Lecture 4. Conditional Heteroskedastic Model

1. Model Setup

1.1. Empirical Regularities

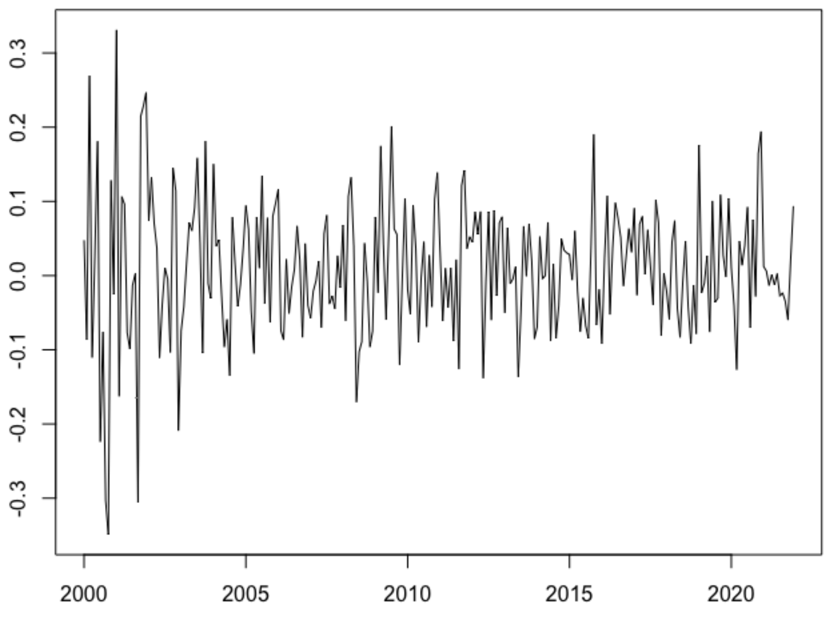
There are empirical regularities for stock return volatility as follows:

- High (low) volatile periods are followed by high (low) volatile periods, which is referred to as volatility clustering. This implies that stock returns are characterized with time-varying volatility conditional on changes in past volatiles. Conditional heteroskedastic models are concerned with the evolution of volatility over time.

- Volatility reacts differently to a price increase (i.e., good news) and a price drop (i.e., bad news) with the latter having a greater impact, which is referred to as leverage effect. According to Black (1976), for instance, bad news tends to drive down the stock price and thus increase the leverage (i.e., the debt-equity ratio) of the stock, thereby causing the stock to be more volatile (i.e., more risky).

- Volatility evolves over time in a continuous manner, i.e., volatility jumps are rare.

Example 1.1. Monthly log returns for the Samsung Electronics stock from January 2000 to December 2021



1.2. General Specification

Let r_t be the log return at time t. Hereafter, we assume

$$\begin{aligned} r_t &= \mu_t + \varepsilon_t \\ \mu_t &= \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} \\ \sigma_t &= f(I_{t-1}), \end{aligned}$$

where $\varepsilon_t = \sigma_t z_t$, $z_t \sim WN(0,1)$, and I_{t-1} is the set of past observations accumulated up to time $t-1$. Notice that both μ_t and σ_t are fully described by past observations.

- Different conditional heteroskedastic models propose different specifications for the function $f(I_{t-1})$.

Theorem 1.2. *It shows*

$$\begin{aligned} E[r_t|I_{t-1}] &= \mu_t \\ Var[r_t|I_{t-1}] &= \sigma_t^2, \end{aligned}$$

both of which are time varying. For this reason, μ_t is referred to as mean equation and σ_t is referred to as volatility equation.

Proof. Since z_t is a white noise process with zero mean and unit variance, we have $E[z_t|I_{t-1}] = E[z_t] = 0$ and $Var[z_t|I_{t-1}] = Var[z_t] = 1$. Based on the fact that μ_t and σ_t are functions of variables in I_{t-1}, we show

$$\begin{aligned} E[r_t|I_{t-1}] &= E[\mu_t + \sigma_t z_t|I_{t-1}] \\ &= E[\mu_t|I_{t-1}] + E[\sigma_t z_t|I_{t-1}] \\ &= \mu_t + \sigma_t E[z_t|I_{t-1}] \\ &= \mu_t \end{aligned}$$

and

$$\begin{aligned} Var[r_t|I_{t-1}] &= Var[\sigma_t z_t|I_{t-1}] \\ &= \sigma_t^2 Var[z_t|I_{t-1}] \\ &= \sigma_t^2. \end{aligned}$$

□

2. Autoregressive Conditional Heteroskedasticity Model

Definition 2.1. The Autoregressive Conditional Heteroskedasticity (ARCH) model of Engle (1982), denoted by ARCH(m), specifies $\sigma_t^2 = f(I_{t-1})$ as

$$\sigma_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + a_2\varepsilon_{t-2}^2 + \cdots + a_m\varepsilon_{t-m}^2, \tag{2.1}$$

where $a_0 > 0$ and $a_i \geq 0$ for $i > 0$.

- The ARCH(m) model implies

$$Var[r_t|I_{t-1}] = a_0 + a_1\varepsilon_{t-1}^2 + a_2\varepsilon_{t-2}^2 + \cdots + a_m\varepsilon_{t-m}^2.$$

  This suggests that the conditional variance of r_t is determined by past squared shocks, i.e., ε_{t-k}^2 for $k = 1,\ldots,m$; as a result, the great (small) uncertainty of r_t tends to be followed by the past great (small) uncertainty, which conforms to the notion of volatility clustering.

Remark 2.2. In an ARMA(p, q)-ARCH(m) model, unknown parameters—i.e., $\{\phi_0,\ldots,\phi_p\}$, $\{\theta_1,\ldots,\theta_q\}$, and $\{a_0,a_1,\ldots,a_m\}$—are estimated with conditional MLE. In practice, z_t is assumed to follow an iid standard normal distribution, an iid Student t distribution, or an iid generalized error distribution.

Remark 2.3. Define $\eta_t = \varepsilon_t^2 - \sigma_t^2$ and assume it quite small, i.e., $\varepsilon_t^2 \approx \sigma_t^2$. We write (2.1) as

$$\varepsilon_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + a_2\varepsilon_{t-2}^2 + \cdots + a_m\varepsilon_{t-m}^2 + \eta_t,$$

which implies that ε_t^2 "approximately" follows an AR(m) process in that η_t is not a white noise error. Let $\hat{\varepsilon}_t$ be an estimate of ε_t, i.e., $\hat{\varepsilon}_t = r_t - \hat{\mu}_t$ where $\hat{\mu}_t$ is an estimate of $E[r_t]$. In practice, we use the PACF of $\hat{\varepsilon}_t^2 = (r_t - \bar{r})^2$ to determine the ARCH order m.

Theorem 2.4. *For $k \geq 1$, we have*

$$E[\sigma_{t+k}^2 z_{t+k}^2|I_t] = E[\sigma_{t+k}^2|I_t].$$

Proof. The ARCH(m) model implies

$$\sigma_{t+k}^2 = a_0 + a_1\sigma_{t+k-1}^2 z_{t+k-1}^2 + a_2\sigma_{t+k-2}^2 z_{t+k-2}^2 + \cdots a_m\sigma_{t+k-m}^2 z_{t+k-m}^2,$$

which means that σ^2_{t+k} is a function of $\{\sigma^2_{t+k-1}, \cdots, \sigma^2_{t+k-m}\}$ and $\{z^2_{t+k-1}, \cdots, z^2_{t+k-m}\}$; σ^2_{t+k-1} is a function of $\{\sigma^2_{t+k-2}, \cdots, \sigma^2_{t+k-1-m}\}$ and $\{z^2_{t+k-2}, \cdots, z^2_{t+k-1-m}\}$; and so on. As a result, σ^2_{t+k} turns out to be a function of $\{z^2_{t+k-1}, z^2_{t+k-2}, \cdots\}$ only. This suggests that σ^2_{t+k} and z^2_{t+k} are independent. Recall that if X and Y are independent, then $E[XY] = E[X]E[Y]$. Therefore, we know

$$\begin{aligned} E[\sigma^2_{t+k} z^2_{t+k} | I_t] &= E[\sigma^2_{t+k}|I_t]E[z^2_{t+k}|I_t] \\ &= E[\sigma^2_{t+k}|I_t]E[z^2_{t+k}] \\ &= E[\sigma^2_{t+k}|I_t]. \end{aligned}$$

□

Remark 2.5. For an ARCH(m) model, we compute the forecasts of σ^2_t as follows:

$$\begin{aligned} \sigma^2_T[1] &= E[\sigma^2_{T+1}|I_T] \\ &= E[a_0 + a_1\varepsilon^2_T + a_2\varepsilon^2_{T-1} + \cdots + a_m\varepsilon^2_{T+1-m}|I_T] \\ &= a_0 + a_1\varepsilon^2_T + a_2\varepsilon^2_{T-1} + \cdots + a_m\varepsilon^2_{T+1-m}, \end{aligned}$$

$$\begin{aligned} \sigma^2_T[2] &= E[\sigma^2_{T+2}|I_T] \\ &= E[a_0 + a_1\varepsilon^2_{T+1} + a_2\varepsilon^2_T + \cdots + a_m\varepsilon^2_{T+2-m}|I_T] \\ &= a_0 + a_1E[\varepsilon^2_{T+1}|I_T] + a_2\varepsilon^2_T + \cdots + a_m\varepsilon^2_{T+2-m} \\ &= a_0 + a_1E[z^2_{T+1}\sigma^2_{T+1}|I_T] + a_2\varepsilon^2_T + \cdots + a_m\varepsilon^2_{T+2-m} \\ &= a_0 + a_1E[\sigma^2_{T+1}|I_T] + a_2\varepsilon^2_T + \cdots + a_m\varepsilon^2_{T+2-m} \\ &= a_0 + a_1\sigma^2_T[1] + a_2\varepsilon^2_T + \cdots + a_m\varepsilon^2_{T+2-m}, \end{aligned}$$

and so on.

3. Generalized Autoregressive Conditional Heteroskedasticity Model

Definition 3.1. The Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model of Bollerslev (1986), denoted by GARCH(m, n), specifies σ^2_t as

$$\sigma^2_t = a_0 + \sum_{i=1}^{m} a_i\varepsilon^2_{t-i} + \sum_{j=1}^{n} b_j\sigma^2_{t-j},$$

where $a_0 > 0$, $a_i \geq 0$, $b_j \geq 0$, and $\sum_{i=1}^{\max(m,n)}(a_i + b_i) < 1$.

- In a GARCH(m, n) model, the conditional variance of r_t depends on the squared shocks ε_{t-i}^2 in the previous m periods as well as the conditional variance σ_{t-j}^2 in the previous n periods.

Remark 3.2. An ARCH(m) model often requires a long lag length m, so a large number of parameters should be estimated. In many applications, however, a GARCH(1, 1) model is enough to obtain a good fit for a financial time series.

Example 3.3. Consider the GARCH(1, 1) model

$$\sigma_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + b_1\sigma_{t-1}^2, \tag{3.1}$$

where $a_0 > 0$, $a_1 \geq 0$, $b_1 \geq 0$, and $a_1 + b_1 < 1$. Let $\eta_t = \varepsilon_t^2 - \sigma_t^2$ and assume it quite small. Then, (3.1) implies

$$\begin{aligned}
\varepsilon_t^2 &= a_0 + a_1\varepsilon_{t-1}^2 + b_1\sigma_{t-1}^2 + \eta_t \\
&= a_0 + a_1\varepsilon_{t-1}^2 + b_1(\varepsilon_{t-1}^2 - \eta_{t-1}) + \eta_t \\
&= a_0 + (a_1 + b_1)\varepsilon_{t-1}^2 + \eta_t - b_1\eta_{t-1}.
\end{aligned}$$

The squared shock ε_t^2 can be expressed as the ARMA(1, 1) model, although η_t is not a white noise process. In many cases, the GARCH coefficient b_1 is found to be around 0.9.

Remark 3.4. Based on the GARCH(1, 1) model in (3.1), we compute the forecasts of σ_t^2 as follows:

$$\begin{aligned}
\sigma_T^2[1] &= E[a_0 + a_1\varepsilon_T^2 + b_1\sigma_T^2 | I_T] \\
&= a_0 + a_1\varepsilon_T^2 + b_1\sigma_T^2,
\end{aligned}$$

$$\begin{aligned}
\sigma_T^2[2] &= E[a_0 + a_1\varepsilon_{T+1}^2 + b_1\sigma_{T+1}^2 | I_T] \\
&= a_0 + a_1 E[\sigma_{T+1}^2 z_{T+1}^2 | I_T] + b_1 E[\sigma_{T+1}^2 | I_T] \\
&= a_0 + a_1 E[\sigma_{T+1}^2 | I_T] + b_1 E[\sigma_{T+1}^2 | I_T] \\
&= a_0 + (a_1 + b_1) E[\sigma_{T+1}^2 | I_T] \\
&= a_0 + (a_1 + b_1)\sigma_T^2[1],
\end{aligned}$$

and so on.

Definition 3.5. An integrated GARCH(1, 1) model, denoted by IGARCH(1, 1), specifies σ_t^2 as

$$\sigma_t^2 = a_0 + a_1 \varepsilon_{t-1}^2 + (1-a_1)\sigma_{t-1}^2,$$

which is a special case of the GARCH(1, 1) model in (3.1) with $a_1 + b_1 = 1$.

- For simplicity, assume $a_0 = 0$. Then, an IGARCH(1, 1) model implies

$$\begin{aligned}
\sigma_t^2 &= a_1 \varepsilon_{t-1}^2 + (1-a_1)(a_1 \varepsilon_{t-2}^2 + (1-a_1)\sigma_{t-2}^2) \\
&= a_1(\varepsilon_{t-1}^2 + (1-a_1)\varepsilon_{t-2}^2) + (1-a_1)^2 \sigma_{t-2}^2 \\
&\vdots \\
&= a_1 \left[\varepsilon_{t-1}^2 + (1-a_1)\varepsilon_{t-2}^2 + (1-a_1)^2 \varepsilon_{t-3}^2 + \cdots \right],
\end{aligned}$$

  which is the exponential smoothing formation of ε_{t-k}^2 with a_1 being a discounting factor. The closer a_1 is to zero, the more weight is put on $\varepsilon_{t-2}^2, \varepsilon_{t-3}^2, \ldots$., while the closer a_1 is to one, the more weight is put on ε_{t-1}^2.

Remark 3.6. In computing value at risk (VaR), J.P. Morgan's RiskMetrics® methodology is based on an IGARCH(1, 1) model.

4. Leverage Effect

A stylized fact of financial volatility is that bad news tends to have a larger impact on volatility than good news. This asymmetric news impact is referred to as the leverage effect.

Definition 4.1. The exponential GARCH model of Nelson (1991), denoted by EGARCH(*m*, *n*), specifies σ_t^2 as

$$h_t = a_0 + \sum_{i=1}^{m} (a_i z_{t-i} + \gamma_i(|z_{t-i}| - E[|z_{t-i}|])) + \sum_{j=1}^{n} b_j h_{t-j},$$

where $h_t = \ln \sigma_t^2$.

Example 4.2. Consider the EGARCH(1, 1) model

$$h_t = a_0 + a_1 z_{t-1} + \gamma_1(|z_{t-1}| - E[|z_{t-1}|]) + b_1 h_{t-1}. \tag{4.1}$$

Using $h_t - b_1 h_{t-1} = \ln(\sigma_t^2/\sigma_{t-1}^{2b_1})$ and $E[|z_t|] = \sqrt{2/\pi}$ for the standard normal random variable z_t, we write (4.1) as

$$\ln\left(\frac{\sigma_t^2}{\sigma_{t-1}^{2b_1}}\right) = a_0 + a_1 z_{t-1} + \gamma_1\left(|z_{t-1}| - \sqrt{\frac{2}{\pi}}\right),$$

which implies

$$\begin{aligned}
\sigma_t^2 &= \sigma_{t-1}^{2b_1}\exp(a_0^*)\exp(a_1 z_{t-1} + \gamma_1|z_{t-1}|) \\
&= \sigma_{t-1}^{2b_1}\exp(a_0^*)\begin{cases}\exp\left((\gamma_1 + a_1)z_{t-1}\right) & \text{for } z_{t-1} > 0 \\ \exp\left((\gamma_1 - a_1)(-z_{t-1})\right) & \text{for } z_{t-1} < 0,\end{cases}
\end{aligned}$$

where $a_0^* = a_0 - \gamma_1\sqrt{2/\pi}$. If bad news has a larger impact, we expect a_1 to be negative.

Definition 4.3. A threshold GARCH model, denoted by TGARCH(m, n), specifies σ_t^2 as

$$\sigma_t^2 = a_0 + \sum_{i=1}^{m}(a_i\varepsilon_{t-i}^2 + \eta_i D_{t-i}\varepsilon_{t-i}^2) + \sum_{j=1}^{n} b_j\sigma_{t-j}^2,$$

where D_{t-i} is the indicator equal to one for $\varepsilon_{t-i} < 0$ and 0 otherwise.

- In a TGARCH(m, n) model, depending on whether ε_{t-i} is above or below the threshold value of zero, ε_{t-i}^2 has different effects on the conditional variance σ_t^2.

Example 4.4. Consider the TGARCH(1, 1) model

$$\begin{aligned}
\sigma_t^2 &= a_0 + (a_1\varepsilon_{t-1}^2 + \eta_1 D_{t-1}\varepsilon_{t-1}^2) + b_1\sigma_{t-1}^2 \\
&= a_0 + (a_1 + \eta_1 D_{t-1})\varepsilon_{t-1}^2 + b_1\sigma_{t-1}^2.
\end{aligned}$$

When good news occurs (i.e., $\varepsilon_{t-1} > 0$), the total effect of ε_{t-1}^2 on σ_t^2 is $a_1\varepsilon_{t-1}^2$. When bad news occurs (i.e., $\varepsilon_{t-i} < 0$), the total effect of ε_{t-1}^2 is $(a_1 + \eta_1)\varepsilon_{t-1}^2$. If the bad news has a larger impact, then the value of η_1 is expected to be positive.

Definition 4.5. The GARCH model of Glosten, Jagannathan, and Runkle (1993), denoted by GJRGARCH(m, n), specifies σ_t^2 as

$$\sigma_t^2 = a_0 + \sum_{i=1}^{m}(a_i\varepsilon_{t-i}^2 + \gamma_i D_{t-i}\varepsilon_{t-i}^2) + \sum_{j=1}^{n} b_j\sigma_{t-j}^2,$$

where D_{t-i} is the indicator equal to one for $\varepsilon_{t-i} < \mu$ and 0 if $\varepsilon_{t-i} \geq \mu$.

- Notice that the threshold value in a GJRGARCH(m, n) model is μ instead of zero. The threshold value μ is unknown and thus should be estimated.

Remark 4.6. Among the models being capable of modeling the leverage effect, the choice of a particular model can be made by the news impact curve of Engle and Ng (1993). The news impact curve represents the functional relationship between the conditional variance at time t and the shock at time $t-1$, holding constant the information dated $t-2$ and earlier.

5. Nonparametric Approach

Let r_t^m be the monthly log return at month t and $r_{t,d}$ be the daily log return during the month. Assume that there are n trading days in the month. Then, we obtain

$$r_t^m = \sum_{d=1}^{n} r_{t,d}$$

and

$$Var[r_t^m] = \sum_{d=1}^{n} Var[r_{t,d}] + \sum Cov[r_{t,d_1}, r_{t,d_2}]. \tag{5.1}$$

- If the daily log return $r_{t,d}$ is assumed to follow a white noise process, then (5.1) simplifies to

  $$Var[r_t^m] = nVar[r_{t,d}].$$

  The variance of monthly return is estimated as

  $$\begin{aligned} \hat{\sigma}_{m,t}^2 &= n\hat{\sigma}_{t,d}^2 \\ &= n\left(\frac{\sum_{d=1}^{n}(r_{t,d}-\bar{r}_t)^2}{n-1}\right), \end{aligned}$$

  where $\bar{r}_t$ is the sample mean of daily returns. This volatility measure is referred to as the realized volatility of monthly returns.

Remark 5.1. With a time series of $\hat{\sigma}_{m,t}$, we apply an appropriate model and then compute the s-step forecast $\hat{\sigma}_T[s]$ from the fitted model.

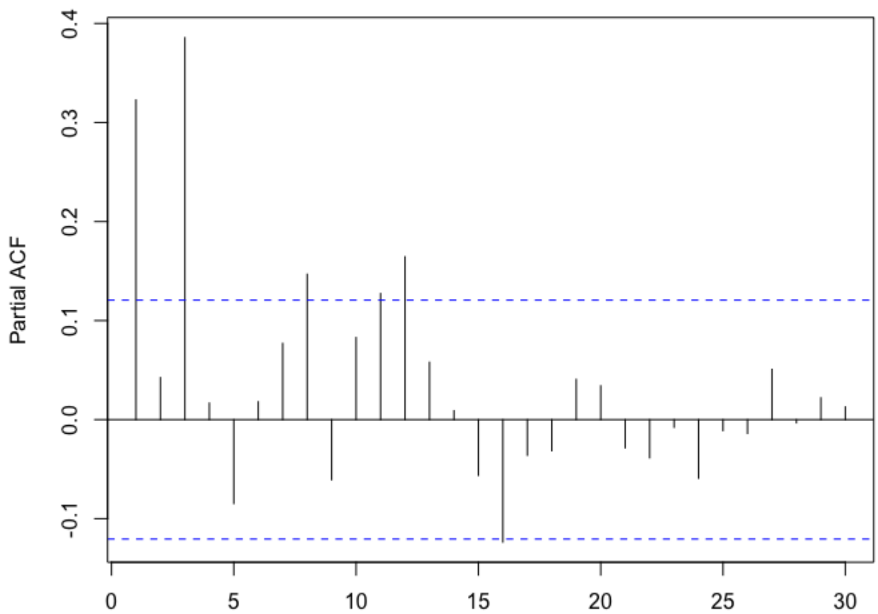
6. R Code

Example 6.1. For the monthly log returns for the Samsung Electronics stock over the period from from January 2000 to December 2021, we specify the mean equation μ_t as an ARMA(0, 0) model of the form

$$\mu_t = \phi_0$$

and assume $z_t \overset{iid}{\sim} N(0,1)$.

```
> mydat <- read.csv("data2_2.csv", header = T)
> rtn <- log(mydat$TRD_RTN + 1)
> rtn <- ts(rtn, freq = 12, start = c(2000, 1))
# Fitting an ARCH(m) Model
> e2 <- (rtn - mean(rtn))^2
> pacf(c(e2), lag = 30, main = "")
```



- Based on the PACF plot, we select an ARCH(12) model.

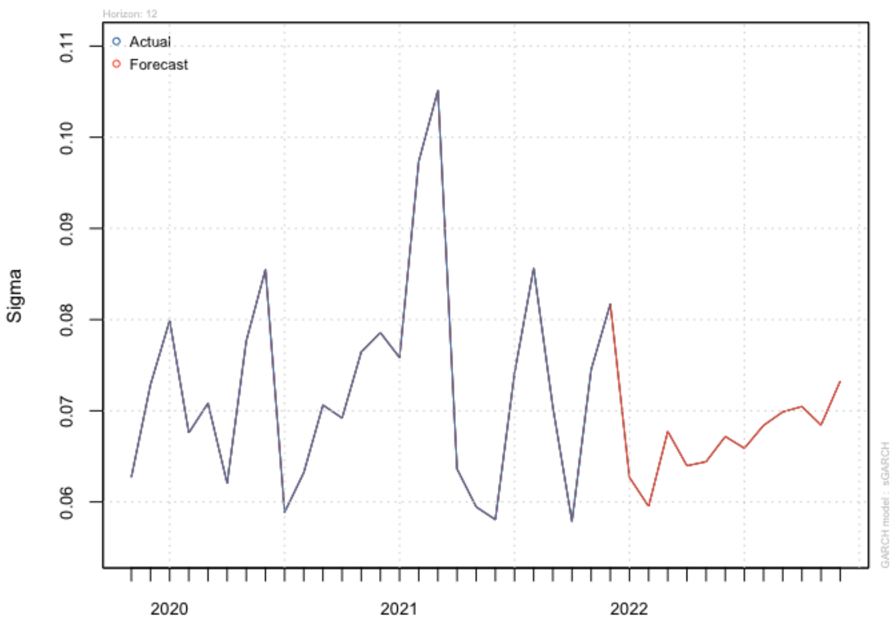
```
> library(rugarch)
> model <- ugarchspec(mean.model = list(armaOrder = c(0, 0)),
                      variance.model = list(garchOrder = c(12, 0)))
> fit <- ugarchfit(spec = model, data = rtn)
> fit
Optimal Parameters
------------------------------------
        Estimate Std.Error  t value Pr(>|t|)
mu      0.009009  0.004607 1.955550 0.050518
```

```
omega   0.002876  0.000711 4.041691 0.000053
alpha1  0.028111  0.078358 0.358758 0.719776
alpha2  0.066025  0.089840 0.734921 0.462387
(...)
alpha12 0.091285  0.079361 1.150248 0.250042
```

- The ARMA(0, 0)-ARCH(12) model is estimated as

$$\mu_t = 0.009009$$
$$\sigma_t^2 = 0.002876 + 0.028111\varepsilon_{t-1}^2 + 0.066025\varepsilon_{t-2}^2 + \cdots + 0.091285\varepsilon_{t-12}^2.$$

```
> fcst <- ugarchforecast(fit, n.ahead = 12)
> plot(fcst, which = 3)
```



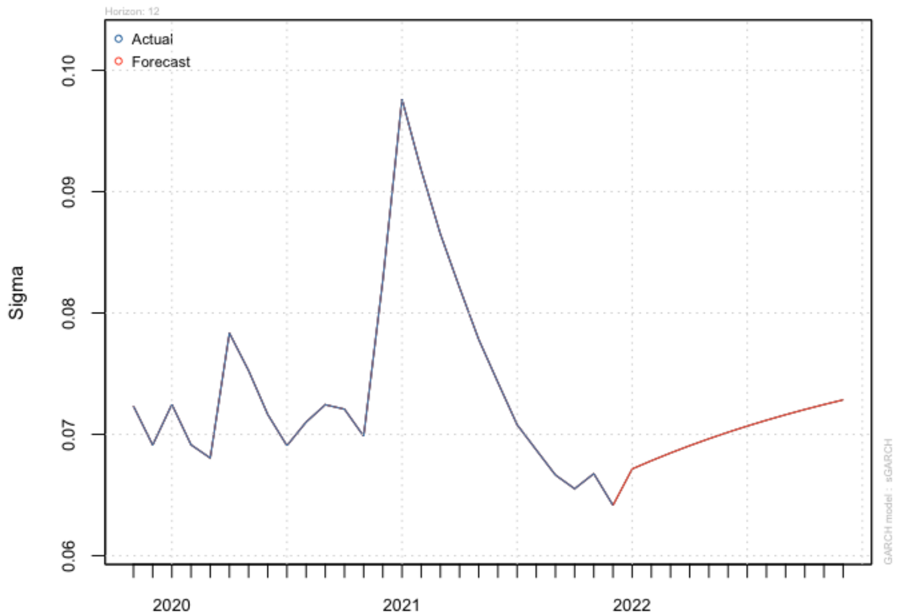

```
# Fitting a GARCH(1, 1) Model
> model <- ugarchspec(mean.model = list(armaOrder = c(0, 0)),
                      variance.model = list(garchOrder = c(1, 1)))
> fit <- ugarchfit(spec = model, data = rtn)
> fit
       Estimate Std. Error t value Pr(>|t|)
mu     0.010367   0.004576  2.2655 0.023481
```

```
omega  0.000306   0.000198  1.5469 0.121896
alpha1 0.100420   0.042207  2.3792 0.017349
beta1  0.851923   0.054479 15.6378 0.000000
```

- The ARMA(0, 0)-GARCH(1, 1) model is estimated as

$$\mu_t = 0.010367$$
$$\sigma_t^2 = 0.000306 + 0.1004206\varepsilon_{t-1}^2 + 0.851923\sigma_{t-1}^2.$$

```
> fcst <- ugarchforecast(fit, n.ahead = 12)
> plot(fcst, which = 3)
```




```
# Fitting an IGARCH(1, 1) Model
> model <- ugarchspec(mean.model = list(armaOrder = c(0, 0)),
                      variance.model = list(model = "iGARCH",
                                            garchOrder = c(1, 1)))
> fit <- ugarchfit(spec = model, data = rtn)
> fit
       Estimate Std.Error t value Pr(>|t|)
mu     0.010544  0.004499  2.3433 0.019114
omega  0.000098  0.000094  1.0432 0.296867
```

```
alpha1 0.137893  0.048172  2.8625 0.004203
beta1  0.862107        NA      NA       NA
```

- The ARMA(0, 0)-IGARCH(1, 1) model is estimated as

$$\begin{aligned} \mu_t &= 0.010544 \\ \sigma_t^2 &= 0.000098 + 0.137893\varepsilon_{t-1}^2 + 0.862107\sigma_{t-1}^2. \end{aligned}$$

Example 6.2. For the monthly log returns for the Samsung Electronics stock over the period from from January 2000 to December 2021, we specific the mean equation μ_t as an ARMA(0, 0) model of the form

$$\mu_t = \phi_0$$

and assume $z_t \overset{iid}{\sim} N(0,1)$.

```
# Fitting an EGARCH(1, 1) Model
> model <- ugarchspec(mean.model = list(armaOrder = c(0, 0)),
                      variance.model = list(model = "eGARCH",
                                            garchOrder = c(1, 1)))
> fit1 <- ugarchfit(spec = model, data = rtn)
> fit1
        Estimate Std.Error t value Pr(>|t|)
mu      0.006445  0.004819  1.3373 0.181130
omega  -0.121636  0.084340 -1.4422 0.149245
alpha1 -0.092041  0.043530 -2.1144 0.034479
beta1   0.976891  0.016805 58.1313 0.000000
gamma1  0.171246  0.079516  2.1536 0.031271
```

- The ARMA(0, 0)-EGARCH(1, 1) model is estimated as

$$\begin{aligned} \mu_t &= 0.006445 \\ h_t &= -0.121636 - 0.092041 z_{t-i} + 0.171246(|z_{t-i}| - E[|z_{t-i}|]) + 0.976891 h_{t-1}. \end{aligned}$$

  Since the estimate of a_1 is negative and statistically significant, we find the leverage effect.

```
> pos.shock <- exp((coef(fit1)[5]+coef(fit1)[3])*2)
> neg.shock <- exp((coef(fit1)[5]-coef(fit1)[3])*2)
```

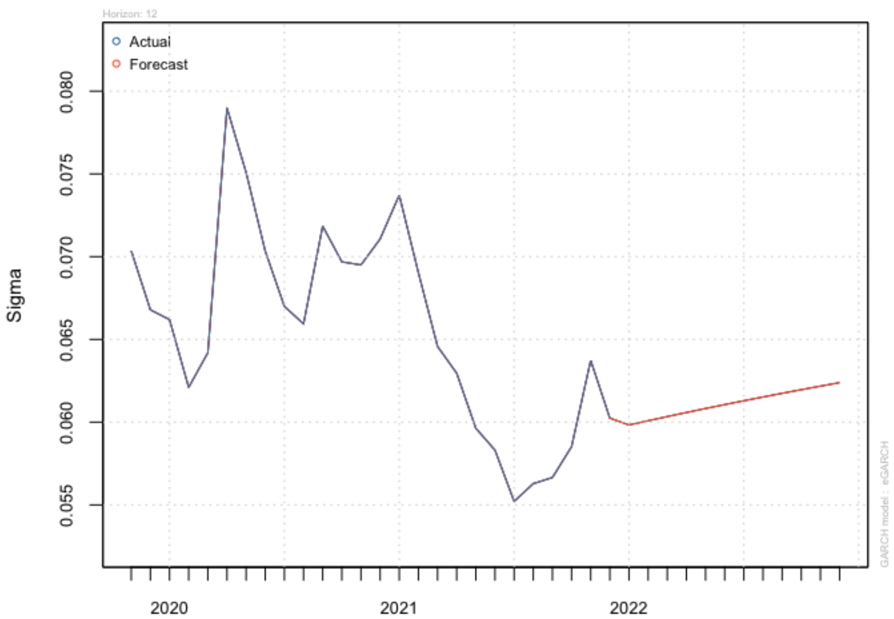
```
> as.numeric(neg.shock/pos.shock)
[1] 1.445078
```

- For a standard normal shock with two standard deviations, we obtain

$$\frac{\sigma_t^2(z_{t-1}=-2)}{\sigma_t^2(z_{t-1}=2)} = \frac{\exp((\gamma_1 - a_1)2)}{\exp((\gamma_1 + a_1)2)} = 1.445078.$$

  So, the impact of a negative shock is about 45% higher than that of a positive shock of the same size.

```
> fcst <- ugarchforecast(fit1, n.ahead = 12)
> plot(fcst, which = 3)
```




```
# Fitting an ARMA(0, 0)-TGARCH(1, 1) Model
> model <- ugarchspec(mean.model = list(armaOrder = c(0, 0)),
                      variance.model = list(model = "fGARCH",
                                            submodel = "TGARCH",
                                            garchOrder = c(1, 1)))
> fit2 <- ugarchfit(spec = model, data = rtn)
> fit2
       Estimate Std.Error t value Pr(>|t|)
mu     0.006049  0.002893  2.0907 0.036556
```

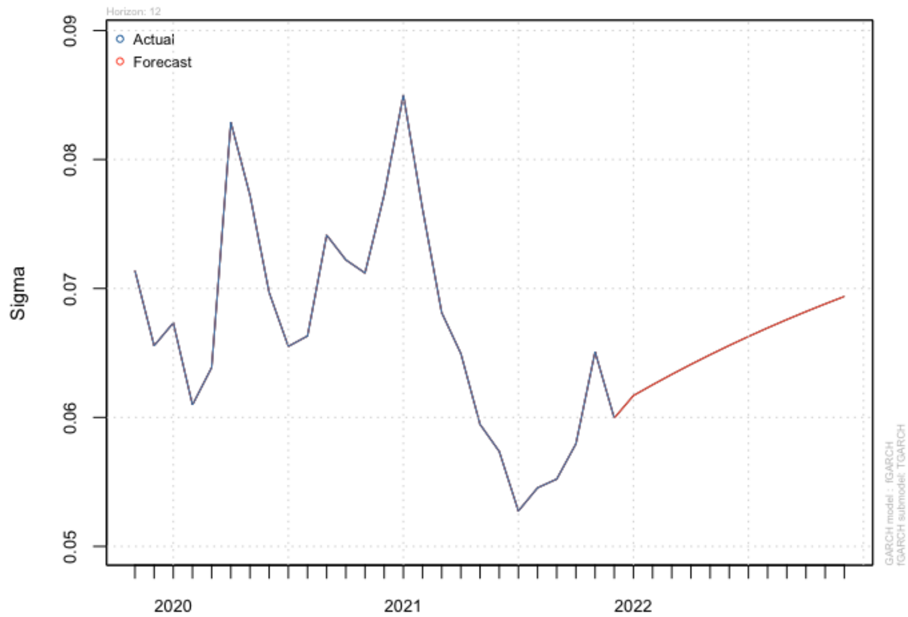
```
omega   0.003107  0.002520   1.2332 0.217503
alpha1 0.137514  0.043216   3.1820 0.001462
beta1   0.853475  0.049325 17.3030 0.000000
eta11   0.384104  0.217887   1.7629 0.077925
```

- The ARMA(0, 0)-TGARCH(1, 1) model is estimated as

$$\mu_t = 0.006049$$
$$\sigma_t^2 = 0.003107 + (0.137514 + 0.384104 D_{t-1})\varepsilon_{t-1}^2 + 0.853475\sigma_{t-1}^2.$$

  The leverage effect is present but insignificant at the 5% level.

```
> fcst <- ugarchforecast(fit2, n.ahead = 12)
> plot(fcst, which = 3)
```




```
# Fitting an ARMA(0, 0)-GJRGARCH(1, 1) Model
> model <- ugarchspec(mean.model = list(armaOrder = c(0, 0)),
                      variance.model = list(model = "gjrGARCH",
                                            garchOrder = c(1, 1)))
> fit3 <- ugarchfit(spec = model, data = rtn)
> fit3
       Estimate Std.Error  t value Pr(>|t|)
```

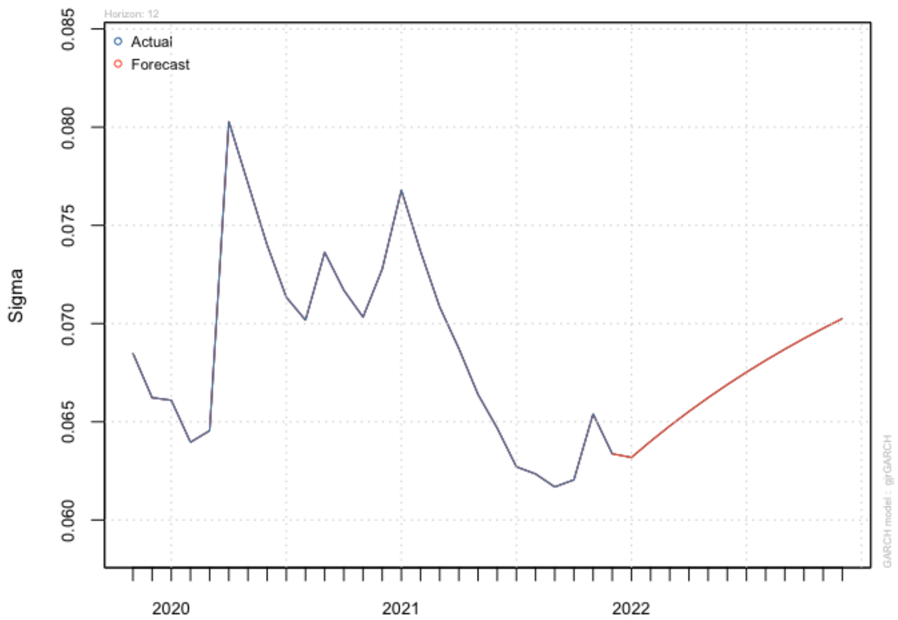
```
mu      0.008048  0.004759  1.69123 0.090793
omega   0.000268  0.000172  1.55313 0.120393
alpha1  0.028704  0.048831  0.58781 0.556660
beta1   0.875089  0.051232 17.08095 0.000000
gamma1  0.110766  0.068874  1.60823 0.107786
```

- The ARMA(0, 0)-GJRGARCH(1, 1) model is estimated as

$$\mu_t = 0.008048$$
$$\sigma_t^2 = 0.000268 + (0.028704 + 0.110766 I_{t-1})\varepsilon_{t-1}^2 + 0.875089\sigma_{t-1}^2.$$

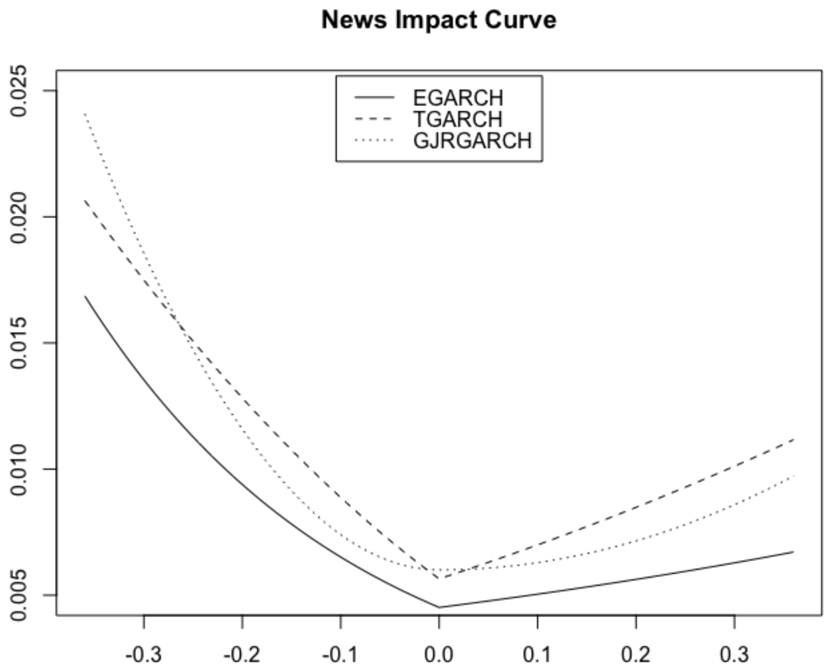
The leverage effect is present but insignificant at the 5% level.

```
> fcst <- ugarchforecast(fit2, n.ahead = 12)
> plot(fcst, which = 3)
```




```
> ni_egarch <- newsimpact(z = NULL, fit1)
> ni_tgarch <- newsimpact(z = NULL, fit2)
> ni_gjrgarch <- newsimpact(z = NULL, fit3)
> plot(ni_egarch$zx, ni_egarch$zy, type="l", xlab = "", ylab = "",
       ylim = c(0.005, 0.025), main = "News Impact Curve")
> lines(ni_tgarch$zx, ni_tgarch$zy, lty = 2)
```

```
> lines(ni_gjrgarch$zx, ni_gjrgarch$zy, lty = 3)
> legend("top", c("EGARCH", "TGARCH", "GJRGARCH"), lty = c(1, 2, 3),
        inset = 0.01)
```

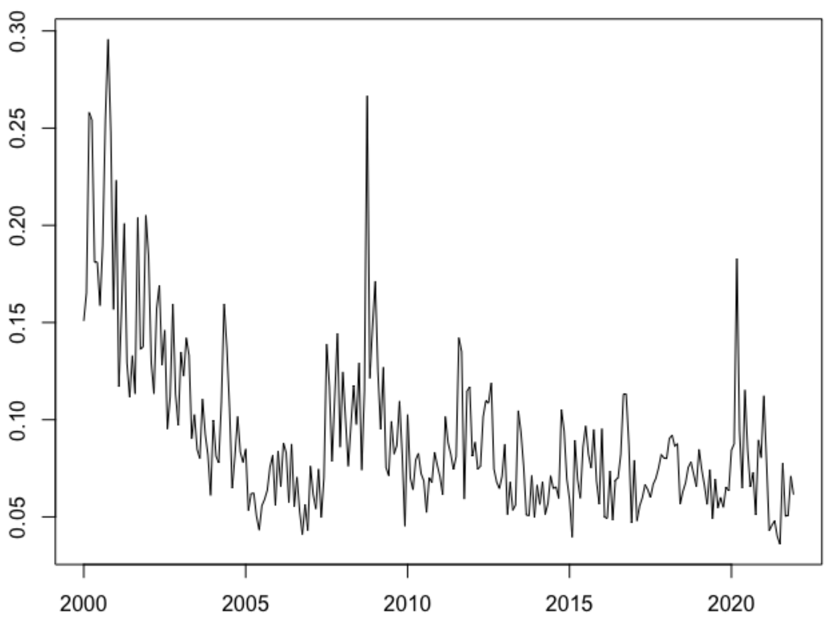


- The news impact curves are asymmetric in the EGARCH, TGARCH, and GJRGARCH models as expected. Compared with the EGARCH and TGARCH models, the GJR-GARCH model seems more appropriate to emphasize the leverage effect.

Example 6.3. We use daily log returns to compute the realized volatility of monthly log returns of the Samsung Electronics stock.

```
> mydat <- read.csv("data4_1.csv", header = T)
> head(mydat)
    TRD_DD ADJ_PRC
1 20000104    6110
2 20000105    5580
(...)
6 20000111    5770
> mydat <- cbind(mydat, RTN = c(NA, diff(log(mydat$ADJ_PRC))))
> mydat <- mydat[-1, ]
> mydat <- cbind(mydat,
```

```
                TRD_MM = substring(as.character(mydat$TRD_DD), 1, 6))
> head(mydat)
    TRD_DD ADJ_PRC          RTN  TRD_MM
2 20000105    5580 -0.090737997  200001
3 20000106    5620  0.007142888  200001
(...)
7 20000112    5720 -0.008703275  200001
> trd_mm <- unique(mydat$TRD_MM)
> result <- NULL
> for (i in 1:length(trd_mm)){
   dat <- subset(mydat, TRD_MM == trd_mm[i])
   n <- dim(dat)[1]
   vol <- sqrt(var(dat$RTN)*n)
   result <- rbind(result, vol)
}
> vol <- ts(result, start = c(2000, 1), freq = 12)
> plot(vol, xlab = "", ylab = "")
```




```
> computeAIC <- function(p, q){
   fit <- arima(vol, order = c(p, 0, q))
```

```
   fit$aic}
> computeAIC(0, 0)
[1] -902.0579
> computeAIC(0, 1)
[1] -1026.727
(...)
> computeAIC(3, 3)
[1] -1136.766
> fit <- arima(vol, order = c(3, 0, 3))
```

- Based on the AIC, we select an ARMA(3, 3) model for modeling the realized volatility of stock returns.

```
> fcst.x <- predict(fit, 12)$pred
> new.x <- ts(c(vol, rep(NA, 12)), start = c(2000, 1), frequency = 12)
> new.x <- window(new.x, start = c(2020, 1))
> plot(new.x, xlab = "", ylab = "")
> lines(fcst.x, lty = 2)
> legend("topright", c("Observations", "Forecasts"), lty = c(1, 2),
         inset = 0.01)
```