2-Stage Stochastic Linear Problem and Polyhedral Geometry

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Contents

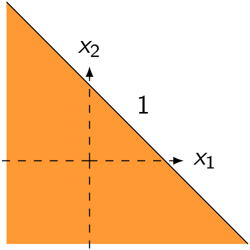
Linear Programming

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leqslant b \end{aligned}$$

Example: $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ & \\ & \\ & \\ & \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ \\ \\ \\ \\ \\ \\ \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 \leqslant 1 \quad & (1) \\ & (2) \\ & (3) \\ & (4) \\ & (5) \\ & (6) \\ & (7) \end{aligned}$$



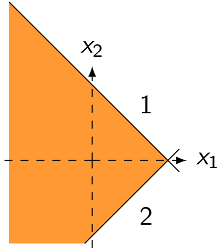
Linear Programming

$$\begin{array}{ll} \min\limits_{x \in \mathbb{R}^n} & c^\top x \\ \text{s.t.} & Ax \leqslant b \end{array}$$

Example: $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ & \\ & \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ \\ \\ \\ \\ \end{pmatrix}$$

$x_1 + x_2 \leqslant 1$	(1)
$x_1 - x_2 \leqslant 1$	(2)
	(3)
	(4)
	(5)
	(6)
	(7)

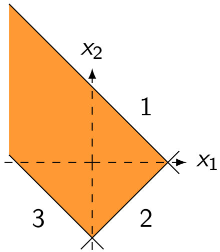


Linear Programming

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leqslant b \end{aligned}$$

Example: $P = \{x \in \mathbb{R}^n \,|\, Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ & \\ & \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \\ \\ \\ \\ \end{pmatrix} \quad \begin{aligned} x_1 + x_2 &\leqslant 1 && (1) \\ x_1 - x_2 &\leqslant 1 && (2) \\ -x_1 - x_2 &\leqslant 1 && (3) \\ & && (4) \\ & && (5) \\ & && (6) \\ & && (7) \end{aligned}$$



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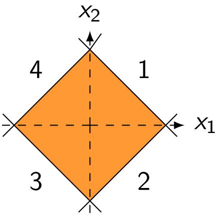
Example: $P = \{x \in \mathbb{R}^n \,|\, Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ & \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \\ \\ \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &\leqslant 1 && (1) \\ x_1 - x_2 &\leqslant 1 && (2) \\ -x_1 - x_2 &\leqslant 1 && (3) \\ -x_1 + x_2 &\leqslant 1 && (4) \\ & && (5) \\ & && (6) \\ & && (7) \end{aligned}$$



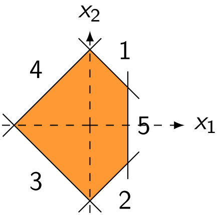
Linear Programming

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leqslant b \end{aligned}$$

Example: $P = \{x \in \mathbb{R}^n \,|\, Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 0 \\ & \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0.5 \\ \\ \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &\leqslant 1 && (1) \\ x_1 - x_2 &\leqslant 1 && (2) \\ -x_1 - x_2 &\leqslant 1 && (3) \\ -x_1 + x_2 &\leqslant 1 && (4) \\ x_1 &\leqslant 0.5 && (5) \\ & && (6) \\ & && (7) \end{aligned}$$



Linear Programming

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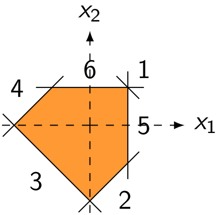
$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ & \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0.5 \\ 0.5 \\ \\ \end{pmatrix} \quad \begin{aligned} x_1 + x_2 &\leqslant 1 && (1) \\ x_1 - x_2 &\leqslant 1 && (2) \\ -x_1 - x_2 &\leqslant 1 && (3) \\ -x_1 + x_2 &\leqslant 1 && (4) \\ x_1 &\leqslant 0.5 && (5) \\ x_2 &\leqslant 0.5 && (6) \\ & && (7) \end{aligned}$$



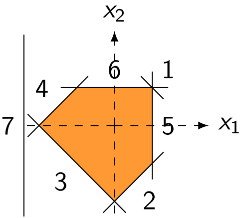
Linear Programming

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leqslant b \end{aligned}$$

Example: $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0.5 \\ 0.5 \\ -1.2 \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &\leqslant 1 & (1) \\ x_1 - x_2 &\leqslant 1 & (2) \\ -x_1 - x_2 &\leqslant 1 & (3) \\ -x_1 + x_2 &\leqslant 1 & (4) \\ x_1 &\leqslant 0.5 & (5) \\ x_2 &\leqslant 0.5 & (6) \\ x_1 &\geqslant -1.2 & (7) \end{aligned}$$



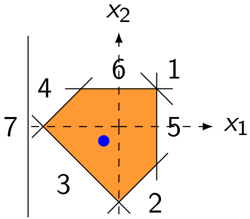
Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \emptyset$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, \qquad\qquad\qquad\qquad\qquad\qquad \}$$

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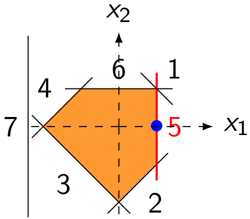

$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{5\}$

To ease the notation, we write:

$\mathcal{I}(A,b) = \{\emptyset, 5, \qquad\qquad\qquad\qquad \}$

Active constraints

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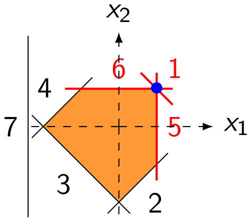
$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{1, 5, 6\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, \qquad \}$$

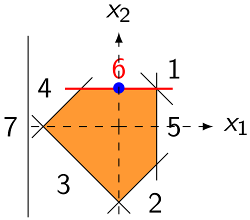
Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{6\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, \qquad \}$$

Active constraints

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We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

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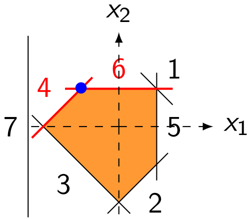
with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{4, 6\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, \qquad\qquad \}$$

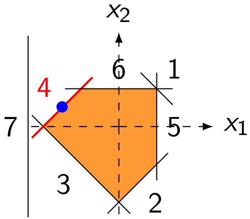
Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

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To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, \qquad\qquad\}$$

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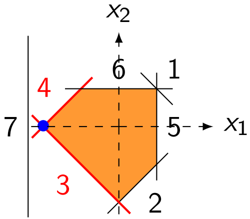

$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{3, 4\}$

To ease the notation, we write:

$$\mathcal{I}(A,b) = \{\emptyset, 5, 156, 6, 46, 4, 34, \qquad \}$$

Active constraints

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We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

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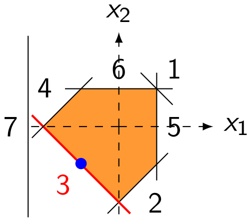

$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{3\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, 34, 3, \qquad \}$$

Active constraints

Definition

We denote by $\mathcal{I}(A, b)$*, the collection of sets of active constraints as :*

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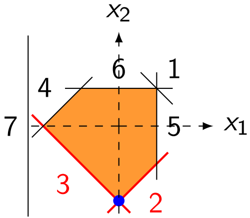
with $I_{A,b}(x) := \{i \in [q] \mid A_i x = b_i\}$



$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{2, 3\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, 34, 3, 23, \qquad \}$$

Active constraints

Definition

We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

$$\mathcal{I}(A, b) = \{I_{A,b}(x) \mid Ax \leqslant b\}$$

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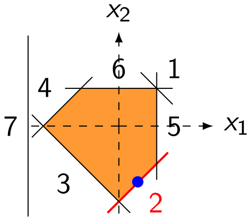

$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{2\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \{\emptyset, 5, 156, 6, 46, 4, 34, 3, 23, 2, \quad \}$$

Active constraints

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We denote by $\mathcal{I}(A, b)$, the collection of sets of active constraints as :

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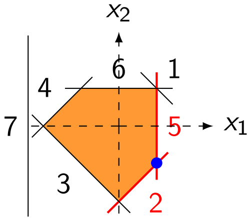


$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

$I_{A,b}(x) = \{2, 5\}$

To ease the notation, we write:

$$\mathcal{I}(A, b) = \big\{\emptyset, 5, 156, 6, 46, 4, 34, 3, 23, 2, 25\big\}$$

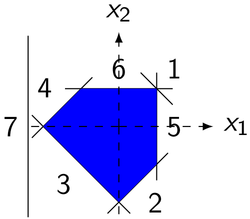
Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$
Example for $I = \emptyset$



P, P^I

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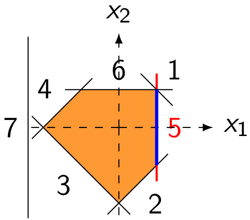
Example for $I = \{5\}$



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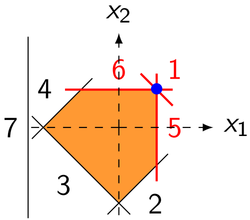
Example for $I = \{1, 5, 6\}$



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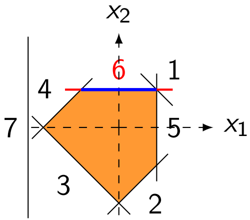
Example for $I = \{6\}$



P, P^I

Faces

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Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \,|\, A_I x = b_I\}$$

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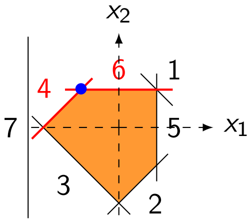
Example for $I = \{4, 6\}$



P, P^I

Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

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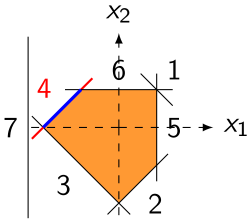
Example for $I = \{4\}$



P, P^I

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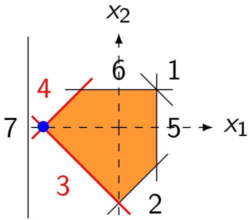
Example for $I = \{3, 4\}$



P, P^I

Faces

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Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

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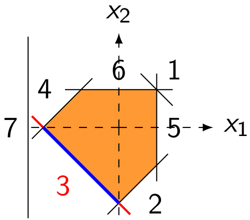
We have $\dim(P^I) = n - \mathrm{rg}(A_I)$
Example for $I = \{3\}$



P, P^I

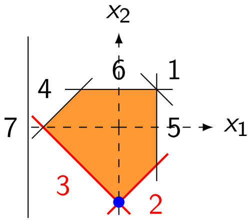
Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \,|\, A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$
Example for $I = \{2, 3\}$



P, P^I

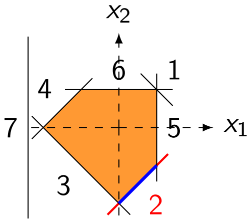
Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

$$P^I = \{x \in P \mid A_I x = b_I\}$$

We have $\dim(P^I) = n - \mathrm{rg}(A_I)$
Example for $I = \{2\}$



P, P^I

Faces

Definition

Let $I \in \mathcal{I}(A, b)$, we denote by P^I the face of P such that:

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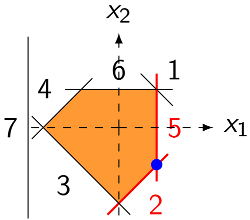
We have $\dim(P^I) = n - \mathrm{rg}(A_I)$
Example for $I = \{2, 5\}$



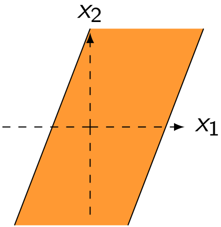
P, P^I

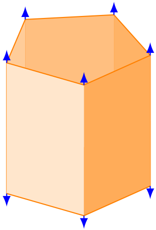
Polyhedra without any vertex ?

Definition (Lineality space)

$\mathrm{Lin}(C) := \{u \in C \mid \forall t \in \mathbb{R},\ \forall x \in c,\ x + tu \in C\}.$

$$\mathrm{Lin}\left(\{x \,|\, Ax \leqslant b\}\right) = \mathrm{Ker}(A)$$





Bases and Vertices

Let $P = \{x \in \mathbb{R}^n | Ax \leqslant b\}$ with $A \in \mathbb{R}^{p \times n}$ and $b \in \mathbb{R}^p$.

Definition

A basis B is a subset of $[p]$ such that $A_B = (A_{i,j})_{i \in B, 1 \leqslant j \leqslant n}$ is invertible.
A vertex of P is a face of dimension 0. $\mathrm{Vert}(P)$ is the set of vertices.

$$\begin{aligned} \mathrm{Vert}(P) \neq \emptyset &\iff A \text{ admits at least one basis} \\ &\iff \mathrm{rg}(A) = n \\ &\iff \mathrm{Lin}(P) = \{0\} \end{aligned}$$

Under this assumption,
For every $I \in \overline{\mathcal{I}(A, b)}$, we can extract a basis B_I and $P^I = \{A_{B_I}^{-1} b_{B_I}\}$.
If $c \notin \mathrm{Lin}(P)^\perp = \mathrm{Im}(A^\top)$, $\min_{x \in P} c^\top x = -\infty$.
Otherwise, we can write $P = P_0 + \mathrm{Lin}(P)$ with $\mathrm{Lin}(P_0) = \{0\}$:
We make this assumption without loss of generality

Bases and Vertices

Let $P = \{x \in \mathbb{R}^n | Ax \leqslant b\}$ with $A \in \mathbb{R}^{p \times n}$ and $b \in \mathbb{R}^p$.

Definition

A basis B is a subset of $[p]$ such that $A_B = (A_{i,j})_{i \in B, 1 \leqslant j \leqslant n}$ is invertible.
A vertex of P is a face of dimension 0. $\mathrm{Vert}(P)$ is the set of vertices.

$$\begin{aligned} \mathrm{Vert}(P) \neq \emptyset &\iff A \text{ admits at least one basis} \\ &\iff \mathrm{rg}(A) = n \\ &\iff \mathrm{Lin}(P) = \{0\} \end{aligned}$$

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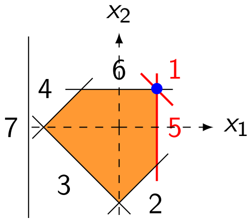
Simplex method

Geometrically:
follow a path on the polyhedron from vertex to vertex



Combinatorially:
pivoting from basis to basis

$$B_1 = \{1, 5\}$$

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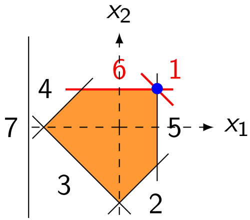

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$$B_2 = \{1, 6\}$$

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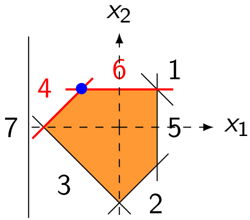

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$$B_3 = \{4, 6\}$$

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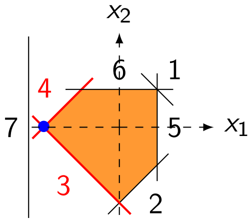

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$$B_1 = \{1,5\}$$
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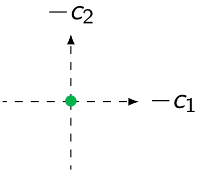
Normal fan $\mathcal{N}(P)$

Definition

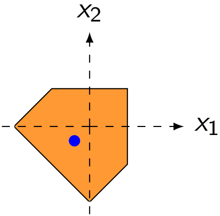
The normal fan of the polyhedron P is

$$\mathcal{N}(P) := \{N_P(x) \,|\, x \in P\}$$

with $N_P(x) = \{c \,|\, \forall x' \in P,\ c^\top(x' - x) \leqslant 0\}$ *the normal cone of P on x.*



$N_P(y)$



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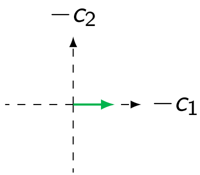

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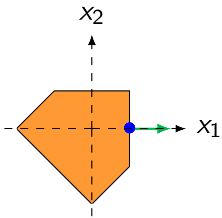

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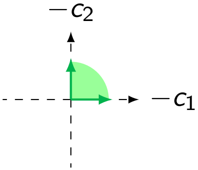

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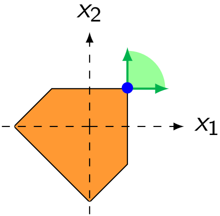

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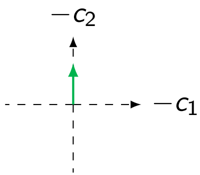

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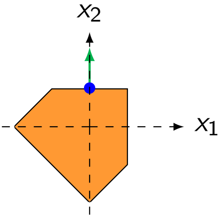

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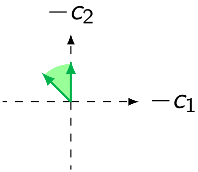

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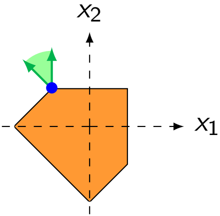

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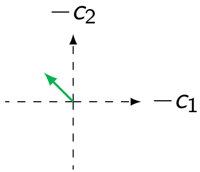

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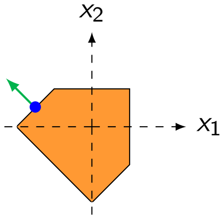

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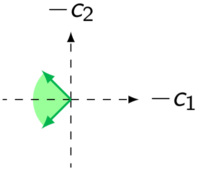

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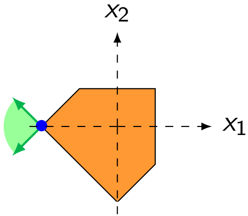

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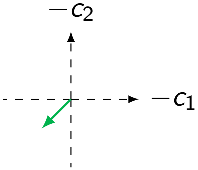

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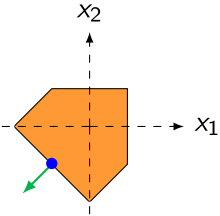

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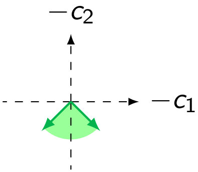

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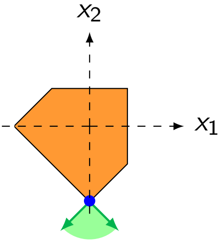

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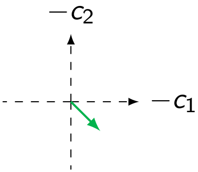

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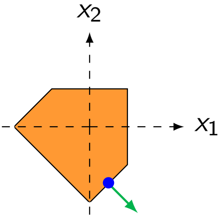

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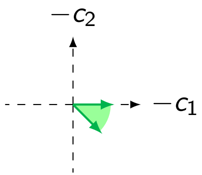

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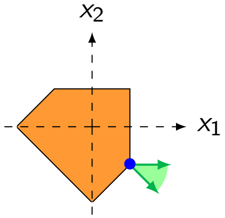

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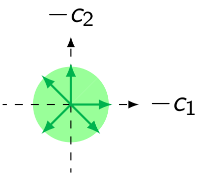
Proposition

$\{\mathrm{ri}(N) \mid N \in \mathcal{N}(P)\}$ *is a partition of* $\operatorname{supp} \mathcal{N}(P)$ *(*$= \mathbb{R}^m$ *if* P *is bounded).*



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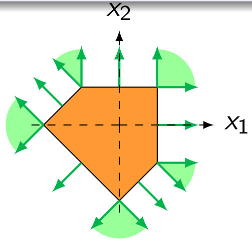


P and $\mathcal{N}(P)$

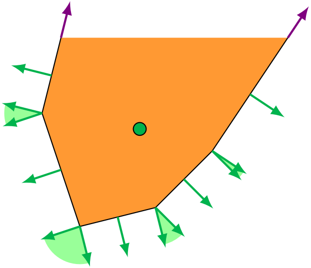
Definition (Recession cone)

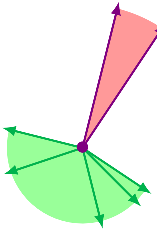
$\text{rc}(C) := \{u \in C \mid \forall t \in \mathbb{R}_+,\ \forall x \in c,\ x + tu \in C\}.$

Let $P = \{x \mid Ax \leqslant b\}$

$$\text{rc}(P) = \{u \mid Au \leqslant 0\}$$

$$-\infty < \begin{cases} \inf_{x\in\mathbb{R}^n} & c^\top x \\ \text{s.t.} & Ax \leqslant b \end{cases} \iff -c \in \text{rc}(P)^\star = \text{Cone}(A^\top) = \text{supp}\left(\mathcal{N}(P)\right)$$

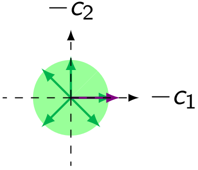




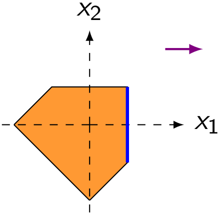
$\mathcal{N}(P)$: partition of cost coherent with the min

For any $N \in \mathcal{N}(P)$ and $-c \to \arg\min_{x \in P} c^\top x$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{x \in P} c^\top x$ is a face of P.



Cost $-c$ and $\mathcal{N}(P)$



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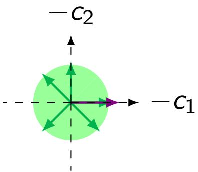

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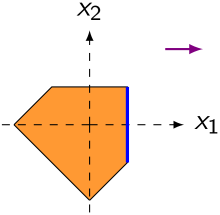

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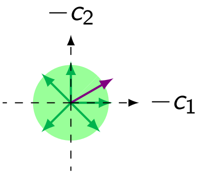

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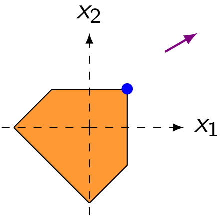

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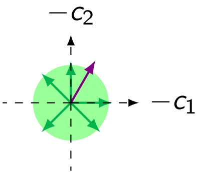

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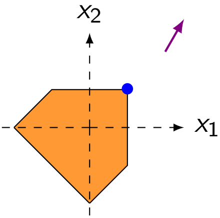

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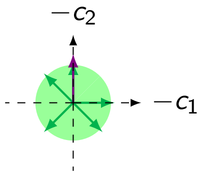

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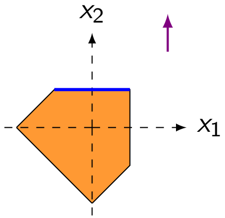

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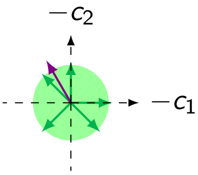

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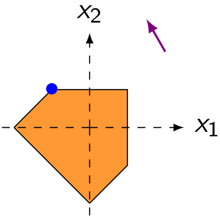

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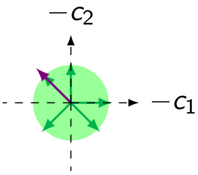

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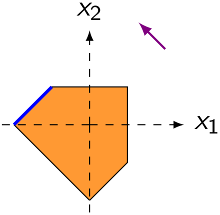

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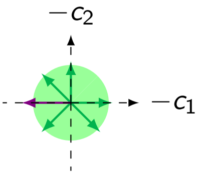

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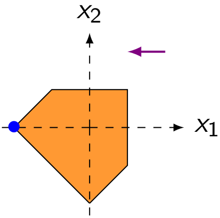

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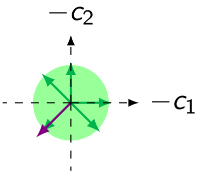
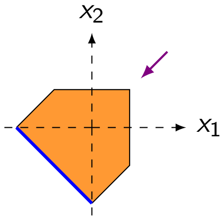


Cost $-c$ and $\mathcal{N}(P)$

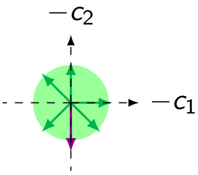


P

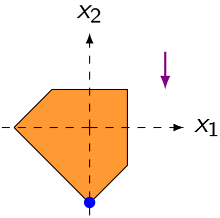
$\mathcal{N}(P)$: partition of cost coherent with the min

For any $N \in \mathcal{N}(P)$ and $-c \to \arg\min_{x \in P} c^\top x$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{x \in P} c^\top x$ is a face of P.



$\mathcal{N}(P)$

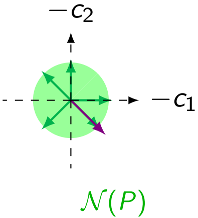


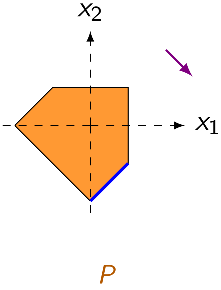
P

$\mathcal{N}(P)$: partition of cost coherent with the min

For any $N \in \mathcal{N}(P)$ and $-c \to \arg\min_{x\in P} c^\top x$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{x\in P} c^\top x$ is a face of P.

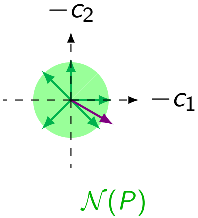


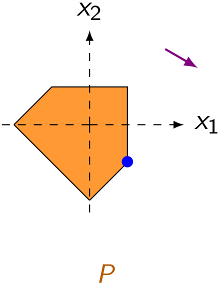


$\mathcal{N}(P)$: partition of cost coherent with the min

For any $N \in \mathcal{N}(P)$ and $-c \to \arg\min_{x \in P} c^\top x$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{x \in P} c^\top x$ is a face of P.

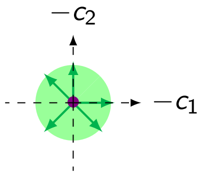




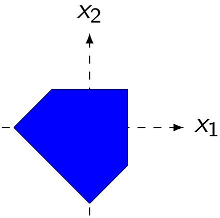
$\mathcal{N}(P)$: partition of cost coherent with the min

For any $N \in \mathcal{N}(P)$ and $-c \to \arg\min_{x \in P} c^\top x$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{x \in P} c^\top x$ is a face of P.



$\mathcal{N}(P)$

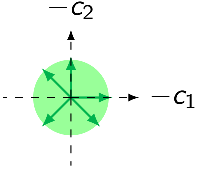


P

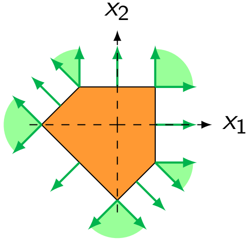
$\mathcal{N}(P)$: partition of cost coherent with the min

For any $N \in \mathcal{N}(P)$ and $-c \to \arg\min_{x \in P} c^\top x$ is constant for all $-c \in \mathrm{ri}(N)$.

$\arg\min_{x \in P} c^\top x$ is a face of P.



$\mathcal{N}(P)$



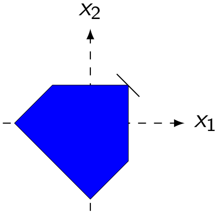
P

Correspondences

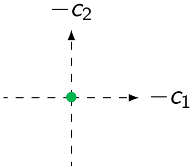
$$\begin{tikzcd}
& \boxed{F \in \mathcal{F}(P)\setminus\{\emptyset\}, \supset} \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x\in \mathrm{ri}(F)"'] \arrow[dr, bend right, "N_P(x) \text{ for } x\in \mathrm{ri}(F)"'] & \\
\boxed{I \in \mathcal{I}(A,b), \subset} \arrow[ur, bend right, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & \boxed{N \in \mathcal{N}(P), \subset} \arrow[ul, bend right, "\arg\min_{x\in P} c^\top x \text{ for } c\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in \mathrm{supp}\, \mathcal{I}(A,b) \mid A_i^\top \in N\}"]
\end{tikzcd}$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \emptyset$



F



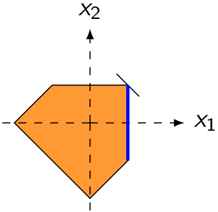
N

Correspondences

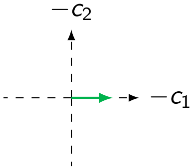
$$
\begin{tikzcd}[column sep=small]
 & {\color{blue}F \in \mathcal{F}(P)}\setminus\{\emptyset\}, \supset \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x \in \mathrm{ri}(F)"'] \arrow[dr, bend right, "N_P(x) \text{ for } x \in \mathrm{ri}(F)"'] & \\
{\color{red}I \in \mathcal{I}(A,b)}, \subset \arrow[ur, bend right, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & {\color{green}N \in \mathcal{N}(P)}, \subset \arrow[ul, bend right, "\arg\min_{x \in P} c^\top x \text{ for } c \in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i \in \mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top \in N\}"]
\end{tikzcd}
$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{5\}$



F



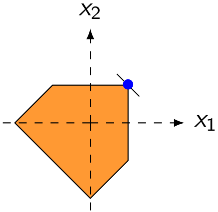
N

Correspondences

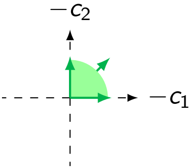
$$\begin{tikzcd}
 & F \in \mathcal{F}(P)\setminus\{\emptyset\}, \supset \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x\in\mathrm{ri}(F)"'] \arrow[dr, bend left, "N_P(x) \text{ for } x\in\mathrm{ri}(F)"] & \\
I \in \mathcal{I}(A,b), \subset \arrow[ur, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & N \in \mathcal{N}(P), \subset \arrow[ul, bend right, "\arg\min_{x\in P} c^\top x \text{ for } c\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in\mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top\in N\}"]
\end{tikzcd}$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{1, 5, 6\}$



F



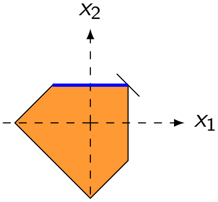
N

Correspondences

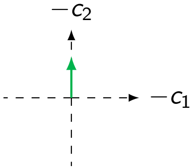
$$\begin{tikzcd}
 & \color{blue}{F \in \mathcal{F}(P)}\setminus\{\emptyset\}, \supset \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x\in\mathrm{ri}(F)"'] \arrow[dr, bend left, "N_P(x) \text{ for } x\in\mathrm{ri}(F)"'] & \\
\color{red}{I \in \mathcal{I}(A,b)}, \subset \arrow[ur, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & \color{green}{N \in \mathcal{N}(P)}, \subset \arrow[ul, bend right, "\arg\min_{x\in P} c^\top x \text{ for } c\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in\mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top \in N\}"]
\end{tikzcd}$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{6\}$

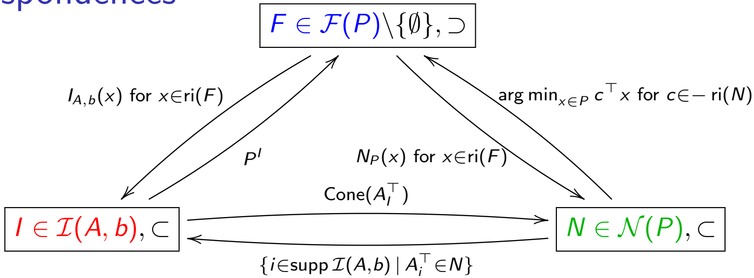


F



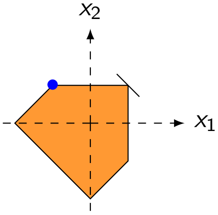
N

Correspondences

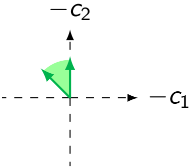


$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{4, 6\}$



F



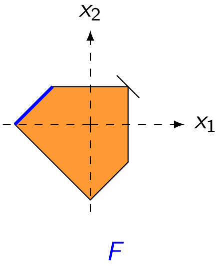
N

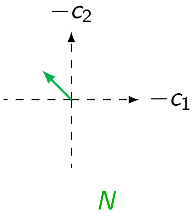
Correspondences

$$
\begin{tikzcd}
 & F \in \mathcal{F}(P)\backslash\{\emptyset\}, \supset \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x \in \mathrm{ri}(F)"'] \arrow[dr, "N_P(x) \text{ for } x \in \mathrm{ri}(F)"'] & \\
I \in \mathcal{I}(A,b), \subset \arrow[ur, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & N \in \mathcal{N}(P), \subset \arrow[ul, bend right, "\arg\min_{x \in P} c^\top x \text{ for } c \in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i \in \mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top \in N\}"]
\end{tikzcd}
$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{4\}$



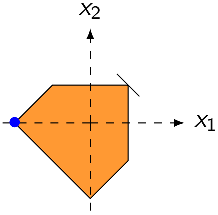


Correspondences

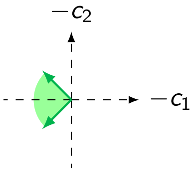
$$
\begin{tikzcd}
 & F \in \mathcal{F}(P)\backslash\{\emptyset\}, \supset \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x\in \mathrm{ri}(F)"'] \arrow[dr, bend left, "N_P(x) \text{ for } x\in \mathrm{ri}(F)"] & \\
I \in \mathcal{I}(A,b), \subset \arrow[ur, bend right, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & N \in \mathcal{N}(P), \subset \arrow[ul, bend left, "\arg\min_{x\in P} c^\top x \text{ for } c\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in \mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top \in N\}"]
\end{tikzcd}
$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{3, 4\}$



F



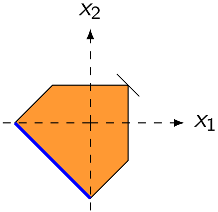
N

Correspondences

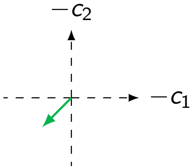
$$\begin{tikzcd}[column sep=large, row sep=huge]
& F \in \mathcal{F}(P)\backslash\{\emptyset\}, \supset \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x\in\mathrm{ri}(F)"'] \arrow[dr, bend left, "N_P(x) \text{ for } x\in\mathrm{ri}(F)"] & \\
I \in \mathcal{I}(A,b), \subset \arrow[ur, bend right, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & N \in \mathcal{N}(P), \subset \arrow[ul, bend left, "\arg\min_{x\in P} c^\top x \text{ for } c\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in\mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top\in N\}"]
\end{tikzcd}$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{3\}$



F



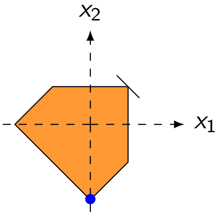
N

Correspondences

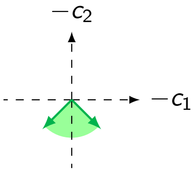
$$\begin{tikzcd}[column sep=small]
 & \boxed{\color{blue}{F \in \mathcal{F}(P)}\setminus\{\emptyset\}, \supset} \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x\in \mathrm{ri}(F)"'] \arrow[dr, bend left, "N_P(x) \text{ for } x\in \mathrm{ri}(F)"] & \\
\boxed{\color{red}{I \in \mathcal{I}(A,b)}, \subset} \arrow[ur, bend right, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & \boxed{\color{green}{N \in \mathcal{N}(P)}, \subset} \arrow[ul, bend left, "\arg\min_{x\in P} c^\top x \text{ for } c\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in \mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top \in N\}"]
\end{tikzcd}$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{2, 3\}$



F



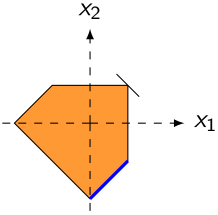
N

Correspondences

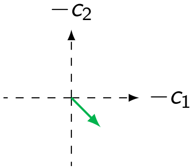
$$
\begin{tikzcd}
& F \in \mathcal{F}(P)\setminus\{\emptyset\}, \supset \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x\in\mathrm{ri}(F)"'] \arrow[dr, bend left, "N_P(x) \text{ for } x\in\mathrm{ri}(F)"'] & \\
I \in \mathcal{I}(A,b), \subset \arrow[ur, bend right, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & N \in \mathcal{N}(P), \subset \arrow[ul, bend right, "\arg\min_{x\in P} c^\top x \text{ for } c\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in\mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top\in N\}"]
\end{tikzcd}
$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{2\}$



F



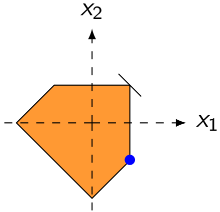
N

Correspondences

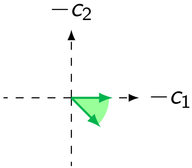
$$
\begin{tikzcd}
 & \boxed{F \in \mathcal{F}(P)\setminus\{\emptyset\}, \supset} \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x\in\mathrm{ri}(F)"'] \arrow[dr, "N_P(x) \text{ for } x\in\mathrm{ri}(F)"'] & \\
\boxed{I \in \mathcal{I}(A,b), \subset} \arrow[ur, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & \boxed{N \in \mathcal{N}(P), \subset} \arrow[ul, bend right, "\arg\min_{x\in P} c^\top x \text{ for } c\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in\mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top \in N\}"]
\end{tikzcd}
$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$I = \{2, 5\}$



F



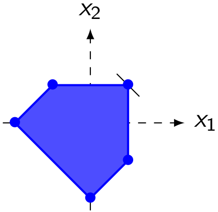
N

Correspondences

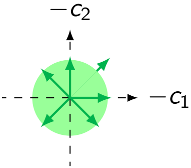
$$\begin{tikzcd}
 & \boxed{F \in \mathcal{F}(P)\backslash\{\emptyset\}, \supset} \arrow[dl, bend right, "I_{A,b}(x) \text{ for } x\in \mathrm{ri}(F)"'] \arrow[dr, bend left, "N_P(x) \text{ for } x\in \mathrm{ri}(F)"] & \\
\boxed{I \in \mathcal{I}(A,b), \subset} \arrow[ur, bend right, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & \boxed{N \in \mathcal{N}(P), \subset} \arrow[ll, bend left, "\{i\in \mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top \in N\}"] \arrow[ul, bend right, "\arg\min_{x\in P} c^\top x \text{ for } c\in -\mathrm{ri}(N)"']
\end{tikzcd}$$

$$\mathrm{rg}(A_I) = n - \dim(F) = \dim(N)$$

$\mathcal{I}(A, b) =$
$\{\emptyset, 5, 156, 6, 46, 4,$
$34, 3, 23, 2, 25\}$



$\mathcal{F}(P)$

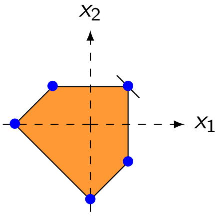


$\mathcal{N}(P)$

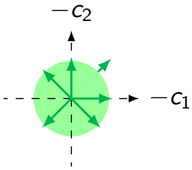
Correspondences

$$\begin{tikzcd}
& v \in \mathrm{Vert}(P) \arrow[dl, bend right, "I_{A,b}(v)"'] \arrow[dr, bend left, "N_P(v)"] & \\
I \in \overline{\mathcal{I}(A,b)} \arrow[ur, bend right, "P^I"'] \arrow[rr, bend left, "\mathrm{Cone}(A_I^\top)"] & & N \in \overline{\mathcal{N}(P)} \arrow[ul, bend right, "\arg\min_{x\in P} c^\top x \text{ for } c\in - \mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in \mathrm{supp}\,\mathcal{I}(A,b) \mid A_i^\top \in N\}"]
\end{tikzcd}$$

$\overline{\mathcal{I}(A,b)} = \{156, 46, 34, 23, 25\}$



$\mathrm{Vert}(P)$



$\overline{\mathcal{N}(P)}$

Contents

2-Stage Stochastic Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^\top x + \mathbb{E}\left[\begin{array}{ll} \min\limits_{y \in \mathbb{R}^m} & \mathbf{q}^\top y \\ \text{s.t.} & Tx + Wy \leqslant h \\ & \end{array}\right] \qquad \text{(2SLP)}$$

$$\text{s.t.} \quad Ax \leqslant b$$

where $T \in \mathbb{R}^{p \times n}$, $W \in \mathbb{R}^{p \times m}$ and $h \in \mathbb{R}^p$.
We can assume $A = 0$ and $b = 0$:
We set

$$\widetilde{T} := \begin{pmatrix} T \\ A \end{pmatrix}, \quad \widetilde{W} := \begin{pmatrix} W \\ 0 \end{pmatrix} \quad \text{and } \widetilde{h} = \begin{pmatrix} h \\ b \end{pmatrix}$$

2-Stage Stochastic Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^\top x + \mathbb{E}\begin{bmatrix} \min\limits_{y \in \mathbb{R}^m} & \mathbf{q}^\top y & \\ \text{s.t.} & Tx + Wy & \leqslant h \\ & Ax & \leqslant b \end{bmatrix}$$

where $T \in \mathbb{R}^{p \times n}, W \in \mathbb{R}^{p \times m}$ and $h \in \mathbb{R}^p$.
We can assume $A = 0$ and $b = 0$:
We set

$$\widetilde{T} := \begin{pmatrix} T \\ A \end{pmatrix}, \quad \widetilde{W} := \begin{pmatrix} W \\ 0 \end{pmatrix} \quad \text{and } \widetilde{h} = \begin{pmatrix} h \\ b \end{pmatrix}$$

2-Stage Stochastic Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^\top x + \mathbb{E}\left[\begin{array}{ll} \min\limits_{y \in \mathbb{R}^m} & \mathbf{q}^\top y \\ \text{s.t.} & Tx + Wy \leqslant h \\ & Ax + 0y \leqslant b \end{array}\right]$$

where $T \in \mathbb{R}^{p \times n}, W \in \mathbb{R}^{p \times m}$ and $h \in \mathbb{R}^p$.
We can assume $A = 0$ and $b = 0$:
We set

$$\widetilde{T} := \begin{pmatrix} T \\ A \end{pmatrix}, \quad \widetilde{W} := \begin{pmatrix} W \\ 0 \end{pmatrix} \quad \text{and } \widetilde{h} = \begin{pmatrix} h \\ b \end{pmatrix}$$

2-Stage Stochastic Linear Programming

$$\min_{x \in \mathbb{R}^n} \quad c^\top x + \mathbb{E}\left[\begin{array}{ll} \min\limits_{y \in \mathbb{R}^m} & \mathbf{q}^\top y \\ \text{s.t.} & \widetilde{T}x + \widetilde{W}y \leqslant \widetilde{h} \\ & \end{array}\right]$$

where $T \in \mathbb{R}^{p \times n}, W \in \mathbb{R}^{p \times m}$ and $h \in \mathbb{R}^p$.
We can assume $A = 0$ and $b = 0$:
We set

$$\widetilde{T} := \begin{pmatrix} T \\ A \end{pmatrix}, \quad \widetilde{W} := \begin{pmatrix} W \\ 0 \end{pmatrix} \quad \text{and } \widetilde{h} = \begin{pmatrix} h \\ b \end{pmatrix}$$

2-Stage Stochastic Linear Programming

$$\min_{x \in \mathbb{R}^n} c^\top x + V(x) \tag{2SLP}$$

where

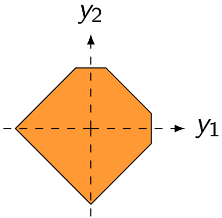
$$V(x) := \mathbb{E}\left[\begin{array}{ll} \min\limits_{y \in \mathbb{R}^m} & \mathbf{q}^\top y \\ \text{s.t.} & Tx + Wy \leqslant h \end{array}\right]$$

Fiber P_x

$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{q}^\top y\Big] \quad \text{where} \quad P_x := \{y \in \mathbb{R}^m \mid Tx + Wy \leqslant h\}$$

We assume $\operatorname{supp}(\mathbf{q}) \subset -\operatorname{Cone}(W^\top)$ i.e. $V(x) > -\infty$. Example:

$$T = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix} \quad W = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



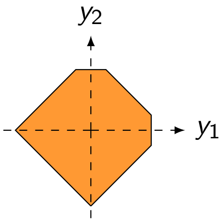
P_x for $x = 0.8$

Fiber P_x

$$V(x) = \mathbb{E}\Big[\min_{y\in P_x} \mathbf{q}^\top y\Big] \quad \text{where} \quad P_x := \{y \in \mathbb{R}^m \mid Tx + Wy \leqslant h\}$$

We assume $\mathrm{supp}(\mathbf{q}) \subset -\mathrm{Cone}(W^\top)$ i.e. $V(x) > -\infty$. Example:

$$\begin{aligned}
y_1 + y_2 &\leqslant 1 && (1)\\
y_1 - y_2 &\leqslant 1 && (2)\\
-y_1 - y_2 &\leqslant 1 && (3)\\
-y_1 + y_2 &\leqslant 1 && (4)\\
y_1 &\leqslant x && (5)\\
y_2 &\leqslant x && (6)\\
x &\leqslant 1.5 && (7)
\end{aligned}$$



P_x for $x = 0.8$

Expectation to final sum

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
 & \boxed{F \in \mathcal{F}(P_x)\backslash\{\emptyset\}, \supset} \arrow[dl, bend right=15, "I_{W,h-Tx}(y) \text{ for } y \in \mathrm{ri}(F)"'] \arrow[dr, bend left=15, "N_{P_x}(y) \text{ for } y \in \mathrm{ri}(F)"] & \\
\boxed{I \in \mathcal{I}(W, h - Tx), \subset} \arrow[ur, bend right=15, "P_x^I"'] \arrow[rr, bend left=5, "\mathrm{Cone}(W_I^\top)"] & & \boxed{N \in \mathcal{N}(P_x), \subset} \arrow[ul, bend left=15, "\arg\min_{y \in P_x} q^\top y \text{ for } q \in -\mathrm{ri}(N)"'] \arrow[ll, bend left=5, "\{i \in \mathrm{supp}\,\mathcal{I}(W, h-Tx) \mid W_i^\top \in N\}"]
\end{tikzcd}
$$

$$
\begin{aligned}
V(x) &= \mathbb{E}\Big[\min_{y \in P_x} \mathbf{q}^\top y\Big] \\
&= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -\mathrm{ri}\, N}\big] y_N(x) \quad \text{with } y_N(x) \in \cap_{q \in -N} \arg\min_{y \in P_x} q^\top y
\end{aligned}
$$

Expectation to final sum

$$\begin{tikzcd}[column sep=huge, row sep=huge]
 & \boxed{F \in \mathcal{F}(P_x)\backslash\{\emptyset\}, \supset} \arrow[dl, bend right, "I_{W,h-Tx}(y) \text{ for } y \in \mathrm{ri}(F)"'] \arrow[dr, bend left=10, "N_{P_x}(y) \text{ for } y \in \mathrm{ri}(F)"'] & \\
\boxed{I \in \mathcal{I}(W, h - Tx), \subset} \arrow[ur, bend right=10, "P_x^I"'] \arrow[rr, bend left=10, "\mathrm{Cone}(W_I^\top)"] & & \boxed{N \in \mathcal{N}(P_x), \subset} \arrow[ul, bend right, "\arg\min_{y \in P_x} q^\top y \text{ for } q \in -\mathrm{ri}(N)"'] \arrow[ll, bend left=10, "\{i \in \mathrm{supp}\, \mathcal{I}(W,h-Tx) \mid W_i^\top \in N\}"]
\end{tikzcd}$$

$$\begin{aligned}
V(x) &= \mathbb{E}\Big[\min_{y \in P_x} \mathbf{q}^\top y\Big] \\
&= \sum_{N \in \mathcal{N}(P_x)} \mathbb{E}\left[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -\mathrm{ri}\, N}\right] y_N(x) \quad \text{with } y_N(x) \in \cap_{q \in -N} \arg\min_{y \in P_x} q^\top y \\
&= \sum_{F \in \mathcal{F}(P_x)} \mathbb{E}\left[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -\mathrm{ri}\, N_{P_x}(F)}\right] y_F \quad \text{with } y_F \in F
\end{aligned}$$

Expectation to final sum

$$
\begin{tikzcd}
 & \boxed{F \in \mathcal{F}(P_x)\backslash\{\emptyset\}, \supset} \arrow[dl, bend right, "I_{W,h-Tx}(y) \text{ for } y\in \mathrm{ri}(F)"'] \arrow[dr, bend left, "N_{P_x}(y) \text{ for } y\in \mathrm{ri}(F)"] & \\
\boxed{I \in \mathcal{I}(W, h-Tx), \subset} \arrow[ur, bend right, "P_x^I"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & \boxed{N \in \mathcal{N}(P_x), \subset} \arrow[ul, bend left, "\arg\min_{y\in P_x} q^\top y \text{ for } q\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in \mathrm{supp}\, \mathcal{I}(W,h-Tx) \mid W_i^\top \in N\}"]
\end{tikzcd}
$$

$$
\begin{aligned}
V(x) &= \mathbb{E}\Big[\min_{y\in P_x} \mathbf{q}^\top y\Big] \\
&= \sum_{N\in\mathcal{N}(P_x)} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q}\in -\mathrm{ri}\, N}\big] y_N(x) \quad \text{with } y_N(x) \in \cap_{q\in -N} \arg\min_{y\in P_x} q^\top y \\
&= \sum_{F\in\mathcal{F}(P_x)} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q}\in -\mathrm{ri}\, N_{P_x}(F)}\big] y_F \quad \text{with } y_F \in F \\
&= \sum_{I\in\mathcal{I}(W,h-Tx)} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q}\in -\mathrm{ri}\, \mathrm{Cone}(W_I^\top)}\big] y_I(x) \quad \text{with } y_I(x) \in P_x^I
\end{aligned}
$$

Expectation to final sum

$$
\begin{tikzcd}
 & \boxed{v \in \mathrm{Vert}(P_x)} \arrow[dl, bend right, "I_{W,h-Tx}(v)"'] \arrow[dr, bend left, "N_{P_x}(v)"] & \\
\boxed{I \in \overline{\mathcal{I}(W,h-Tx)}} \arrow[ur, bend right, "P_x^I"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & \boxed{N \in \overline{\mathcal{N}(P_x)}, \subset} \arrow[ul, bend right, "\arg\min_{y\in P_x} q^\top y \text{ for } q \in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i \in \mathrm{supp}\, \mathcal{I}(W,h-Tx) \mid W_i^\top \in N\}"]
\end{tikzcd}
$$

If $\mathbf{q}$ has a density,

$$
\begin{aligned}
V(x) &= \mathbb{E}\Big[\min_{y\in P_x} \mathbf{q}^\top y\Big] \\
&= \sum_{N\in\overline{\mathcal{N}(P_x)}} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q}\in -N}\big] y_N(x) \quad \text{ with } y_N(x) \in \cap_{q\in -N} \operatorname*{arg\,min}_{y\in P_x} q^\top y \\
&= \sum_{v\in\mathrm{Vert}(P_x)} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q}\in -N_{P_x}(F)}\big] v \\
&= \sum_{I\in\overline{\mathcal{I}(W,h-Tx)}} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q}\in -\mathrm{Cone}(W_I^\top)}\big] y_I(x) \quad \text{ with } y_I(x) \in P_x^I
\end{aligned}
$$

Expectation to final sum

$$\begin{tikzcd}[column sep=huge, row sep=huge]
& \boxed{v \in \mathrm{Vert}(P_x)} \arrow[dl, bend right, "I_{W,h-Tx}(v)"'] \arrow[dr, bend left, "N_{P_x}(v)"'] & \\
\boxed{I \in \overline{\mathcal{I}(W, h-Tx)}} \arrow[ur, bend right, "P_x^I"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & \boxed{N \in \overline{\mathcal{N}(P_x)}, \subset} \arrow[ul, bend left, "\arg\min_{y \in P_x} q^\top y \text{ for } q \in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i \in \mathrm{supp}\, \mathcal{I}(W,h-Tx) \mid W_i^\top \in N\}"]
\end{tikzcd}$$

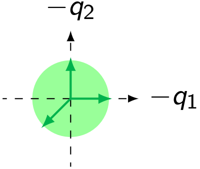
If $\mathbf{q}$ has a density,

$$\begin{aligned}
V(x) &= \mathbb{E}\big[\min_{y \in P_x} \mathbf{q}^\top y\big] \\
&= \sum_{N \in \overline{\mathcal{N}(P_x)}} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -N}\big] y_N(x) \quad \text{with } y_N(x) \in \cap_{q \in -N} \arg\min_{y \in P_x} q^\top y \\
&= \sum_{v \in \mathrm{Vert}(P_x)} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -N_{P_x}(F)}\big] v \\
&= \sum_{I \in \overline{\mathcal{I}(W,h-Tx)}} \mathbb{E}\big[\mathbf{q}^\top \mathbb{1}_{\mathbf{q} \in -\mathrm{Cone}(W_I^\top)}\big] W_{B_I}^{-1}(h_{B_I} - T_{B_I} x) \text{ with basis } B_I \subset I
\end{aligned}$$

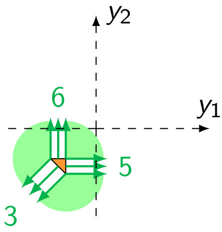
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

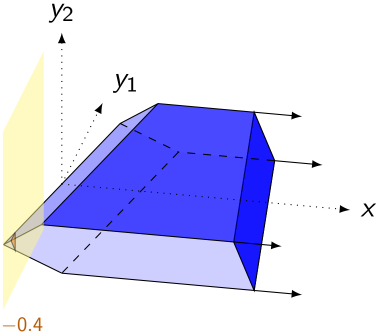
For $x = -0.4$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

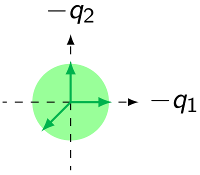


P and P_x

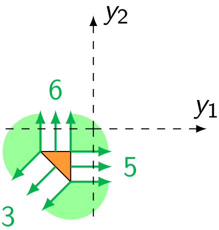
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

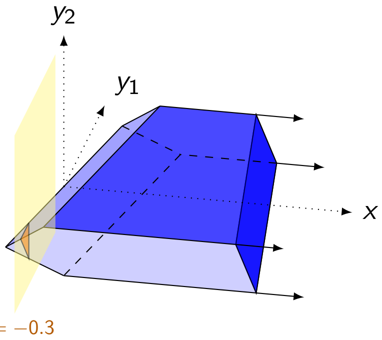
$$\text{For } x = -0.3, \quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

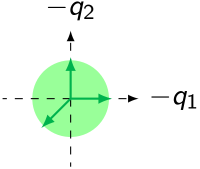


P and P_x

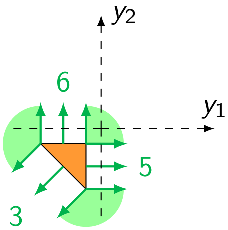
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

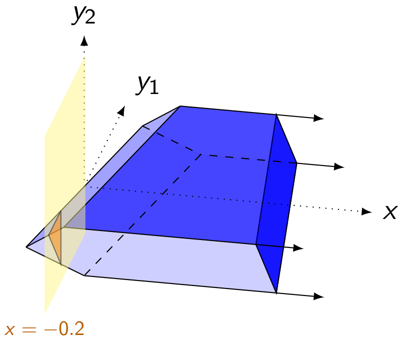
For $x = -0.2$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

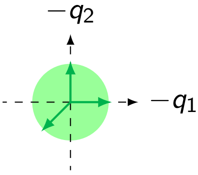


P and P_x

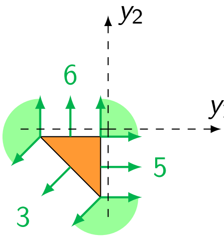
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

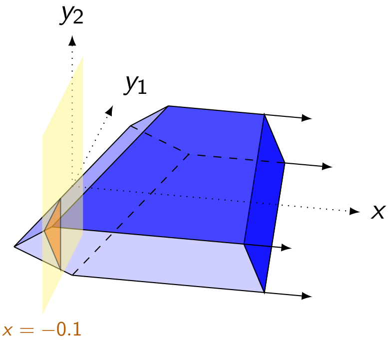
$$\text{For } x = -0.1, \quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

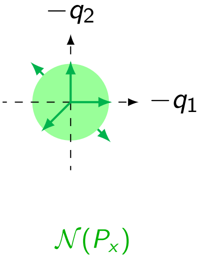


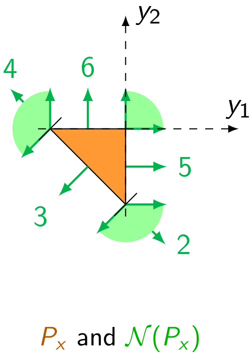
P and P_x

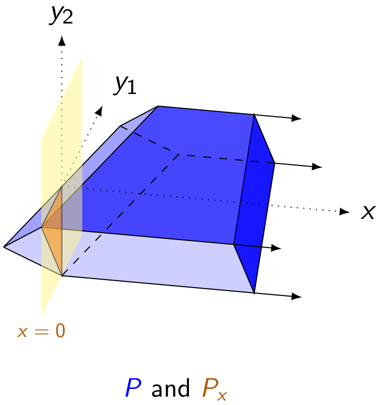
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

$$\text{For } x = 0, \quad \overline{\mathcal{I}(W, h - Tx)} = \{523, 346, 65\}$$



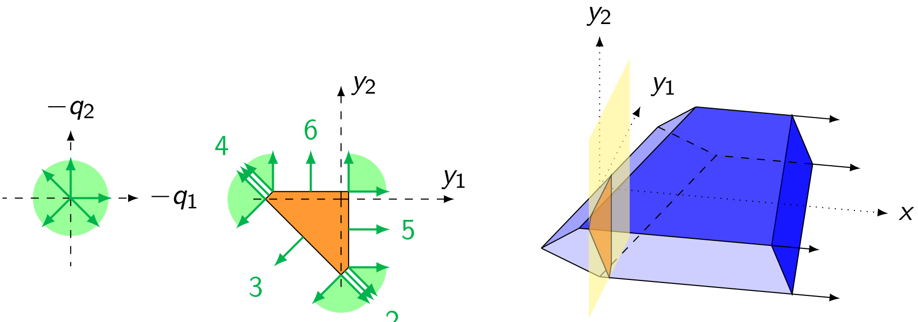




$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

$$\text{For } x = 0.1, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$$



$\mathcal{N}(P_x)$

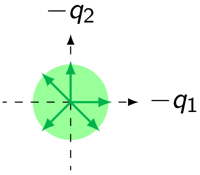
P_x and $\mathcal{N}(P_x)$

P and P_x

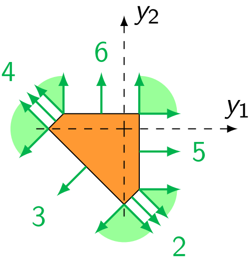
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

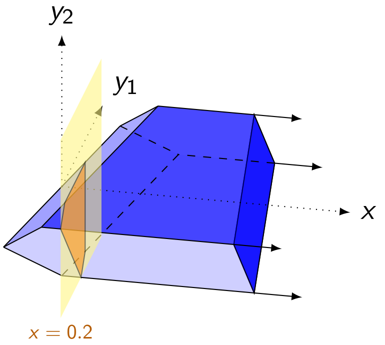
For $x = 0.2$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

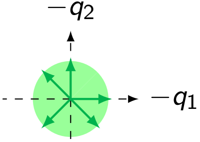


P and P_x

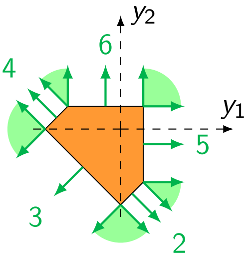
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

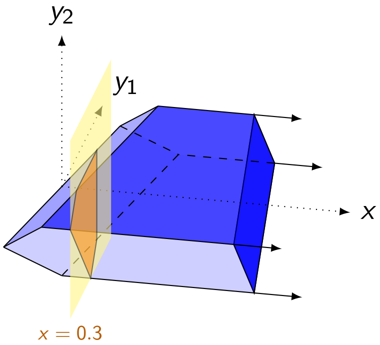
$$\text{For } x = 0.3, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

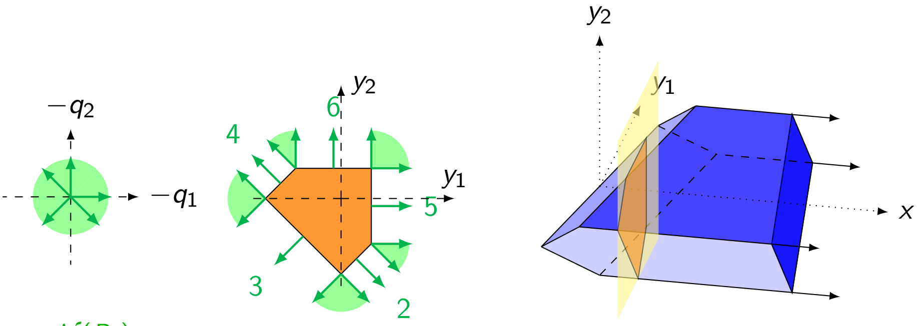


P and P_x

$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

$$\text{For } x = 0.4, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$$



$\mathcal{N}(P_x)$

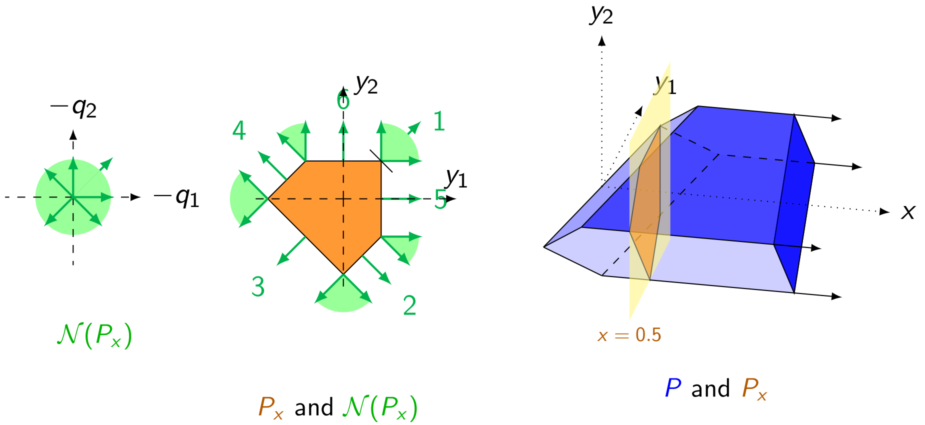
P_x and $\mathcal{N}(P_x)$

P and P_x

$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

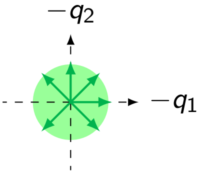
$$\text{For } x = 0.5, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 615\}$$



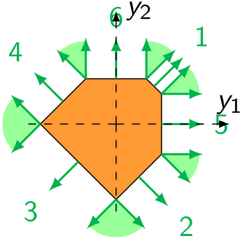
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

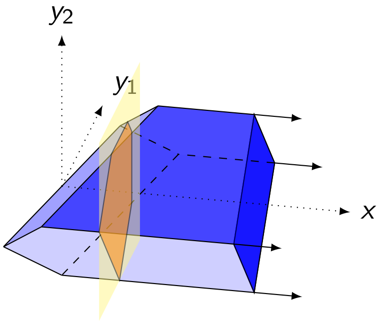
$$\text{For } x = 0.6, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

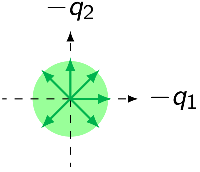


P and P_x

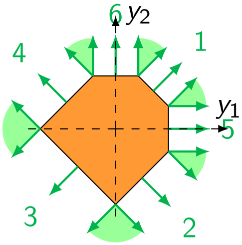
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

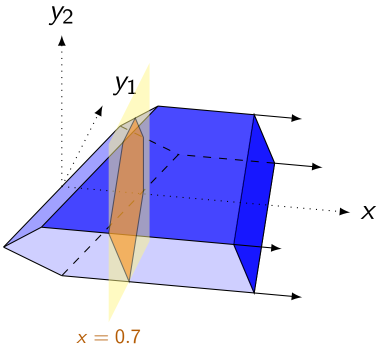
$$\text{For } x = 0.7, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

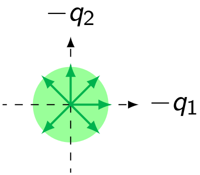


P and P_x

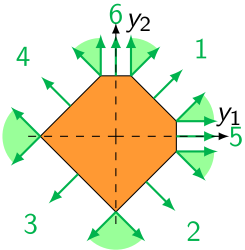
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

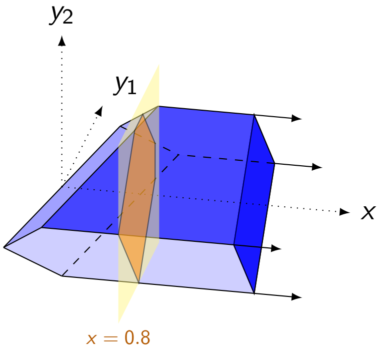
$$\text{For } x = 0.8, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

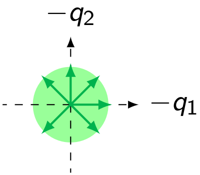


P and P_x

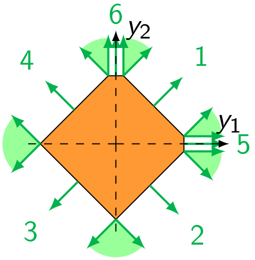
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

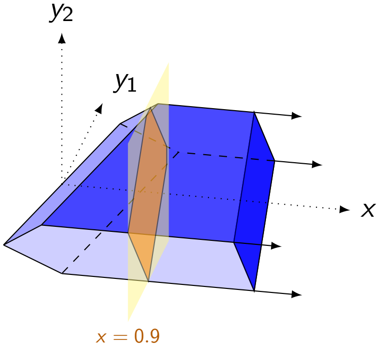
$$\text{For } x = 0.9, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

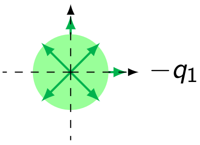


P and P_x

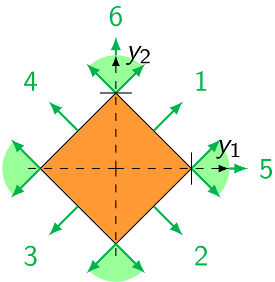
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

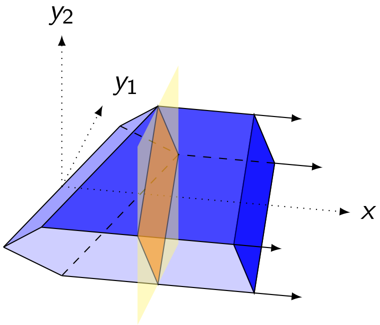
For $x = 1$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{152, 23, 34, 461\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

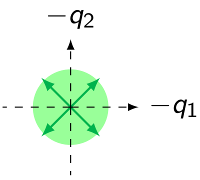


P and P_x

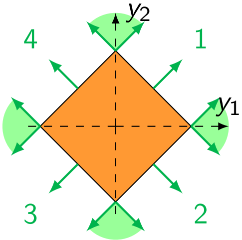
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

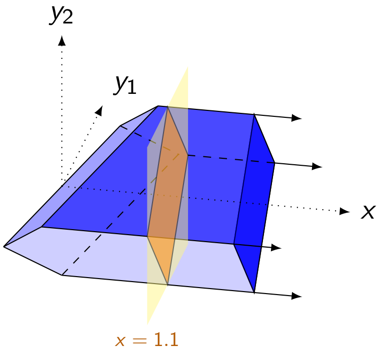
For $x = 1.1$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

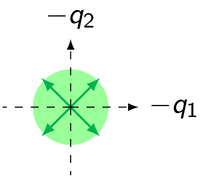


P and P_x

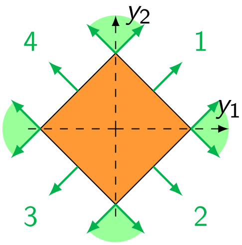
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

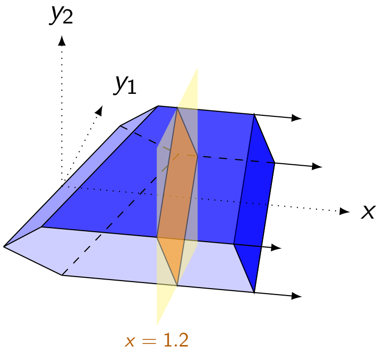
$$\text{For } x = 1.2, \quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

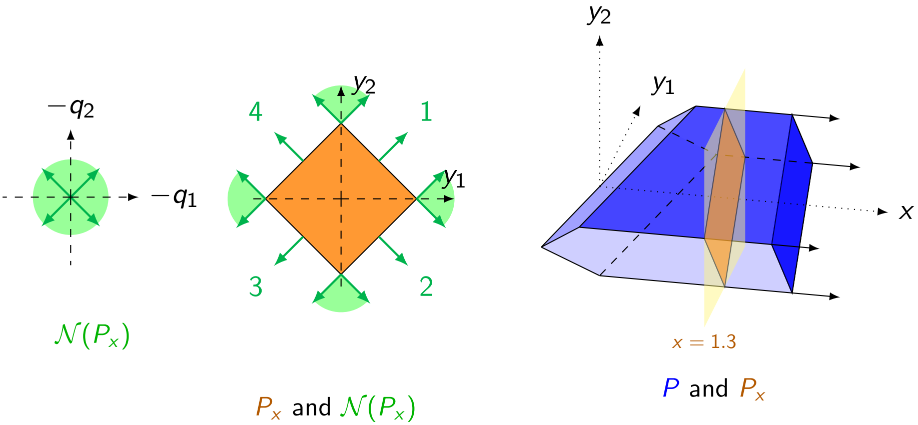


P and P_x

$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

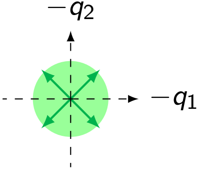
$$\text{For } x = 1.3, \quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$$



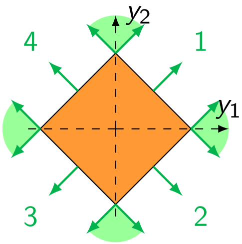
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

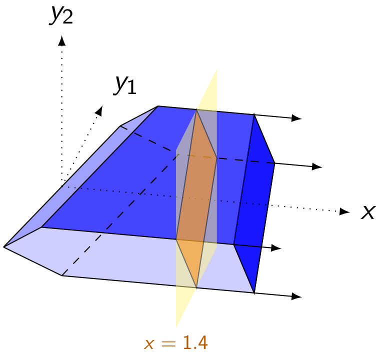
$$\text{For } x = 1.4, \quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

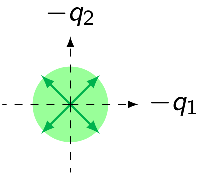


P and P_x

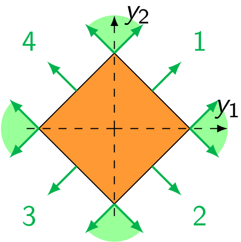
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

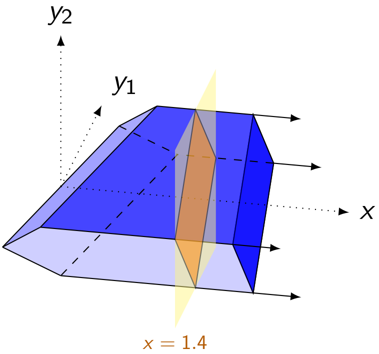
For $x = 1.4$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

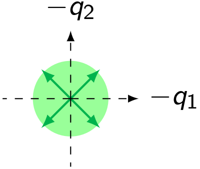


P and P_x

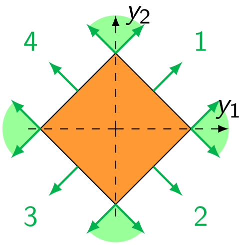
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

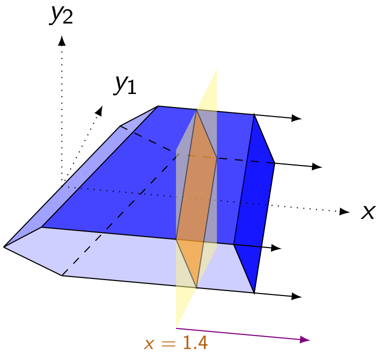
$$\text{For } x = 1.4, \quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

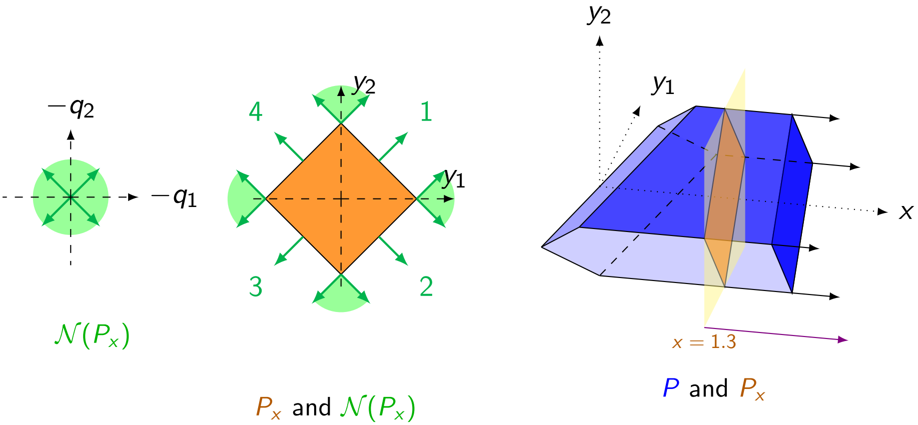


P and P_x

$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

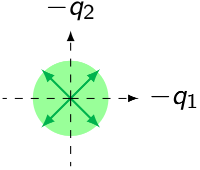
$$\text{For } x = 1.3, \quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$$



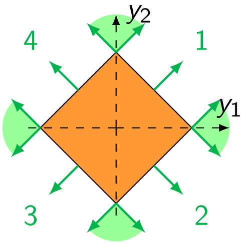
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

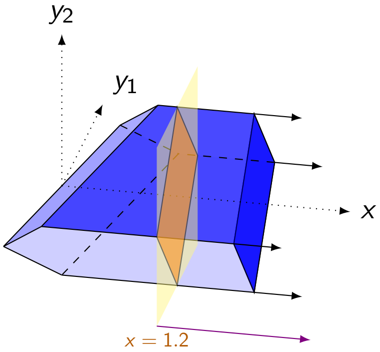
$$\text{For } x = 1.2, \quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

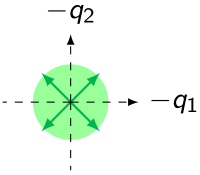


P and P_x

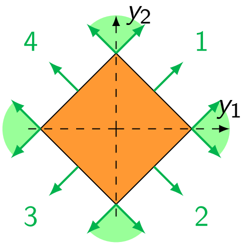
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

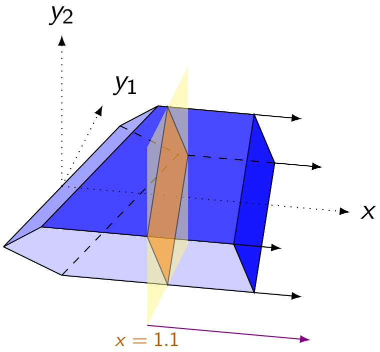
$$\text{For } x = 1.1, \quad \overline{\mathcal{I}(W, h - Tx)} = \{12, 23, 34, 41\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

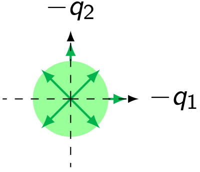


P and P_x

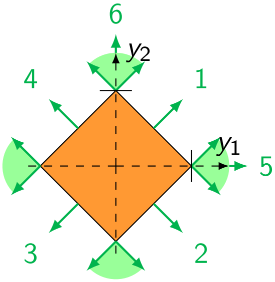
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

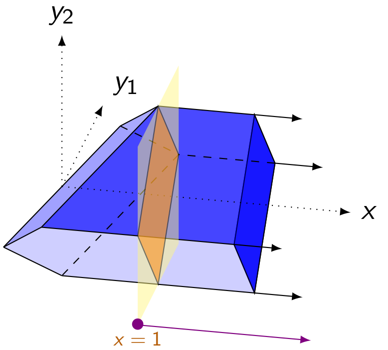
$$\text{For } x = 1, \quad \overline{\mathcal{I}(W, h - Tx)} = \{152, 23, 34, 461\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

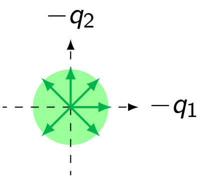


P and P_x

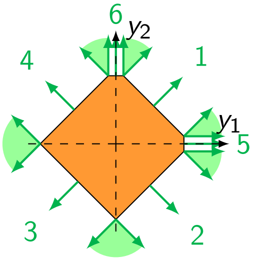
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

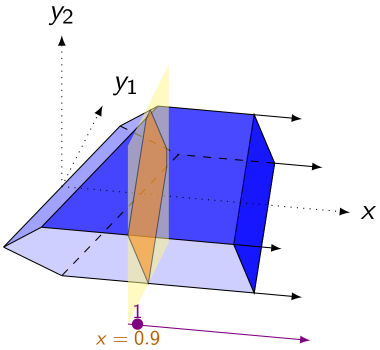
$$\text{For } x = 0.9, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

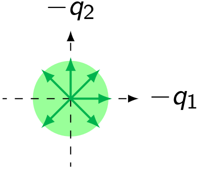


P and P_x

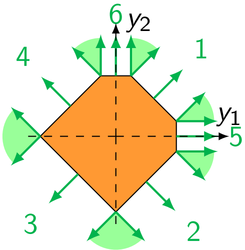
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

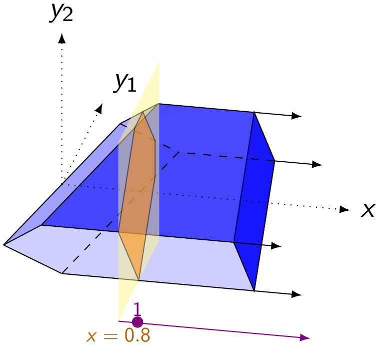
$$\text{For } x = 0.8, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

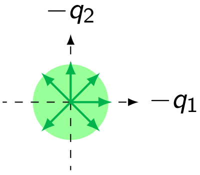


P and P_x

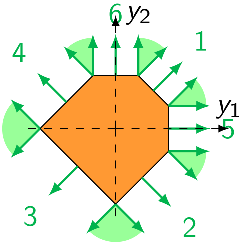
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

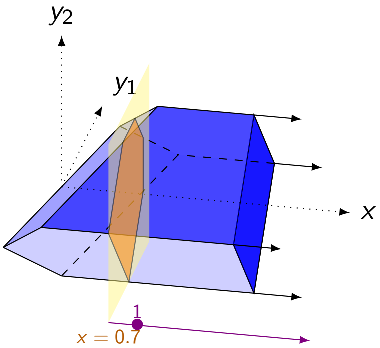
$$\text{For } x = 0.7, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

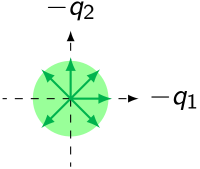


P and P_x

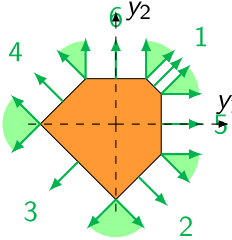
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

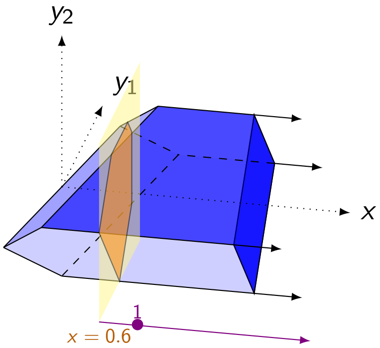
$$\text{For } x = 0.6, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 61, 15\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

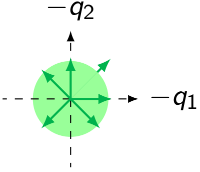


P and P_x

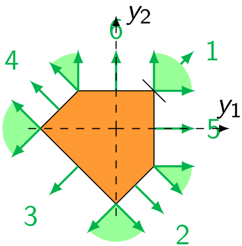
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

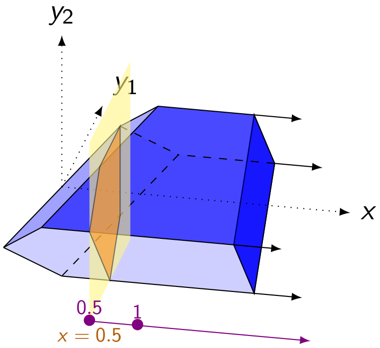
$$\text{For } x = 0.5, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 615\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



P and P_x

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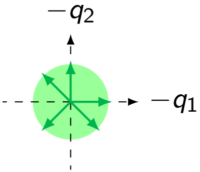
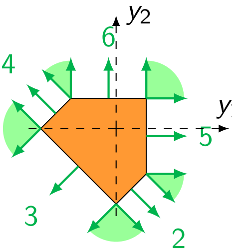
$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

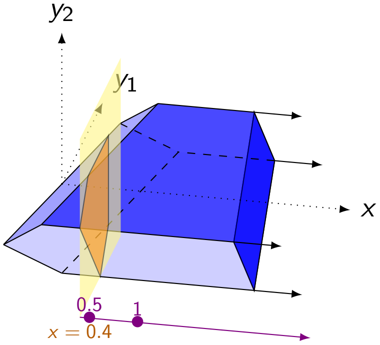
$$\text{For } x = 0.4, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

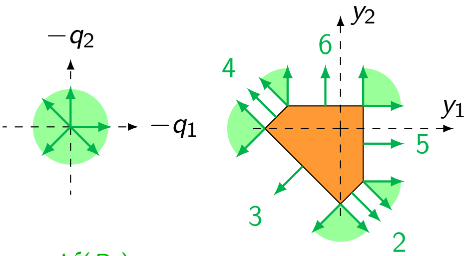


P and P_x

$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

$$\text{For } x = 0.3, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$$



$\mathcal{N}(P_x)$

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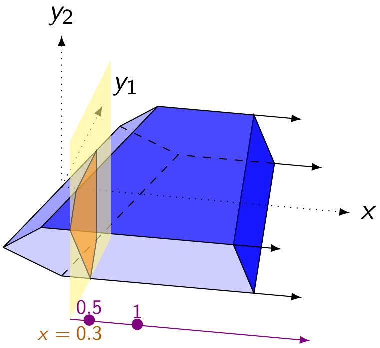
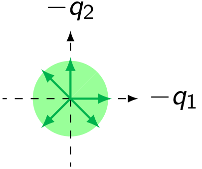


P and P_x

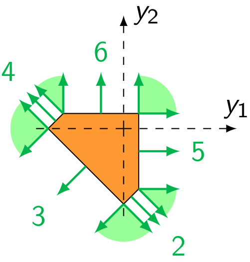
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

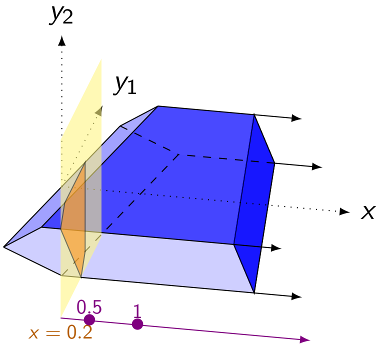
For $x = 0.2$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

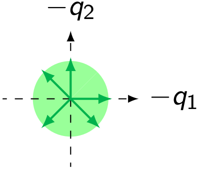


P and P_x

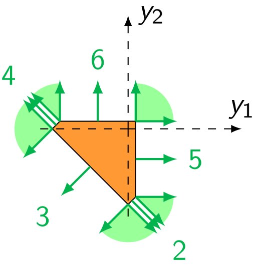
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

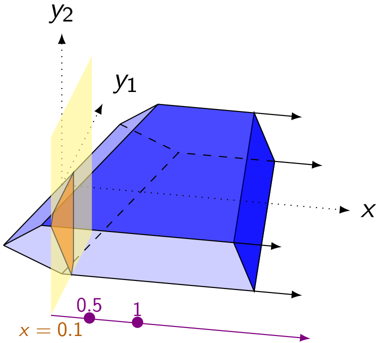
$$\text{For } x = 0.1, \quad \overline{\mathcal{I}(W, h - Tx)} = \{52, 23, 34, 46, 65\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

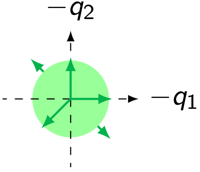


P and P_x

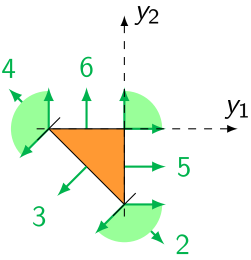
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

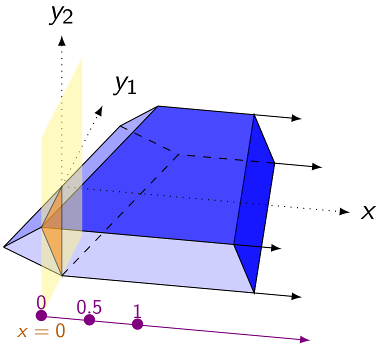
For $x = 0$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{523, 346, 65\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

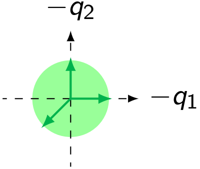


P and P_x

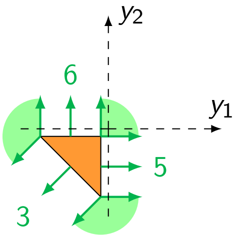
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

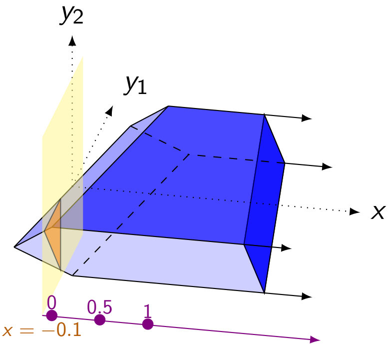
For $x = -0.1$, $\quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

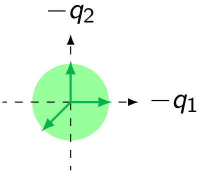


P and P_x

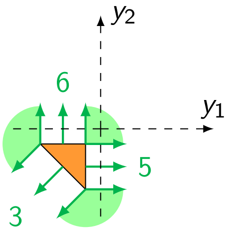
$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

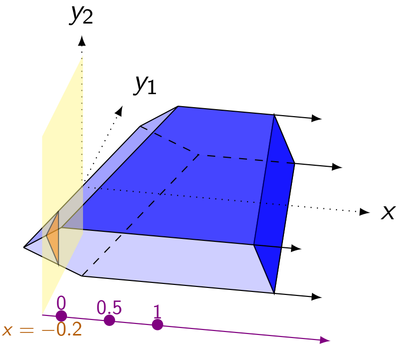
$$\text{For } x = -0.2, \quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



P and P_x

$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

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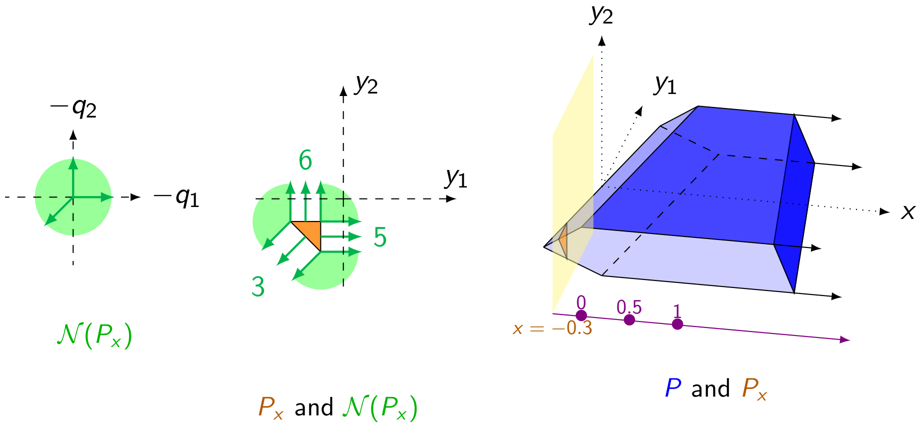
$$\text{For } x = -0.3, \quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$$



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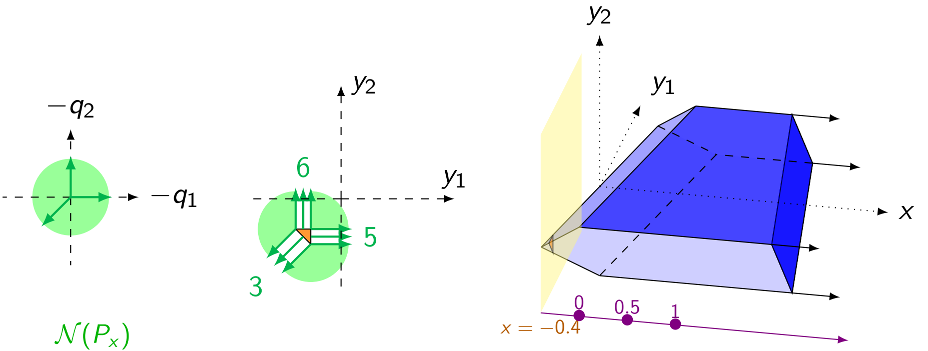
$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

$$\text{For } x = -0.4, \quad \overline{\mathcal{I}(W, h - Tx)} = \{53, 36, 65\}$$

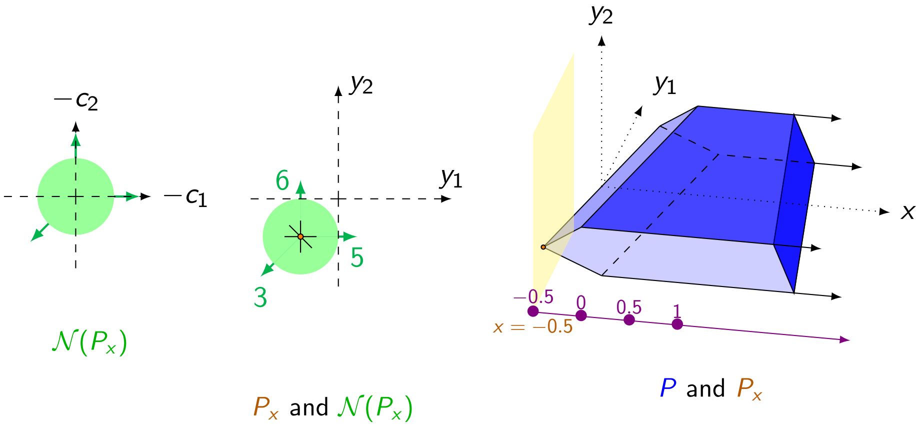


$\mathcal{N}(P_x)$ $\quad$ P_x and $\mathcal{N}(P_x)$ $\quad$ P and P_x

$\mathcal{N}(P_x)$ and $\mathcal{I}(W, h - Tx)$ are piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

$$\text{For } x = -0.5, \quad \overline{\mathcal{I}(W, h - Tx)} = \{536\}$$

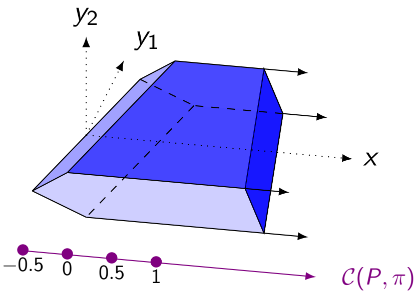


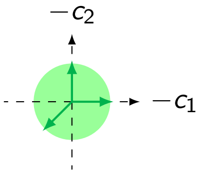
What are the constant regions of $\mathcal{N}(P_x)$, $\mathcal{I}(W, h - Tx)$?

Lemma

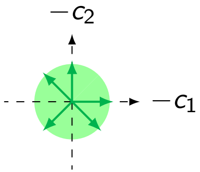
There exists a collection $\mathcal{C}(P, \pi)$ whose relative interior of cells are the constant regions of $x \to \mathcal{N}(P_x)$ and $x \to \mathcal{I}(W, h - Tx)$.

For $\sigma \in \mathcal{C}(P, \pi)$ and $x, x' \in \mathrm{ri}(\sigma)$,
$\mathcal{N}(P_x) = \mathcal{N}(P_{x'}) = \mathcal{N}_\sigma$
$\mathcal{I}(W, h - Tx) = \mathcal{I}(W, h - Tx') = \mathcal{I}_\sigma$

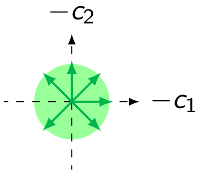




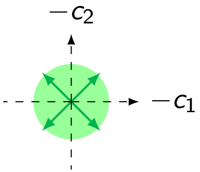
$\mathcal{N}_\sigma$ for $\sigma = [-0.5, 0]$



$\mathcal{N}_\sigma$ for $\sigma = [0, 0.5]$



$\mathcal{N}_\sigma$ for $\sigma = [0.5, 1]$



$\mathcal{N}_\sigma$ for $\sigma = [1, +\infty)$

Chamber complex

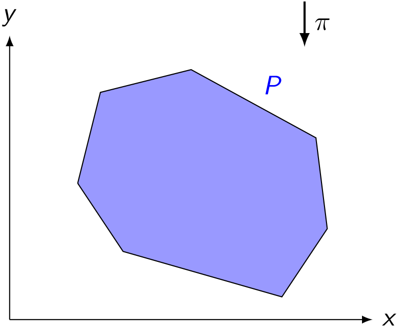
Definition

The *chamber complex* $\mathcal{C}(P,\pi)$ of P along π is

$$\mathcal{C}(P,\pi) := \{\sigma_{P,\pi}(x) \mid x \in \pi(P)\}$$

where

$$\sigma_{P,\pi}(x) := \bigcap_{F \in \mathcal{F}(P) \text{ s.t. } x \in \pi(F)} \pi(F)$$



where $\mathcal{F}(P)$ is the set of faces of P and π is the projection $(x,y) \to x$

$$\pi(E) := \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m,\ (x,y) \in E\}$$

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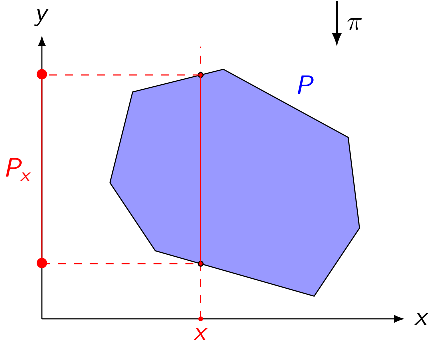

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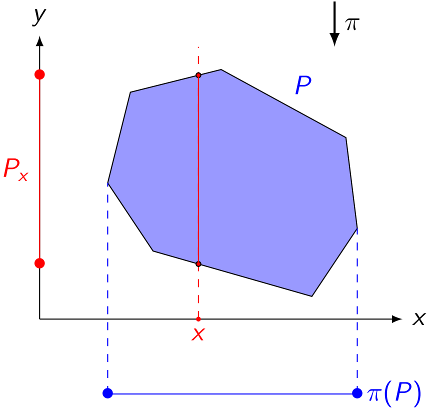

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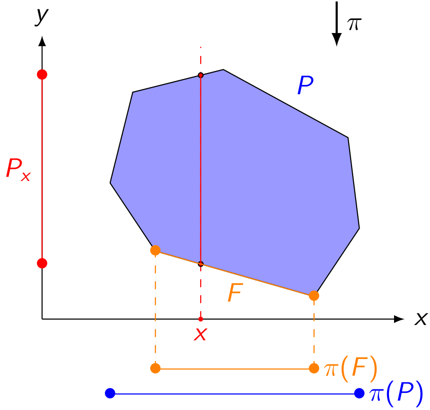

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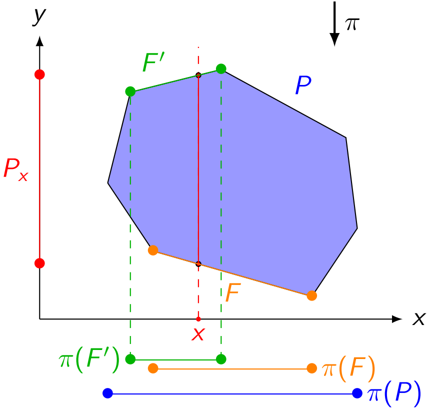

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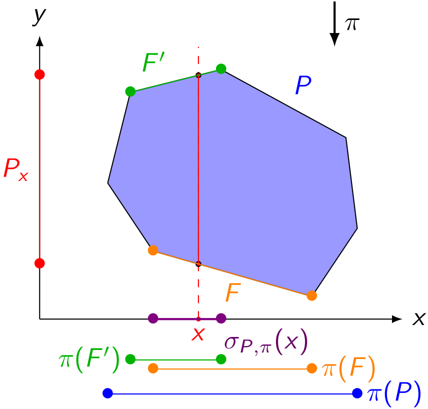

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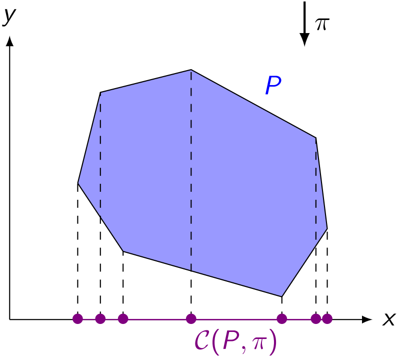

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H-representation of projection of faces

Let $I \in \mathcal{I}\big((T, W), h\big)$ be a set of indices

$$x \in \pi(P^I) \iff \begin{cases} \exists y \in \mathbb{R}^m, & \qquad (x, y) \in P^I \\ & \end{cases}$$

H-representation of projection of faces

Let $I \in \mathcal{I}\big((T, W), h\big)$ be a set of indices

$$x \in \pi(P^I) \iff \begin{cases} \exists y \in \mathbb{R}^m, & T_I x + W_I y = h_I \\ \quad \forall j \in [q] \backslash I, & T_j x + W_j y \leqslant h_j \end{cases} \Longleftarrow I \in \mathcal{I}(W, h - Tx)$$

H-representation of projection of faces

Let $I \in \mathcal{I}\big((T, W), h\big)$ be a set of indices

$$x \in \operatorname{ri} \pi(P^I) \iff \begin{cases} \exists y \in \mathbb{R}^m, & T_I x + W_I y = h_I \\ \quad \forall j \in [q] \backslash I, & T_j x + W_j y < h_j \end{cases} \iff I \in \mathcal{I}(W, h - Tx)$$

H-representation of projection of faces

Let $I \in \mathcal{I}\big((T, W), h\big)$ be a set of indices from which we can extract a basis (i.e. $\mathrm{rg}(W_I^\top) = m$) and let B such a basis

$$x \in \operatorname{ri} \pi(P^I) \iff \begin{cases} \exists y \in \mathbb{R}^m, & T_B x + W_B y = h_B \\ \forall i \in I \backslash B, & T_i x + W_i y = h_i \\ \forall j \in [q] \backslash I, & T_j x + W_j y < h_j \end{cases} \iff I \in \mathcal{I}(W, h - Tx)$$

H-representation of projection of faces

Let $I \in \mathcal{I}\big((T, W), h\big)$ be a set of indices from which we can extract a basis (i.e. $\mathrm{rg}(W_I^\top) = m$) and let B such a basis

$$x \in \operatorname{ri} \pi(P^I) \iff \begin{cases} \exists y \in \mathbb{R}^m, & y = W_B^{-1}(h_B - T_B x) \\ \forall i \in I \backslash B, & T_i x + W_i y = h_i \\ \forall j \in [q] \backslash I, & T_j x + W_j y < h_j \end{cases} \iff I \in \mathcal{I}(W, h - Tx)$$

H-representation of projection of faces

Let $I \in \mathcal{I}\big((T, W), h\big)$ be a set of indices from which we can extract a basis (i.e. $\mathrm{rg}(W_I^\top) = m$) and let B such a basis

$$x \in \operatorname{ri} \pi(P^I) \iff \begin{cases} \forall i \in I \backslash B, & T_i x + W_i W_B^{-1}(h_B - T_B x) = h_i \\ \forall j \in [q] \backslash I, & T_j x + W_j W_B^{-1}(h_B - T_B x) < h_j \end{cases}$$

H-representation of projection of faces

Let $I \in \mathcal{I}\big((T, W), h\big)$ be a set of indices from which we can extract a basis (i.e. $\mathrm{rg}(W_I^\top) = m$) and let B such a basis

$$x \in \mathrm{ri}\left(\pi(P^I)\right) \iff \begin{cases} \forall i \in I \backslash B, & (v_i^B)^\top x = u_i^B \\ \forall j \in [q] \backslash I, & (v_j^B)^\top x < u_j^B \end{cases} \iff I \in \mathcal{I}(W, h - Tx)$$

where

$$v_i^B := \left(T_i - W_i W_B^{-1} T_B\right)^\top$$
$$u_i^B := h_i - W_i W_B^{-1} h_B$$

H-representation of chambers

Let $\sigma \in \mathcal{C}(P, \pi)$

$$x \in \bigcap_{I \in \overline{\mathcal{I}_\sigma}} \operatorname{ri}\left(\pi(P^I)\right) \iff \begin{cases} \forall I \in \overline{\mathcal{I}_\sigma}, \\ \forall i \in I \backslash B_I, \quad (v_i^{B_I})^\top x = u_i^{B_I} \\ \forall j \in [q] \backslash I, \quad (v_j^{B_I})^\top x < u_j^{B_I} \end{cases} \iff \mathcal{I}(W, h - Tx) = \mathcal{I}_\sigma$$

where

$$\begin{aligned} v_i^B &:= (T_i - W_i W_B^{-1} T_B)^\top \\ u_i^B &:= h_i - W_i W_B^{-1} h_B \end{aligned}$$

with B_I basis $\subset I$ and

$$\begin{aligned} \mathcal{G}_\sigma &:= \{F \in \mathcal{F}(P) \,|\, \sigma \subset \pi(F)\} \\ \mathcal{I}_\sigma &:= \{I \in \mathcal{I}\big((T, W), h\big) \,|\, \sigma \subset \pi(P^I)\} \end{aligned}$$

We have $\sigma = \cap_{G \in \mathcal{G}_\sigma} \pi(G) = \cap_{I \in \mathcal{I}_\sigma} \pi(P^I)$

H-representation of chambers

Let $\sigma \in \mathcal{C}(P, \pi)$

$$x \in \mathrm{ri}(\sigma) \iff \begin{cases} \forall I \in \overline{\mathcal{I}_\sigma}, \\ \forall i \in I \backslash B_I, \quad (v_i^{B_I})^\top x = u_i^{B_I} \\ \forall j \in [q] \backslash I, \quad (v_j^{B_I})^\top x < u_j^{B_I} \end{cases} \iff \mathcal{I}(W, h - Tx) = \mathcal{I}_\sigma$$

where

$$\begin{aligned} v_i^B &:= (T_i - W_i W_B^{-1} T_B)^\top \\ u_i^B &:= h_i - W_i W_B^{-1} h_B \end{aligned}$$

with B_I basis $\subset I$ and

$$\begin{aligned} \mathcal{G}_\sigma &:= \{F \in \mathcal{F}(P) \,|\, \sigma \subset \pi(F)\} \\ \mathcal{I}_\sigma &:= \{I \in \mathcal{I}\big((T, W), h\big) \,|\, \sigma \subset \pi(P^I)\} \end{aligned}$$

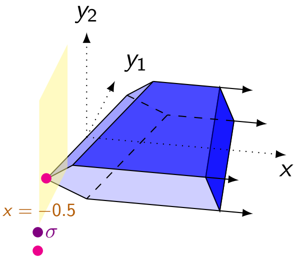
We have $\sigma = \cap_{G \in \underline{\mathcal{G}_\sigma}} \pi(G) = \cap_{I \in \overline{\mathcal{I}_\sigma}} \pi(P^I)$

Correspondences

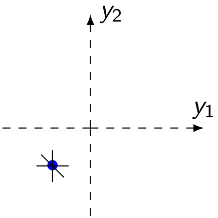
$$
\begin{tikzcd}
\boxed{G \in \mathcal{G}_\sigma, \supset} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend left, "I_{(T,W),h}(G)"] & & \boxed{F \in \mathcal{F}(P_x) \backslash \{\emptyset\}, \supset} \arrow[ll, bend right, "G \in \mathcal{F}(P) \text{ minimal s.t.} \{x\} \times F \subset G"'] \arrow[dl, bend right, "I_{W,h-Tx}(F)"'] \arrow[dr, bend right, "N_{P_x}(F)"'] & \\
& \boxed{I \in \mathcal{I}_\sigma, \subset} \arrow[ul, bend left, "P^I"] \arrow[ur, bend right, "P_x^I"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & \boxed{N \in \mathcal{N}_\sigma, \subset} \arrow[ll, bend left, "\{i \in \mathrm{supp}\, \mathcal{I}_\sigma \mid W_i^\top \in N\}"] \arrow[ul, bend right, "\arg\min_{y \in P_x} q^\top x \text{ for } q \in -\mathrm{ri}(N)"']
\end{tikzcd}
$$

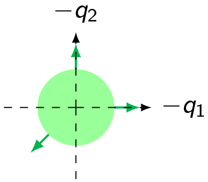
Correspondences

$$\begin{tikzcd}
G \in \underline{\mathcal{G}_\sigma} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & \mathrm{Vert}(P_x) \arrow[ll, bend right, "G\in\mathcal{F}(P) \text{ minimal s.t.} (x,v)\in G"'] \arrow[dl, bend right, "I_{W,h-Tx}(v)"'] \arrow[dr, bend right, "N_{P_x}(v)"'] & \\
& I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "P^I_x"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ul, bend right, "\arg\min_{y\in P_x} q^\top x \text{ for } q\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in\mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top\in N\}"]
\end{tikzcd}$$



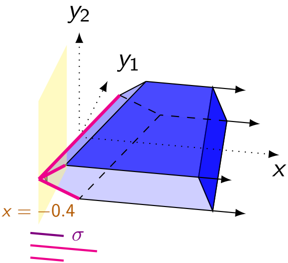
$\overline{\mathcal{I}_\sigma} = \{356\}$



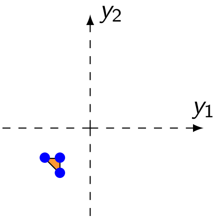


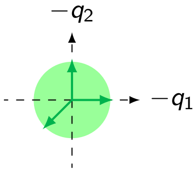
Correspondences

$$
\begin{tikzcd}
\boxed{G \in \underline{\mathcal{G}_\sigma}} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & \boxed{\mathrm{Vert}(P_x)} \arrow[ll, bend right, "G \in \mathcal{F}(P) \text{ minimal s.t.} (x,v) \in G"'] \arrow[dl, bend right, "I_{W,h-Tx}(v)"'] \arrow[dr, bend right, "N_{P_x}(v)"'] & \\
& \boxed{I \in \overline{\mathcal{I}_\sigma}} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "P^I_x"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & \boxed{N \in \overline{\mathcal{N}_\sigma}} \arrow[ul, bend right, "\arg\min_{y \in P_x} q^\top x \text{ for } q \in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i \in \mathrm{supp}\, \mathcal{I}_\sigma \mid W_i^\top \in N\}"]
\end{tikzcd}
$$



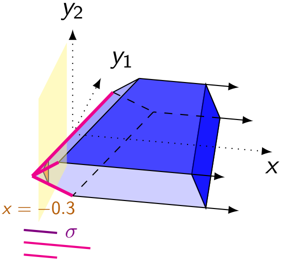
$\overline{\mathcal{I}_\sigma} = \{53, 36, 65\}$



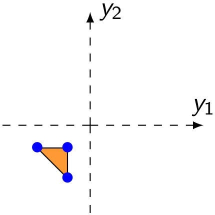


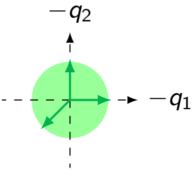
Correspondences

$$\begin{tikzcd}[column sep=large, row sep=huge]
\boxed{G \in \underline{\mathcal{G}_\sigma}} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & \boxed{\mathrm{Vert}(P_x)} \arrow[ll, bend right, "G\in\mathcal{F}(P) \text{ minimal s.t.} (x,v)\in G"'] \arrow[dl, bend right, "I_{W,h-Tx}(v)"'] \arrow[dr, bend right, "N_{P_x}(v)"'] & \\
& \boxed{I \in \overline{\mathcal{I}_\sigma}} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "P^I_x"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & \boxed{N \in \overline{\mathcal{N}_\sigma}} \arrow[ul, bend right, "\arg\min_{y\in P_x} q^\top x \text{ for } q\in-\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in\mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top\in N\}"]
\end{tikzcd}$$



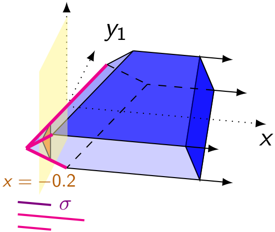
$\overline{\mathcal{I}_\sigma} = \{53, 36, 65\}$



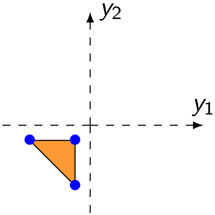


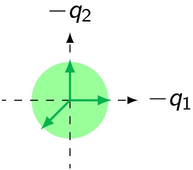
Correspondences

$$\begin{tikzcd}
G \in \underline{\mathcal{G}_\sigma} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & \mathrm{Vert}(P_x) \arrow[ll, bend right, "G \in \mathcal{F}(P) \text{ minimal s.t.} (x,v) \in G"'] \arrow[dl, bend right, "I_{W,h-Tx}(v)"'] \arrow[dr, bend right, "N_{P_x}(v)"'] & \\
& I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "P^I_x"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ll, bend left, "\{i \in \mathrm{supp}\, \mathcal{I}_\sigma \mid W_i^\top \in N\}"] \arrow[ul, bend right, "\arg\min_{y \in P_x} q^\top x \text{ for } q \in -\mathrm{ri}(N)"']
\end{tikzcd}$$



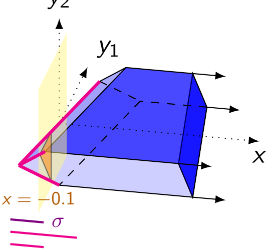
$\overline{\mathcal{I}_\sigma} = \{53, 36, 65\}$



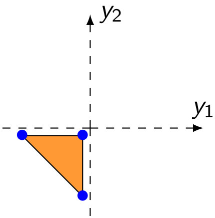


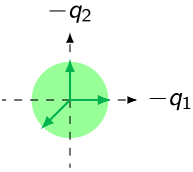
Correspondences

$$
\begin{tikzcd}
\color{magenta}{G \in \underline{\mathcal{G}_\sigma}} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & \color{blue}{\mathrm{Vert}(P_x)} \arrow[ll, bend right, "G \in \mathcal{F}(P) \text{ minimal s.t.} (x,v) \in G"'] \arrow[dl, bend right, "I_{W,h-Tx}(v)"'] \arrow[dr, bend left, "N_{P_x}(v)"'] & \\
& \color{red}{I \in \overline{\mathcal{I}_\sigma}} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "P_x^I"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & \color{green}{N \in \overline{\mathcal{N}_\sigma}} \arrow[ul, bend right, "\arg\min_{y \in P_x} q^\top x \text{ for } q \in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i \in \mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top \in N\}"]
\end{tikzcd}
$$



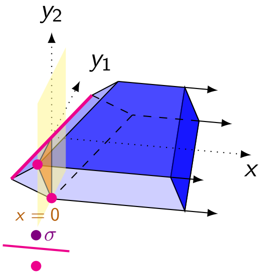
$\overline{\mathcal{I}_\sigma} = \{53, 36, 65\}$



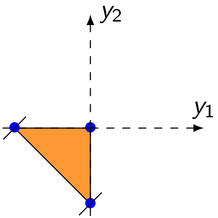


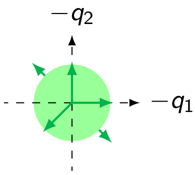
Correspondences

$$
\begin{tikzcd}[column sep=large, row sep=huge]
G \in \mathcal{G}_\sigma \arrow[rr, bend right=10, "G_x"'] \arrow[dr, bend right=10, "I_{(T,W),h}(G)"'] & & \mathrm{Vert}(P_x) \arrow[ll, bend right=10, "G\in\mathcal{F}(P) \text{ minimal s.t.} (x,v)\in G"'] \arrow[dl, bend right=10, "I_{W,h-Tx}(v)"'] \arrow[dr, bend right=10, "N_{P_x}(v)"'] & \\
 & I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right=10, "P^I"'] \arrow[ur, bend right=10, "P^I_x"'] \arrow[rr, bend left=5, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ul, bend right=10, "\arg\min_{y\in P_x} q^\top x \text{ for } q\in -\mathrm{ri}(N)"'] \arrow[ll, bend left=5, "\{i\in\mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top \in N\}"]
\end{tikzcd}
$$

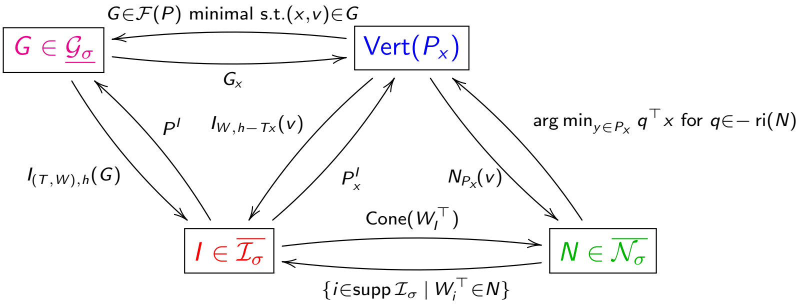


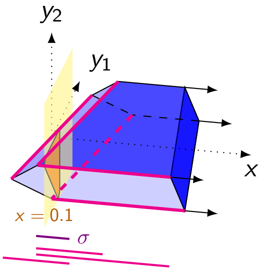
$\overline{\mathcal{I}_\sigma} = \{523, 346, 65\}$



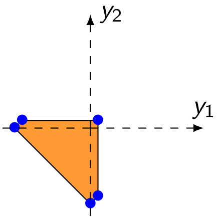


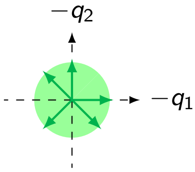
Correspondences





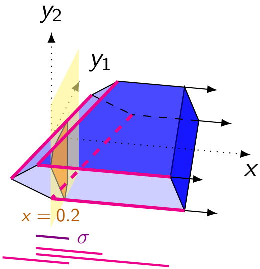
$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 65\}$



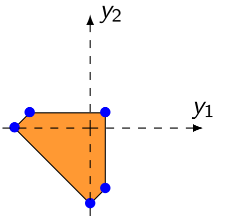


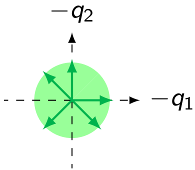
Correspondences

$$
\begin{tikzcd}
\underline{G \in \mathcal{G}_\sigma} \arrow[rr, bend right=10, "G_x"'] \arrow[dr, bend right=10, "I_{(T,W),h}(G)"'] & & \mathrm{Vert}(P_x) \arrow[ll, bend right=10, "G \in \mathcal{F}(P) \text{ minimal s.t.} (x,v) \in G"'] \arrow[dl, bend right=10, "I_{W,h-Tx}(v)"'] \arrow[dr, bend right=10, "N_{P_x}(v)"'] & \\
& I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right=10, "P^I"'] \arrow[ur, bend right=10, "P^I_x"'] \arrow[rr, bend left=5, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ul, bend right=10, "\arg\min_{y \in P_x} q^\top x \text{ for } q \in -\mathrm{ri}(N)"'] \arrow[ll, bend left=5, "\{i \in \mathrm{supp}\, \mathcal{I}_\sigma \mid W_i^\top \in N\}"]
\end{tikzcd}
$$

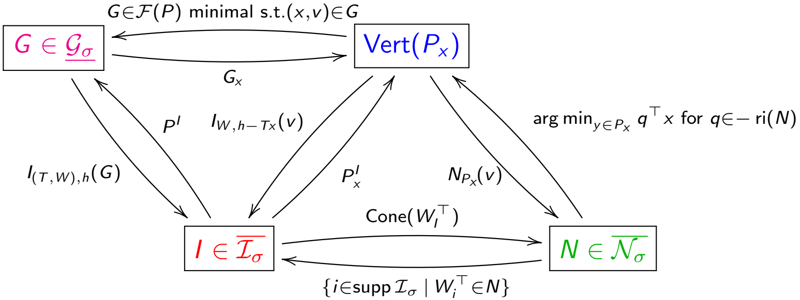


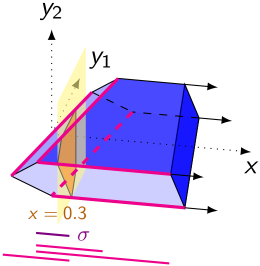
$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 65\}$



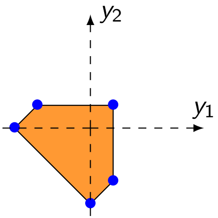


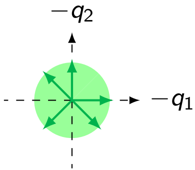
Correspondences





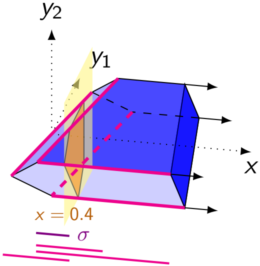
$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 65\}$



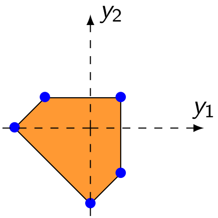


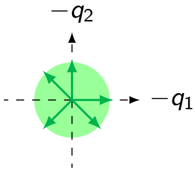
Correspondences

$$
\begin{tikzcd}[column sep=large, row sep=huge]
G \in \underline{\mathcal{G}_\sigma} \arrow[rr, bend right=5, "G_x"'] \arrow[dr, bend right=10, "I_{(T,W),h}(G)"'] & & \mathrm{Vert}(P_x) \arrow[ll, bend right=5, "G\in\mathcal{F}(P)\ \text{minimal s.t.}(x,v)\in G"'] \arrow[dl, bend right=10, "I_{W,h-Tx}(v)"'] \arrow[dr, bend right=10, "N_{P_x}(v)"'] & \\
 & I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right=10, "P^I"'] \arrow[ur, bend right=10, "P^I_x"'] \arrow[rr, bend left=5, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ul, bend right=10, "\arg\min_{y\in P_x} q^\top x\ \text{for}\ q\in-\mathrm{ri}(N)"'] \arrow[ll, bend left=5, "\{i\in\mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top\in N\}"]
\end{tikzcd}
$$



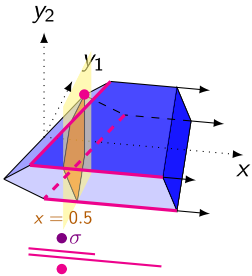
$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 65\}$



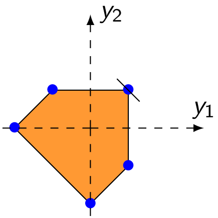


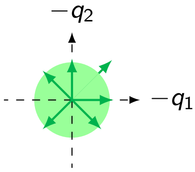
Correspondences

$$\begin{tikzcd}[column sep=large, row sep=huge]
G \in \underline{\mathcal{G}_\sigma} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & \mathrm{Vert}(P_x) \arrow[ll, bend right, "G\in\mathcal{F}(P) \text{ minimal s.t.} (x,v)\in G"'] \arrow[dl, bend right, "I_{W,h-Tx}(v)"'] \arrow[dr, bend right, "N_{P_x}(v)"'] & \\
& I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "P^I_x"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ul, bend right, "\arg\min_{y\in P_x} q^\top x \text{ for } q\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in\mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top \in N\}"]
\end{tikzcd}$$



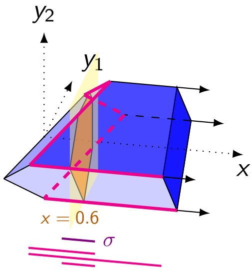
$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 615\}$



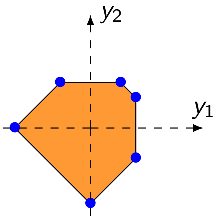


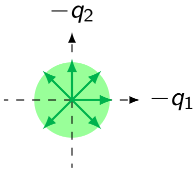
Correspondences

$$
\begin{tikzcd}[column sep=large, row sep=huge]
\boxed{G \in \mathcal{G}_\sigma} \arrow[rr, bend right=10, "G_x"'] \arrow[dr, bend right=10, "I_{(T,W),h}(G)"'] & & \boxed{\mathrm{Vert}(P_x)} \arrow[ll, bend right=10, "G\in\mathcal{F}(P) \text{ minimal s.t.} (x,v)\in G"'] \arrow[dl, bend right=10, "I_{W,h-Tx}(v)"'] \arrow[dr, bend right=10, "N_{P_x}(v)"'] & \\
& \boxed{I \in \overline{\mathcal{I}_\sigma}} \arrow[ul, bend right=10, "P^I"'] \arrow[ur, bend right=10, "P^I_x"'] \arrow[rr, bend left=10, "\mathrm{Cone}(W_I^\top)"] & & \boxed{N \in \overline{\mathcal{N}_\sigma}} \arrow[ul, bend right=10, "\arg\min_{y\in P_x} q^\top x \text{ for } q\in -\mathrm{ri}(N)"'] \arrow[ll, bend left=10, "\{i\in \mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top \in N\}"]
\end{tikzcd}
$$



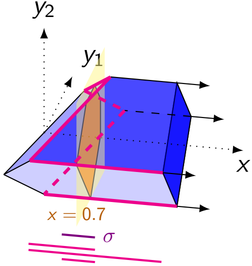
$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 61, 15\}$



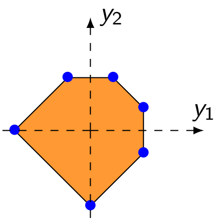


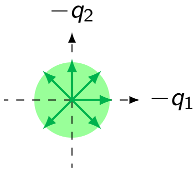
Correspondences

$$\begin{tikzcd}[column sep=large, row sep=huge]
{\color{magenta} G \in \underline{\mathcal{G}_\sigma}} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & {\color{blue} \mathrm{Vert}(P_x)} \arrow[ll, bend right, "G\in\mathcal{F}(P) \text{ minimal s.t.} (x,v)\in G"'] \arrow[dl, bend right, "I_{W,h-Tx}(v)"'] \arrow[dr, bend right, "N_{P_x}(v)"'] & \\
& {\color{red} I \in \overline{\mathcal{I}_\sigma}} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "P^I_x"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & {\color{green} N \in \overline{\mathcal{N}_\sigma}} \arrow[ul, bend right, "\arg\min_{y\in P_x} q^\top x \text{ for } q\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in \mathrm{supp}\, \mathcal{I}_\sigma \mid W_i^\top \in N\}"]
\end{tikzcd}$$



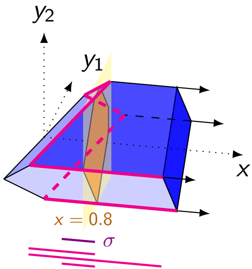
$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 61, 15\}$



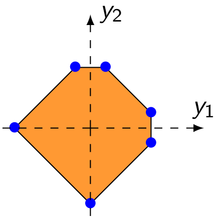


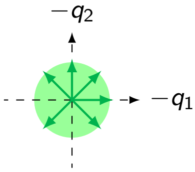
Correspondences

$$
\begin{tikzcd}
\boxed{G \in \underline{\mathcal{G}_\sigma}} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & \boxed{\mathrm{Vert}(P_x)} \arrow[ll, bend right, "G\in\mathcal{F}(P) \text{ minimal s.t.} (x,v)\in G"'] \arrow[dl, bend right, "I_{W,h-Tx}(v)"'] \arrow[dr, bend right, "N_{P_x}(v)"'] & \\
& \boxed{I \in \overline{\mathcal{I}_\sigma}} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "P^I_x"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & \boxed{N \in \overline{\mathcal{N}_\sigma}} \arrow[ul, bend right, "\arg\min_{y\in P_x} q^\top x \text{ for } q\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in\mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top \in N\}"]
\end{tikzcd}
$$



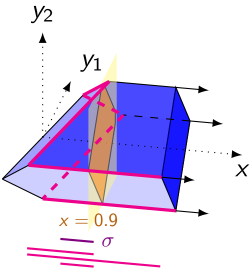
$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 61, 15\}$



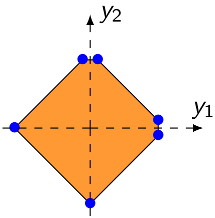


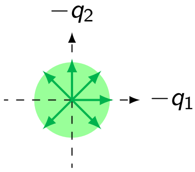
Correspondences

$$\begin{tikzcd}[column sep=large, row sep=huge]
\boxed{G \in \underline{\mathcal{G}_\sigma}} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & \boxed{\mathrm{Vert}(P_x)} \arrow[ll, bend right, "G \in \mathcal{F}(P) \text{ minimal s.t.} (x,v) \in G"'] \arrow[dl, bend right, "I_{W,h-Tx}(v)"'] \arrow[dr, bend left, "N_{P_x}(v)"] & \\
& \boxed{I \in \overline{\mathcal{I}_\sigma}} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "P^I_x"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & \boxed{N \in \overline{\mathcal{N}_\sigma}} \arrow[ul, bend left, "\arg\min_{y \in P_x} q^\top x \text{ for } q \in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i \in \mathrm{supp}\, \mathcal{I}_\sigma \mid W_i^\top \in N\}"]
\end{tikzcd}$$



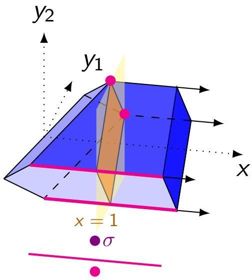
$\overline{\mathcal{I}_\sigma} = \{52, 23, 34, 46, 61, 15\}$



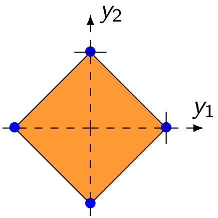


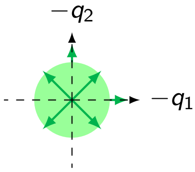
Correspondences

$$\begin{tikzcd}[column sep=large]
G \in \underline{\mathcal{G}_\sigma} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & \mathrm{Vert}(P_x) \arrow[ll, bend right, "G\in\mathcal{F}(P)\text{ minimal s.t.}(x,v)\in G"'] \arrow[dl, bend right, "I_{W,h-Tx}(v)"'] \arrow[dr, bend right, "N_{P_x}(v)"'] & \\
& I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "P^I_x"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ul, bend right, "\arg\min_{y\in P_x} q^\top x \text{ for } q\in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i\in\mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top\in N\}"]
\end{tikzcd}$$



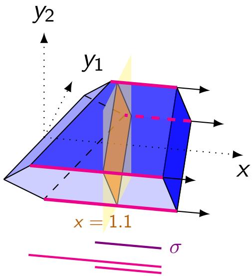
$\overline{\mathcal{I}_\sigma} = \{152, 23, 34, 461\}$



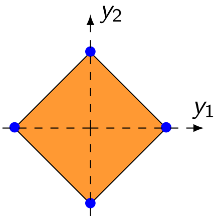


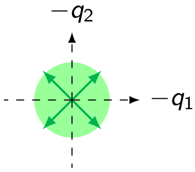
Correspondences

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
G \in \mathcal{G}_\sigma \arrow[rr, bend right=10, "G_x"'] \arrow[dr, bend right=10, "I_{(T,W),h}(G)"'] & & \mathrm{Vert}(P_x) \arrow[ll, bend right=10, "G\in\mathcal{F}(P) \text{ minimal s.t.} (x,v)\in G"'] \arrow[dl, bend right=10, "I_{W,h-Tx}(v)"'] \arrow[dr, bend right=10, "N_{P_x}(v)"'] & \\
& I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right=10, "P^I"'] \arrow[ur, bend right=10, "P^I_x"'] \arrow[rr, bend left=10, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ul, bend right=10, "\arg\min_{y\in P_x} q^\top x \text{ for } q\in -\mathrm{ri}(N)"'] \arrow[ll, bend left=10, "\{i\in\mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top\in N\}"]
\end{tikzcd}
$$



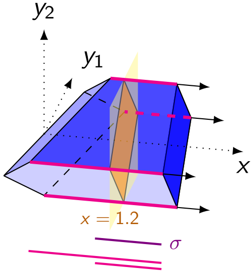
$\overline{\mathcal{I}_\sigma} =$
$\{12, 23, 34, 41\}$



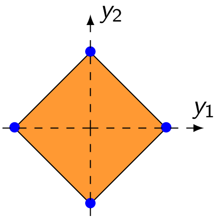


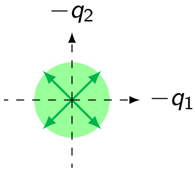
Correspondences

$$
\begin{tikzcd}
G \in \underline{\mathcal{G}_\sigma} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & \mathrm{Vert}(P_x) \arrow[ll, bend right, "G \in \mathcal{F}(P) \text{ minimal s.t.} (x,v) \in G"'] \arrow[dl, bend right, "I_{W,h-Tx}(v)"'] \arrow[dr, bend right, "N_{P_x}(v)"'] & \\
& I \in \overline{\mathcal{I}_\sigma} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "P^I_x"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & N \in \overline{\mathcal{N}_\sigma} \arrow[ul, bend right, "\arg\min_{y \in P_x} q^\top x \text{ for } q \in -\mathrm{ri}(N)"'] \arrow[ll, bend left, "\{i \in \mathrm{supp}\, \mathcal{I}_\sigma \mid W_i^\top \in N\}"]
\end{tikzcd}
$$



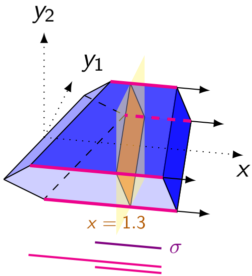
$\overline{\mathcal{I}_\sigma} =$
$\{12, 23, 34, 41\}$



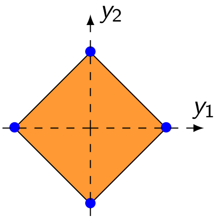


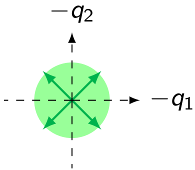
Correspondences

$$
\begin{tikzcd}
\color{magenta}{G \in \underline{\mathcal{G}_\sigma}} \arrow[rr, bend right, "G_x"'] \arrow[dr, bend right, "I_{(T,W),h}(G)"'] & & \color{blue}{\mathrm{Vert}(P_x)} \arrow[ll, bend right, "G\in\mathcal{F}(P) \text{ minimal s.t.} (x,v)\in G"'] \arrow[dl, bend right, "I_{W,h-Tx}(v)"'] \arrow[dr, bend right, "N_{P_x}(v)"'] & \\
& \color{red}{I \in \overline{\mathcal{I}_\sigma}} \arrow[ul, bend right, "P^I"'] \arrow[ur, bend right, "P^I_x"'] \arrow[rr, bend left, "\mathrm{Cone}(W_I^\top)"] & & \color{green}{N \in \overline{\mathcal{N}_\sigma}} \arrow[ll, bend left, "\{i\in \mathrm{supp}\,\mathcal{I}_\sigma \mid W_i^\top \in N\}"] \arrow[ul, bend right, "\arg\min_{y\in P_x} q^\top x \text{ for } q\in -\mathrm{ri}(N)"']
\end{tikzcd}
$$

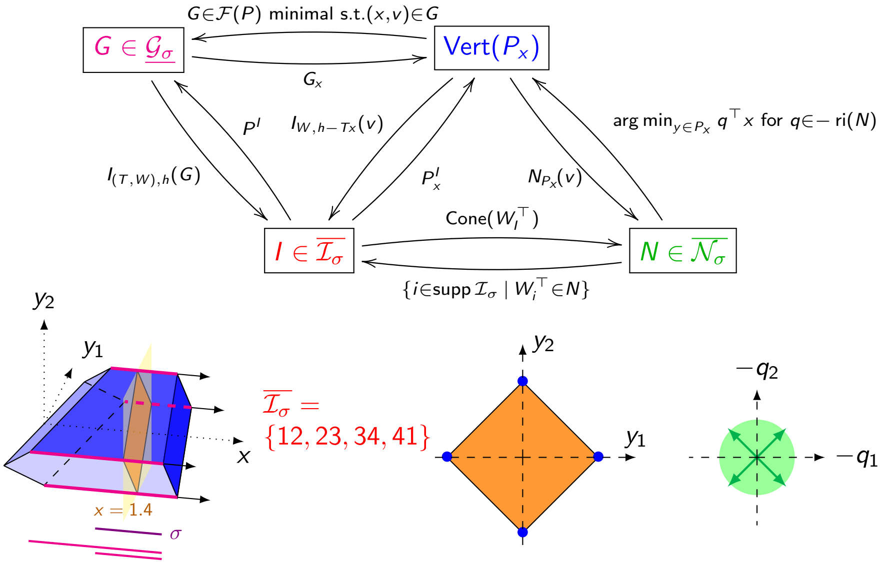


$\overline{\mathcal{I}_\sigma} =$
$\{12, 23, 34, 41\}$



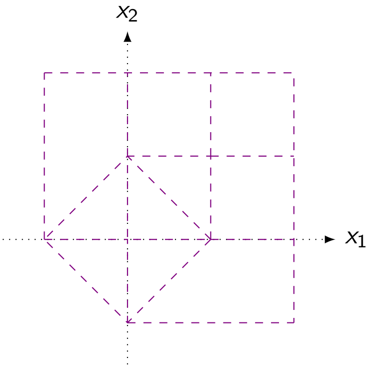


Correspondences



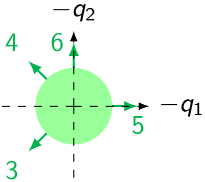
Simplex for 2SLP

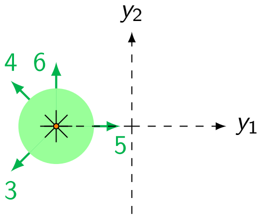
$$y_1 + y_2 \leqslant 1 \tag{1}$$
$$y_1 - y_2 \leqslant 1 \tag{2}$$
$$-y_1 - y_2 \leqslant 1 \tag{3}$$
$$-y_1 + y_2 \leqslant 1 \tag{4}$$
$$y_1 \leqslant x_1 \tag{5}$$
$$y_2 \leqslant x_2 \tag{6}$$
$$x_1 \leqslant 2 \tag{7}$$
$$x_2 \leqslant 2 \tag{8}$$

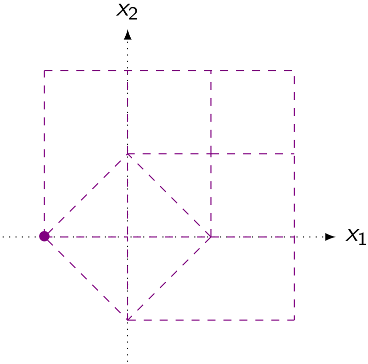


Simplex for 2SLP

$\overline{\mathcal{I}} = \{3456\}$

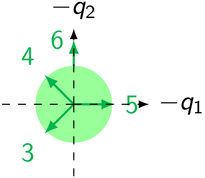


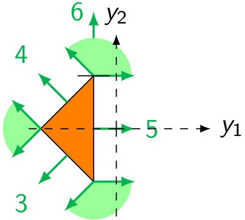


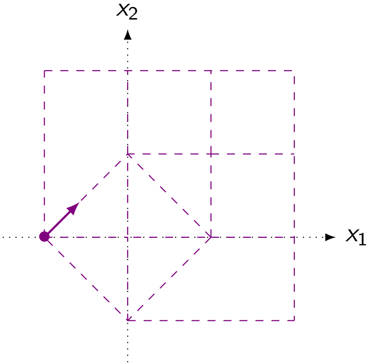


Simplex for 2SLP

$\overline{\mathcal{I}} = \{34, 35, 456\}$

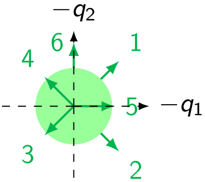


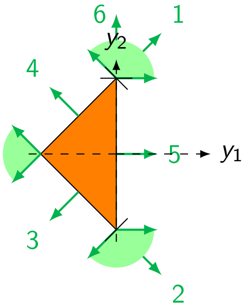


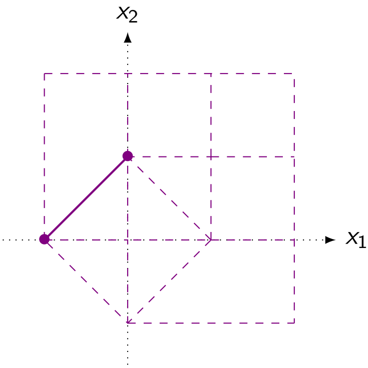


Simplex for 2SLP

$\overline{\mathcal{I}} = \{34, 235, 1456\}$

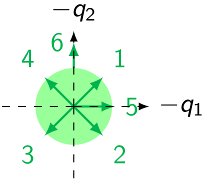


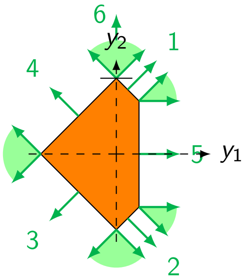


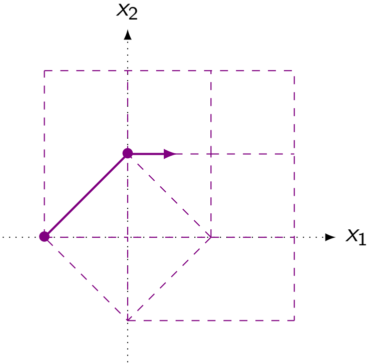


Simplex for 2SLP

$\overline{\mathcal{I}} = \{34, 23, 25, 146, 15\}$

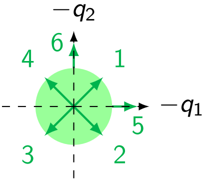


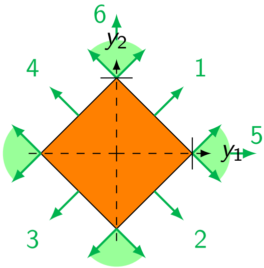


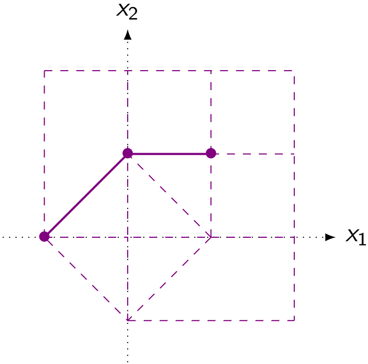


Simplex for 2SLP

$$\overline{\mathcal{I}} = \{34, 23, 125, 146\}$$

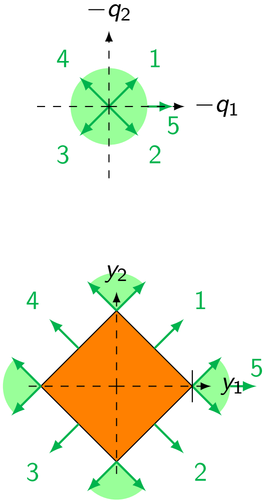


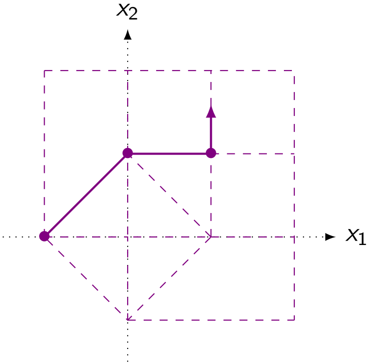




Simplex for 2SLP

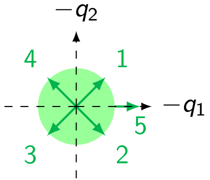
$$\overline{\mathcal{I}} = \{34, 23, 125, 14\}$$

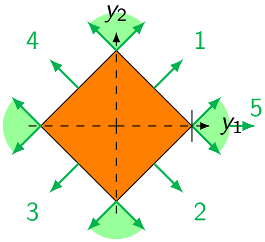


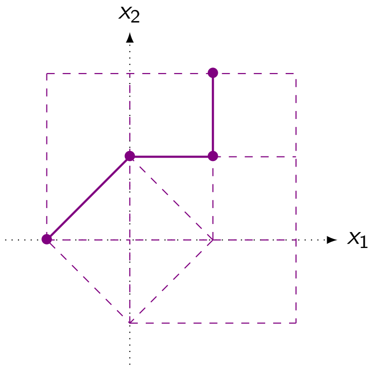


Simplex for 2SLP

$\overline{\mathcal{I}} = \{348, 238, 1258, 148\}$

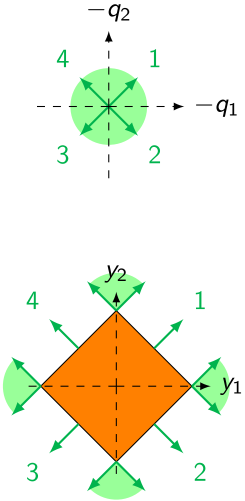


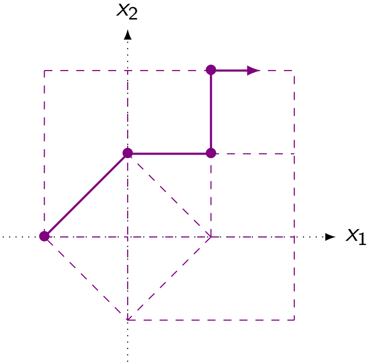




Simplex for 2SLP

$$\overline{\mathcal{I}} = \{348, 238, 128, 148\}$$





Simplex for 2SLP

$\overline{\mathcal{I}} = \{3478, 2378, 1278, 1478\}$