UNIVERSITY OF DAR ES SALAAM

COLLEGE OF ICT

IS 143

Discrete Structure

Lecture 8

Instructor

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Content

- Introduction to Course
- Proposition, Sets, Relations and Functions
- Algorithm and Basic Logics
- Proof Techniques
- Basics Of Counting (Mathematical Reasoning)
- Graphs And Trees
- Discrete Probability

Mathematical Reasoning - Probability

- Probability theory plays an important role in Mathematical reasoning in deriving algorithm of nondeterministic events

- In computer science, it is used in complexity theory (average complexity of algorithms, expert systems for medical diagnosis, etc)

- Unlike in deterministic algorithms, in probabilistic algorithms, the output of a program may change given the same input (random choices are taken!)

Introduction to Discrete Probability

- Introduction

  - In the eighteenth century, Laplace defined the probability of an event as the number of successful outcomes divided by the number of possible outcomes

  - The probability of obtaining an odd number when rolling a dice is equal to $3/6 = ½$ (assume die is fair!)

Introduction to Discrete Probability

- Finite probability
  - Experiment is a procedure that yields one of the given set of possible outcomes
  - Sample space of the experiment is the set of possible outcomes
  - An event is a subset of the sample space

Introduction to Discrete Probability

- Definition 1:

the probability of an event E, which is a subset of a finite sample space S of equally likely outcomes, is

$$p(E) = \frac{|E|}{|S|}.$$

Introduction to Discrete Probability

- **Example:**

  A bowl contains 4 blue balls and 5 red balls. What is the probability that a ball chosen from the bowl is blue?

Solution:

to calculate the probability, note that there are 9 possible outcomes and 4 of these possible outcomes produce a blue ball. Hence, the probability that a blue ball is chosen is 4/9.

Introduction to Discrete Probability

- Example: Find the probability that a hand of 5 cards in poker contains 4 cards of one kind.

  Solution: By the product rule, the number of hands of 5 cards with 4 cards of one kind is the product of the number of ways to pick one kind, the number of ways to pick the 4 of this kind out of the 4 in the deck of this kind, and the number of ways to pick the 5th card. This is C(13, 1) C(4, 4) C(48, 1).
  Since there is a total of C(52, 5) different hands of 5 cards, the probability that a hand contains 4 cards of one kind is

$$\frac{C(13,1)C(4,4)C(48,1)}{C(52,5)} = \frac{13*1*48}{2{,}598{,}960} \approx 0.00024.$$

Introduction to Discrete Probability

- Probability of combinations of events
  - Theorem 1:

Let E be an event in a sample space S. The probability of the event $\overline{E}$, the complementary event of E, is given by

$$p(\overline{E}) = 1 - p(E).$$

Proof: To find the probability of the event $\overline{E}$, note that

$$|\overline{E}| = |S| - |E|.$$

Hence,

$$p(\overline{E}) = \frac{|S| - |E|}{|S|} = 1 - \frac{|E|}{|S|} = 1 - p(E)$$

Introduction to Discrete Probability

- Example: A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?

  Solution: Let E be the event that at least one of the 10 bits is 0. Then $\overline{E}$ is the event that all the bits are 1s. Since the sample space S is the set of all bit strings of length 10. It follows that

$$p(E) = 1 - p(\overline{E}) = 1 - \frac{|\overline{E}|}{|S|} = 1 - \frac{1}{2^{10}} = 1 - \frac{1}{1024} = \frac{1023}{1024}.$$

Introduction to Discrete Probability

- Theorem 2:

Let E_1 and E_2 be events in the sample space S. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2).$$

Proof: Using the formula for the number of elements in the union of two sets, it follows that

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|.$$

Hence,

$$p(E_1 \cup E_2) = \frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|}$$

$$= \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|} = p(E_1) + p(E_2) - p(E_1 \cap E_2).$$

Introduction to Discrete Probability

- Example: What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

  Solution:

  E_1 = event that integer selected is divisible by 2
  E_2 = event that integer is divisible by 5
  $E_1 \cup E_2$ = event that integer divisible by either 2 or 5
  $E_1 \cap E_2$ = event that integer divisible by both 2 & 5, or equivalently, that is divisible by 10
  Since $|E_1| = 50$, $|E_2| = 20$ and $|E_1 \cap E_2| = 10$,

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

$$= \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = \frac{3}{5}.$$

Probability Theory

- Assigning probabilities
  - Let S be the sample space of an experiment with a finite or countable number of outcomes.
    p(s) is the probability assigned to each outcome s

a) $0 \le p(s) \le 1 \quad \forall s \in S$
and

b) $$\sum_{s \in S} p(s) = 1$$

There are n possible outcomes, $x_1, x_2, \ldots, x_n$, the two conditions to be checked are:

a) $0 \le p(x_i) \le 1 \quad \forall i = 1, 2, \ldots, n$
and

b) $$\sum_{i=1}^{i=n} p(x_i) = 1$$

The function p from the set of all events of the sample S is called probability distribution

Probability Theory (cont.)

- **Example:** What probabilities should we assign to the outcomes H(heads) and T(tails) when a fair coin is flipped? What probabilities should be assigned to these events when the coin is biased so that heads comes up twice as often as tails?

Solution:

- unbiased coin: p(H) = p(T) = ½
- biased coin since:

$$\begin{cases} p(H) = 2p(T) \\ p(H) + p(T) = 1 \end{cases} \Rightarrow p(T) = \frac{1}{3} \quad and \quad p(H) = \frac{2}{3}.$$

Probability Theory (cont.)

- Definition 2:

The probability of the event E is the sum of the probabilities of the outcome in E. That is,

$$p(E) = \sum_{s \in E} p(s)$$

Probability Theory (cont.)

- <u>Example</u>:
  Suppose that a die is biased so that 3 appears twice as often as each other number but that the other 5 outcomes are equally likely. What is the probability that an odd number appears when we roll this die?

  <u>Solution</u>: We want to find the probability of the event

  $$E = \{1, 3, 5\}$$

  $p(1) = p(2) = p(4) = p(5) = p(6) = 1/7$
  $p(3) = 2/7$

  It follows that:
  $p(E) = p(1) + p(3) + p(5) = 1/7 + 2/7 + 1/7 = 4/7$

Probability Theory (cont.)

- Conditional probability
  - Definition 3:

Let E and F be events with $p(F) > 0$. The **conditional probability** of E given F, denoted $p(E|F)$, is defined as

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}.$$

Probability Theory (cont.)

- Example: A bit string of length 4 is generated at random so that each of the 16 bit strings of length 4 is equally likely. What is the probability that it contains at least 2 consecutive 0s, given that its first bit is a 0? (We assume that 0 bits and 1 bits are equally likely)

Solution:

E = event that a bit string of length 4 contains at least 2 consecutive 0s.

F = event that the first bit of a bit string of length 4 is a 0.

The probability that a bit string of length 4 has at least 2 consecutive 0s, given that its first bit is equal 0, equals

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}.$$

Probability Theory (cont.)

Solution (cont.):

Since $E \cap F = \{0000, 0001, 0010, 0011, 0100\}$, we see that $p(E \cap F) = 5/16$. Since there are 8 bit strings of length 4 that start with a 0, we have
$p(F) = 8/16 = ½$.

Consequently,

$$p(E \mid F) = \frac{5/16}{1/2} = \frac{5}{8}.$$

Probability Theory (cont.)

- Random variables

  - They are numerical values associated with the outcome of an experiment

  - Definition 5:

A **random variable** is a function from the sample space of an experiment to the set of real numbers. That is, a random variable assigns a real number to each possible outcome.

Remark: Note that a random variable is a function. It is not a variable, and it is not random!

Probability Theory (cont.)

- Example: Suppose that a coin is flipped 3 times. Let $X(t)$ be the random variable that equals the number of heads that appear when t is the outcome. Then $X(t)$ takes the following values:

$$X(HHH) = 3,$$

$$X(HHT) = X(HTH) = X(THH) = 2,$$

$$X(TTH) = X(THT) = X(HTT) = 1,$$

$$X(TTT) = 0.$$

Expected Value & Variance

- Introduction

  - **The expected value** is very useful in computing the average-case complexity of algorithms

  - Another useful measure of a random variable is its **variance**

  - The variance tells us how spread out the values of this random variable are

Expected Value & Variance (cont.)

- Expected values

  - Definition 1:

The **expected value (or expectation)** of the random variable X(s) on the sample space S is equal to

$$E(X) = \sum_{s \in S} p(s) X(s).$$

Expected Value & Variance (cont.)

- Example: Expected value of a die
  Let X be the number that comes up when a die is rolled. What is the expected value of X?

  Solution: The random variable X takes the values 1, 2, 3, 4, 5, or 6, each with probability 1/6. It follows that

$$E(X)=\frac{1}{6}*1+\frac{1}{6}*2+\frac{1}{6}*3+\frac{1}{6}*4+\frac{1}{6}*5+\frac{1}{6}*6=\frac{21}{6}=\frac{7}{2}.$$

Expected Value & Variance (cont.)

- Theorem 1:

If X is a random variable and p(X =r) is the probability that X = r, so that

$$\boldsymbol{p(X = r) = \sum_{s \in S, X(s)=r} p(s)}$$

then

$$E(X) = \sum_{r \in X(S)} p(X = r) * r.$$

Expected Value & Variance (cont.)

- Example: What is the expected value of the sum of the numbers that appear when a pair of fair dice is rolled?

  Solution:
  Let X be the random variable equal to the sum of the numbers that appear when a pair of dice is rolled.
  We have 36 outcomes for this experiment.
  The range of X is {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}.

Expected Value & Variance (cont.)

Solution (cont.):

p(X = 2) = p(X = 12) = 1/36,

p(X = 3) = p(X = 11) = 2/36 = 1/18,

p(X = 4) = p(X = 10) = 3/36 = 1/12,

p(X = 5) = p(X = 9) = 4/36 = 1/9,

p(X = 6) = p(X = 8) = 5/36,

p(X = 7) = 6/36 = 1/6.

$$\Rightarrow E(X) = 2*\frac{1}{36} + 3*\frac{1}{18} * 4*\frac{1}{12} + 5*\frac{1}{9} + 6*\frac{5}{36} + 7*\frac{1}{6}$$

$$+ 8*\frac{5}{36} + 9*\frac{1}{9} + 10*\frac{1}{12} + 11*\frac{1}{18} + 12*\frac{1}{36}$$

$$= 7.$$

Expected Value & Variance (cont.)

- Linearity of expectations
  - Theorem 3:

If X_i, $i = 1,2, \ldots, n$ with n a positive integer, are random variables on S, and if a and b are real numbers, then

$$E(X_1 + X_2 + \ldots + X_n)$$
$$= E(X_1) + E(X_2) + \ldots + E(X_n)$$

i.e. $E(aX + b) = aE(X) + b.$

Expected Value & Variance (cont.)

- Average-case computational complexity
  - Computing the average-case computational complexity of an algorithm can be interpreted as computing the expected value of a random variable
  - **Principle:** Let the sample space of an experiment be the set of possible inputs a_j,
    ($j = 1, 2, .., n$), and let $X(a_j)$ computes the **number of operations** used by the algorithm when a_j is an input.

Expected Value & Variance (cont.)

- Then, the average-case complexity of the algorithm is:

$$E(X) = \sum_{j=1}^{n} p(X(a_j)) * (X(a_j)) = expected\ \ value\ \ of\ \ X.$$

- The average-case computational complexity of an algorithm is usually more difficult to determine than its worst-case computational complexity

- Example: Exercise to think about, compute Average-case complexity of the linear search algorithm.

Expected Value & Variance (cont.)

- Variance
  - The expected value of a random variable provides the average value, but the variance tells us how widely its values are distributed
  - Definition 4:

Let X be a random variable on a sample space S. The **variance** of X, denoted by V(X), is

$$V(X) = \sum_{s \in S} (X(s) - E((X))^2 p(s).$$

The **standard deviation** of X, denoted σ(X), is defined to be

$$\sqrt{V(X)}.$$

Expected Value & Variance (cont.)

- Theorem 6:

> If X is a random variable on a sample S, then
> $V(X) = E(X^2) - E(X)^2$.

Proof: Note that

$$
\begin{aligned}
V(X) &= \sum_{s \in S} (X(s) - E(X))^2 p(s) \\
&= \sum_{s \in S} X(s)^2 p(s) - 2E(X) \sum_{s \in S} X(s) p(s) + E(X)^2 \sum_{s \in S} p(s) \\
&= E(X^2) - 2E(X)E(X) + E(X)^2 \\
&= E(X^2) - E(X)^2
\end{aligned}
$$

Expected Value & Variance (cont.)

- Example: Variance of the value of a die

What is the variance of the random variable X, where X is the number that comes up when a die is rolled?

Solution:

$$V(X) = E(X^2) – E(X)^2.$$

We know that $E(X) = 7/2$. To find $E(X^2)$ note that X^2 takes the values $i^2 = 1, 2, \ldots, 6$, each with probability $1/6$.

$$\Rightarrow E(X^2) = \frac{1}{6}\left(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2\right) = \frac{91}{6}.$$

Conclusion:

$$V(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}.$$

Assignment 4

1) Give a big-O estimate for $(x^2 + x(\log x)^3) \cdot \left(2^x + x^3\right)$

2) Give a big-O estimate for the number additions used in this segment of an algorithm.

$$
\begin{aligned}
&t := 0\\
&\textbf{for } i := 1 \textbf{ to } n\\
&\quad \textbf{for } j := 1 \textbf{ to } n\\
&\qquad t := t + i + j
\end{aligned}
$$

3) The final exam of IS143 consists of 50 true/false questions, each worth two points, and 25 multiple-choice questions, each worth four points. The probability that Asha answers a true/false question correctly is 0.9, and the probability that she answers a multiple-choice question correctly is 0.8. What is her expected score on the final?

Assignment 4

4) Given a real number x, what is the average-case computational complexity of the linear search if a linear-search algorithm to locate x by successively comparison to each element in the list of n distinct real numbers. Assume the probability that x is in the list is p and it is equally likely that x is any of the n elements in the list?

Instruction:

- Submission deadline Wednesday, 19th June 2019, **1600 Hrs.**