

# 2-Stage Stochastic Linear Problem and Polyhedral Geometry

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# Linear Programming

$$\begin{array}{ll}\min_{x \in \mathbb{R}^n} & c^\top x \\ \text{s.t.} & Ax \leq b\end{array}$$

Example:  $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$

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(1)

(2)

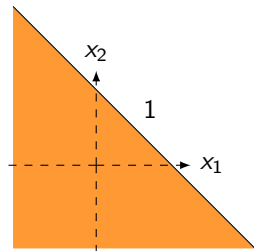
(3)

(4)

(5)

(6)

(7)

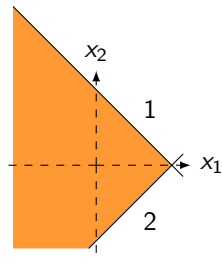


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$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{array}{ll} x_1 + x_2 \leq 1 & (1) \\ x_1 - x_2 \leq 1 & (2) \end{array}$$



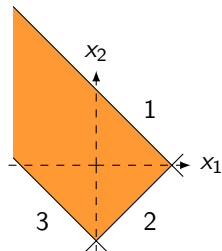
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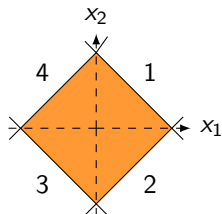
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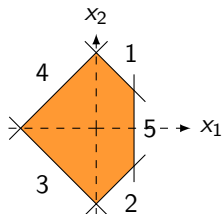


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$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0.5 \end{pmatrix}$$
$$\begin{array}{ll}x_1 + x_2 \leq 1 & (1) \\ x_1 - x_2 \leq 1 & (2) \\ -x_1 - x_2 \leq 1 & (3) \\ -x_1 + x_2 \leq 1 & (4) \\ x_1 \leq 0.5 & (5) \\ & (6) \\ & (7)\end{array}$$

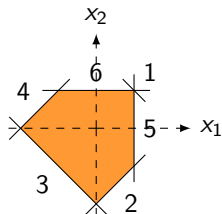


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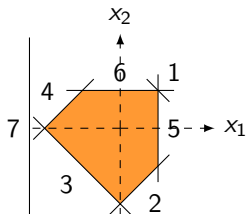


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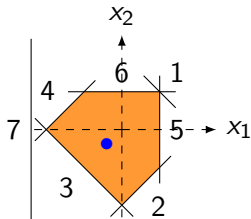
# Active constraints

## Definition

We denote by  $\mathcal{I}(A, b)$ , the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leq b\}$$

with  $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$$I_{A,b}(x) = \emptyset$$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, \quad \quad \quad \}$$

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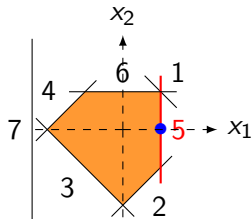
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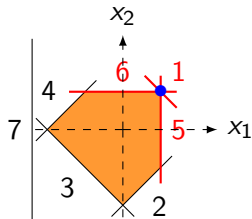
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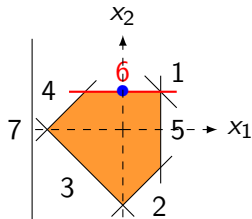
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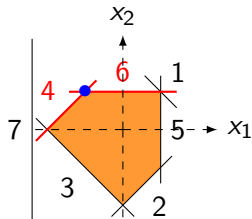
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$$I_{A,b}(x) = \{4, 6\}$$

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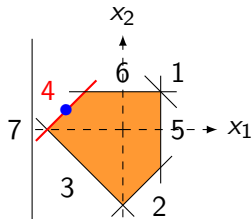
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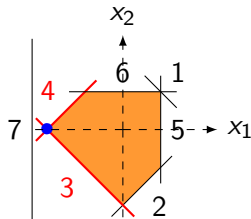
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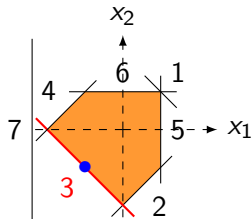
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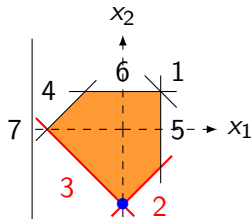
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$$I_{A,b}(x) = \{2, 3\}$$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, 34, 3, 23, \quad \}$$

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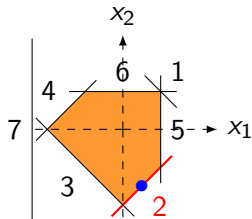
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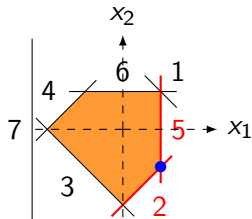
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$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, 34, 3, 23, 2, 25\}$$

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# Faces

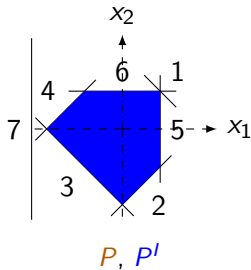
## Definition

Let  $I \in \mathcal{I}(A, b)$ , we denote by  $P^I$  the face of  $P$  such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have  $\dim(P^I) = n - \text{rg}(A_I)$

Example for  $I = \emptyset$



# Faces

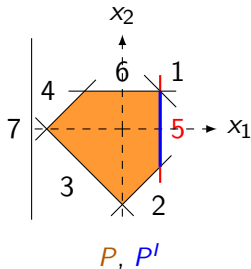
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Example for  $I = \{5\}$



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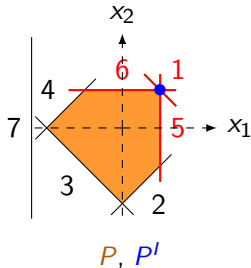
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Example for  $I = \{1, 5, 6\}$



# Faces

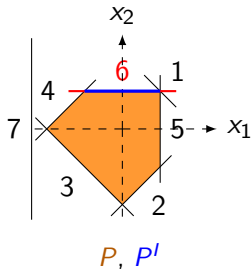
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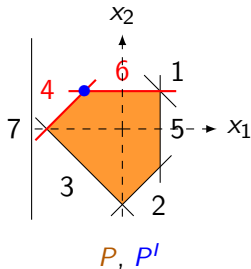
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Example for  $I = \{4, 6\}$



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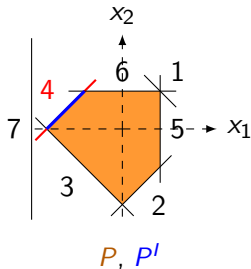
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Example for  $I = \{4\}$



# Faces

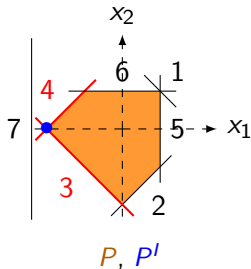
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Example for  $I = \{3, 4\}$



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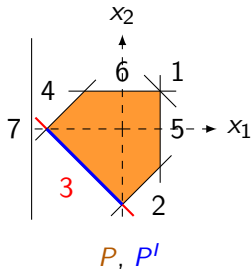
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# Faces

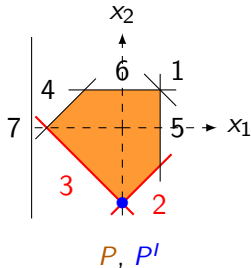
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Example for  $I = \{2, 3\}$



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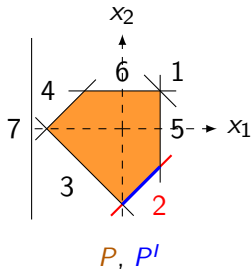
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Example for  $I = \{2\}$



# Faces

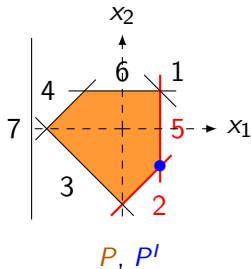
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Example for  $I = \{2, 5\}$

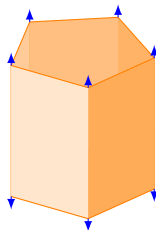
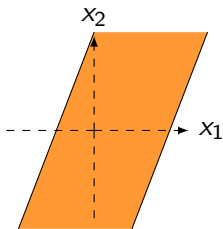


# Polyhedra without any vertex ?

## Definition (Lineality space)

$\text{Lin}(C) := \{u \in C \mid \forall t \in \mathbb{R}, \forall x \in c, x + tu \in C\}.$

$$\text{Lin}(\{x \mid Ax \leq b\}) = \text{Ker}(A)$$





# Bases and Vertices

Let  $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$  with  $A \in \mathbb{R}^{p \times n}$  and  $b \in \mathbb{R}^p$ .

## Definition

*A basis  $B$  is a subset of  $[p]$  such that  $A_B = (A_{i,j})_{i \in B, 1 \leq j \leq n}$  is invertible.  
A vertex of  $P$  is a face of dimension 0.  $\text{Vert}(P)$  is the set of vertices.*

$$\begin{aligned}\text{Vert}(P) \neq \emptyset &\iff A \text{ admits at least one basis} \\ &\iff \text{rg}(A) = n \\ &\iff \text{Lin}(P) = \{0\}\end{aligned}$$

Under this assumption,

For every  $I \in \overline{\mathcal{I}(A, b)}$ , we can extract a basis  $B_I$  and  $P^I = \{A_{B_I}^{-1} b_{B_I}\}$ .

If  $c \notin \text{Lin}(P)^\perp = \text{Im}(A^\top)$ ,  $\min_{x \in P} c^\top x = -\infty$ .

Otherwise, we can write  $P = P_0 + \text{Lin}(P)$  with  $\text{Lin}(P_0) = \{0\}$ :

We make this assumption without loss of generality

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Otherwise, we can write  $P = P_0 + \text{Lin}(P)$  with  $\text{Lin}(P_0) = \{0\}$ :

We make this assumption without loss of generality

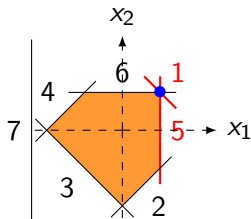
# Simplex method

Geometrically:

follow a path on the polyhedron from  
vertex to vertex

Combinatorially:

pivoting from basis to basis



$$B_1 = \{1, 5\}$$

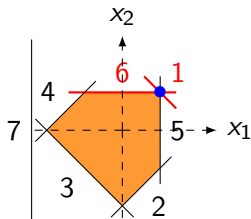
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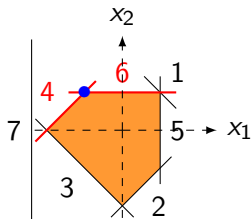
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$$B_3 = \{4, 6\}$$

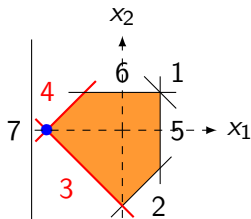
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Combinatorially:

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$$B_3 = \{4, 6\}$$

$$B_2 = \{3, 4\}$$

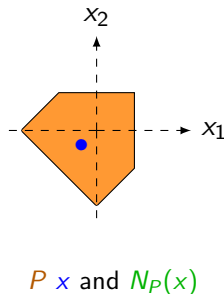
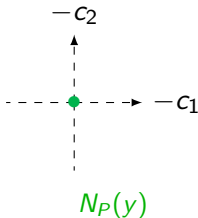
# Normal fan $\mathcal{N}(P)$

## Definition

The normal fan of the polyhedron  $P$  is

$$\mathcal{N}(P) := \{N_P(x) \mid x \in P\}$$

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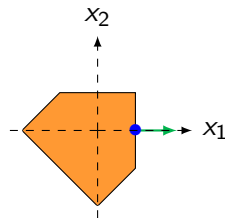
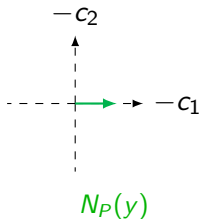
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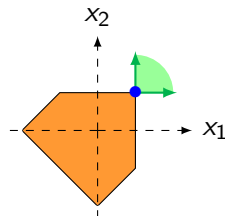
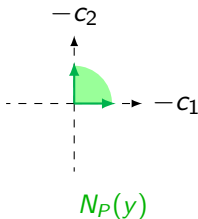
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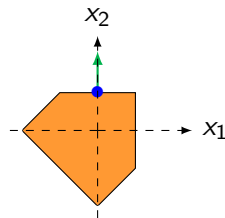
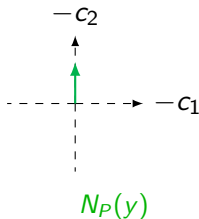
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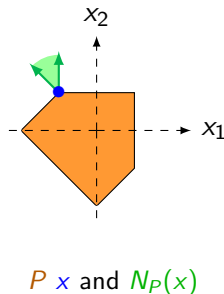
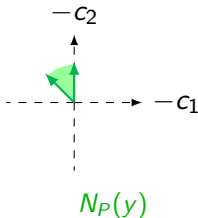
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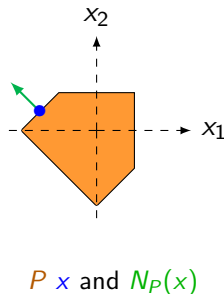
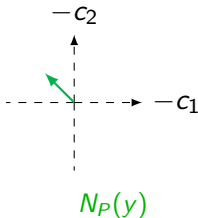
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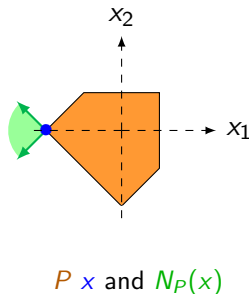
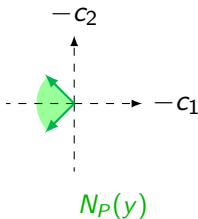
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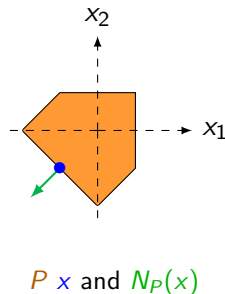
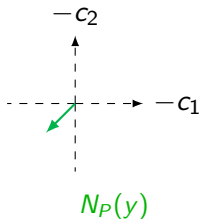
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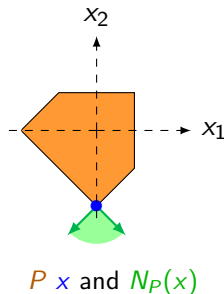
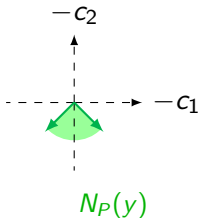
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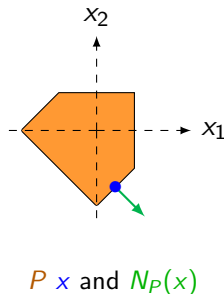
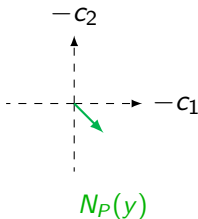
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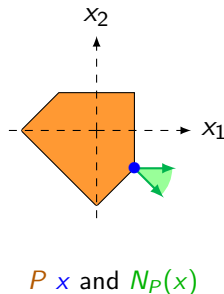
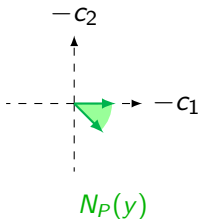
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## Proposition

$\{\text{ri}(N) \mid N \in \mathcal{N}(P)\}$  is a partition of  $\text{supp } \mathcal{N}(P)$  ( $= \mathbb{R}^m$  if  $P$  is bounded).



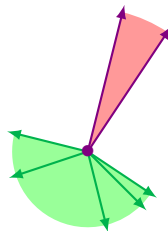
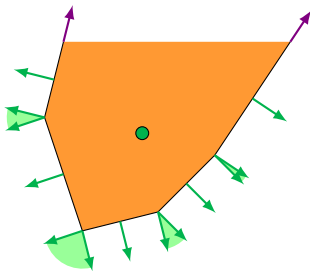
## Definition (Recession cone)

$$\text{rc}(C) := \{u \in C \mid \forall t \in \mathbb{R}_+, \forall x \in C, x + tu \in C\}.$$

$$\text{Let } P = \{x \mid Ax \leq b\}$$

$$\text{rc}(P) = \{u \mid Au \leq 0\}$$

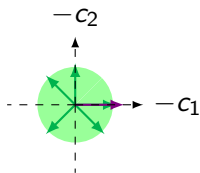
$$-\infty < \begin{cases} \inf_{x \in \mathbb{R}^n} & c^\top x \\ \text{s.t.} & Ax \leq b \end{cases} \iff -c \in \text{rc}(P)^* = \text{Cone}(A^\top) = \text{supp}(\mathcal{N}(P))$$



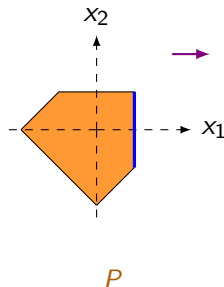
## $\mathcal{N}(P)$ : partition of cost coherent with the min

For any  $N \in \mathcal{N}(P)$  and  $-\mathbf{c} \rightarrow \arg \min_{x \in P} \mathbf{c}^\top x$  is constant for all  $-\mathbf{c} \in \text{ri}(N)$ .

$\arg \min_{x \in P} \mathbf{c}^\top x$  is a face of  $P$ .



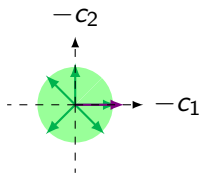
Cost  $-\mathbf{c}$  and  $\mathcal{N}(P)$



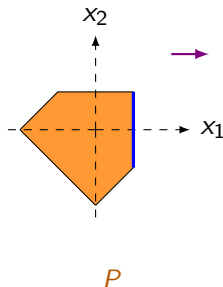
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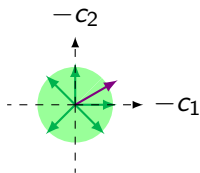
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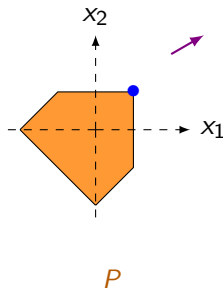
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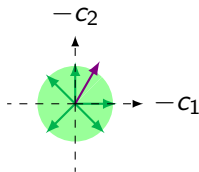
Cost  $-c$  and  $\mathcal{N}(P)$



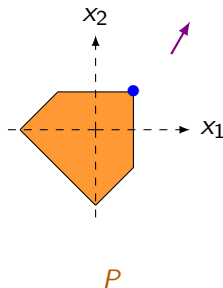
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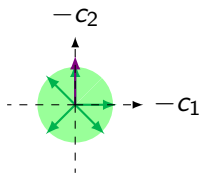
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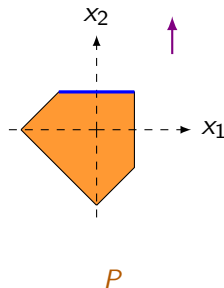
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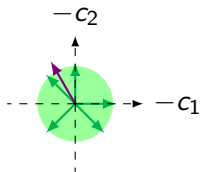




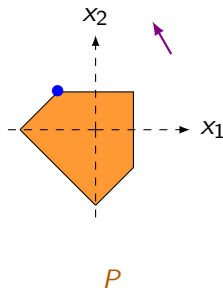
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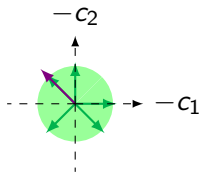
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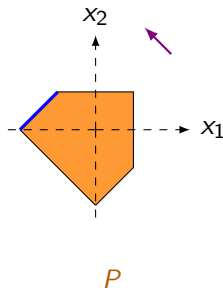
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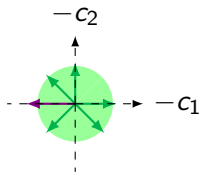
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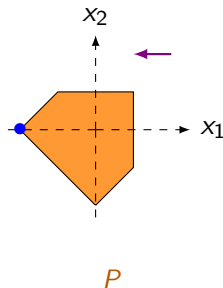
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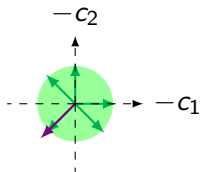
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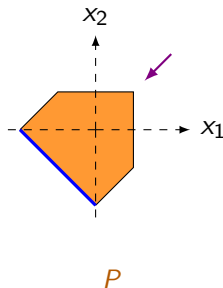
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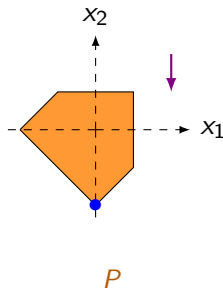
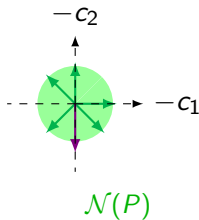
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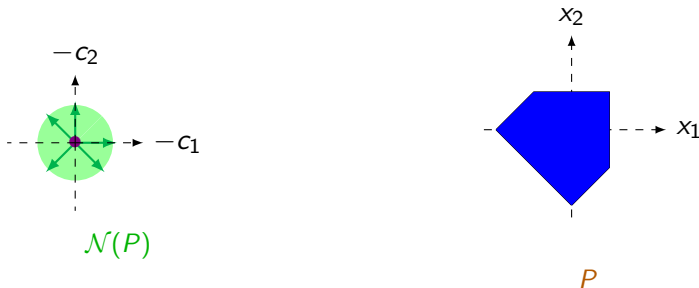
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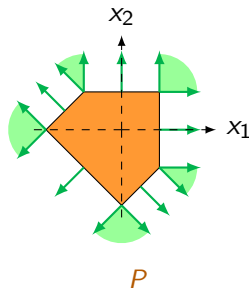
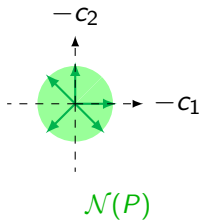




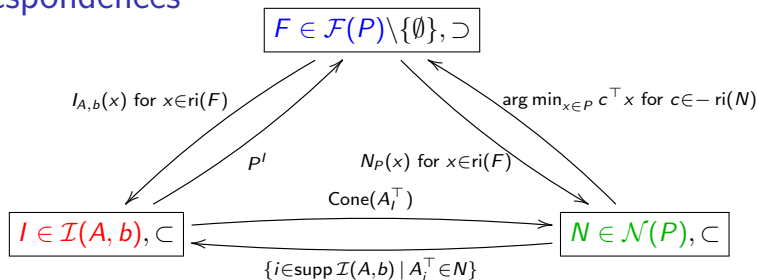
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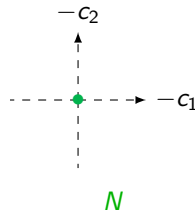
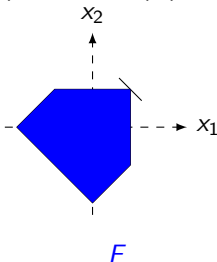


# Correspondences

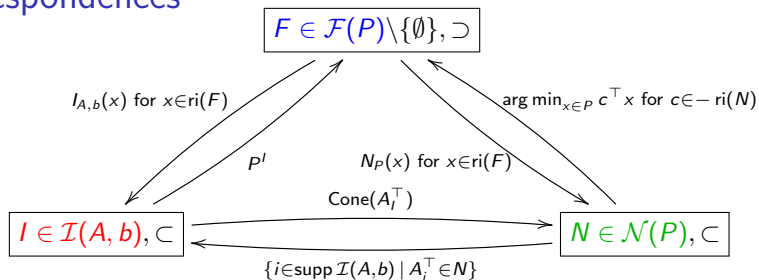


$$\text{rg}(A_I) = n - \dim(F) = \dim(N)$$

$$I = \emptyset$$

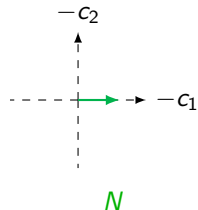
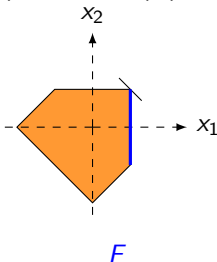


# Correspondences

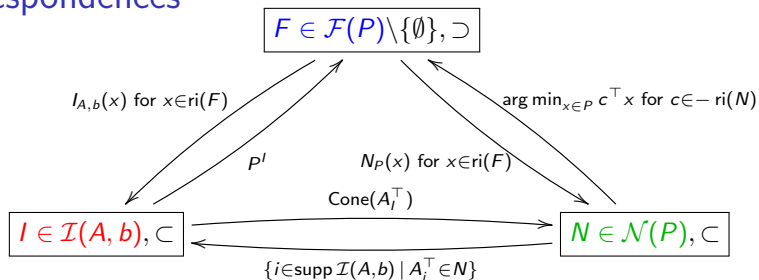


$$\text{rg}(A_I) = n - \dim(F) = \dim(N)$$

$$I = \{5\}$$

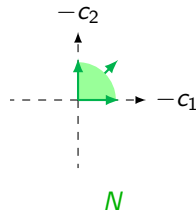
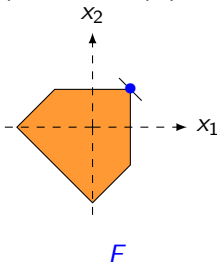


# Correspondences

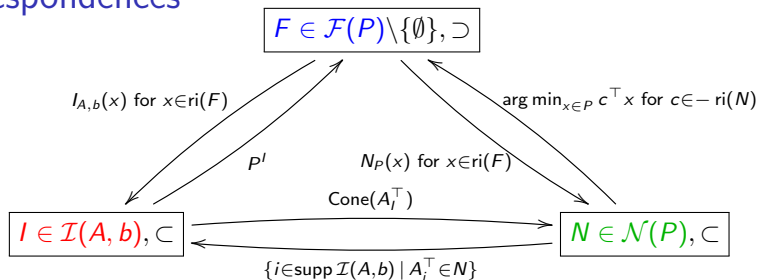


$$\text{rg}(A_I) = n - \dim(F) = \dim(N)$$

$$I = \{1, 5, 6\}$$

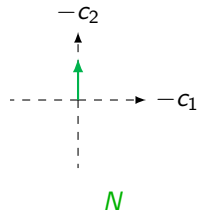
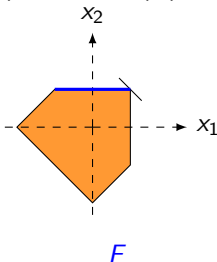


# Correspondences

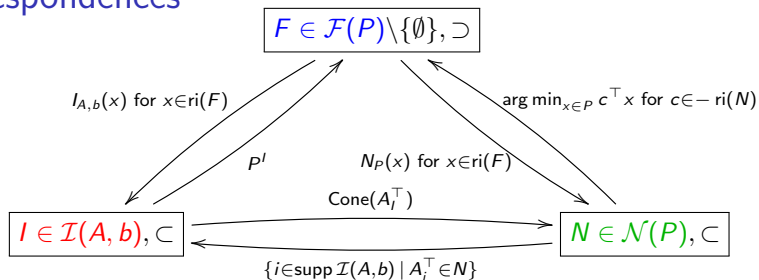


$$\text{rg}(A_I) = n - \dim(F) = \dim(N)$$

$$I = \{6\}$$

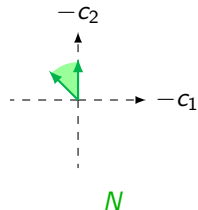
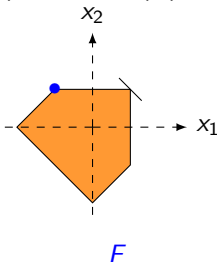


# Correspondences

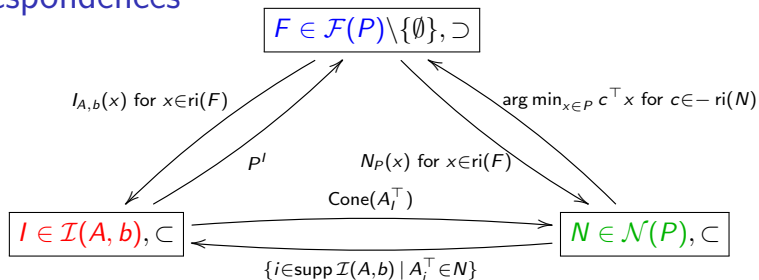


$$\text{rg}(A_I) = n - \dim(F) = \dim(N)$$

$$I = \{4, 6\}$$

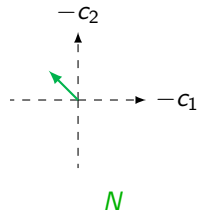
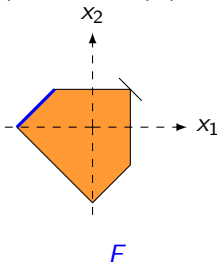


# Correspondences

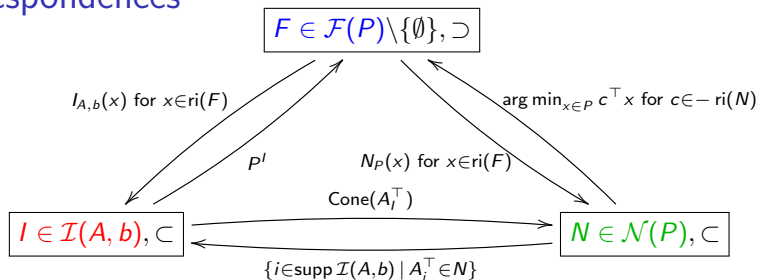


$$\text{rg}(A_I) = n - \dim(F) = \dim(N)$$

$$I = \{4\}$$

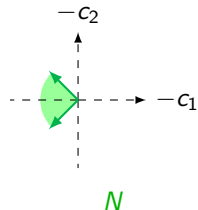
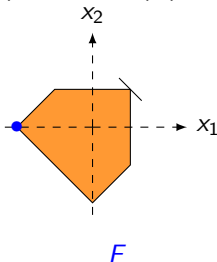


# Correspondences



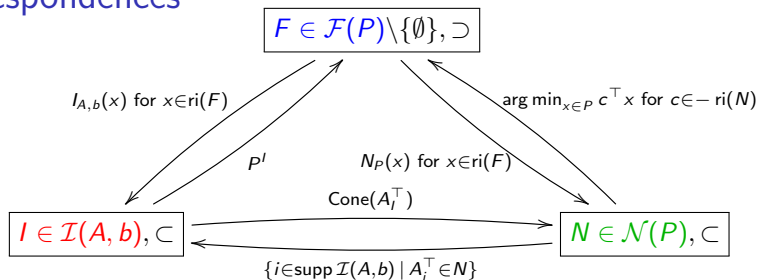
$$\text{rg}(A_I) = n - \dim(F) = \dim(N)$$

$$I = \{3, 4\}$$



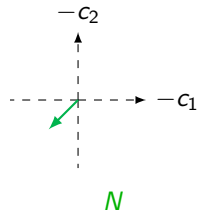
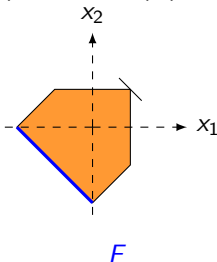


# Correspondences

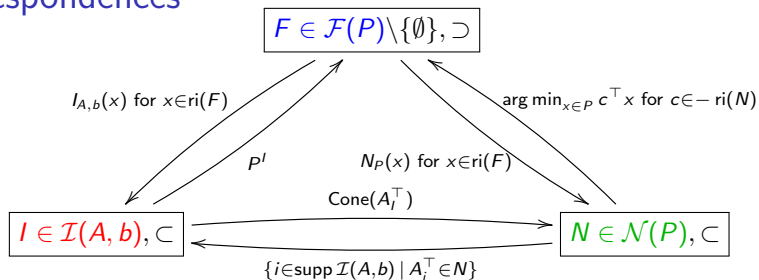


$$\text{rg}(A_I) = n - \dim(F) = \dim(N)$$

$$I = \{3\}$$

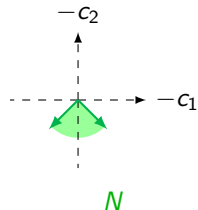
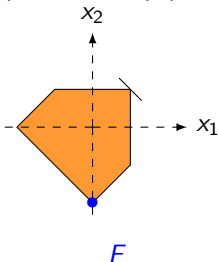


# Correspondences

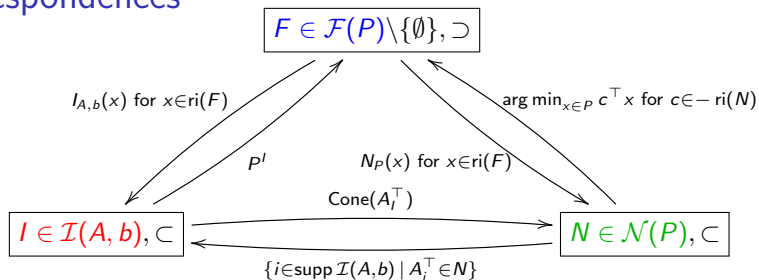


$$\text{rg}(A_I) = n - \dim(F) = \dim(N)$$

$$I = \{2, 3\}$$

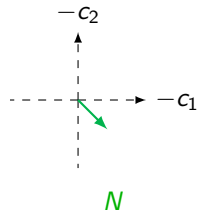
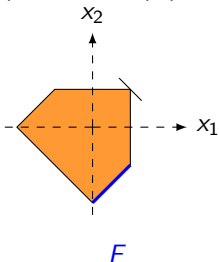


# Correspondences

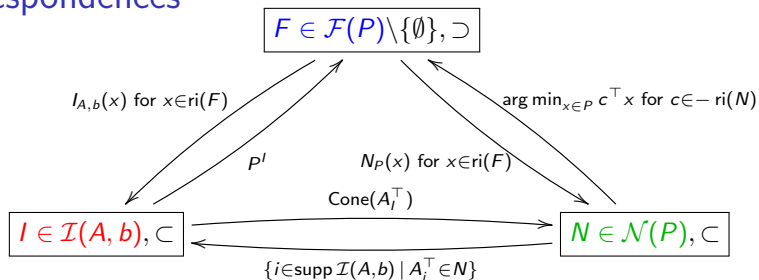


$$\text{rg}(A_I) = n - \dim(F) = \dim(N)$$

$$I = \{2\}$$

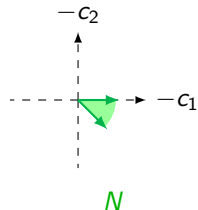
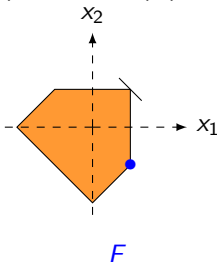


# Correspondences

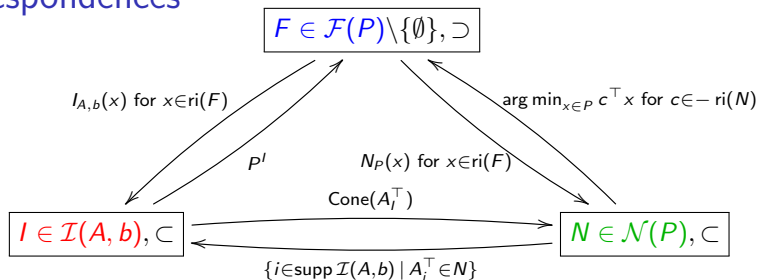


$$\text{rg}(A_I) = n - \dim(F) = \dim(N)$$

$$I = \{2, 5\}$$

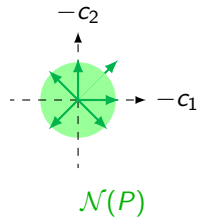
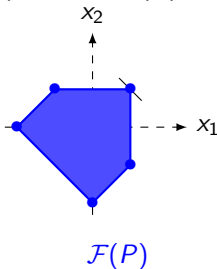


# Correspondences

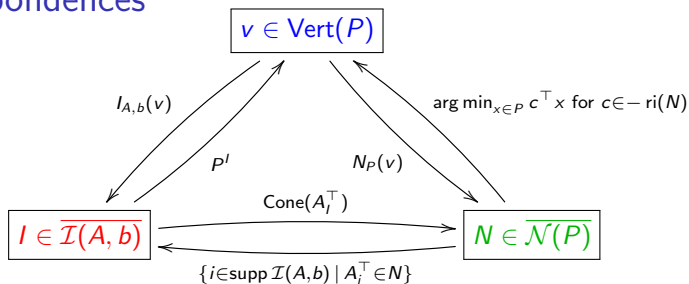


$$\text{rg}(A_I) = n - \dim(F) = \dim(N)$$

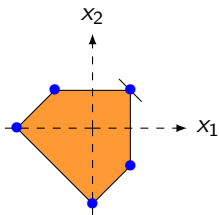
$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, 34, 3, 23, 2, 25\}$$



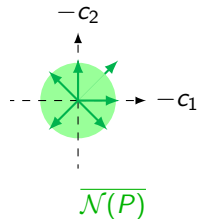
# Correspondences



$$\overline{\mathcal{I}(A, b)} = \{156, 46, 34, 23, 25\}$$



$\text{Vert}(P)$



# Contents

## 1 Linear Programming

- Active constraints
- Normal fan
- Correspondences

## 2 2-Stage Stochastic Linear Programming

- Reduction to finite sum
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- Simplex for 2SLP

## 2-Stage Stochastic Linear Programming

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x + \mathbb{E} \left[ \begin{array}{ll} \min_{y \in \mathbb{R}^m} & \mathbf{q}^\top y \\ \text{s.t.} & Tx + Wy \leq h \end{array} \right] \\ \text{s.t.} \quad & Ax \leq b \end{aligned} \quad (2\text{SLP})$$

where  $T \in \mathbb{R}^{p \times n}$ ,  $W \in \mathbb{R}^{p \times m}$  and  $h \in \mathbb{R}^p$ .

We can assume  $A = 0$  and  $b = 0$ :

We set

$$\tilde{T} := \begin{pmatrix} T \\ A \end{pmatrix}, \quad \tilde{W} := \begin{pmatrix} W \\ 0 \end{pmatrix} \quad \text{and} \quad \tilde{h} = \begin{pmatrix} h \\ b \end{pmatrix}$$



## 2-Stage Stochastic Linear Programming

$$\min_{x \in \mathbb{R}^n} c^\top x + \mathbb{E} \left[ \begin{array}{ll} \min_{y \in \mathbb{R}^m} & \mathbf{q}^\top y \\ \text{s.t.} & Tx + Wy \leq h \\ & Ax \leq b \end{array} \right]$$

where  $T \in \mathbb{R}^{p \times n}$ ,  $W \in \mathbb{R}^{p \times m}$  and  $h \in \mathbb{R}^p$ .

We can assume  $A = 0$  and  $b = 0$ :

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$$\tilde{T} := \begin{pmatrix} T \\ A \end{pmatrix}, \quad \tilde{W} := \begin{pmatrix} W \\ 0 \end{pmatrix} \quad \text{and} \quad \tilde{h} = \begin{pmatrix} h \\ b \end{pmatrix}$$

## 2-Stage Stochastic Linear Programming

$$\min_{x \in \mathbb{R}^n} c^\top x + \mathbb{E} \left[ \begin{array}{ll} \min_{y \in \mathbb{R}^m} & \mathbf{q}^\top y \\ \text{s.t.} & Tx + Wy \leq h \\ & Ax + 0y \leq b \end{array} \right]$$

where  $T \in \mathbb{R}^{p \times n}$ ,  $W \in \mathbb{R}^{p \times m}$  and  $h \in \mathbb{R}^p$ .

We can assume  $A = 0$  and  $b = 0$ :

We set

$$\tilde{T} := \begin{pmatrix} T \\ A \end{pmatrix}, \quad \tilde{W} := \begin{pmatrix} W \\ 0 \end{pmatrix} \quad \text{and} \quad \tilde{h} = \begin{pmatrix} h \\ b \end{pmatrix}$$

## 2-Stage Stochastic Linear Programming

$$\min_{x \in \mathbb{R}^n} c^\top x + \mathbb{E} \left[ \begin{array}{ll} \min_{y \in \mathbb{R}^m} & \mathbf{q}^\top y \\ \text{s.t.} & \tilde{T}x + \tilde{W}y \leq \tilde{h} \end{array} \right]$$

where  $T \in \mathbb{R}^{p \times n}$ ,  $W \in \mathbb{R}^{p \times m}$  and  $h \in \mathbb{R}^p$ .

We can assume  $A = 0$  and  $b = 0$ :

We set

$$\tilde{T} := \begin{pmatrix} T \\ A \end{pmatrix}, \quad \tilde{W} := \begin{pmatrix} W \\ 0 \end{pmatrix} \quad \text{and} \quad \tilde{h} = \begin{pmatrix} h \\ b \end{pmatrix}$$

# 2-Stage Stochastic Linear Programming

$$\min_{x \in \mathbb{R}^n} c^\top x + V(x) \quad (2SLP)$$

where

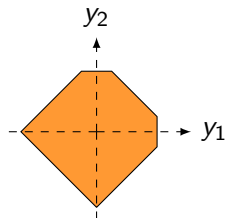
$$V(x) := \mathbb{E} \left[ \min_{y \in \mathbb{R}^m} \mathbf{q}^\top y \right. \\ \left. \text{s.t. } T x + W y \leq h \right]$$

## Fiber $P_x$

$$V(x) = \mathbb{E} \left[ \min_{y \in P_x} \mathbf{q}^\top y \right] \quad \text{where} \quad P_x := \{y \in \mathbb{R}^m \mid T x + W y \leq h\}$$

We assume  $\text{supp}(\mathbf{q}) \subset -\text{Cone}(W^\top)$  i.e.  $V(x) > -\infty$ . Example:

$$T = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix} \quad W = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



$P_x$  for  $x = 0.8$

## Fiber $P_x$

$$V(x) = \mathbb{E} \left[ \min_{y \in P_x} \mathbf{q}^\top y \right] \quad \text{where} \quad P_x := \{y \in \mathbb{R}^m \mid T x + W y \leq h\}$$

We assume  $\text{supp}(\mathbf{q}) \subset -\text{Cone}(W^\top)$  i.e.  $V(x) > -\infty$ . Example:

$$y_1 + y_2 \leq 1 \quad (1)$$

$$y_1 - y_2 \leq 1 \quad (2)$$

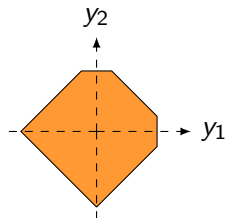
$$-y_1 - y_2 \leq 1 \quad (3)$$

$$-y_1 + y_2 \leq 1 \quad (4)$$

$$y_1 \leq x \quad (5)$$

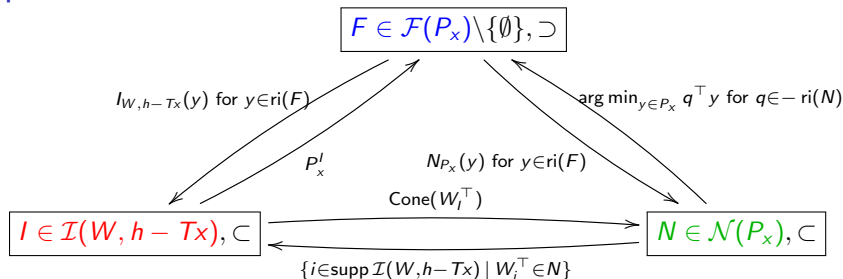
$$y_2 \leq x \quad (6)$$

$$x \leq 1.5 \quad (7)$$



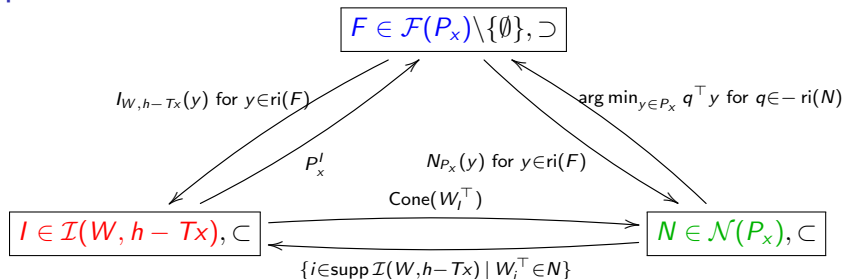
$P_x$  for  $x = 0.8$

# Expectation to final sum



$$\begin{aligned}
 V(x) &= \mathbb{E} \left[ \min_{y \in P_x} q^\top y \right] \\
 &= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E} \left[ q^\top \mathbf{1}_{q \in -\text{ri } N} \right] y_N(x) \quad \text{with } y_N(x) \in \bigcap_{q \in -N} \arg \min_{y \in P_x} q^\top y
 \end{aligned}$$

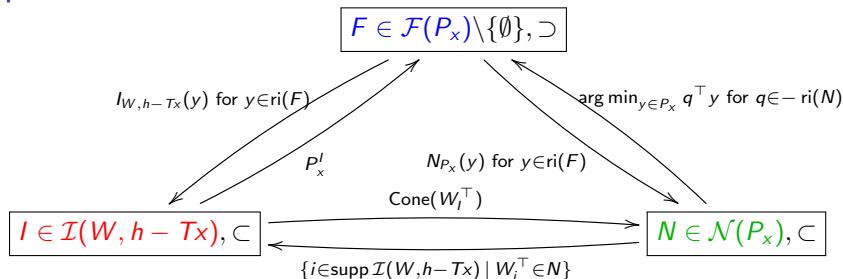
# Expectation to final sum



$$\begin{aligned}
 V(x) &= \mathbb{E} \left[ \min_{y \in P_x} q^\top y \right] \\
 &= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E} [q^\top \mathbf{1}_{q \in -\text{ri } N}] y_N(x) \quad \text{with } y_N(x) \in \cap_{q \in -N} \arg \min_{y \in P_x} q^\top y \\
 &= \sum_{F \in \mathcal{F}(P_x)} \mathbb{E} [q^\top \mathbf{1}_{q \in -\text{ri } N_{P_x}(F)}] y_F \quad \text{with } y_F \in F
 \end{aligned}$$

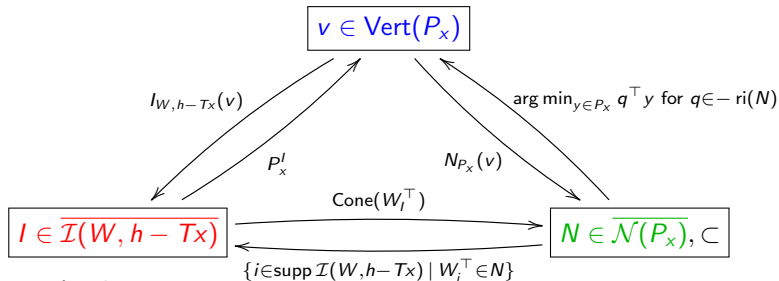


# Expectation to final sum



$$\begin{aligned}
 V(x) &= \mathbb{E} \left[ \min_{y \in P_x} q^\top y \right] \\
 &= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E} [q^\top \mathbf{1}_{q \in -\text{ri } N}] y_N(x) \quad \text{with } y_N(x) \in \cap_{q \in -N} \arg \min_{y \in P_x} q^\top y \\
 &= \sum_{F \in \mathcal{F}(P_x)} \mathbb{E} [q^\top \mathbf{1}_{q \in -\text{ri } N_{P_x}(F)}] y_F \quad \text{with } y_F \in F \\
 &= \sum_{I \in \mathcal{I}(W, h - Tx)} \mathbb{E} [q^\top \mathbf{1}_{q \in -\text{ri } \text{Cone}(W_i^\top)}] y_I(x) \quad \text{with } y_I(x) \in P_x^I
 \end{aligned}$$

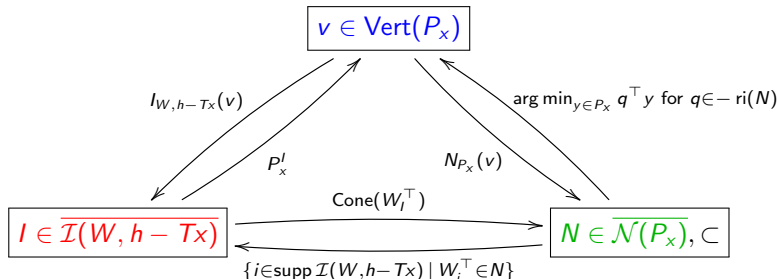
# Expectation to final sum



If  $\mathbf{q}$  has a density,

$$\begin{aligned}
 V(x) &= \mathbb{E} \left[ \min_{y \in P_x} \mathbf{q}^\top y \right] \\
 &= \sum_{N \in \overline{\mathcal{N}(P_x)}} \mathbb{E} \left[ \mathbf{q}^\top \mathbf{1}_{\mathbf{q} \in -N} \right] y_N(x) \quad \text{with } y_N(x) \in \cap_{q \in -N} \arg \min_{y \in P_x} q^\top y \\
 &= \sum_{v \in \text{Vert}(P_x)} \mathbb{E} \left[ \mathbf{q}^\top \mathbf{1}_{\mathbf{q} \in -N_{P_x}(F)} \right] v \\
 &= \sum_{I \in \overline{\mathcal{I}(W, h - Tx)}} \mathbb{E} \left[ \mathbf{q}^\top \mathbf{1}_{\mathbf{q} \in -\text{Cone}(W_I^\top)} \right] y_I(x) \quad \text{with } y_I(x) \in P_x^I
 \end{aligned}$$

# Expectation to final sum



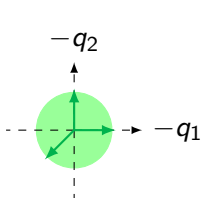
If  $\mathbf{q}$  has a density,

$$\begin{aligned}
 V(x) &= \mathbb{E} \left[ \min_{y \in P_x} \mathbf{q}^\top y \right] \\
 &= \sum_{N \in \overline{\mathcal{N}(P_x)}} \mathbb{E} \left[ \mathbf{q}^\top \mathbf{1}_{\mathbf{q} \in -N} \right] y_N(x) \quad \text{with } y_N(x) \in \cap_{q \in -N} \arg \min_{y \in P_x} q^\top y \\
 &= \sum_{v \in \text{Vert}(P_x)} \mathbb{E} \left[ \mathbf{q}^\top \mathbf{1}_{\mathbf{q} \in -N_{P_x}(F)} \right] v \\
 &= \sum_{I \in \overline{\mathcal{I}(W, h - Tx)}} \mathbb{E} \left[ \mathbf{q}^\top \mathbf{1}_{\mathbf{q} \in -\text{Cone}(W_I^\top)} \right] W_{B_I}^{-1} (h_{B_I} - T_{B_I} x) \quad \text{with basis } B_I \subset I
 \end{aligned}$$

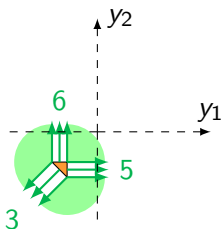
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

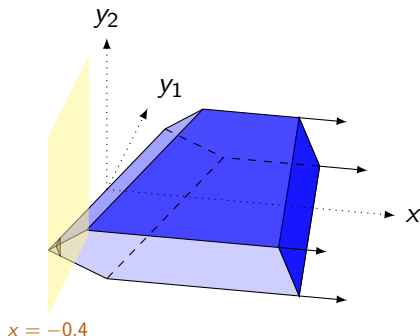
For  $x = -0.4$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$



$\mathcal{N}(P_x)$



$P_x$  and  $\mathcal{N}(P_x)$

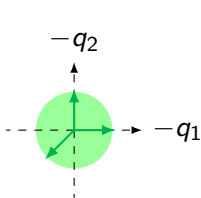


$P$  and  $P_x$

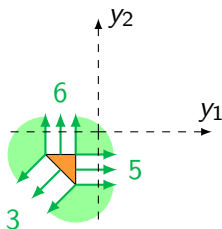
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

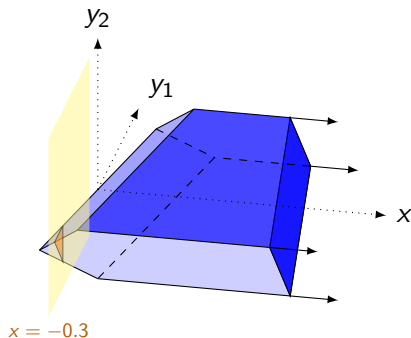
For  $x = -0.3$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$



$\mathcal{N}(P_x)$



$P_x$  and  $\mathcal{N}(P_x)$

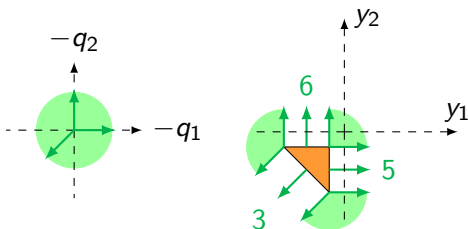


$P$  and  $P_x$

$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

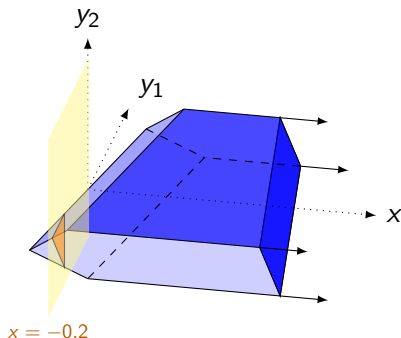
$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

For  $x = -0.2$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$



$\mathcal{N}(P_x)$

$P_x$  and  $\mathcal{N}(P_x)$

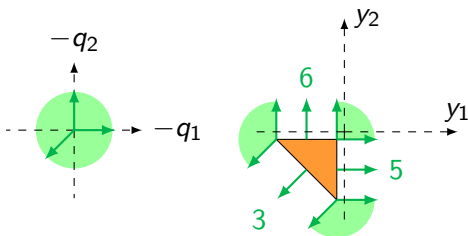


$P$  and  $P_x$

$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

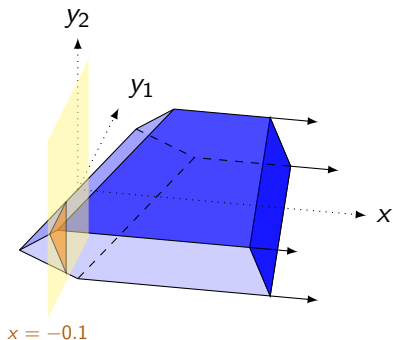
$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

For  $x = -0.1$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$



$\mathcal{N}(P_x)$

$P_x$  and  $\mathcal{N}(P_x)$

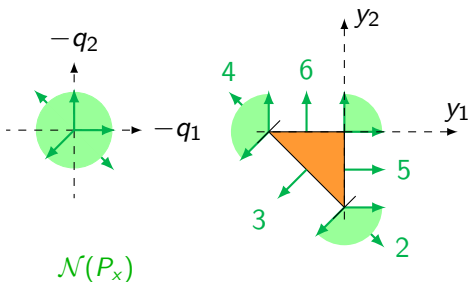


$P$  and  $P_x$

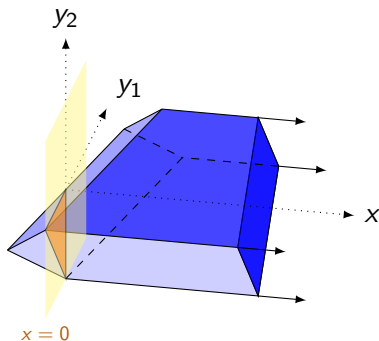
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

For  $x = 0$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{523, 346, 65\}$



$P_x$  and  $\mathcal{N}(P_x)$



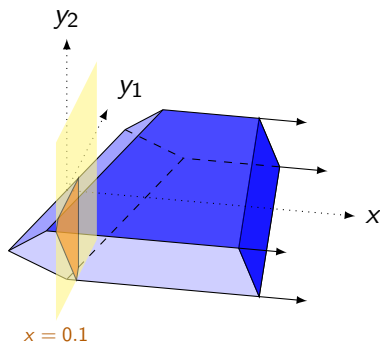
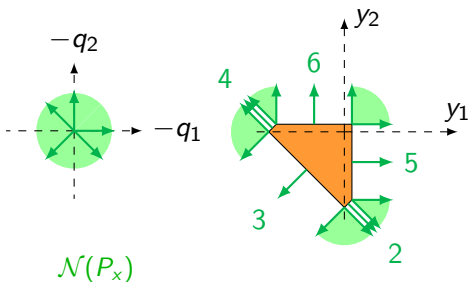
$P$  and  $P_x$



$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

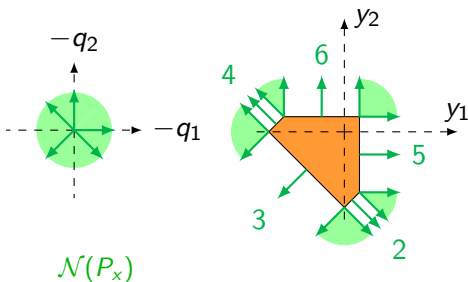
For  $x = 0.1$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$



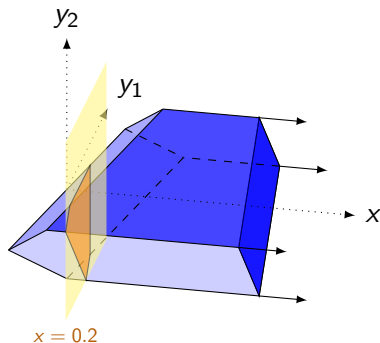
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

For  $x = 0.2$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$



$P_x$  and  $\mathcal{N}(P_x)$

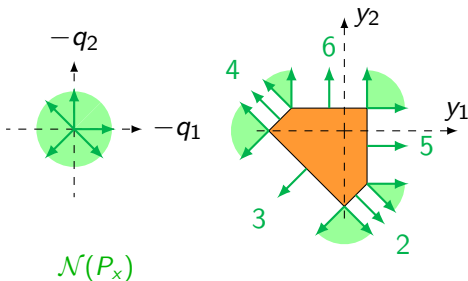


$P$  and  $P_x$

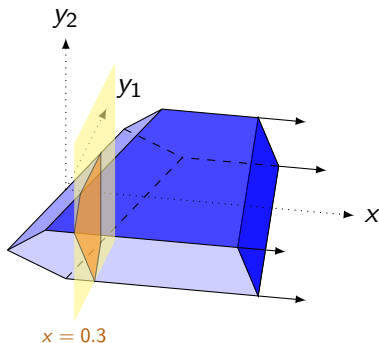
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

For  $x = 0.3$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$



$P_x$  and  $\mathcal{N}(P_x)$

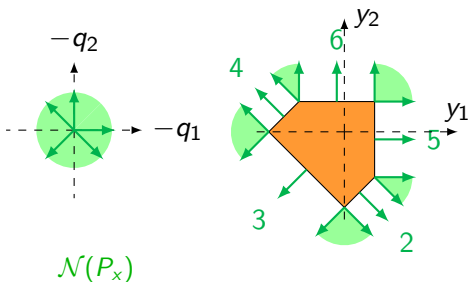


$P$  and  $P_x$

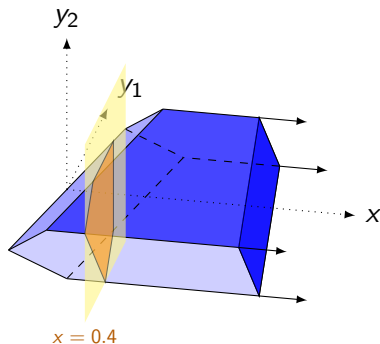
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

For  $x = 0.4$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$



$P_x$  and  $\mathcal{N}(P_x)$

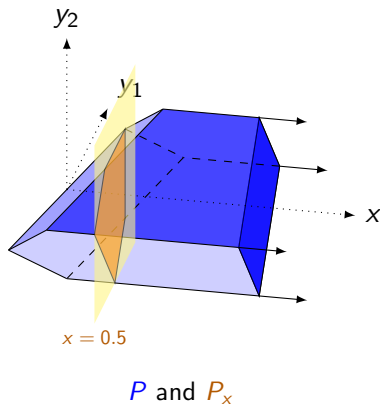
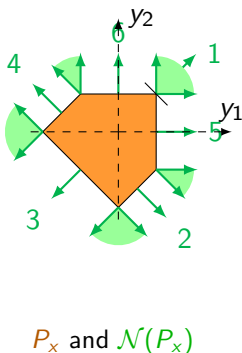
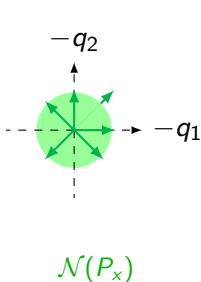


$P$  and  $P_x$

$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

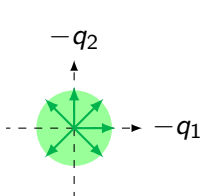
For  $x = 0.5$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 615\}$



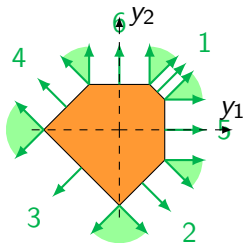
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

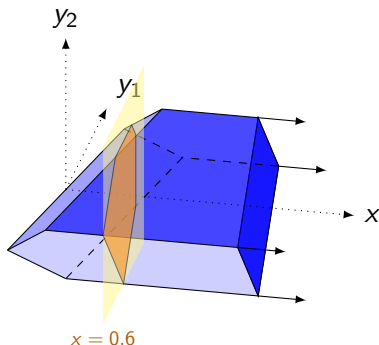
For  $x = 0.6$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$



$\mathcal{N}(P_x)$



$P_x$  and  $\mathcal{N}(P_x)$

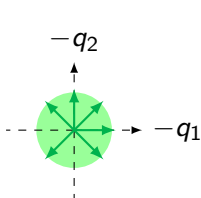


$P$  and  $P_x$

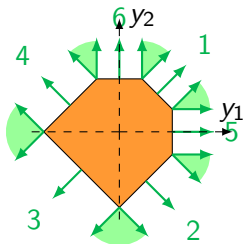
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

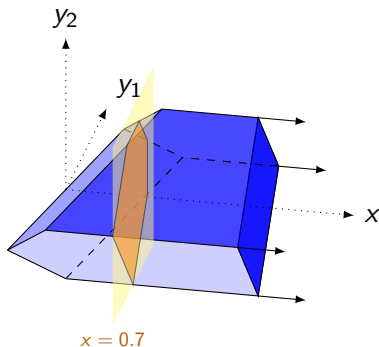
For  $x = 0.7$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$



$\mathcal{N}(P_x)$



$P_x$  and  $\mathcal{N}(P_x)$

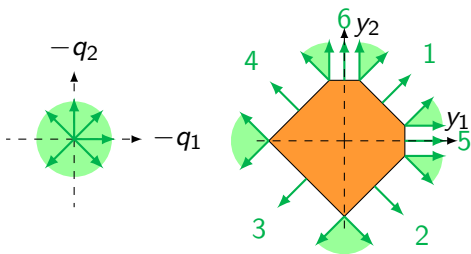


$P$  and  $P_x$

$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

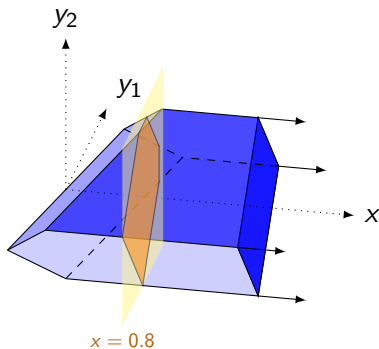
$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

For  $x = 0.8$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$



$\mathcal{N}(P_x)$

$P_x$  and  $\mathcal{N}(P_x)$



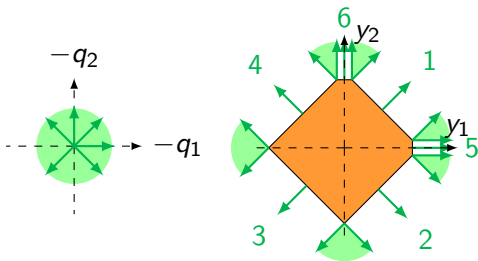
$P$  and  $P_x$



$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

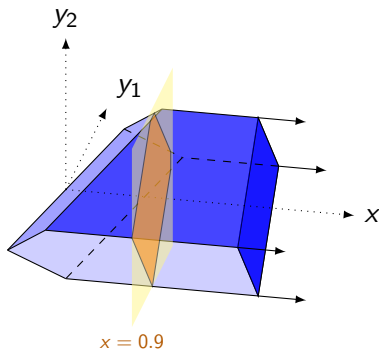
$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

For  $x = 0.9$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$



$\mathcal{N}(P_x)$

$P_x$  and  $\mathcal{N}(P_x)$

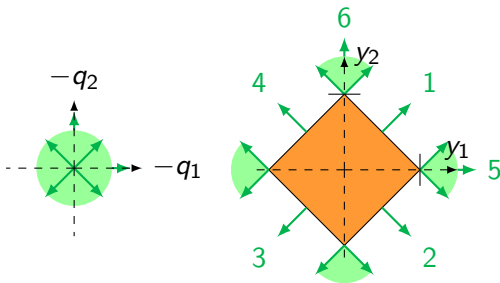


$P$  and  $P_x$

$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

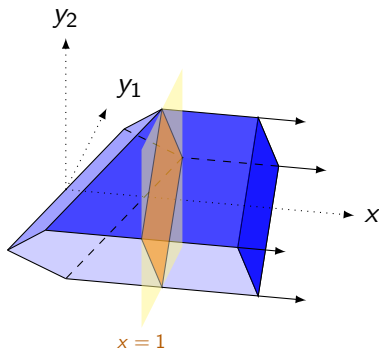
$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

For  $x = 1$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{152, 23, 34, 461\}$



$\mathcal{N}(P_x)$

$P_x$  and  $\mathcal{N}(P_x)$

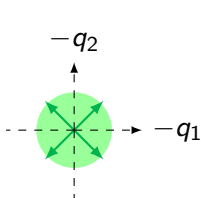


$P$  and  $P_x$

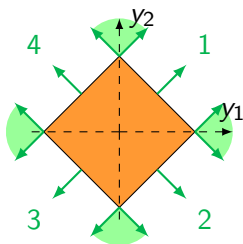
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

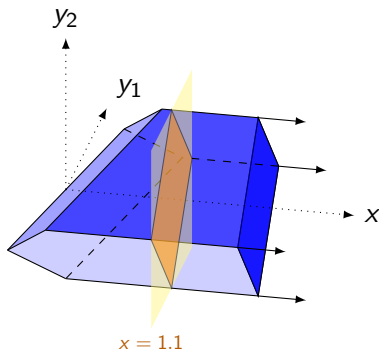
For  $x = 1.1$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$



$\mathcal{N}(P_x)$



$P_x$  and  $\mathcal{N}(P_x)$

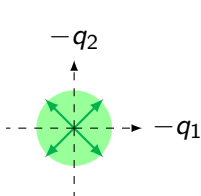


$P$  and  $P_x$

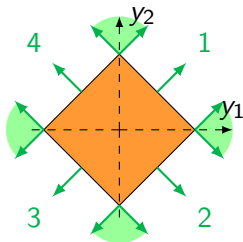
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

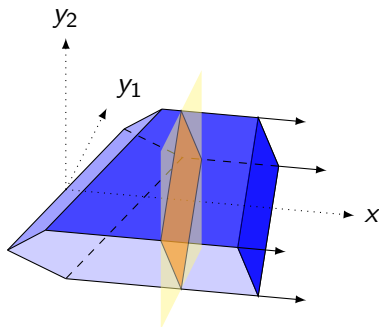
For  $x = 1.2$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$



$\mathcal{N}(P_x)$



$P_x$  and  $\mathcal{N}(P_x)$



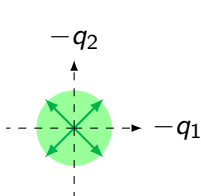
$x = 1.2$

$P$  and  $P_x$

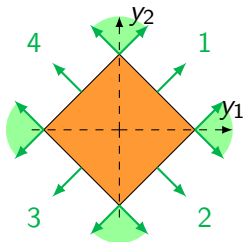
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

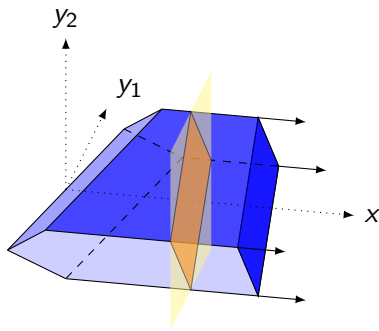
For  $x = 1.3$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$



$\mathcal{N}(P_x)$



$P_x$  and  $\mathcal{N}(P_x)$



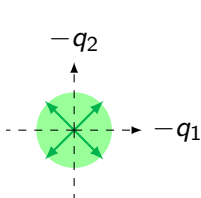
$x = 1.3$

$P$  and  $P_x$

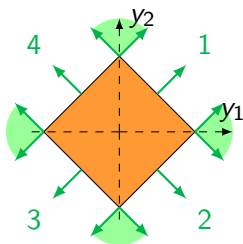
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

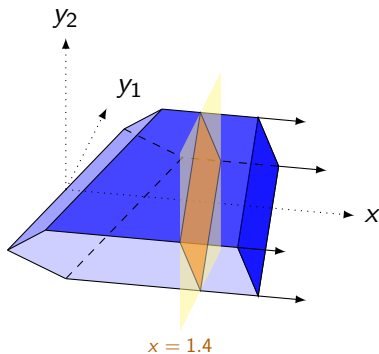
For  $x = 1.4$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$



$\mathcal{N}(P_x)$



$P_x$  and  $\mathcal{N}(P_x)$

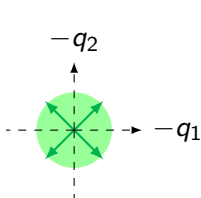


$P$  and  $P_x$

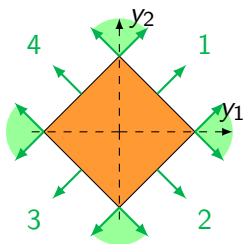
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

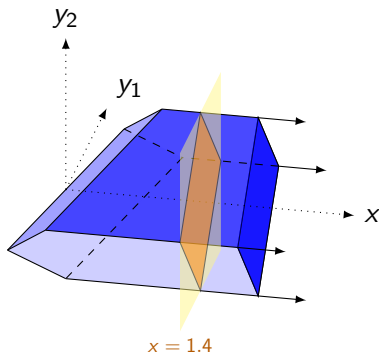
For  $x = 1.4$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$



$\mathcal{N}(P_x)$



$P_x$  and  $\mathcal{N}(P_x)$

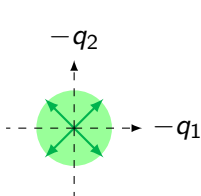


$P$  and  $P_x$

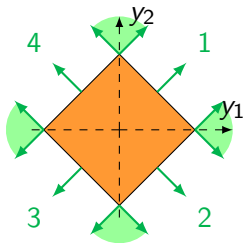
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

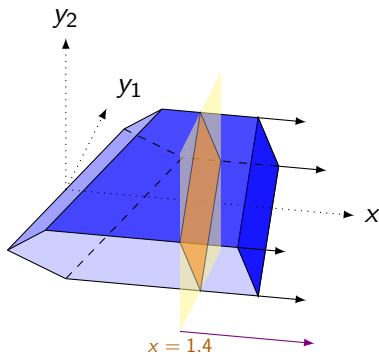
For  $x = 1.4$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$



$\mathcal{N}(P_x)$



$P_x$  and  $\mathcal{N}(P_x)$



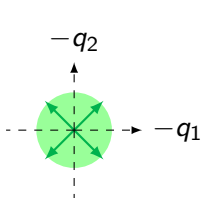
$P$  and  $P_x$



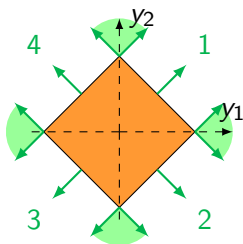
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

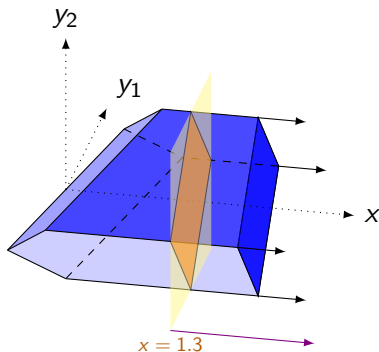
For  $x = 1.3$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$



$\mathcal{N}(P_x)$



$P_x$  and  $\mathcal{N}(P_x)$

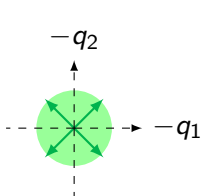


$P$  and  $P_x$

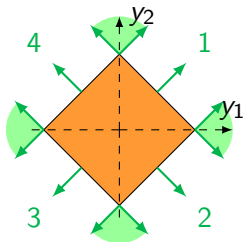
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

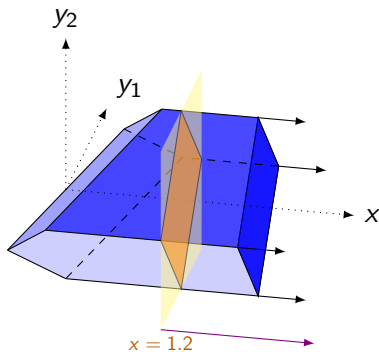
For  $x = 1.2$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$



$\mathcal{N}(P_x)$



$P_x$  and  $\mathcal{N}(P_x)$

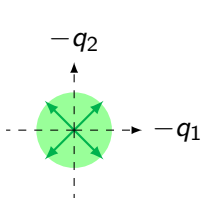


$P$  and  $P_x$

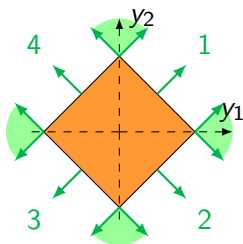
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

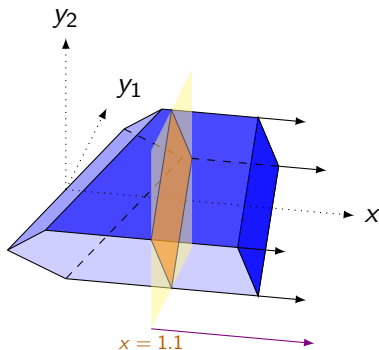
For  $x = 1.1$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$



$\mathcal{N}(P_x)$



$P_x$  and  $\mathcal{N}(P_x)$

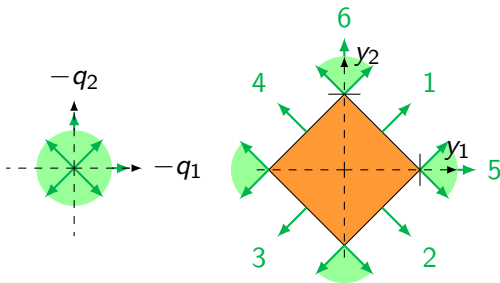


$P$  and  $P_x$

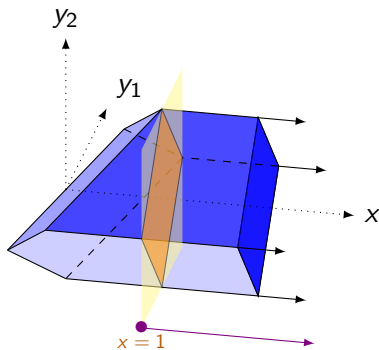
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

For  $x = 1$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{152, 23, 34, 461\}$



$P_x$  and  $\mathcal{N}(P_x)$

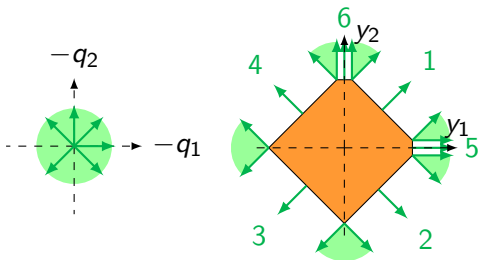


$P$  and  $P_x$

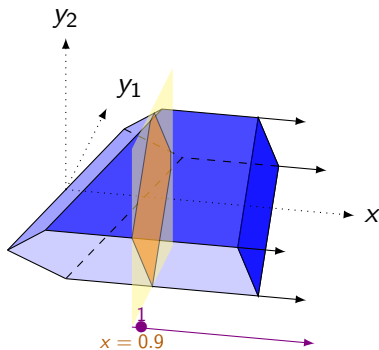
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

For  $x = 0.9$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$



$P_x$  and  $\mathcal{N}(P_x)$

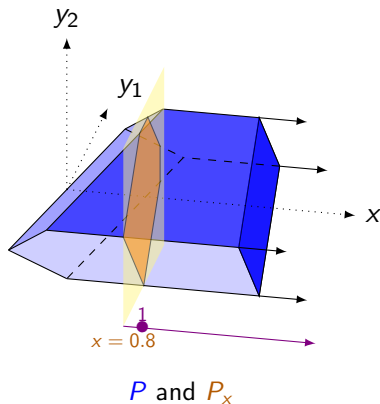
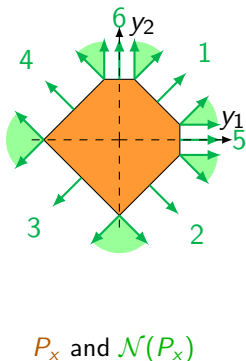
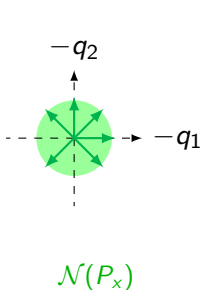


$P$  and  $P_x$

$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

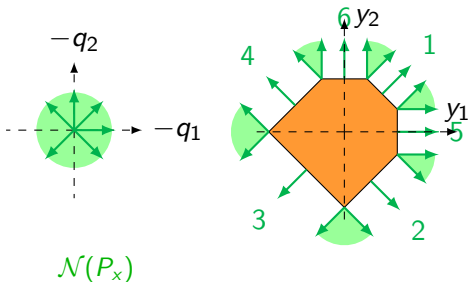
For  $x = 0.8$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$



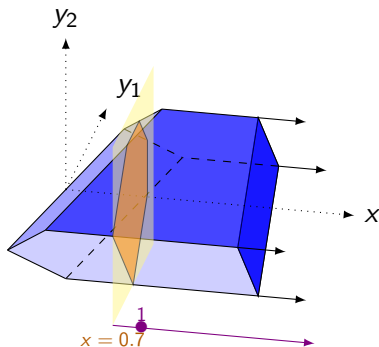
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

For  $x = 0.7$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$



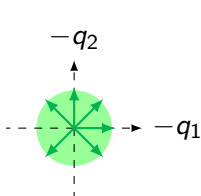
$P_x$  and  $\mathcal{N}(P_x)$



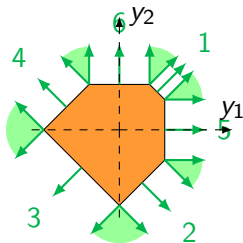
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

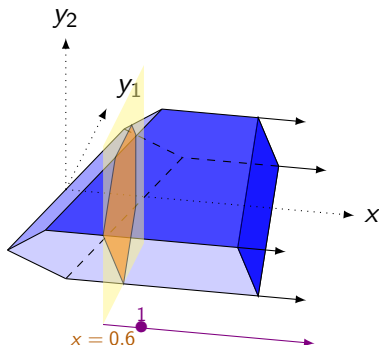
For  $x = 0.6$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$



$\mathcal{N}(P_x)$



$P_x$  and  $\mathcal{N}(P_x)$



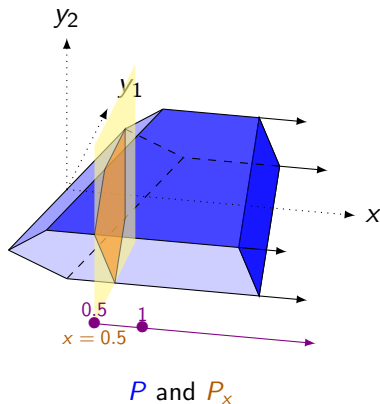
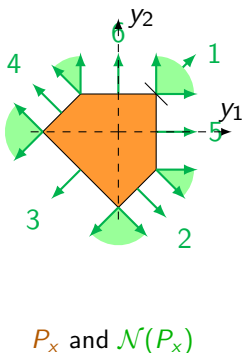
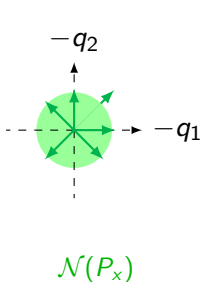
$P$  and  $P_x$



$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

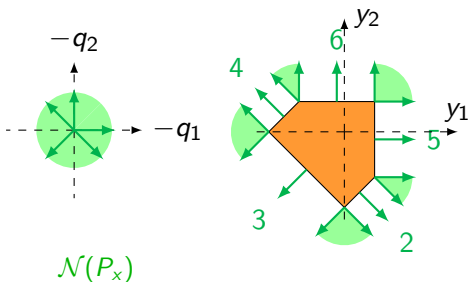
For  $x = 0.5$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 615\}$



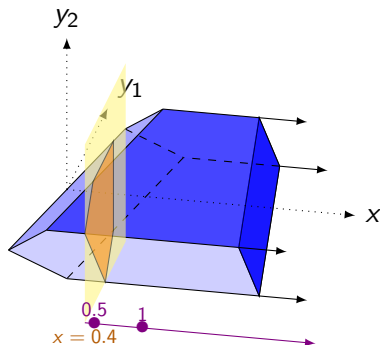
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

For  $x = 0.4$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$



$P_x$  and  $\mathcal{N}(P_x)$

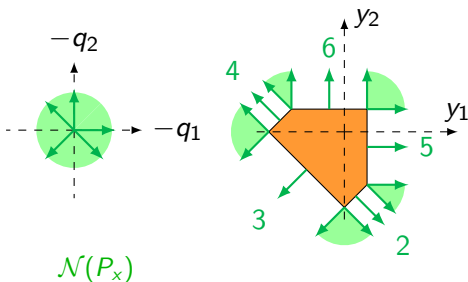


$P$  and  $P_x$

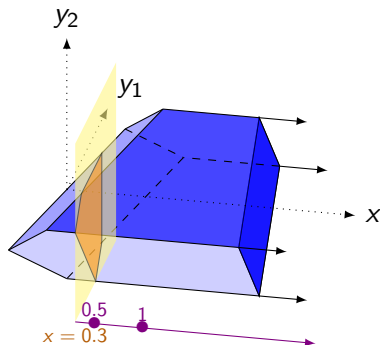
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

For  $x = 0.3$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$



$P_x$  and  $\mathcal{N}(P_x)$

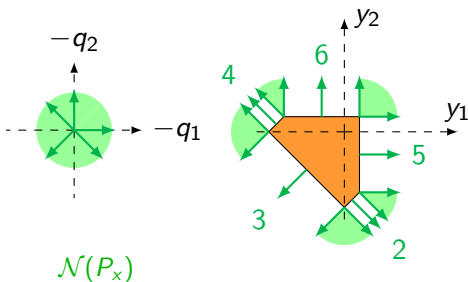


$P$  and  $P_x$

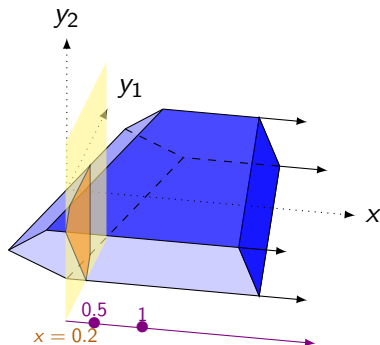
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

For  $x = 0.2$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$



$P_x$  and  $\mathcal{N}(P_x)$

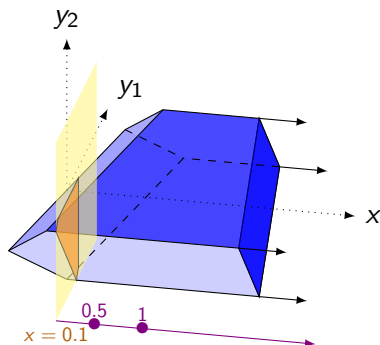
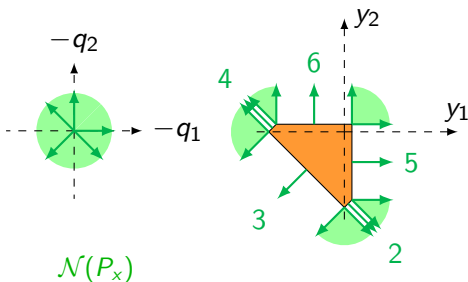


$P$  and  $P_x$

$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

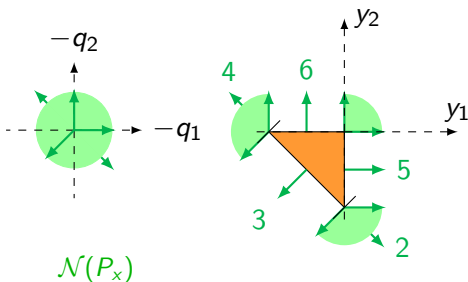
For  $x = 0.1$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$



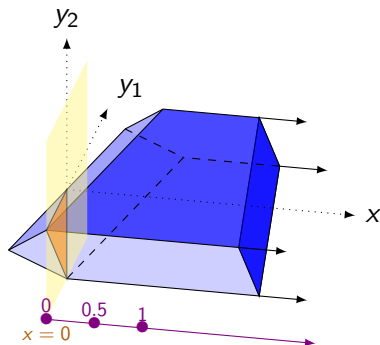
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

For  $x = 0$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{523, 346, 65\}$



$P_x$  and  $\mathcal{N}(P_x)$

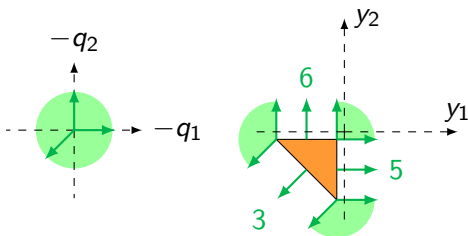


$P$  and  $P_x$

$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

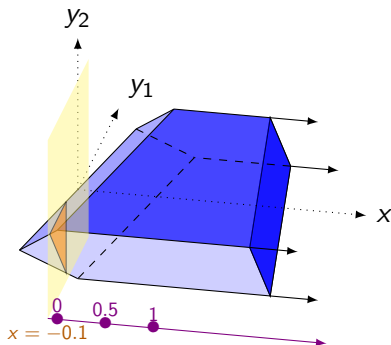
$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

For  $x = -0.1$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$



$\mathcal{N}(P_x)$

$P_x$  and  $\mathcal{N}(P_x)$

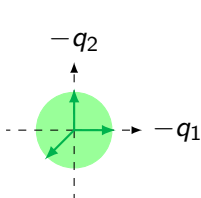


$P$  and  $P_x$

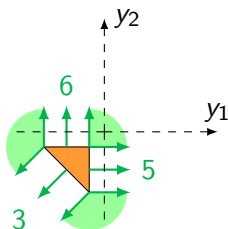
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

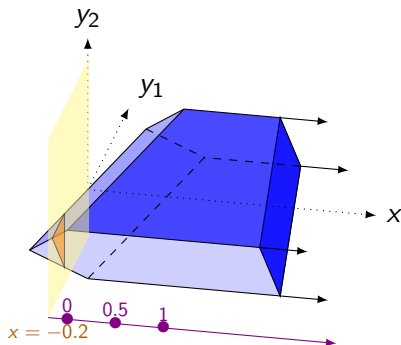
For  $x = -0.2$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$



$\mathcal{N}(P_x)$



$P_x$  and  $\mathcal{N}(P_x)$



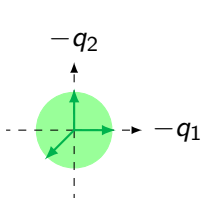
$P$  and  $P_x$



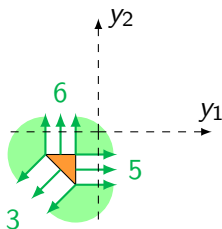
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

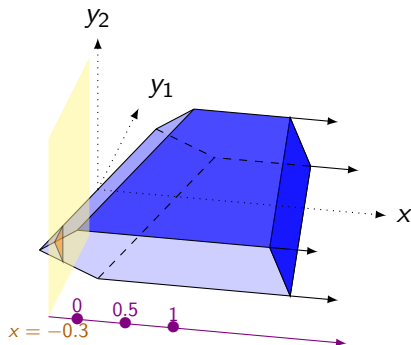
For  $x = -0.3$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$



$\mathcal{N}(P_x)$



$P_x$  and  $\mathcal{N}(P_x)$

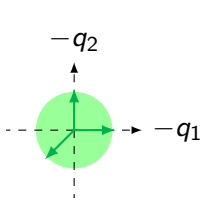


$P$  and  $P_x$

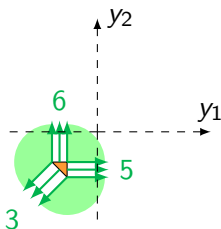
$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

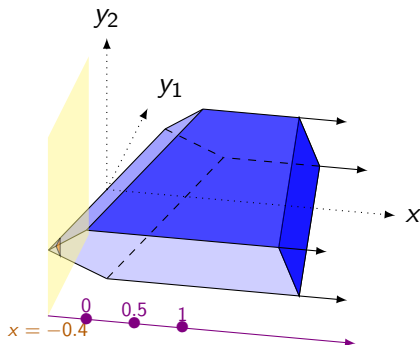
For  $x = -0.4$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$



$\mathcal{N}(P_x)$



$P_x$  and  $\mathcal{N}(P_x)$

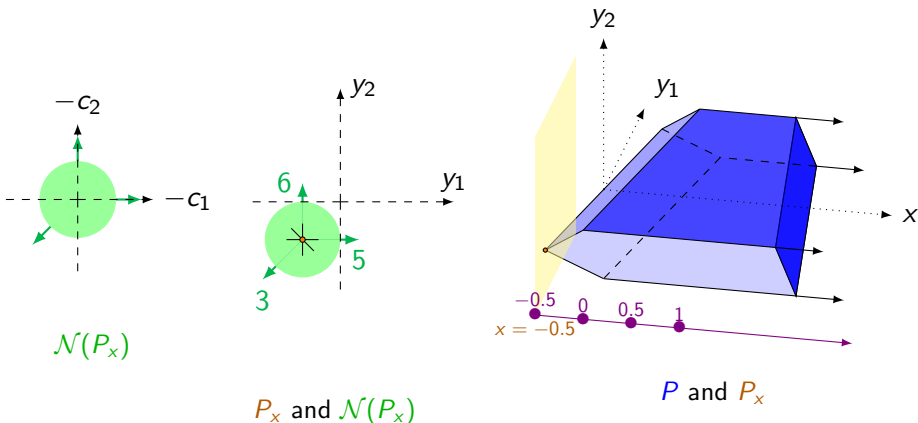


$P$  and  $P_x$

$\mathcal{N}(P_x)$  and  $\mathcal{I}(W, h - Tx)$  are piecewise constant with  $x$ .

$$P := \{(x, y) \mid Tx + Wy \leq h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leq h\}$$

For  $x = -0.5$ ,  $\overline{\mathcal{I}(W, h - Tx)} = \{536\}$

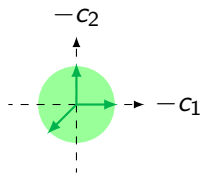
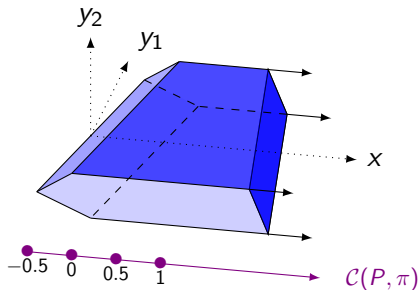


What are the constant regions of  $\mathcal{N}(P_x)$ ,  $\mathcal{I}(W, h - Tx)$ ?

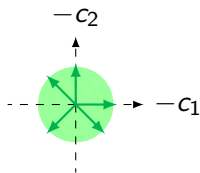
## Lemma

There exists a collection  $\mathcal{C}(P, \pi)$  whose relative interior of cells are the constant regions of  $x \rightarrow \mathcal{N}(P_x)$  and  $x \rightarrow \mathcal{I}(W, h - Tx)$ .

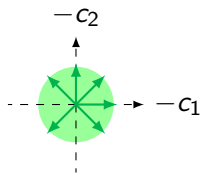
For  $\sigma \in \mathcal{C}(P, \pi)$  and  $x, x' \in \text{ri}(\sigma)$ ,  
 $\mathcal{N}(P_x) = \mathcal{N}(P_{x'}) = \mathcal{N}_\sigma$   
 $\mathcal{I}(W, h - Tx) = \mathcal{I}(W, h - Tx') = \mathcal{I}_\sigma$



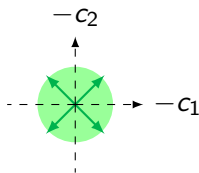
$\mathcal{N}_\sigma$  for  $\sigma = [-0.5, 0]$



$\mathcal{N}_\sigma$  for  $\sigma = [0, 0.5]$



$\mathcal{N}_\sigma$  for  $\sigma = [0.5, 1]$



$\mathcal{N}_\sigma$  for  $\sigma = [1, +\infty)$

# Chamber complex

## Definition

The *chamber complex*  $\mathcal{C}(P, \pi)$  of  $P$  along  $\pi$  is

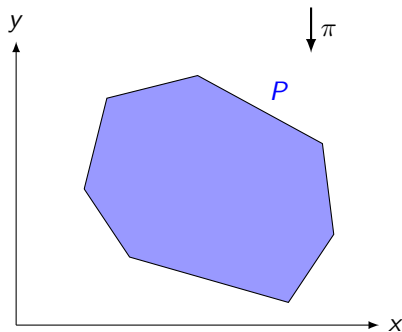
$$\mathcal{C}(P, \pi) := \{\sigma_{P, \pi}(x) \mid x \in \pi(P)\}$$

where

$$\sigma_{P, \pi}(x) := \bigcap_{F \in \mathcal{F}(P) \text{ s.t. } x \in \pi(F)} \pi(F)$$

where  $\mathcal{F}(P)$  is the set of faces of  $P$   
and  $\pi$  is the projection  $(x, y) \rightarrow x$

$$\pi(E) := \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m, (x, y) \in E\}$$



# Chamber complex

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The *chamber complex*  $\mathcal{C}(P, \pi)$  of  $P$  along  $\pi$  is

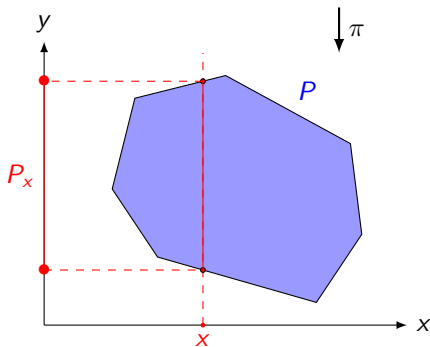
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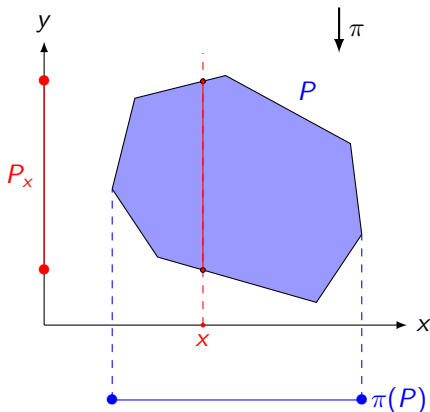
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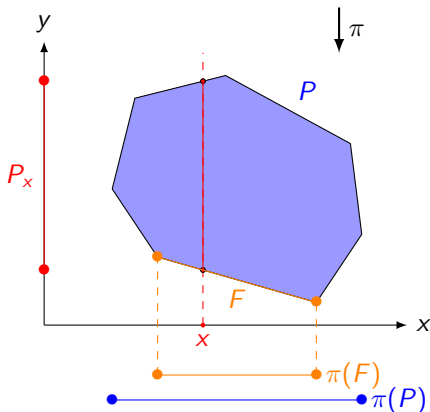
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# Chamber complex

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The *chamber complex*  $\mathcal{C}(P, \pi)$  of  $P$  along  $\pi$  is

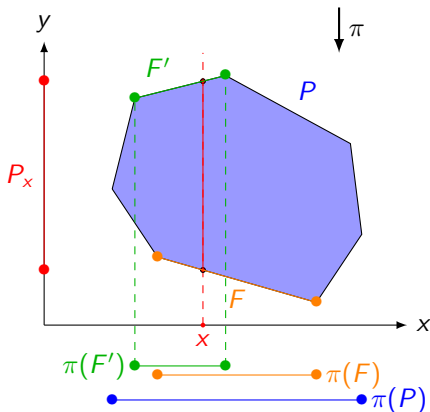
$$\mathcal{C}(P, \pi) := \{\sigma_{P, \pi}(x) \mid x \in \pi(P)\}$$

where

$$\sigma_{P, \pi}(x) := \bigcap_{F \in \mathcal{F}(P) \text{ s.t. } x \in \pi(F)} \pi(F)$$

where  $\mathcal{F}(P)$  is the set of faces of  $P$   
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# Chamber complex

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The *chamber complex*  $\mathcal{C}(P, \pi)$  of  $P$  along  $\pi$  is

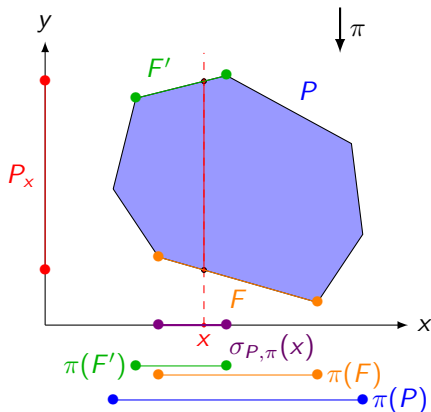
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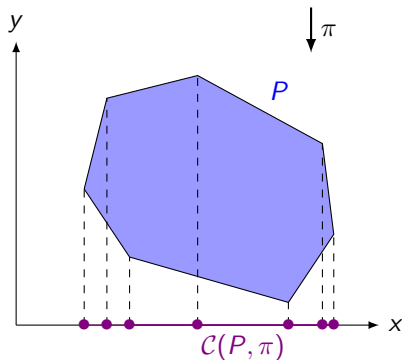
$$\mathcal{C}(P, \pi) := \{\sigma_{P, \pi}(x) \mid x \in \pi(P)\}$$

where

$$\sigma_{P, \pi}(x) := \bigcap_{F \in \mathcal{F}(P) \text{ s.t. } x \in \pi(F)} \pi(F)$$

where  $\mathcal{F}(P)$  is the set of faces of  $P$   
and  $\pi$  is the projection  $(x, y) \rightarrow x$

$$\pi(E) := \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m, (x, y) \in E\}$$



# H-representation of projection of faces

Let  $I \in \mathcal{I}((T, W), h)$  be a set of indices

$$x \in \pi(P^I) \iff \begin{cases} \exists y \in \mathbb{R}^m, & (x, y) \in P^I \end{cases}$$

# H-representation of projection of faces

Let  $I \in \mathcal{I}((T, W), h)$  be a set of indices

$$x \in \pi(P^I) \iff \begin{cases} \exists y \in \mathbb{R}^m, & T_I x + W_I y = h_I \\ \forall j \in [q] \setminus I, & T_j x + W_j y \leq h_j \end{cases} \iff I \in \mathcal{I}(W, h - T x)$$

# H-representation of projection of faces

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## H-representation of projection of faces

Let  $I \in \mathcal{I}((T, W), h)$  be a set of indices from which we can extract a basis (i.e.  $\text{rg}(W_I^\top) = m$ ) and let  $B$  such a basis

$$x \in \text{ri } \pi(P^I) \iff \begin{cases} \exists y \in \mathbb{R}^m, & T_B x + W_B y = h_B \\ \forall i \in I \setminus B, & T_i x + W_i y = h_i \\ \forall j \in [q] \setminus I, & T_j x + W_j y < h_j \end{cases} \iff I \in \mathcal{I}(W, h - Tx)$$

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$$x \in \text{ri } \pi(P^I) \iff \begin{cases} \exists y \in \mathbb{R}^m, & y = W_B^{-1}(h_B - T_B x) \\ \forall i \in I \setminus B, & T_i x + W_i y = h_i \\ \forall j \in [q] \setminus I, & T_j x + W_j y < h_j \end{cases} \iff I \in \mathcal{I}(W, h - T x)$$



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## H-representation of projection of faces

Let  $I \in \mathcal{I}((T, W), h)$  be a set of indices from which we can extract a basis (i.e.  $\text{rg}(W_I^\top) = m$ ) and let  $B$  such a basis

$$x \in \text{ri}(\pi(P^I)) \iff \begin{cases} \forall i \in I \setminus B, & (v_i^B)^\top x = u_i^B \\ \forall j \in [q] \setminus I, & (v_j^B)^\top x < u_j^B \end{cases} \iff I \in \mathcal{I}(W, h - Tx)$$

where

$$v_i^B := (T_i - W_i W_B^{-1} T_B)^\top$$

$$u_i^B := h_i - W_i W_B^{-1} h_B$$

# H-representation of chambers

Let  $\sigma \in \mathcal{C}(P, \pi)$

$$x \in \bigcap_{I \in \overline{\mathcal{I}_\sigma}} \text{ri}(\pi(P^I)) \iff \begin{cases} \forall I \in \overline{\mathcal{I}_\sigma}, \\ \forall i \in I \setminus B_I, \quad (v_i^{B_I})^\top x = u_i^{B_I} \\ \forall j \in [q] \setminus I, \quad (v_j^{B_I})^\top x < u_j^{B_I} \end{cases} \iff \mathcal{I}(W, h - Tx) = \mathcal{I}_\sigma$$

where

$$\begin{aligned} v_i^B &:= (T_i - W_i W_B^{-1} T_B)^\top \\ u_i^B &:= h_i - W_i W_B^{-1} h_B \end{aligned}$$

with  $B_I$  basis  $\subset I$  and

$$\begin{aligned} \mathcal{G}_\sigma &:= \{F \in \mathcal{F}(P) \mid \sigma \subset \pi(F)\} \\ \mathcal{I}_\sigma &:= \{I \in \mathcal{I}((T, W), h) \mid \sigma \subset \pi(P^I)\} \end{aligned}$$

We have  $\sigma = \bigcap_{G \in \mathcal{G}_\sigma} \pi(G) = \bigcap_{I \in \mathcal{I}_\sigma} \pi(P^I)$

# H-representation of chambers

Let  $\sigma \in \mathcal{C}(P, \pi)$

$$x \in \text{ri}(\sigma) \iff \begin{cases} \forall I \in \overline{\mathcal{I}_\sigma}, \\ \forall i \in I \setminus B_I, \quad (v_i^{B_I})^\top x = u_i^{B_I} \\ \forall j \in [q] \setminus I, \quad (v_j^{B_I})^\top x < u_j^{B_I} \end{cases} \iff \mathcal{I}(W, h - Tx) = \mathcal{I}_\sigma$$

where

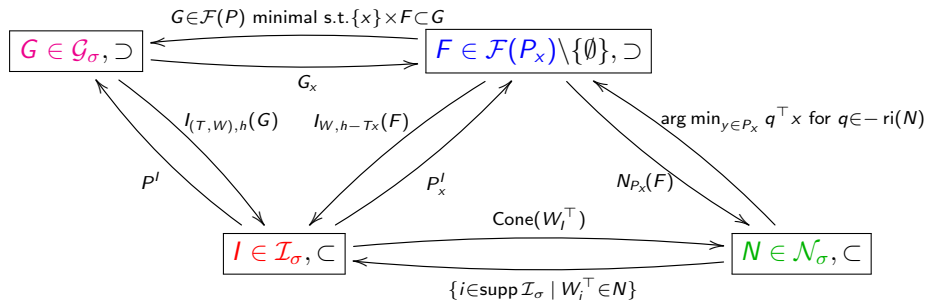
$$\begin{aligned} v_i^B &:= (T_i - W_i W_B^{-1} T_B)^\top \\ u_i^B &:= h_i - W_i W_B^{-1} h_B \end{aligned}$$

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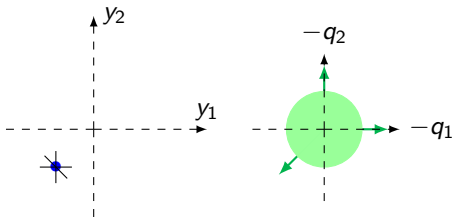
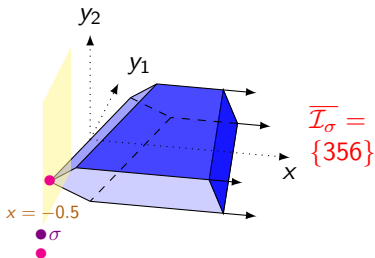
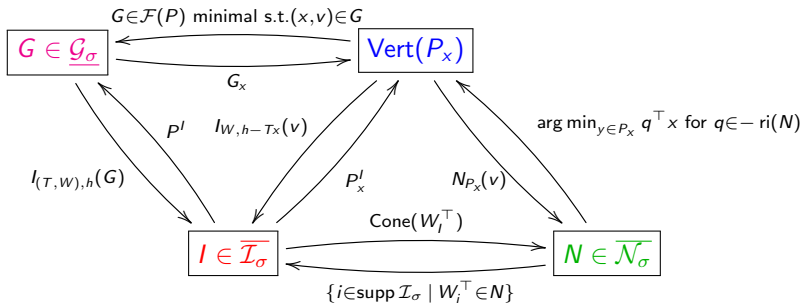
$$\begin{aligned} \mathcal{G}_\sigma &:= \{F \in \mathcal{F}(P) \mid \sigma \subset \pi(F)\} \\ \mathcal{I}_\sigma &:= \{I \in \mathcal{I}((T, W), h) \mid \sigma \subset \pi(P^I)\} \end{aligned}$$

We have  $\sigma = \bigcap_{G \in \underline{\mathcal{G}}_\sigma} \pi(G) = \bigcap_{I \in \overline{\mathcal{I}}_\sigma} \pi(P^I)$

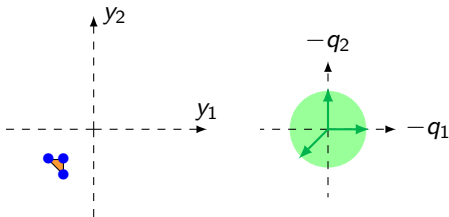
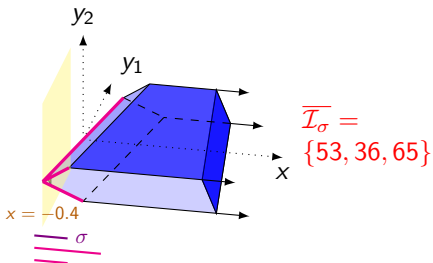
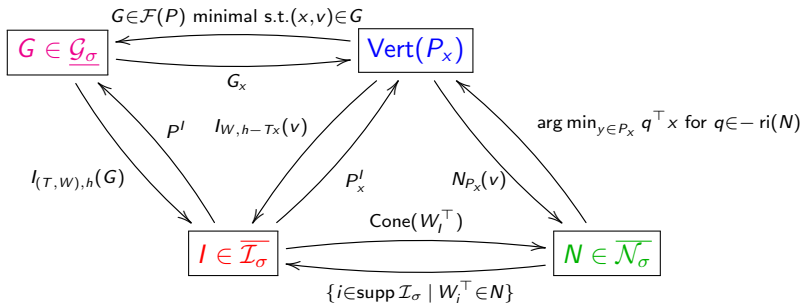
# Correspondences



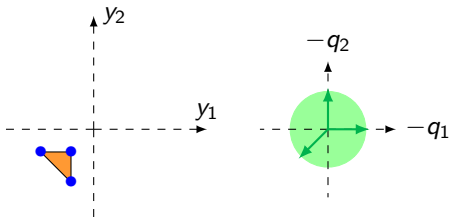
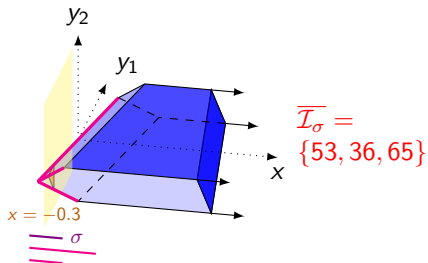
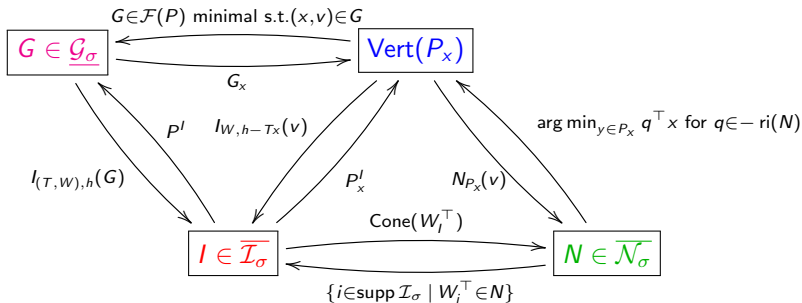
# Correspondences



# Correspondences

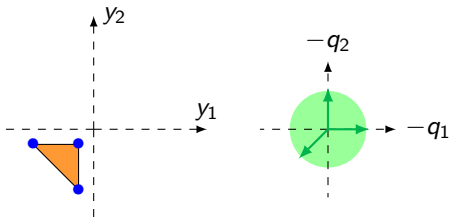
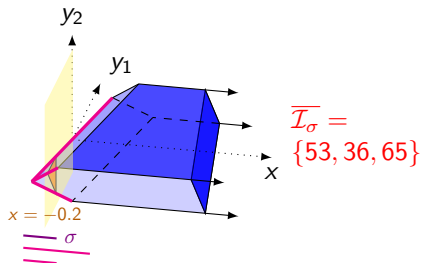
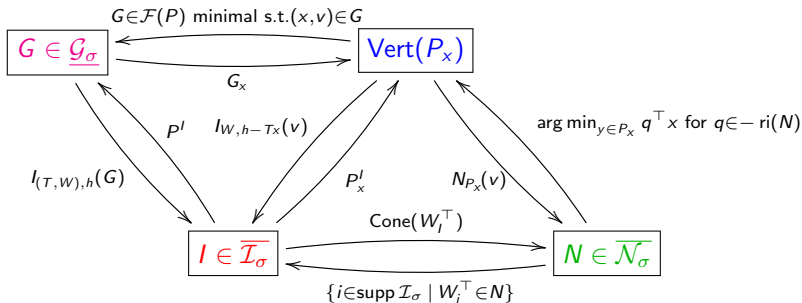


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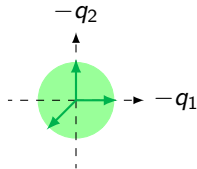
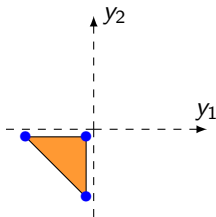
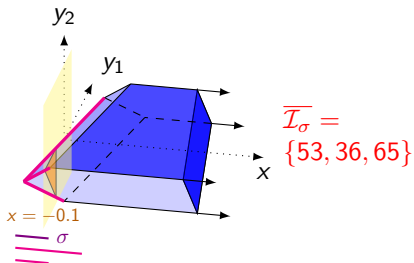
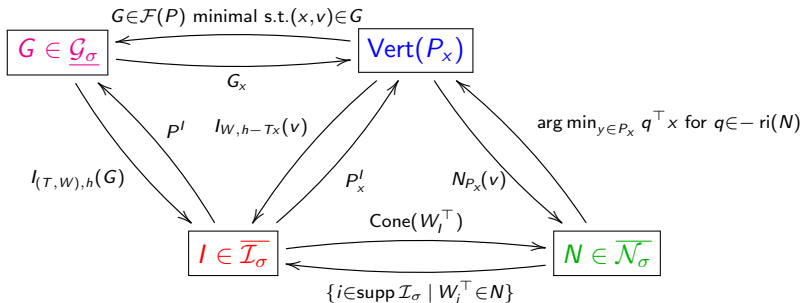




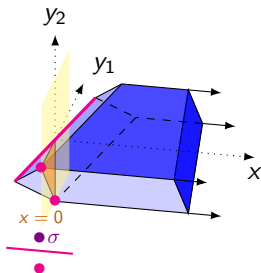
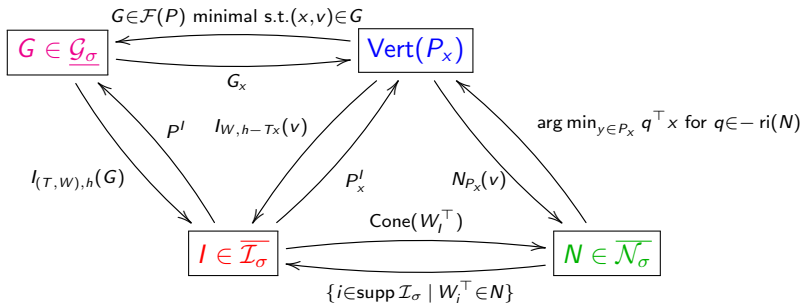
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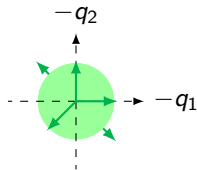
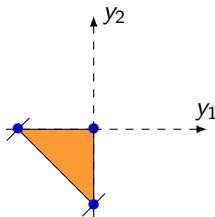
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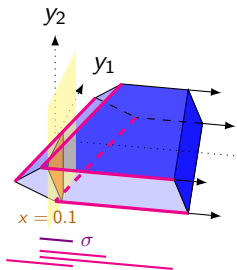
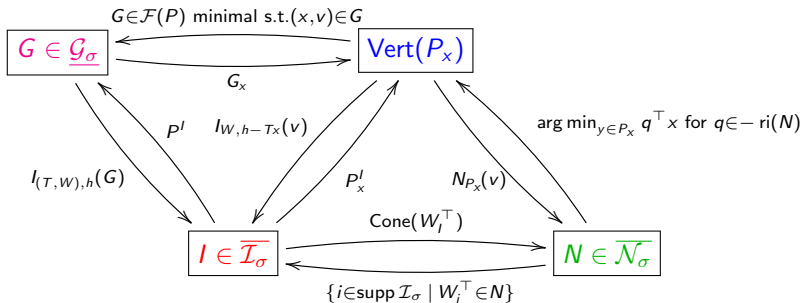
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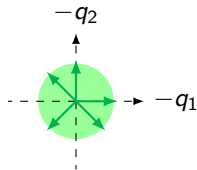
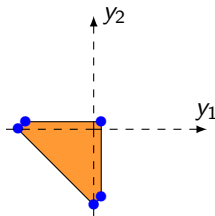
$$\overline{\mathcal{I}}_\sigma = \{523, 346, 65\}$$



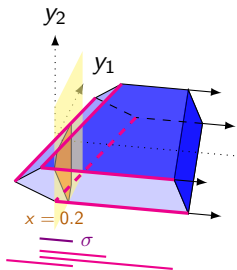
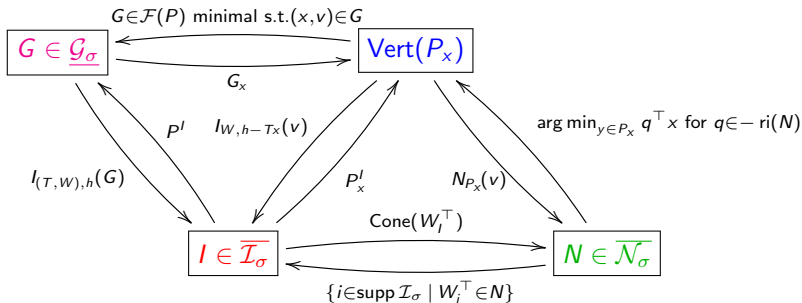
# Correspondences



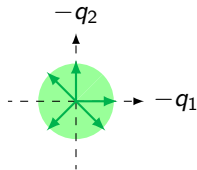
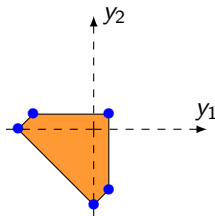
$\overline{\mathcal{I}}_\sigma =$   
 $\{52, 23, 34,$   
 $46, 65\}$



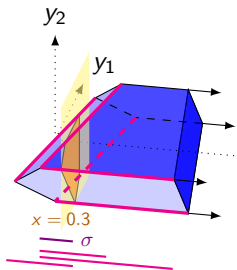
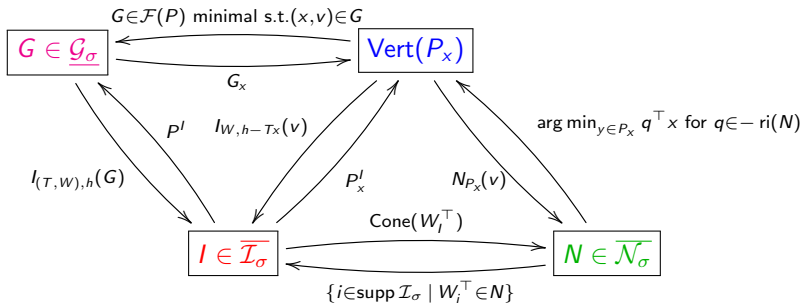
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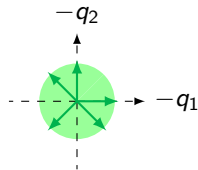
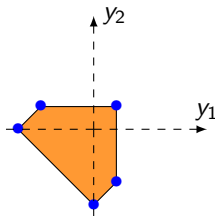
$\overline{\mathcal{I}}_\sigma =$   
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 $46, 65\}$



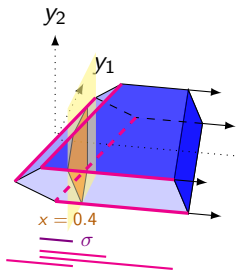
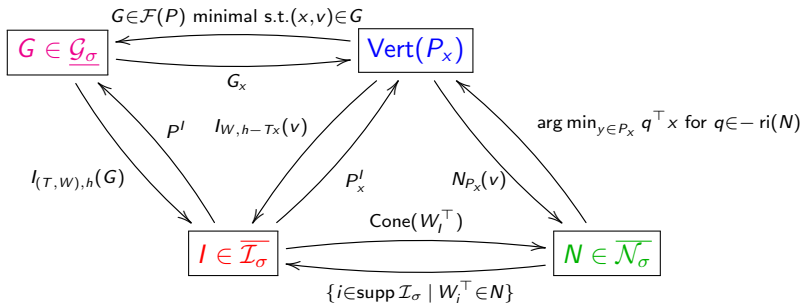
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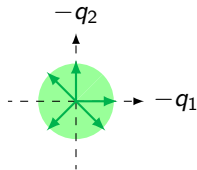
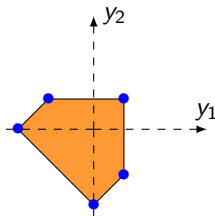
$\overline{\mathcal{I}}_\sigma =$   
 $\{52, 23, 34,$   
 $46, 65\}$



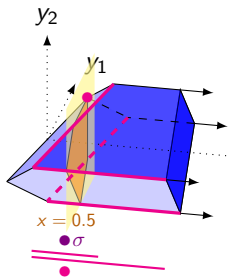
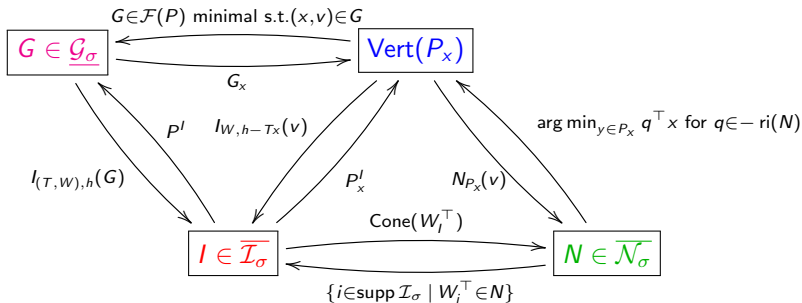
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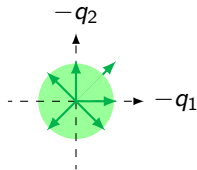
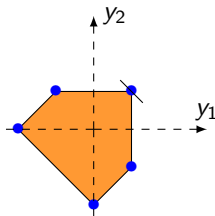
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# Correspondences

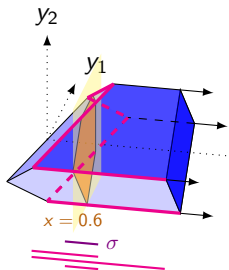
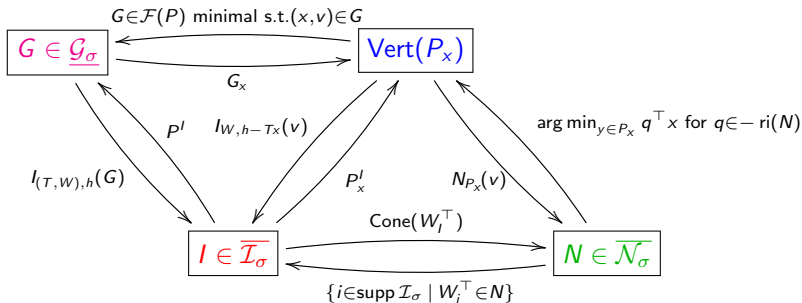


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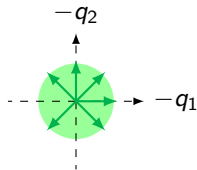
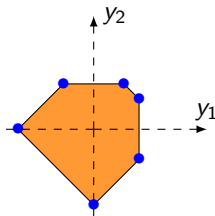




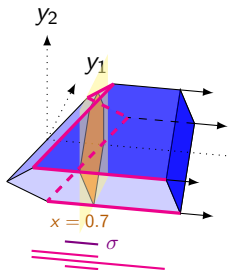
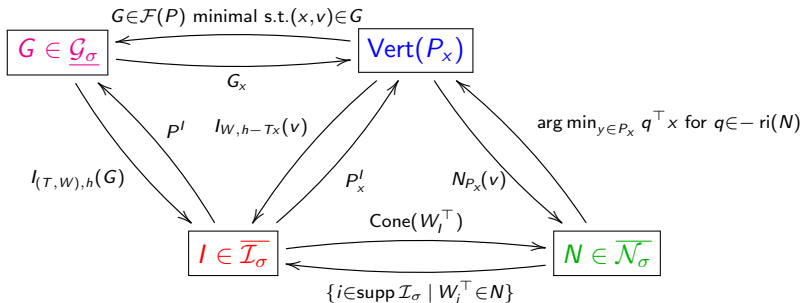
# Correspondences



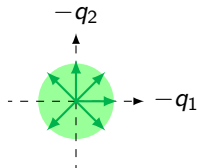
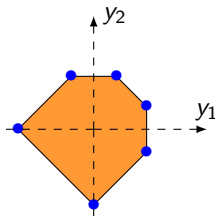
$$\overline{\mathcal{I}}_\sigma = \{52, 23, 34, 46, 61, 15\}$$



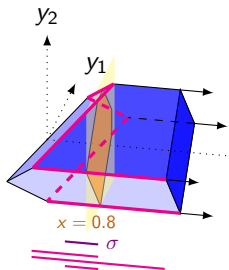
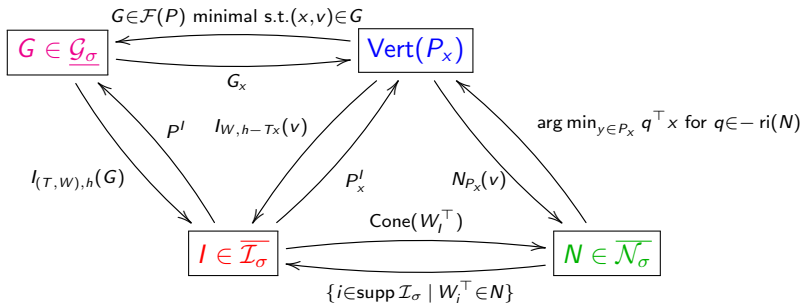
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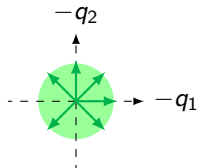
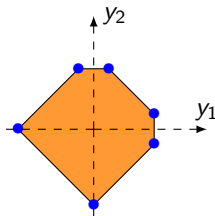
$$\overline{\mathcal{I}}_\sigma = \{52, 23, 34, 46, 61, 15\}$$



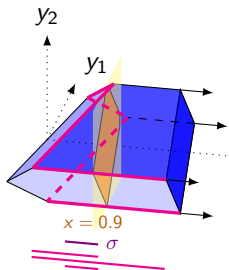
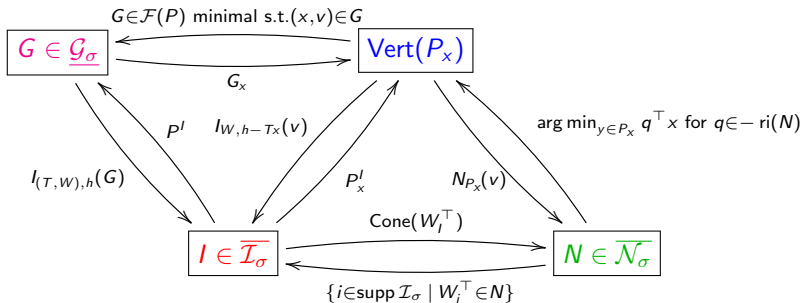
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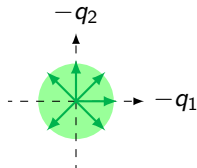
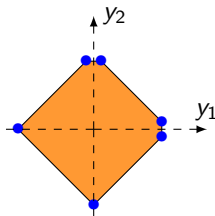
$\overline{\mathcal{I}}_\sigma =$   
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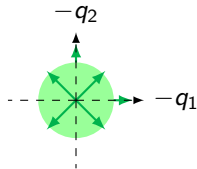
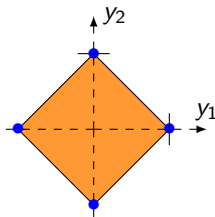
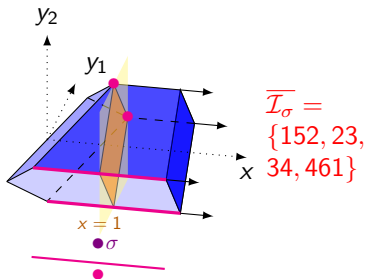
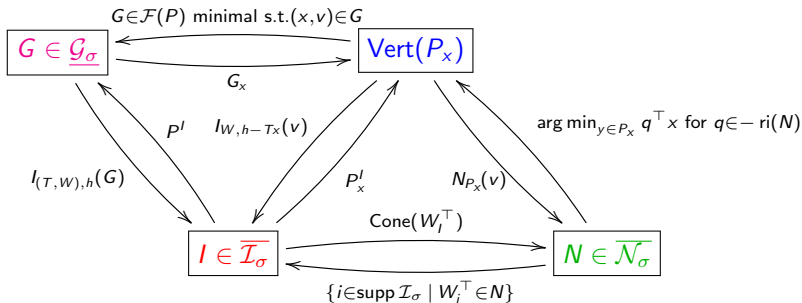
# Correspondences



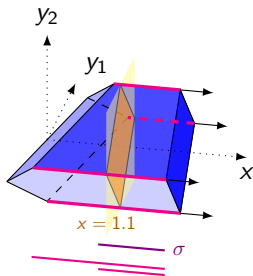
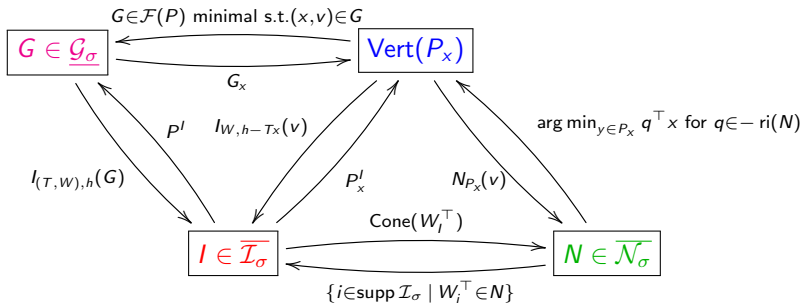
$$\overline{\mathcal{I}}_\sigma = \{52, 23, 34, 46, 61, 15\}$$



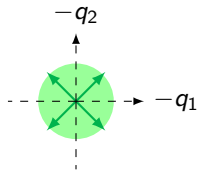
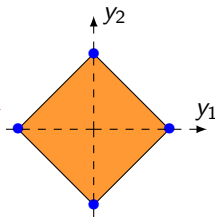
# Correspondences



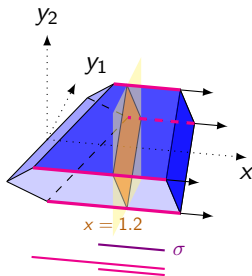
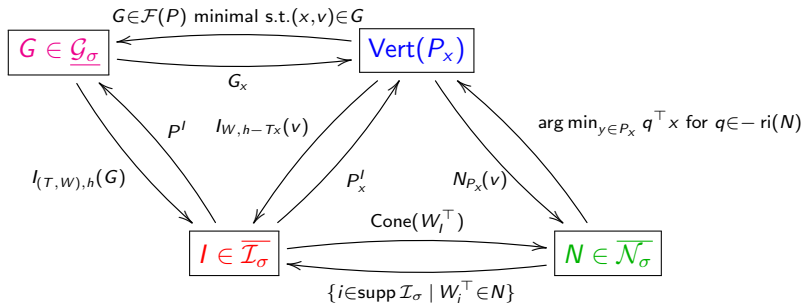
# Correspondences



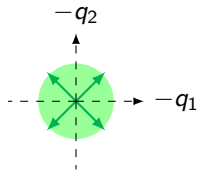
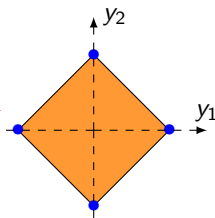
$$\overline{\mathcal{I}}_\sigma = \{12, 23, 34, 41\}$$



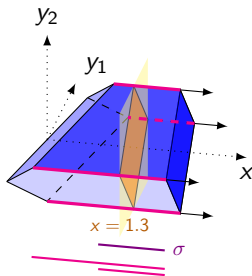
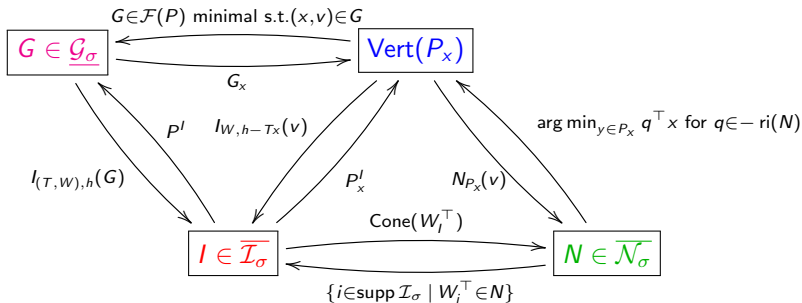
# Correspondences



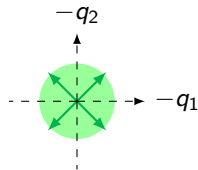
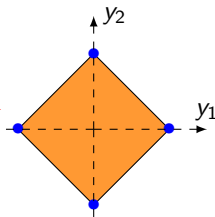
$$\overline{\mathcal{I}}_\sigma = \{12, 23, 34, 41\}$$



# Correspondences

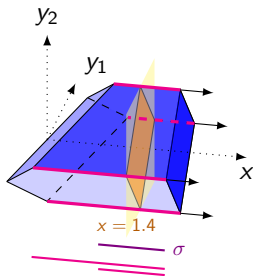
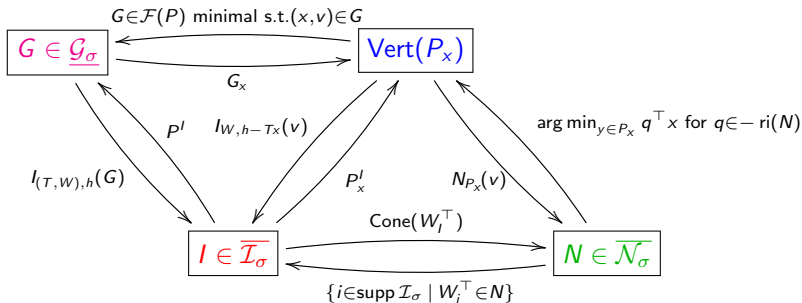


$$\overline{\mathcal{I}}_\sigma = \{12, 23, 34, 41\}$$

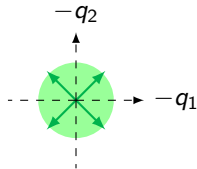
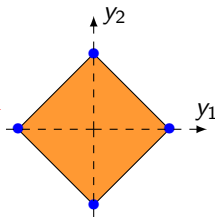




# Correspondences



$$\overline{\mathcal{I}}_\sigma = \{12, 23, 34, 41\}$$



# Simplex for 2SLP

$$y_1 + y_2 \leq 1$$

$$y_1 - y_2 \leq 1$$

$$-y_1 - y_2 \leq 1$$

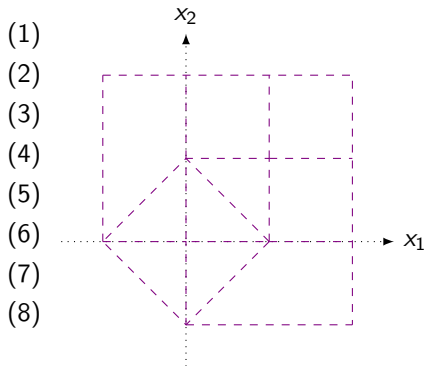
$$-y_1 + y_2 \leq 1$$

$$y_1 \leq x_1$$

$$y_2 \leq x_2$$

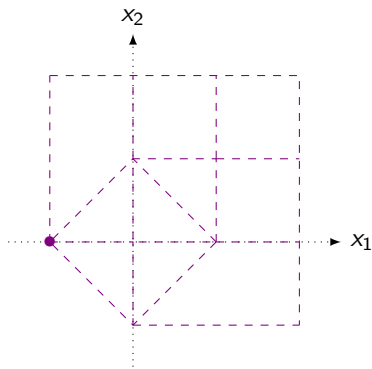
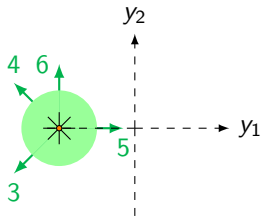
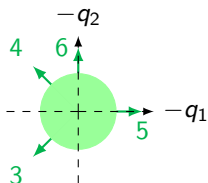
$$x_1 \leq 2$$

$$x_2 \leq 2$$



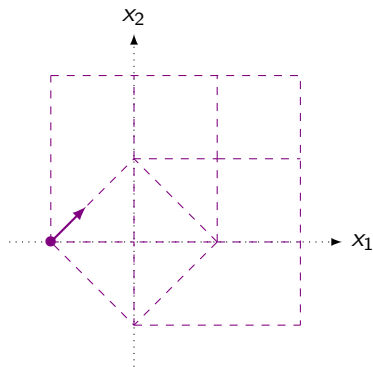
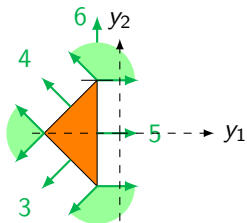
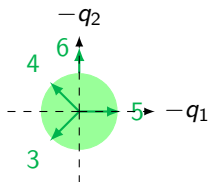
# Simplex for 2SLP

$$\overline{\mathcal{I}} = \{3456\}$$



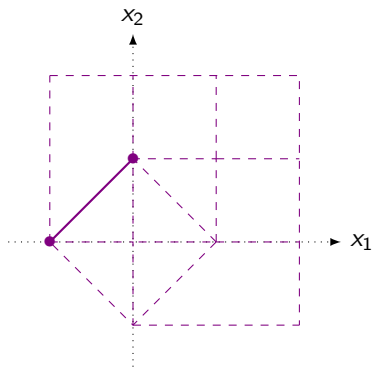
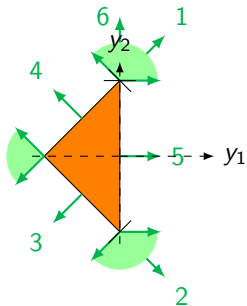
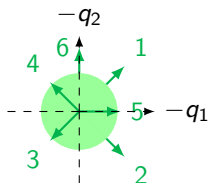
# Simplex for 2SLP

$$\overline{\mathcal{I}} = \{34, 35, 456\}$$



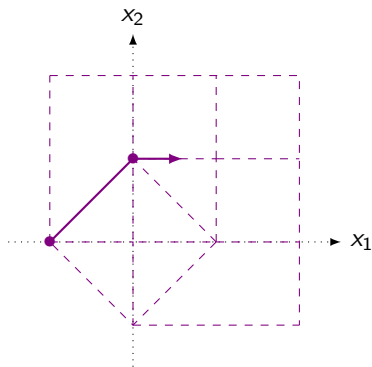
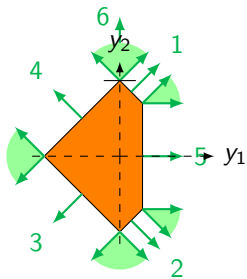
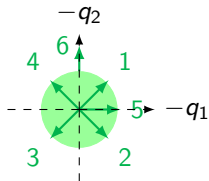
# Simplex for 2SLP

$$\bar{\mathcal{I}} = \{34, 235, 1456\}$$



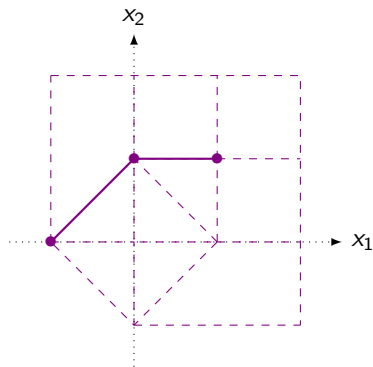
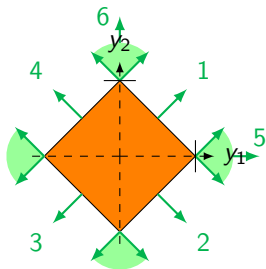
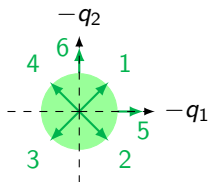
# Simplex for 2SLP

$$\overline{\mathcal{I}} = \{34, 23, 25, 146, 15\}$$



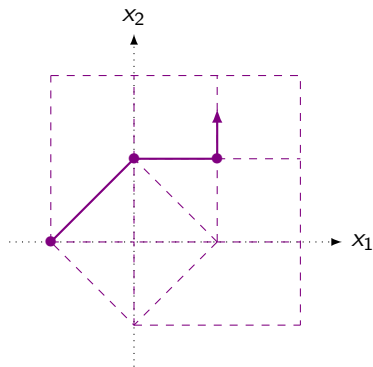
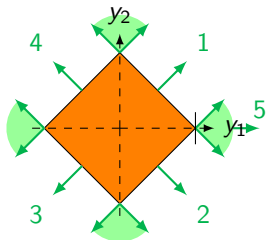
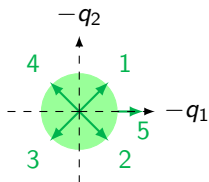
# Simplex for 2SLP

$$\overline{\mathcal{I}} = \{34, 23, 125, 146\}$$



# Simplex for 2SLP

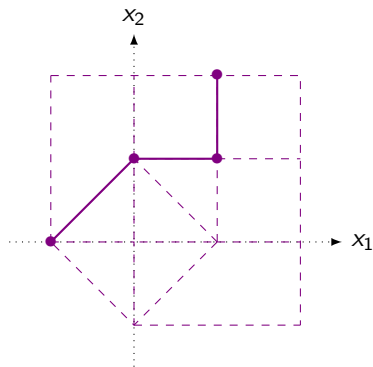
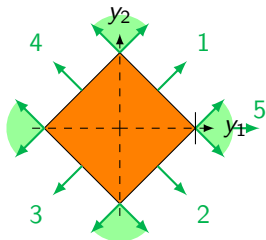
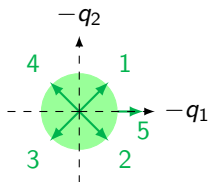
$$\overline{\mathcal{I}} = \{34, 23, 125, 14\}$$





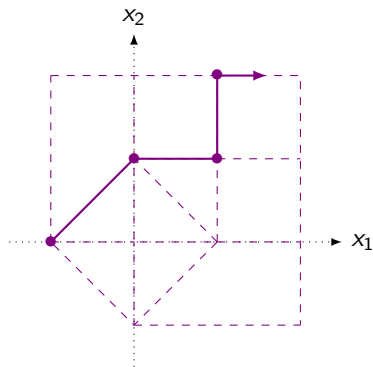
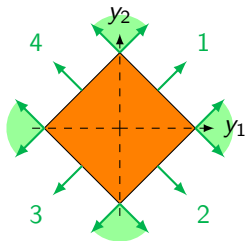
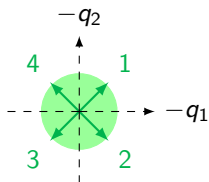
# Simplex for 2SLP

$$\bar{\mathcal{I}} = \{348, 238, 1258, 148\}$$



# Simplex for 2SLP

$$\bar{\mathcal{I}} = \{348, 238, 128, 148\}$$



# Simplex for 2SLP

$$\bar{\mathcal{I}} = \{3478, 2378, 1278, 1478\}$$

