Section A

1. Task 1: Linear Mathematical Model of the Pendulum System

1.1. Introduction

The purpose of this task is to develop a linear approximation model of a pendulum system as depicted in Figure 1. The model describes the relationship between the torque, T, and the angle, θ,between the rod and the vertical plane. Following assumptions were used to derive the linear approximation mathematical model:

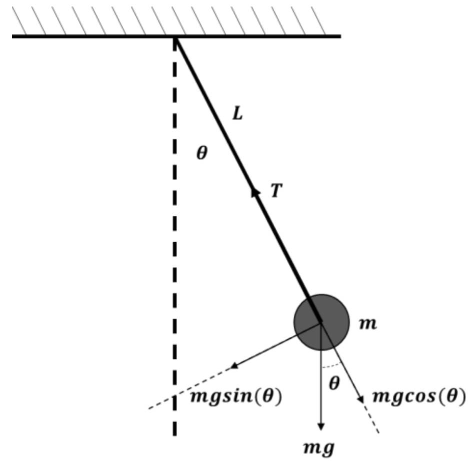

Figure 1 - Simple Pendulum System FBD

- Pendulum system adheres to the law of conservation of energy.
- Friction forces, including drag or mechanical friction in the joint, are negligible, representing an ideal pendulum oscillating freely around the pivot point.
- There are no external torques acting on the system.
- The rod length (L) is a made of a rigid material and maintains tension without bending, with the mass concentrated at the end of the rod, hence, neglected. Total mass of pendulum system = m.

1.2. Method

Equations of Motion & Taylor Series Approximation to Derive Linear Model

Newton's equation of motion for rotating objects is expressed as Equation 1 [1]:

$$\Sigma\tau = \frac{d\boldsymbol{H}}{dt} = J \cdot \ddot{\theta} \tag{1}$$

Where $\boldsymbol{H}$ is the angular momentum, τ is the torque applied to the mass, J is the moment of inertia of the system and $\ddot{\theta}$ is the angular acceleration. The torque generated by the pendulum is due to its weight, expressed by the cross product of the position vector and the force ($\tau = \vec{r} \times \vec{f}$), given by Equation 2:

$$|\tau| = |r|\,|f|\sin\theta = m \cdot g \cdot L \cdot \sin\theta \tag{2}$$

For a system where the mass is concentrated at a single point, the moment of inertia, $J = m \cdot L^2$. As the derivative of the angular momentum and the torque are both vector quantities equal to each other, their magnitudes are also equal. Therefore, the equation of motion of the pendulum is:

$$-m \cdot g \cdot L \cdot \sin\theta = m \cdot L^2\,\ddot{\theta} \tag{3}$$

Simplifying to obtain the nonlinear equation of motion for the pendulum system gives Equation 4:

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 \tag{4}$$

Equation 4 is linearized under the small angle approximation, where $\sin\theta \approx \theta$, giving Equation 5. [10]

$$\ddot{\theta} = -\frac{g}{L}\,\theta \tag{5}$$

In matrix form, the state space representation is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Where the state variables are $x_1 = \theta$ and $x_2 = \dot{\theta}$; the linearized equation can be re-expressed in terms of state variables as $\dot{x}_1 = x_2$ and $\dot{x}_2 = -\frac{g}{l} \cdot x_1$.

Comparison of Linear and Nonlinear Pendulum Models in Simulink

In Figure 2, the Simulink model contrasts linear and nonlinear pendulum behaviors at various angles with gravity set to 9.81 m/s^2, the pendulum length at 1 meter and at rest.

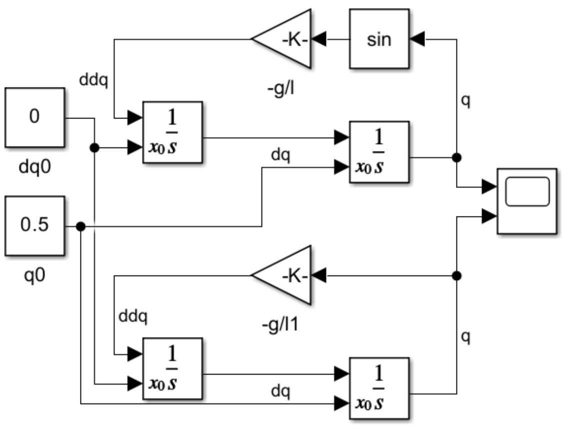

Figure 2 - MATLAB Simulink Diagram Comparing Linear & Non Linear Model

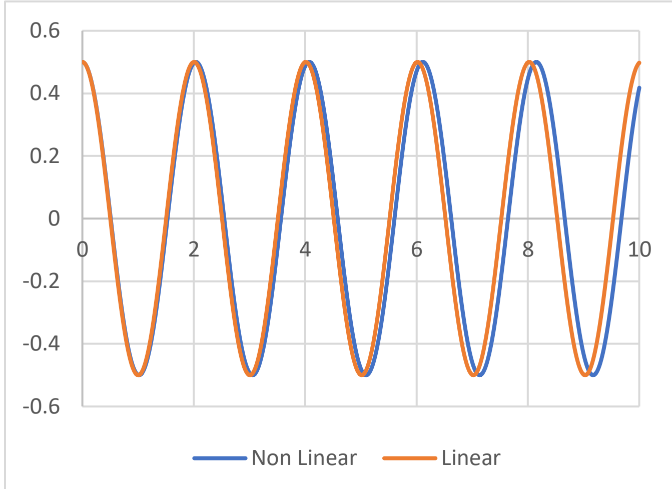

Figure 3 - Linear vs Non Linear Model Response at 0.5 Radians

Starting from rest, the linear model diverges from the nonlinear at 0.5 radians, with discrepancies in oscillation period and amplitude, indicating the linear model's limits beyond small angles. At smaller angles, such as 0.2 radians, these differences diminish, confirming the linear model's applicability within the small-angle approximation.

Considering External Torque

If an external torque $T_{external}$ is considered, then, according to Newton's rotational law, the *total torque equals the moment of inertia multiplies by the angular acceleration [2]*, which is the sum of the pendulum torque and the external torque, as shown in Equation 6.

$$T = J \cdot \ddot{\theta} \quad where \quad T = T_{pendulum} + T_{external} \tag{6}$$

Equation 8, rearranged from obtained equation 7, represents the total torque on the system.

$$-m \cdot g \cdot L \cdot \theta + T_{external} = m \cdot L^2 \cdot \ddot{\theta} \tag{7}$$

$$\ddot{\theta} = \frac{1}{m\,L^2}\,T_{external} - \frac{g}{L}\,\theta \tag{8}$$

In matrix form, the state space representation for a simple pendulum system with external torque considered is:

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m\,L^2} \end{bmatrix} T_{external}$$

2. Task 2: Simplified Car Suspension

2.1. Introduction

The system in consideration is a simplified car suspension, consisting of three masses M_1 (wheel), M_2 (chassis), and M_3 (car seat), with associated displacements y and z.

2.2. Assumptions

- All springs in the car suspension system obey Hooke's law [4], thus considered ideal.
- Dampers exhibit a linear damping characteristic.
- The inertial frame of reference is assumed to be constant (or in motion with constant velocity), therefore, Newton's 2nd law can be applied [3].
- The seat-back friction C_3 is connected of M_3 (seat) and moves as a function of the displacement of M_3 and M_2 (chassis); the movement of the seat dictates the displacement of C_3.
- Constant relationship between the set of coordinates (y_1, y_2, y_3, etc.) [3].
- The system is linear, with forces proportional to displacements and their derivatives.

2.3. Method

Newton's & Hooke's Laws to Determine ODEs

Utilizing Newton's 2nd law of motion ($\Sigma F = m\ddot{y}$, etc. [5]) and Hooke's law ($F_s = -kx, F_d = C\ddot{x}$), the equations of motion of the car suspension system were derived. Free body diagrams for masses $M_1, M_2, \& M_3$ in Figure 4 were drawn and ordinary differential equations 9, 10, and 11 were obtained.

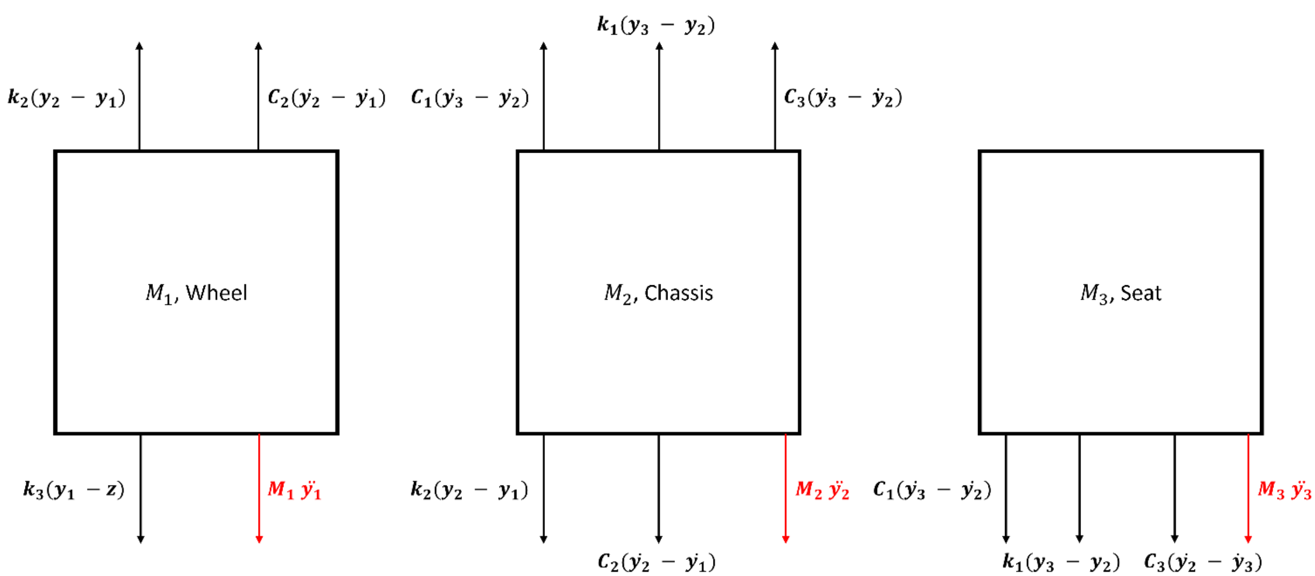

Figure 4 – Free Body Diagram of Car Suspension with Masses M1, M2, and M3; Resultant in red, System descending; positive Y upward.

$$-k_3(y_1 - z) + k_2(y_2 - y_1) + C_2(\dot{y}_2 - \dot{y}_1) = M_1\,\ddot{y}_1 \tag{9}$$

$$-k_2(y_2 - y_1) - C_2(\dot{y}_2 - \dot{y}_1) + k_1(y_3 - y_2) + C_1(\dot{y}_3 - \dot{y}_2) + C_3(\dot{y}_3 - \dot{y}_2) = M_2\,\ddot{y}_2 \tag{10}$$

$$-k_1(y_3 - y_2) - C_1(\dot{y}_3 - \dot{y}_2) - C_3(\dot{y}_3 - \dot{y}_2) = M_3\,\ddot{y}_3 \tag{11}$$

Where $y_1, y_2, \& y_3$ represent the vertical displacements, z is the displacement from the ground profile, $k_1, k_2, \& k_3$ are the spring constant, and $C_1, C_2, \& C_3$ are the damper coefficients.

LaPlace Transform

Equations 9, 10, and 11 were converted into the s-domain by performing LaPlace transform, assuming zero initial conditions, producing Equations 12, 13, and 14.

$$-k_3(Y_1 - Z) + k_2(Y_2 - Y_1) + C_2 s(Y_2 - Y_1) = M_1 s^2 Y_1 \tag{12}$$

$$-k_2(Y_2 - Y_1) - C_2 s(Y_2 - Y_1) + k_1(Y_3 - Y_2) + C_1 s(Y_3 - Y_2) + C_3 s(Y_3 - Y_2) = M_2 s^2 Y_2 \tag{13}$$

$$-k_1(Y_3 - Y_2) - C_1 s(Y_3 - Y_2) - C_3 s(Y_3 - Y_2) = M_3 s^2 Y_3 \tag{14}$$

Equations 15, 16, and 17 are obtained by solving Equations 12, 13, and 14 for Y_1, Y_2, Y_3.

$$Y_1 = \frac{k_3 Z + Y_2(k_2 + C_2 s)}{M_1 s^2 + C_2 s + k_3 + k_2} \tag{15}$$

$$Y_2 = \frac{Y_1(k_2 + C_2 s) + Y_3(k_1 + C_1 s + C_3 s)}{M_2 s^2 + s(C_2 + C_1 - C_3) + k_2 + k_1} \tag{16}$$

$$Y_3 = \frac{Y_2(k_1 + C_1 s + C_3 s)}{M_3 s^2 + s(C_1 + C_3) + k_1} \tag{17}$$

Transfer Functions

The LaPlace transform as transfer functions are shown by Equations 18 to 25.

$$Y_{1,\ Wheel}(s) = [GY_1(s) \times Y_2(s)] + [G_Z(s) \times Z(s)] \tag{18}$$

$$Y_{2,\ Chassis}(s) = [GY_{23}(s) \times Y_3(s)] + [GY_{21}(s) \times Y_1(s)] \tag{19}$$

$$Y_{3,\ Seat}(s) = GY_3(s) \times Y_2(s) \tag{20}$$

Where:

$$GY_1(s) = \frac{C_2 s + k_2}{M_1 s^2 + C_2 s + k_3 + k_2} \tag{21}$$

$$GY_Z(s) = \frac{k_3}{M_1 s^2 + C_2 s + k_3 + k_2} \tag{22}$$

$$GY_{21}(s) = \frac{C_2 s + k_2}{M_2 s^2 + s(C_2 + C_1 + C_3) + k_2 + k_1} \tag{23}$$

$$GY_{23}(s) = \frac{s(C_1 + C_3) + k_1}{M_2 s^2 + s(C_1 + C_2 + C_3) + k_2 + k_1} \tag{24}$$

$$GY_3(s) = \frac{s(C_1 + C_3) + k_1}{M_3 s^2 + s(C_1 + C_3) + k_1} \tag{25}$$

Matrix & State Space Representation

The system in matrix form with transfer functions is represented by Equation 26.

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 0 & GY_1(s) & 0 \\ GY_{21}(s) & 0 & GY_{23}(s) \\ 0 & GY_3(s) & 0 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} + \begin{bmatrix} G_Z(s) \\ 0 \\ 0 \end{bmatrix} Z(s) \qquad 26$$

To represent the system in state-space format, the state variables were defined as:

$$x_1 = y_1;\ x_2 = \dot{y_1};\ x_3 = y_2;\ x_4 = \dot{y_2};\ x_5 = y_3;\ x_6 = \dot{y_3}$$

$$\dot{x} = Ax + Bu; y = Cx + Du$$

$$Input = u = z$$

$$Output = \ Y_1,\ Y_2,\ Y_3$$

The state Equations 27, 28, 29, and 30 are obtained by expressing each second-order ODE as a set of two first-order ODEs.

$$\dot{x}_1 = x_2;\ \dot{x}_3 = x_4;\ \dot{x}_5 = x_6 \qquad 27$$

$$\dot{x_2} = \frac{1}{M_1}\left(k_2(x_3 - x_1) - k_3(x_1 - z) + C_2(x_4 - x_2)\right) \qquad 28$$

$$\dot{x_4} = \frac{1}{M_2}\left(k_1(x_5 - x_3) - k_2(x_3 - x_1) + C_1(x_6 - x_4) - C_2(x_4 - x_2) + C_3(x_6 - x_4)\right) \qquad 29$$

$$\dot{x_6} = \frac{1}{M_3}\left(-k_1(x_5 - x_3) - C_1(x_6 - x_4) - C_3(x_6 - x_4)\right) \qquad 30$$

In matrix form, the Equations 27 to 30 provide the state- space model represented by Equation 31.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_3 + k_2}{M_1} & -\frac{C_2}{M_1} & \frac{k_2}{M_1} & \frac{C_2}{M_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k_2}{M_2} & \frac{C_2}{M_2} & -\frac{k_1 + k_2}{M_2} & -\frac{C_1 + C_2 + C_3}{M_2} & \frac{k_1}{M_2} & \frac{C_1 + C_3}{M_2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{k_1}{M_3} & \frac{C_1 + C_3}{M_3} & -\frac{k_1}{M_3} & -\frac{C_1 + C_3}{M_3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k_3}{M_1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} z \qquad 31$$

Since initial conditions were assumed to be zero, the direct transmission matrix D is a zero matrix (which means there is no direct feedthrough from the input to the output), and the output matrix C will be an identity matrix because the outputs are directly the states of the displacements:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Section B

3. Task 3: Signal Flow Diagram of Simplified Car Suspension

A signal flow graph is a visual representation of mathematical relationships (algebraic equations) between variables in a linear system, comprising of nodes (representing variables) and directed branches (representing transfer functions); nodes are required to be sequentially arranged from input to output, where each node's value is the cumulative sum of its input signals. [6] [7]

Signal flow graphs are particularly useful analyzing the internal behavior of a system and modelling feedback control due to the clear direction of signals. [8]

Block diagrams are another graphical representation of a system similar to signal flow graphs, but provide more detail, require intuitive understanding with more focus on the transfer functions' behavior, and take up relatively more space and time. [9]

The contrast between the two representation methods is demonstrated in the subsequent figures. Figure 5, featuring the signal flow graph, succinctly displays the relationship among system variables, offering a streamlined view in comparison to the more complex and space-taking block diagram in Figure 6.

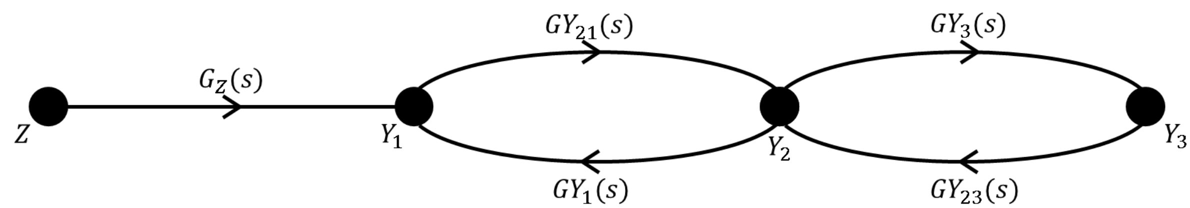

Figure 5 - Signal Flow Graph for Car Suspension System

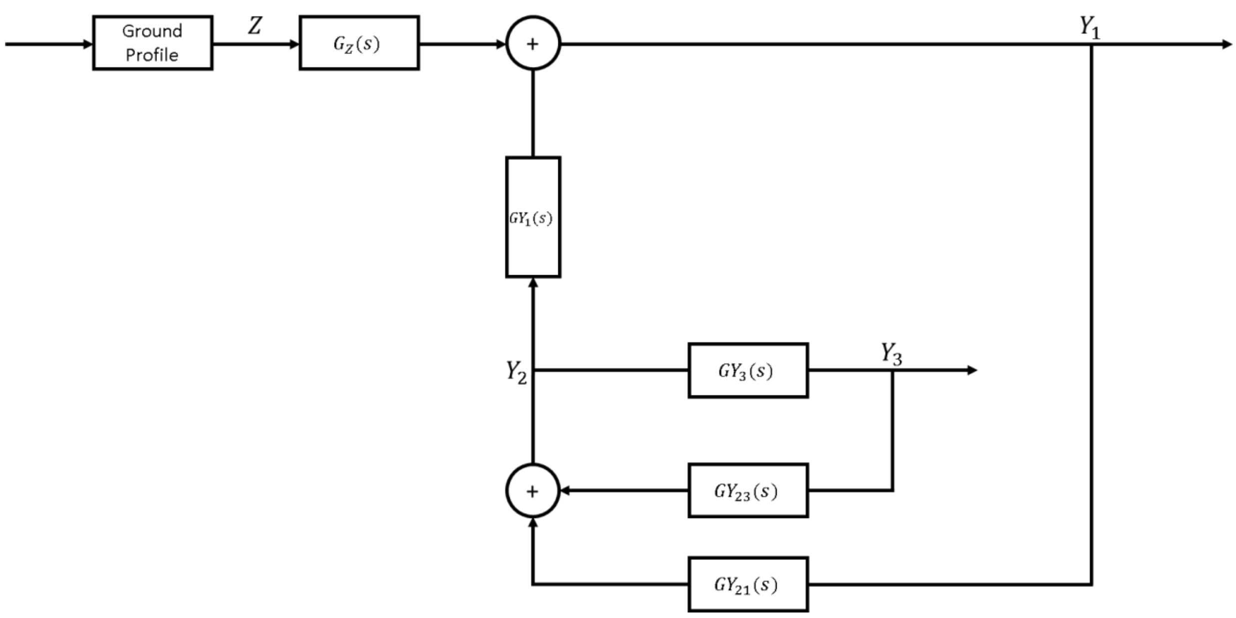

Figure 6 - Block Diagram for Car Suspension System

The block diagram and signal flow graph were derived from the transfer functions from Equations 18 to 25.

4. Task 4: Depth Control for a Torpedo

For the first task, the system was run with parameters $k_1 = 10, k_2 = 1, \& \ k_3 = 1$ and later $k_2 = 0.8$, with Figures 7 and 8 showing the Simulink model and comparing the response at $k_2 = 1$ and $k_2 = 0.8$.

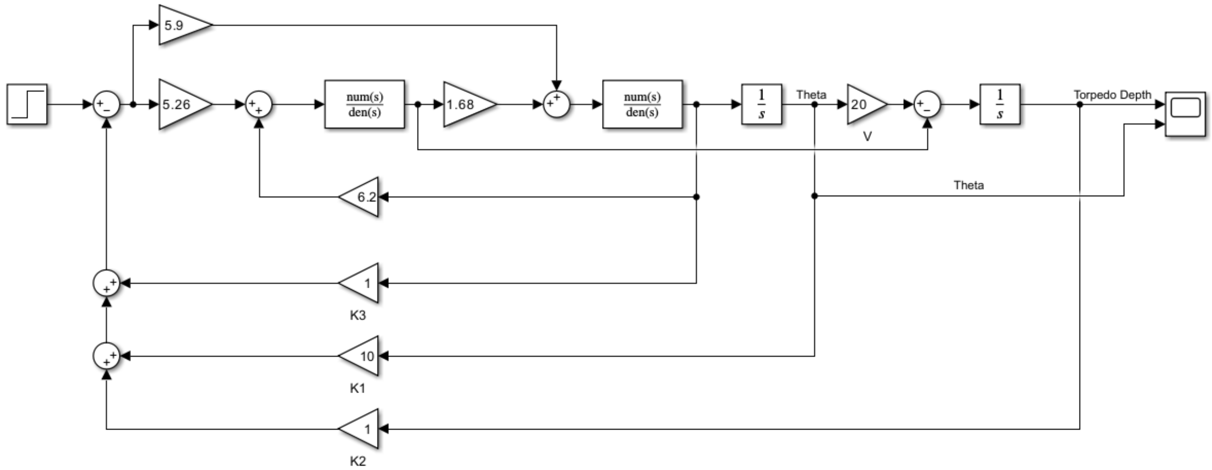

Figure 7 - MATLAB Simulink Closed Loop Linear Model of Torpedo Depth Control

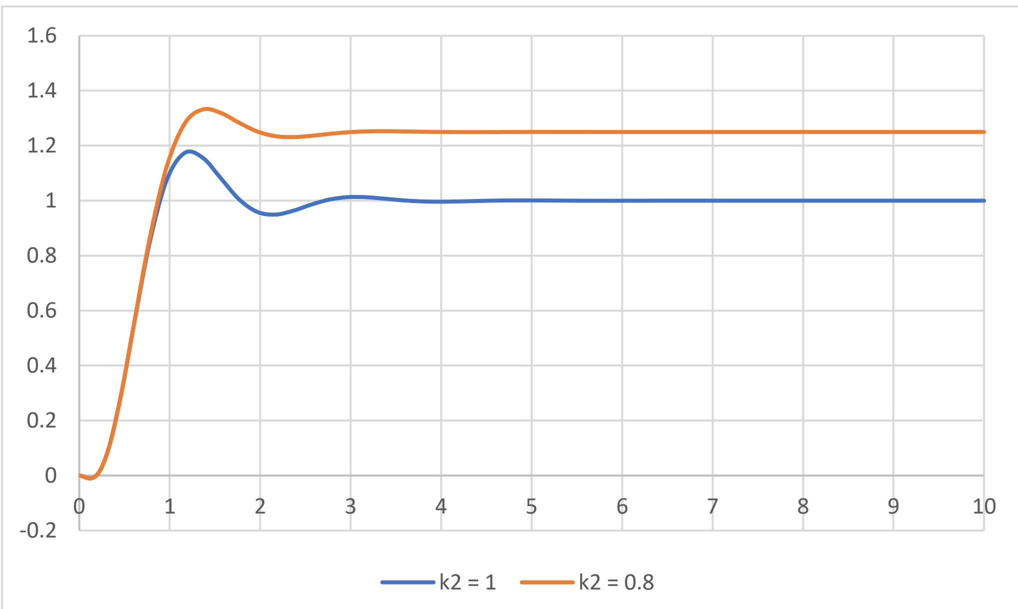

Figure 8 – Compared Response of System for k_2 = 1 and k_2 = 0.1 (v = 20 m/s^2)

Initially, the system had a 17.5% overshoot and a 3.8-second settling time. Lowering k_2 from 1 to 0.8 cut the overshoot to 5.5% and settling time to 3 seconds but incurred a 25% steady-state error with respect to the reference signal. The reduction in k_2 decreased overshoot and settling time without affecting rise time, demonstrating the gain's influence on transient response, crucial for tuning desired performance. (The system's theta output is the same as the nonlinear system – shown in Figure 12)

Effect of Speed of Torpedo

The speed of the torpedo was increased to 40 m/s^2 from 20 m/s^2 and its effect on the step response was investigated for task 2; Figure 9 shows the effect of the change in speed.

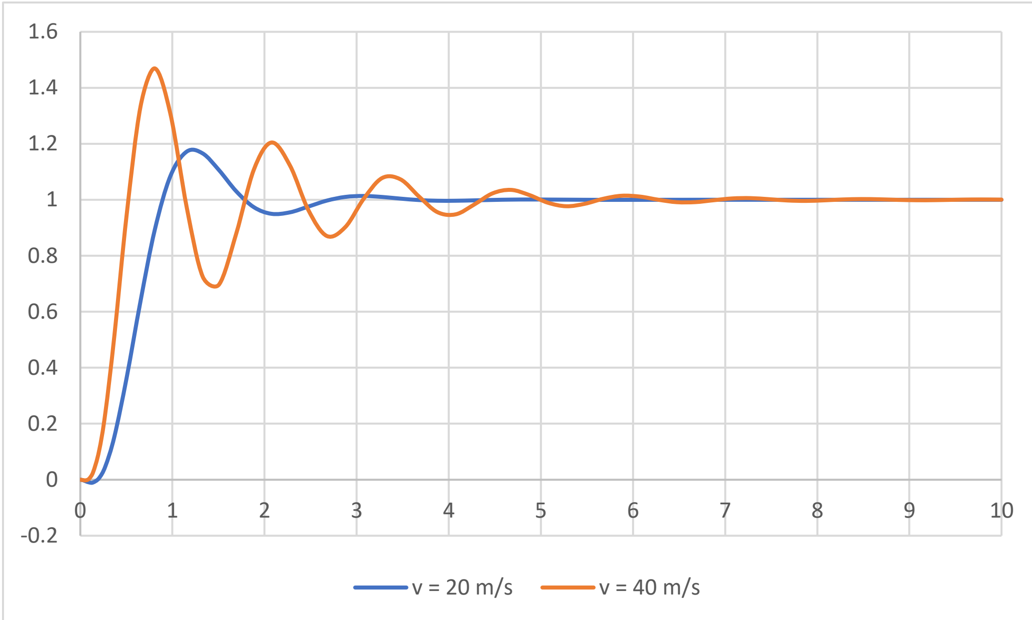

Figure 9 - System Step Response for v = 20 m/s^2 vs. v = 40 m/s^2

It was evident that the increase in speed of the torpedo significantly impacted the transient performance, including rise time, settling time, and overshoot. TABLE 1 compares the performance of the torpedo for the speeds.

Table 1 - Transient Response of Torpedo Depth Compared for v = 20 m/s^2 vs. v = 40 m/s^2

Transient Performance	$v = 20\ m/s$	$v = 40\ m/s$
Rise Time	0.495	0.287
Peak Response	17.2	46.8
Settling Time	3.5	7.4

The speed increase to 40 m/s^2 increased the peak response and settling time but decreased rise time. Adjusted $k_1 = 18$ and $k_3 = 1.5$ values mitigated the underdamped response; Figure 10 displays the improved step response and theta output.

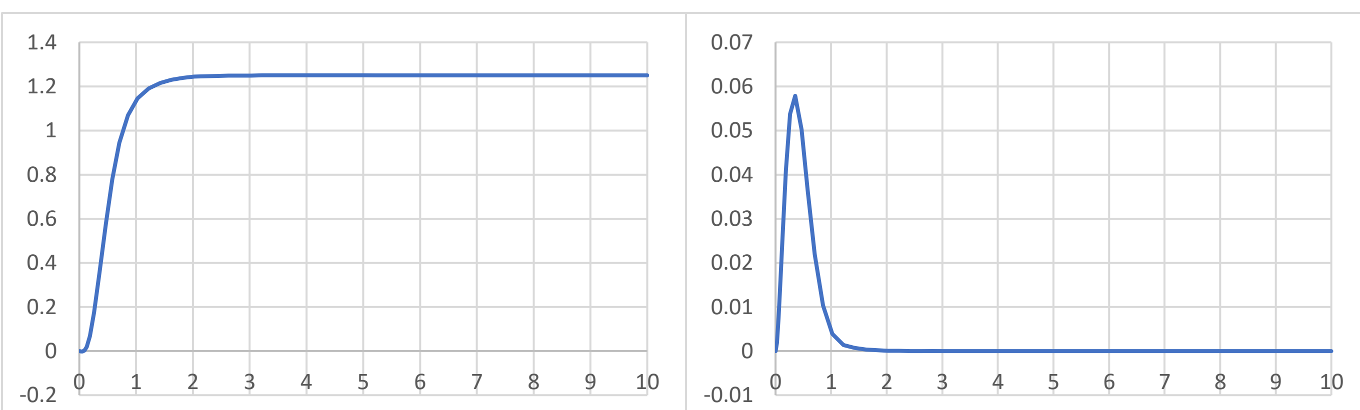

Figure 10 - Stable System Response: k_1 = 18 and k_3 = 1.5; constant $k_2 = 0.8$ yields 25% steady-state error, stable at 1.25 instead of 1 (shown in Figure 8)

The step response had no overshoot, nor any steady-state error and a settling time of 2 seconds. The system also did not show any signs of being overdamped, suggesting that the adjusted gain values effectively coped with the change in speed of the torpedo.

Nonlinear Response

The block diagram in Figure 11 shows the nonlinear system.

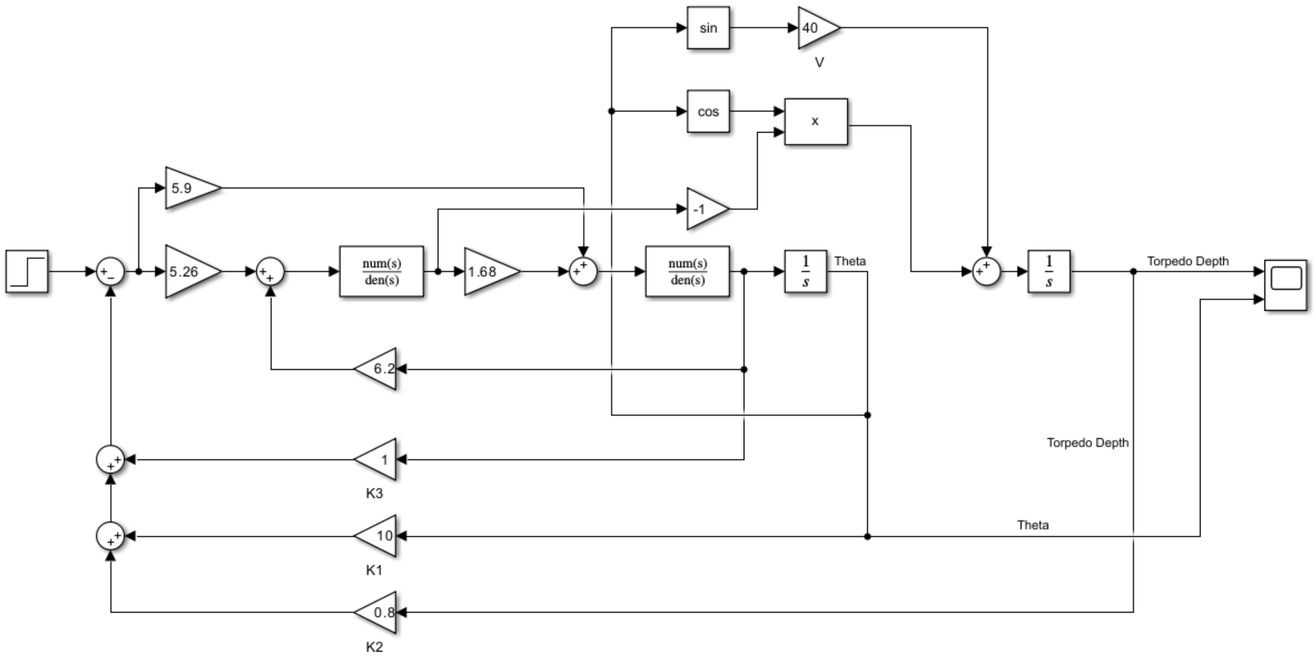

Figure 11 - Block Diagram of Non-Linearized Torpedo Depth Control, with k_1 = 10, k_2 = 0.8, k_3 = 1, and v = 20 m/s^2

The theta output and step response from the nonlinear system shown in Figure 12 is very similar to the linearized model with the same parameters (Figures 8 and 9) as theta is capped between $\pm 5°$. The difference between the linear and nonlinear system is miniscule (< 0.0001%) for small pitch angles and velocities. However, as the operating conditions move away from those nominal conditions, deviations between the linear and nonlinear model responses are emphasized, as shown in Figure 13 for a velocity of 150 m/s^2. In conclusion, the linear approximation used is effective, specifically for smaller pitch angles.

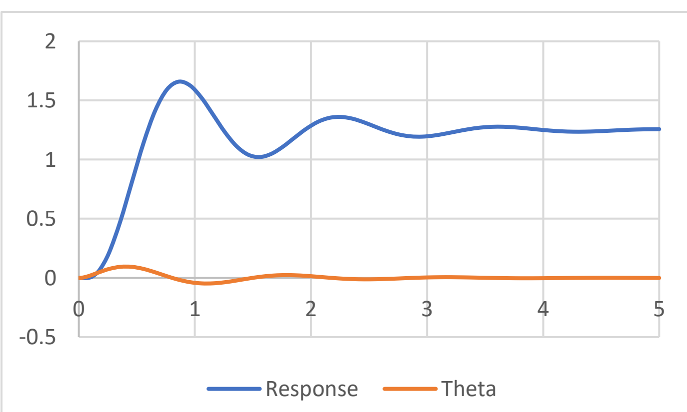

Figure 12 - Non-Linear System Response (Theta & Step Response) for k_2 = 0.8

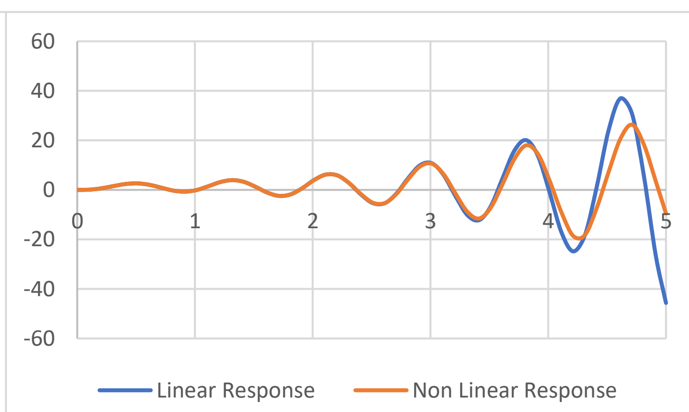

Figure 13 - Linear vs. Non-Linear System Response for v = 150 m/s^2

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