UNIVERSITY OF DAR ES SALAAM

COLLEGE OF ICT

IS 143

Discrete Structure

Lecture 4

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Content

- Introduction to Course
- Proposition, Sets, Relations and Functions
- Algorithm and Basic Logics
- Proof Techniques
- Basics Of Counting (Mathematical Reasoning)
- Graphs And Trees
- Discrete Probability

Relations

- Relations & their properties

  - Definition 1

    Let A and B be sets. A binary relation from A to B is a subset of A * B.

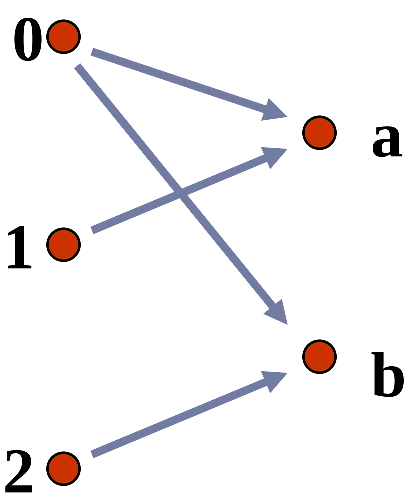
    In other words, a binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.

Relations (cont.)

- Notation:

  $aRb \Leftrightarrow (a, b) \in R$

  $aRb \Leftrightarrow (a, b) \notin R$



R	a	b
0	X	X
1	X	
2		X

Relations (cont.)

- Relations on a set
  - Definition 2

    A relation on the set A is a relation from A to A.

  - Example: A = set {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a divides b}

    Solution: Since (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b

    R = {(1,1), (1,2), (1.3), (1.4), (2,2), (2,4), (3,3), (4,4)}

Relations (cont.)

- Properties of Relations
  - Definition 3

    A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.

Relations (cont.)

- Example (a): Consider the following relations on {1, 2, 3, 4}

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$
$R_2 = \{(1,1), (1,2), (2,1)\}$
$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)\}$
$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$
$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$
$R_6 = \{(3,4)\}$

Which of these relations are reflexive?

Relations (cont.)

Solution:

R_3 and R_5: reflexive $\Leftarrow$ both contain all pairs of the form (a, a): (1,1), (2,2), (3,3) & (4,4).

R_1, R_2, R_4 and R_6: not reflexive $\Leftarrow$ not contain all of these ordered pairs. (3,3) is not in any of these relations.

R_1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}
R_2 = {(1,1), (1,2), (2,1)}
R_3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)}
R_4 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}
R_5 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}
R_6 = {(3,4)}

Relations (cont.)

- Definition 4:

A relation R on a set A is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.

A relation R on a set A such that $(a, b) \in R$ and $(b, a) \in R$ only if $a = b$, for all $a, b \in A$, is called antisymmetric.

Relations (7.1) (cont.)

- Example: Which of the relations from example (a) are symmetric and which are antisymmetric?

Solution:

- R_2 & R_3: **symmetric** $\Leftarrow$ each case (b, a) belongs to the relation whenever (a, b) does.
For R_2: only thing to check that both (1,2) & (2,1) belong to the relation
For R_3: it is necessary to check that both (1,2) & (2,1) belong to the relation.
None of the other relations is symmetric: find a pair (a, b) so that it is in the relation but (b, a) is not.

R_1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}
R_2 = {(1,1), (1,2), (2,1)}
R_3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)}
R_4 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}
R_5 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}
R_6 = {(3,4)}

Relations (cont.)

Solution (cont.):

- R_4, R_5 and R_6: **antisymmetric** $\Leftarrow$ for each of these relations there is no pair of elements a and b with $a \neq b$ such that both (a, b) and (b, a) belong to the relation.
- None of the other relations is antisymmetric.: find a pair (a, b) with $a \neq b$ so that (a, b) and (b, a) are both in the relation.

R_1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}
R_2 = {(1,1), (1,2), (2,1)}
R_3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)}
R_4 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}
R_5 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}
R_6 = {(3,4)}

Relations (cont.)

- Definition 5:

A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b,c) \in R$, then $(a, c) \in R$, for all $a, b, c \in R$.

Relations (cont.)

- **Example:** Which of the relations in example (a) are transitive?
- R_4 , R_5 & R_6 : transitive $\Leftarrow$ verify that if (a, b) and (b, c) belong to this relation then (a, c) belongs also to the relation
  R_4 transitive since (3,2) and (2,1), (4,2) and (2,1), (4,3) and (3,1), and (4,3) and (3,2) are the only such sets of pairs, and (3,1) , (4,1) and (4,2) belong to R_4.
  Same reasoning for R_5 and R_6.
- R_1 : not transitive $\Leftarrow$ (3,4) and (4,1) belong to R_1, but (3,1) does not.
- R_2 : not transitive $\Leftarrow$ (2,1) and (1,2) belong to R_2, but (2,2) does not.
- R_3 : not transitive $\Leftarrow$ (4,1) and (1,2) belong to R_3, but (4,2) does not.

R_1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}
R_2 = {(1,1), (1,2), (2,1)}
R_3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)}
R_4 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}
R_5 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}
R_6 = {(3,4)}

Relations (cont.)

- Combining relations
  - Example:
    Let A = {1, 2, 3} and B = {1, 2, 3, 4, }.

    The relations

    $R_1 = \{(1,1), (2,2), (3,3)\}$ and

    $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$

    can be combined to obtain:

    $R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\}$

    $R_1 \cap R_2 = \{(1,1)\}$

    $R_1 - R_2 = \{(2,2), (3,3)\}$

    $R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$

Relations (cont.)

- Definition 6:

Let R be a relation from a set A to a set B and S a relation from B to a set C.

The composite of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

Relations (cont.)

- Example: What is the composite of the relations R and S where R is the relation from {1,2,3} to {1,2,3,4} with R = {(1,1), (1,4), (2,3), (3,1), (3,4)} and S is the relation from {1,2,3,4} to {0,1,2} with S = {(1,0), (2,0), (3,1), (3,2), (4,1)}?

  Solution: $S \circ R$ is constructed using all ordered pairs in R and ordered pairs in S, where the second element of the ordered in R agrees with the first element of the ordered pair in S.
  For example, the ordered pairs (2,3) in R and (3,1) in S produce the ordered pair (2,1) in $S \circ R$. Computing all the ordered pairs in the composite, we find

  $$S \circ R = ((1,0), (1,1), (2,1), (2,2), (3,0), (3,1)\}$$

N-ary Relations & their Applications

- Relationship among elements of more than 2 sets often arise: n-ary relations

- Like relationship between (Airline, flight number, starting point, destination, departure time, arrival time)

N-ary Relations & their Applications (cont.)

- N-ary relations
  - Definition 1:

Let $A_1, A_2, \ldots, A_n$ be sets. An n-ary relation on these sets is a subset of $A_1 * A_2 * \ldots * A_n$ where A_i are the domains of the relation, and n is called its degree.

- Example: Let R be the relation on N * N * N consisting of triples (a, b, c) where a, b, and c are integers with $a<b<c$.

  Then $(1,2,3) \in R$, but $(2,4,3) \notin R$.

  The degree of this relation is 3. Its domains are equal to the set of integers.

N-ary Relations & their Applications (cont.)

- Databases & Relations

  - Relational database model has been developed for information processing

  - A database consists of records, which are n-tuples made up of fields

  - The fields contains information such as:
    - Name
    - Student #
    - Major
    - Grade point average of the student

N-ary Relations & their Applications (cont.)

- The relational database model represents a database of records or n-ary relation

- The relation is R(Student-Name, Id-number, Major, GPA)

N-ary Relations & their Applications (cont.)

- Example of records

(Smith, 3214, Mathematics, 3.9)
(Stevens, 1412, Computer Science, 4.0)
(Rao, 6633, Physics, 3.5)
(Adams, 1320, Biology, 3.0)
(Lee, 1030, Computer Science, 3.7)

N-ary Relations & their Applications (cont.)

TABLE A: Students

Students Names	ID #	Major	GPA
Smith	3214	Mathematics	3.9
Stevens	1412	Computer Science	4.0
Rao	6633	Physics	3.5
Adams	1320	Biology	3.0
Lee	1030	Computer Science	3.7

N-ary Relations & their Applications (cont.)

- Operations on n-ary relations

  - There are varieties of operations that are applied on n-ary relations in order to create new relations that answer eventual queries of a database

  - Definition 2:

Let R be an n-ary relation and C a condition that elements in R may satisfy. Then the selection operator s_C maps n-ary relation R to the n-ary relation of all n-tuples from R that satisfy the condition C.

N-ary Relations & their Applications (cont.)

- Example:

  if s_C = "Major = "computer science" $\wedge$ GPA > 3.5" then the result of this selection consists of the 2 four-tuples:

  (Stevens, 1412, Computer Science, 4.0)

  (Lee, 1030, Computer Science, 3.7)

N-ary Relations & their Applications (cont.)

- Definition 3:

The projection $P_{i_1, i_2, \ldots, i_m}$ maps the n-tuple $(a_1, a_2, \ldots, a_n)$ to the m-tuple $(a_{i_1}, a_{i_2}, \ldots, a_{i_m})$ where $m \leq n$.

In other words, the projection $P_{i_1, i_2, \ldots, i_m}$ deletes $n - m$ of the components of n-tuple, leaving the i_1th, i_2th, …, and i_mth components.

N-ary Relations & their Applications (cont.)

- **Example:** What relation results when the projection $P_{1,4}$ is applied to the relation in Table A?

Solution: When the projection $P_{1,4}$ is used, the second and third columns of the table are deleted, and pairs representing student names and GPA are obtained. Table B displays the results of this projection.

TABLE B: GPAs

Students Names	GPA
Smith	3.9
Stevens	4.0
Rao	3.5
Adams	3.0
Lee	3.7

N-ary Relations & their Applications (cont.)

- Definition 4:

Let R be a relation of degree m and S a relation of degree n. The join $J_p(R,S)$, where $p \leq m$ and $p \leq n$, is a relation of degree $m + n - p$ that consists of all $(m + n - p)$-tuples $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$, where the m-tuple $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p)$ belongs to R and the n-tuple $(c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$ belongs to S.

N-ary Relations & their Applications (cont.)

- Example: What relation results when the operator J_2 is used to combine the relation displayed in tables C and D?

TABLE C: Teaching Assignments

Professor	Dpt	Course #
Cruz	Zoology	335
Cruz	Zoology	412
Farber	Psychology	501
Farber	Psychology	617
Grammer	Physics	544
Grammer	Physics	551
Rosen	Computer Science	518
Rosen	Mathematics	575

TABLE D: Class Schedule

Dpt	Course #	Room	Time
Computer Science	518	N521	2:00 PM
Mathematics	575	N502	3:00 PM
Mathematics	611	N521	4:00 PM
Physics	544	B505	4:00 PM
Psychology	501	A100	3:00 PM
Psychology	617	A110	11:00 AM
Zoology	335	A100	9:00 AM
Zoology	412	A100	8:00 AM

N-ary Relations & their Applications (cont.)

Solution: The join J_2 produces the relation shown in Table E

Table E: Teaching Schedule

Professor	Dpt	Course #	Room	Time
Cruz	Zoology	335	A100	9:00 AM
Cruz	Zoology	412	A100	8:00 AM
Farber	Psychology	501	A100	3:00 PM
Farber	Psychology	617	A110	11:00 AM
Grammer	Physics	544	B505	4:00 PM
Rosen	Computer Science	518	N521	2:00 PM
Rosen	Mathematics	575	N502	3:00 PM

Representing Relations

- Relations can be represented through matrices

$$m_{ij} = \begin{cases} 1 & if\ (a_i, b_j) \in R \\ 0 & otherwise \end{cases}$$

  - Example: Suppose that the relation R on a set is represented by the matrix:

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

Representing Relations

- **Example:** Suppose that the relation R on a set is represented by the matrix:

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution:

- Since all the diagonal elements of this matrix are equal to 1, then **R is reflexive**.

- Since the two side of diagonal reflect each other then R is Symmetric

- Since M_R is symmetric, **R is not antisymmetric**.

Representing Relations

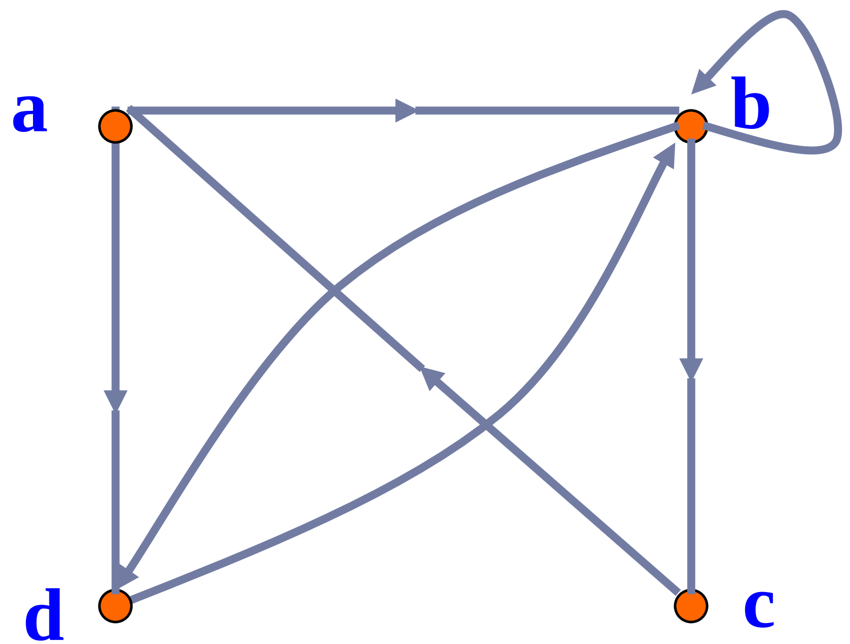
- Relations can also represented using diagraphs
  - Definition 1:

A directed graph, or diagraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs).

The vertex a is called the initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.

Representing Relations

- **Example:** The directed graph with vertices a, b, c and d , and edges (a,b), (a,d), (b,b), (b,d), (c,a) and (d,b). The edge (b,b) is called a loop.



Equivalence Relations

- Equivalence classes

  - Definition 1:

Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the equivalence class of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we will delete the subscript R and write [a] for this equivalence class.

Equivalence Relations

- **Example:** What are the equivalences classes of 0 and 1 for congruence modulo 4?

  Solution:
  The equivalence class of 0 contains all the integers a such that a ≡ 0 (mod 4). Hence, the equivalence class of 0 for this relation is

  $$[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$$

- The equivalence class of 1 contains all the integers a such that a ≡ 1 (mod 4). The integers in this class are those that have a remainder of 1 when divided by 4. Hence, the equivalence class of 1 for this relation is

  $$[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$$

Equivalence Relations

- Equivalence classes & partitions

  - Theorem 1:

Let R be an equivalence relation on a set A. These statements are equivalent:

i. $a \, R \, b$
ii. $[a] = [b]$
iii. $[a] \cap [b] \neq \varnothing$

Equivalence Relations

- Theorem 2:

Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Equivalence Relations

- Example: List the ordered pairs in the equivalence relation R produced by the partition $A_1 = [1,2,3\}$, $A_2 = \{4,5\}$ and $A_3 = \{6\}$ of $S = \{1,2,3,4,5,6\}$

  Solution: The subsets in the partition are the equivalences classes of R. The pair $(a,b) \in R$ if and only if a and b are in the same subset of the partition.

  The pairs (1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2) and $(3,3) \in R \Leftarrow A_1 = [1,2,3\}$ is an equivalence class. The pairs (4,4), (4,5), (5,4) and $(5,5) \in R \Leftarrow A_2 = \{4,5\}$ is an equivalence class. The pair $(6,6) \in R \Leftarrow \{6\}$ is an equivalence class.

  No pairs other than those listed belongs to R.

Examples

- Let A = {1, 2, 3, 4, 5, 6}, construct matrix representation of the relation R on A for the following as:

$(a)\;\; R = \{(j, k) \mid j \text{ is a multiple of } k\}$

$(b)\;\; R = \{(j, k) \mid (j - k)^2 \in A\}$

$(c)\;\; R = \{(j, k) \mid (j \text{ divides } k\}$

$(d)\;\; R = \{(j, k) \mid j\,k \text{ is a prime}$

Examples

- Let A = {1, 2, 3} determine whether the relation R whose matrix M_R is given is an equivalence relation:

$$(a) \quad M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

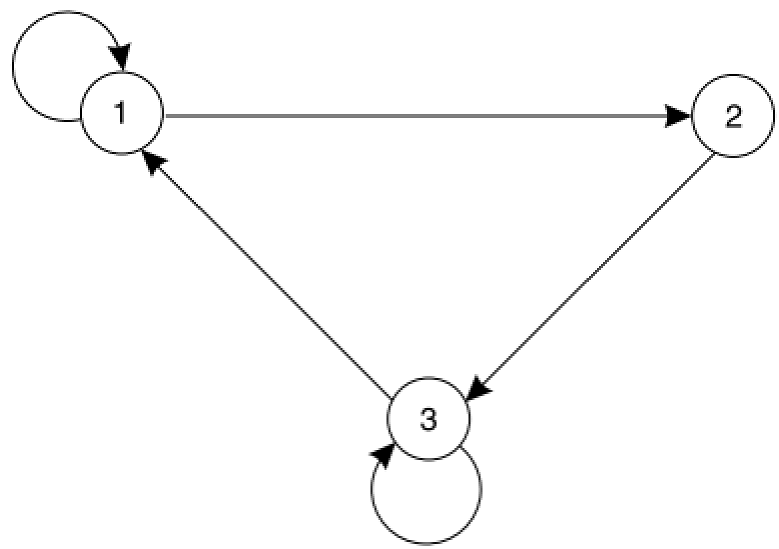
$$\text{(c)} \quad M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

$$(b) \quad M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\text{(d)} \quad M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Examples

- Determine whether the relation whose digraph is given below is an equivalence relation.



- Let A = {1, 2, 3} and R = {(1, 1), (1, 4), (4, 1), (4, 4), (2, 2), (2, 3), (3, 2), (3, 3)}. Write the Matrix of R and sketch its graph. Determine whether R is equivalence.