UB	Gap
1	378.667	382.711	1.0567%
2	380.122	381.100	0.2567%
3	380.601	380.844	0.0640%
4	380.842	380.893	0.0007%
5	380.843	380.856	0.0004%
6	380.844	380.847	0.0002%

Results given by GAPM for LandS problem[2]

[2] illustration from Ramirez-Pico and Moreno

Numerical Results - ProdMix

k	x_k	z_L^k	z_U^k	Gap	$\lvert\mathcal{P}_k^{\max}\rvert$
1	$(1333.33, 66.67)$	-18666.67	-16939.71	9.3%	4
2	$(1441.41, 59.57)$	-17873.01	-17383.73	2.7%	9
3	$(1399.05, 57.91)$	-17789.88	-17659.19	0.74%	16
4	$(1379.98, 56.64)$	-17744.67	-17708.00	0.20%	25
5	$(1371.36, 55.71)$	-17718.96	-17709.05	0.056%	36
6	$(1375.55, 56.21)$	-17713.74	-17711.37	0.013%	49

Table: Results for problem Prod-Mix

To compare our approach with SAA, we solved the same problem 100 times, each with 10 000 scenarios randomly drawn, yielding a 95% confidence interval centered in -17711, with radius 2.2.

Conclusion

	$\boldsymbol{A}$	$(\boldsymbol{B}, \boldsymbol{b})$	$\boldsymbol{c}$
Local	×	✓	✓
Uniform	×	×	✓

- Links with fundamental polyhedral geometry, regular subdivisions and fiber polytope (Chap. 3 and 4).

- *Uniform and universal* exact quantization for $\boldsymbol{c}$ in MSLP (Chap.4).
  ➥ New complexity results.

- *Local* exact quantization for $\boldsymbol{B}$ and $\boldsymbol{b}$.
  ➥ Adaptive Partition-based Methods (APM) for general distribution: solves small 2SLP with high precision (Chap. 5).

- Extension of Stochastic Dual Dynamic Programming algorithms and more generally all Trajectory Following Dynamic Programming algorithm for non finitely supported distribution (Chap. 6).

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Perspectives (Chap. 7)

- Higher order simplex algorithm on the chamber complex for 2SLP.
- 2-time scale MSLP, nested fiber polyhedra and convex bodies.
- Reintroduce approximation or sampling.
- Exact quantization for stochastic integer linear problems
- Understanding the complexity of MSLP.

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Thank you for listening ! Any question ?

M. Forcier, S. Gaubert, V. Leclère
Exact quantization of multistage stochastic linear problems.
arXiv preprint arXiv:2107.09566 (2021).

M. Forcier, V. Leclère
Generalized adaptive partition-based method for two-stage stochastic linear programs: convergence and generalization.
Operation Research Letters, to appear (2022).

M. Forcier, V. Leclère
Convergence of Trajectory Following Dynamic Programming algorithms for multistage stochastic problems without finite support assumptions
HAL Id : hal-03683697 (2022).

Contents

Local exact quantization and adapted partition

Local exact quantization

random cost

Recall that for a fixed x,

$$\mathbb{E}\big[\min_{y \in P_x} \boldsymbol{c}^\top y\big] = \sum_{N \in \mathcal{N}(P_x)} p_N \min_{y \in P_x} \check{c}_N{}^\top y$$

where,

$$p_N := \mathbb{P}\big[\boldsymbol{c} \in -\operatorname{ri} N\big]$$
$$\check{c}_N := \mathbb{E}\big[\boldsymbol{c} \mid \boldsymbol{c} \in -\operatorname{ri} N\big]$$

$$P_x := \{y \in \mathbb{R}^m \mid Ay + Bx \leqslant b\}$$

GAPM

random constraints

Similarly, for a given q, and all x,

$$\begin{aligned} V(x) &:= \mathbb{E}\big[Q(x, \boldsymbol{\xi})\big] \\ &= \mathbb{E}\big[\max_{\lambda \in D_q} (\boldsymbol{h} - \boldsymbol{T}x)^\top \lambda\big] \\ &= \sum_{N \in \mathcal{N}(D_q)} p_N \max_{\lambda \in D_q} \psi_{N,x}{}^\top \lambda \end{aligned}$$

where,

$$p_N := \mathbb{P}\big[\boldsymbol{h} - \boldsymbol{T}x \in \operatorname{ri} N\big]$$
$$\psi_{N,x} := \mathbb{E}\big[\boldsymbol{h} - \boldsymbol{T}x \mid \boldsymbol{h} - \boldsymbol{T}x \in \operatorname{ri} N\big]$$

$$D_q := \{\lambda \in \mathbb{R}^l \mid W^\top \lambda \leqslant q\}$$

An explicit adapted partition

Consider $x \in \mathbb{R}^n$ and $N \in \mathcal{N}(D_q)$ a normal cone of D_q. We define

$$E_{N,x} := \{\xi \in \Xi \mid h - Tx \in \operatorname{ri} N\}$$

Theorem (FL 2021)

$\mathcal{R}_x := \{E_{N,x} \mid N \in \mathcal{N}(D_q)\}$ *is an adapted partition to* x
i.e. $V_{\mathcal{R}_x}(x) = V(x)$

Proof:

$$\begin{aligned} V(x) &:= \mathbb{E}\big[Q(x,\xi)\big] \\ &= \sum_{N \in \mathcal{N}(D)} \mathbb{P}\big[\boldsymbol{h} - \boldsymbol{T}x \in \operatorname{ri} N\big] \min_{\lambda \in D} \mathbb{E}\big[\boldsymbol{h} - \boldsymbol{T}x \,|\, \boldsymbol{h} - \boldsymbol{T}x \in \operatorname{ri} N\big]^\top \lambda \\ &= \sum_{N \in \mathcal{N}(D)} \mathbb{P}\big[\boldsymbol{\xi} \in E_{N,x}\big] Q\Big(\mathbb{E}\big[\boldsymbol{\xi} \,|\, \boldsymbol{\xi} \in E_{N,x}\big], x\Big) = V_{\mathcal{R}_x}(x) \end{aligned}$$

An explicit adapted partition

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Numerical Results - ProdMix

k	z_L^k	z_U^k	$z_U^k - z_L^k$	Total time	$\lvert\mathcal{P}^k\rvert$
1	-18666.67	-16939.71	1726.96	0.57 s	4
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6	-17713.74	-17711.37	2.37	23.7 s	49
8	-17711.71	-17711.56	0.15	50.0 s	81
10	-17711.57	-17711.56	0.01	88.0 s	121

Table: Results for problem Prod-Mix

Comparison with SAA : we solved the same problem 100 times, each with 10 000 scenarios randomly drawn
$\rightsquigarrow$ 95% confidence interval centered in -17711, with radius 2.2.
$\rightsquigarrow$ required 2058s of computation.

Contents

Synthesis of local and uniform quantization results

	$\boldsymbol{W}$	$(\boldsymbol{T}, \boldsymbol{h})$	$\boldsymbol{q}$
Local	$\emptyset$	$\mathcal{R}_x$	$\mathcal{N}(P_x)$
Uniform	$\emptyset$	$\emptyset$	$\bigwedge_{\sigma \in \mathcal{C}(P,\pi)} \mathcal{N}_\sigma$

Stochastic cost and recourse

- We have shown a local exact quantization result for random $\boldsymbol{T}, \boldsymbol{h}$, and deterministic q, W.
- If $\boldsymbol{q}$ and $\boldsymbol{W}$ are finitely supported random variable:
  1. compute an exact quantization $\mathcal{N}_\xi$ for every element of the support;
  2. take the common refinement.

We have seen that we can deal with non-finitely supported $\boldsymbol{q}$ through the chamber complexes.

➥ Can we do the same here ?

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Adapted partition for general q

We define coupling constraint and fiber for the dual.

$$\begin{aligned} D_q &:= \left\{\lambda \in \mathbb{R}^\ell \quad \middle| \quad W^\top \lambda \leqslant q\right\} \\ \Delta &:= \left\{(\lambda, q) \in \mathbb{R}^\ell \times \mathbb{R}^m \quad \middle| \quad W^\top \lambda \leqslant q\right\} \\ \mathcal{R}_{x,q} &:= \left\{E_{N,x} \quad \middle| \quad N \in \mathcal{N}(D_q)\right\} \end{aligned}$$

Recall that $q \mapsto \mathcal{N}(D_q)$ is piecewise constant on $\mathcal{C}(\Delta, \pi_\lambda^{\lambda,q})$ and so is $\mathcal{R}_{x,q}$.
➡ we can take the common refinement of a finite number of $\mathcal{R}_{x,q}$!!

More precisely:

- The chamber complex $\mathcal{C}(\Delta, \pi_\lambda^{\lambda,q}) = \Sigma\text{-fan}(W)$[3].
- For $S \in \Sigma\text{-fan}(W)$ define $\mathcal{R}_{x,S} := \mathcal{R}_{x,q}$ for any $q \in \mathrm{ri}(S)$.

➡ $\left\{ \mathrm{ri}(S) \times R \,\middle|\, S \in \Sigma\text{-fan}(W), R \in \mathcal{R}_{x,S} \right\}$ is an adapted partition to x.

[3] The well studied secondary fan of W

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Contents

Dual problem

$$V(x) := \mathbb{E}\left[\begin{array}{ll} \inf\limits_{y} & \boldsymbol{c}^\top y \\ \text{s.t.} & Ax + By \leqslant b \end{array}\right] = \mathbb{E}\big[\inf_{y \in P_x} \boldsymbol{c}^\top y\big]$$

where $P_x = \{x \mid Ax + By \leqslant b\}$

$$V(x) := \mathbb{E}\left[\begin{array}{ll} \sup\limits_{\mu} & (Ax - b)^\top \mu \\ \text{s.t.} & B^\top \mu + \boldsymbol{c} = 0 \\ & \mu \geqslant 0 \end{array}\right] = \mathbb{E}\big[\sup_{\mu \in D_{\boldsymbol{c}}} (Ax - b)^\top \mu\big]$$

where $D_c = \{\mu \mid B^\top \mu + c = 0, \mu \geqslant 0\}$

Fiber Polyhedron

Minkowski sum :

$$E + F = \{x + x' \,|x \in E, x' \in F\}$$

Definition

The *fiber polyhedron* E of the bundle $(D_c)_{c\in\mathrm{supp}(\boldsymbol{c})}$ is the Minkowsky integral of all the fiber at c when c varies according to its probability distribution:

$$E := \int D_c \mathbb{P}(dc) = \Big\{ \int \mu(c)\mathbb{P}(dc) \,|\, \mu(c) \in D_c \quad a.s., \ \mu \in L_\infty(\mathbb{R}^m, \mathbb{R}^l) \Big\}$$

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The Fiber Polyhedron is a finite Minkowski sum

Theorem

There exists a chamber complex $\mathcal{R}$ depending on A such that

$$E = \int D_c \mathbb{P}(dc) = \sum_{R \in \mathcal{R}} \check{p}_R D_{\check{c}_R}$$

where $\check{p}_R := \mathbb{P}\big[\boldsymbol{c} \in \mathrm{ri}(R)\big]$ *and* $\check{c}_R := \mathbb{E}\big[\boldsymbol{c} \,|\, \boldsymbol{c} \in \mathrm{ri}(R)\big]$.

Alternative proof of the quantization result

$$V(x) = \sigma_E(Ax-b) = \sum_{R \in \mathcal{R}} \check{p}_R \sigma_{D_{\check{c}_R}}(Ax-b) = \sum_{R \in \mathcal{R}} \check{p}_R \min_{y \in P_x} \check{c}_R^\top y$$

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Nested Fiber Polyhedra for Multistage

$$V_t(x_{t-1}) = \mathbb{E}\left[\begin{array}{l} \min\limits_{x_t \in \mathbb{R}^{n_t}} \boldsymbol{c}_t^\top x_t + V_{t+1}(x_t) \\ \text{s.t. } A_t x_t + B_t x_{t-1} \leqslant b_t \end{array}\right]$$

Definition

We define by induction the following nested fiber polyhedra

$$D_{t,c_t} := \{\mu_t | \mu_t \geqslant 0, A_t^\top \mu_t + c_t = 0\} \qquad \forall t \in [T]$$

$$F_{T,c_T} := D_{T,c_T}$$

$$E_t := \mathbb{E}\left[F_{t,\boldsymbol{c}_t}\right] \qquad \forall t \in [T]$$

$$F_{t,c_t} := \{(\mu_t, \lambda_{[t+1:T]}) \,|\, \mu_t \in D_{t,c_t + B_{t+1}^\top \lambda_{t+1}},\ \lambda_{[t+1:T]} \in E_{t+1}\} \quad \forall t \in [T-1]$$

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\end{aligned}$$

$$V_t(x_{t-1}) = \sigma_{E_t}(B_t x_{t-1} - b_t, -b_{[t+1:T]})$$

2-time scale MSLP reduced to Quadratic Problem

We consider a MSLP where the dynamic depends on parameters p we have to optimize

$$
\begin{aligned}
\min_{p \in \mathbb{R}^m, (\boldsymbol{x}_t) \in \mathbb{R}^{n_t}} \quad & q^\top p + \mathbb{E}\big[\sum_{t=1}^{T} \boldsymbol{c}_t^\top \boldsymbol{x}_t \big] \\
\text{s.t.} \quad & Dp \leqslant d \\
& A_t \boldsymbol{x}_t + B_t \boldsymbol{x}_{t-1} + C_t p \leqslant h_t \quad \text{a.s.} & \forall t \in [T] \\
& \boldsymbol{x}_t \preceq \sigma(\boldsymbol{c}_1, \cdots, \boldsymbol{c}_t) & \forall t \in [T]
\end{aligned}
$$

If we know the fiber polyhedron, it reduces to a finite dimensional quadratic problem

$$
\begin{aligned}
& \min_{p \in \mathbb{R}^m} q^\top p + \sup_{(\lambda_t)_{t \in [T]}} \sum_{t=1}^{T} (C_t p - h_t)^\top \lambda_t \\
& \text{s.t. } Dp \leqslant d \\
& \qquad (\lambda_1, \cdots, \lambda_T) \in E_1
\end{aligned}
$$

Contents

Contents

Linear Programming

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leqslant b \end{aligned}$$

Example: $P = \{x \in \mathbb{R}^n \,|\, Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ & \\ & \\ & \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ \\ \\ \\ \\ \end{pmatrix}$$

$x_1 + x_2 \leqslant 1$ (1)

(2)

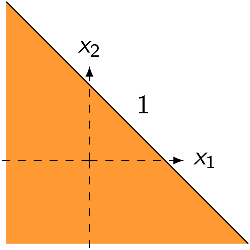
(3)

(4)

(5)

(6)

(7)



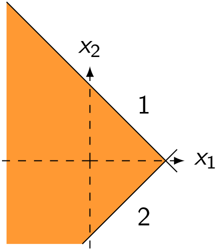
Linear Programming

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leqslant b \end{aligned}$$

Example: $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ & \\ & \\ & \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ \\ \\ \\ \\ \\ \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &\leqslant 1 && (1) \\ x_1 - x_2 &\leqslant 1 && (2) \\ & && (3) \\ & && (4) \\ & && (5) \\ & && (6) \\ & && (7) \end{aligned}$$



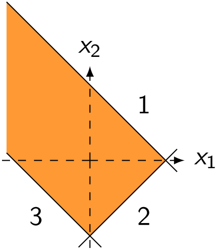
Linear Programming

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leqslant b \end{aligned}$$

Example: $P = \{x \in \mathbb{R}^n \,|\, Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ \\ \\ \\ \\ \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \\ \\ \\ \\ \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &\leqslant 1 && (1) \\ x_1 - x_2 &\leqslant 1 && (2) \\ -x_1 - x_2 &\leqslant 1 && (3) \\ & && (4) \\ & && (5) \\ & && (6) \\ & && (7) \end{aligned}$$



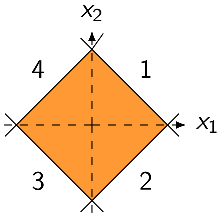
Linear Programming

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leqslant b \end{aligned}$$

Example: $P = \{x \in \mathbb{R}^n \,|\, Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ & \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \\ \\ \\ \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &\leqslant 1 && (1) \\ x_1 - x_2 &\leqslant 1 && (2) \\ -x_1 - x_2 &\leqslant 1 && (3) \\ -x_1 + x_2 &\leqslant 1 && (4) \\ & && (5) \\ & && (6) \\ & && (7) \end{aligned}$$



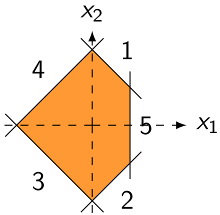
Linear Programming

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leqslant b \end{aligned}$$

Example: $P = \{x \in \mathbb{R}^n \,|\, Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 0 \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0.5 \\ \\ \\ \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &\leqslant 1 && (1) \\ x_1 - x_2 &\leqslant 1 && (2) \\ -x_1 - x_2 &\leqslant 1 && (3) \\ -x_1 + x_2 &\leqslant 1 && (4) \\ x_1 &\leqslant 0.5 && (5) \\ & && (6) \\ & && (7) \end{aligned}$$



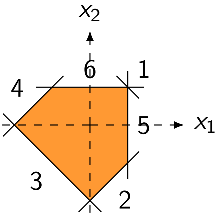
Linear Programming

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leqslant b \end{aligned}$$

Example: $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0.5 \\ 0.5 \\ \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &\leqslant 1 && (1) \\ x_1 - x_2 &\leqslant 1 && (2) \\ -x_1 - x_2 &\leqslant 1 && (3) \\ -x_1 + x_2 &\leqslant 1 && (4) \\ x_1 &\leqslant 0.5 && (5) \\ x_2 &\leqslant 0.5 && (6) \\ & && (7) \end{aligned}$$



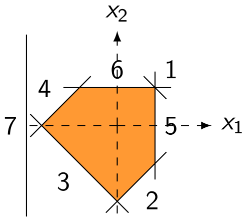
Linear Programming

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leqslant b \end{aligned}$$

Example: $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0.5 \\ 0.5 \\ -1.2 \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &\leqslant 1 && (1) \\ x_1 - x_2 &\leqslant 1 && (2) \\ -x_1 - x_2 &\leqslant 1 && (3) \\ -x_1 + x_2 &\leqslant 1 && (4) \\ x_1 &\leqslant 0.5 && (5) \\ x_2 &\leqslant 0.5 && (6) \\ x_1 &\geqslant -1.2 && (7) \end{aligned}$$



Contents

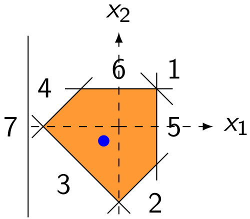
Active constraints

Definition

We denote by $\mathcal{I}(A,b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A,b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \emptyset$

To ease the notation, we write:

$$\mathcal{I}(A,b) = \{\emptyset, \qquad \}$$

Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

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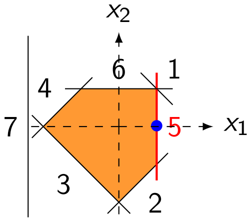


$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{5\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, \qquad\qquad\qquad\qquad \}$$

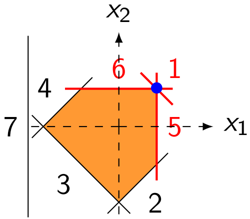
Active constraints

Definition

We denote by $\mathcal{I}(A,b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A,b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{1,5,6\}$

To ease the notation, we write:

$$\mathcal{I}(A,b) = \{\emptyset, 5, 156, \qquad \}$$

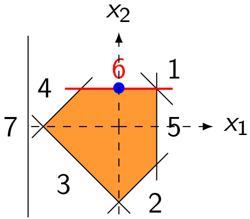
Active constraints

Definition

We denote by $\mathcal{I}(A,b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A,b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{6\}$

To ease the notation, we write:

$$\mathcal{I}(A,b) = \{\emptyset, 5, 156, 6, \qquad\qquad \}$$

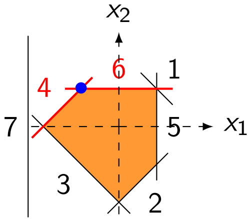
Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{4, 6\}$

To ease the notation, we write:

$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, \qquad\qquad \}$

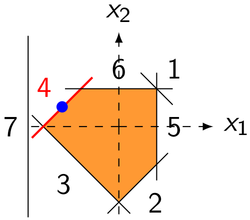
Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{4\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, \qquad \}$$

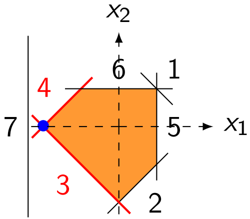
Active constraints

Definition

We denote by $\mathcal{I}(A,b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A,b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{3,4\}$

To ease the notation, we write:

$$\mathcal{I}(A,b) = \{\emptyset, 5, 156, 6, 46, 4, 34, \qquad \}$$

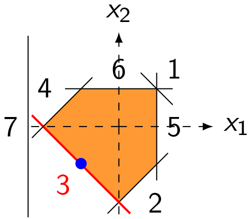
Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{3\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, 34, 3, \qquad\qquad \}$$

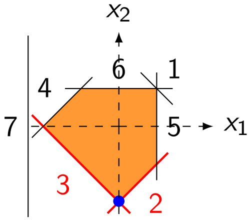
Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{2, 3\}$

To ease the notation, we write:

$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, 34, 3, 23, \qquad \}$

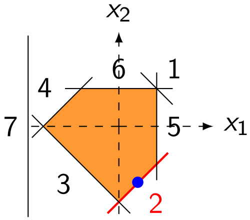
Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{2\}$
To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, 34, 3, 23, 2, \quad \}$$

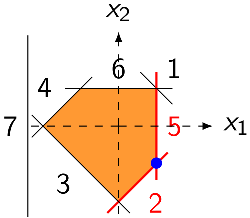
Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{2, 5\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, 34, 3, 23, 2, 25\}$$

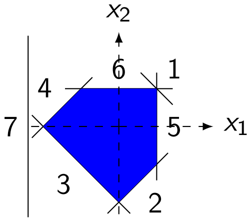
Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$
Example for $I = \emptyset$



P, P^I

Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

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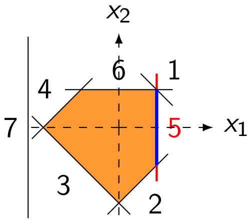
Example for $I = \{5\}$



P, P^I

Faces

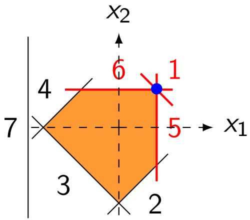
Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$

Example for $I = \{1, 5, 6\}$



P, P^I

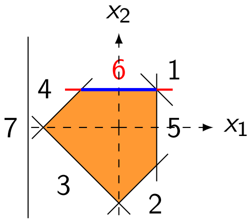
Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$
Example for $I = \{6\}$



P, P^I

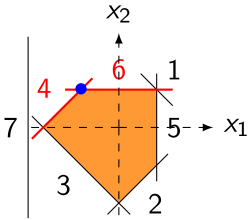
Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$
Example for $I = \{4, 6\}$



P, P^I

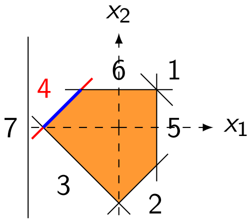
Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$
Example for $I = \{4\}$



P, P^I

Faces

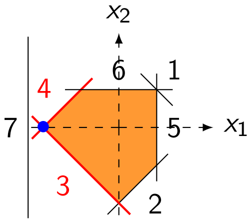
Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$

Example for $I = \{3, 4\}$



P, P^I

Faces

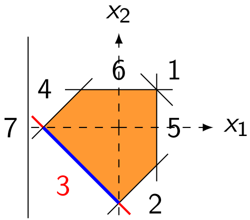
Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$

Example for $I = \{3\}$



P, P^I

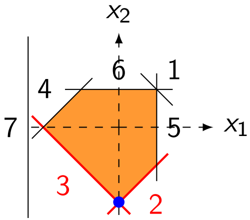
Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$
Example for $I = \{2, 3\}$



P, P^I

Faces

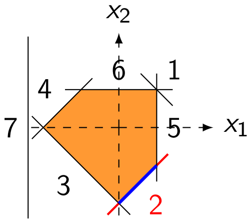
Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$

Example for $I = \{2\}$



P, P^I

Faces

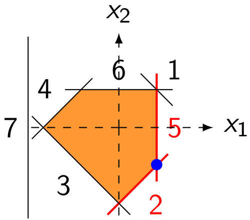
Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$

Example for $I = \{2, 5\}$

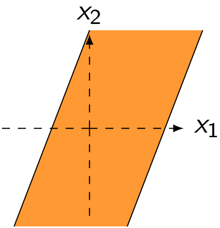


P, P^I

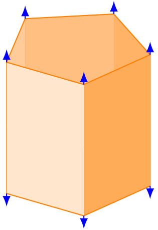
Lineality space, vertices and bases

Definition (Lineality space)

$\mathrm{Lin}(C) := \{u \in C \mid \forall t \in \mathbb{R},\ \forall x \in c,\ x + tu \in C\}.$



If
$P = \{x \in \mathbb{R}^n | Ax \leqslant b\}$,
then $\mathrm{Lin}(P) = \mathrm{Ker}(A)$



Definition (Bases and vertices)

A basis B is a subset of $[p]$ such that $A_B = (A_{i,j})_{i \in B, 1 \leqslant j \leqslant n}$ is invertible.
A vertex of P is a face of dimension 0. $\mathrm{Vert}(P)$ is the set of vertices.

$\mathrm{Vert}(P) \neq \emptyset \Leftrightarrow A$ admits at least one basis $\Leftrightarrow \mathrm{rg}(A) = n \Leftrightarrow \mathrm{Lin}(P) = \{0\}$

We make this assumption without loss of generality.

Contents

Link with regular subdivisions

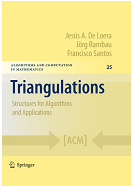
Definition (DLRS10)

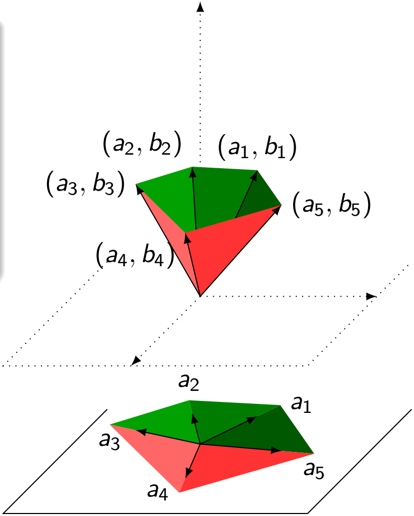
$$\mathcal{S}(A^\top, b) := \{ I_F \mid F \in \mathcal{F}_{\text{low}}(LC_{A^\top, b}) \}$$

$$LC_{A^\top, b} := \text{Cone}\left(\begin{pmatrix} a_i \\ b_i \end{pmatrix}_{i \in [q]} \right)$$

$$I_F := \{ i \in [q] \mid (a_i, b_i) \in F \}.$$

$$\mathcal{S}(A^\top, b) = \mathcal{I}(A, b)$$





$$\mathcal{I}(W^\top q) = \mathcal{I}_{com} \cup \{ \{5\}, \{4, 5\}, \{1, 5\} \}$$

Link with regular subdivisions

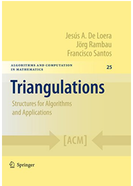
Definition (DLRS10)

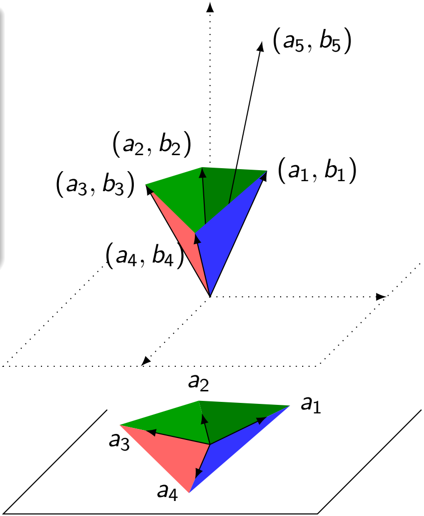
$$\mathcal{S}(A^\top, b) := \{I_F \mid F \in \mathcal{F}_{\mathrm{low}}(LC_{A^\top,b})\}$$

$$LC_{A^\top,b} := \mathrm{Cone}\left(\begin{pmatrix} a_i \\ b_i \end{pmatrix}_{i\in[q]}\right)$$

$$I_F := \{i \in [q] \mid (a_i, b_i) \in F\}.$$

$$\mathcal{S}(A^\top, b) = \mathcal{I}(A, b)$$





$$\mathcal{I}(W^\top q) = \mathcal{I}_{com} \cup \{\{1, 4\}\}$$

Link with regular subdivisions

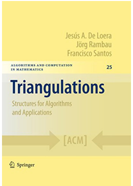
Definition (DLRS10)

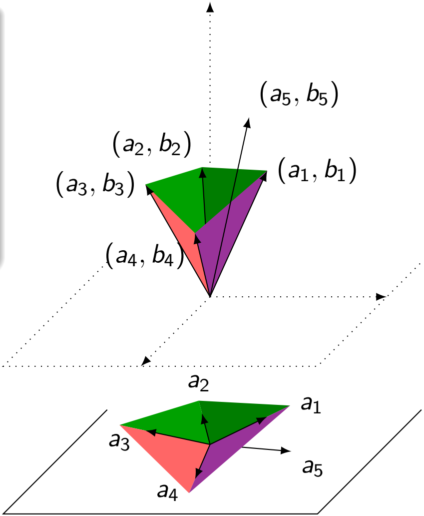
$$\mathcal{S}(A^\top, b) := \{I_F \mid F \in \mathcal{F}_{\text{low}}(LC_{A^\top,b})\}$$

$$LC_{A^\top,b} := \text{Cone}\left(\begin{pmatrix} a_i \\ b_i \end{pmatrix}_{i\in[q]}\right)$$

$$I_F := \{i \in [q] \mid (a_i, b_i) \in F\}.$$

$$\mathcal{S}(A^\top, b) = \mathcal{I}(A, b)$$





$$\mathcal{I}(W^\top q) = \mathcal{I}_{com} \cup \{\{1, 4, 5\}\}$$

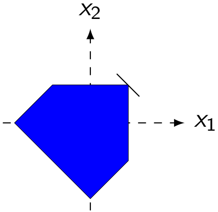
Contents

Correspondences

$$\begin{tikzcd}[column sep=huge, row sep=huge]
 & \boxed{F \in \mathcal{F}(P)\setminus\{\emptyset\}, \supset} \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x\in \mathrm{ri}(F)"'] \arrow[dr, bend left, "N_P(x) \text{ for } x\in \mathrm{ri}(F)"] & \\
\boxed{I \in \mathcal{I}(A,b), \subset} \arrow[ur, bend right, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & \boxed{N \in \mathcal{N}(P), \subset} \arrow[ul, bend left, "\arg\min_{x\in P} c^\top x \text{ for } c\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in \mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top \in N\}"]
\end{tikzcd}$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \emptyset$



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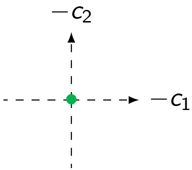


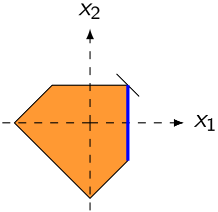
N

Correspondences

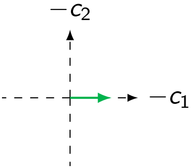
$$\begin{tikzcd}
 & F \in \mathcal{F}(P)\setminus\{\emptyset\}, \supset \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x\in\mathrm{ri}(F)"'] \arrow[dr, bend left, "N_P(x) \text{ for } x\in\mathrm{ri}(F)"] & \\
I \in \mathcal{I}(A,b), \subset \arrow[ur, bend right, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & N \in \mathcal{N}(P), \subset \arrow[ul, bend right, "\arg\min_{x\in P} c^\top x \text{ for } c\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in\mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top \in N\}"]
\end{tikzcd}$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{5\}$



F



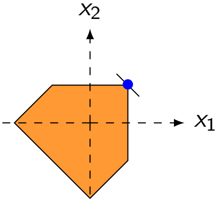
N

Correspondences

$$
\begin{tikzcd}[column sep=small]
 & \boxed{F \in \mathcal{F}(P)\setminus\{\emptyset\}, \supset} \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x\in \mathrm{ri}(F)"'] \arrow[dr, bend right, "N_P(x) \text{ for } x\in \mathrm{ri}(F)"'] & \\
\boxed{I \in \mathcal{I}(A,b), \subset} \arrow[ur, bend right, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & \boxed{N \in \mathcal{N}(P), \subset} \arrow[ul, bend right, "\arg\min_{x\in P} c^\top x \text{ for } c\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in \mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top \in N\}"]
\end{tikzcd}
$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{1, 5, 6\}$



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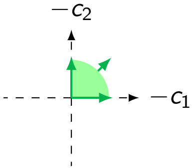


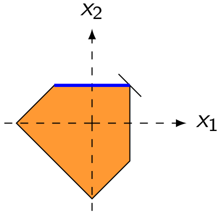
N

Correspondences

$$\begin{tikzcd}
& F \in \mathcal{F}(P)\setminus\{\emptyset\}, \supset \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x\in \mathrm{ri}(F)"'] \arrow[dr, bend left, "N_P(x) \text{ for } x\in \mathrm{ri}(F)"] & \\
I \in \mathcal{I}(A,b), \subset \arrow[ur, bend right, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & N \in \mathcal{N}(P), \subset \arrow[ul, bend right, "\arg\min_{x\in P} c^\top x \text{ for } c\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in \mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top \in N\}"]
\end{tikzcd}$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{6\}$



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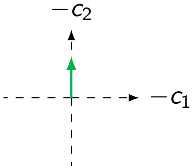


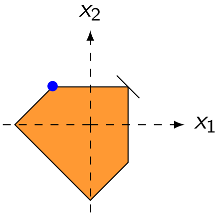
N

Correspondences

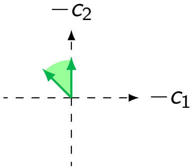
$$\begin{tikzcd}
 & \boxed{{\color{blue}F \in \mathcal{F}(P)}\setminus\{\emptyset\}, \supset} \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x\in\mathrm{ri}(F)"'] \arrow[dr, bend left, "N_P(x) \text{ for } x\in\mathrm{ri}(F)"] & \\
\boxed{{\color{red}I \in \mathcal{I}(A,b)}, \subset} \arrow[ur, bend right, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & \boxed{{\color{green}N \in \mathcal{N}(P)}, \subset} \arrow[ul, bend right, "\arg\min_{x\in P} c^\top x \text{ for } c\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in\mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top\in N\}"]
\end{tikzcd}$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{4, 6\}$



F



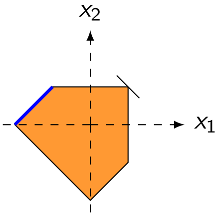
N

Correspondences

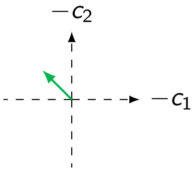
$$
\begin{tikzcd}[column sep=large, row sep=huge]
 & F \in \mathcal{F}(P)\setminus\{\emptyset\}, \supset \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x\in \mathrm{ri}(F)"'] \arrow[dr, bend left, "N_P(x) \text{ for } x\in \mathrm{ri}(F)"] & \\
I \in \mathcal{I}(A,b), \subset \arrow[ur, bend right, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & N \in \mathcal{N}(P), \subset \arrow[ul, bend right, "\arg\min_{x\in P} c^\top x \text{ for } c\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in \mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top \in N\}"]
\end{tikzcd}
$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{4\}$



F



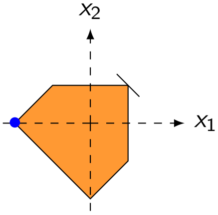
N

Correspondences

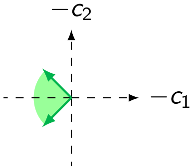
$$\begin{tikzcd}[column sep=large, row sep=huge]
& \color{blue}{F \in \mathcal{F}(P)}\setminus\{\emptyset\}, \supset \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x\in \mathrm{ri}(F)"'] \arrow[dr, bend left, "N_P(x) \text{ for } x\in \mathrm{ri}(F)"] & \\
\color{red}{I \in \mathcal{I}(A,b)}, \subset \arrow[ur, bend right, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & \color{green}{N \in \mathcal{N}(P)}, \subset \arrow[ul, bend left, "\arg\min_{x\in P} c^\top x \text{ for } c\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in \mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top \in N\}"]
\end{tikzcd}$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{3, 4\}$



F



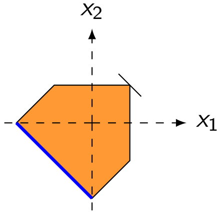
N

Correspondences

$$\begin{tikzcd}
 & \boxed{F \in \mathcal{F}(P)\setminus\{\emptyset\}, \supset} \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x\in\mathrm{ri}(F)"'] \arrow[dr, bend left, "N_P(x) \text{ for } x\in\mathrm{ri}(F)"] & \\
\boxed{I \in \mathcal{I}(A,b), \subset} \arrow[ur, bend right, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & \boxed{N \in \mathcal{N}(P), \subset} \arrow[ul, bend right, "\arg\min_{x\in P} c^\top x \text{ for } c\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in \mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top \in N\}"]
\end{tikzcd}$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{3\}$



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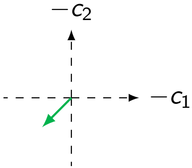


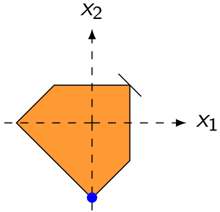
N

Correspondences

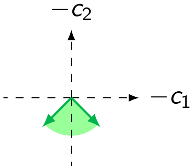
$$\begin{tikzcd}
& F \in \mathcal{F}(P)\setminus\{\emptyset\}, \supset \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x\in \mathrm{ri}(F)"'] \arrow[dr, bend left, "N_P(x) \text{ for } x\in \mathrm{ri}(F)"] & \\
I \in \mathcal{I}(A,b), \subset \arrow[ur, bend right, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & N \in \mathcal{N}(P), \subset \arrow[ul, bend right, "\arg\min_{x\in P} c^\top x \text{ for } c\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in \mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top \in N\}"]
\end{tikzcd}$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{2, 3\}$



F



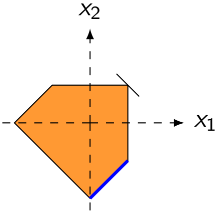
N

Correspondences

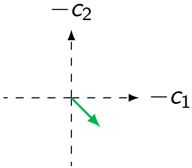
$$
\begin{tikzcd}[column sep=huge, row sep=huge]
 & F \in \mathcal{F}(P)\setminus\{\emptyset\}, \supset \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x \in \mathrm{ri}(F)"'] \arrow[dr, bend right, "N_P(x) \text{ for } x \in \mathrm{ri}(F)"'] & \\
I \in \mathcal{I}(A,b), \subset \arrow[ur, bend right, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & N \in \mathcal{N}(P), \subset \arrow[ul, bend right, "\arg\min_{x\in P} c^\top x \text{ for } c \in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i \in \mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top \in N\}"]
\end{tikzcd}
$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{2\}$



F



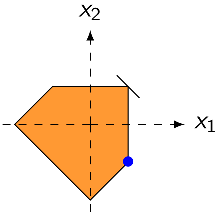
N

Correspondences

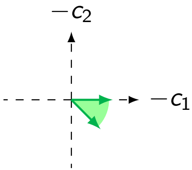
$$\begin{tikzcd}
 & \boxed{F \in \mathcal{F}(P)\setminus\{\emptyset\}, \supset} \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x\in \mathrm{ri}(F)"'] \arrow[dr, bend left, "N_P(x) \text{ for } x\in \mathrm{ri}(F)"] & \\
\boxed{I \in \mathcal{I}(A,b), \subset} \arrow[ur, bend right, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & \boxed{N \in \mathcal{N}(P), \subset} \arrow[ul, bend right, "\arg\min_{x\in P} c^\top x \text{ for } c\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in \mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top \in N\}"]
\end{tikzcd}$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{2, 5\}$

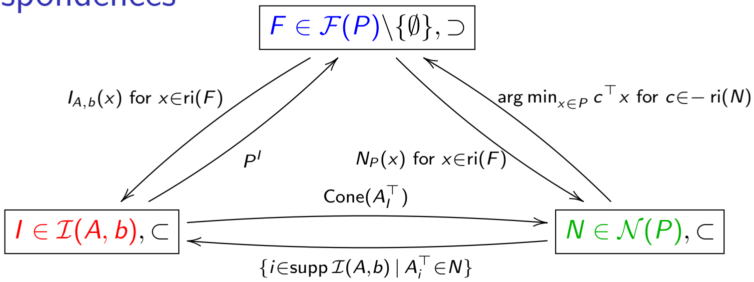


F



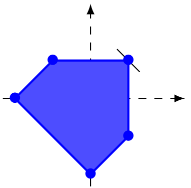
N

Correspondences

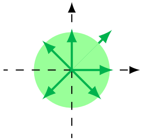


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$\mathcal{I}(A,b) =$
$\{\emptyset, 5, 156, 6, 46, 4,$
$34, 3, 23, 2, 25\}$



$\mathcal{F}(P)$

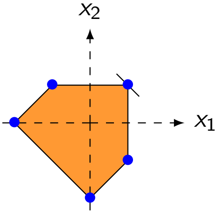


$\mathcal{N}(P)$

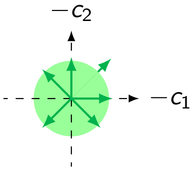
Correspondences

$$\begin{tikzcd}
& v \in \mathrm{Vert}(P) \arrow[dl, bend right, "I_{A,b}(v)"'] \arrow[dr, bend left, "N_P(v)"] & \\
I \in \overline{\mathcal{I}(A,b)} \arrow[ur, bend right, "A_{B_I}^{-1} b_{B_I}"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & N \in \overline{\mathcal{N}(P)} \arrow[ul, bend right, "\arg\min_{x\in P} c^\top x \text{ for } c \in - \mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i \in \mathrm{supp}\, \mathcal{I}(A,b) \mid A_i^\top \in N\}"]
\end{tikzcd}$$

$\overline{\mathcal{I}(A,b)} =$
$\{156, 46, 34, 23, 25\}$



$\mathrm{Vert}(P)$



$\overline{\mathcal{N}(P)}$

Contents

Proof of normal equivalence

$$\textcolor{magenta}{\mathcal{G}_x} := \{G \in \mathcal{F}(P) \mid x \in \mathrm{ri}\,(\pi(G))\}$$

Let $\sigma \in \mathcal{C}(P, \pi)$, for all $x, x' \in \mathrm{ri}(\sigma)$, we have

$$\mathcal{G}_\sigma := \mathcal{G}_x = \mathcal{G}_{x'}$$

$$\begin{tikzcd}
\boxed{G \in \mathcal{G}_x} \arrow[rr, bend right, "G_x"'] \arrow[dr, shift left, "I_{(T,W),h}(G)"] & & \boxed{F \in \mathcal{F}(P_x) \setminus \{\emptyset\}} \arrow[ll, bend right, "G \in \mathcal{F}(P) \text{ minimal s.t.} \{x\} \times F \subset G"'] \arrow[dl, shift right, "I_{W,h-Tx}(F)"'] \arrow[dr, shift left, "N_{P_x}(F)"'] & \\
& \boxed{I \in \mathcal{I}(W, h - Tx)} \arrow[ul, shift left, "P^I"] \arrow[ur, shift right, "P^I_x"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & \boxed{N \in \mathcal{N}(P_x)} \arrow[ul, shift left, "\cap_{q \in -N} \arg\min_{y \in P_x} q^\top y"'] \arrow[ll, bend left, "\{i \in \mathrm{supp}\, \mathcal{I}_\sigma \mid W_i^\top \in N\}"]
\end{tikzcd}$$

By the correspondences,

$$\mathcal{I}_\sigma := \mathcal{I}(W, h - Tx) = \mathcal{I}(W, h - Tx') \qquad \mathcal{N}_\sigma := \mathcal{N}(P_x) = \mathcal{N}(P_{x'})$$

Proof of normal equivalence

$$\mathcal{G}_x := \{ G \in \mathcal{F}(P) \,|\, x \in \operatorname{ri}\left(\pi(G)\right) \}$$

Let $\sigma \in \mathcal{C}(P, \pi)$, for all $x, x' \in \operatorname{ri}(\sigma)$, we have

$$\mathcal{G}_\sigma := \mathcal{G}_x = \mathcal{G}_{x'}$$

$$\begin{tikzcd}
\boxed{G \in \mathcal{G}_x} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend left, "I_{(T,W),h}(G)"] & & \boxed{F \in \mathcal{F}(P_x) \setminus \{\emptyset\}} \arrow[ll, bend right, "G \in \mathcal{F}(P) \text{ minimal s.t.} \{x\} \times F \subset G"'] \arrow[dl, bend right, "I_{W,h-Tx}(F)"'] \arrow[dr, bend right, "N_{P_x}(F)"'] & \\
& \boxed{I \in \mathcal{I}(W, h - Tx)} \arrow[ul, bend left, "P^I"] \arrow[ur, bend right, "P^I_x"'] \arrow[rr, bend left, "\operatorname{Cone}(W_I^\top)"] & & \boxed{N \in \mathcal{N}(P_x)} \arrow[ul, bend right, "\cap_{q \in -N} \arg\min_{y \in P_x} q^\top y"'] \arrow[ll, bend left, "\{i \in \operatorname{supp} \mathcal{I}_\sigma \,|\, W_i^\top \in N\}"]
\end{tikzcd}$$

By the correspondences,

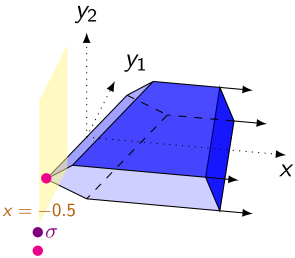
$$\mathcal{I}_\sigma := \mathcal{I}(W, h - Tx) = \mathcal{I}(W, h - Tx') \qquad \mathcal{N}_\sigma := \mathcal{N}(P_x) = \mathcal{N}(P_{x'})$$

Correspondences

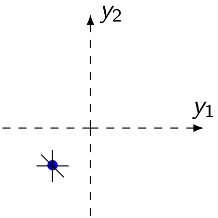
$$
\begin{tikzcd}[column sep=large, row sep=huge]
\boxed{G \in \mathcal{G}_\sigma, \supset} \arrow[rr, bend right=10, "G_x"'] \arrow[dr, bend left=10, "I_{(T,W),h}(G)"] & & \boxed{F \in \mathcal{F}(P_x) \setminus \{\emptyset\}, \supset} \arrow[ll, bend right=10, "G \in \mathcal{F}(P) \text{ minimal s.t.} \{x\} \times F \subset G"'] \arrow[dl, bend right=10, "I_{W,h-Tx}(F)"'] \arrow[dr, bend right=10, "N_{P_x}(F)"'] & \\
& \boxed{I \in \mathcal{I}_\sigma, \subset} \arrow[ul, bend left=10, "P^I"] \arrow[ur, bend right=10, "P^I_x"'] \arrow[rr, bend left=5, "\mathrm{Cone}(W_I^\top)"] & & \boxed{N \in \mathcal{N}_\sigma, \subset} \arrow[ul, bend right=10, "\arg\min_{y \in P_x} q^\top x \text{ for } q \in -\mathrm{ri}(N)"'] \arrow[ll, bend left=5, "\{i \in \mathrm{supp}\, \mathcal{I}_\sigma \mid W_i^\top \in N\}"]
\end{tikzcd}
$$

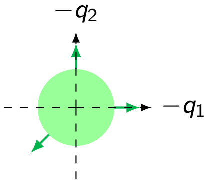
Correspondences

$$
\begin{tikzcd}
G \in \underline{\mathcal{G}_\sigma} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & v \in \mathrm{Vert}(P_x) \arrow[ll, bend right, "G\in\mathcal{F}(P) \text{ minimal s.t.} (x,v)\in G"'] \arrow[dl, bend right, "I_{W,h\text{-}Tx}(v)"'] \arrow[dr, bend right, "N_{P_x}(v)"'] & \\
& I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "W_{B_I}^{-1}(h_{B_I}\text{-}T_{B_I}x)"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ul, bend right, "\arg\min_{y\in P_x} q^\top x \text{ for } q\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in \mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top \in N\}"]
\end{tikzcd}
$$



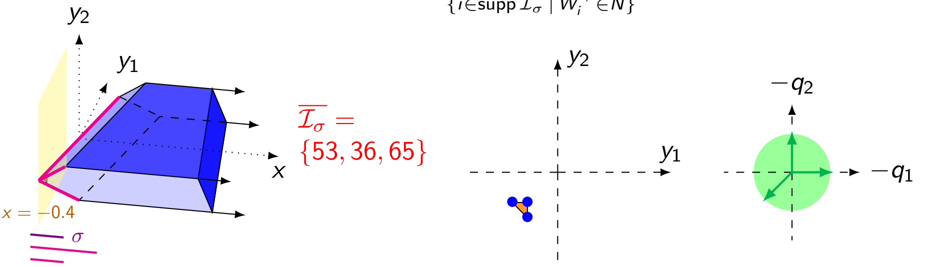
$\overline{\mathcal{I}_\sigma} = \{356\}$





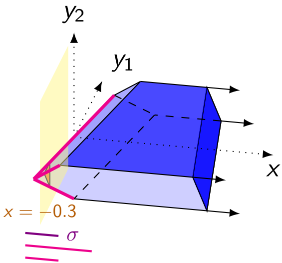
Correspondences

$$\begin{tikzcd}[column sep=large, row sep=huge]
G \in \mathcal{G}_\sigma \arrow[rr, bend right=10, "G_x"'] \arrow[dr, bend right=10, "I_{(T,W),h}(G)"'] & & v \in \mathrm{Vert}(P_x) \arrow[ll, bend right=10, "G\in\mathcal{F}(P) \text{ minimal s.t.} (x,v)\in G"'] \arrow[dl, bend right=10, "I_{W,h\text{-}Tx}(v)"'] \arrow[dr, bend left=10, "N_{P_x}(v)"] & \\
& I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right=10, "P^I"'] \arrow[ur, bend right=10, "W_{B_I}^{-1}(h_{B_I}\text{-}T_{B_I}x)"'] \arrow[rr, bend left=10, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ll, bend left=10, "\{i\in \mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top \in N\}"] \arrow[ul, bend left=10, "\arg\min_{y\in P_x} q^\top x \text{ for } q\in -\mathrm{ri}(N)"']
\end{tikzcd}$$

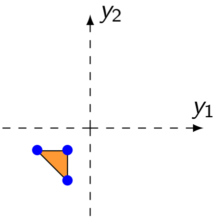


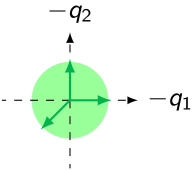
Correspondences

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{G \in \underline{\mathcal{G}_\sigma}} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & \boxed{v \in \mathrm{Vert}(P_x)} \arrow[ll, bend right, "G\in\mathcal{F}(P) \text{ minimal s.t.} (x,v)\in G"'] \arrow[dl, bend right, "I_{W,h\text{-}Tx}(v)"'] \arrow[dr, bend right, "N_{P_x}(v)"'] & \\
& \boxed{I \in \overline{\mathcal{I}_\sigma}} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "W_{B_I}^{-1}(h_{B_I}\text{-}T_{B_I}x)"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & \boxed{N \in \overline{\mathcal{N}_\sigma}} \arrow[ul, bend right, "\arg\min_{y\in P_x} q^\top x \text{ for } q\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in\mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top\in N\}"]
\end{tikzcd}$$



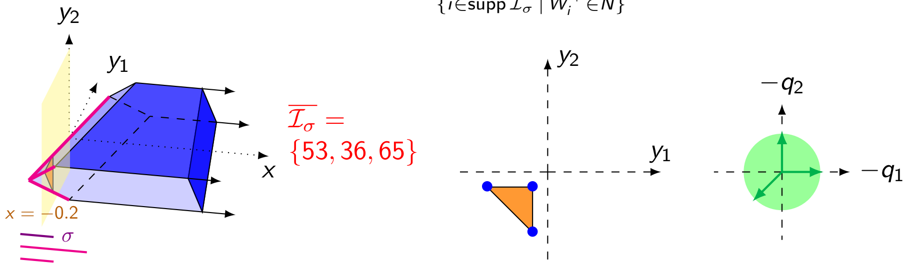
$\overline{\mathcal{I}_\sigma} =$
$\{53, 36, 65\}$





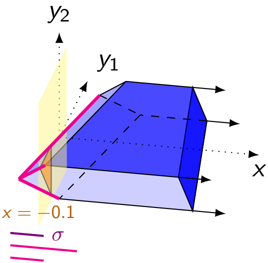
Correspondences

$$
\begin{tikzcd}
G \in \mathcal{G}_\sigma \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & v \in \mathrm{Vert}(P_x) \arrow[ll, bend right, "G\in\mathcal{F}(P) \text{ minimal s.t.} (x,v)\in G"'] \arrow[dl, bend right, "I_{W,h\text{-}Tx}(v)"'] \arrow[dr, bend right, "N_{P_x}(v)"'] & \\
& I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "W_{B_I}^{-1}(h_{B_I}\text{-}T_{B_I}x)"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ul, bend right, "\arg\min_{y\in P_x} q^\top x \text{ for } q\in-\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in\mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top\in N\}"]
\end{tikzcd}
$$

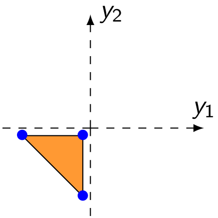


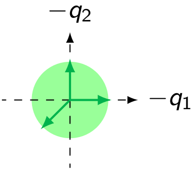
Correspondences

$$
\begin{tikzcd}
G \in \underline{\mathcal{G}_\sigma} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & v \in \mathrm{Vert}(P_x) \arrow[ll, bend right, "G{\in}\mathcal{F}(P)\text{ minimal s.t.}(x,v){\in}G"'] \arrow[dl, bend right, "I_{W,h\text{-}Tx}(v)"'] \arrow[dr, bend right, "N_{P_x}(v)"'] & \\
& I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "W_{B_I}^{-1}(h_{B_I}\text{-}T_{B_I}x)"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ul, bend right, "\arg\min_{y\in P_x} q^\top x\text{ for }q{\in}-\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i{\in}\mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top{\in}N\}"]
\end{tikzcd}
$$

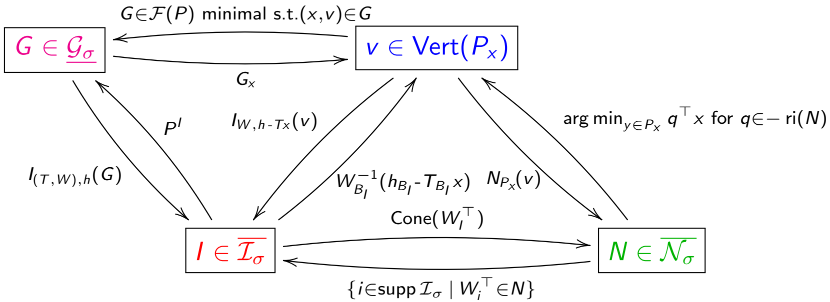


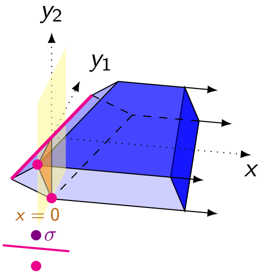
$\overline{\mathcal{I}_\sigma} =$
$\{53, 36, 65\}$



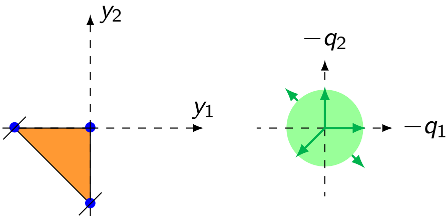


Correspondences



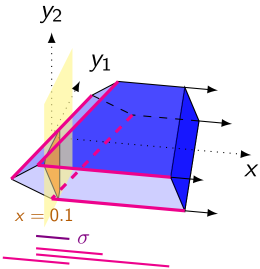


$\overline{\mathcal{I}_\sigma} = \{523, 346, 65\}$

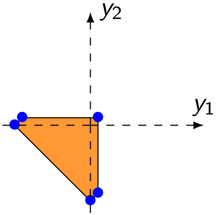


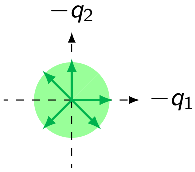
Correspondences

$$
\begin{tikzcd}
G \in \mathcal{G}_\sigma \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & v \in \mathrm{Vert}(P_x) \arrow[ll, bend right, "G\in\mathcal{F}(P) \text{ minimal s.t.} (x,v)\in G"'] \arrow[dl, bend right, "I_{W,h\text{-}Tx}(v)"'] \arrow[dr, bend left, "N_{P_x}(v)"] & \\
& I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "W_{B_I}^{-1}(h_{B_I}\text{-}T_{B_I}x)"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ul, bend right, "\arg\min_{y\in P_x} q^\top x \text{ for } q\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in\mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top \in N\}"]
\end{tikzcd}
$$



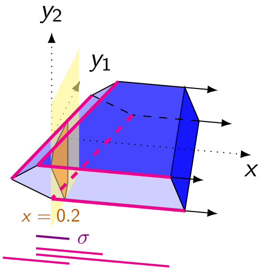
$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 65\}$



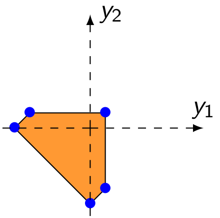


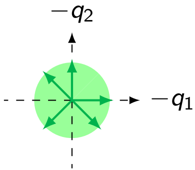
Correspondences

$$
\begin{tikzcd}
G \in \mathcal{G}_\sigma \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & v \in \mathrm{Vert}(P_x) \arrow[ll, bend right, "G\in\mathcal{F}(P)\ \text{minimal s.t.}\,(x,v)\in G"'] \arrow[dl, bend right, "I_{W,h\text{-}Tx}(v)"'] \arrow[dr, bend left, "N_{P_x}(v)"] & \\
& I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "W_{B_I}^{-1}(h_{B_I}\text{-}T_{B_I}x)"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ll, bend left, "\{i\in\operatorname{supp}\mathcal{I}_\sigma \mid W_i^\top\in N\}"] \arrow[ul, bend right, "\arg\min_{y\in P_x} q^\top x\ \text{for}\ q\in -\mathrm{ri}(N)"']
\end{tikzcd}
$$



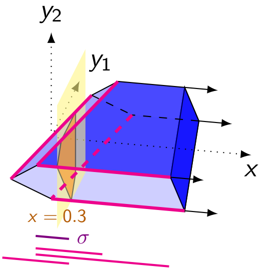
$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 65\}$



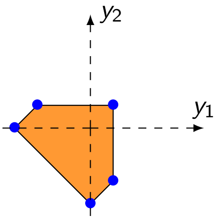


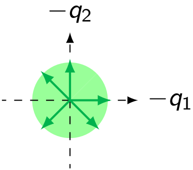
Correspondences

$$
\begin{tikzcd}
\boxed{G \in \underline{\mathcal{G}_\sigma}} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & \boxed{v \in \mathrm{Vert}(P_x)} \arrow[ll, bend right, "G\in\mathcal{F}(P) \text{ minimal s.t.} (x,v)\in G"'] \arrow[dl, bend right, "I_{W,h\text{-}Tx}(v)"'] \arrow[dr, bend left, "N_{P_x}(v)"] & \\
& \boxed{I \in \overline{\mathcal{I}_\sigma}} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "W_{B_I}^{-1}(h_{B_I}\text{-}T_{B_I}x)"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & \boxed{N \in \overline{\mathcal{N}_\sigma}} \arrow[ll, bend left, "\{i\in\mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top\in N\}"] \arrow[ul, bend left, "\arg\min_{y\in P_x} q^\top x \text{ for } q\in-\mathrm{ri}(N)"]
\end{tikzcd}
$$



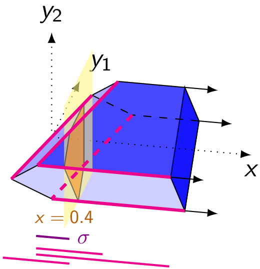
$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 65\}$



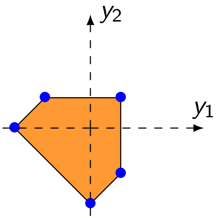


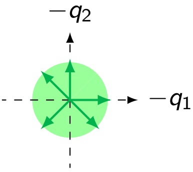
Correspondences

$$
\begin{tikzcd}
G \in \underline{\mathcal{G}_\sigma} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & v \in \mathrm{Vert}(P_x) \arrow[ll, bend right, "G \in \mathcal{F}(P) \text{ minimal s.t.} (x,v) \in G"'] \arrow[dl, bend right, "I_{W,h\text{-}Tx}(v)"'] \arrow[dr, bend right, "N_{P_x}(v)"'] & \\
& I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "W_{B_I}^{-1}(h_{B_I}\text{-}T_{B_I}x)"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ul, bend right, "\arg\min_{y \in P_x} q^\top x \text{ for } q \in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i \in \mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top \in N\}"]
\end{tikzcd}
$$



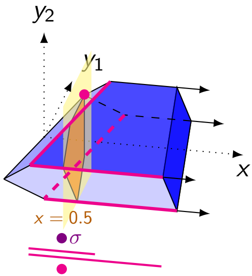
$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 65\}$



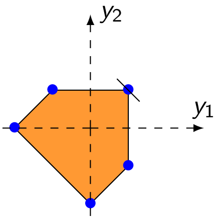


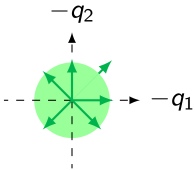
Correspondences

$$
\begin{tikzcd}[column sep=large, row sep=huge]
G \in \mathcal{G}_\sigma \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & v \in \mathrm{Vert}(P_x) \arrow[ll, bend right, "G\in\mathcal{F}(P)\text{ minimal s.t.}(x,v)\in G"'] \arrow[dl, bend right, "I_{W,h\text{-}Tx}(v)"'] \arrow[dr, bend left, "N_{P_x}(v)"] & \\
& I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "W_{B_I}^{-1}(h_{B_I}\text{-}T_{B_I}x)"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ul, bend left, "\arg\min_{y\in P_x} q^\top x \text{ for } q\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in\mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top\in N\}"]
\end{tikzcd}
$$



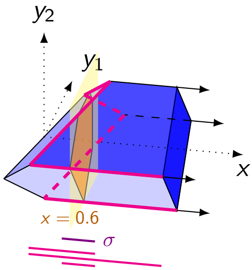
$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 615\}$



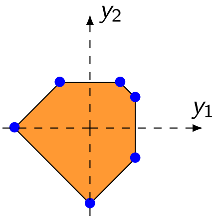


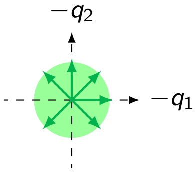
Correspondences

$$\begin{tikzcd}
\boxed{G \in \mathcal{G}_\sigma} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & \boxed{v \in \mathrm{Vert}(P_x)} \arrow[ll, bend right, "G \in \mathcal{F}(P) \text{ minimal s.t.} (x,v) \in G"'] \arrow[dl, bend right, "I_{W,h\text{-}Tx}(v)"'] \arrow[dr, bend right, "N_{P_x}(v)"'] & \\
& \boxed{I \in \overline{\mathcal{I}_\sigma}} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "W_{B_I}^{-1}(h_{B_I}\text{-}T_{B_I}x)"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & \boxed{N \in \overline{\mathcal{N}_\sigma}} \arrow[ul, bend right, "\arg\min_{y \in P_x} q^\top x \text{ for } q \in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i \in \mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top \in N\}"]
\end{tikzcd}$$

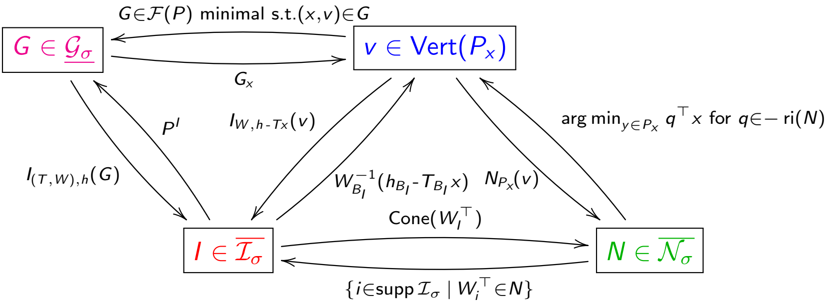


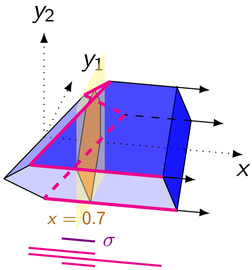
$\overline{\mathcal{I}_\sigma} =$
$\{52, 23, 34,$
$46, 61, 15\}$



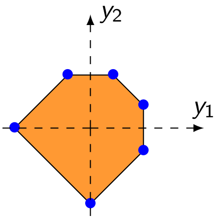


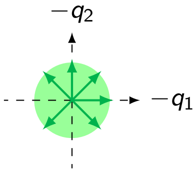
Correspondences





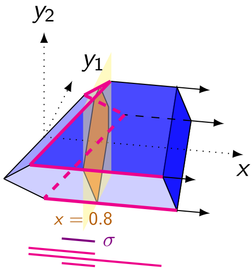
$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 61, 15\}$



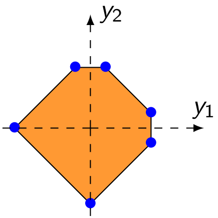


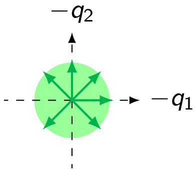
Correspondences

$$\begin{tikzcd}
G \in \mathcal{G}_\sigma \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & v \in \mathrm{Vert}(P_x) \arrow[ll, bend right, "G \in \mathcal{F}(P) \text{ minimal s.t.} (x,v) \in G"'] \arrow[dl, bend right, "I_{W,h\text{-}Tx}(v)"'] \arrow[dr, bend right, "N_{P_x}(v)"'] & \\
& I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "W_{B_I}^{-1}(h_{B_I}\text{-}T_{B_I}x)"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ul, bend right, "\arg\min_{y \in P_x} q^\top x \text{ for } q \in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i \in \mathrm{supp}\, \mathcal{I}_\sigma \mid W_i^\top \in N\}"]
\end{tikzcd}$$

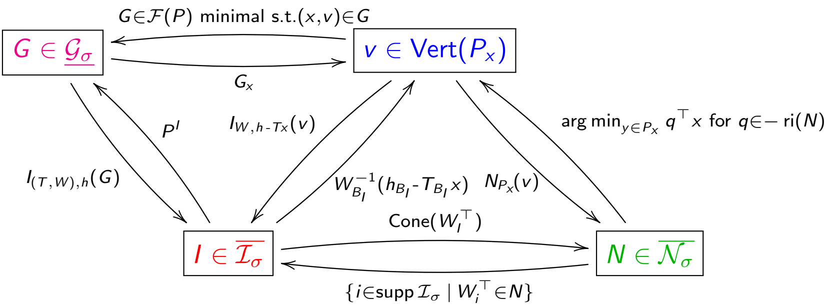


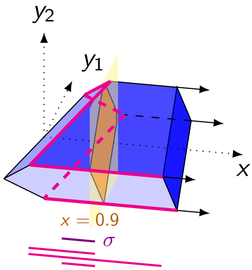
$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 61, 15\}$



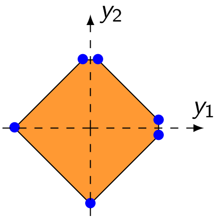


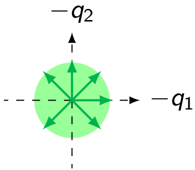
Correspondences





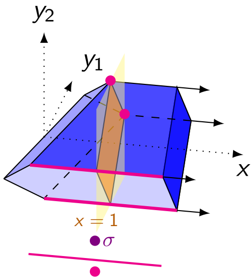
$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 61, 15\}$



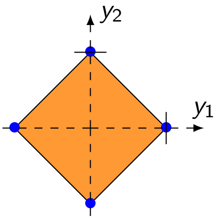


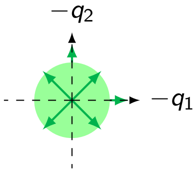
Correspondences

$$\begin{tikzcd}
G \in \mathcal{G}_\sigma \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & v \in \mathrm{Vert}(P_x) \arrow[ll, bend right, "G\in\mathcal{F}(P) \text{ minimal s.t.} (x,v)\in G"'] \arrow[dl, bend right, "I_{W,h\text{-}Tx}(v)"'] \arrow[dr, bend right, "N_{P_x}(v)"'] & \\
& I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "W_{B_I}^{-1}(h_{B_I}\text{-}T_{B_I}x)"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ul, bend right, "\arg\min_{y\in P_x} q^\top x \text{ for } q\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in\mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top\in N\}"]
\end{tikzcd}$$



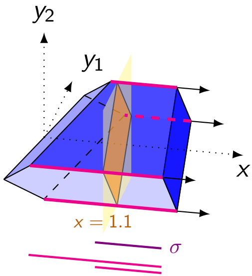
$\overline{\mathcal{I}_\sigma} = \{152, 23, 34, 461\}$



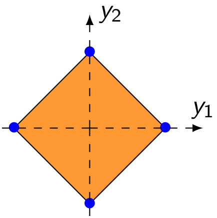


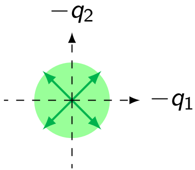
Correspondences

$$\begin{tikzcd}[column sep=huge, row sep=huge]
G \in \mathcal{G}_\sigma \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & v \in \mathrm{Vert}(P_x) \arrow[ll, bend right, "G \in \mathcal{F}(P) \text{ minimal s.t.} (x,v) \in G"'] \arrow[dl, bend right, "I_{W,h\text{-}Tx}(v)"'] \arrow[dr, bend right, "N_{P_x}(v)"'] & \\
& I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "W_{B_I}^{-1}(h_{B_I}\text{-}T_{B_I}x)"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ul, bend right, "\arg\min_{y \in P_x} q^\top x \text{ for } q \in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i \in \mathrm{supp}\, \mathcal{I}_\sigma \mid W_i^\top \in N\}"]
\end{tikzcd}$$



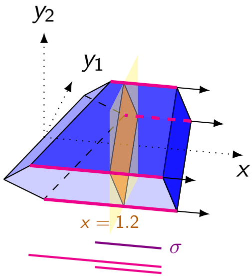
$\overline{\mathcal{I}_\sigma} =$
$\{12, 23, 34, 41\}$



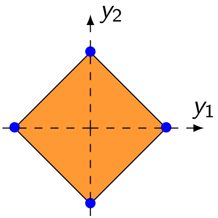


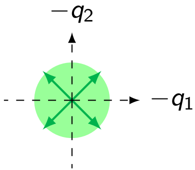
Correspondences

$$
\begin{tikzcd}
\color{magenta}{G \in \underline{\mathcal{G}_\sigma}} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & \color{blue}{v \in \mathrm{Vert}(P_x)} \arrow[ll, bend right, "G \in \mathcal{F}(P) \text{ minimal s.t.} (x,v) \in G"'] \arrow[dl, bend right, "I_{W,h\text{-}Tx}(v)"'] \arrow[dr, bend left, "N_{P_x}(v)"] & \\
& \color{red}{I \in \overline{\mathcal{I}_\sigma}} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "W_{B_I}^{-1}(h_{B_I}\text{-}T_{B_I}x)"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & \color{green}{N \in \overline{\mathcal{N}_\sigma}} \arrow[ul, bend left, "\arg\min_{y \in P_x} q^\top x \text{ for } q \in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i \in \mathrm{supp}\, \mathcal{I}_\sigma \mid W_i^\top \in N\}"]
\end{tikzcd}
$$



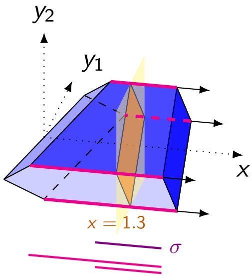
$\overline{\mathcal{I}_\sigma} =$
$\{12, 23, 34, 41\}$



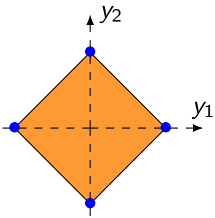


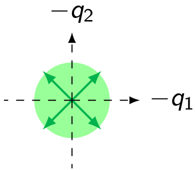
Correspondences

$$
\begin{tikzcd}
G \in \underline{\mathcal{G}_\sigma} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & v \in \mathrm{Vert}(P_x) \arrow[ll, bend right, "G \in \mathcal{F}(P) \text{ minimal s.t.} (x,v) \in G"'] \arrow[dl, bend right, "I_{W,h\text{-}Tx}(v)"'] \arrow[dr, bend right, "N_{P_x}(v)"'] & \\
& I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "W_{B_I}^{-1}(h_{B_I}\text{-}T_{B_I}x)"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ul, bend right, "\arg\min_{y \in P_x} q^\top x \text{ for } q \in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i \in \mathrm{supp}\, \mathcal{I}_\sigma \mid W_i^\top \in N\}"]
\end{tikzcd}
$$



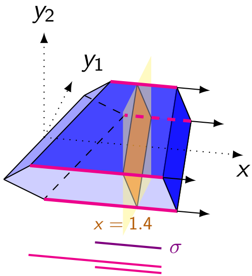
$\overline{\mathcal{I}_\sigma} =$
$\{12, 23, 34, 41\}$



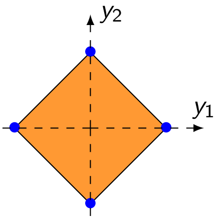


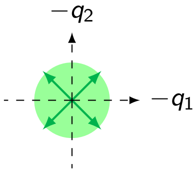
Correspondences

$$\begin{tikzcd}[column sep=huge, row sep=huge]
G \in \mathcal{G}_\sigma \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & v \in \mathrm{Vert}(P_x) \arrow[ll, bend right, "G \in \mathcal{F}(P) \text{ minimal s.t.} (x,v) \in G"'] \arrow[dl, bend right, "I_{W,h\text{-}Tx}(v)"'] \arrow[dr, bend right, "N_{P_x}(v)"'] & \\
& I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "W_{B_I}^{-1}(h_{B_I}\text{-}T_{B_I}x)"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ul, bend right, "\arg\min_{y \in P_x} q^\top x \text{ for } q \in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i \in \mathrm{supp}\, \mathcal{I}_\sigma \mid W_i^\top \in N\}"]
\end{tikzcd}$$



$\overline{\mathcal{I}_\sigma} =$
$\{12, 23, 34, 41\}$





H-representation of projection of faces

Let $I \in \mathcal{I}\big((T, W), h\big)$ be a set of indices

$$x \in \pi(P^I) \iff \begin{cases} \exists y \in \mathbb{R}^m, & (x, y) \in P^I \\ & \end{cases}$$

H-representation of projection of faces

Let $I \in \mathcal{I}\big((T, W), h\big)$ be a set of indices

$$x \in \pi(P^I) \iff \begin{cases} \exists y \in \mathbb{R}^m, & T_I x + W_I y = h_I \\ \quad \forall j \in [q] \backslash I, & T_j x + W_j y \leqslant h_j \end{cases} \Longleftarrow I \in \mathcal{I}(W, h - Tx)$$

H-representation of projection of faces

Let $I \in \mathcal{I}\big((T, W), h\big)$ be a set of indices

$$x \in \operatorname{ri} \pi(P^I) \iff \begin{cases} \exists y \in \mathbb{R}^m, & T_I x + W_I y = h_I \\ \quad \forall j \in [q] \backslash I, & T_j x + W_j y < h_j \end{cases} \iff I \in \mathcal{I}(W, h - Tx)$$

H-representation of projection of faces

Let $I \in \mathcal{I}\big((T, W), h\big)$ be a set of indices from which we can extract a basis (i.e. $\mathrm{rg}(W_I^\top) = m$) and let B such a basis

$$x \in \operatorname{ri} \pi(P^I) \iff \begin{cases} \exists y \in \mathbb{R}^m, & T_B x + W_B y = h_B \\ \forall i \in I \backslash B, & T_i x + W_i y = h_i \\ \forall j \in [q] \backslash I, & T_j x + W_j y < h_j \end{cases} \iff I \in \mathcal{I}(W, h - Tx)$$

H-representation of projection of faces

Let $I \in \mathcal{I}\big((T, W), h\big)$ be a set of indices from which we can extract a basis (i.e. $\mathrm{rg}(W_I^\top) = m$) and let B such a basis

$$x \in \operatorname{ri} \pi(P^I) \iff \begin{cases} \exists y \in \mathbb{R}^m, & y = W_B^{-1}(h_B - T_B x) \\ \forall i \in I \backslash B, & T_i x + W_i y = h_i \\ \forall j \in [q] \backslash I, & T_j x + W_j y < h_j \end{cases} \iff I \in \mathcal{I}(W, h - Tx)$$

H-representation of projection of faces

Let $I \in \mathcal{I}\big((T, W), h\big)$ be a set of indices from which we can extract a basis (i.e. $\mathrm{rg}(W_I^\top) = m$) and let B such a basis

$$x \in \operatorname{ri} \pi(P^I) \iff \begin{cases} \forall i \in I \backslash B, & T_i x + W_i W_B^{-1}(h_B - T_B x) = h_i \\ \forall j \in [q] \backslash I, & T_j x + W_j W_B^{-1}(h_B - T_B x) < h_j \end{cases}$$

H-representation of projection of faces

Let $I \in \mathcal{I}\big((T, W), h\big)$ be a set of indices from which we can extract a basis (i.e. $\mathrm{rg}(W_I^\top) = m$) and let B such a basis

$$x \in \mathrm{ri}\left(\pi(P^I)\right) \iff \begin{cases} \forall i \in I \backslash B, & (v_i^B)^\top x = u_i^B \\ \forall j \in [q] \backslash I, & (v_j^B)^\top x < u_j^B \end{cases} \iff I \in \mathcal{I}(W, h - Tx)$$

where

$$v_i^B := (T_i - W_i W_B^{-1} T_B)^\top$$
$$u_i^B := h_i - W_i W_B^{-1} h_B$$

H-representation of chambers

Let $\sigma \in \mathcal{C}(P, \pi)$

$$x \in \bigcap_{I \in \overline{\mathcal{I}_\sigma}} \mathrm{ri}\left(\pi(P^I)\right) \iff \begin{cases} \forall I \in \overline{\mathcal{I}_\sigma}, \\ \forall i \in I \backslash B_I, \quad (v_i^{B_I})^\top x = u_i^{B_I} \\ \forall j \in [q] \backslash I, \quad (v_j^{B_I})^\top x < u_j^{B_I} \end{cases} \iff \mathcal{I}(W, h - Tx) = \mathcal{I}_\sigma$$

where

$$\begin{aligned} v_i^B &:= (T_i - W_i W_B^{-1} T_B)^\top \\ u_i^B &:= h_i - W_i W_B^{-1} h_B \end{aligned}$$

with B_I basis $\subset I$ and

$$\begin{aligned} \mathcal{G}_\sigma &:= \{F \in \mathcal{F}(P) \mid \mathrm{ri}(\sigma) \subset \mathrm{ri}\left(\pi(F)\right)\} \\ \mathcal{I}_\sigma &:= \{I \in \mathcal{I}\big((T, W), h\big) \mid \mathrm{ri}(\sigma) \subset \mathrm{ri}\left(\pi(P^I)\right)\} \end{aligned}$$

We have $\sigma = \cap_{G \in \mathcal{G}_\sigma} \pi(G) = \cap_{I \in \mathcal{I}_\sigma} \pi(P^I)$

H-representation of chambers

Let $\sigma \in \mathcal{C}(P, \pi)$

$$x \in \mathrm{ri}(\sigma) \iff \begin{cases} \forall I \in \overline{\mathcal{I}_\sigma}, \\ \forall i \in I \backslash B_I, \quad (v_i^{B_I})^\top x = u_i^{B_I} \\ \forall j \in [q] \backslash I, \quad (v_j^{B_I})^\top x < u_j^{B_I} \end{cases} \iff \mathcal{I}(W, h - Tx) = \mathcal{I}_\sigma$$

where

$$v_i^B := (T_i - W_i W_B^{-1} T_B)^\top$$
$$u_i^B := h_i - W_i W_B^{-1} h_B$$

with B_I basis $\subset I$ and

$$\mathcal{G}_\sigma := \{F \in \mathcal{F}(P) \mid \mathrm{ri}(\sigma) \subset \mathrm{ri}\left(\pi(F)\right)\}$$
$$\mathcal{I}_\sigma := \{I \in \mathcal{I}\big((T, W), h\big) \mid \mathrm{ri}(\sigma) \subset \mathrm{ri}\left(\pi(P^I)\right)\}$$

We have $\sigma = \cap_{G \in \underline{\mathcal{G}_\sigma}} \pi(G) = \cap_{I \in \overline{\mathcal{I}_\sigma}} \pi(P^I)$