Lecture 3. Trending Time-Series Data

1. Nonstationarity

Definition 1.1. A time series x_t is a trend-stationary process if it has the form

$$x_t = \alpha + \delta t + u_t,$$

where $\delta \neq 0$ and u_t is a stationary process with $E[u_t] = 0$ and $Var[u_t] = \sigma_u^2$. The term $\alpha + \delta t$ is referred to as a deterministic time trend.

- The mean of x_t is $E[x_t] = \alpha + \delta t$ that depends on $t > 0$, so the trend-stationary process x_t is nonstationary.
- The variance of x_t is $Var[x_t] = \sigma_u^2$ that is constant over time, so the trend-stationary process x_t exhibits trend reversion in that x_t never deviates too far away from the deterministic time trend $\alpha + \delta t$.

Remark 1.2. If $\delta = 0$, then $E[x_t] = \alpha$ and $Var[x_t] = \sigma_u^2$, both of which are time invariant; consequently, $x_t = \alpha + u_t$ is stationary.

Example 1.3. Consider a trend-stationary AR(1) process x_t of the form

$$x_t = \alpha + \delta t + u_t \tag{1.1}$$

$$u_t = \phi_1 u_{t-1} + \varepsilon_t, \tag{1.2}$$

where $\delta \neq 0$, $|\phi_1| < 1$, and $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$. From (1.2), it shows

$$u_t = \frac{\varepsilon_t}{1 - \phi_1 L} = \varepsilon_t + \phi_1 \varepsilon_{t-1} + \phi_1^2 \varepsilon_{t-2} + \cdots \tag{1.3}$$

because $|\phi_1| < 1$, which implies that $E[u_t] = 0$; and

$$Var[u_t] = \sigma_\varepsilon^2 + \phi_1^2 \sigma_\varepsilon^2 + \phi_1^4 \sigma_\varepsilon^2 + \cdots = \frac{\sigma_\varepsilon^2}{1 - \phi_1^2}$$

because $|\phi_1^2| < 1$. From (1.1), we have

$$E[x_t] = \alpha + \delta t$$

$$Var[x_t] = \frac{\sigma_\varepsilon^2}{1 - \phi_1^2},$$

which shows that x_t is nonstationary. Combining (1.1) and (1.3) leads to

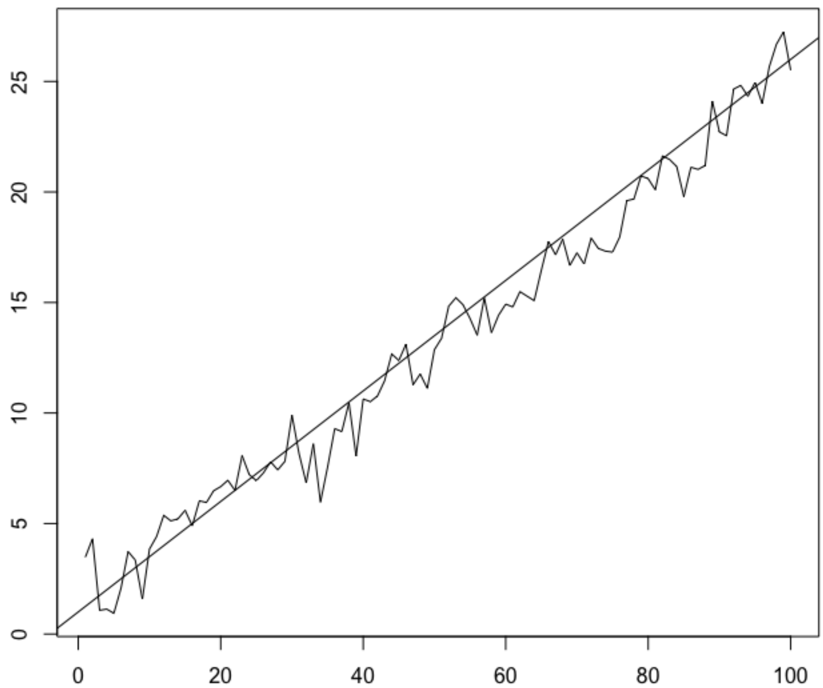
$$x_t = \alpha + \delta t + \varepsilon_t + \phi_1 \varepsilon_{t-1} + \phi_1^2 \varepsilon_{t-2} + \cdots,$$

which implies $\partial x_{t+k}/\partial \varepsilon_t = \phi_1^k$ for any $k \geq 1$. Therefore, the shock to the trend-stationary AR(1) process x_t is temporary in that the impact of the shock ε_t on x_{t+k} decays over time, i.e., $\lim_{k\to\infty} \phi_1^k = 0$.

Example 1.4. Simulation from a trend stationary AR(1) process x_t of the form

$$x_t = 1 + 0.25t + u_t,$$

where $u_t = 0.75u_{t-1} + \varepsilon_t$ and $\varepsilon_t \overset{iid}{\sim} N(0,1)$ for $t = 1, \ldots, 100$



Definition 1.5. A time series x_t is a unit-root process if it has the form

$$x_t = \delta + x_{t-1} + u_t, \tag{1.4}$$

where u_t is a zero-mean stationary process. The term δ is referred to as a drift. Alternatively, the unit-root process x_t is given by

$$\Delta x_t = \delta + u_t,$$

where $\Delta x_t = x_t - x_{t-1}$.

- With the fixed starting value x_0, we obtain

$$\begin{aligned} x_t &= \delta + (\delta + x_{t-2} + u_{t-1}) + u_t \\ &= 2\delta + x_{t-2} + u_t + u_{t-1} \\ &\vdots \\ &= x_0 + \delta t + \sum_{i=1}^{t} u_i. \end{aligned}$$

  This implies that (a) $E[x_t] = x_0 + \delta t$ depends on t if $\delta \neq 0$ (i.e., $E[x_t]$ grows linearly over time) but $E[x_t] = x_0$ is time invariant if $\delta = 0$ (i.e., $E[x_t]$ remains constant over time) and (b) $Var[x_t] = Var\left[\sum_{i=1}^{t} u_i\right]$ depends on t. Therefore, the unit-root process x_t is always nonstationary regardless of whether the drift δ is zero or nonzero.

Remark 1.6. Since $Var[u_t]$ is constant, $Var[x_t]$ explodes as t increases. Consequently, the unit-root process of x_t can deviate far away from the deterministic trend $x_0 + \delta t$ over time, which is contrast to the trend reversion of a trend-stationary process.

Remark 1.7. If a time series x_t exhibits a time trend, then x_t is always nonstationary and follows either a trend stationary process with $\delta \neq 0$ or a unit-root process with $\delta \neq 0$. If a time-series x_t does not exhibit a time trend, then x_t can be either stationary (in the form of a trend-stationary process with $\delta = 0$) or nonstationary (in the form of a unit-root process with $\delta = 0$).

Example 1.8. A random walk process of x_t with drift is defined by

$$x_t = \delta + x_{t-1} + \varepsilon_t,$$

where $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$. This is a special case of the unit-root nonstationary process of x_t when u_t is a white noise error, i.e., $u_t = \varepsilon_t$, in (1.4). With the fixed starting value x_0, we obtain

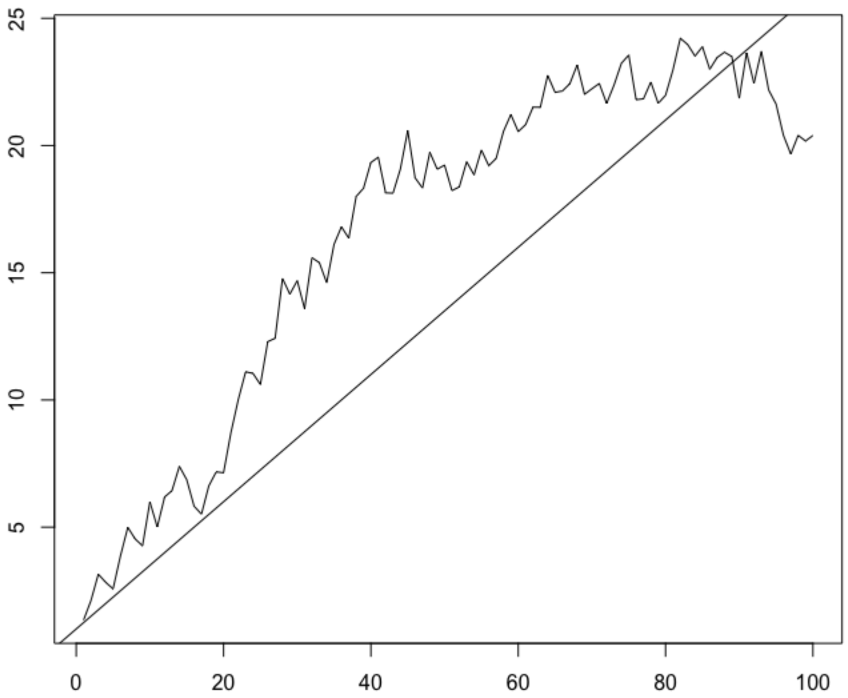
$$x_t = x_0 + \delta t + \sum_{i=1}^{t} \varepsilon_i,$$

so $\partial x_t / \partial \varepsilon_{t-k} = 1$ for any $k \geq 1$. This means that the impact of the shock ε_t on the random walk process of x_{t+k} does not decay over time, or equivalently, the shock has a permanent effect.

Example 1.9. Simulation with $x_0 = 1$ from the random walk process

$$x_t = 0.25 + x_{t-1} + \varepsilon_t,$$

where $\varepsilon_t \overset{iid}{\sim} N(0,1)$ for $t = 1, \ldots, 100$.



2. ARIMA Model

Definition 2.1. For a time series of x_t, an ARIMA(p, d, q) model is defined by

$$(1-L^d)x_t = \phi_0 + \sum_{i=1}^{p} \phi_i (1-L^d)x_{t-i} + \varepsilon_t + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j},$$

where $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$. For $x_t \sim$ARIMA(p, 1, q), it shows $\Delta x_t \sim$ARMA(p, q).

- If Δx_t is stationary and $E[\Delta x_t] = \mu$, then we write the ARIMA(p, 1, q) model is given by

$$(\Delta x_t - \mu) = \sum_{i=1}^{p} \phi_i (\Delta x_{t-i} - \mu) + \varepsilon_t + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}.$$

Theorem 2.2. *Suppose that u_t follows a zero-mean ARMA(p+1, q) process of the form*

$$(1 - \phi_1 L - \cdots - \phi_{p+1} L^{p+1})u_t = (1 + \theta_1 L + \cdots + \theta_q L^q)\varepsilon_t, \tag{2.1}$$

where $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$. If one of the solutions to the AR characteristic equation $\phi(z) = 0$ is a unit root (i.e., equal to one in modulus) and other p solutions are greater than one in modulus, *then it shows that (a) u_t is nonstationary and (b) Δu_t follows a stationary zero-mean ARMA(p, q) process.*

We consider a linear model of the form

$$x_t = c(t) + u_t,$$

where u_t follows a zero-mean ARMA($p+1$, q) process. Three cases emerge as follows:

- Case I: Suppose that $c(t) = \alpha$ and u_t is stationary. In this case, x_t does not exhibit a time trend because $E[x_t] = \alpha$; and follows a stationary ARMA($p+1$, q) model of the form

$$(x_t - \alpha) = \sum_{i=1}^{p+1} \phi_i (x_{t-i} - \alpha) + \varepsilon_t + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}.$$

- Case II: Suppose that $c(t) = \alpha + \delta t$ and u_t is stationary. Then x_t exhibits a time trend because $E[x_t] = \alpha + \delta t$; and follows a trend-stationary process that is nonstationary. In this case, a time-trended series of x_t, i.e., $x_t^* = x_t - \alpha - \delta t$, follows a stationary zero-mean ARMA($p+1$, q) process of the form

$$x_t^* = \sum_{i=1}^{p+1} \phi_i x_{t-i}^* + \varepsilon_t + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}.$$

  or equivalently

$$\begin{aligned} x_t &= \alpha + \delta t + u_t \\ u_t &= \sum_{i=1}^{p+1} \phi_i u_{t-i} + \varepsilon_t + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}. \end{aligned}$$

- Case III: Suppose that there is a unit root in the AR characteristic equation $\phi(z) = 0$. Then x_t is nonstationary because u_t is nonstationary. In this case, a first-differenced series of x_t, i.e., $\Delta x_t = x_t - x_{t-1}$, is a stationary ARMA($p$, q) process because (a) $\Delta x_t = \Delta u_t$ if $c(t) = \alpha$ and (b) $\Delta x_t = \delta + \Delta u_t$ if $c(t) = \alpha + \delta t$; and $\Delta u_t \sim$ARMA(p, q) with $E[\Delta u_t] = 0$. Put differently, x_t is a unit-root process with the stationary error Δu_t and follows the following ARIMA(p, 1, q) model:

$$(\Delta x_t - E[\Delta x_t]) = \sum_{i=1}^{p} \phi_i (\Delta x_{t-i} - E[\Delta x_t]) + \varepsilon_t + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j},$$

where (a) $E[\Delta x_t] = 0$ if $c(t) = \alpha$ and (b) $E[\Delta x_t] = \delta$ if $c(t) = \alpha + \delta t$. Notice that the time-trended series, $x_t - c(t)$, is not stationary in this case.

3. Unit-Root Test

3.1. Motivation

- Suppose that there is no clear time trend in the data of x_t. We test the null hypothesis that x_t is a unit-root process with $\delta = 0$. If the null hypothesis is not rejected, we cannot apply an ARMA model to x_t because x_t is nonstationary and the ARMA model must be applied to a stationary time series; instead, we apply an ARIMA(p, 1, q) model to x_t with $E[\Delta x_t] = 0$. If the null hypothesis is rejected, then x_t is stationary; consequently, we apply an ARMA model to x_t with $E[x_t] = \alpha$.

- Suppose that x_t is a trending time-series process which is nonstationary. We test the null hypothesis that x_t is a unit-root process with $\delta \neq 0$. If the null hypothesis is not rejected, we apply an ARIMA(p, 1, q) model to x_t with $E[\Delta x_t] = \delta$. If the null hypothesis is rejected, we apply an ARMA model to the time-trended series, i.e., $x_t^* = x_t - \alpha - \delta t$.

3.2. Dickey-Fuller Unit-Root Test

Consider an AR(1) process x_t of the form

$$x_t = c(t) + \phi_1 x_{t-1} + \varepsilon_t, \tag{3.1}$$

where $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$. If $\phi_1 = 1$, x_t follows a unit-root process; while if $|\phi_1| < 1$, x_t follows an AR(1) process with the deterministic term $c(t)$.

Theorem 3.1. *Consider $H_0 : \phi_1 = 1$ versus $H_1 : |\phi_1| < 1$ in (3.1). A test statistic is given by*

$$t = \frac{\hat{\phi}_1 - 1}{se(\hat{\phi}_1)},$$

where $\hat{\phi}_1$ is the OLS estimate of ϕ_1 and $se(\hat{\phi}_1)$ is its standard error. Dickey and Fuller (1979) show that the test statistic has a Dickey-Fuller (DF) distribution under H_0 and has an asymptotic standard normal distribution under H_1.

- In practice, it is important to specify $c(t)$, so that the alternative hypothesis appropriately reflects the trend properties of the observed data. Two cases emerge: $c(t) = \alpha$ if the data exhibit no time trend and $c(t) = \alpha + \delta t$ if the data exhibit a time trend.

3.3. Augmented Dickey-Fuller Unit-Root Test

Contrary to the simple AR(1) assumption of the Dickey-Fuller test, a typical time series has a complicated dynamic structure. Said and Dickey (1984) augment the basic Dickey-Fuller test to accommodate general AR$(p+1)$ models. Their test is referred to as the augmented Dickey-Fuller (ADF) test.

Theorem 3.2. *(Dickey-Fuller transformation) An AR($p+1$) model of the form*

$$x_t = \phi_0 + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_{p+1} x_{t-p-1} + \varepsilon_t$$

can be written as

$$x_t = \phi_0 + \rho x_{t-1} + \psi_1 \Delta x_{t-1} + \psi_2 \Delta x_{t-2} + \cdots + \psi_p \Delta x_{t-p} + \varepsilon_t,$$

where $\rho = \phi_1 + \cdots + \phi_{p+1}$ and ψ_j is a linear combination of ϕ_is.

- For instance, the Dickey-Fuller transformation of an AR(2) model is

$$\begin{aligned} x_t &= \phi_0 + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \varepsilon_t \\ &= \phi_0 + \phi_1 x_{t-1} + \phi_2 x_{t-1} - \phi_2 x_{t-1} + \phi_2 x_{t-2} + \varepsilon_t \\ &= \phi_0 + (\phi_1 + \phi_2) x_{t-1} - \phi_2 (x_{t-1} - x_{t-2}) + \varepsilon_t \\ &= \phi_0 + \rho x_{t-1} + \psi_1 \Delta x_{t-1} + \varepsilon_t. \end{aligned}$$

Consider an AR$(p+1)$ model of the form

$$x_t = c(t) + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_{p+1} x_{t-p-1} + \varepsilon_t, \tag{3.2}$$

where $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$. The Dickey-Fuller transformation implies that (3.2) can be written as

$$x_t = c(t) + \rho x_{t-1} + \sum_{j=1}^{p} \psi_j \Delta x_{t-j} + \varepsilon_t$$

or

$$\Delta x_t = c(t) + \pi x_{t-1} + \sum_{j=1}^{p} \psi_j \Delta x_{t-j} + \varepsilon_t, \tag{3.3}$$

where $\pi = \rho - 1$. If there is a unit root (i.e., $\rho = 1$), then $\pi = 0$.

Theorem 3.3. *In testing for $H_0 : \pi = 0$ versus $H_1 : \pi \neq 0$ in (3.3), a test statistic is given by*

$$t = \frac{\hat{\pi}}{se(\hat{\pi})},$$

where $\hat{\pi}$ is the OLS estimate of π and $se(\hat{\pi})$ is its standard error. It shows that the test statistic has a DF distribution under H_0 and has an asymptotic standard normal distribution under H_1.

Remark 3.4. In (3.3), p lagged difference terms Δx_{t-j} approximate the AR structure of Δx_t, and the value of p should be set so that ε_t is serially uncorrelated. If p is too small, then the remaining serial correlation in the error will bias the test. If p is too large, then the power of test (i.e., the probability that H_0 is rejected when H_1 is true) will suffer.

Remark 3.5. Ng and Perron (1995) suggest the data dependent lag length selection procedure for choosing p for the ADF test, which follows two steps. First, we set an upper bound p_{max} as

$$p_{max} = \left[12 \cdot \left(\frac{T}{100} \right)^{1/4} \right],$$

where $[x]$ denotes the integer part of x. Second, we estimate the ADF test regression (3.3) with $p = p_{max}$. If the absolute value of the t-ratio of the the last lagged difference is greater than 1.6, then set $p = p_{max}$; otherwise, we reduce the lag length by one and repeat the process.

4. Seasonality

Definition 4.1. For a seasonal time series of x_t with periodicity s, the operation $\Delta_s = 1 - L^s$ is called a seasonal differencing and removes a seasonality. The conventional difference $\Delta = 1 - L$ is referred to as a regular differencing, which is used to remove a linear trend.

- When $s = 4$, for instance, it shows

$$\Delta_4 x_t = (1 - L^4) x_t = x_t - x_{t-4}.$$

Definition 4.2. A seasonal ARIMA$(p,d,q) \times (P,1,Q)_s$ model of x_t has the form

$$\phi(L)\Phi(L^s)(1 - L^d)(1 - L^s)x_t = \theta(L)\Theta(L^s)\varepsilon_t,$$

where $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$, s is the periodicity of the times series x_t, and

$$\begin{aligned}
\text{non-seasonal AR componet: } \phi(L) &= 1 - \phi_1 L - \cdots - \phi_p L^p \\
\text{non-seasonal MA componet: } \theta(L) &= 1 + \theta_1 L + \cdots + \theta_q L^q \\
\text{seasonal AR componet: } \Phi(L^s) &= 1 - \Phi_1 L^s - \cdots - \Phi_P L^{sP} \\
\text{seasonal MA componet: } \Theta(L^s) &= 1 + \Theta_1 L^s + \cdots + \Theta_Q L^{sQ}.
\end{aligned}$$

It is common to set $P = 0$ and $Q = 1$ in practice.

- To correct for a seasonality, we apply the seasonal ARIMA(p, 0, q)$\times(P,1,Q)_s$ model to a non-trending time series and the seasonal ARIMA(p, 1, q)$\times(P,1,Q)_s$ model to a trending time series.

Example 4.3. A seasonal ARIMA$(1,0,1) \times (0,1,1)_4$ model is written as

$$(1-\phi_1 L)(1-L^4)x_t = (1+\theta_1 L)(1+\Theta_1^4)\varepsilon_t$$

and a seasonal ARIMA$(1,1,1) \times (0,1,1)_4$ model is written as

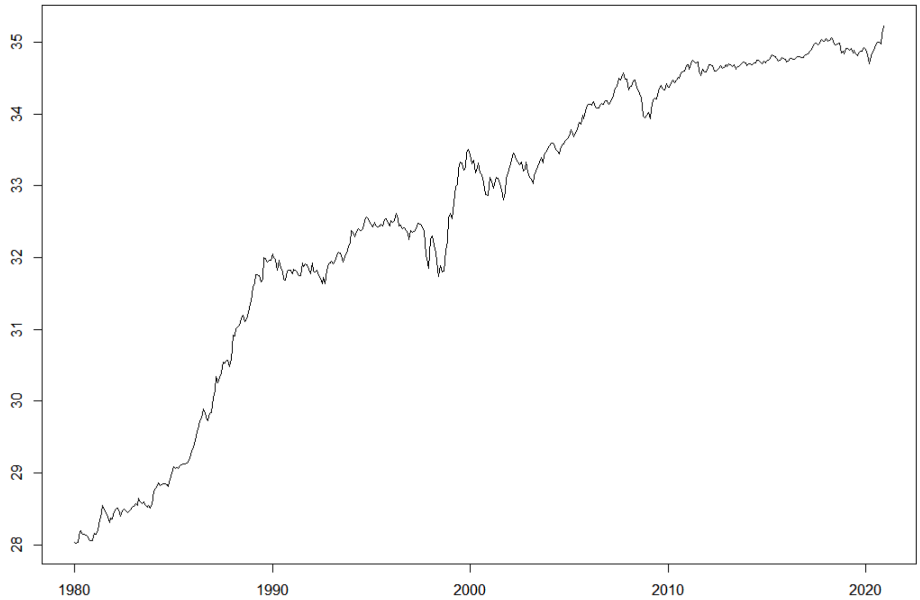
$$(1-\phi_1 L)(1-L)(1-L^4)x_t = (1+\theta_1 L)(1+\Theta_1^4)\varepsilon_t.$$

5. R Code

Example 5.1. Log series of the monthly market value of all stocks listed on the KOSPI market from January 1980 to December 2020

```
> library(urca)
> library(fUnitRoots)
> library(forecast)
> mydat <- read.csv("data3_1.csv", header = T)
> head(mydat)
    TRD_DD   KOSPI
1 19800131 28.0360
2 19800229 28.0263
3 19800331 28.0350
4 19800430 28.1848
5 19800531 28.1933
6 19800630 28.1478
```

```
> kospi <- ts(mydat$KOSPI, start = c(1980, 1), freq = 12)
> kospi1 <- window(kospi, end = c(2020, 12))
> plot(kospi1, xlab = "", ylab = "")
```



- There is an increasing trend, so x_t would be either a unit-root process with $\delta > 0$ or a trend-stationary process with $c(t) = \alpha + \delta t$.

```
> pmax <- floor(12*(length(kospi1)/100)^(1/4))
> fit <- ur.df(kospi1, type = "trend", lags = (pmax - 1))
> summary(fit)
Coefficients:
                Estimate  Std.Error t value Pr(>|t|)
(Intercept)    3.061e-01  1.695e-01   1.805   0.0717 .
z.lag.1       -9.640e-03  5.833e-03  -1.653   0.0991 .
tt             8.834e-05  8.726e-05   1.012   0.3119
z.diff.lag1    1.123e-01  4.662e-02   2.409   0.0164 *
z.diff.lag2    2.505e-02  4.685e-02   0.535   0.5931
(...)
z.diff.lag16 -8.753e-02  4.675e-02  -1.872   0.0618 .
```

- To check whether x_t is a unit-root nonstationary, we use $c(t) = \alpha + \delta t$. The absolute value of the t-ratio of the last lagged difference Δx_{t-16} is greater than 1.6, so $p = 16$.

```
> adfTest(kospi1, lags = (pmax - 1), type = "ct")
Dickey-Fuller:  -1.6527
P VALUE: 0.7252
```

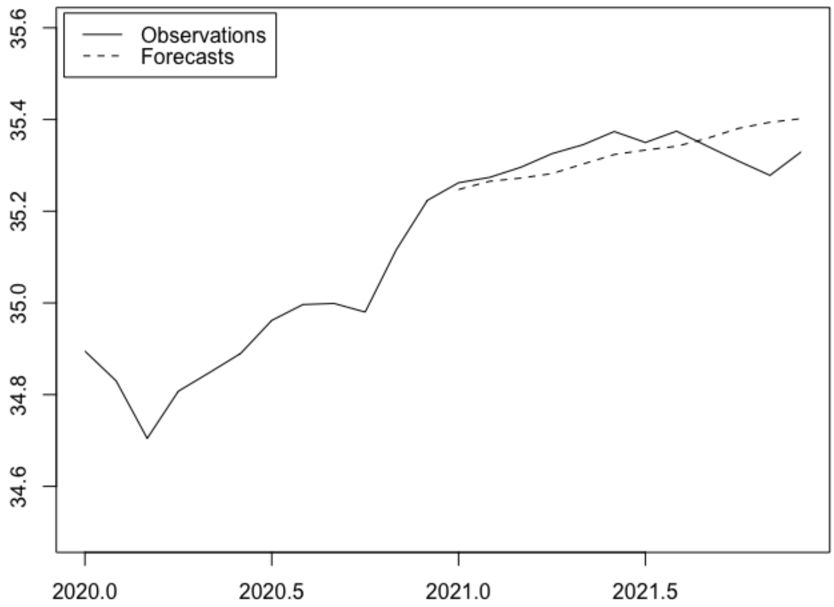
- The unit-root hypothesis cannot be rejected at the 5% level, which suggests that x_t can be appropriately described by an ARIMA(p, 1, q) process with $E[\Delta x_t] = \delta$.

```
# Unit-Root Process
> computeAIC <- function(p, q){
   fit <- Arima(kospi1, order = c(p, 1, q), include.constant = T)
   fit$aic
}
> computeAIC(0, 0)
[1] -1105.239
(...)
> computeAIC(3, 3)
[1] -1113.74
> fit <- Arima(kospi1, order = c(3, 1, 2), include.constant = T)
> fit
Coefficients:
         ar1      ar2    ar3      ma1    ma2  drift
      0.2899 -0.9655 0.0845 -0.1757 1.0000 0.0146
s.e. 0.0454  0.0182 0.0450  0.0060 0.0104 0.0039
```

- The ARIMA(3, 1, 2) model is estimated as

$$\begin{aligned}(\Delta x_t - 0.0146) &= 0.2899(\Delta x_{t-1} - 0.0146) - 0.9655(\Delta x_{t-2} - 0.0146)\\ &\quad +0.0845(\Delta x_{t-3} - 0.0146) + \varepsilon_t - 0.1757\varepsilon_{t-1} + \varepsilon_{t-2}.\end{aligned}$$

```
> fcst.x <- forecast(fit, 12)$mean
> new.x <- window(kospi, start = c(2020, 1))
> plot(new.x, xlab = "", ylab = "", ylim = c(34.5, 35.6))
> lines(fcst.x, lty = 2)
> legend("topleft", c("Observations", "Forecasts"), lty = c(1, 2),
         inset = 0.01)
```

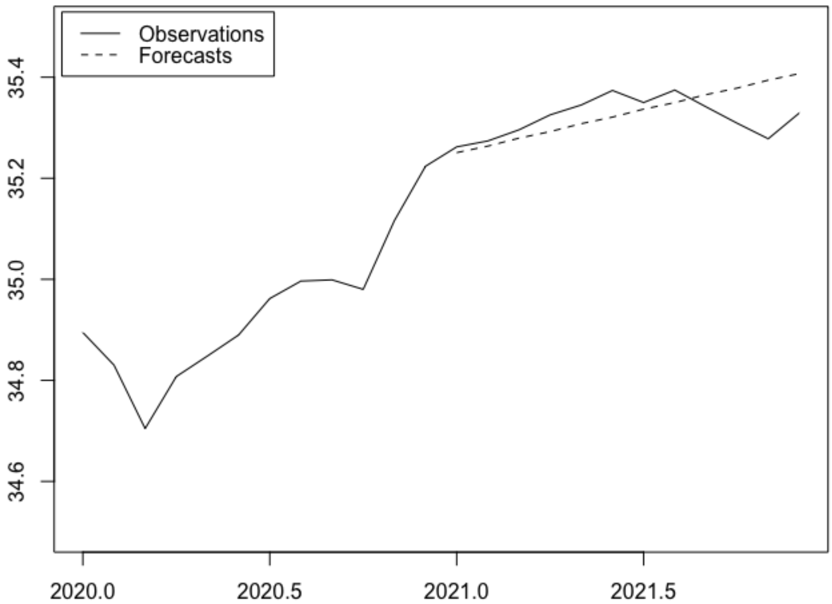



```
# Trend-Stationary Process
> tdx <- 1:length(kospi1)
> computeAIC <- function(p, q){
   fit <- arima(kospi1, order = c(p, 0, q), xreg = tdx)
   fit$aic
}
> computeAIC(0, 0)
[1] 998.6413
(...)
> computeAIC(3, 3)
[1] -1101.649
Warning message:  In arima(kospi1, order = c(p, 0, q), xreg = tdx) :
possible convergence problem:  optim gave code = 1
> (fit <- arima(kospi1, order = c(3, 0, 1), xreg = tdx, method = "ML"))
Coefficients:
        ar1    ar2     ar3    ma1 intercept    tdx
     0.1332 0.9883 -0.1347 0.9881   28.7000 0.0145
s.e. 0.0485 0.0145  0.0452 0.0209    0.6701 0.0020
```

- Suppose that x_t might be a trend-stationary process. In this case, the fitted model is given by

$$x_t = 28.7000 + 0.0145t + u_t$$
$$u_t = 0.1332u_{t-1} + 0.9883u_{t-2} - 0.1347u_{t-3} + \varepsilon_t + 0.9881\varepsilon_{t-1}.$$

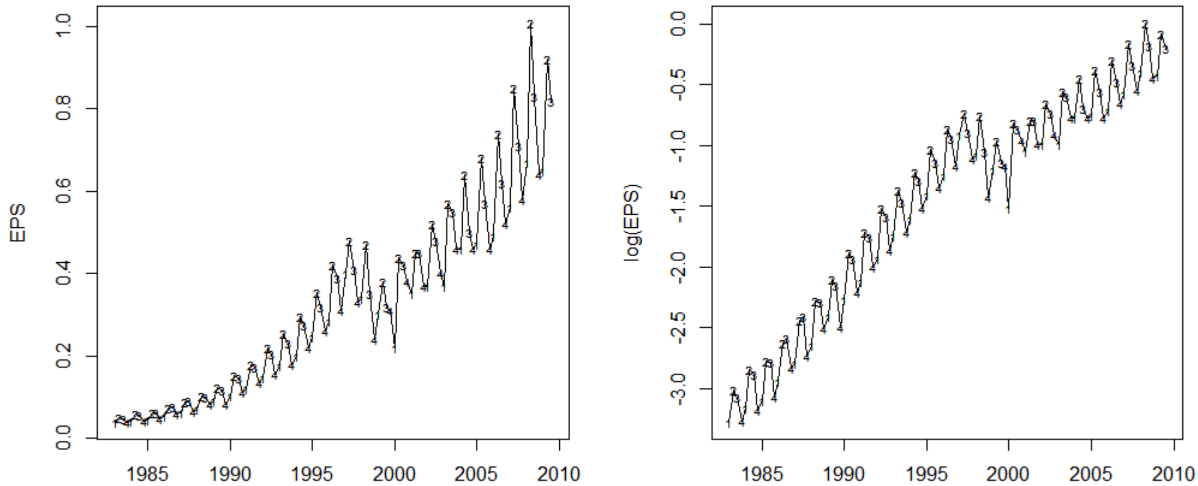
```
> new.tdx <- (length(kospi1) + 1):(length(kospi1) + 12)
> fcst.x <- predict(fit, newxreg = new.tdx, n.ahead = 12)$pred
> new.x <- window(kospi, start = c(2020, 1))
> plot(new.x, xlab = "", ylab = "", ylim = c(34.5, 35.5))
> lines(fcst.x, lty = 2)
> legend("topleft", c("Observations", "Forecasts"), lty = c(1, 2),
         inset = 0.01)
```



Example 5.2. Consider quarterly earnings per share (EPS) of the Coca-Cola Company from 1Q 1983 to 3Q 2009.

```
> library(forecast)
> mydat <- read.table("data3_2.txt", header = T)
> head(mydat)
     pends  anntime  value
1 19830331 19830426 0.0375
2 19830630 19830725 0.0492
```

```
3 19830930 19831102 0.0463
4 19831231 19840214 0.0379
5 19840331 19840419 0.0425
6 19840630 19840720 0.0583
> EPS <- ts(mydat$value, start = c(1983, 1), freq = 4)
> eps <- log(EPS)
> par(mfrow = c(1, 2))
> plot(EPS, xlab = "", ylab = "EPS")
> c1 <- c("1", "2", "3", "4")
> points(EPS, pch = c1, cex = 0.6)
> plot(eps, xlab = "", ylab = "log(EPS)")
> points(eps, pch = c1, cex = 0.6)
```



- Two observations emerge. First, quarterly EPS shows a strong seasonality; specifically, the seasonal pattern repeats itself every year and the periodicity of the series is 4. Second, EPS grows exponentially, while log EPS grows linearly. Indeed, log transformation stabilizes the variability of the series in that compared with the left plot, the increasing pattern in variability disappears in the right plot. Hence, we apply a seasonal ARIMA$(p,1,q) \times (0,1,1)_4$ model to log EPS.

```
> computeAIC <- function(p, q){
   fit <- Arima(eps, order = c(p, 1, q),
                 seasonal = list(order = c(0, 1, 1), period = 4))
```

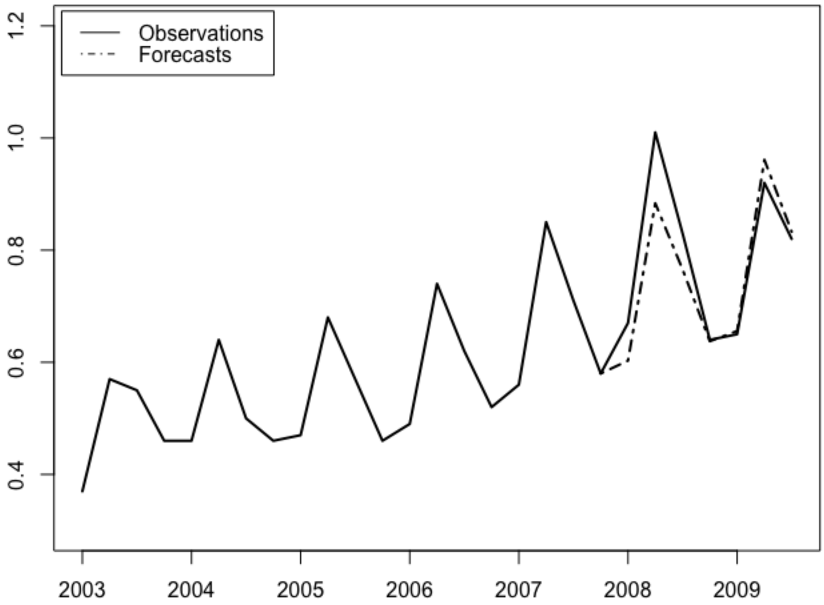
```
   fit$aic
}
> computeAIC(0, 0)
[1] -188.5548
(...)
> computeAIC(3, 3)
[1] -201.5322
> fit <- Arima(eps, order = c(3, 1, 2),
               seasonal = list(order = c(0, 1, 1), period = 4))
> fit
Coefficients:
        ar1      ar2      ar3      ma1     ma2     sma1
     0.7561  -0.5361  -0.1876  -1.2340  0.9252  -0.8437
s.e. 0.1451   0.1136   0.1107   0.1061  0.1088   0.0679
```

- The seasonal ARIMA model is estimated as

$$\phi(L)(1-L)(1-L^4)x_t = \theta(L)(1-0.8437L^4)\varepsilon_t,$$

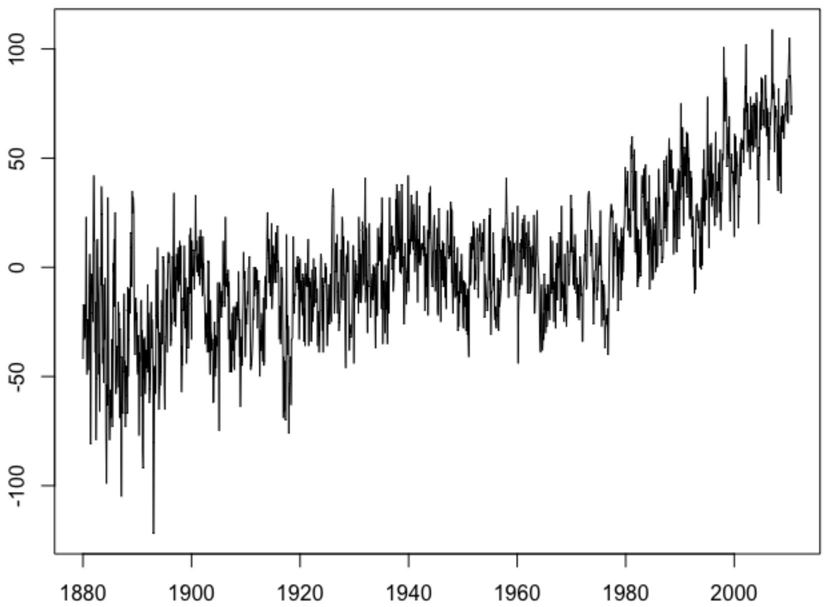
where $\phi(L) = 1-0.7561L+0.5361L^2+0.1876L^3$ and $\theta(L) = 1-1.2340L+0.9252L^2$.

```
> eps1 <- window(eps, end = c(2007, 4))
> fit1 <- Arima(eps1, order = c(0, 1, 1),
                seasonal = list(order = c(0, 1, 1), period = 4))
> fcst.x <- forecast(fit1, 7)$mean
> EPS1 <- window(EPS, start = c(2003, 1))
> plot(EPS1, lwd = 2, xlab = "", ylab = "", ylim = c(0.3, 1.3))
> Fcst.x <- exp(fcst.x)
> lines(ts(c(EPS1[20], Fcst.x), start = c(2007, 4), freq = 4),
        lty = 4, lwd = 2)
> legend("topleft",
         c("Observations", "Forecasts", "95% confidence interval"),
         lty = c(1, 4, 3), lwd = c(2, 2, 1), inset = 0.01)
```



Example 5.3. Consider the monthly global temperature anomalies from January 1880 to August 2010.

```
> x <- scan(file = "data3_3.txt")
Read 1568 items
> x <- ts(x, start = c(1880, 1), freq = 12)
> plot(x, xlab = "", ylab = "")
```



```
> library(forecast)
> fit <- auto.arima(x, ic = "aic")
> fit
Series:  x
ARIMA(2,1,2) with drift
Coefficients:
         ar1      ar2      ma1    ma2  drift
      1.0072 -0.1443 -1.5636 0.5716 0.0638
s.e.  0.1516  0.0918  0.1454 0.1416 0.0250
> plot(forecast(fit, 1200), main = "")
```