UNIVERSITY OF DAR ES SALAAM

COLLEGE OF ICT

IS 143

Discrete Structure

Lecture 3

Instructor

Dr. Joseph Cosmas

Kijitonyama Campus
Block A,
Room No. A023

Joseph.cosmas@udsm.ac.tz

Content

- Introduction to Course
- Proposition, Sets, Relations and Functions
- Algorithm and Basic Logics
- Proof Techniques
- Basics Of Counting (Mathematical Reasoning)
- Graphs And Trees
- Discrete Probability

Sets

- A set is a collection or group of objects or *elements* or *members*.
  - A set is said to contain its elements.
  - There must be an underlying universal set U, either specifically stated or understood.

Sets (cont.)

- Notation:
  - List the elements between braces:

    $$S = \{a, b, c, d\}=\{b, c, a, d, d\}$$

    - Note: listing an object more than once does not change the set.
    - Ordering means nothing.
  - Specification by predicates:

    $$S= \{x \mid P(x)\},$$

    - S contains all the elements from U which make the predicate P true.
  - Brace notation with ellipses:

    $$S = \{ \ldots, -3, -2, -1\},$$

  the negative integers.

Sets (cont.)

- Common Universal Sets
  - **R** = reals
  - **N** = natural numbers = {0, 1, 2, 3, . . . }, the *counting* numbers
  - **Z** = all integers = {. . , -3, -2, -1, 0, 1, 2, 3, 4, . .}
  - Z^+ is the set of positive integers
- Notation:

x is a member of S or x is an element of S:

$$x \in S.$$

x is not an element of S:

$$x \notin S.$$

Sets (cont.)

- Subsets
  - Definition: The set A is a *subset* of the set B, denoted $A \subseteq B$, iff

$$\forall x \, [x \in A \rightarrow x \in B]$$

  - Definition: The void set, the null set, the empty set, denoted $\emptyset$, is the set with no members.

    Note: the assertion $x \in \emptyset$ is <u>always</u> false.

    Hence

$$\forall x \, [x \in \emptyset \rightarrow x \in B]$$

    is always true.

  - Therefore, $\emptyset$ is a subset of every set.

Note: a set, say B, is always a subset of itself.

Sets (cont.)

- Definition:

  If $A \subseteq B$ but $A \neq B$ then we say A is a *proper* subset of B, denoted $A \subset B$.

- Definition:

  The set of all subset of a set A, denoted P(A), is called the *power set* of A.

- Example:

  If $A = \{a, b\}$ then

  $$P(A) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$$

Sets (cont.)

- Definition:

  The number of (distinct) elements in A, denoted $|A|$, is called the *cardinality* of A.

  If the cardinality is a natural number (in N), then the set is called *finite*, else *infinite*.

- Example:

  $A = \{a, b\}$,

  $| A | = |\{a, b\}| = 2$,

  $| P(A) | = |P(\{a, b\})| = 4$.

  - ✓ If A is finite then P(A) is also finite.
  - ✓ **Useful Fact**: $|A| = n$ implies $|P(A)| = \mathbf{2^n}$

Sets (cont.)

- Definition: The *Cartesian product* of A with B, denoted A x B, is the set of <u>ordered pairs</u> $\{<a, b> \mid a \in A \wedge b \in B\}$

Notation: $\underset{i=1}{\overset{n}{\times}} A_i = \{< a_1, a_2, \ldots, a_n > a_i \in A_i\}$

Note: The Cartesian product of anything with $\varnothing$ is $\varnothing$. (why?)

- Example:

  A = {a,b}, B = {1, 2, 3}

  AxB = {<a, 1>, <a, 2>, <a, 3>, <b, 1>, <b, 2>, <b, 3>}

- What is BxA? AxBxA?
- If |A| = m and |B| = n, **what is |AxB|**?

Set Operations (cont.)

- Propositional calculus and set theory are both instances of an algebraic system called a

  Boolean Algebra.

  - The operators in set theory are defined in terms of the corresponding operator in propositional calculus
  - As always there must be a universe U. All sets are assumed to be subsets of U

Set Operations (cont.)

- Definition:

  Two sets A and B are equal, denoted $A = B$, iff

  $$\forall x\ [x \in A \leftrightarrow x \in B].$$

  - Note: From a previous logical equivalence we have

    $$A = B \text{ iff } \forall x\ [(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

    or

    $$A = B \text{ iff } A \subseteq B \text{ and } B \subseteq A$$

Set Operations (cont.)

- Definitions:
  - The *union* of A and B, denoted A U B, is the set $\{x \mid x \in A \vee x \in B\}$
  - The *intersection* of A and B, denoted $A \cap B$, is the set

    $$\{x \mid x \in A \wedge x \in B\}$$

    Note: If the intersection is void, A and B are said to be *disjoint*.
  - The complement of A, denoted , is the set $\{x \mid \neg(x \in A)\}$

    Note: Alternative notation is A^c, and $\{x \mid x \notin A\}$.
  - The difference of A and B, or the complement of B relative to A, denoted A - B, is the set $A \cap \overline{B}$

    Note: The (absolute) complement of A is U - A.
- The *symmetric difference* of A and B, denoted $A \oplus B$, is the set

  $$(A - B) \cup (B - A)$$

Set Operations (cont.)

- **Examples:**

  U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

  A= {1, 2, 3, 4, 5},

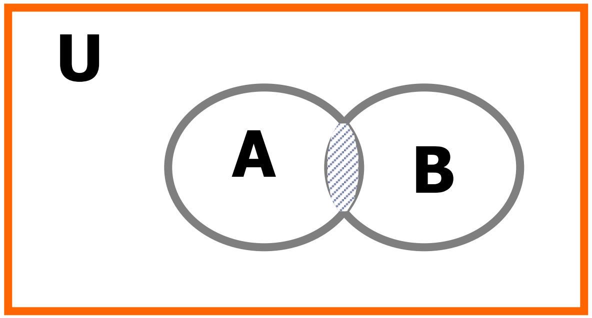
  B = {4, 5, 6, 7, 8}.

  Then

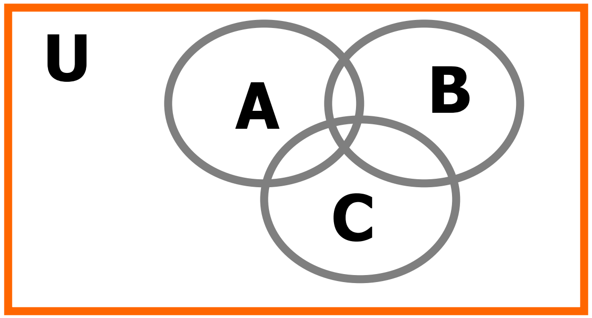
  - $A \cup B =$
  - $A \cap B =$
  - $\overline{\boldsymbol{A}} =$
  - $\overline{\boldsymbol{B}} =$
  - $A - B =$
  - $B - A =$
  - $A \oplus B =$

Set Operations (cont.)

- Venn Diagrams
  - A useful geometric visualization tool (for 3 or less sets)
  - The Universe U is the rectangular box
  - Each set is represented by a circle and its interior
  - All possible combinations of the sets must be represented



For 2 sets



For 3 sets

  - Shade the appropriate region to represent the given set operation.

Set Operations (cont.)

- Set Identities

  - Set identities correspond to the logical equivalences.

  - Example:

    The complement of the union is the intersection of the complements:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

Proof: To show:

$$\forall x \, [x \in \overline{A \cup B} \leftrightarrow x \in \overline{A} \cap \overline{B}]$$

To show two sets are equal we show for all x that x is a member of one set if and only if it is a member of the other.

Set Operations (cont.)

- We now apply an important rule of inference (defined later) called

Universal Instantiation

In a proof we can eliminate the universal quantifier which binds a variable if we do not assume anything about the variable other than it is an arbitrary member of the Universe. We can then treat the resulting predicate as a proposition.

Set Operations (cont.)

- We say

'Let x be arbitrary.'

<u>Then</u> we can treat the predicates as propositions:

Assertion	Reason
$x \in \overline{A \cup B} \Leftrightarrow x \notin [A \cup B]$	Def. of complement
$x \notin A \cup B \Leftrightarrow \neg[x \in A \cup B]$	Def. of $\notin$
$\Leftrightarrow \neg[x \in A \vee x \in B]$	Def. of union
$\Leftrightarrow \neg x \in A \wedge \neg x \in B$	DeMorgan's Laws
$\Leftrightarrow x \notin A \wedge x \notin B$	Def. of $\notin$
$\Leftrightarrow x \in \overline{A} \wedge x \in \overline{B}$	Def. of complement
$\Leftrightarrow x \in \overline{A} \cap \overline{B}$	Def. of intersection

Set Operations (cont.)

- Hence

$$x \in \overline{A \cup B} \leftrightarrow x \in \overline{A} \cap \overline{B}$$

  is a tautology.

Since

- x was arbitrary
- we have used only logically equivalent assertions and definitions

Set Operations (cont.)

- we can apply another rule of inference called

Universal Generalization

We can apply a universal quantifier to bind a variable if we have shown the predicate to be true for all values of the variable in the Universe.

and claim the assertion is true for all x, i.e.,

$$\forall x \, [x \in \overline{\boldsymbol{A} \cup \boldsymbol{B}} \leftrightarrow x \in \overline{\boldsymbol{A}} \cap \overline{\boldsymbol{B}}]$$

Set Operations (cont.)

- Note: As an alternative which might be easier in some cases, use the identity

$$A = B \Leftrightarrow [A \subseteq B \text{ and } B \subseteq A]$$

- Example:

Show $A \cap (B - A) = \varnothing$

The void set is a subset of every set. Hence,

$$A \cap (B - A) \supseteq \varnothing$$

Therefore, it suffices to show

$$A \cap (B - A) \subseteq \varnothing \quad \text{or} \quad \forall x\, [x \in A \cap (B - A) \rightarrow x \in \varnothing]$$

So as before we say 'let x be arbitrary'.

Set Operations (cont.)

- Example (cont.)

  Show $x \in A \cap (B - A) \rightarrow x \in \varnothing$ is a tautology.

  But the consequent is always false.

  Therefore, the antecedent better always be false also.

  Apply the definitions:

Assertion	Reason
$x \in A \cap (B - A) \Leftrightarrow x \in A \wedge x \in (B - A)$	Def. of $\cap$
$\Leftrightarrow x \in A \wedge (x \in B \wedge x \notin A)$	Def. of $-$
$\Leftrightarrow (x \in A \wedge x \notin A) \wedge x \in B$	Props of 'and'
$\Leftrightarrow 0 \wedge x \in B$	Table 6
$\Leftrightarrow 0$	Domination

Set Operations (cont.)

- Union and Intersection of Indexed Collections

  - Let $A_1, A_2, \ldots, A_n$ be an indexed collection of sets.
  - Union and intersection are associative (because 'and' and 'or' are) we have:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

and

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

Set Operations (cont.)

- **Examples**

Let

$$A_i = [i, \infty), 1 \leq i < \infty$$

$$\bigcup_{i=1}^{n} A_i = [1, \infty)$$

$$\bigcap_{i=1}^{n} A_i = [n, \infty)$$

Examples

- **Assignment details:**
  - Book Title: Discrete Mathematical Structure (shared earlier)
  - Book Author: Shanker G. Rao
  - Page: 67
  - Exercise 2.2
  - Question no. 2, 4, 7, 8, 25, 26