Mathematics in Trades and Life

Mathematics in Trades and Life

Mark Fitch
University of Alaska Anchorage

Megan Ossiander-Gobeille
University of Alaska Anchorage

15 November 2024

Edition: Fall 2024

Website: uaa.alaska.edu

Acknowledgements

This text began as class presentation notes from Megan Gobeille. Its development was greatly accelerated with the support of a sabbatical by the University of Alaska Anchorage. The following people have provided valuable input.

- Megan Ossiander-Gobeille
- Deb Crawford

For Students

This book is intended for use as lessons in a course that emphasizes building the skills to read and use mathematics (such as in a technical manual), and to recognize mathematical concepts in things you see and read in life.

This book is written to demonstrate effective thought processes, and provide the reader with practice learning to analyze examples and apply examples to similar, not solely identical, problems.

The general approach is to provide initial examples, present general concepts used in the examples, and then provide a chance to practice the concept or skill. There are two types of special examples. One demonstrates the entire thought process used to figure out what to do. These are much longer and ill suited to use as examples when working on an assignment. Another special example are those that illustrate using an example (examples on how to use an example). These are also longer as they show the thought process for connecting pieces of a question to a sufficiently similar problem.

By design there are exercises for which there is no step-by-step example in the section. This book is not written as a how to manual for specific applications. When you encounter these problems, as needed, refer to the examples illustrating thought processes. Mimic the thinking skills. You will go farther in life with general thinking skills than you will with memorizing specific processes.

The topics include

1. Interpret data in various formats and analyze mathematical models
2. Read and use mathematical models in a technical document
3. Communicate results in mathematical notation and in language appropriate to the technical field

You will learn to work with the following mathematical concepts.

1. Precision and accuracy
   (a) Rounding (skill)
   (b) Significant Figures (skill)
   (c) Determining appropriate rounding from context (critical thinking)
2. Proportions
   (a) Set up and solve proportions (skill)
   (b) Calculate Percentages (skill)
   (c) Understand and interpret percentages (critical thinking)
   (d) Unit conversion (skill)
3. Rates
   (a) Identify rates as linear, quadratic, exponential, or other (critical thinking)
   (b) Identify data varrying directly or indirectly (critical thinking)

4. Solving
   (a) Solve linear, rational, quadratic, and exponential equations and formulas (skill)
   (b) Solve a system of linear equations (skill)
5. Models
   (a) Read and interpret models (critical thinking)
   (b) Use models including linear, quadratic, exponential/logarithmic, and trigonometric (skill)
6. Geometric Reasoning
   (a) Identify shapes and apply their properties (skill)
   (b) Analyze right triangles (skill)
   (c) Analyze non-right triangles (skill)
   (d) Identify properties of sine and cosine functions (skill)

For Faculty

This book is designed to be used for classes supporting trade programs in a variety of fields and also to satisfy baccalaureate general education requirements.

Because of the focus on trade programs, choices of topics and the extent to which they are covered is more limited than in other texts. Also examples are designed to look like how a person in a trade would actually perform the calculations. This includes the assumption that all arithmetic will be performed by a device and hence all numbers will be in decimal notation.

The inclusion of general education outcomes motivates inclusion of the “life” portion of the book: these are applications that are not restricted to use in one or more trades. They enable any person to process simple phenomena they experience. The quantitative literacy sections also contribute to the general education goals. Many topics will conclude with activities designed to help readers see limitations of models and learn to process mathematical information with an appropriately critical eye.

This book is also designed to be used in conjunction with active learning pedagogies rather than as a reference text. That is, it has been designed for students to read the examples, work initial problems to help them identify questions, and to then seek help while working on exercises. It can be easily incorporated into flipped classrooms and asynchronous learning. It intentionally does not provide an example to mimic for each question in the exercises. Rather each section provides an simple example, a more complex example, and some explanation about what is important. Then exercises are provided for them to test their ability to recognize and use the mathematics from that section.

Another motivation of this text was that existing texts fall into the trap of providing examples to mimic without requiring the student to understand why it worked and hence are not enabled to process new questions. The student statment “I didn’t see an example for this problem” is emblamatic of this issue.

The check points (exercises in the reading) are self-grading (using MyOpenMath) with feedback so the reader can determine what if any questions they need to ask. Videos, where included, are presentations of the introduction of the concept in that section. Homework by default is live, online problems that provide feedback. The scores on these cannot be saved in an LMS however. If you wish to use MyOpenMath, the problems, and even a shell, can be provided. If desired the PDF version can be used which does not have live homework.

The projects are an integral part of the general education goals. These are intended to be assigned after relevant material is covered. They require students to use topics from that chapter to perform calculations and then interpret the results of their work. This is part of achieving general education outcomes. The projects also provide a opportunity for students to express calculations using standard mathematical notation and to communicate mathematical results in clear language.

For context, here is a brief history. The first version of this text was written to transition to an OER for MATH A104 Technical Mathematics at the Univesity of Alaska Anchorage. Commercially available texts emphasized memorizing problem types with limited critical thinking. As such they could not be used to satisfy the general education requirement. There was also a desire to reduce costs including for textbooks and online homework systems. As a result this book was created with matching homework in MyOpenMath.

Served disciplines at UAA included auto/diesel, heavy equipment mechanics, welding and non-destructive testing, aviation mechanics, piloting, air traffic controll, and medical certification programs.

The general education outcome around which this was designed was: Quantitative courses develop abilities to reason mathematically and analyze quantitative and qualitative data to reach sound conclusions for success in undergraduate study and professional life. The indicators were

- Interprets info in mathematical form (equations, graphs, diagrams, tables, words)
- Represents and/or converts relevant quant info and explain its assumptions and limits
- Applies mathematical forms (equations, graphs, diagrams, tables, words) to quantitative problems to reach sound conclusions
- Communicates quantitative results appropriate to the problem or context

Contents

Chapter 1

Cross Cutting Topics

1.1 Units

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Unit conversion (skill)
- Set up and solve proportions (skill)

This book is designed to present mathematics in various contexts including a variety of trades. As a result numbers will frequently be connected to units such as length (feet, meters), time (seconds) or others. These units are part of the arithmetic, and so we must learn what they mean and how to perform this arithmetic. Sometimes the units even suggest to us how to setup the arithmetic.

In this section we will introduce the units so we can understand them when we read technical material (Item 2) and use them correctly to communicate results to others (Item 3). We will also learn to convert units (Item 2.d) which involves our first use of proportions (Item 2.a).

We measure many things such as distance, time, and weight. We describe these measurements in terms of units like miles, hours, and pounds. But have you ever stopped to think about how these units are defined?

The story of some of these units is lost in history. For example dividing the day into 24 units began with ancient Egyptians. They did not record, that we know of, the reason for choosing 24 units as opposed to 30 or any other number.

Other units, such as the metric (also called the International System of Units or simply SI) are much more modern. Initially many units were based on something physical. For example, one calorie is the amount of heat it takes to raise the temperature of one gram of water 1° C. The meter was originally defined as one ten-millionth of the distance from the equator to the north pole. The problem with this type of measurement is that it is neither fixed (depends on where on the equator your begin) nor easy to measure.

Thus modern definitions were developed. The length of a meter was changed to mean the length of a bar of metal kept in special storage in France. The bar had been carefully constructed and was used to confirm other measurement devices were correctly calibrated. It was changed yet again to be based on wavelengths of radiation. These are uniform no matter where they are done, so they can be used by many people to construct simple measurement tools.

1.1.1 Types of Measurement

First we will look at units for different types of measurement in the two major systems. In a specific trade you will need to memorize the units you use most often. For this class you should ask your instructor which units must be memorized and which you may look up when working on problems.

Note that the U.S. Customary system (related to the British Imperial system) is non-uniform, so there are multiple names for some types of measurements. This is in contrast to the SI (metric) which has one name for each property and prefixes to indicate the size. Table 1.1.1 lists names of units for both systems. It is important to be able to recognize which unit (name) goes with which type of measurement (e.g., length, volume, ...).

Table 1.1.1 Units of Measure

Measuring	US Customary	Metric
Length	inch (in) foot (ft) yard (yd) mile (mi) nautical mile (nm)	meter (m)
Volume	fluid ounce (oz) cup (c) pint (pt) quart (qt) gallon (g)	liter (L or ℓ)
Weight	ounce (oz) pound (lb)	gram (g)
Temperature	degrees Fahrenheit (F)	degrees Celsius (C)
Pressure	inches of mercury (inHg) pounds per square inch (psi)	bar Pascal (Pa)
Time	second (s) minute (min) hour (hr)	

One name: two meanings Note that fluid ounces and weight ounces are not the same unit. Ten (10) fluid ounces of milk does not weigh ten (10) ounces. You must determine which ounce is referenced by the context. This can be tricky in recipes, which is a good reason to use SI units.

Note a gram is a unit of mass rather than than a unit of weight. Pound and ounce on the other hand are units of weight. Nevertheless gram is often used to describe weight because it is easy to switch between it and weight. Namely, mass can be obtained by dividing by the acceleration due to gravity (see a physics book for more information). The official unit for weight (a force) is a Newton, but we will not use that in this book.

1.1.2 U.S. Customary

Often we need to convert between units within the U.S. Customary system. This section provides the information needed for conversion and examples of performing them. It is an example of units suggesting how we setup the calculation.

Why would we need to convert units? This can occur because measurements were taken with different scales. For instance, we cannot add 3 inches to 2.2 feet without changing one to make the units match.

Why are there multiple units in the first place? There are different units for different scales (e.g., inches for small lengths and miles for long distances). This is a result of the U.S. Customary system being developed from the British Imperial system which was based on disparate measurements from multiple centuries ago (look up unit names in an etymological dictionary for fun). Converting between units therefore requires remembering special numbers for conversion. Most of these you likely know.

Table 1.1.2 Converting within U.S. Customary

Measuring	Unit 1	Unit 2
Length	1 nm	6076 ft
	1 mi	5280 ft
	1 yd	3 ft
	1 ft	12 in
Area	1 acre	43,560 ft^2
Volume	1 g	4 qts
	1 qt	2 pts
	1 pt	2 c
	1 c	8 oz
Weight	1 ton	2000 lbs
	1 lb	16 oz
Time	1 year	365 days
	1 day	24 hrs
	1 hr	60 mins
	1 min	60 secs

Of course a year is not always the same number of days. In each circumstance it is important to determine whether we can use the common approximation of 365 days without injury or loss.

Review each of these examples to see how to convert from one U.S. Customary unit to another.

Example 1.1.3 How many quarts is 2.3 gallons?

From Table 1.1.2 we know (or can look up) that each gallon is 4 quarts. This means we have $\frac{4 \text{ quarts}}{1 \text{ gallon}}$. This suggests that we can multiply 2.3 by this ratio, because the gallons will divide out.

$$2.3 \text{ gallons} \cdot \frac{4 \text{ quarts}}{\text{gallon}} = 9.2 \text{ quarts}$$

□

Sometimes to convert units we need to convert more than once.

Example 1.1.4 How many cups is 1.5 quarts?

While we do not have an entry for cups per quart in the conversion table, we do have entries for quarts to pints and pints to cups. This suggests we can put these two conversions together to convert quarts to cups.

$1.5 \text{ quarts} \cdot \frac{2 \text{ pints}}{\text{quart}} = 3.0 \text{ pints}$. Now we can convert the pints to cups. $3.0 \text{ pints} \cdot \frac{2 \text{ cups}}{\text{pint}} = 6.0 \text{ cups}$.

When we know that we need multiple conversions we can calculate them all at once.

$$1.5 \text{ quarts} \cdot \frac{2 \text{ pints}}{\text{quart}} \cdot \frac{2 \text{ cups}}{\text{pint}} = 6.0 \text{ cups}$$

When we use more than one conversion, we are careful to set up the ratios so that the intermediate units (pints in this case) are divided out leaving us with only the desired unit. □

Nothing restricts this process to two at a time.

Example 1.1.5 How many cups is 1.7 gallons?

Solution. The conversion table does not include the number of cups per gallon. However, we can start by converting gallons to quarts. Then looking at the table again, we can convert quarts to pints, and finally we can convert pints to cups.

As with the previous example we can treat each conversion as a ratio of units. We setup the units so that multiplying will result in units dividing out.

$$1.7 \text{ gallons} \cdot \frac{4 \text{ quarts}}{\text{gallon}} \cdot \frac{2 \text{ pints}}{\text{quart}} \cdot \frac{2 \text{ cups}}{\text{pint}} = 27.2 \text{ cups}$$

□

Example 1.1.6 How many days is 17 hours?

Solution. Here we are going from a small unit (hours) to a bigger one (days). This does not change our process. We can still multiply the amount by the unit conversion. Because we want to end up with days we use $\frac{1 \text{ day}}{24 \text{ hours}}$

$$17 \text{ hours} \cdot \frac{1 \text{ day}}{24 \text{ hours}} \approx 0.7083 \text{ days}$$

□

Standalone

Check that you can perform a unit conversion using this Checkpoint.

Checkpoint 1.1.7 Convert the measurement. You may find it useful to use this table[1].

64 cups = ____ gal

1.1.3 Metric (SI)

Just as with U.S. Customary units, we often need to convert between SI units. This section provides the information needed for conversion and examples of performing them. Again, units suggest how we setup the calculations.

Rather than have different names for different scales, metric uses one name of the unit (e.g., liter) and then uses prefixes to indicate size. These can be converted easily, because each prefix is a power of ten (uniform).

You will need to memorize a few of the prefixes. As with units which ones depends on your work. Ask your instructor which prefixes you should memorize for this course.

[1] `mital.uaa.alaska.edu/section-units.html#table-customary-convert`

Table 1.1.8 Metric Prefixes

Multiple	Prefix
10^{12}	tera (T)
10^9	giga (G)
10^6	mega (M)
10^3	kilo (k)
10^2	hecto (h)
10	deka (da)
1/10	deci (d)
$1/10^2$	centi (c)
$1/10^3$	milli (m)
$1/10^6$	micro (μ)
$1/10^9$	nano (n)
$1/10^{12}$	pico (p)

This next table illustrates these prefixes in the context of length (meters). Notice how it is easier to avoid the fractions (last three entries).

Table 1.1.9 Metric Conversion

Scaled Unit	Base Unit
kilometer (km)	$10^3 = 1000$ meters
hectometer	$10^2 = 100$ meters
10 decimeters	meter
$10^2 = 100$ centimeters	meter
$10^3 = 1000$ millimeters	meter

Review each of these examples to see how to convert from one SI unit to another.

Example 1.1.10 How many centimeters is 3.8 meters?

From Table 1.1.8 we know $10^2 = 100$ centimeters is 1 meter, that is, $\frac{100 \text{ cm}}{1 \text{ m}}$. This suggests that we can multiply 3.8 by the ratio which will cause the meters units to divide out.

$$3.8 \text{ m} \cdot \frac{100 \text{ cm}}{\text{m}} = 3.8 \cdot 100 \text{ cm} = 380 \text{ cm}.$$

Note, because centi is a power of ten (10^2) the result is shifting the decimal place two positions. 3.8 meters becomes 380 centimeters.

Using this idea we can convert 0.76 meters to 76 centimeters by just shifting the decimal (no additional process necessary). □

Example 1.1.11 How many kilotons is 2.3 megatons?

We know one kiloton is 10^3 tons and one megaton is 10^6 tons. These are three powers apart ($6 - 3 = 3$), which means we shift the decimal position three places. Because we are converting from a large unit to a smaller unit, we move the decimal place to the right (make the number bigger). 2.3 megatons is 2,300 kilotons. □

Example 1.1.12 How many centiliters is 13.6 milliliters?

Solution. We know $10^2 = 100$ centiliters is 1 liter and $10^3 = 1000$ milliliters is 1 liter. This means we shift the decimal $2 - (3) = -1$. Because we are moving from a smaller unit to a larger unit, we move the decimal place to the left (shown by the negative) to make the number smaller. 13.6 milliliters is 1.36 centiliters. □

Checkpoint 1.1.13 Convert the units below. You may find it useful to use this table[2].

1420 mL = ______ L

[2] `mital.uaa.alaska.edu/section-units.html#table-metric-sizes`

1.1.4 Converting between Systems

Commonly we end up with measurements in both U.S. Standard system and SI. We will need to convert all units to one system before using them together. This process is the same as converting one Standard unit to another (e.g., converting miles to feet). This section provides the information needed for conversion and examples of performing them. It is an example of units suggesting how we setup the calculation.

Table 1.1.14 U.S. Customary to SI

Measuring	Standard	SI
Length	1 nm	1.85200 km
Length	1 mi	1.60934 km
	1 ft	0.304800 m
	1 in	2.54000 cm
Volume	1 gal	3.78541 L
	1 oz	29.5735 mL
Weight	1 lb	0.453592 kg
	1 oz	28.3495 g

Table 1.1.15 SI to U.S. Customary

Measuring	SI	Standard
Length	1 km	0.621371 mi
	1 m	3.28084 ft
	1 cm	0.393701 in
Volume	1 L	0.264172 gal
	1 mL	0.0338140 oz
Weight	1 kg	2.20462 lb
	1 g	0.0352740 oz

Review each of these examples to see how to convert between U.S. Customary units and SI units.

Example 1.1.16 How many kilometers is 26.2 miles?

Solution. From Table 1.1.14 we know each mile is 1.60934 km; this means there is

$$\frac{1.60934 \text{ km}}{1 \text{ mi}}.$$

The ratio suggests that we can multiply 23.6 miles by the ratio, because the miles will divide out.

$$26.2 \text{ miles} \cdot \frac{1.60934 \text{ km}}{\text{mi}} = 42.164708 \approx 42.2 \text{ km}$$

We have rounded here to 3 digits to match the original number (26.2). A reason to do so is presented in Section 1.2. □

Example 1.1.17 How many inches is 15 centimeters?

Solution. From Table 1.1.15 we know each centimeter is 0.393701 inches; this means there is

$$\frac{0.393701 \text{ in}}{1 \text{ cm}}.$$

The ratio suggests that we can multiply 15 centimeters by the ratio, because the centimeters will divide out.

$$15 \text{ cm} \cdot \frac{0.393701 \text{ in}}{\text{cm}} = 5.905515 \text{ in} \approx 5.9 \text{ in}$$

We have rounded here to 2 digits to match the original number (15). A reason to do so is presented in Section 1.2. □

Example 1.1.18 How many inches is 1 meter?

Solution. From Table 1.1.15 we know each meter is 3.28084 feet. From Table 1.1.2 that each foot is 12 inches. We use the method of setting up a product of ratios so that the units divide out. We start with meters, so the first ratio must be feet per meters. We want to end with inches so the second ratio must be inches per feet.

$$1 \text{ m} \cdot \frac{3.28084 \text{ ft}}{1 \text{ m}} \cdot \frac{12 \text{ in}}{1 \text{ ft}} = 39.37008 \text{ in} \approx 39 \text{ in}$$

We have rounded here to 2 digits to match the original number (15). A reason to do so is presented in Section 1.2. □

Checkpoint 1.1.19 While purchasing gas on vacation in Canada, Erin wonders how many gallons she is purchasing. You may find it useful to use this table[3].

50 L = ____ gal

1.1.5 Converting Compound Units

Some units, such as speeds, are compound. For example speed is distance per time. This section provides examples of converting compound units.

Example 1.1.20 How many meters per second is 15 miles per hour?

We start with $\frac{15 \text{ mi}}{1 \text{ hr}}$. We can convert miles to feet and feet to meters (multi-step conversion like Example 1.1.4). The conversion ratios suggest we can multiply the 15 mi/hr by the conversion ratios.

$$\frac{15 \text{ mi}}{1 \text{ hr}} \cdot \frac{5280 \text{ ft}}{1 \text{ mi}} \cdot \frac{0.3048 \text{ m}}{1 \text{ ft}} \approx \frac{24140 \text{ m}}{1 \text{ hr}}$$

We can use the same method (multiplying by conversion ratios to divide out units) to also conver hours to seconds.

$$\frac{24140 \text{ m}}{1 \text{ hr}} \cdot \frac{1 \text{ hr}}{60 \text{ min}} \cdot \frac{1 \text{ min}}{60 \text{ sec}} \approx 6.706 \frac{\text{m}}{\text{sec}}.$$

Note, we could perform that conversion in on step by multiplying all the conversion ratios at once. □

Example 1.1.21 How many pounds does a tablespoon of water weigh? Note that one gallon of pure water weighs 8 lbs. Also a tablespoon is a half fluid ounce.

Solution. We need to convert the gallons into tablespoons. Because conversions are ratios, we multiply 8 lbs by the necessary conversions.

$$\frac{8 \text{ lbs}}{1 \text{ gal}} \cdot \frac{1 \text{ gal}}{4 \text{ qts}} \cdot \frac{1 \text{ qt}}{2 \text{ pints}} \cdot \frac{1 \text{ pint}}{2 \text{ cups}} \cdot \frac{1 \text{ cup}}{16 \text{ oz}} \cdot \frac{1 \text{ oz}}{2 \text{ tbs}} = \frac{1\text{lb}}{64 \text{ tbs}} = 0.015625 \frac{\text{lbs}}{\text{tbs}}.$$

□

Checkpoint 1.1.22 Convert 63 miles per hour to feet per second. You may wish to use this table[4].

63 miles per hour = _____ feet per second. Round your answer to the nearest tenth.

Another kind of compound unit is square units such as square feet or seconds squared. When converting these we must account for the square.

Example 1.1.23 Convert 2 acres to units of square miles.

Solution. First we note that an acre is $43,560 \text{ ft}^2$. From Table 1.1.2 we know that there are 5280 ft per mile. These conversion ratios suggest that we can multiply the 2 acres by the ratios to obtain the result in square miles.

$$2 \text{ acres} \cdot \frac{43,560 \text{ ft}^2}{\text{acre}} \cdot \frac{\text{mi}}{5280 \text{ ft}} \cdot \frac{\text{mi}}{5280 \text{ ft}} =$$

[3] mital.uaa.alaska.edu/section-units.html#table-si-to-customary
[4] mital.uaa.alaska.edu/section-units.html#table-customary-convert

$$
\begin{aligned}
2 \text{ acres} \cdot \frac{43,560 \text{ ft}^2}{\text{acre}} \cdot \left(\frac{\text{mi}}{5280 \text{ ft}}\right)^2 &= \\
2 \text{ acres} \cdot \frac{43,560 \text{ ft}^2}{\text{acre}} \cdot \frac{\text{mi}^2}{5280^2 \text{ ft}^2} &= \frac{1}{320} \text{ mi}^2 \\
&= 0.003125
\end{aligned}
$$

It is not necessary to write all of the steps above if you understand how the final conversion line is obtained. The steps are included here to show how the squares show up in the final conversion. □

Checkpoint 1.1.24 Convert 181 square inches to square feet. Round your answer to the nearest hundredth.
_____ ft^2

1.1.6 Exercises

1. **Units.** Select the correct units to complete the conversion below.
gallons ∗ ______________ = miles
Answer:

(a) miles/gallons

(b) miles

(c) gallons

(d) gallons/miles

(e) 1/gallons

(f) 1/miles

2. **Units.** Select the correct units to correctly complete the calculation below.
(people)∗(______________) = (dollars)
Answer:

(a) people/dollars

(b) 1/people

(c) dollars

(d) dollars/people

(e) 1/dollars

(f) people

3. **Units.** Convert the measurement. You may find it useful to use this table[5].
64 cups = ____ gal

4. **Units.** Mellody mixed 6 gallons of lemonade and poured it into three 7-quart jugs. How many cups of lemonade were left over after she filled the jugs?
This table[6] may be useful for this problem.
_ cups

5. **Units.** Convert $4\frac{9}{10}$ hours to minutes. Enter your answer as an integer or a reduced fraction.

$4\frac{9}{10}$ hours =______ minutes

[5] `mital.uaa.alaska.edu/section-units.html#table-customary-convert`
[6] `mital.uaa.alaska.edu/section-units.html#table-customary-convert`

6. **Units.** Convert the measurement.
3 days = ________ sec

7. **Units.** A corn stalk grew 8 inches in the first month after it planted, since then it grow another 3 feet.
What is the total height of the corn in feet and inches? __________________ ft __________________ in
What is the to total height of the corn stalk in inches? ____________________
What is the total height of the corn stalk in feet? ____________________ Round your answer to 2 decimal places.

8. **Units.** Select the unit that best fits the scenario
The box of ice cream contains 4

(a) fluid ounce(s)

(b) cup(s)

(c) gallon(s)

9. **Units.** Add the following weights:
18 lb 2 oz + 21 lb 15 oz + 23 lb 12 oz

_______________ pounds ____________________ ounces

10. **Units.** You are in charge of drinks for a community barbecue. You need to supply at least 120 cups of beverage to provide enough for the projected number of people that will attend. So far, you have received the following donations:

- Enough mix to make 3 gallons of lemonade
- 6 bottles of fruit juice that each contain 64 fl. oz.

How many cups of beverage do you have?
____
Will you have enough for the barbecue?

(a) no

(b) yes

11. **Units.** Convert 63 miles per hour to feet per second. You may wish to use this table[7].
63 miles per hour = _____ feet per second. Round your answer to the nearest tenth.

12. **Units.** Convert 514 square inches to square feet.
Round your answer to the nearest hundredth.
_____ square feet

13. **Units.** Jean's bedroom is 11 feet by 10 feet. She has chosen a carpet which costs $30.45 per square yard. This includes installation. Determine her cost to carpet her room.
$ ____________________
How much would she have saved if she went with the carpet that costs $30.10 per square yard instead?
$ ____________________

14. **Units.** Odina is making a quilt and she has determined she needs 1276 square inches of green fabric and 822 square inches of gray. How many square yards of each material will she need to purchase from the fabric store?
The store only sells fabric by the by the quarter yard.
The green fabric: ____________________ square yards
The gray fabric: ____________________ square yards

[7] mital.uaa.alaska.edu/section-units.html#table-customary-convert

How many total yards of fabric will she have to buy?
____________________ square yards

15. **Units.** A unit of measure sometimes used in surveying is the *link;* 1 link is about 8 inches. About how many links are there in 7 feet? Do not round your answer.
There are ___ links in 7 feet.

16. **Units.** 208 in. to yards, feet, and inches
__ yds __ ft __ in

17. **Units.** David has 9 yd. of material that he will cut into strips 16 in. wide to make mats. How many mats can David make?
___

18. **Units.** Part 1 of 2$\$81,0003050$What is the hourly pay for this job? We will answer the question by converting \$81,000 per year into dollars per hour.

If we begin with the fraction $\frac{\$81000}{\text{year}}$, we can multiply by two unit fractions to complete the conversion. What are these fractions? Choose the correct fractions in the calculation below:

$\frac{\$81000}{\text{year}} \times$

(a) 50 weeks/1 year

(b) 1 week / 50 years

(c) 50 years/1 week

(d) 1 year/50 weeks

$\times$

(a) 30 weeks/1 hour

(b) 1 week/30 hours

(c) 30 hours/1 week

(d) 1 hour/30 weeks

Part 2 of 2
Hourly pay = ____________________ dollars/hour

19. **Units.** Part 1 of 2

Eduardo was driving at 101 feet per second on the freeway the other day. If the speed limit is 65 miles per hour, was he driving too fast? Answer the question by converting 101 feet per second into miles per hour.

To answer this question, we will convert the numerator into miles and the denominator into hours.

If we begin with the fraction $\frac{101\text{ feet}}{1\text{ second}}$, we can multiply by three unit fractions to complete the conversion. What are these fractions? Choose the correct fractions in the calculation below:

$\frac{101\text{ feet}}{1\text{ second}} \times$

(a) 60 min/1 sec

(b) 60 sec/1 min

(c) 1 min/60 sec

(d) 1 sec/60 min

$\times$

(a) 60 min/1 hr

(b) 60 hrs/1 min

(c) 1 min/60 hrs

(d) 1 hr/60 min

×

(a) 5280 mi/1 ft

(b) 1 mi/5280 ft

(c) 5280 ft/1 mi

(d) 1 ft/5280 mi

Part 2 of 2
$\frac{101 \text{ feet}}{1 \text{ second}} \times \frac{60 \text{ sec}}{1 \text{ min}} \times \frac{60 \text{ min}}{1 \text{ hr}} \times \frac{1 \text{ mi}}{5280 \text{ ft}}$
Was Eduardo driving too fast?
Answer:

(a) Yes

(b) No

______65

20. Units. Part 1 of 37.38 $\frac{\text{dollars}}{\text{yard}^2}$Note about this unit[8]yards^{2}Averie is sure she saw this advertised online for 0.46$\frac{\text{cents}}{\text{inch}^2}$, and she wants to know if it is a better deal in the store or online. She will take a minute to convert the price from $\frac{\text{dollars}}{\text{yard}^2}$ to $\frac{\text{cents}}{\text{inch}^2}$ to see which is the better deal.

Averie knows the following facts:

- 1 yard = 3 feet
- 1 ft = 12 inches
- 1 dollar = 100 cents

$\frac{7.38 \text{ dollars}}{\text{yard}^2} *$

(a) 1yd / 3ft

(b) 1ft / 3yd

(c) 3ft / 1yd

(d) 3yd / 1ft

*

(a) 3ft / 1yd

(b) 1yd / 3ft

(c) 1ft / 3yd

(d) 3yd / 1ft

*

(a) 1ft / 12in

(b) 1in / 12ft

(c) 12ft / 1in

(d) 12in / 1ft

*

(a) 1ft / 12in

(b) 12ft / 1in

(c) 1in / 12ft

(d) 12in / 1ft

*

(a) 1cent / 100dollars

(b) 100cents / 1dollar

(c) 1dollar / 100cents

(d) 100dollars / 1cent

Part 2 of 3

$$\frac{7.38 \text{ dollars}}{\text{yard}^2} * \frac{1 \text{ yd}}{3 \text{ ft}} * \frac{1 \text{ yd}}{3 \text{ ft}} * \frac{1 \text{ ft}}{12 \text{ in}} * \frac{1 \text{ ft}}{12 \text{ in}} * \frac{100 \text{ cents}}{1 \text{ dollar}}$$

What is the final result, rounded to two decimal places?

Answer: ____________________ $\frac{\text{cents}}{\text{inch}^2}$

Part 3 of 3

Answer:

(a) Online

(b) The Store

21. Units. Select the unit that best fits the scenario

The pancake recipe called for 300

(a) milliliter(s)

(b) liter(s)

(c) centimeter(s)

(d) kilogram(s)

(e) meter(s)

(f) kilometer(s)

22. Units. Select the unit that best fits the scenario

The book weighs 1

(a) milligram(s)

[8] #

(b) gram(s)

(c) kilogram(s)

(d) centimeter(s)

(e) meter(s)

(f) liter(s)

(g) milliliter(s)

23. Units. Select the unit that best fits the scenario
My friend weighs 95

(a) milligram(s)

(b) gram(s)

(c) kilogram(s)

24. Units. How many millimeters are there in a meter? ____________________
How many liters are in a decaliter? ____________________
How many centigrams are in there in a gram? ____________________
Which prefix indicates a bigger quantity? hecto
kilo
Which prefix indicates a bigger quantity? deci
deca
Which prefix indicates a bigger quantity? kilo
mega
Which prefix indicates a bigger quantity? centi
milli

25. Units. Convert the measurement
5690 mL = ______ L

26. Units. Convert the measurement
2.1 g = ______ mg

27. Units. Convert the measurement
7 kg = ______ g

28. Units. Convert the measurement
2 m = ______ cm

29. Units. A bottle of Vitamin E contains 90 soft gels, each containing 20 mg of vitamin E. How many total grams of vitamin E are in this bottle?
There are __ grams of Vitamin E in the bottle.

30. Units. Convert the measurement
342 cm = ______ m

31. Units. Convert the measurement using the rules of SI prefixes.
1.5 terabyte = __________ gigabyte

32. Units. Convert 1.88 square meters to square centimeters.
_____ square centimeters

33. Units. A small rectangular panel measures 1.4 cm by 2.5 cm. What is its area in square millimeters?
_____ square millimeters

1.2 Accuracy and Precision

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Rounding (skill)
- Significant Figures (skill)
- Determining appropriate rounding from context (critical thinking)

While calculating devices will produce a lot of decimal places, these are not always meaningful nor useful. This section presents different purposes for rounding numbers, examples of using each one, and examples of interpreting numbers we consume in work and life.

First, we will consider what lots of decimal places do and do not mean which will lead to definitions, then we will present a method for reasonably tracking precision, then consider other motivations and matching methods for rounding, and later consider the importance of managing error.

1.2.1 Explanation

When working with measurements, we care about the reasonableness of the results. Suppose four people measure the length of a piece of wood and come up with 1.235 m, 1.236 m, 1.237 m, and 1.234 m. We might conclude that we are confident it is 1.23 m long but we are not certain about the millimeters position. This leads to the concepts of accuracy and precision.

Definition 1.2.1 Accuracy. The **accuracy** of a measurement is how close the measurement is to the actual value. ◊

Example 1.2.2 If the board referenced above is actually 1.2364 m long then all four measurements are accurate to the second decimal place. The second measurement (1.236 m) is accurate to the third decimal place. □

Example 1.2.3 Note $\frac{22}{7} \approx 3.142857$ is an approximation for π. Because π to six decimals places (not rounded) is 3.141592, the approximation 22/7 is accurate to only 2 decimal places (i.e., 3.14).

Note, π is not a measurement, rather it is defined theoretically. Thus we can produce an approximation that is as accurate as we have time and will to do. If curious, ask the nearest calculus instructor for details. □

Note, if we are measuring something, it is because it is not possible to know the actual value. In the example of measuring the board all we can do is use measuring tools and our use of all such tools has a margin of error. The actual length of the board is a mystery. Because of this we cannot determine the exact value of many kinds of data nor determine how accurate our measurement is. Instead we will settle for repeatability. If we get the same result often enough, we can convince ourselves that it is accurate.

Definition 1.2.4 Error. The **error** of a measurement is the difference between the reported measurement and the actual value. ◊

Example 1.2.5 The number of people at an outdoor concert was 2453. If someone estimated that the number of people was 2500, then that estimate is accurate to 1000's place, but has an error of only $2500 - 2453 = 47$. □

Definition 1.2.6 Precision. The **precision** of a measurement is the size of the smallest unit in it. ◊

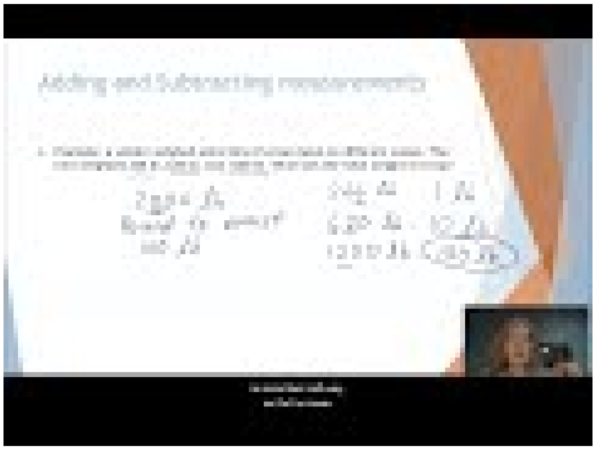
Note we can have high precision with low accuracy. That is, just because we write a lot of decimal places does not mean that number is close to the actual value of the measurement.

Example 1.2.7 The answer to a homework question is 5.7632. If a response given is 5.7647 what are the precision and accuracy of the response?

Precision is effectively the number of decimal places. This is precise to 4 decimal places (the 10,000th position).

Because the response matches the actual value to 2 decimal places (the 100ths position), it is accurate to 2 decimal places.

Because $5.7647 - 5.7632 = 0.0015$, the error is 0.0015. □

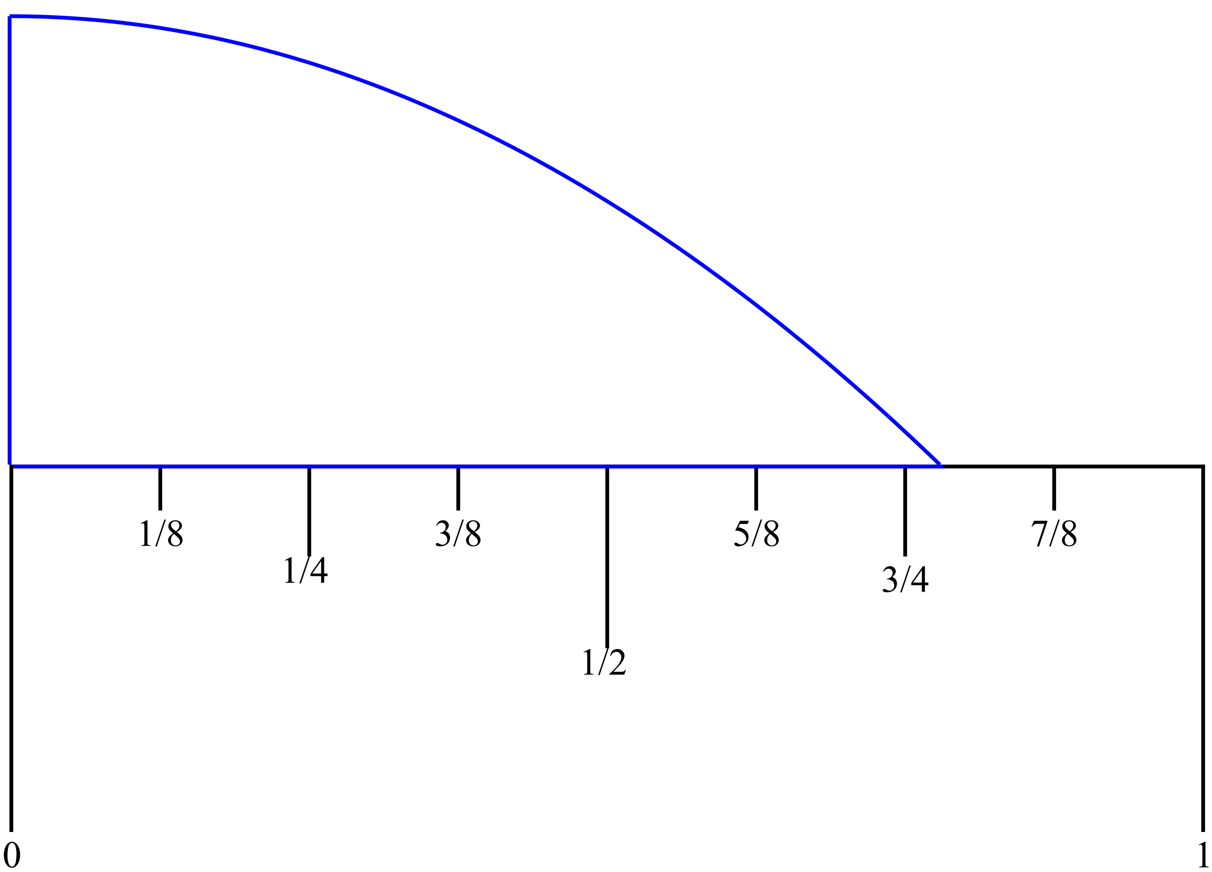


Standalone

Figure 1.2.8 Introduction to Precision and Accuracy

How do we end up with parts of measurements which are not accurate? Consider the following.

Example 1.2.9



When measuring the width of the blue, curved shape using the ruler (measurement in inches), it is clearly longer than 3/4'' and less than 7/8''. The right side of the shape appears to be less than half way between 3/4'' and 7/8''. Because it appears to be closer to 3/4'', we can state the width is 3/4''. Because the ruler does not have finer markings (e.g., 16ths or 32nds), we cannot be more precise.

We know this measurement is accurate to the nearest 1/8'', because the ruler has those marked and, in this case, we can be confident it is closer to the left side.

To estimate the error we note that the right edge is less than half way between the markings. Half way would be 13/16'' or 1/16'' farther. Thus we can state that the shape is 3/4'' wide with an error that is less than 1/16''. □

While other tools for measurement can be more precise, every tool has a limit to its precision similar to this example. We should always be aware of the limitations of measurements when we use them.

1.2.2 Significant Figures

It makes no sense to write numbers that are more precise than they are accurate. For example writing 3.142857 (from the approximation $\pi \approx \frac{22}{7}$ Example 1.2.3) makes no sense, because it is only accurate to the hundreds position (i.e., 3.14). It also makes no sense to perform arithmetic on digits that are not accurate. This section presents a reasonable way of tracking meaningful precision and rounding to maintain it. This will be used in many of the problems involving data from measurments for the rest of the text.

When writing down measurements we need a way to indicate how precise the measurement is. **Significant digits**, also called **significant figures** or simply "sig figs", are a way to do this.

The rules for writing numbers with significant digits have two parts: non-zero digits, and zero digits.

1. All non-zero digits are signficant.
2. Zeros between non-zeros are significant.
3. Any zeros written to the right of the decimal point are significant.
4. If zeros between non-zero digits (on left) and the decimal point (on right) are supposed to be significant, a line is drawn over top of the last significant digit.
5. If the zero to the left of the decimal is significant, the decimal point may be used with no digits to the right (the bar is an easier to read choice however)
6. For numbers less than 1, zeros between the decimal point (on left) and non-zero digits (on right) are not significant.

We can summarize these rules as: write only digits that you mean, and if it is ambiguous, clarify.

Significant digits apply to numbers resulting from measurements. That is, they apply when there is doubt about the accuracy of the number. These will be mixed with exact numbers (numbers with infinite precision). For example the 1/2 in the area of a triangle (Area $= 1/2bh$) is an exact number.

Example 1.2.10 Writing Significant Digits. Each of these numbers is written with five (5) significant digits.

- 10267
- 1.2400
- $7201\bar{0}$
- 53010.
- 2834100
- 0.0010527

□

Checkpoint 1.2.11 How many significant digits does 104 have? ___
How many significant digits does $1,0\bar{0}0$ have? ___
How many significant digits does 0.00571 have? ___

We also need rules for arithmetic with significant digits. These are based on two principles

- A result of arithmetic cannot be more precise than the least precise measurement.
- Arithmetic does not increase accuracy.

For addition and subtraction the result (sum or difference) has the same precision as the least precise number added or subtracted. After adding or subtracting we round to the farthest left of the last significant digits.

Example 1.2.12 Subtraction with Significant Digits. $11050 - 723 \approx 10330$. This is because the last significant digit of 11050 is the 10's position (with the 5 in it) whereas the last significant digit of 723 is the 1's position (with the 3 in it). We do not know the 1's position of 11050, so we cannot know the 1's position in the result. □

Example 1.2.13 Addition with Significant Digits. $311 + 8310 + 202200 \approx 210800$. This is because the farthest left, last significant digit is in the 100's position in 202200. The extra precision of the other two numbers is not useful. □

The significant digits addition/subtraction rule basically says that adding precise data to imprecise data does not increase the precision of the imprecise data. For those who are curious, an explanation of why this rule works is in this video.

Standalone

Exact numbers may be mixed in calculations with addition/subtraction. For example suppose we are converting temperature from Fahrenheit to Celsius based on a thermometer reading. The formula is $C = \frac{5}{9}(F - 32)$. The 32 and 5/9 are exact numbers (part of the definition of the Fahrenheit and Celsius systems). The F (measured temperature) would have limited precision and therefore determine the precision of the result.

Checkpoint 1.2.14 Calculate each of the following.
$251 + 42.08 + 320 =$ ____________________
$42.08 - 66 =$ ____________________

For multiplication and division the result (product or quotient) has the same number of significant digits as the least number of signficant digits of the input numbers.

Example 1.2.15 Division with Significant Digits. $11050/722 \approx 15.3$. This is because 722 has only 3 significant digits. □

Example 1.2.16 Multiplication with Significant Digits. $17 \times 14\bar{0} \times 3.178 \approx 7600$. This is because 17 has only two significant digits. □

The significant digit multiplication/division rule basically says that digits that were multiplied by imprecise data cannot be precise. For those who are curious, an explanation of why this rule works is in this video.

Standalone

Exact numbers may be mixed in calculations with multiplication and division. The following example illustrates how we determine the resulting number of significant digits when exact numbers are mixed with measurements.

Example 1.2.17 Suppose we are converting temperature from Fahrenheit to Celsius based on a thermometer reading. The formula is $C = \frac{5}{9}(F - 32)$. The 32 and 5/9 are exact numbers (part of the definition of the Fahrenheit and Celsius systems).

If we read the temperature as 44.7° F then the conversion is as follows.

$$\begin{aligned} C &= \frac{5}{9}(F - 32) \\ C &= \frac{5}{9}(44.7 - 32) \\ &= \frac{5}{9}(12.7) \text{ Maintains precisions to tenths place} \\ &\approx 7.055555 \text{ Maintains 3 sigfigs} \\ &\approx 7.06 \text{ So, round to 3 sigfigs} \end{aligned}$$

In the subtraction step the 32 is exact so the precision is determined by solely 44.7 (tenths place). In the multipication step the 5/9 is exact so the precision is determined by the 12.7 (three significant digits). □

Checkpoint 1.2.18 Calculate the following. Round using significant digits.

$466 \times 29.88 =$ ____________________

$12.15 \div 79 =$ ____________________

The rounding for significant digit rules is applied at the end of a calculation. That is if we have a mix of addition, subtraction, multiplication, and division then we do all of the operations, track the significant digits that should apply for each operation and apply the rounding at the end.

This example also illustrates tracking significant digits and illustrates a way you might keep track of which digits are significant in each step. The rightmost (smallest place value) digit that is significant is underlined. This is not a mathematical notation, but rather a convenient way to track the significant digits.

Example 1.2.19 Multi-Step Arithmetic with Significant Digits. Evaluate $11,728 + 39(17.9 + 1.23)$.

$$\begin{aligned} 11,72\underline{8} + 3\underline{9}(17.\underline{9} + 1.2\underline{3}) &= \\ 11,72\underline{8} + 3\underline{9}(19.\underline{1}3) &= \\ 11,72\underline{8} + 7\underline{4}6.07 &= 12,4\underline{7}4.07 \\ &\approx 12,470. \end{aligned}$$

By order of operations we first calculate $17.9 + 1.23 = 19.13$. Note that the result 19.13 is significant only to the first decimal place because 17.9 is only significant to the first decimal place. Second by order of operations we calculate $39 \cdot 19.13 = 746.07$. This is the product of a number with 2 significant digits and one with 3 significant digits so the result should have only 2 significant digits which would be the 10's place (the 4). The last calculation is $11,728 + 746.07 = 12,474.07$. 11,728 is significant to the one's place but the 746.07 is significant only to the 10's place. This means the final result is rounded to the 10's place so $11,728 + 39(17.9 + 1.23) \approx 12,470$. □

Example 1.2.20 Calculate

$$21 \cdot 9 - \frac{1}{2} \cdot 9(21 - 5 - 4).$$

The 1/2 is an exact number here. The rest are measurements in centimeters.

$$\begin{aligned} 2\underline{1} \cdot \underline{9} - \frac{1}{2} \cdot \underline{9}(2\underline{1} - \underline{5} - \underline{4}) &= \\ 2\underline{1} \cdot \underline{9} - \frac{1}{2} \cdot \underline{9}(1\underline{2}) &= && \text{Subtraction maintains precision to units} \\ \underline{1}89 - \underline{5}4 &= && \text{Multiplication maintains only 1 sigfig} \\ \underline{1}35 &\approx 100 && \text{Subtraction maintains only to hundreds} \end{aligned}$$

When we round we lose 35 units. If this seems like a lot consider the following. If all these measurements were rounded up (e.g., 21 was rounded from 20.5), then the value could have been

$$20.5 \cdot 8.5 - \frac{1}{2} 8.5(20.5 - 4.5 - 3.5) = 121.125.$$

If all the measurements were rounded down (e.g., 21 was rounded from 21.4), the the value could have been

$$21.4 \cdot 9.4 - \frac{1}{2} 9.4(21.4 - 5.4 - 4.4) = 146.64.$$

Using these two possibilities we see that the value could be anywhere from 121.125 to 146.64. The one hundred is the only digit about which we can be certain. That is why it is the only significant digit and why we eliminate (replace with zero) all the others.

If losing this much precision is a problem, then we need to obtain more precise measurements. □

Significant digts communicates which digits are definitely meaningful. There are other motivations for rounding which lead to keeping more digits. This will be addressed in the next section.

Checkpoint 1.2.21 $46.9 \cdot 9.1 + 19.2 \cdot 5.4 =$ ____________________

1.2.3 Rounding

Significant digits uses rounding to remove non-useful precision. This section presents various motivations for rounding and types of rounding and motivations for each.

Table 1.2.22 Reasons for Rounding

Reality Constraints	For example we cannot buy partial packages or have fractional people
Remove Detail	For example when describing the population of a nation
Control Error	When used in significant digits

The reason for rounding determines how we do it. Consider the following reality constraints requiring rounding. For example if we need 21 eggs and eggs are sold in cartons of one dozen (12) eggs, we need $21/12 = 1.75$ cartons. Since we cannot purchase part of a carton, we must round 1.75 up to 2, and purchase 2 cartons.

Note in this example reality requires us to round up to the nearest integer. We round to an integer because we cannot purchase fractional cartons of eggs. We had to round up, because rounding down would leave us with insufficient eggs (and we are hungry).

Suppose you have a bank account containing \$11,410 that accrues 1.65% interest. The bank calculates the payment should be $\$11,410 \cdot 0.0165 = \$188.265$. The bank will pay you \$188.26. They round to the nearest one hundredth because cents is a unit which can be paid. They round down, because they like paying less.

For removal of detail consider reporting the population of a country. We might report the population as over 9 million rather than 9,904,607. There are multiple motivations for this rounding. Note the population is likely changing multiple times per day, so more precision in the number does not equal more accuracy. Also, because of the scale (millions) the detail about how many ones, tens, hundred, and thousands looses meaning.

When reporting on salary ranges we might report a range between \$60,000 and \$80,000. That the range is actually \$61,233.57 to \$80,290.11 is unlikely to change a decision. The applicant will ask about the exact salary after deciding the position is a good fit. A common usage of removing detail is when we care about the scale of things rather than the count.

Rounding to control error is the use of significant digits.

Before considering context, we will practice rounding numbers. Note we can round to any digit. We can round up, down, or to the nearest number (what is meant by "round" if neither up nor down are specified). Context or instructions will specify which digit and which type of rounding.

Example 1.2.23 Rounding Up/Down.

(a) Round 72481 down to the nearest hundred.

Solution. 72400 is rounding down: we leave the 4 (hundred position) alone and "truncate" (turn to 0) all digits to the right. Note $72400 \leq 72481$.

(b) Round 72481 up to the nearest hundred.

Solution. 72500 is rounding up: we increase the 4 to a 5 and "truncate" (turn to 0) all digits to the right. Note $72500 \geq 72481$.

(c) Round 72481 the nearest hundred.

Solution. Because 72481 is closer to 72500 than it is to 72400, we round to 72500. We can recognize that we should round up because the tens position is $8 \geq 5$ which means rounding up results in a closer number. We could also recognize the need to round up by calculating $500 - 481 = 19$ and $481 - 400 = 81$ and noticing that $19 \leq 81$ (round up is closer).

□

Example 1.2.24 Rounding Up/Down.

(a) Round 72481 down to the nearest thousand.

Solution. 72000 is rounding down: we leave the 2 (thousands position) alone and "truncate" (turn to 0) all digits to the right. Note $72000 \leq 72481$.

(b) Round 72481 up to the nearest thousand.

Solution. 73000 is rounding up: we increase the 2 to a 3 and "truncate" (turn to 0) all digits to the right. Note $73000 \geq 72481$.

(c) Round 72481 to the nearest thousand.

Solution. Because 72481 is closer to 72000 than it is to 73000, we round to 72000. We can recognize that we should round up because the hundreds position is $4 < 5$ which makes it closer to go down. We could also recognize the need to round down by calculating $3000 - 2481 = 519$ and $2481 - 2000 = 481$ and noticing that $481 < 519$.

□

Example 1.2.25 Rounding to Different Precisions. Round 72321.83 to the specified precision.

- Thousands: 72000
- Ones:72322
- Tenths: 72321.8

□

Checkpoint 1.2.26 Round 334389 as indicated below.
To the 10's position: ____________________
To the 100's position: ____________________
Up in the 100's position: ____________________
Down in the 100's position: ____________________

Next we need to consider when to use each type of rounding.

Example 1.2.27 Some floors are covered in carpet tiles. These are squares of carpet that are tiled to cover a floor. Suppose the carpet tiles are square with side length 20''. If a room is 50 feet by 38 feet, how many carpet square do we need?

First lets figure out how many tiles will go across the 50 feet. Note 50 feet is $50\text{ ft} \times \frac{12\text{ in}}{\text{ft}} = 600\text{ in}$. This will require laying $\frac{600\text{ in}}{20\text{ in}} = 30$ tiles across.

Next, lets figure out how many tiles will go across the 38 feet. Note 38 feet is $38\text{ ft} \times \frac{12\text{ in}}{\text{ft}} = 456\text{ in}$. This will require laying $\frac{456\text{ in}}{20\text{ in}} = 22.8$ tiles across. For each 0.8 of a tile we must cut a tile leaving only 0.2 of a tile left. This is too small to use elsewhere. Thus for each of these we will use a whole tile resulting in needing 23 tiles across (rounding up to have enough).

Finally we can count the number of tiles which is $30 \times 23 = 690$ tiles.

If you are wondering why we do not use four of the 0.2 parts of a tile to fill a space, it is because that would look bad. Also, with so many seams it is more likely to pull up. Knowledge from carpet installation is required to determine this rounding. □

Example 1.2.28 Suppose you baked three (3) dozen cookies and are distributing them equally between 7 people. How many cookies does each person receive?

There are $3 \cdot 12 = 36$ cookies. Each person can have $36/7 \approx 5.1$ cookies. Because cutting cookies into pieces is how the cookie crumbles, we must round this down to 5 cookies per person.

Curious minds want to know what happens with the rest of the cookies. Notice there will be $7 \cdot 5 = 35$ cookies given away leaving just one cookie which the baker can enjoy. □

1.2.4 Greatest Possible Error

We have acknowledged that measurements will always have error. We have considered ways to round that are practical for the circumstances. Part of this depends on controlling the error. This section presents how to calculate the maximum error (worst case scenario). Typically we use this to ensure that error will not cause problems.

Because our rule for rounding is digits 0-4 round down and digits 5-9 round up, rounding will always have a greatest possible error of 5 in the position to the right of the one rounded. Consider the following.

Example 1.2.29 What is the greatest possible error if 130 was rounded to the nearest 10?

One possibility is that 130 was rounded down. Then the original number was one of 130, 131, 132, 133, or 134. 134 is the farthest away from 130 at $134 - 130 = 4$.

The other possibility is that 130 was rounded up. Then the original number was one of 125, 126, 127, 128, or 129. 125 is the farthest away at $130 - 125 = 5$.

Thus the greatest possible error was 5 from the case that 125 was rounded up.

Note in this solution we assumed the number rounded was an integer. However, if we allowed for 134.927 and 125.01 the result would be the same. the extra digits don't change the rounding. □

Example 1.2.30 What is the greatest possible error if 9.31 was rounded to the nearest hundredth?

Solution. The largest possible error is if 9.31 was rounded up from 9.305. Thus the greatest possible error is 5 one thousandths. □

Example 1.2.31 What is the greatest possible error if 223 was rounded up to the nearest one?

Solution. 223 could have been rounded up from 222.1. But it could also have been rounded up from 222.01 or anything else. Thus the greatest possible error is less than 1 ($223 - 222 = 1$). □

Notice we have to know what type of rounding was used. In most measurements (i.e., significant digits) standard rounding will be used. For example think about measuring on a ruler: if the object is not exactly on one of the lines, you will choose the closest one. Moving to the closest one is rounding.

Checkpoint 1.2.32 What is the greatest possible error of 86000 if it was rounded to the nearest 1000? ___

Checkpoint 1.2.33 What is the greatest possible error of $425,\bar{0}00$? ___

1.2.5 Exercises

1. **Significant Digits.** How many significant figures does 11,100,000,000 kg have?

____

2. **Count Significant Digits.** Tell how many significant digits there are in each measurement.

(a) 23,000.0 km _____

(b) 2041 m _____

(c) 0.03 gal _____

(d) 68,000,000 mi _____

3. **Count Significant Digits.** How many significant digits does 6970. have?

1
2
3
4
5
6
7
8

4. **Count Significant Digits.** Determine how many significant figures are in each measurement. If the measurement is exact, select "exact". Exact means there is no error in measurement.

(a) 1400 m

i. 1
ii. 2
iii. 3
iv. 4
v. 5
vi. 6
vii. 7
viii. 8
ix. 9
x. 10
xi. exact

(b) \$45.43

   i. 1
   ii. 2
   iii. 3
   iv. 4
   v. 5
   vi. 6
   vii. 7
   viii. 8
   ix. 9
   x. 10
   xi. exact

(c) 654.07 mL

   i. 1
   ii. 2
   iii. 3
   iv. 4
   v. 5
   vi. 6
   vii. 7
   viii. 8
   ix. 9
   x. 10
   xi. exact

(d) 1ft 2in

   i. 1
   ii. 2
   iii. 3
   iv. 4
   v. 5
   vi. 6
   vii. 7
   viii. 8
   ix. 9
   x. 10
   xi. exact

(e) 1400.0 m

   i. 1
   ii. 2
   iii. 3

iv. 4

v. 5

vi. 6

vii. 7

viii. 8

ix. 9

x. 10

xi. exact

5. Count Significant Digits. Determine the accuracy (i.e., the number of significant digits) of this number: 0.075
123
4
5

6. Count Significant Digits. Determine the accuracy (i.e., the number of significant digits) of this number: $89\overline{0},000$
123
4
5

7. Significant Digits Arithmetic. Calculate the product below, and express the result with the correct number of significant figures.
$5.032 \times 5.3 =$ _____

8. Significant Digits Arithmetic. Calculate the quotient below, and express the result with the correct number of significant figures.
16 &;div; 6.22 = _____

9. Significant Digits Arithmetic. Calculate the sum below, and express the result with the correct number of significant figures.
$0.189 + 124 + 65.76 =$ _____

10. Significant Digits Rounding. Round off the approximate number as indicated.
14.69; 2 significant digits

____

11. Greatest Error. Determine the GPE (i.e., the greatest possible measurement error) of this number: 0.4008

(a) ± 0.5

(b) ± 0.05

(c) ± 0.005

(d) ± 0.0005

(e) ± 0.00005

12. Greatest Error. Given the measurement 4.6 gal, find the following.
Precision to nearest ____________________ (thousand, hundred, ten, whole, tenth, hundreth, thousandth)
Accuracy ____________________ (number of significant digits)
Greatest possible error ____________________ gal

13. Greatest Error. Given the measurement 0.009 ft, find the following.
Precision to nearest ____________________ (thousand, hundred, ten, whole, tenth, hundreth, thousandth)
Accuracy ____________________ (number of significant digits)

Greatest possible error ____________________ ft

14. Greatest Error. Given the measurement 339 psi, find the following.
Precision to nearest ____________________ (thousand, hundred, ten, whole, tenth, hundreth, thousandth)
Accuracy ____________________ (number of significant digits)
Greatest possible error ____________________ psi

1.3 Working with Applications

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Solve *linear*, rational, quadratic, and exponential equations and formulas (skill)
- Read and interpret models (critical thinking)
- Use models including linear, quadratic, exponential/logarithmic, and trigonometric (skill)

In life when we figure out processes at work or in science we often express the result in mathematical notation. This includes equations, functions, and other options. These are collectively known as models. They allow us to communicate what we know and calculate results as needed. To succeed in many jobs and to fully enjoy life we need to be proficient at reading and using models.

This section begins by presenting models and illustrates calculating some results from them. It progresses to solving equations (models) as a review of algebra skills. Finally we present tips on how to identify and use models arising in applications. These topics are continued with specific models in later sections.

1.3.1 Calculating Results using Models

Model 1.3.1 Ohm's Law. *Ohm's Law relates three properties of electricity: voltage, current, and resistance.* ***Voltage****, measured in volts (V), is analagous to the amount of pressure to move the electrons.* ***Current****, measured in amperes (amps), is how much electricity is moving.* ***Resistance*** *measured in Ohms (Ω), is, as it sounds, the resistance of a material to letting electricity flow.*

The relationship is

$$V = IR$$

where

- *V is voltage,*
- *I is current, and*
- *R is the resistance.*

Example 1.3.2 Suppose the current is 3.0 amps and the resistance is 8.0 ohms. What is the voltage? These are measurements in a science model which means we will use significant digits rounding.

Using the model we can calculate

$$V = 3.0 \cdot 8.0 = 24.$$

Thus in this system there must be 24 volts.

If the current is increased to 6.0 amps on the 8.0 ohm circuit, then

$$V = 6.0 \cdot 8.0 = 48.$$

To double the amps, we would need to double the voltage.

Similarly if we know that the current is 1.7 amps and the resistance is 6.0 ohms, then we can calculate

$$V = 1.7 \cdot 6.0 = 10.2 \approx 10.$$

Thus in this system there must be 10 volts. □

Model 1.3.3 Lift Equation. *Lift is the force that keeps aircraft in the air.The lift equation explains factors that control the strength of lift produced by an airfoil (think wing or propellor). The factors included are air density, surface area of the airfoil, the coefficient of lift, and velocity. Air density is the amount of air per volume; you may see this as highs and lows on a weather map. It is also related to pressurizing aircraft flying at high altitude. The coefficient of lift incorporates multiple factors that are part of the design of the airfoil and how it is in use during flight.*

The lift equation is

$$L = \frac{1}{2}\rho s C_L v^2$$

where

- *L is the lift in units of lbs or Newtons)*
- *ρ is air density in units of slugs per cubic feet or kilograms per cubic meter*
- *s surface area in units of square feet or square meters*
- *C_L is the coefficient of lift which is unitless*
- *v is velocity in units of feet per second or meters per second*

Example 1.3.4 Suppose the air density is 0.002378 slugs per cubic feet, surface area is $125\,\mathrm{ft}^2$, $C_L = 1.5617$, and velocity is $84.4\,\frac{\mathrm{ft}}{\mathrm{s}}$. Calculate the lift. This is a science model with measurements, so we will use significant digits rounding.

$$\begin{aligned} L &= \frac{1}{2} \cdot 0.002378 \cdot 125 \cdot 1.5617 \cdot 84.4^2 \\ &\approx 16\underline{5}3.386439 \\ &\approx 1650. \end{aligned}$$

Under these circumstances this airfoil can lift 1650 lbs.

If the air density is reduced to 0.001988 slugs per cubic feet, what is the lift? This represents the same aircraft flying 6000 feet higher (hence lower air density).

$$\begin{aligned} L &= \frac{1}{2} \cdot 0.001988 \cdot 125 \cdot 1.5617 \cdot 84.4^2 \\ &13\underline{8}2.225501 \\ &\approx 1380. \end{aligned}$$

Notice that without changing other factors (like velocity), it can lift (hold in the air) $1650 - 1380 = 270$ lbs less. □

Model 1.3.5 Ideal Gas Law. *The ideal gas law is a relationship between the volume, pressure, temperature, and number of molecules of an ideal gas. The relationship is*

$$PV = nRT$$

where

- *P is the pressure in units of atmospheres (atm) or Pascals (Pa)*
- *V is the volume in units of cubic feet or cubic meters*

- *n is the number of moles (number of molecules, see a chemistry text for details)*
- *R is a constant specific to each gas (e.g., oxygen and nitrogen have different ones) in units that match the other values*
- *T is the temperature in degrees Rankine or Kelvin (these are shifted versions of Fahrenheit and Celsius).*

When the number of molecules remains fixed, such as in a closed container, this law can be used to produce the equation

$$\frac{P_1V_1}{T_1+273} = \frac{P_2V_2}{T_2+273}.$$

where

- *P_1, V_1, and T_1 are the initial pressure, volume, and temperature, and*
- *P_2, V_2, and T_2 are the pressure, volume, and temperature at another time.*

Note in both forms of the law the units can be other than those listed (especially different scale like centimeters rather than meters). However, they must always match including the constant R which is looked up in reference books.

For rounding note 273 is an exact number because it is part of the definition of the Kelvin temperature scale.

Example 1.3.6 Suppose the initial conditions are $P_1 = 101.3$ Pa, $V_1 = 0.125$ m^3, and $T_1 = 10.2°$ C. Also $V_2 = 0.125$ m^3 and $T_2 = 50.7°$ C. Calculate the new pressure. This is a science model with measurements, so we will use significant digits rounding.

$$\begin{aligned}
\frac{101.3\cdot 0.125}{10.2+273} &= \frac{P_2\cdot 0.125}{50.7+273} \text{ Addition maintains precision to tenths}\\
\frac{101.3\cdot 0.125}{10.2+273} &= \frac{P_2\cdot 0.125}{323.\underline{7}} \text{ Multiplication maintains 3 sigfigs}\\
\frac{12.\underline{6}625}{283.\underline{2}} &= \frac{P_2\cdot 0.125}{323.\underline{7}} \text{ Division maintains 3 sigfigs}\\
0.044\underline{7}122 &\approx 0.00038\underline{6}160P_2 \text{ Divide to isolate } P_2\\
\frac{0.044\underline{7}122}{0.00038\underline{6}160} &\approx \frac{0.00038\underline{6}160P_2}{0.00038\underline{6}160} \text{ Division maintains 3 sigfigs}\\
11\underline{5}.786721 &\approx P_2\\
116 &\approx P_2.
\end{aligned}$$

If instead $T_2 = -10.3°$ C, then we have the following.

$$\begin{aligned}
\frac{101.3\cdot 0.125}{10.2+273} &= \frac{P_2\cdot 0.125}{-10.3+273}\\
\frac{12.\underline{6}625}{283.\underline{2}} &= \frac{P_2\cdot 0.125}{262.\underline{7}}\\
0.044\underline{7}122 &\approx 0.00047\underline{5}828P_2\\
\frac{0.044\underline{7}122}{0.00047\underline{5}828} &\approx \frac{0.00047\underline{5}828P_2}{0.00047\underline{5}828}\\
93.\underline{9}6714779 &\approx P_2\\
94.0 &\approx P_2
\end{aligned}$$

□

Checkpoint 1.3.7 Recall Ohm's Law states $V = IR$.

Calculate the voltage if the current is $I = 2.2$ and the resistance is $R = 10$. $V =$ ____

Checkpoint 1.3.8 Note $\frac{V_1P_1}{T_1+273} = \frac{V_2P_2}{T_2+273}$ with temperature in Celsius.

If $V_1 = V_2 = 0.250$, $P_1 = 2085$, $T_1 = 11.0$, and $T_2 = 28.0$, what is P_2? ______

1.3.2 Calculating Results Requiring Solving

The previous section illustrated calculating model results without solving. This section presents additional example requiring limited solving and finishes with solving before any values have been substituted.

It does not matter if the value we desire is by itself, we can solve using arithmetic.

Example 1.3.9 Recall the model for lift. Suppose we know the weight of the aircraft ($w = 239\bar{0}$ lbs), density of air ($\rho = 0.001869$ slugs/ft^3), wing surface area ($s = 165$ ft^2), and velocity ($v = 91.1$ ft/sec). Noting that lift must equal weight, what must the coefficient of lift be? This is a science model with measurements, so we will use significant digits rounding.

$$\begin{aligned}
L &= \frac{1}{2}\rho s C_L v^2 \\
239\bar{0} &= \frac{1}{2}(0.001869)(165)C_L(91.1)^2. \text{ Multiply.} \\
239\bar{0} &\approx 12\underline{7}9.675938C_L. \text{ Multiplication maintains 3 sigfigs} \\
\frac{239\bar{0}}{12\underline{7}9.675938} &\approx \frac{12\underline{7}9.675938C_L}{12\underline{7}9.675938} \text{ Divide to isolate } C_L. \\
1.8\underline{6}7660342 &\approx C_L. \text{ Division maintains 3 sigfigs} \\
1.87 &\approx C_L.
\end{aligned}$$

□

The desired value from the model may be in a denominator. We can solve for this using multiplication and division.

Example 1.3.10 Recall from Model 1.3.5 that under simplifying assumptions

$$\frac{P_1V_1}{T_1+273} = \frac{P_2V_2}{T_2+273}.$$

Suppose we know the initial conditions ($P_1 = 1.00$ atm, $V_1 = 1.35$ ft^3, $T_1 = 51.2°$ F) and also $P_2 = 1.00$ atm and $V_2 = 1.39$ ft^3. What must the new temperature (T_2) be? This is a science model with measurements, so we will use significant digits rounding.

$$\begin{aligned}
\frac{1.00 \cdot 1.35}{51.2+273} &= \frac{1.00 \cdot 1.39}{T_2+273}. \text{ Perform arithmetic} \\
\frac{1.3\underline{5}}{324.\underline{2}} &= \frac{1.3\underline{9}}{T_2+273}. \text{ More arithmetic} \\
0.0041\underline{6}40962 &\approx \frac{1.3\underline{9}}{T_2+273}. \text{ Multiply to move } T_2 \text{ out of denominator.} \\
0.0041\underline{6}40962(T_2+273) &\approx \frac{1.3\underline{9}}{T_2+273}(T_2+273). \\
0.0041\underline{6}40962(T_2+273) &\approx 1.3\underline{9}. \text{ Divide to undo multiplication.} \\
\frac{0.0041\underline{6}40962(T_2+273)}{0.0041\underline{6}40962} &\approx \frac{1.3\underline{9}}{0.0041\underline{6}40962}. \text{ Division maintains 3 sigfigs} \\
T_2+273 &\approx 33\underline{3}.80593. \text{ Subtract to undo addition.} \\
T_2+273-273 &\approx 33\underline{3}.8059259-273. \\
T_2 &\approx 6\underline{0}.80593. \text{ Subtraction maintains precision to units}
\end{aligned}$$

$$T_2 \approx 61.$$

Note we use significant digits for rounding because this is a science model. □

The previous examples solved for a variable in a model after substituting numbers for the other variables. The next examples illustrate solving first. Note this process is the same as solving after substituting (same algebra) though there may be more steps. We might wish to solve this way, so it is easier to use the model multiple times.

Example 1.3.11 Solve the equation $V = IR$ for R. Note, this model is explained in Model 1.3.1.

$$\begin{aligned} V &= IR. \\ \frac{V}{I} &= \frac{IR}{I} && \text{Divide to undo multiplication.} \\ \frac{V}{I} &= R. \end{aligned}$$

□

Example 1.3.12 Solve the lift equation $L = \frac{1}{2}\rho S C_L v^2$ for S.

$$\begin{aligned} L &= \frac{1}{2}\rho S C_L v^2. \\ 2L &= 2\frac{1}{2}\rho S C_L v^2. && \text{Multiply to undo division.} \\ 2L &= \rho S C_L v^2. \\ \frac{2L}{\rho} &= \frac{\rho S C_L v^2}{\rho}. && \text{Divide to undo multiplication.} \\ \frac{2L}{\rho} &= S C_L v^2. \\ \frac{2L}{\rho C_L} &= \frac{S C_L v^2}{C_L}. \\ \frac{2L}{\rho C_L} &= S v^2. \\ \frac{2L}{\rho C_L v^2} &= \frac{S v^2}{v^2}. \\ \frac{2L}{\rho C_L v^2} &= S. \end{aligned}$$

Notice that the steps are the same algebra as if there were numbers. Also we could divide by v^2 and we are not concerned with the square as part of solving for S. □

Checkpoint 1.3.13 Solve the lift equation $L = \frac{1}{2}\rho S C_L v^2$ for ρ.

1.3.3 Process Overview

Above we started with a model and were asked to do something with it. Normally we will start with a problem which does not identify a model to use. Very few problems you encounter in lifte come pre-labled with models. This section presents how to start with a problem and work to a solution by identifying the model first.

Our first task is to read the problem to understand it.

- Read the problem description a few times.
    - If you can paraphrase it, you understand it enough.

  - Drawing a picture and labeling parts may help.

- Identify what we are asked to do.
- Identify the information we are given. Note distinctions like measurements and rates.
- Identify any units. These often help us set up a model.
- Write everything! We do not model in our heads.

Next we write the mathematical model (equation or function).

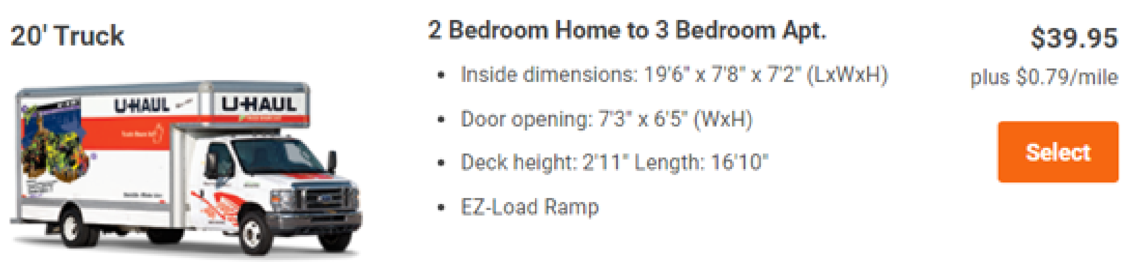
- Use the description to determine which application type (e.g., percent, proportion, linear model, etc.). Note units can suggest this (e.g., meters and meters squared indicate something was squared).
- Do not insert any numbers yet.
- Do not do any calculations yet.

Now we will have a model that matches our situation and possibly some numbers to insert.

- Insert numbers into the model. You may have to calculate some of these (e.g., you are given two points but not the slope you need).
- Solve for the desired value. Note it may help to do some calculations with the numbers first.
- State your answer and use units appropriately.

Finally we should check that our answer makes sense. We should not have negative prices (usually) or distances larger than the earth (when working with terrestrial problems).

Example 1.3.14 You moved across town and rented a 20 foot moving truck for the day. You want to make sure the bill you received is correct. If you paid \$81.03, for how many miles were you charged? Assume there were no extra fees.



Solution. We want to compare the bill we received to the price listed in the add. The question is about how many miles (not how much money).

We are given the price per mile (\$0.79 per mile). There is also a fixed cost for the rental (\$39.95). Adding the fixed cost and the milage cost will give us the total.

Our model is $C = \$39.95 + \$0.79m$ where C is the total cost and m is the number of miles.

We know the total cost, which will leaves m in the equation, the number of miles, which is what we want to calculate. We can use the solving technique in Section 3.1

$$\begin{aligned}
\$81.03 &= \$39.95 + \$0.79m \\
-\$39.95 + \$81.03 &= -\$39.95 + \$39.95 + \$0.79m \\
\$41.08 &= \$0.79m \\
\frac{\$41.08}{\$0.79} &= \frac{\$0.79}{\$0.79}m \\
52 &= m.
\end{aligned}$$

The charge is for 52 miles. □

Standalone

Figure 1.3.15 Using math modeling for rental truck

1.3.4 Exercises

1. Minimum Grade. A student in a Pre-Calculus class has test scores of 80, 70, 76, and 63. The final exam is worth 4 test grades.

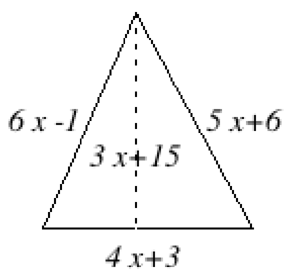
Write a linear equation that models this problem, where x is the grade on the final exam and y is the student's grade in the course. The whole grade is based on these tests and the final.

____

What grade is needed on the final to earn a B (average score of 80%)?

____

2. Triangle Properties.



Consider the triangle shown on the picture. Find the value of x, given that the perimeter of the triangle is 113 unit.

$x =$ _______ unit

3. Bike Rental. Amanda rented a bike from Ted's Bikes.

It costs $9 for the helmet plus $2.25 per hour.

If Amanda paid about $25.88, how many hours did she rent the bike?

a) Let $h =$ the number of hours she rented the bike. Write the equation you would use to solve this problem.

____

b) Now solve your equation

Amanda rented the bike for ____________________ hours. (Round your answer to the nearest tenth of an hour.)

4. Pilot Training. Part 1 of 5

The cost of a private pilot course is \$1221. The flight portion costs \$597 more than the ground school portion. What is the cost of the flight portion alone?

a) Let x represent the cost of the ground school portion. Write a variable expression to represent the cost of the flight portion.

____

Part 2 of 5

Fill in the boxes using the information from the problem and your expressions from part a:

Table 1.3.16

Cost of Ground School	+	Cost of flight portion	=	Total Cost
____________________	+	____________________	=	____________________

Part 3 of 5

c) Solve the equation $x + (x + 597) = 1221$ to answer the question.

$x =$ ____________________

Part 4 of 5

d) Since $x = 312$, this tells us:

(a) Ground school portion costs \$312

(b) Flight school portion costs \$312

Part 5 of 5

e) We used the expression $x+597$ to represent the cost of the flight portion. Knowing that $x = 312$, what is the cost of the flight portion alone?

Flight portion costs \$____________________

5. Thunder. In a thunderstorm, the formula:

$M = \dfrac{t}{5.7}$

gives the approximate distance, M, in miles, from a lightning strike if it takes t seconds to hear the thunder after seeing the lightning. If you are 9.4 miles away from the lightning flash, how long will it take the sound of the thunder to reach you.

Answer: It will take ______ seconds for the sound to reach you.

6. Speeding. In a Northwest Washington County, speeding fines are determined by the formula:

$F = 14(s - 40) + 75$

where F is the cost, in dollars, of the fine if a person is caught driving at a speed of s miles per hour.

If a fine comes to \$173, how fast was the person speeding?

Answer: The person's speed was ______ miles per hour.

7. Area of Triangle. The area of a triangle is given by the formula $A = \dfrac{1}{2}bh$, where A is the area of the triangle, b is its base and h is its height.

Solve the formula, $A = \dfrac{1}{2}bh$, for h.

$h =$ ____________

Find the height of the triangle with base of 7 meters and area of 96 square meters. Write your answer as a decimal, rounded to the nearest hundredth, when necessary.

height = ______ meters

8. Unit cost. You and your classmates create 23 paintings to sell at the PTA auction to raise money for your school. In order to pay for the materials you bought to make the paintings, you need to sell each of the paintings for \$4. When transporting the paintings to the auction, one of your parents accidentally drops four of the paintings in the street and they are run over by a Amtrak train. What is the new minimum price you need to set for the remaining paintings in order to pay for the materials?

Keep in mind that you do not want to collect less money than you paid for the materials. This may

affect how you round your answer.
$____________________

9. Paychecks. Your weekly paycheck is 25 percent less than your coworker's. Your two paychecks total 755. Find the amount of each paycheck.
Your coworker's is : $ __________ and yours is $ __________ .
Given your answers to the nearest cent

10. Mixture. The radiator in a car is filled with a solution of 75 per cent antifreeze and 25 per cent water. The manufacturer of the antifreeze suggests that for summer driving, optimal cooling of the engine is obtained with only 50 per cent antifreeze.
If the capacity of the radiator is 3.9 liters, how much coolant (in liters) must be drained and replaced with pure water to reduce the antifreeze concentration to 50 per cent?
Round your answer to two significant figures.
____________________ L

11. Area of Rectangle. A rectangular garden is 30 ft wide. If its area is 1800ft^2, what is the length of the garden?
Your answer is : ____________________ ft

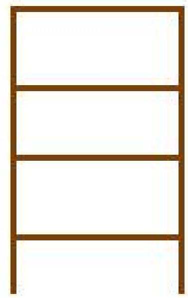
12. Shelving. A bookshelf containing 7 bookshelves is to be constructed. The floor-to-ceiling clearance is 7 ft 3.0 in. Each shelf is 1.0 in thick. An equal space is to be left between the shelves, the top shelf and the ceiling, and the bottom shelf and the floor. (There is no shelf on the ceiling or floor.)
What space should be between each shelf and the next? Round your answer to the nearest tenth of an inch.
____________________ in

13. Fax Cost. An online fax company, EFaxIt.com, has a customer plan where a subscriber pays a monthly subscription fee of $17.25 dollars and can send/receive 130 fax pages at no additional cost. For each page sent or received past the 130 page limit, the customer must pay an overage fee of $0.05 per page. The following expression gives the total cost, in dollars, to send p pages beyond the plan's monthly limit.
$C = 0.05p + 17.25$
If the monthly bill under this plan comes out to be $19.15, what was the total number of pages that were sent or received?
Answer: The total number of pages sent/received was ____________________.

14. Population Decrease. The current population of a small city is 34500 people. Due to a loss of jobs, the population is decreasing by an average of 275 people per year. How many years (from now) will it take for the population to decrease to 31475 people?
A) Write an equation you can use to answer this question. Be sure all the numbers given above appear in your equation. Use x as your variable and use no commas in your equation.
The equation is ____________________
B) Solve your equation in part [A] for x.
Answer: $x =$ ____________________

15. Sales. After a 35% reduction, you purchase a new bike for $302.25. What was the price of the bike before the reduction?
A) First write an equation you can use to answer this question. Use x as your variable and express any percents in decimal form in the equation.
The equation is ____________________
B) Solve your equation in part [A] to find the original price of the bike.
Answer: The original price of the bike was ____________________ dollars.

16. Bookcase. A bookcase is to have 4 shelves including the top as pictured below.



The width is to be 17 feet less than 3 times the height. Find the width and the height if the carpenter expects to use 30 feet of lumber to make it.

Width: _____ feet

Height: _____ feet

17. Knitting. It takes Rylla 18 hours to knit a scarf. She can only knit for 1.5 hours per day. How many days will it take her to knit the scarf?

Part 1: Let x be the number of days it will take her to knit the scarf. Choose the correct translation of this problem into an equation:

(a) 1.5 x=18

(b) x=(1.5)(18)

(c) 18-1.5=x

Part 2: Solve for x.

____

18. Rental Cost. A rental car company charges $40 plus 30 cents per each mile driven.

Part1. Which of the following could be used to model the total cost of the rental where m represents the miles driven.

(a) $C = 0.3m + 40$

(b) $C = 30m + 40$

(c) $C = 3m + 40$

Part 2. The total cost of driving 125 miles is;

$____________________

19. Mixture. You need a 5% alcohol solution. On hand, you have a 45 mL of a 40% alcohol mixture. How much pure water will you need to add to obtain the desired solution?

A) Write an equation using the information as it is given above that can be used to solve this problem. Use x as your variable to represent the amount of pure water you need to use. Equation: ______________________________

B) You will need

_____ mL of pure water

to obtain

_____ mL of the desired 5% solution.

20. Mixture. 14.0 liters of fuel containing 3.4% oil is available for a certain two-cycle engine. This fuel is to be used for another engine requiring a 6.1% oil mixture. How many liters of oil must be added?

Give your answer to 3 significant digits.

____

21. Wire Cutting. A wire 22 cm long is cut into two pieces. The longer piece is 4 cm longer than the shorter piece.

Find the length of the shorter piece of wire

____________________ cm

1.4 Project: Literal Formula

Project 1 Literal Formula. Most math books define the area of a circle as follows: $A = \pi r^2$, where A is the area of the circle and r is the radius of a circle. A text used in UAA's Automotive Diesel program defines the area of a circle as $A = 0.7854d^2$, where A is the area of the circle and d is the diameter of the circle.

The purpose of this project is to determine when each formula is most useful.

(a) What is the mathematical relationship between radius and diameter? Your answer can be a sentence or an equation.

(b) Show mathematically how to get from the formula $A = \pi r^2$ to the formula $A = 0.7854d^2$. This should take you multiple steps.

(c) Explain in words what you did in each step to change the first formula into the second. What assumptions did you have to make? Anyone reading this answer should be able to replicate the math by just reading your answer. That is, talk me through all the steps.

(d) Did you have any false starts or did you see how to change the formula right away? There is no wrong answer here; I just want you to think about your process.

(e) For this problem, you will need a tape measure or a ruler. If doing this on a device it must be a computer and ensure you are at 100% magnification. Your phone or a scaled version will distort the results. First *measure* the radius of the circle in Figure 1.4.1. Then *measure* the diameter of the circle below and record your answer. ***Do not calculate the diameter!*** This must be measured, not calculated. Try to be as precise as is reasonably possible. Include units.

Was it easier to measure the radius or the diameter?

(f) What is one reason why it might be more practical on a job to use the formula $A = 0.7854d^2$ instead of $A = \pi r^2$? If it helps, you may wish to ask yourself why the auto diesel students in particular use this less traditional formula.

(g) Determine how many significant figures are in each measurement. If the number is not a measurement or the measurement has no error then it is called 'exact'.

- π
- 0.7854

(h) Which of the two formulas is more accurate? Which is more precise? Give a reason to back up your answer.

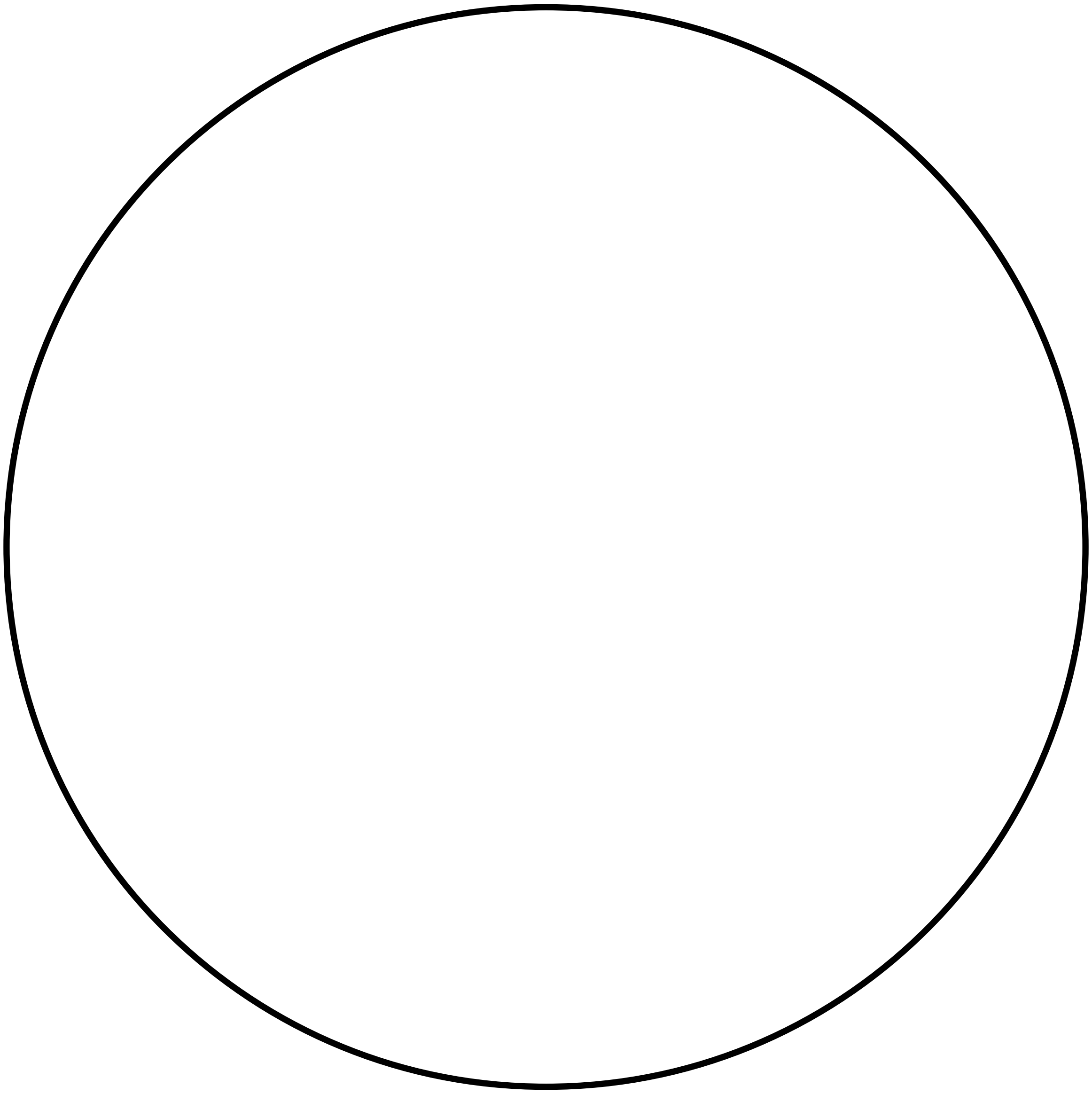


Figure 1.4.1 Circle

Chapter 2

Ratios

2.1 Percents

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Calculate Percentages (skill)
- Understand and interpret percentages (critical thinking)

Percentages are an often convenient way to express the relative size of two quantities such as the number of people who like lemon meringue pie to the number of those who like pie.

We will learn how to calculate a percent (Subsection 2.1.1, Item 2.b), how to convert between percents and the numbers (also Subsection 2.1.1, Item 2.b), how to describe growth in terms of percents (Subsection 2.1.2, Item 2.c), and how to recognize what a percent does and cannot tell us (Subsection 2.1.3, Item 2.c).

2.1.1 Calculating Percents

Definition 2.1.1 Percent. A **percent** is a ratio of part of something to the whole of that thing that is written as parts per hundred. ◊

Example 2.1.2 Calculate a Percent. In a class there are 34 students. Of them 21 are female. In this case female students is part of the whole (all students). Thus the percent is calculated as

$$\frac{\text{part}}{\text{whole}} = \frac{21}{34} \approx 0.6176.$$

This number says there are 61 hundreths (remembering our numbering system), so the percent is written as 61.76%.

Rounding to two (2) decimal places was chosen to illustrate how we convert a ratio in decimal form to a percent. If we were reporting this information, we would most likely round to 62%. This would convey the same meaning, because the difference between 61.76% and 62% for 34 people is less than one person. □

Standalone

Generally, we calculate a percent by

$$100 \times \frac{\text{part}}{\text{whole}}.$$

Example 2.1.3 In the class there are 34 students. Of them 13 are male. The percent is calculated as

$$100 \times \frac{13}{34} = 100 \times 0.3824 = 38.24\%.$$

□

Now that we have presented two examples of calculating a percent from counts, use the check point below to test that you can setup and calculate one yourself.

Checkpoint 2.1.4 In another class there are 84 students and 48 are female. What percent of the students are female? ______

In the first pair of examples we had a whole class of 34 students with 21 female and 13 male. Of course 21+13 = 34, that is the two parts add up to the whole. Because of this 61.76% + 38.24% = 100% as well.

Sometimes we are given the size of the whole and a percent, and we are interested in calculating how many are in the part.

Example 2.1.5 In a class of 22 students, 18% are Alaska Native. How many students are Alaska Native?

We use the same setup as before, but we do not know the part yet.

$$\begin{aligned} 100 \cdot \frac{P}{22} &= 18\% \text{ Divide to isolate } P \\ \frac{P}{22} &= \frac{18}{100} \text{ Multiply to isolate } P \\ \frac{P}{22} &= 0.18 \\ P &= 22 \cdot 0.18 \\ &= 3.96. \end{aligned}$$

Notice that 3.96 does not make sense as a result when counting people, so we expect that the correct result is 4. We can confirm this by checking that

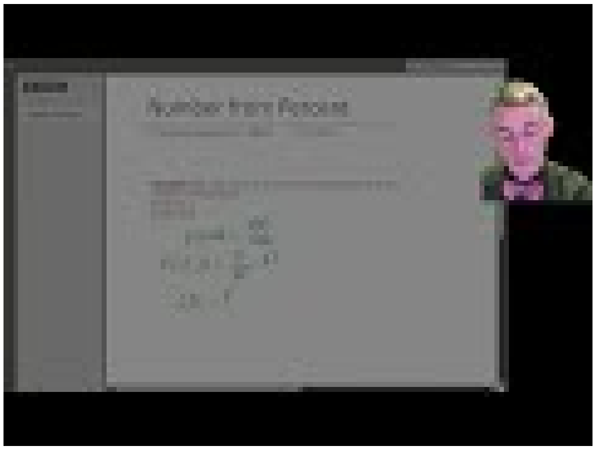
$$\begin{aligned} \frac{\text{part}}{\text{whole}} &= \\ \frac{4}{22} &= 0.18181818 \\ &\approx 18.18\% \end{aligned}$$

This suggests that the original 18% was rounded. Likely it was rounded to the ones position out of convenience.

We can solve this another way. We know that a percent is a number out of 100, so we can skip a step from the previous example.

$$\begin{aligned} \frac{P}{22} &= 0.18 \\ P &= 22 \cdot 0.18 \\ P &= 3.96 \end{aligned}$$

□



Standalone

Checkpoint 2.1.6 There are 44 students in a class. Below are percents for each racial group tracked. Calculate the number of students in each group.

Table 2.1.7

Group	Percent	Number
Alaska Native	6.25%	___
Asian	12.5%	___
Black	6.25%	___
White	71.88%	___
Other	0%	___

Sometimes we know the size of a part and what percent it is of the whole. From this information we can calculate the size of the whole.

Example 2.1.8 In a class 2 Alaska Native students make up 6.25% of the class. How many students are in the class?

Again we use the same setup, but we don't yet know the whole.

$$\begin{aligned} \frac{\text{part}}{\text{whole}} &= \text{percent}. \\ \frac{2}{W} &= 0.0625. \\ \frac{2}{W} \cdot W &= 0.0625 \cdot W. \text{ Move } W \text{ out of the denominator} \\ 2 &= 0.0625 \cdot W. \\ \frac{2}{0.0625} &= \frac{0.0625 \cdot W}{0.0625}. \text{ Divide to isolate } W \\ \frac{2}{0.0625} &= W. \\ 32 &= W. \end{aligned}$$

□

Example 2.1.9 How to Use an Example: Percents. Consider the following question.

If the first chapter of a certain book is 18 pages long and makes up 2% of the book, how many pages does the entire book have?

Because we see "2%" we recognize this as a percent problem. Without more information we can begin writing our steps. In Example 2.1.8 the first step is writing the definition of percent.

$$\frac{\text{part}}{\text{whole}} = \text{percent}$$

In the example 0.0625 is written on the right (in place of percent). In this problem we know the percent is 2, and we know that in a calculation we write the percent as a decimal. Thus our next step is

$$\frac{\text{part}}{\text{whole}} = \frac{2}{100}$$

In the next step in the example the entries for part and whole are entered. In this problem the 18 pages is stated as one chapter and is contrasted to the "entire" book. Thus the 18 is the part. As with the example, the whole is not known so we leave it as a variable.

$$\frac{18}{\text{whole}} = \frac{2}{100}$$

Finally in the example they solve for the variable. Note the steps of solving may vary depending on what we know, so rather than follow the rest of the example, we apply our algebra skills. For convenience we will write W instead of "whole".

$$\begin{aligned} \frac{18}{W} &= \frac{2}{100} \\ \frac{18}{W}W &= \frac{2}{100}W \\ 18 &= \frac{2}{100}W \\ \frac{100}{2}\cdot 18 &= \frac{100}{2}\cdot\frac{2}{100}W \\ 900 &= W \end{aligned}$$

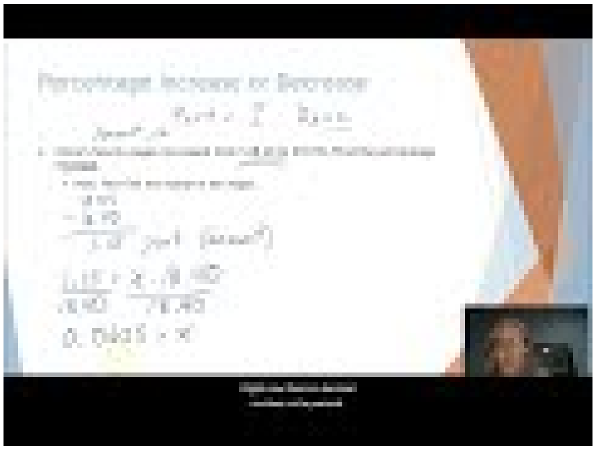
Thus we know the entire book has 900 pages. □

Standalone

In this next check point the terminology is different but something is still part of a whole and the amount can be calculated using the same approach as above.

Checkpoint 2.1.10 Find the number of millilitres of alcohol needed to prepare 140 mL of solution that is 6% alcohol. ______

This video covers the topics above.



Standalone

2.1.2 Percent Increase/Decrease

A common use of percents is to indicate how much something has increased (or decreased) from one time to the next.

Example 2.1.11 In spring there were 22 students in a class. In the following fall there were 34 students in the same class. What was the percent increase? Round the percent to the nearest unit.

This was an increase of 34-22=12 students. We can calculate what percent the increase of 12 is with respect to the original (spring) class size of 22.

$$100 \times \frac{12}{22} \approx 54.545454 \approx 55\%.$$

□

We say that the class size had a **percent increase** of 55%. Note this says the **increase** was 55% of the previous **whole**.

We can think of this in another way.

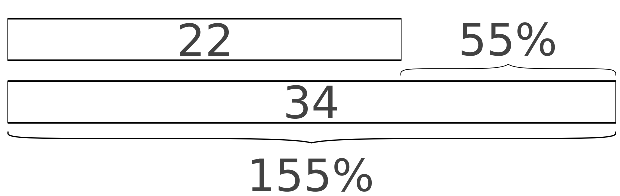
Example 2.1.12 In spring there were 22 students in a class. In the following fall there were 34 students in the same class. What is the percent increase in the fall?

We calculate the percentage the fall class size is with respect to the spring class size.

$$100 \times \frac{34}{22} = 155\%.$$

Because the fall class size (in the role of "part") is greater than the spring class size (in the role of whole), the percent ends up being greater than 100%. For percent increase we should always expect a percent greater than 100%.

Because this is 55% greater than 100%, the percent increase was 55% over the previous semester.



□

Checkpoint 2.1.13 What is the percent increase if enrollment in a class was 60 in spring and 90 in the following fall? _____

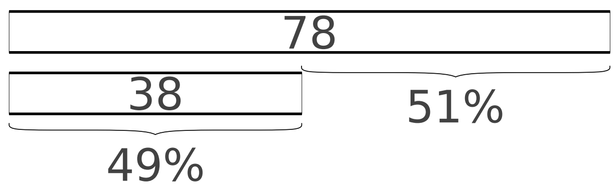
Round to the nearest percent (ones).

Example 2.1.14 What is the percent increase or decrease if enrollment in a class was 78 in fall and 38 in the following spring? Round the percent to the nearest unit.

Because 38 is less than 78 this is a decrease. Similar to the percent increase we can calculate the decrease first and then calculate the percent. $78 - 38 = 40$. Thus the percent decrease is

$$100 \cdot \frac{40}{78} \approx 0.51282051 \approx 51\%.$$

As with the percent increase we can also start by simply computing what percent the fall enrollment is with respect to the prior spring enrollment. The ratio is $100 \cdot \frac{38}{78} \approx 0.48717949 \approx 0.49$. Because the new enrollment *is* 49% of the previous enrollment the decrease is 100%-49%=51%.



□

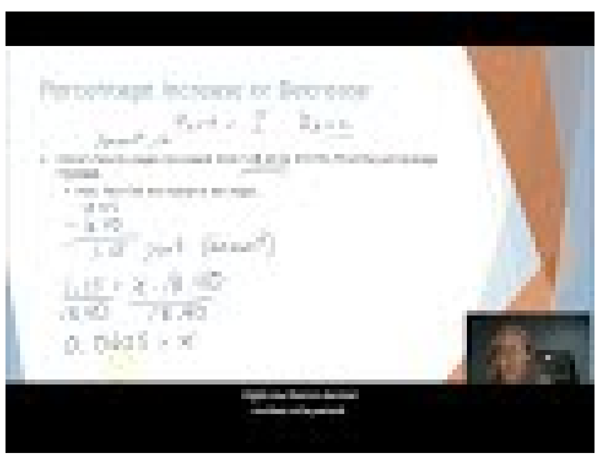
Checkpoint 2.1.15 Suppose enrollment in a class was 58 in the fall and 56 in the following spring.

It was a ____________________ percent

1. decrease

2. increase

This video covers percent increase topics.



Standalone

2.1.3 Limitations

We use percents because they can make the difference in scale between two quantities clear to us. However presenting a percent by itself can be deceptive.

Example 2.1.16 Which of the following do you suppose represents a greater reduction in students?

Percent reduction	Total
18%	495
1.85%	54
60%	5

Solution.

Percent reduction	Total	Number reduced
18%	495	90
1.85%	54	1
60%	5	3

The 18% of 495 represents the largest number of students. The 60% is a higher percent, but because the total is so small it represents very few students. A percent is more useful if we also know the total number.

Did you calculate 89 for 18% of 495? Compare the following to see why both are reasonable responses. 89 is what percent of 495? 90 is what percent of 495? □

While percent is defined a parts per one hundred, there are times when percents, sensibly, add to more than one hundred.

Example 2.1.17 Table 2.1.18 contains data from the 2020 U.S. Census. It contains the percent of the state population who checked the box for that race. Note the total is 149.4%. The reason is that a person can select more than one race. As a result a large number of people are counted more than once. Naturally the total is greater than 100 as a result.

When interpreting percents and data in general we should ask about the assumptions are before we draw conclusions. □

Table 2.1.18 Declared Race in Alaska

Race	Percent
Alaska Native/Native American	21.9%
Native Hawaiian/Pacific Islander	2.5%
Asian	8.4%
Black	40.8%
White	70.4%
Other	5.4%

2.1.4 Exercises

Exercise Group. Questions about the definition, terminology, and notation.

1. **Decimal to Percentage.** The decimal 0.65 is equivalent to what percent?
   ____ %
   (Do not enter the % sign)
2. **Fraction to Percentage.** The fraction $\frac{3}{4}$ is equivalent to what percent? __________ %
   (Do not enter the % sign)
3. **Percent of Whole.** 33 is what percent of 20.
   ____%

Exercise Group. Use percentages in various settings.

4. **Application of Percent.** There are 18000 students attending the community college. Find the percent of students that attend classes in the evening if there are 2431 evening students.
   __________ %
   Round to units. Do not type the %
5. **Percent of Whole.** There are $14,000$ students attending a private college. There are $3,500$ evening students. What percent of all students are the evening students?
   Evening students are _% of all the college's students.
6. **Use Percent to Calculate Total.** If the first chapter of a certain book is 24 pages long and makes up 4% of the book, how many pages does the entire book have?
   __________ pages

7. Calculate Original from Percent. This week Darnell got a promotion at work that came with a 3 % pay increase. If now his monthly salary is $ 1802.5 , how much was he making before the raise?

Enter your answer as a decimal. If needed, round to the nearest penny. Do not use commas.
$ ________

8. Calculate Whole from Part and Percent. Lennon is running a race. He has completed 18 km, which happens to be 90% of the total race. How long is the race? ____________________ km

9. Percent Decrease. Bryce's car insurance bill this year has decreased by 11%. He will be paying $98.34 less this year than last. How much did he pay last year?

____

10. Percents as Fractions. Out of the last 49 days, it rained 16 days. What percent of these days did it rain? ____________________ %
Round to 2 decimal places.

11. Percents as Fractions. In 27.27 % of the last 33 days it rained. How many days did it not rain? ____________________ days
Round to the nearest whole.

12. Percents as Fractions. In 50 % of the last 32 days it rained. How many days did it not rain? ____________________ days
Round to the nearest whole.

13. Percents as Fractions. An e-book regularly costs $9.99. How much does it cost if it's on sale for 62% of the regular cost?
____________________ dollars
Round to the nearest cent.

14. Percents as Fractions. Laura invested $5300 in stocks which were later sold for $7300. What percent of the initial investment were they sold for?
____________________ %
Round to 2 decimal places.

15. Percents as Fractions. A city has a population 160,000 in a state with population 5,900,000. What percent of the state live in this city?
____________________ %
Round to 2 decimal places.

16. Repeated Percents. Breanna loves cookies so much that after returning from Costco with some cookies, she eats 40% of the cookies she just bought. The next day she eats 40% of the remaining cookies and continues to eat another 40% each day. What percent of the cookies will Breanna have left after five days? (Round your answer to the nearest whole percent.)
____________________ %

17. Percent Increase or Decrease. Identify as an increase or decrease. Then find the percent of increase or decrease. If necessary, round to the nearest percent.
Original: 190
New: 150
_____%

(a) Increasing

(b) Decreasing

Exercise Group. Add problems asking students if the total should be 100% or greater.
Add problems asking students which percent and which amount are larger.

18. Dummy entry. Identify as an increase or decrease. Then find the percent of increase or decrease. If necessary, round to the nearest percent.
Original: 190
New: 150

_____%

(a) Increasing

(b) Decreasing

2.2 Mixtures

This section addresses the following topics.

- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Calculate Percentages (skill)
- Understand and interpret percentages (critical thinking)

This section continues the topic of percents through a set of applications.

There are many situations where we desire to mix two or more substances together in precise ratio. These include mixing medicines in substances (like water) and mixing ingredients in a recipe. This section presents how to calculate the ratio of substances after mixing multiple solutions, and the reverse problem how to calculate the amount of each solution to mix for a desired ratio of substances.

2.2.1 Calculate the Result

In some circumstances we know the concentrations of substances in multiple solutions and how much of each has been combined. From this we can calculate the concentration of substances in the resulting solution.

2.2.1.1 Mix Multiple Solutions

Example 2.2.1 Suppose we have a container with a solution that is 22% sugar and the rest water and another container with a solution that is 16% sugar and the rest water. If we combine 150 g of the first solution and 250 g of the second solution, what is the percent of sugar in the resulting solution? Round the percent to two decimal places.

To calculate a percent we need the amount of the part and the total amount. We can calculate the total directly: $T = 150\text{ g}+250\text{ g} = 400\text{ g}$. Note we know we can add these because both represented total amounts and they have the same units (g).

To calculate the part we need to know how much (rather than what percent) sugar was obtained from the two solutions. We can calculate that from the given percents and amounts.

$$\begin{aligned} 150\text{ g} \cdot 0.22 &= 33\text{ g}. \\ 250\text{ g} \cdot 0.16 &= 40\text{ g}. \\ P &= 33\text{ g} + 40\text{ g} \\ &= 73\text{ g}. \end{aligned}$$

Thus the percent of sugar in the mixture is $P/T = \frac{73\text{ g}}{400\text{ g}} = 18.25\%$. □

Example 2.2.2 Suppose we have 11.3 lbs of 4140 steel which is a type of steel containing 40% carbon and 9.2 lbs of 4150 steel which contains 50% carbon. If we melt and mix these two metals, what is the resulting percent carbon? Round the final percent to two decimal places.

To calculate a percent we need the amount of the part and the total amount. We can calculate the total directly: $T = 11.3\text{ lbs} + 9.2\text{ lbs} = 20.5\text{ lbs}$. Note we know we can add these because both represented total amounts and they have the same units (g).

To calculate the part we need to know how much (rather than what percent) carbon was obtained from the two metals. We can calculate that from the given percents and amounts.

$$\begin{aligned} 11.3 \text{ lbs} \cdot 0.40 &\approx 4.52 \text{ lbs}. \\ 9.2 \text{ lbs} \cdot 0.50 &= 4.6 \text{ lbs}. \\ P &= 4.52 \text{ lbs} + 4.6 \text{ lbs} \\ &= 9.12 \text{ lbs}. \end{aligned}$$

Thus the percent of carbon in the resulting metal is $P/T = \frac{9.12 \text{ lbs}}{20.5 \text{ lbs}} \approx 0.44487805 \approx 44.49\%$. □

Checkpoint 2.2.3 Suppose 5.0 oz of a solution that is 70.% alcohol is mixed with 2.0 oz of a solution that is 99% alcohol. The resulting solution is what percent alcohol? _____

2.2.1.2 Dilute a Solution

This section presents a slight variation of the mixture calculation problem. In this case we are adding only water resulting in diluting the solution.

The next problem is producing a isopropyl alcohol with a lower concentration of alcohol than the original solution. We begin with 16.0 oz of a 91.0% isopropyl alcohol solution. The other ingredient is water.

Example 2.2.4 Dilute Alcohol. Suppose we begin with 16.0 oz of a 91.0% isopropyl alcohol and add 4.00 oz of water to this mixture. What will the percent alcohol of the resulting solution be? These are measurements, so we will round using significant digits.

The percent alcohol is the amount of alcohol (part) divided by the total volume (whole). Only the original solution has alcohol so based on the meaning of percent the volume of alchol is $0.910(16.0 \text{ oz}) = 14.\underline{5}6$ oz. We are adding 4.00 oz total to the solution, thus the total volume is $16.0 \text{ oz} + 4.00 \text{ oz} = 20.0$ oz oz. The final percent alcohol is

$$\frac{14.\underline{5}6 \text{ oz}}{20.0 \text{ oz}} = 0.72\underline{8}$$

or 72.8%. □

If we added more water would the percent alcohol be greater or less? If we are using all of the alcohol solution, the amount of water we add determines the percent alcohol.

Checkpoint 2.2.5 Suppose 6.5 cups of lemonade is 18% lemon. If 4.2 oz of sparkling water is added, what is the percent lemon in the resulting drink? _____%

2.2.2 Producing a Desired Solution

In the previous section we calculated the result of mixing two solutions. In this section the goal is to figure out how much water to add to achieve a specific concentration. That is, the previous section calculated in this section we solve.

Example 2.2.6 Calcluate Dilution. If we start with 16.0 oz of 91.0% alcohol solution, how much water do we add to get (at least) 25.0 oz of a 55.0% alcohol solution? These are measurements, so we will round using significant digits.

How much solution total does this produce? Be aware it will not necessarily be 25.0 oz.

This is a percent problem with the total alcohol unchanged and adding only some amount w of water. Because the result is specified as a percent we need w to be such that $\frac{\text{part}}{\text{whole}} = \frac{A}{16.0+w} = 0.550$ where A is the amount of alcohol. We do not use the 25.0 oz at this time. We will address that at the end.

Because all of the alcohol comes from the initial solution, the amount of alcohol is $(0.910)16.0 \text{ oz} = 14.56$ oz. Thus we setup

$$\begin{aligned} \frac{(0.910)16.0 \text{ oz}}{16.0 \text{ oz} + w \text{ oz}} &= 0.550. \\ \frac{14.\underline{5}6 \text{ oz}}{16.0 \text{ oz} + w \text{ oz}} &= 0.550 \text{ Multiply to move } w \text{ out of the denominator} \end{aligned}$$

$$\begin{aligned}
14.\underline{5}6 \text{ oz} &= 0.550(16.0 \text{ oz} + w \text{ oz}) \text{ Distribute}\\
14.\underline{5}6 \text{ oz} &= (0.550)16.0 \text{ oz} + (0.550)w \text{ oz Multiply}\\
14.\underline{5}6 \text{ oz} &= 8.80 \text{ oz} + (0.550)w \text{ oz Subtract to isolate } w\\
5.\underline{7}6 \text{ oz} &= (0.550)w \text{ oz Divide to isolate } w\\
\frac{5.\underline{7}6 \text{ oz}}{0.550} &= w\\
1\underline{0}.47272727 \text{ oz} &\approx w\\
10 \text{ oz} &\approx w
\end{aligned}$$

We end up with $16 \text{ oz} + 10 \text{ oz} = 27$ oz of new solution. Because we have at least the 25 oz we need, and it is the correct concentration, we have achieved our goal. If we had added less water to get exactly 25 oz we would have had a more concentrated solution.

Also note that the percent alcohol is $\frac{0.910(16.0 \text{ oz})}{16.0 \text{ oz}+10 \text{ oz}} = 0.56$ or 56%. This is not exactly 55% because of the rounding in the calculations. □

Checkpoint 2.2.7 If we have 16.0 oz of a 91.0% alcohol solution left in a bottle and we want 24.0 oz of a 59.0% alcohol solution, how much water should we add? _____

Standalone

2.2.3 Exercises

1. **Mixture.** You need a 5% alcohol solution. On hand, you have a 45 mL of a 40% alcohol mixture. How much pure water will you need to add to obtain the desired solution?

A) Write an equation using the information as it is given above that can be used to solve this problem. Use x as your variable to represent the amount of pure water you need to use. Equation: ______________________________

B) You will need
_____ mL of pure water
to obtain
_____ mL of the desired 5% solution.

2. **Mixture.** 14.0 liters of fuel containing 3.4% oil is available for a certain two-cycle engine. This fuel is to be used for another engine requiring a 6.1% oil mixture. How many liters of oil must be added?

Give your answer to 3 significant digits.

____

3. **Medical Proportion.** Quinidine gluconate is a liquid mixture, part medicine and part water, which is administered intravenously. There are 110.0 mg of quinidine gluconate in each cubic centimeter (cc) of the liquid mixture. Dr. Alverez orders 506 mg of quinidine gluconate to be administered daily to a patient with malaria.

How much of the solution would have to be administered to achieve the recommended daily dosage?

____________________ cc

4. Medical Proportion. Albuterol is a medicine used for treating asthma. It comes in an inhaler that contains 16 mg of albuterol mixed with a liquid. One actuation (inhalation) from the mouthpieces delivers a 90 μg dose of albuterol. (Reminder: 1 mg = 1000 μg .)

a.) Dr. Olson orders 2 inhalations 4 times per day. How many micrograms of albuterol does the patient inhale per day?

____________________ μg

b.) How many actuations are contained in one inhaler?

____________________ actuations

c.) Alicia is going away for 5 months and wants to take enough albuterol to last for that time. Her physician has prescribed 2 inhalations 4 times per day. How many inhalers will Alicia need to take with her for the 5 period? Assume 30-day months.

Hint: she can't bring a fraction of an inhaler, and she does not want to run out of medicine while away.

____

5. Medical Ratio. Amoxicillin is a common antibiotic prescribed for children. It is a liquid suspension composed of part amoxicillin and part water.

In one formulation there are 200 mg of amoxicillin in 6 cubic centimeters (cc's) of the liquid suspension. Dr. Scarlotti prescribes 400 mg per day for a 2-yr old child with an ear infection.

How much of the amoxicillin liquid suspension would the child's parent need to administer in order to achieve the recommended daily dosage?

____

6. Medical Proportion. Diphenhydramine HCL is an antihistamine available in liquid form, part medication and part water. One formulation contains 20 mg of medication in 4 mL of liquid. An allergist orders 40-mg doses for a high school student. How many milliliters should be in each dose?

____________________ mL

7. Percent Concentration. How many mL of sodium hydroxide are required to prepare 550 mL of a 15.5% solution? Assume the sodium hydroxide dissolves in the solution and does not contribute to the overall volume.

____________________ mL

8. Dilution Ratio. You are asked to make a 1/4 dilution using 7 mL of serum. How much diluent do you need to use?

____________________ mL

9. Dilution Ratio. A clinical lab technician determines that a minimum of 65 mL of working reagent is needed for a procedure. To prepare a $\frac{1}{11}$ dilution ratio of the reagent from a stock solution, one should measure 65 mL of the reagent and ____________________ mL of the diluent.

10. Dilution Ratio. A patient's glucose result is suspected to be outside the range of the analyzer, so the techs decide to dilute the sample before running it. 25 microliters of serum is added to 75 microliters of diluent and the diluted sample is analyzed. The analyzer reads that the glucose value of the diluted sample is 40 $\frac{mg}{dL}$.

What was the ratio the sample was diluted to?

____

What is the glucose value of the original sample?

____________________ $\frac{mg}{dL}$

11. Dilution Ratio. A thyroid peroxidase antibody test was performed on a 45 year old man. The dilution sequence was 50 μL serum added to 350 μL of diluent in tube 1. Then 40 μL from tube 1 was added to 80 μL of diluent in tube 2. Finally 45 μL from tube 2 was added to 360 μL of diluent in tube 3.

All dilution ratios should be given as fractions.

a.) What is the dilution ratio in tube 1?

b.) What is the dilution ratio in tube 2?

c.) What is the dilution ratio in tube 3?

d.) What is the overall (serial) dilution ratio?

2.3 Ratios

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Set up and solve proportions (skill)

A ratio expresses a fixed relationship between two quantities. This section illustrates using ratios to calculate amounts subject to a ratio and presents how to recognize what a ratio does and cannot tell us. Note, percents are simply ratios scaled to parts per 100, that is, what we know about percents is applicable here.

2.3.1 Example Ratios

Example 2.3.1 Simple syrup consists of one cup of sugar and one cup of water which is heated until the sugar is disolved. There are multiple ratios that express this combination. $\frac{1 \text{ cup sugar}}{1 \text{ cup water}}$, $\frac{7.05 \text{ oz}}{8 \text{ oz}}$ ratio of sugar to water by weight, $\frac{7.05 \text{ oz}}{15.05 \text{ oz}}$ ratio of sugar to simple syrup.

Each of these ratios indicates that there is a fixed amount of sugar relative to the water or resulting syrup. That is, is if we made simple syrup with 2 cups of sugar we would need 2 cups of water because

$$\frac{2 \text{ cups sugar}}{2 \text{ cups water}} = \frac{1 \text{ cup sugar}}{1 \text{ cup water}}$$

by reducing fractions. □

Example 2.3.2 In a neighborhood there are 7 dogs and 12 cats. To express the relative number dogs and cats we can write the ratio

$$\frac{7 \text{ dogs}}{12 \text{ cats}}$$

Note that $\frac{7}{12} \approx 0.58 < 1$. Because it is less than one the ratio tells us that there are fewer dogs than cats in this neighborhood.

We could also write $\frac{12 \text{ cats}}{7 \text{ dogs}}$ to express the exact same relationship. Note that $\frac{12}{7} \approx 1.7 > 1$. Because it is greater than one the ratio tells us that there are more cats than dogs in the neighborhood (same result as above). □

Example 2.3.3 Rates are expressed as ratios. The following rates are all written as ratios.

$$\frac{35 \text{ miles}}{1 \text{ hour}}$$

$$\frac{8 \text{ gallons}}{1 \text{ minute}}$$

Notice that these (and most rates) are expressed with the denominator being one. This makes it easier to calculate as will be seen below. Frequently we will skip writing the one in the denominator, e.g., $\frac{35 \text{ miles}}{\text{hour}}$.

Rates are not required to have a denominator of one and sometimes a different denominator is easier for calculation. For example

$$\frac{3 \text{ nm}}{2\text{min}} = \frac{1.5 \text{ nm}}{1 \text{ min}}$$

□

2.3.2 Using Ratios

Just like percents (which are ratios written in decimal form) if we know a ratio and one of the amounts we can calculate the other amount. The method is the same as with percents, namely multiplying the correct number or solving an equation.

Example 2.3.4 Ratio: Airspeed. The Diamond DA-20 cruises at the rate (speed) of

$$\frac{110 \text{ nautical miles}}{1 \text{ hour}}.$$

Cruise is a portion of flight in which the speed is typically constant. How far can the plane cruise in 2.5 hours?

The units suggest we can multiply these.

$$\frac{110 \text{ nm}}{1 \text{ hr}} \cdot 2.5 \text{ hr} = 275 \text{ nm}.$$

This works because we are multiplying by hours and dividing by hours which leaves us with nautical miles as desired. □

There is another approach to the same question.

Example 2.3.5 Ratio: Airspeed by Table. The Diamond DA-20 cruises at the rate (speed) of

$$\frac{110 \text{ nautical miles}}{1 \text{ hour}}.$$

How far can the plane cruise in 2.5 hours?

Because this ratio remains the same during cruise we can break the problem down into pieces. Based on the ratio (speed) in the first hour the plane will fly 110 nm. In the second hour it will fly another 110 nm. In the last half hour it will fly half of the distance which is $110/2 = 55$ nm. Thus in 2.5 hours it will fly $110 + 110 + 55 = 275$ nm.

Some manuals provide tables to make this method convenient. See Table 2.3.6 for a table of this sort. How far can the DA-20 cruise in 2.7 hours? We need to write 2.7 as the sum of numbers in the table. One way is $2.7 = 2 + 0.5 + 0.2$. From the table we know it will fly 220 nm in 2 hours, 55 nm in an additional 0.5 hours, and 22 nm in the final 0.2 hours. Thus it will fly $220 + 55 + 22 = 297$ nm.

While this method makes sense, it takes more time than simply multiplying. □

Table 2.3.6 Airspeed and Distance

Time	Distance
0.1 hour	11 nm
0.2 hour	22 nm
0.3 hour	33 nm
0.4 hour	44 nm
0.5 hour	55 nm
1 hour	110 nm
2 hour	220 nm
3 hour	330 nm

Example 2.3.7 Water is flowing out of a hose at a rate of 11 gallons per minute. How many gallons have come out after 2.7 minutes?

Solution. We can set this up like the calculation in Example 2.3.4. In that example multiplied the ratio (nm/hr) by the number of hours which gave us nm. Here we multiply the ratio (gal/min) by the number of minutes which will give us gallons.

$$\frac{11 \text{ gal}}{1 \text{ min}} \cdot 2.7 \text{ min} = 29.7 \text{ gal}.$$

□

Example 2.3.8 Based on data from the FDA the average amount of mercury found in fresh or frozen salmon is 0.022 ppm (parts per million). This means there are 0.022 mg of mercury in one liter of salmon. If a meal portion of salmon is 0.0020 liters how much mercury is consumed?

Solution. We can use the ratio $\frac{0.022 \text{ mg}}{1 \ \ell}$. We apply this ratio to the given volume of 0.0020 liters.

$$\frac{0.022 \text{ mg}}{1 \ \ell} \cdot 0.0020 \ \ell = 0.000044 \text{ mg}$$

□

Checkpoint 2.3.9 Calculate how far a plane flying at 110 nm/hour would travel in 3.4 hours. ____________

The method for ratios that have a number other than one in the denominator is the same.

Example 2.3.10 A saline solution intended for nasal rinsing has a ratio of 2.5 g of salt (sodium chloride) per 240 mL of pure water. How much salt is needed to make a half liter of this saline solution? Your scale is precise to a gram. Round appropriately.

We can apply the given ratio (2.5 g/240 mL) to the given amount (0.5 L). First it will be convenient to convert a half liter to milliliters. This is also a ratio problem. See Table 1.1.8 for the conversion ratio.

$$\frac{1000 \text{ mL}}{1 \text{ L}} \cdot 0.5 \text{ L} = 500 \text{ mL}.$$

The units suggest that we can multiply the saline solution ratio by the desired volume.

$$\frac{2.5 \text{ g}}{240\text{mL}} \cdot 500 \text{ mL} \approx 5.208333335 \approx 5 \text{ g}.$$

We round to units, because we cannot measure more precisely. □

Example 2.3.11 At 90 nm/hr a plane travels 3 nm/2 min. How far will it travel in 6 minutes?

Solution. We can multiply the given ratio (nm/hr) by the given amount (min) to calculate distance (nm).

$$\frac{3 \text{ nm}}{2 \text{ min}} \cdot 6 \text{ min} = 9 \text{ nm}.$$

□

Checkpoint 2.3.12 One formulation of amoxicillin, a drug used to treat infections in infants, contains 125 mg of amoxicillin per 5.00 mL of liquid. How much amoxicillin is in 11.0 mL of liquid? ____________

2.3.3 Understanding Ratios

Like percents ratios tell us a relationship between two quantities but do not tell us how much. For example if a cookie recipe calls for 2 cups of milk for every 3 eggs, we do not know how many eggs are needed for a dozen cookies. We would also need to know either how many cups of milk per dozen or how many eggs per dozen.

Ratios may be based on rounded numbers. For example the unit convesion $\frac{8 \text{ kilometers}}{5 \text{ miles}}$ is convenient for quick calculations. However, $8/5 = 1.6 \approx 1.609344$ which is a more accurate conversion rate. If we are trying to convert “Tempo 30” (a speed limit of 30 kph) to mph this ratio is fine. If we are sending a satellite to another planet, we will need a more accurate conversion.

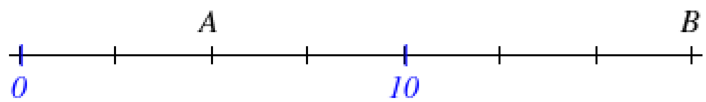
Rounding may occur to make the ratio easier to comprehend. For example according to an article by the U.S. Census Bureau[1] 1 in 6 people in the U.S. was aged 65 and over. Because this ratio uses small numbers it is easy to understand. It is much easier to read and use than a more precise estimate of $\frac{57822315}{333287562}$.

2.3.4 Exercises

1. Enter the ratio as a fraction in lowest terms
2 ft to 82 in.

____

2. Identify the decimals labeled with the letters A and B on the scale below.



Letter A represents the number ____
Letter B represents the number ____

3. Write the ratio as a ratio of whole numbers in lowest terms
\$0.20 to \$0.50

____

4. Consider the rectangle with width 9 ft and length 30 ft, write a ratio of the length to the width.

____

5. If you spend 4 hours a week studying for English and 4.5 hours studying for math what is the ratio of time spent studying in math to studying for English?

____

6. An employee pays \$550 towards health insurance, while the employer pays \$325. What is the ratio of the employers contribution to the employees contribution?

____

7. Enter the ratio as a fraction in lowest terms (no decimals).
0.11 inches to 1.2 inches

____

8. At a recent Pac-12 sporting event, there were 51,600 Beavers fans and 8,000 Sun Devils fans. Write each ratio as as a reduced fraction.
A) The ratio of Beavers fans to Sun Devils fans. ______
B) The ratio of Beavers fans to total fans. ______

9. In a recent survey, 48 percent of people claimed to like math. Write this ratio as a reduced fraction.

______

10. Ananya spent 23 hours playing video games in a week. The table shows the number of hours Ananya played each video game.

Table 2.3.13

video games	hours:
Minecraft	12
Battlefield	9
Call of Duty	2

What is the ratio of Minecraft to total hours Ananya plays video games in a week?
__ : __

[1] www.census.gov/library/stories/2023/05/2020-census-united-states-older-population-grew.html

11.

12.34 ounces = 350 grams.Use that conversion factor to determine the weight in grams of a 10 ounce box of Granola.

Round your answer to the nearest whole gram. 10 ounces = _____ grams

12.

12 fluid ounces (fl oz),355 milliliters (ml).In Zambia they sell 440 ml cans. How many fluid ounces is that?

*Round your answer to 2 decimal places.*440 ml =___fl oz

13. A few winters ago, it was very cold in northern Maine, and Stacy thought about leaving the kitchen faucet running overnight (so the water pipes wouldn't freeze). Stacy's roommate Reshanda was a little concerned that they would be wasting a lot of water, so they performed an experiment.

Reshanda turned the kitchen faucet on so it was dripping water at a constant rate. Then she held up a 1/4 teaspoon under the faucet, and it filled in 6 seconds. So, the water was "flowing" at a rate of 0.25 teaspoons per 6 seconds.

Question: They were going to sleep and planned to get up 10 hours later to turn off the faucet. How many *gallons of water* would have gone down the drain in that time?

[Answer this question by converting 10 hours into gallons. Give your answer as a decimal number, rounded correctly to the nearest thousandth of a gallon.]

- 0.25 teaspoons = 6 seconds
- 128 fluid ounces = 1 gallon
- 1 hour = 60 minutes
- 1 minute = 60 seconds
- 1 tablespoon = 3 teaspoons
- 2 tablespoons = 1 fluid ounce

In all, approximately ________ gallons of water will flow down the drain in 10 hours.

14. A 41 oz bottle of dish soap sells for $2.67. A 46 oz bottle of dish soap sells for $8.02.
(round all answers to four decimal places)
The unit price of the 41 oz bottle is $____ per oz
The unit price of the 46 oz bottle is $____ per oz
Which of the two is a better deal?

(a) The 41 oz bottle for $2.67

(b) The 46 oz bottle for $8.02

15. You can purchase a 12 fl oz bottle of window cleaner for $2.14 or a 19 fl oz bottle for $3.35. Which bottle of window cleaner is the better deal? What is the unit price of this bottle?

(a) 19 fl oz bottle for 17.63 cents per fl oz

(b) 19 fl oz bottle for 5.67 cents per fl oz

(c) 12 fl oz bottle for 17.83 cents per fl oz

(d) 12 fl oz bottle for 5.61 cents per fl oz

(e) None of these

16. Add some about interpreting ratios here.

2.4 Proportions

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Set up and solve proportions (skill)

In some circumstances a ratio is fixed. For example when we scale a recipe the ratio of flour to water must remain the same. This means the original ratio (in recipe) must equal the ratio used (when doubling for example). These circumstances are called **proportions**. This section presents a variety of problems in which this is useful and, indirectly, reviews the algebra needed to solve them.

Fixed ratios make sense in examples like conversion of units. For example 1 gallon is always 4 quarts. In contrast rates often change: your average speed may be 25 mph, but you must have driven slower and faster during that drive. For the ratios that do not change we can write equations and solve for properties.

2.4.1 Proportion Examples

Proportion problems start with a fixed ratio. Because it is fixed we can write the first ratio equals the second ratio. This gives us an equation to solve. There are multiple ways to solve these, each of which is demonstrated below.

Standalone

The first example shows a straight forward proportion with the simplest solving method. This is like solving a percent problem.

Example 2.4.1 Cheesecake Groceries: Double. A particular cheesecake recipe calls for 150 g of eggs and 1500 g of cream cheese. How many grams of eggs and how many grams of cream cheese do we need to double the recipe?

This means everything will be in ratio of 2 (needed)/1 (in the recipe). We must perform the calculation for eggs and cream cheese separately. We want the ratio of amount of eggs in the doubled recipe to the amount of eggs in the original recipe to be 2/1. Thus

$$\frac{2}{1} = \frac{E \text{ g}}{150 \text{ g}}.$$

Because the quantity to solve is in the numerator we can simply multiply to isolate that quantity (variable).

$$\begin{aligned}
\frac{2}{1} &= \frac{E \text{ g}}{150 \text{ g}}. \\
2 &= \frac{E \text{ g}}{150 \text{ g}}. \\
2 \cdot 150 \text{ g} &= \frac{E \text{ g}}{150 \text{ g}} \cdot 150 \text{ g}. \text{ Multiply to isolate } E \\
2 \cdot 150 \text{ g} &= E \\
300 \text{ g} &= E \text{ of eggs.}
\end{aligned}$$

Because scales are accurate to a gram, we do not need to round.

In commercial recipes (and quality home cooking) weights are used because items like eggs are not uniform in mass. If we always use 3 eggs, it might be more or less than we need which will mess up the food. ☐

This example may seem overly simple because doubling is easy, but the arithmetic is the same for any scaling.

Example 2.4.2 Guido needs 6 dozen cookies. A recipe makes 4 dozen. If that recipe calls for 300 g of flour, how much flour does he need for the 6 dozen cookies?

First, we determine the ratio for scaling. We want 6 dozen and the recipe makes 4 dozen so our ratio is

$6/4 = 3/2$. Thus to determine the amount of flour needed we setup the proportion

$$\begin{aligned}\frac{3}{2} &= \frac{F\text{ g}}{300\text{ g}}\\ \frac{3}{2}\cdot 300\text{ g} &= \frac{F\text{ g}}{300\text{ g}}300\text{ g}\\ 450\text{ g} &= F\end{aligned}$$

□

The following example shows methods for handling proportions when the desired quantity (variable) ends up in the denominator.

Example 2.4.3 Cheesecake Groceries: Unknown. A particular cheesecake recipe calls for 150 g of eggs and 1500 g of cream cheese. If we have 350 g of egg how much cream cheese do we need?

We know that the egg to cream cheese ratio must be 150/1500. We notice $150/1500 = 1/10$. This means we need to solve

$$\frac{1}{10} = \frac{350\text{ g}}{C\text{ g}}.$$

Because a ratio expresses the relationship between two quantities, it is not important which is numerator or denominator. Thus it is equally valid to write

$$\begin{aligned}\frac{10}{1} &= \frac{C\text{ g}}{350\text{ g}}.\\ \frac{10}{1}\cdot 350\text{ g} &= \frac{C\text{ g}}{350\text{ g}}\cdot 350\text{ g}.\\ 3500\text{ g} &= C\text{ of cream cheese.}\end{aligned}$$

□

The following example shows an alternate way to solve for the desired quantity when it is in the denominator.

Example 2.4.4 Proportion: Solving. According to the airplane flight manual a Diamond DA-20 cruises at the rate of

$$\frac{110\text{ nautical miles}}{1\text{ hour}}.$$

How long will it take to travel 236 nm? Round to tenths of an hour (a standard in aviation).

Because cruise speed is a fixed ratio we can write

$$\begin{aligned}\frac{110\text{ nm}}{\text{hour}} &= \frac{236\text{ nm}}{t\text{ hours}}.\\ \frac{\text{hour}}{110\text{ nm}} &= \frac{t\text{ hours}}{236\text{ nm}}.\text{ Inverse is still true.}\\ \frac{\text{hour}}{110\text{ nm}}(236\text{ nm}) &= \frac{t\text{ hours}}{236\text{ nm}}(236\text{ nm}).\text{ Multiply to isolate } t\\ \frac{236}{110}\text{ hours} &= t\\ 2.14545\text{ hours} &\approx t\\ 2.1\text{ hours} &\approx t.\end{aligned}$$

□

Checkpoint 2.4.5 If the Diamond DA-20 climbs at the rate of $\frac{450\ \text{feet}}{1.0\ \text{minute}}$ how long will it take it to climb $2,500$ ft? ____________________

2.4.2 Multiple Proportions

When we experience math in the wild, problems do not come labeled with solving methods. We must recognize the math and apply our knowledge appropriately. The next example illustrates identifying ratios (proportions) more than once when answering a question.

Example 2.4.6 Suppose a Diamond DA-20 climbs at the rate of $\frac{550\text{ feet}}{1.0\text{ minute}}$ and is traveling $\frac{72.8\text{ nm}}{\text{hour}}$ across the ground during this climb. How far forward in nautical miles does the plane fly during a climb of 3500 feet? Round the final distance to one decimal place.

Because we are told how far the plane climbs, we must use the rate of climb ratio first. As before we can calculate how long it will take to climb.

$$3500\text{ ft} \cdot \frac{1\text{ min}}{550\text{ ft}} \approx 6.36363636\text{ min}.$$

Now that we know how long the plane flies during this climb, we can use that time with the ground speed ratio to calculate how far foward it flies. However, we must first convert the speed to feet per minute or the time to hours. We use conversions from Table 1.1.2.

$$\frac{72.8\text{ nm}}{\text{hr}} \cdot \frac{6076\text{ ft}}{\text{nm}} \cdot \frac{1\text{ hr}}{60\text{ min}} \approx \frac{7372.213333\text{ ft}}{\text{min}}$$

Finally we can use this rate and the time of climb to calculate the desired distance.

$$\begin{aligned} \frac{7372.213333\text{ ft}}{\text{min}} &= \frac{s\text{ ft}}{6.36363636\text{ min}} \\ \frac{7372.213333\text{ ft}}{\text{min}} \cdot (6.36363636\text{ min}) &= \frac{s\text{ ft}}{6.36363636\text{ min}} \cdot (6.36363636\text{ min}) \\ 46914.08482 &\approx s\text{ ft}. \end{aligned}$$

Now we convert this to nautical miles.

$$(46914.08482\text{ ft}) \cdot \frac{1\text{ nm}}{6076\text{ ft}} \approx 7.72121212\text{ nm} \approx 7.7\text{ nm}$$

□

Example 2.4.7 A recipe for hush puppies calls for 150 g of flour for 340 g of buttermilk. If we have 465 g of flour and 918 g of buttermilk, how much of the flour and buttermilk can we use? Which one constrains us (limits size of our batch)? Note quality kitchen scales are accurate to a single gram. Round appropriately.

The ingredients must remain in the ratio $\frac{340\text{ g buttermilk}}{150\text{ g flour}} = \frac{34\text{ g buttermilk}}{15\text{ g flour}}$. We can select either ingredient and see how much the ratio tells us we need of the other ingredient.

Suppose we use all 465 g of flour. Then we can setup the proportion

$$\begin{aligned} \frac{34\text{ g buttermilk}}{15\text{ g flour}} &= \frac{B\text{ g}}{465\text{ g flour}} \\ \frac{34\text{ g buttermilk}}{15\text{ g flour}} \cdot (465\text{ g flour}) &= \frac{B\text{ g}}{465\text{ g flour}} \cdot (465\text{ g flour}) \\ 1054\text{ g} &= B. \end{aligned}$$

Notice that this is more buttermilk than we have. That means the buttermilk is the limiting ingredient. We will be able to use all of the buttermilk, but only some of the flour. To determine how much we setup the proportion but this time solve for flour.

We will use all 918 g of buttermilk. Then we can setup the proportion

$$\begin{aligned} \frac{15\text{ g flour}}{34\text{ g buttermilk}} &= \frac{F\text{ g}}{918\text{ g buttermilk}} \\ \frac{15\text{ g flour}}{34\text{ g buttermilk}} \cdot (918\text{ g buttermilk}) &= \frac{F\text{ g}}{918\text{ g buttermilk}} \cdot (918\text{ g buttermilk}) \end{aligned}$$

$$405 \text{ g} = F.$$

Thus we can use all 918 g of buttermilk and 405 of the 465 g of flour. Note we have rounded everything to one gram because that is as accurate as we can measure with our scale. If a single recipe uses 340 g of buttermilk, then we will be making

$$\frac{918 \text{ g}}{340 \text{ g}} = 2.7$$

times as much. □

Checkpoint 2.4.8 A cheesecake recipe calls for 150 g of eggs and 750 g of cream cheese. If you currently have 200 g of eggs and 1058 g of cream cheese, determine which ingredient is the limiting one (will use all of it) and the amount of each you will use.

Limiting ingredient:

1. cream cheese

2. eggs

Eggs: ______ g Cream cheese: ______ g

2.4.3 Similar polygons

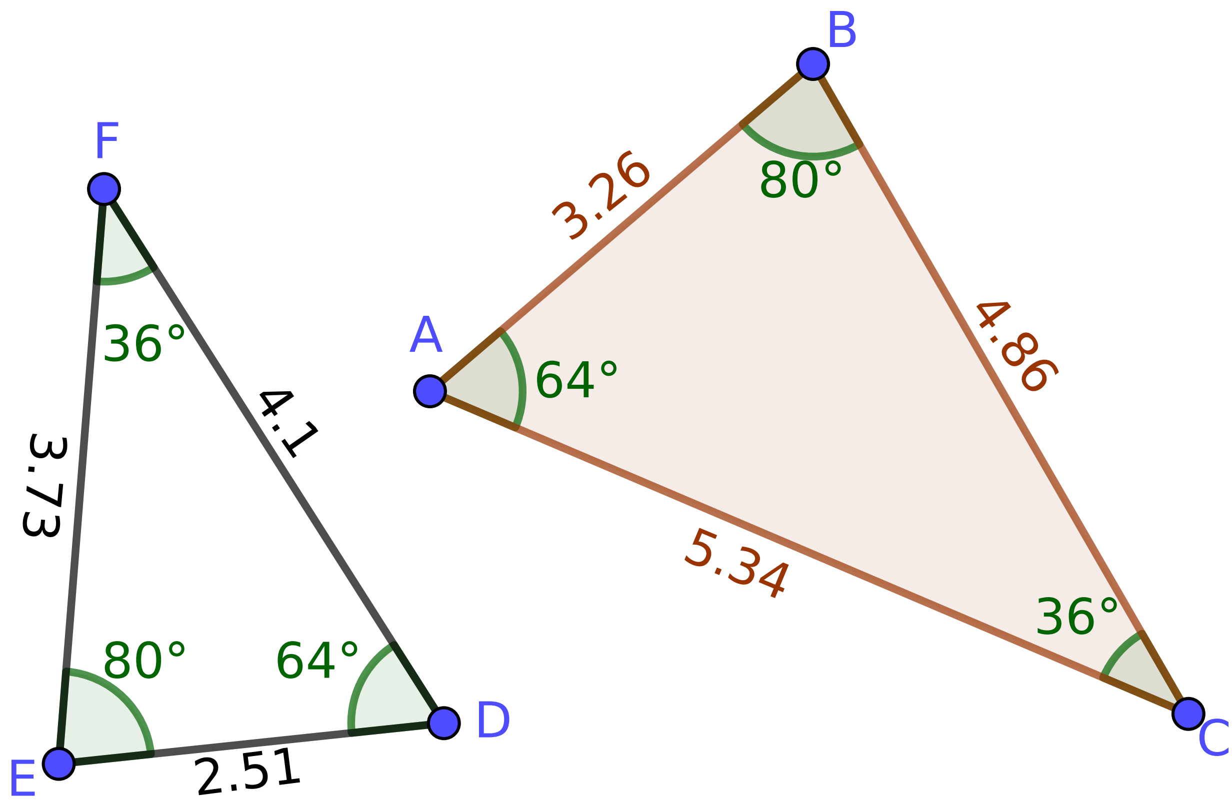
This section presents a geometric fact which is expressed as proportions. This geometry can be used to solve for distances or lengths in some circumstances. First we define and illustrate the geometric fact.

2.4.3.1 Explaining Similar Triangles

Definition 2.4.9 Similar Triangles. Two triangles are **similar** if and only if corresponding angles are the same. ◊

When triangles are similiar their corresponding side lengths are proportional. Two sides from the different triangles are **corresponding** if they are across from angles of the same measurement. This is illustrated in the following example.

Example 2.4.10



The triangles $\triangle ABC$ and $\triangle DEF$ are similar. Notice that the angles at A and D have the same measure (64°). The same is true of the angles at B and E (both 80°) and the angles at C and F (both 36°).

BC is the side opposite the angle at A and EF is the side opposite the angle at D. Because A and D have the same measure, the sides opposite them are corresponding sides.

Similarly CA is the side opposite the angle at B and FD is the side opposite the angle at E. Because B and E have the same measure, the sides opposite them are corresponding sides.

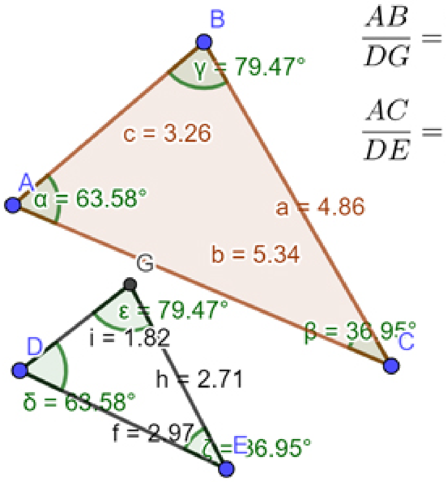
What is the third pair of corresponding sides?

As a result the following ratios of sides are the same

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}.$$

You can confirm this by dividing the lengths ($\frac{3.26}{2.51} = \frac{4.86}{3.73} = \frac{5.34}{4.1}$). □

From a single example we might think this was just a special case. To convince yourself use the following interactive example.



$$\frac{AB}{DG} = \frac{3.26}{1.82} = 1.8$$

$$\frac{AC}{DE} = \frac{5.34}{2.97} = 1.8$$

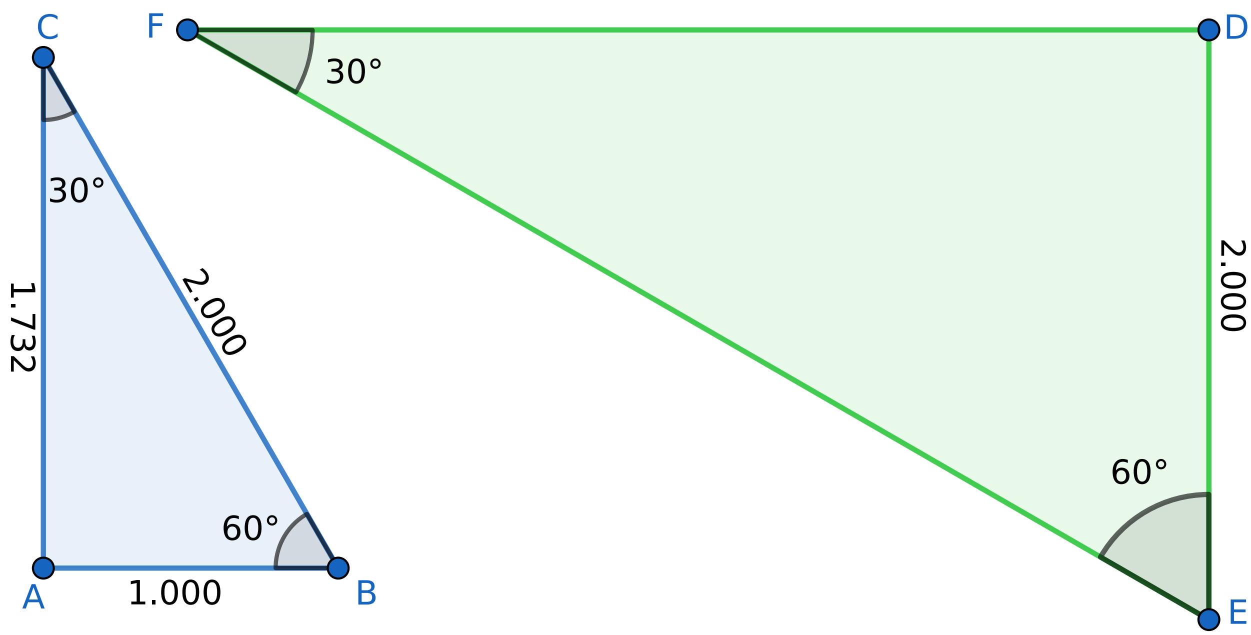
Standalone
Embed

Figure 2.4.11 Similar Triangles

2.4.3.2 Calculating Using Similar Triangles

We can use the proportionaliy of similar triangle sides lengths to calculate the lengths using the same technique as Example 2.4.4.

Example 2.4.12



Suppose triangle ABC has angles $90°, 60°, 30°$ with the lengths of the sides opposite them $2.000, 1.732, 1.000$. If triangle DEF also has angles $90°, 60°, 30°$ it is similar. Suppose the length of the side DE opposite the $30°$ angle at point F is 2.000.

First, we identify the corresponding sides. $\overline{AB}$ and $\overline{DE}$ are opposite $30°$ angles so they are corresponding. $\overline{CA}$ and $\overline{FD}$ are opposite $60°$ angles so they are corresponding. Finally, $\overline{BC}$ and $\overline{EF}$ are opposite $90°$ angles so they are corresponding. This means

$$\frac{\overline{AB}}{\overline{DE}} = \frac{\overline{CA}}{\overline{FD}} = \frac{\overline{BC}}{\overline{EF}}.$$

Because we know the ratio $\frac{\overline{AB}}{\overline{DE}}$, we can use the proportion to solve for the other two side lengths on triangle DEF. We invert the ratios for easier solving.

$$\begin{aligned}
\frac{\overline{DE}}{\overline{AB}} &= \frac{\overline{DF}}{\overline{CA}} \\
\frac{2.000}{1.000} &= \frac{\overline{DF}}{1.732}. \\
\frac{2.000}{1.000} \cdot 1.732 &= \frac{\overline{DF}}{1.732} \cdot 1.732. && \text{Multiply to isolate } \overline{DF} \\
3.464 &= \overline{DF}.
\end{aligned}$$

We can calculate the length of the third side in the same way.

$$\begin{aligned}
\frac{\overline{DE}}{\overline{AB}} &= \frac{\overline{EF}}{\overline{BC}}. \\
\frac{2.000}{1.000} &= \frac{\overline{EF}}{2.000}. \\
\frac{2.000}{1.000} \cdot 2.000 &= \frac{\overline{EF}}{2.000} \cdot 2.000. \\
4.000 &= \overline{EF}.
\end{aligned}$$

□

Checkpoint 2.4.13 Suppose triangle A has angles 60°, 80°, and 40° and the lengths of the sides opposite are 10, 11.37, and 7.42 respectively. If triangle B has the same angle measures and the side opposite the angle of measure 60° is length 51, what are the other two side lengths?

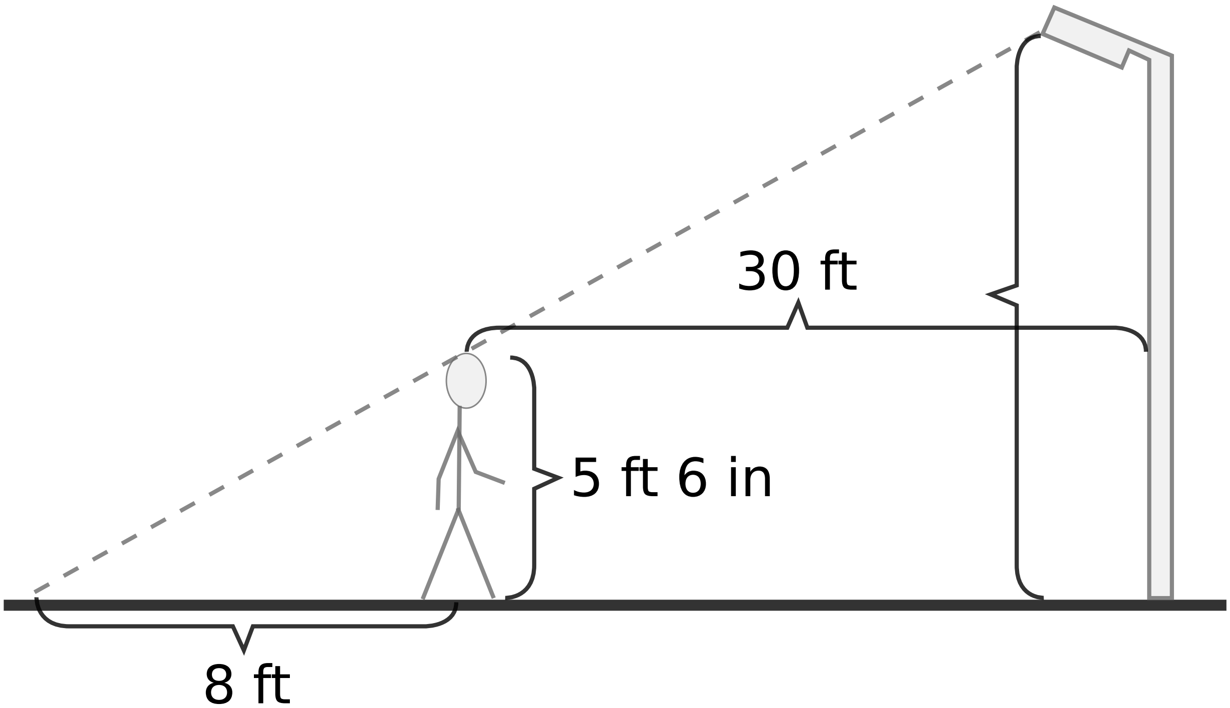
Length opposite angle of measure 80°: ______

Length opposite angle of measure 40°: ______

2.4.3.3 Similarity in Applications

Similar triangles can be found in a variety of circumstances. This example recognizes similar triangles in a context which has been used many time in history for indirect measurement.

Example 2.4.14



A person is standing 30 ft from a light pole. The shadow cast by the light is 8 ft long. If the person is 5 ft 6 in tall, how high is the point on the light that is casting the shadow?

In this image we have two (right) triangles that will be useful. The smaller one has legs of length 8 ft and 5 ft 6 in. The third side (hypotenuse) is the dashed gray line, but we will not need it. The other triangle has a leg that is the entire bottom (length 8 ft plus 30 ft). The other leg is the height of the light. The angles of the two triangles are the same. Because they are both right triangles (we are supposing the light post is straight up and the person is standing straight up), the angles at the persons feet and the base of the light are the same. They share the angle on the left (between dashed line and ground). The third angle must match because the first two do.

Because these have the same angles, they are similar, and we can use the proportionality of corresponding side lengths. Before we do, we will convert the height of the person to decimal. 5 ft 6 in is 5.5 ft.

$$\begin{aligned} \frac{5.5 \text{ ft}}{8 \text{ ft}} &= \frac{h}{38 \text{ ft}}. \\ \frac{5.5 \text{ ft}}{8 \text{ ft}} \cdot (38 \text{ ft}) &= \frac{h}{38 \text{ ft}} \cdot (38 \text{ ft}). \\ 26.125 \text{ ft} &= h. \end{aligned}$$

If these measurements were taken with a tape measure we can reasonably suppose they are accurate to the nearest inch. We can convert the 0.125 ft into inches.

$$0.125 \text{ ft} \cdot \frac{12 \text{ in}}{1 \text{ ft}} = 1.5 \text{ in}.$$

Thus the height to that point on the light is approximately 26 ft and 2 in.

We use this type of measurement because it is simpler: we can measure across the ground much more easily than we can climb the pole and measure its height. It was not important that the person chose to be 30 ft from the pole. If they chose to be 20 ft, then the shadow would also be shorter (proportional). If we are too close it will be hard to accurately measure the short shadow (ever try to measure a shadow?). ☐

2.4.3.4 Similarity Beyond Triangles

Shapes other than triangles can be similar. For example there are similar rectangles and similar pentagons. To be similar they must have the same number of sides, corresponding angles must be the same, and corresponding sides must be in the same ratio. Note that just having the same angles is insufficient: any two rectangles have all the same angles (right angles) but not every pair is similar.

Similar Polygons. One way to define similar polygons that avoids the trap illustrated by rectangles is to break the shape down into triangles (e.g., a rectangle can be split into two triangles). Then we require that every such triangle has the same angles. Note how we break a shape down into triangles is not unique. Take two, non-similar rectangles and find the triangles that don't match.

One place where similar shapes (beyond triangles) is used is scale drawing and scale models. If you ever built a model of a car or a plane or some such there was most likely a scale given. For example they may be 1/32 scale. This means that one inch on the model is 32 inches on the actual object.

2.4.4 Exercises

Exercise Group. Solve each of these proportions.

1. Find the unknown number in the proportion

$$\frac{x}{60} = \frac{2}{12}$$

____

2. Solve for the variable in $\dfrac{x}{7.8} = \dfrac{3.76}{9.4}$

$x =$__________

3. Find the unknown number in the proportion

$$\frac{6}{12} = \frac{2}{x}$$

____

Exercise Group. Identify a proportion in each application. Set it up, and solve for the requested value(s).

4. Cellular phone service that charges per-minute will charge $75 for 270 minutes. How much would 1033 minutes cost?

Round your answer to the nearest cent.

$_____

5. Ben goes to the grocery store at a rate of 4 times a week. How many times would he be expected to go to the grocery store in 8 weeks? Use x as the variable.

Table 2.4.15

Translate to a proportion:	____________________
$x =$___ times in 8 weeks	

6. A carpet store charges $341.00 to install 62 square yards of carpet. Assuming they charge the same rate per square yard regardless of the amount of carpet installed, how much would they charge to install 100 square yards of carpet? Use the variable x in setting up the proportion.

What is the unit price for installation per square yard of carpet?

Table 2.4.16

Translate to a proportion:	____________________
They would charge $___ to install 100 square yards of carpet.	

$___ per square yard

7. Gwen's Gravel Company supplied a homeowner with 8 cubic yards of gravel for his driveway at a cost of \$612.00. Assuming they charge the same rate per cubic yard regardless of the amount of gravel supplied, what would they charge for 32 cubic yards of gravel?

Table 2.4.17

Translate to a proportion:	____________________
\$_____ for 32 cubic yards of gravel	

8. If a 30-acre alfalfa field produces 90 tons of hay, how many acres would be needed for a field to produce 156 tons of hay?

Table 2.4.18

Translate to a proportion:	____________________
A field would need to be _____ acres to produce 156 tons of hay.	

9. A recipe for lemon tea cookies calls for $1\frac{1}{4}$ cups of flour for every $\frac{3}{4}$ cup of sugar. How many cups of sugar are needed if $1\frac{2}{3}$ cups of flour are used?

For $1\frac{2}{3}$ cups of flour you need __________ cups of sugar.

10. A label reads: "2.5 mL of solution for injection contains 1,000 mg of streptomycin sulfate." How many millilitres are needed to give 700 mg of streptomycin?

____

11. A floor plan has a 64 : 1 scale. On the drawing, one of the rooms measures $2\frac{3}{4}$" by $2\frac{7}{8}$". Show answers to the nearest .01 The actual dimensions would be: ____________________ feet by: ____________________ feet. The area of the room would be ____________________ square feet.

12. While planning a hiking trip, you examine a map of the trail you are going on hike. The scale on the map shows that 2 inches represents 5 miles.
If the trail measures 12 inches on the map, how long is the trail?
____________________ miles

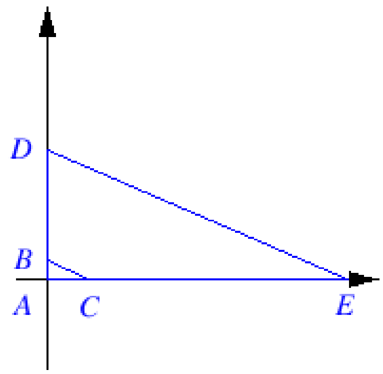
Exercise Group. Use the property of side ratios for similar triangles to find the values requested.

13. The side lengths of ΔABC are: $AB = 2$ $BC = 9$ $AC = 8$
The side lengths of ΔRST are: $RS = 10$ $ST = 45$ $RT = 16$
Simplify the given corresponding side ratios:
$\frac{RS}{AB} =$ _____ $\frac{ST}{BC} =$ _____ $\frac{RT}{AC} =$ _____
Is $\Delta ABC \sim \Delta RST$?

(a) No

(b) Yes

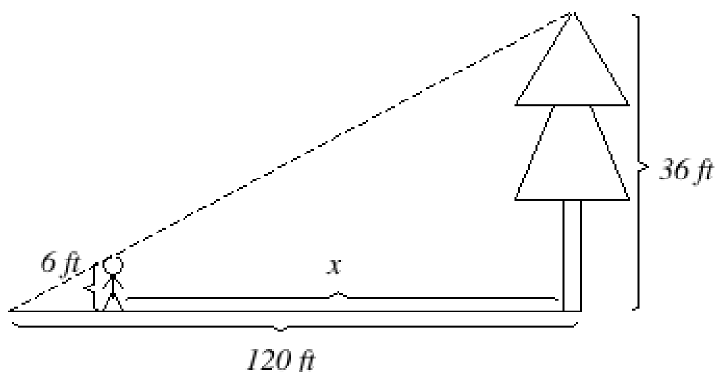
14.



Find the coordinates of point E so that $\Delta ABC \sim \Delta ADE$
$A = (0,0), B = (0,1), C = (7,0), D = (0,7)$
$E = (\_\_ ,\_\_)$

Exercise Group. Identify similar triangles in each application, then use the property of side ratios to find the requested value(s).

15. Suppose you are standing such that a 36-foot tree is directly between you and the sun. If you are 6 feet tall and the tree casts a 120-foot shadow, how far away from the tree can you stand and still be completely in the shadow of the tree?

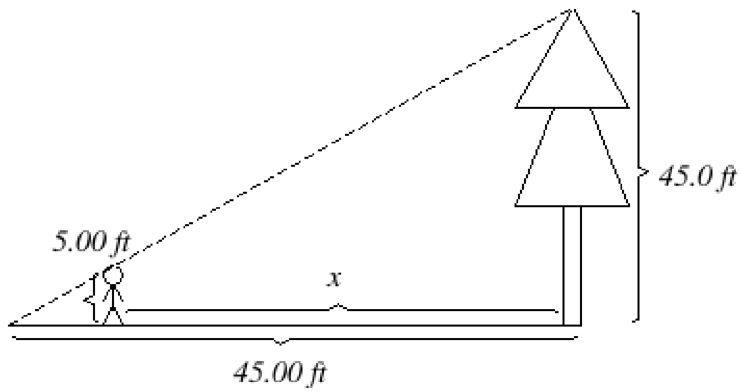


The distance between you and the tree is ________________ ft (If needed, round to 1 decimal place.)

16. A stick 1.0 meter long casts a shadow 0.70 meters long. A building casts a shadow 21.0 meters long. How tall is the building? Use the rules of working with significant figures to round.
__________________ meters

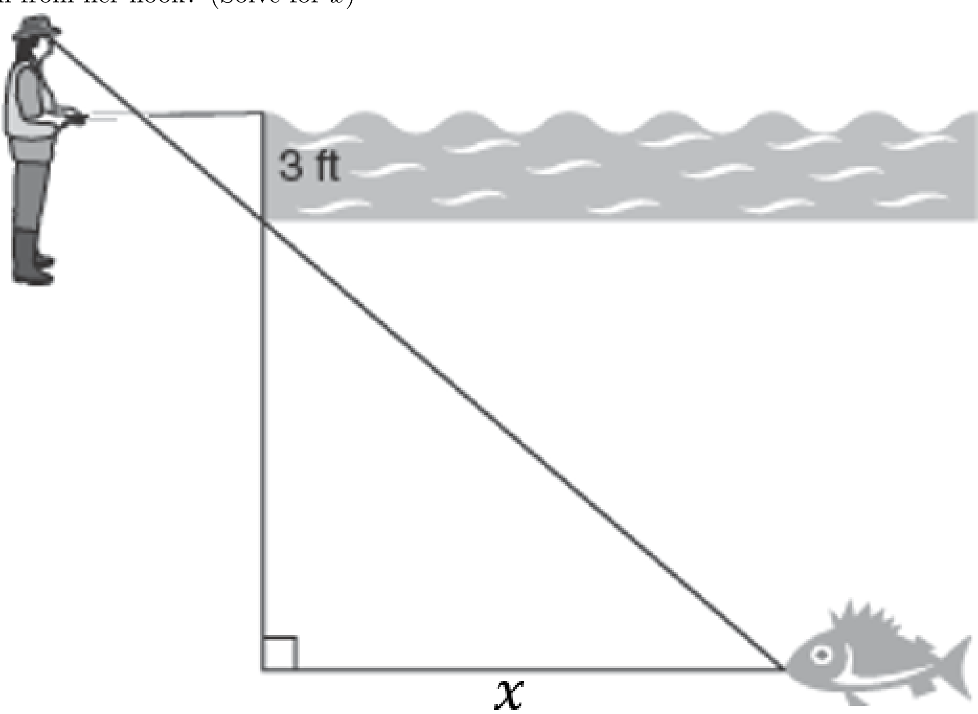
17. Suppose you are standing such that a 45.0-foot tree is directly between you and the sun. If you are 5.00 feet tall and the tree casts a 45.00-foot shadow, how far away from the tree can you stand

and still be completely in the shadow of the tree?



The distance between you and the tree is _____ ft (Use the rules of working with significant figures to round.)

18. Victoria holds a fishing pole with fishing line extended according to the picture below. How far is the fish from her hook? (Solve for x)



$5ft19.5ftx =$ _____

(a) ft

(b) in

(c) cm

(d) m

(e) °

2.5 Medical Ratios

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Setup and solve proportions (skill)

This section uses medical applications, primarily determine medicine dosages, to illustrate the use of ratios including change of units and proportions. In each example look for how the ratio is recognized and how the information provided is used to setup a calculation.

A common application of ratios in medicine is creating drugs of a desired strength. For example some drugs need to be administered based on the body weight of the patient. This requires the medical personnel to mix the drug they have on hand to the needed strength.

We will work the following three types of medical problems.

- Measure drug concentration
- Dilute a drug to a lower concentration
- Determine how much drug to use

2.5.1 Rounding in Medical Applications

Rounding for medical applications will be dictated by two factors. First, if an amount is more precise than human and/or machine can measure, then that precision is of no value. For example, if the syringe pumps are capable of measuring a hundredth of a milliliter, then we round to a hundredth. Second, a hospital or other medical provider may have a policy.

Just as with significant digits, rounding will occur solely at the end of a calculation. Intermediate rounding could change a number. Rounding at the end is a practical necessity as noted above.

2.5.2 Terminology

This section defines terminology used in medicine and sciences about solution concentrations that we need for the ratio examples in this section.

The active ingredient in a drug is often added to a inactive ingredient (often liquid) to administer it. This liquid is known as a **diluent**. The diluent might be water, saline solution, or other substances.

The substance (active ingredient for medicine) which may be a powder or another liquid to which the diluent is added is called the **solute**. The solute is disolved in the diluent. For example salt (solute) is disolved in water (diluent) to make saline solution.

Even if the drug can be administered directly (e.g., is already liquid) we sometimes need to dilute the **stock solution** (undiluted drug) for ease of use.

In some problems the drug mixture will be divided into parts. These parts are sometimes called **aliquots**. For example when testing substances (like blood samples) we may divide the sample drawn into multiple aliquots, one for each test to be run.

The most important concept is measuring how concentrated a solution is. This enables providing sufficient and safe amounts of drugs. There are three common ways concentration is written. These three are examples of how ratios can present the relationship between quantities in different ways. Being able to change between the different presentations of concentration will demonstrate your ability to understand and use ratios accurately.

Definition 2.5.1 Dilution Ratio. The **Dilution Ratio** is the ratio of solute (drug) to diluent.

If the solute is a liquid, then this is in units of volume per volume (e.g., mL per mL). For example a dilution ratio of 1:4 means 1 mL of drug to 4 mL of diluent giving 5 mL of solution.

This expression of concentration is unlikely to be used for dry solutes. ◊

Definition 2.5.2 Dilution Factor. The **Dilution Factor** is the ratio of solute (drug) to the resulting solution.

If the solute is liquid, then this is in units of volume per volume. For example a dilution factor of 5 means 1 unit of the drug in every 5 units of solution implying 4 units of diluent (1/4 dilution ratio).

If the solute is solid (e.g, powder) then this is in units of mass per volume. For example, 5 g of drug in a total of 100 mL of solution. Note we do not care how much diluent was added (hence we cannot calculate dilution ratio). This dilution factor can be achieved by putting in the dry ingredient, then adding part of the diluent to disolve the dry ingredient, then pouring in enough additional diluent to reach the desired volume. ◊

Definition 2.5.3 Percent Concentration. The **Percent Concentration** is the ratio of mass of solute (drug) to 100 mL of diluent.

If the solute is liquid, then this is in units of volume per volume. For example if the are 2 mL of drug per 100 mL of solution, then the percenet concentration is 2/100 or 2%.

If the solute is solid (e.g, powder) then this is in units of mass per volume. For example, if there are 2 mg of drug per 100 mL of solution, then the percent concentration is 2/100 or 2%. Note this is neither percent by volume nor percent by mass as would be used in science. ◊

These examples illustrate the meaning of these terms.

Example 2.5.4 A solution is produced from 3 mL of concentrated chloroform and 37 mL of water.

(a) What is the dilution ratio?

The dilution ratio is the ratio of the solute to the diluent. We are given both. The dilution ratio is 3/37.

(b) What is the dilution factor?

The dilution factor is the ratio of the total to the substance. The total is substance plus diluent. Here that is $3 + 37 = 4\bar{0}$. The ratio then is $\frac{4\bar{0}}{3} \approx 13$.

(c) Calculate the percent concentration.

The percent concentration is the ratio of the solute to the total solution by volume written as a percent. We are given the volume of the solute (3 mL) and have calculated the total volume in the previous task ($4\bar{0}$ mL). Thus the percent concentration is $\frac{3}{40} = 0.075$ which is 7.5%.

□

Example 2.5.5 Saline solution consists of the solute salt (sodium chloride) dissolved in the diluent pure water. The saline solution most commonly used in medical applications is 9 g of the solute salt, which is a solid, dissolved in enough water to make 1 liter of solution.

Because the solution is produced by adding enough water (amount not specified) it is easiest to express the concentration as a dilution factor. For medicine these are most commonly expressed in terms of milliliters, so we will convert units (metric terminology). The dilution factor is

$$\frac{9 \text{ g}}{1 \text{ L}} \cdot \frac{1 \text{ L}}{1000 \text{ mL}} = \frac{9 \text{ g}}{1000 \text{ mL}}.$$

We can also calculate the percent concentration. To do this we need to express volume in milliliters (percent concentration is ratio of solute to 100 mL). We can use unit conversion from above. The percent concentration is

$$\begin{aligned} \frac{9 \text{ g}}{1000 \text{ mL}} &= \\ \frac{9 \text{ g}}{1000 \text{ mL}} \cdot \frac{1/10}{1/10} &= \\ \frac{0.9 \text{ g}}{100 \text{ mL}} &= 0.009 \end{aligned}$$

or 0.9%.

Note how comparing the information provided to the definition showed us we needed to perform a unit conversion. The definitions also stated which number is numerator and denominator and what form to use for a final expression (e.g., percent or fraction). □

Example 2.5.6 Clorox® Disinfecting Bleach contains 7.0% sodium hypochlorite which is a liquid. This means the percent concentration is 7.0%. From this information we can calculate the dilution ratio and dilution factor.

One commercially available size of bleach contains 11 oz. To calculate the dilution ratio and dilution factor we need to know the amount of solute (sodium hypochlorite) in the 11 oz. Because it has a percent concentration of 7.0% there is $11 \cdot 0.070 = 0.77$ oz of sodium hypochlorite.

This is all we need for the dilution factor which is $\frac{0.77}{11}$. This would be easier to read if we express it with a denominator of one. We start by converting to a decimal. $\frac{0.77}{11} = 0.7$. Thus the dilution factor is $\frac{0.7}{1}$ expressed in ounces of bleach to ounces of water.

For the dilution ratio we need the amount of water added. Because the total solution is 11 oz and we know 0.77 oz is bleach, the water is $11 - 0.77 = 10.23$ oz. Thus the dilution ratio is $\frac{0.77}{10.23}$.

This ratio is hard to interpret. We could express it with the numerator as one to indicate how much diluent is added to one part solute. We can calculate this multiple ways. The first way is to scale the ratio so the numerator is one.

$$\begin{aligned}\frac{0.77}{10.23} &= \frac{0.77}{10.23} \cdot \frac{1/0.77}{1/0.77}\\ &= \frac{1}{10.23/0.77}\\ &\approx \frac{1}{13.29}.\end{aligned}$$

Another option is to write this as a proportion.

$$\begin{aligned}\frac{0.77}{10.23} &= \frac{1}{R}.\\ \frac{10.23}{0.77} &= \frac{R}{1}.\\ \frac{10.23}{0.77} &= R.\\ 13.28571429 &\approx R.\\ 13.29 &\approx R.\end{aligned}$$

The ratio $\frac{1}{13.29}$ means there is one ounce of bleach for every 13.29 ounces of water. □

Note a pure substance has dilution factor 1/1 (the total volume of the solution is just the volume of the solute). The percent concentration for a pure substance is 100%.

Standalone

Use these Checkpoints to test your ability to calculate these ratios.

2.5.3 Dilution

This section shows how to use knowledge of proportions to perform calculations required in medicine. Dilution ratios or factors tell us a desired ratio, and we know the initial ratio. This pair allows us to setup a proportion.

This first example shows how to produce a solution with a desired dilution factor.

Example 2.5.7 How much diluent do we need to add to produce a solution containing 3.0 mL of concentrated chloroform that will have a dilution factor of 50? Because this is in a medical context, we will round to one milliliter.

The dilution factor is the ratio of the total to the substance. We want that to equal 50, so we can write the proportion

$$\frac{\text{total}}{\text{solute}} = \frac{50}{1}.$$

We are not given the total volume, but the total is the volume of the solute plus the volume of the diluent. We do know the volume of solute (3.0 mL), and the volume of diluent is what we want to calculate. We can call the volume of diluent D. The volume of the solution is $3.0 + D$ where D is the volume of diluent to add.

Because we are starting with 3.0 mL of concentrated chloroform (no dilution) our proportion is

$$\begin{aligned}
\frac{3.0 + D}{3.0} &= \frac{50}{1.0}. \\
3.0 \cdot \frac{3.0 + D}{3.0} &= 3.0 \cdot \frac{50}{1.0}. \text{ Multiply to isolate } D \\
3.0 + D &= 150. \\
-3.0 + 3.0 + D &= -3.0 + 150. \text{ Subtract to isolate } D \\
D &= 147.
\end{aligned}$$

So we need 147 mL of diluent. Notice once we had the proportion set up we needed only algebra. □

This example shows us how to apply a dilution ratio (dilute our solution). We can calculate the resulting dilution factor afterward.

Example 2.5.8 A doctor orders 120 mL of 50% solution of Ensure every two hours. How much Ensure (liquid) and water is needed?

50% is a percent concentration. This means the Ensure should be 50% of the total volume (120 mL). $120 \cdot 0.50 = 60$ mL of Ensure. This leaves $120 - 60 = 60$ mL of water (diluent).

The dilution ratio is 1/1, because there is the same volume of solute (Ensure) and diluent (water). The dilution factor is 1/2, because we have 60 mL of Ensure in 120 mL of solution ($60/120 = 1/2$). □

Working on dilutions is a proportion problem. This next example presents a scenario where we work a dilution problem backwards. Notice that the setup is still a proportion, and the solving is still just the algebra steps needed.

Example 2.5.9 One usage of dilution is to reduce the concentration so that instruments can accurately measure it. Consider trying to measure an acid without dissolving the tools used to measure it.

A sample of a suspected high blood glucose value was obtained. According to the manufacturer of the instrument used to read blood glucose values, the highest glucose result which can be obtained on this particular instrument is 500 mg/dL. When the sample was run, the machine gave an error message (concentration too high).

The serum was then diluted to 1/10 and retested. The machine gave a result of 70 mg/dL. What was the initial concentration?

Note that the ratio is milligrams to decilitres (weight to volume). In these types of problems the amount of substance is so small that it does not affect the volume.

Before we jump into an equation, let's try an experiment. That's right, in math we do not have to know what to do when we start. We will try something, learn from it, and maybe revise our approach based on what we learned.

This is a dilution problem which means we can setup the proportion

$$\frac{\text{solute mg}}{\text{diluent dL}} = \frac{70 \text{ mg}}{\text{dL}}.$$

We are trying to find the amount of blood sugar in the sample, so the solute portion is unknown. We also do not have the size of sample taken. We will experiment to see how this affects the problem.

Suppose we take 1 dL of the original serum. Because the blood sample is so small, we can calculate as if all the volume is the diluent. That is we started with 1 dL and added more to dilute. To dilute to a ratio of 1/10 we need to add $10-1=9$ dL of diluent. No blood glucose was added thus the concentration is changed only by the diluent. Thus the concentration proportion is now

$$\begin{aligned}
\frac{C+0 \text{ mg}}{1+9 \text{ dL}} &= \frac{70 \text{ mg}}{\text{dL}} \\
\frac{C \text{ mg}}{10 \text{ dL}} &= \frac{70 \text{ mg}}{\text{dL}} \\
C \text{ mg} &= \frac{70 \text{ mg}}{\text{dL}}(10 \text{ dL}) \\
C &= 700 \text{ mg}.
\end{aligned}$$

Did this result depend on our selecting 1 dL of the original serum? If we are uncertain we can try the problem again and select 2 dL of the original serum. To figure out the total amount of which 2 is 1/10, we can treat this like Example 2.3.7

$$\begin{aligned}
\frac{1}{10} &= \frac{1}{10}\frac{2}{2} \\
&= \frac{2}{20}.
\end{aligned}$$

This means we need $20-2=18$ dL of diluent to have the desired dilution ratio. Also we will have twice as much of the blood glucose.

$$\begin{aligned}
\frac{2C+0 \text{ mg}}{2+18 \text{ dL}} &= \frac{70 \text{ mg}}{\text{dL}} \\
\frac{2C \text{ mg}}{20 \text{ dL}} &= \frac{70 \text{ mg}}{\text{dL}} \\
2C \text{ mg} &= \frac{70 \text{ mg}}{\text{dL}}(20 \text{ dL}) \\
2C \text{ mg} &= 1400 \text{ mg} \\
\frac{2C \text{ mg}}{2} &= \frac{1400 \text{ mg}}{2}
\end{aligned}$$

$$C = 700 \text{ mg}$$

The result is the same. This makes sense, because we are setting up a proportion, and ratios do not depend on the amount.

We can be confident that the original serum sample had a blood glucose level of 700 mg/dL. □

Sometimes we dilute more than one time. Here we experiment to determine what the effect of **serial dilution** is upon the resulting dilution factor.

Example 2.5.10 Suppose you have a solution consisting of 10 mL of acyl chloride and 90 mL of water. If this is diluted to a dilution ratio of 1/2 and then diluted again to a dilution ratio of 1/3, what is the final dilution ratio?

Solution. We can do the calculations one at a time. First we calculate the original concentration.

$$\begin{aligned}\frac{10 \text{ mL}}{10+90 \text{ mL}} &= \frac{10}{100}\\ &= \frac{1}{10}.\end{aligned}$$

To dilute to a ratio of 1/2 we can calculate the amount of diluent to add as a proportion problem like in Example 2.4.1.

$$\begin{aligned}\frac{1}{2} &= \frac{100 \text{ mL}}{T \text{ mL}}\\ \frac{2}{1} &= \frac{T \text{ mL}}{100 \text{ mL}}\\ 200 \text{ mL} &= T.\end{aligned}$$

The *total* will be 200 mL so we need to add 200 mL − 100 mL = 100 mL of additional diluent. Note at this point the concentration is

$$\frac{10 \text{ mL acyl chloride}}{200 \text{ mL diluent}} = \frac{1}{20}.$$

To dilute again to a ratio of 1/3 we can calculate the amount of diluent to add

$$\begin{aligned}\frac{1}{3} &= \frac{200 \text{ mL}}{T \text{ mL}}\\ 1\cdot(T \text{ mL}) &= 3\cdot(200 \text{ mL})\\ T &= 600 \text{ mL}.\end{aligned}$$

The total will be 600 mL so we need to add 600 mL − 200 mL = 400 mL of additional diluent. Note at this point the concentration is

$$\frac{10 \text{ mL acyl chloride}}{600 \text{ mL diluent}} = \frac{1}{60}.$$

Now we can determine what the resulting dilution ratio after diluting twice (1/2 and then 1/3).

$$\begin{aligned}\frac{1}{10}\cdot F &= \frac{1}{60}\\ 10\cdot\frac{1}{10}\cdot F &= 10\cdot\frac{1}{60}.\\ F &= \frac{1}{6}.\end{aligned}$$

Notice that $\frac{1}{6} = \frac{1}{2}\cdot\frac{1}{3}$, that is, the resulting dilution factor is the product of the serial dilutions. This relationship is always true for serial dilution. □

2.5.4 Dosage

If we know the concentration of a drug, we can determine how much is needed for a given dose. These are proportion problems that require change of units.

In medicine some substances are measured in **International Unit** or IU. For each substance this is defined by the effect of that amount of the drug.

Example 2.5.11 One IU of insulin is 0.0347 mg. A common concentration of insulin is U-100 which is 100 IU/mL. This is produced by combining 100 units of insulin in one mL of diluent.

If a person needs 2 units of insulin, how many mL of solution will that be?

This can be solved as a proportion because are asked for an amount that matches a ratio (concentration) which is 100 IU/mL. Because we are solving for a number of mL, we will write the proportion as

$$\begin{aligned}
\frac{\text{mL}}{100\text{ IU}} &= \frac{v\text{ mL}}{2\text{ IU}} \\
\frac{\text{mL}}{100\text{ IU}} \cdot (2\text{ IU}) &= \frac{v\text{ mL}}{2\text{ IU}} \cdot (2\text{ IU}) \\
\frac{2\text{ mL}}{100} &= v\text{ mL} \\
\frac{1}{50}\text{ mL} &= v. \\
0.02\text{ mL} &= v.
\end{aligned}$$

Because we have a ratio of desired amount to provided amount we can also solve this problem as a percent.

The ratio is desired amount per provided amount. In this case $\frac{2\text{ IU}}{100\text{ IU}} = \frac{1}{50} = 0.02$ which is 2%. We therefore want 2% of the 1 mL (from 100 IU/1 mL) or 0.02 mL. □

Example 2.5.12 A label reads "2.5 mL of solution for injection contains 1000 mg of streptomycin sulfate." How many milliliters are needed to contain 800 mg of streptomycin?

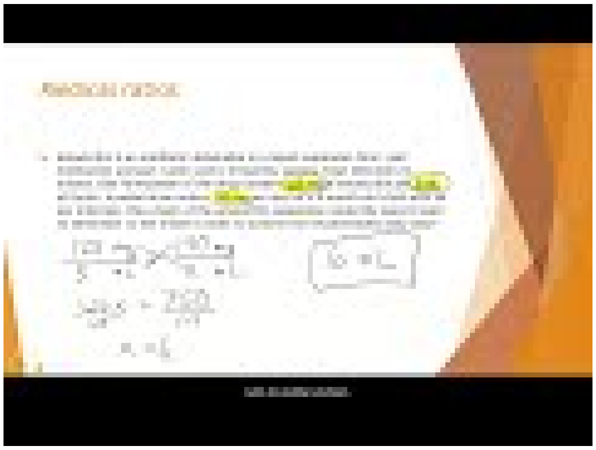
Solution 1. Because a ratio is given (1000 mg/2.5 mL) and we want to scale this down (to 800 mg), we can set this up as a proportion.

We want to solve for volume (mL), so we set up the proportion as follows.

$$\begin{aligned}
\frac{2.5\text{ mL}}{1000\text{ mg}} &= \frac{v\text{ mL}}{800\text{ mg}}. \\
\frac{2.5\text{ mL}}{1000\text{ mg}} \cdot (800\text{ mg}) &= \frac{v\text{ mL}}{800\text{ mg}} \cdot (800\text{ mg}). \\
2\text{ mL} &= v.
\end{aligned}$$

Solution 2. Because we have a ratio of desired amount to provided amount we can also solve this problem as a percent.

The ratio is desired amount/provided amount. In this case $\frac{800\text{ IU}}{1000\text{ IU}} = \frac{4}{5} = 0.8$ which is 80%. We therefore want 80% of the 2.5 mL dose or $0.8 \cdot 2.5 = 2$ mL. □



Standalone

Checkpoint 2.5.13 A physician ordered Lexapro (escitalopram oxalate) 10 mg. Lexapro (escitalopram oxalate) has a concentration of 5 mg per 5 milliliters.

What volume should be administered? ____ mL

A physician may prescribe a medicine and specify a total amount and a speed at which it should be delivered. For IV's this is called **drop factor** and is specified as a number of drops per minute. Medical personnel calculate how long to operate the IV so that the total amount of drug prescribed is delivered in the specified time.

Example 2.5.14 Give 1500 mL of saline solution IV with a drop factor of 10 drops per mL at a rate of 50 drops per minute to an adult patient. Determine how long in hours the IV should be administered.

The rate is specified in drops and the amount is specified in mL which means we need to convert units. This will be done like Example 1.1.20. The units suggest we should multiply the conversion ratios as follows.

$$\frac{50 \text{ drops}}{\text{minute}} \cdot \frac{\text{mL}}{10 \text{ drops}} = \frac{5 \text{ mL}}{\text{minute}}$$

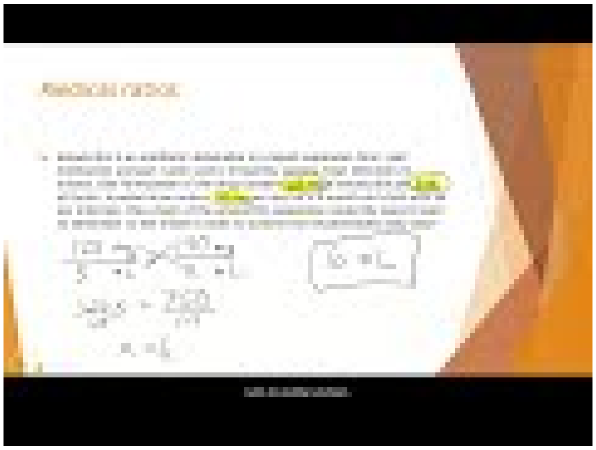
Now that we know the rate in mL, we can set up a proportion so that time calculated per total grams of medication matches the specified rate. Notice how we invert the rate to make the algebra easier.

$$\begin{aligned} \frac{T \text{ min}}{1500 \text{ mL}} &= \frac{1 \text{ min}}{5 \text{ mL}}. \\ \frac{T \text{ min}}{1500 \text{ mL}} \cdot (1500 \text{ mL}) &= \frac{1 \text{ min}}{5 \text{ mL}} \cdot (1500 \text{ mL}). \\ T &= 300 \text{ min}. \end{aligned}$$

The final step is to convert minutes to hours. This is another unit conversion problem using a conversion from Table 1.1.2. The units suggest we can multiply the 300 minutes by the conversion ratio.

$$300 \text{ minutes} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} = 5 \text{ hours}$$

□



Standalone

Example 2.5.15 Amoxicillin is an antibiotic obtainable in a liquid suspension form, part medication and part water, and is frequently used to treat infections in infants. One formulation of the drug contains 125 mg of amoxicillin per 5 mL of liquid. A pediatrician orders 150 mg per day for a 4-month-old child with an ear infection. How much of the amoxicillin suspension would the parent need to administer to the infant in order to achieve the recommended daily dose?

Solution. Here we need to scale the amount (from 125 mg to 150 mg). This is a proportion problem, that is, the ratio of medicine to volume is the same so we can setup an equation based on the drug concentration.

$$\begin{aligned} \frac{125 \text{ mg}}{5 \text{ mL}} &= \frac{150 \text{ mg}}{A \text{ mL}} \\ \frac{5 \text{ mL}}{125 \text{ mg}} &= \frac{A \text{ mL}}{150 \text{ mg}} \\ \frac{5 \text{ mL}}{125 \text{ mg}} \cdot (150 \text{ mg}) &= A \text{ mL} \\ 6 \text{ mL} &= A. \end{aligned}$$

□

Checkpoint 2.5.16 A 5% dextrose solution (D5W) contains 5 g of pure dextrose per 100 mL of solution. A doctor orders 500 mL of D5W IV for a patient. How much dextrose does the patient receive from that IV?

Example 2.5.17 A sample of chloroform water has a dilution factor of 40. If 2 mL of chloroform are needed how many milliliters total are needed?

Solution. A dilution factor of 40 indicates that 1 mL of chloroform is in 40 mL total of solution. We can setup a proportion to answer this.

$$\begin{aligned} \frac{1 \text{ mL}}{40 \text{ mL}} &= \frac{2 \text{ mL}}{T \text{ mL}} \\ \frac{40 \text{ mL}}{1 \text{ mL}} &= \frac{T \text{ mL}}{2 \text{ mL}} \\ \frac{40 \text{ mL}}{1 \text{ mL}} \cdot 2 \text{ mL} &= T \text{ mL} \\ 80 \text{ mL}. \end{aligned}$$

□

2.5.5 Exercises

1. **Medical Ratio.** A 1 litre (1,000 mL) IV bag of dextrose solution contains 70 g of dextrose. Find the ratio of grams per millilitre of dextrose. (Enter your answer in fraction form.) ____________

2. **Medical Ratio.** Find the flow rate (in drops/min) for the given IV (assume a drop factor of 15 drops/mL).

1300 mL in 6.0 h

____________ drops/min

3. **Medical Ratio.** Find the length of time (in h) the IV should be administered (assume a drop factor of 15 drops/mL).

1,000 mL at a rate of 40 drops/min

____________ h

4. **Medical Proportion.** A label reads: "2.5 mL of solution for injection contains 1,000 mg of streptomycin sulfate." How many millilitres are needed to give 700 mg of streptomycin?

____

5. **Medicine to Solution.** Quinidine gluconate is a liquid mixture, part medicine and part water, which is administered intravenously. There are 110.0 mg of quinidine gluconate in each cubic centimeter (cc) of the liquid mixture. Dr. Alverez orders 506 mg of quinidine gluconate to be administered daily to a patient with malaria.

How much of the solution would have to be administered to achieve the recommended daily dosage?

____________ cc

6. **Medical Ratio with Rounding.** Albuterol is a medicine used for treating asthma. It comes in an inhaler that contains 16 mg of albuterol mixed with a liquid. One actuation (inhalation) from the mouthpieces delivers a 90 μg dose of albuterol. (Reminder: 1 mg = 1000 μg .)

a.) Dr. Olson orders 2 inhalations 4 times per day. How many micrograms of albuterol does the patient inhale per day?

____________ μg

b.) How many actuations are contained in one inhaler?

____________ actuations

c.) Alicia is going away for 5 months and wants to take enough albuterol to last for that time. Her physician has prescribed 2 inhalations 4 times per day. How many inhalers will Alicia need to take with her for the 5 period? Assume 30-day months.

Hint: she can't bring a fraction of an inhaler, and she does not want to run out of medicine while away.

____

7. **Concentration.** Amoxicillin is a common antibiotic prescribed for children. It is a liquid suspension composed of part amoxicillin and part water.

In one formulation there are 200 mg of amoxicillin in 6 cubic centimeters (cc's) of the liquid suspension. Dr. Scarlotti prescribes 400 mg per day for a 2-yr old child with an ear infection.

How much of the amoxicillin liquid suspension would the child's parent need to administer in order to achieve the recommended daily dosage?

____

8. **Concentration.** Diphenhydramine HCL is an antihistamine available in liquid form, part medication and part water. One formulation contains 20 mg of medication in 4 mL of liquid. An allergist orders 40-mg doses for a high school student. How many milliliters should be in each dose?

____________ mL

9. **Concentration.** How many mL of sodium hydroxide are required to prepare 550 mL of a 15.5% solution? Assume the sodium hydroxide dissolves in the solution and does not contribute to the overall volume.

____________ mL

10. Dilution Ratio. You are asked to make a 1/4 dilution using 7 mL of serum. How much diluent do you need to use?

____________________ mL

11. Dilution Ratio. A clinical lab technician determines that a minimum of 65 mL of working reagent is needed for a procedure. To prepare a $\frac{1}{11}$ dilution ratio of the reagent from a stock solution, one should measure 65 mL of the reagent and ____________________ mL of the diluent.

12. Dilution Ratio. A patient's glucose result is suspected to be outside the range of the analyzer, so the techs decide to dilute the sample before running it. 25 microliters of serum is added to 75 microliters of diluent and the diluted sample is analyzed. The analyzer reads that the glucose value of the diluted sample is 40 $\frac{mg}{dL}$.

What was the ratio the sample was diluted to?

____

What is the glucose value of the original sample?

____________________ $\frac{mg}{dL}$

13. Serial Dilution. A thyroid peroxidase antibody test was performed on a 45 year old man. The dilution sequence was 50 μL serum added to 350 μL of diluent in tube 1. Then 40 μL from tube 1 was added to 80 μL of diluent in tube 2. Finally 45 μL from tube 2 was added to 360 μL of diluent in tube 3.

All dilution ratios should be given as fractions.

a.) What is the dilution ratio in tube 1?

____

b.) What is the dilution ratio in tube 2?

____

c.) What is the dilution ratio in tube 3?

____

d.) What is the overall (serial) dilution ratio?

____

2.6 Project: False Position

Project 2 Method of False Position. In this project, we are going to learn about an ancient algebraic technique that is built around correcting guesses. We may gain greater appreciation for the value of *wrong* guesses and what we can gain from them.

(a) Solve the following equation any way you would like.

$$x\left(1+\frac{1}{3}+\frac{1}{4}\right)=14.$$

Check your answer using technology.

(b) Notice that 12 is the least common multiple of 3 and 4: the denominators. Distribute 12 in the following expression.

$$12\left(1+\frac{1}{3}+\frac{1}{4}\right).$$

Is this bigger, equal to, or smaller than 14?

(c) We multiplied by a convenient number, which is not quite right. Because it is multiplication we can scale (multiply) our not quite right guess to make it right. Consider

$$y\cdot 12\left(1+\frac{1}{3}+\frac{1}{4}\right)=14.$$

Replace 12 times the sum with your result from the previous step.

Solve the resulting equation for y.

(d) Note that y is the correction to our guess of 12. Calculate $y \cdot 12$.

This will match your original solution. If not, check your calculations.

(e) This method is called *false position* because it guesses a convenient number which is typically false then corrects it. One of the original motivations for this method was the lack of a useful notation for fractions (it dates to the Sumerians and ancient Egyptians).

Many people today use similar methods when dealing with fractions. What is a reason people might distribute a convenient number before doing the solving?

2.7 Project: Arclength Estimation

Project 3 Estimating Arc Lengths. In aviation it is sometimes useful to estimate a distance between points as the length of a circular arc. This results from navigation methods (search for VOR and DME arc if curious). To estimate on the fly they use what is known as the 60:1:1 approximation. It means that 60 miles from a point a one degree arc is approximately one mile in length. Note in aviation the distances would be in nautical miles (nm), but the ratio does not change if we use statute miles (the usual type).

Here we will practice using the method to approximate then check why it works.

(a) *Using the Ratio.*

- **(i)** View Example 2.7.1 to Example 2.7.3.
- **(ii)** What is the arclength of 2 degrees at a distance of 30 miles?
- **(iii)** What is the arclength of 5 degrees at a distance of 30 miles?
- **(iv)** What is the arclength of 10 degrees at a distance of 20 miles?

(b) *Explaining the Ratio.*

- **(i)** Calculate the perimeter of a circle with radius 60 miles using the formula $P = 2\pi r$ where P is the perimeter and r is the radius.
- **(ii)** Calculate the perimeter of a semi-circle (half circle) with radius 60 miles.
- **(iii)** Calculate the perimeter of a quarter of a circle with radius 60 miles.
- **(iv)** Calculate the perimeter of 1/360 of a circle with radius 60 miles.
- **(v)** Note that the previous task is the 60:1:1 ratio (1 degree is 1/360th of a circle). Does your result match (i.e., is the result approximately 1 mile)?

Example 2.7.1 Calculate the arclength of 3 degrees at 60 miles.

Solution. If each degree is one mile then 3 degrees is $A = 3 \cdot 1 = 3$ miles. □

Example 2.7.2 Calculate the arclength of 1 degree at 30 miles.

Solution. At 30 miles we are only half way $(30/60 = 1/2)$, so the length is $A = \frac{1}{2} \cdot 1 = \frac{1}{2}$ miles. □

Example 2.7.3 Calculate the arclength of 4 degree at 18 miles.

Solution. The radius is $(18/60 = 3/10)$ of the usual. Thus each degree is $\frac{3}{10}$ of a mile. This arc is $4°$ so the length is $A = \frac{3}{10} \cdot 4 = 1.2$ miles. □

Chapter 3

Models

3.1 Linear Expressions

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Solve linear, rational, quadratic, and exponential equations and formulas (skill)
- Read and interpret models (critical thinking)
- Use models including linear, quadratic, exponential/logarithmic, and trigonometric (skill)

Section 1.3 presented models in general. This section presents linear models. First, we look at some examples and learn how the pieces of a linear model work. Next, we learn to write linear models given a description of a problem. After that we practice solving for different parts of a linear equation. Section 3.3 will introduce a more in depth look at identifying linear models.

3.1.1 Linear Models

This section presents examples of linear models and provides an explanation for the two parts of a linear model.

A linear model (equation) can be written in the following, equivalent forms.

- $y = mx + b$
- $Ay + Bx + C = 0$

The second form can be solved for y which will make it look like the first form. The models in this section will be in the first form which you may know as slope intercept. The m is commonly described as the slope which is a measure of how steep the line is. In these models we will generalize that to a rate. The b is commonly called the intercept which indicates a shift of the line up or down. This shift will have a meaning in each application.

Model of Temperature Change with Altitude. As a result of atmospheric physics temperature decreases as the distance above the ground increases. For lower altitudes this can be modeled as

$$T_A = T_G - \left(\frac{3.5}{1000}\right) A.$$

- T_A is the expected temperature at the specified altitude.
- T_G is the temperature at ground level.
- A is the specified altitude in number of feet above ground level.
- $\frac{3.5^\circ}{1000\text{ ft}}$ is the rate of temperature decrease.

All temperatures are in Fahrenheit.

Before we can use this model we need to know the parameter T_G. A parameter is not a variable, rather it is a value (number) that we obtain from the circumstances and write into the model (equation) before we do any work.

In contrast the ratio $-\frac{3.5^\circ}{1000\text{ ft}}$ is a constant (not a parameter), because it is a result of atmospheric physics that is not dependent on the location for this simplified model.

Temperature (T_A) and altitude (A) are variables which implies that the model shows a relationship between these two properties.

The model subtracting from the starting temperature results in a decrease of temperature from T_G. This implies that temperature decreases with altitude.

Every linear model (equation) has a rate. In this case $m = -\frac{3.5}{1000}$.

Every linear model has a shift. In this case $b = T_G$.

Example 3.1.1 If the temperature at ground level is 43° what is the temperature 1000 ft above ground level (AGL)? 2000 ft AGL, 3000 ft AGL, 3500 ft AGL?

Because fractions of a degree are not useful in making decisions like what to wear, we will round to units.

Note $T_G = 43^\circ$. We need to calculate T_A for $A = 1000, 2000, 3000, 3500$.

$$\begin{aligned}
T_{1000} &= 43^\circ - \frac{3.5^\circ}{1000\text{ ft}}(1000\text{ ft}) \\
&= 39.5 \\
&= 43^\circ - 3.5^\circ \\
&= 39.5 \\
&\approx 40. \\
T_{2000} &= 43^\circ - \frac{3.5^\circ}{1000\text{ ft}}(2000\text{ ft}) \\
&= 43^\circ - 7.0^\circ \\
&= 36. \\
T_{3000} &= 43^\circ - \frac{3.5^\circ}{1000\text{ ft}}(3000\text{ ft}) \\
&= 43^\circ - 10.5^\circ \\
&= 32.5 \\
&\approx 33. \\
T_{3500} &= 43^\circ - \frac{3.5^\circ}{1000\text{ ft}}(3500\text{ ft}) \\
&= 43^\circ - 12.25^\circ \\
&= 30.75 \\
&\approx 31.
\end{aligned}$$

Notice that we now know that it will be below freezing just above 3000 ft. □

Model of Time to Altitude. A fixed wing aircraft flown optimally climbs from a starting altitude at a fixed climb rate.

$$A_t = A_G + C \cdot t.$$

- A_t is the altitude after t minutes.
- A_G is the starting altitude (likely ground level) in feet mean sea level (MSL).
- C is the rate of climb in feet per minute.
- t is the time since the climb began in minutes.

Before we can use this model we need to know the parameters A_G and C. A parameter is not a variable, rather it is a value (number) that we obtain from the circumstances and write into the model (equation) before we do any work. A_G varies by airport, because they are at different altitudes. The rate C must be obtained for each plane and is often available in the aircraft's Pilot's Operating Handbook (POH).

Final altitude (A_t) and time (t) are the variables which implies that the model shows a relationship between time climbing and how high the plane is.

In this model everything is added which matches the increase of elevation over time (adding makes the altitude bigger).

Every linear model (equation) has a rate (the slope m in the equation). In this case $m = \frac{C \text{ ft}}{1 \text{ min}}$. Every linear model has a shift (the b in the equation), which may be zero. In this case $b = A_G$. $b = A_G = 0$ is possible because an aircraft can take off from sea level (e.g., float planes). This shift makes sense, because the climb starts at the altitude of the ground: the plane was already shifted up by being at that airport.

Example 3.1.2 If a plane begins at 160 ft MSL and is climbing at 700 ft/min, how high will it be after 5 minutes? 10 minutes? 15 minutes?

These calculations are made as part of safety planning. The data is sufficiently accurate that rounding is not necessary. Rather we make conservative estimates of the parameters, so that there is always a safety buffer. In this case a conservative estimate for A_G is to round down: this will give us a lower altitude. If that lower altitude is safe, then altitude 5 feet higher will be safe as well. For the climb rate a conservative estimate is to round down as well. If we can reach an altitude climbing at 700 ft/min, then if we climb at 720 ft/min we will reach that safe altitude a little sooner.

Note

$$A_t = 160 \text{ ft} + \frac{700 \text{ ft}}{\text{min}} \cdot t.$$

The expected altitudes are

$$\begin{aligned}
A_5 &= 160 \text{ ft} + \frac{700 \text{ ft}}{\text{min}} \cdot 5 \text{ min} \\
&= 160 \text{ ft} + 3500 \text{ ft} \\
&= 3660. \\
A_{10} &= 160 \text{ ft} + \frac{700 \text{ ft}}{\text{min}} \cdot 10 \text{ min} \\
&= 160 \text{ ft} + 7000 \text{ ft} \\
&= 7160. \\
A_{15} &= 160 \text{ ft} + \frac{700 \text{ ft}}{\text{min}} \cdot 15 \text{ min} \\
&= 160 \text{ ft} + 10500 \text{ ft} \\
&= 10660.
\end{aligned}$$

Based on these calculations the plane will climb above 4500 ft MSL in between 5 and 10 minutes (closer to 5). □

Model of Fuel Remaining Calculation. When operated at a fixed power setting a vehicle burns the same amount of gas per hour (or other time unit). This leads to the linear model

$$F_t = F_I - r \cdot t.$$

- F_t is the amount of fuel remaining after t minutes.
- F_I is the amount of fuel at the beginning.
- r is the rate (volume per time) at which fuel is being consumed.
- t is the time the vehicle has been operated.

Fuel amounts will be measured in units of volume like gallons or liters. Time will be measured in minutes or hours. The rate r is then in units such as gallons/hour or liters/min.

Before we can use this model we need to know the parameters F_I and r. These parameters are not variables (they remains the same the whole time the model is in use), rather they are values (numbers) that we obtain from the circumstances and write into the model (equation) before we do any work.

The initial fuel F_I is obtained by checking the fuel tanks or fuel gauges. The rate r is often is not shown during operation (fuel gauges show how much is remaining rather than how fast it is used). The rate can sometimes be obtained from vehicle documentation.

Final fuel (F_t) and time (t) are the variables which implies that the model shows a relationship between time flown and fuel available (left in the tanks).

Because fuel decreases the $r \cdot t$ term is subtracted decreasing the amount from F_I.

Every linear model (equation) has a rate. In this case $m = \frac{r \text{ gal}}{1 \text{ hr}}$ (or similar units).

Every linear model has a shift. In this case $b = F_I$. We can think of this shift in terms of the needle on the gas gauge moving up to represent the amount of fuel present.

Example 3.1.3 If a car begins with 20 gallons of fuel and burns 1.55 gallons per hour, how much fuel will it have after 1 hour, 2 hours, 3 hours, 36 minutes?

A gallon is a large amount so we will maintain one decimal place precision. For safety we should always assume a larger fuel burn, so we will round fuel remaining down.

Solution. The model is

$$F_t = 20 \text{ gal} - \frac{1.55 \text{ gal}}{\text{hr}} \cdot t \text{ hr}.$$

Thus

$$\begin{aligned}
F_1 &= 20 \text{ gal} - \frac{1.55 \text{ gal}}{\text{hr}} \cdot 1 \text{ hr} \\
&= 20 \text{ gal} - 1.55 \text{ gal} \\
&= 18.45 \\
&\approx 18.4. \\
F_2 &= 20 \text{ gal} - \frac{1.55 \text{ gal}}{\text{hr}} \cdot 2 \text{ hrs} \\
&= 20 \text{ gal} - 3.10 \text{ gal} \\
&= 16.9. \\
F_3 &= 20 \text{ gal} - \frac{1.55 \text{ gal}}{\text{hr}} \cdot 3 \text{ hrs} \\
&= 20 \text{ gal} - 4.65 \text{ gal} \\
&= 15.35 \\
&\approx 15.3. \\
F_{0.6} &= 20 \text{ gal} - \frac{1.55 \text{ gal}}{\text{hr}} \cdot 0.6 \text{ hrs} \\
&= 20 \text{ gal} - 0.93 \text{ gal} \\
&= 19.07 \\
&\approx 19.0.
\end{aligned}$$

□

Use this Checkpoint to practice using a linear model.

Checkpoint 3.1.4 The expected temperature at a height above ground is given by

$T_A = T_G - \dfrac{3.5}{1000}A$

where T_A is the expected temperature in Fahrenheit

T_G is the temperature at ground level in Fahrenheit

A is the height above ground level in feet

If the temperature on the ground is 73°, what will it be at 4700 feet above ground level? ___

Answers should be rounded to the units place.

3.1.2 Building Linear Models

The previous section presented linear models, and illustrated using the models provided. This section presents problems that can be modeled as linear equations, and illustrates writing the model (equation) before using it.

A linear model has a starting point (shift, b) and rate (ratio, m). We need to identify these and then write the linear model

$$y = mx + b$$

with these values. We should also label units and explain any parameters.

Example 3.1.5 Consider rope that costs \$0.93 per foot with a shipping charge of \$7.64. To produce a model for the cost of each purchase we will start by trying a couple specific orders.

Suppose we are purchasing 20 feet of this rope. The cost for the 20 feet will be $20 \text{ ft} \cdot \frac{\$0.93}{\text{ft}}$, because each foot is \$0.93. This is just like unit conversion: the units (\$/ft and ft) suggest multiplying.

Notice this multiplication is also the same as using a ratio (proportion). We could setup $\frac{\$0.93}{1\text{ft}} = \frac{C}{20 \text{ ft}}$ When we solve this we end up with the same multiplication $20 \text{ ft} \cdot \frac{\$0.93}{\text{ft}}$.

Next we must add the shipping charge. Thus the final cost is $20 \text{ ft} \cdot \frac{\$0.93}{\text{ft}} + \$7.64 = \$26.24$. Note there is no rounding because all numbers are exact (no measurements, so no signficant digits) and no fractions of a cent occurred.

Suppose we are purchasing 100 feet of this rope. The cost for the 100 feet will be

$$100 \text{ ft} \cdot \frac{\$0.93}{\text{ft}}.$$

Then we must add the shipping charge. Thus the final cost is $100 \text{ ft} \cdot \frac{\$0.93}{\text{ft}} + \$7.64 = \$100.64$.

Notice we could do this with any number of feet (unless the shipping charge increases for larger orders). So in general we can write the cost as

$$s \text{ ft} \cdot \frac{\$0.93}{\text{ft}} + \$7.64 = C.$$

Notice that this equation has a ratio (0.93/1), which is the cost per foot, but also has a shift (+7.64), which is the fixed shipping cost. Thus this is another linear equation. □

When cost is set per linear foot, or per square yard, or similar per unit pricing we often end up with a linear model.

Example 3.1.6 At lower altitudes the barometric pressure typically drops 1 inHg for every 1000 feet of elevation gained (the air is less dense higher up). To produce a model for pressure decrease we will start by calculating the pressure for a couple specific cases.

If the pressure on the ground is 29.76 inHg, what do we expect the pressure to be flying at 4500 ft above ground level?

The pressure drop is a ratio $\frac{1 \text{ inHg}}{1000 \text{ ft}}$. The units suggest we can multiply $4500 \text{ ft} \cdot \frac{1 \text{ inHg}}{1000 \text{ ft}} = 4.5 \text{ inHg}$. This is the drop in pressure. To calculate the resulting pressure we need $29.76 \text{ inHg} - 4.5° \text{ inHg} = 25.26 \text{ inHg}$. We retain 2 decimal places because that is the traditional amount for reporting by meteorologists. Written as one calculation this is $T = 29.76 \text{ inHg} - \left(4500 \text{ ft} \cdot \frac{1 \text{ inHg}}{1000 \text{ ft}}\right)$.

If the pressure on the ground is 30.02 inHg what do we expect the pressure to be flying at 6000 ft above ground level?

The pressure drop is a ratio $\frac{1 \text{ inHg}}{1000 \text{ ft}}$. The units suggest we can multiply 6000 ft $\cdot \frac{1 \text{ inHg}}{1000 \text{ ft}} = 6$ inHg. This is the drop in pressure. To calculate the resulting pressure we need 30.02 inHg $-$ 6 inHg $=$ 24.02 inHg. Written as one calculation this is $P = 30.02 \text{ inHg} - \left(6000 \text{ ft} \cdot \frac{1 \text{ inHg}}{1000 \text{ ft}}\right)$.

Notice we could do this same calculation for any altitude. So in general we can write

$$P_A = P_G - \left(A \text{ ft} \cdot \frac{1 \text{ inHg}}{1000 \text{ ft}}\right).$$

P_A is the pressure at the specified altitude. P_G is the pressure at ground level. A is the altitude above ground level. This is a linear equation with a ratio of (-1/1000) which is the drop in pressure with altitude, and a shift of P_G, which is the pressure on the ground. □

Example 3.1.7 We will find a model (equation) that converts temperature in Fahrenheit to temperature in Celsius. Note that every 9 degrees F is only 5 degrees C, so to convert we must scale the degrees. Also they use different values for the starting point (which is the freezing point of water). Fahrenheit starts at 32° and Celsius starts at 0°. Notice we have a ratio and a shift, so this already looks like a linear model.

To begin with we will convert 52° F to Celsius. We will round to units, because this is just an example (no one will be injured in the demonstration of this model).

The first step is to determine how many degrees above freezing. Because 32° F is the freezing temperature of water, 52° F is $52° - 32° = 20°$ above freezing.

The next step is to scale the degrees. The conversion ratio is $\frac{9° \text{ F}}{5° \text{ C}}$. Converting the 20° above freezing is now a unit conversion. The units suggest we can multiply $20° \text{ F} \cdot \frac{5° \text{ C}}{9° \text{ F}} \approx 11° \text{ C}$. Notice we flipped the conversion ratio so the Fahrenheit degrees would divide to one. The result is 11° C above the freezing point of water in Celsius.

Finally we can add the degrees Celsius to the starting point (freezing temperature of water). Because that is 0°, we have $0° + 11° = 11°$ C.

If we write all of that as one step we obtain

$$(52° \text{ F} - 32° \text{ F})\frac{5° \text{ C}}{9° \text{ F}} + 0° \text{ C} = 11° \text{ C}.$$

Notice we could do this with any temperature. So if we call the temperature to convert T we have

$$C = (T - 32)\frac{5}{9} + 0.$$

This may not look like a starting point plus a ratio scaled. If we expand the expression we obtain

$$C = (T - 32)\frac{5}{9} + 0 = \frac{5}{9}T - \frac{160}{9}.$$

So this is a linear model. We prefer the first form of the equation because the numbers have meaning (e.g., 32° is the freezing point of water as opposed to 160/9 which has no useful meaning. □

The temperature conversion example illustrates an idea about models. We describe a linear model as having a ratio and a (that is one) shift. However, in the temperature conversion example there is a shift a ratio and another shift, or two shifts. We showed these can be combined as one shift. That is always true. However, sometimes the version with multiple shifts is easier to understand. This will be true when we look at graphs of quadratics and exponentials (other forms of models).

3.1.3 Solving Linear Equations

The first section demonstrated using linear models to calculate values. However, sometimes we know the result and want to know the input. This requires solving the linear equation. This section reviews solving linear equation starting with non-contextualized examples and then using some of the models presented above.

Before reading farther solve the equation $5x - 7 = 12$. What steps did you use? Why do they work? Example 3.1.8 is an example of solving another linear equation.

Example 3.1.8 Solve $-8x - 3 = 5$.

$$\begin{aligned} -8x - 3 &= 5. \\ -8x - 3 + 3 &= 5 + 3. \\ -8x &= 8. \\ \frac{-8x}{-8} &= \frac{8}{-8}. \\ x &= -1. \end{aligned}$$

Note we added three because it eliminates the -3 (undoes subtraction of 3). We divided by negative eight because it eliminates the -8 (undoes the multiplication by -8). □

Checkpoint 3.1.9 What is the solution to $14 = 9x + 5$? ___

Some linear equations need one more technique. What would you need to solve $17 - 4y = 14 - y$? Below is an example of solving a similar linear equation.

Example 3.1.10 Solve $17 - 4y = 5 + 2y$.

$$\begin{aligned} 17 - 4y &= 5 + 2y. \\ -5 + 17 - 4y &= -5 + 5 + 2y. && \text{Collect constants on one side.} \\ 12 - 4y &= 2y. \\ 12 - 4y + 4y &= 2y + 4y. && \text{Collect the variable on the other side.} \\ 12 &= 2y + 4y. \\ 12 &= (2 + 4)y. && \text{Factoring reduces to one instance of the variable.} \\ 12 &= 6y. \\ \frac{12}{6} &= \frac{6y}{6}. && \text{Arithmetic to isolate } y \\ 2 &= y. \end{aligned}$$

Because there were multiple instances of the variable in the initial equation, we had to combine like terms (factor and add). □

Checkpoint 3.1.11 What is the solution to $7x + 2 = 4x + 11$? ___

Another linear equation is $\frac{x}{3} + \frac{x}{4} = \frac{7}{12}$. How would you solve it?

We can solve this the same as in Example 3.1.10 but because there are factions as coefficients we will use another technique.

Example 3.1.12 Solve $\frac{x}{5} + \frac{2x}{7} = \frac{34}{35}$

$$\begin{aligned} \frac{x}{5} + \frac{2x}{7} &= \frac{34}{35}. \\ 5 \cdot \left(\frac{x}{5} + \frac{2x}{7}\right) &= 5 \cdot \frac{34}{35}. && \text{Eliminate a denominator} \\ \frac{5x}{5} + \frac{10x}{7} &= 5 \cdot \frac{34}{35}. && \text{Distribute} \\ x + \frac{10x}{7} &= \frac{34}{7}. \end{aligned}$$

$$
\begin{aligned}
7 \cdot \left(x + \frac{10x}{7}\right) &= 7 \cdot \frac{34}{7}. && \text{Eliminate the other denominator} \\
7x + \frac{7 \cdot 10x}{7} &= 7 \cdot \frac{34}{7}. && \text{Distribute} \\
7x + 10x &= 34. \\
(7 + 10)x &= 34. && \text{Factor} \\
17x &= 34. \\
\frac{17x}{17} &= \frac{34}{17}. \\
x &= 2.
\end{aligned}
$$

This is referred to as clearing denominators. We are once again eliminating division by multiplying. Always remember to distribute. Note, we could multiply once if we figured out the correct number (it would be 35 in this case), but there are no prizes for doing this fast, so you can do this either way. □

Now that we have practiced solving linear equations, we can use this skill with the models.

Example 3.1.13 Given the temperature model in Model of Temperature Change with Altitude and supposing the temperature at ground level is 65, determine at what altitude we expect the temperature to be freezing. Round to the tens position; our measurements and sensors are not more precise.

In this case the model is $T_A = 65° - \frac{3.5°}{1000 \text{ ft}}A$. We know $T_A = 32°$ and we want to calculate A, the altitude in feet.

$$
\begin{aligned}
32° &= 65° - \frac{3.5°}{1000 \text{ ft}}A \\
-65° + 32° &= -65° + 65° - \frac{3.5°}{1000 \text{ ft}}A \\
-33° &= -\frac{3.5°}{1000 \text{ ft}}A \\
-\frac{1000 \text{ ft}}{3.5°}(-33°) &= -\frac{1000 \text{ ft}}{3.5°}\left(-\frac{3.5°}{1000 \text{ ft}}A\right) \\
9428.571428 \text{ ft} &= A. \\
9430 \text{ ft} &\approx A.
\end{aligned}
$$

□

Example 3.1.14 Given the time to altitude model in Model of Time to Altitude and supposing that we are climbing from 80 ft MSL to 5000 ft MSL with a climb rate of 700 ft/min, how long will it take to complete the climb?

In this case the model is $A_t = 80 \text{ ft} + \frac{700 \text{ ft}}{\text{min}}t$. We know $A_t = 5000$ ft, and we want to know the time t.

$$
\begin{aligned}
5000 \text{ ft} &= 80 \text{ ft} + \frac{700 \text{ ft}}{\text{min}}t. \\
-80 \text{ ft} + 5000 \text{ ft} &= -80 \text{ ft} + 80 \text{ ft} + \frac{700 \text{ ft}}{\text{min}}t. \\
4920 \text{ ft} &= \frac{700 \text{ ft}}{\text{min}}t. \\
\frac{\text{min}}{700 \text{ ft}} \cdot 4920 \text{ ft} &= \frac{\text{min}}{700 \text{ ft}} \cdot \frac{700 \text{ ft}}{\text{min}}t. \\
7.028571429 \text{ min} &= t. \\
8 \text{ min} &= t.
\end{aligned}
$$

We round up as a safety margin: it is better to assume we need 8 minutes and be higher than to hope we can recognize 0.03 of a minute (not quite 2 seconds). □

Use this Checkpoint to try solving a linear model.

Checkpoint 3.1.15 How long can you fly if you start with 48 gallons of fuel, burn 14 gallons per hour, and want land with one hour worth of fuel remaining? ____

Round down to the nearest tenth for a conservative estimate.

3.1.4 Identifying Linear Expressions

All of the equations in this section are linear. What can we use to identify linear expressions or linear equations? Table 3.1.16 shows examples of linear expressions and non-linear expressions.

Table 3.1.16 Linear and Non-linear

Linear	Non-linear
$5x+3$	$5x^2-x+3$
$y=11-\frac{7}{13}x$	$y=\frac{17}{x}$
$7x-9y=8$	$3-2xy=12$

Some equations that may not appear to be linear can be solved using the same methods.

Example 3.1.17 Solve $\frac{11}{x}+2=\frac{18}{x}-5$.

$$\begin{aligned}
\frac{11}{x}+2 &= \frac{18}{x}-5. \\
x\cdot\left(\frac{11}{x}+2\right) &= x\cdot\left(\frac{18}{x}-5\right). && \text{Clearing the denominators} \\
\frac{11x}{x}+2x &= \frac{18x}{x}-5x. && \text{Distribute.} \\
11+2x &= 18-5x. && \text{Now it appears linear.} \\
-11+11+2x &= -11+18-5x. \\
2x &= 7-5x. \\
2x+5x &= 7-5x+5x. \\
7x &= 7. \\
\frac{7x}{7} &= \frac{7}{7}. \\
x &= 1.
\end{aligned}$$

□

In Section 3.3 we will learn to identify linear models from data.

3.1.5 Exercises

1. Solve. Solve the equation below.

$6(x-11)-7=27x-325$

Answer: $x=$ ____________________

2. Solve. Solve $8(x+4)-4=-3(x-6)$ for x algebraically. If your answer is a fraction, write it in reduced, fractional form. Do NOT convert the answer to a decimal.

$x=$____________________

3. Solve. Solve the equation for the given variable:

$\frac{4b+3}{8}=-9$

If your answer is a fraction, write it in fraction from and reduce it completely. Do NOT convert to decimals.

$b=$ ________

4. **Solve.** Solve the equation $\frac{1}{4}y + 5 = \frac{1}{10}y$.

$y =$ ____________________

5. **Solve.** Solve the equation for the given variable. If your answer is a fraction, write it in reduced, fractional form. Do NOT convert the answer to a decimal.

$\frac{y}{3} + \frac{y}{5} = \frac{7}{5}$

Answer: $y =$ ____________________

6. **Solve.** In certain deep parts of oceans, the pressure of sea water, P, in pounds per square foot, at a depth of d feet below the surface, is given by the following equation:

$P = 12 + \frac{4d}{13}$

If a scientific team uses special equipment to measures the pressure under water and finds it to be 288 pounds per square foot, at what depth is the team making their measurements?

Answer: The team is measuring at ______ feet below the surface.

7. **Solve.** Solve for k in the equation: $-\frac{7}{9}k - \frac{6}{5} = -7 - \frac{3}{2}k$.

Round your answer to three decimal places. *Note: round only on the last step!*

$k =$ __________

8. **Solve.** Solve the following formula for x

$y = 2mx + 4b$

$x =$ ____________________

Enter your answer as an expression.

But be careful...to enter an expression like $\frac{a+b}{3+m}$ you need to type *(a+b)/(3+m)*. You need parentheses for both the numerator and denominator.

9. **Solve.** Solve the following formula for m

$c = amt$

$m =$ ____________________

Enter your answer as an expression.

But be careful...to enter an expression like $\frac{a+b}{3+m}$ you need to type *(a+b)/(3+m)*. You need parentheses for both the numerator and denominator.

10. **Solve.** Solve the formula $d = rt$ for r.

$r =$ _______

Enter your answer as an expression.

Be sure to PREVIEW your answer before submitting!

11. **Solve.** Solve the formula $V = \pi r^2 h$ for h. HINT: type π as pi.

$h =$ _______

Enter your answer as an expression.

Be sure to PREVIEW your answer before submitting!

12. **Solve.** Solve the formula $A = \frac{1}{2}bh$ for b.

$b =$ _______

Enter your answer as an expression.

Be sure to PREVIEW your answer before submitting!

13. **Solve.** Solve the formula $A = \frac{1}{2}h(a + b)$ for a.

$a =$ _______

Enter your answer as an expression.

Be sure to PREVIEW your answer before submitting!

14. **Solve.** Solve the formula $S = P(1 + rt)$ for P.

$P =$ __________

Enter your answer as an expression.
Be sure to PREVIEW your answer before submitting!

15. Solve. Solve the formula $A = P + Prt$ for r.
$r =$ __________
Enter your answer as an expression.
Be sure to PREVIEW your answer before submitting!

3.2 Representing Data

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Identify rates as linear, quadratic, exponential, or other (critical thinking)
- Identify data varying directly or indirectly (critical thinking)

We often represent numerical data using tables, diagrams, and graphs. These include various kinds of charts like bar graphs and pie charts, and graphs of functions. We do this to make certain traits of the data easier to notice. Here we will look at how some of these are produced and begin to learn to recognize differences due to rates. More details about rates will be covered in Section 3.3.

3.2.1 Reading Tables of Data

This section illustrates how some data can be stored in tables, how to read data from a table, and how to infer additional data when reasonable.

Tables are useful if we have a limited number of entries, and the data can be organized by two traits. If there are too many entries, the table may be too large to efficiently use. The two traits become headers for the rows and columns. If the table has more entries than we can see on a page or screen, it becomes less easy to use. Technology can make it easier to find the desired row and column such as a spreadsheet with the row and column headings frozen.

Table 3.2.1 Stall speed at 2550 lbs, most rearward center of gravity, speeds KIAS

	Angle of Bank			
Flap Setting	0°	30°	45°	60°
Up	48	52	57	68
Approach	43	46	51	61
Landing	40	43	48	57

We do not need to understand what bank, flap, and stall speed mean to read this table. Indeed a table can be presented precisely to help explain what terms mean. However, to satisfy your curiosity, stall speed refers to the speed at which a wing will produce insufficient lift to keep a plane flying. Falling beneath this speed typically results in the plane lowering its nose to regain speed. Angle of bank refers to how steeply the plane is tipped (left or right) in order to turn. KIAS stands for knots indicated air speed. Indicated airspeed is a speed pilots can see (think speedometer). Flaps are a structure extended for landing and sometimes take-off. Up means they are not in use. Approach and landing refer to varying degrees of extension.

Example 3.2.2 What is the stall speed in a 30° bank angle with flaps up?

We can determine this by looking for the column labeled “30°” and the row labeled “Up”. In that cell is the number 52. Thus the stall speed at that bank angle with flaps up is 52 KIAS. □

Example 3.2.3 In what condition is the stall speed the highest?

If we read all three rows, the largest number we find is 68. That is in the Up row and 60° column. So the stall speed is highest in the steep, 60° turn with the flaps in the up position.

Note, there is no shortcut here for checking the entry in every row and column. □

Example 3.2.4 As the angle of bank increases (from 0° to 60°) what happens to the stall speed?

Solution. If we look in the Up row, the stall speed changes from 48 to 52 to 57 to 68. The first thing we notice is that the stall speed increases.

If we repeat this in the Approach row, we again see the speeds are increasing. The same is true in the Landing row.

Thus we can say that stall speed increases as the angle of bank increases.

In later sections (Section 3.3) we will learn to be more specific about patterns when possible. □

Checkpoint 3.2.5

Table 3.2.6

	0	30	45	60
Up	48	52	57	68
Approach	43	46	51	61
Landing	40	43	48	57

Based on the table, what is the stall speed in a 0 degree turn with flaps in the Approach position? ___

Sometimes we want to know data that is between entries in a table. We can estimate these values if we know or can safely assume some property about the data. This is called **interpolation**. Below we provide examples of interpolation for linear data. Linear data is described in Section 3.1 and Section 3.3.

Example 3.2.7 Interpolation in a Table. What is the stall speed in a 15° bank angle with flaps in the Approach setting?

First, we note that there is no column for 15° bank angle. However we have 0° and 30°. 15° is half way between these two. For this chart it is reasonable to estimate our desired stall speed by calculating the number half way between those in the table.

The two stall speeds are 43 and 46. The number in between (the average) is $(43+46)/2 = 44.5$. Airspeed is reported only to units, so we must round to units. For stall speeds, it is safest to assume a higher stall speed, so we will round to 45 KIAS. □

Example 3.2.8 When we want a value that is half way between two entries in a table, we can simply average them. However, if we want a value somewhere other than half way in between we must perform an additional calculation.

What is the stall speed with 10° bank angle with flaps in the Up setting?

The nearest entries in the table are 0° and 30°. We need to figure out what percent 10 is between 0 and 30. We can use that to find the matching number between the table entries (48 and 52). Percent is part/whole which in this case is

$$\frac{10}{30-0} = \frac{10}{30} = \frac{1}{3}.$$

We do not need to write this as a percent (it would be approximately 33.3%), because we are just using it to multiply in the next step.

We want the speed that is 1/3 of the way between 48 and 52. Again we treat this as a percent problem. We want percent times the whole to calculate the part.

$$\frac{1}{3} \cdot (52-48) = \frac{4}{3}.$$

This result is how far above 48, so the speed is

$$48 + 4/3 \approx 49.3 \approx 50.$$

Again we round up for safety.

Because there are only 4 knots between the entries, it hardly seems worthwhile to do this work, especially because we round up for safety. There are times however, when this process is useful. □

Checkpoint 3.2.9

Table 3.2.10

	0	30	45	60
Up	48	52	57	68
Approach	43	46	51	61
Landing	40	43	48	57

Based on the table, what is the stall speed in a 15 degree turn with flaps in the Approach position? ___ Flight safety requires rounding up stall speeds. As a result, round up to the nearest integer.

3.2.2 Reading Graphs

This section illustrates how some data can be represented in graphs and how to read data from a graph, including some comparisons of graphs.

Graphs that are curves (like lines) are read by finding a vertical heading that matches our question (think row) and reading the corresponding horizontal heading (think column). Note this could be reversed, that is, find a horizontal heading that matches and read the corresponding vertical one.

Example 3.2.11 Figure 3.2.12 presents the maximum engine out glide for an airplane. If that plane is 2400 ft above the ground, how many nautical miles can it glide forward?

First we note that 2400 ft matches the vertical axis. We want to find the line across the graph that represents 2400 ft. Note, no line is labeled 2400, but we do have 2000 and 4000 and there are lines between them. To figure out which of these lines we should use, we must figure out how many feet each minor line represents.

We need to know how much each line represents. We count 10 minor lines between each major line. Because each major line represents 2000 ft, we know the minor lines represent $2000/10 = 200$ ft.

Because $2400 = 2000 + 2(200)$ we want the second minor line above 2000. We follow that to the blue line, then we follow the gray (minor) line down to the bottom. Our result is two minor lines before 4.

We must figure out how much each minor vertical line represents. Each vertical major line is 2 nm. Again there are 10 minor lines between each major line, so we know the minor lines represent $2/10 = 0.2$ nm.

The glide distance for 2400 ft is therefore $4 - 2(0.2) = 3.8$ nm. We subtracted here because it is before 4. □

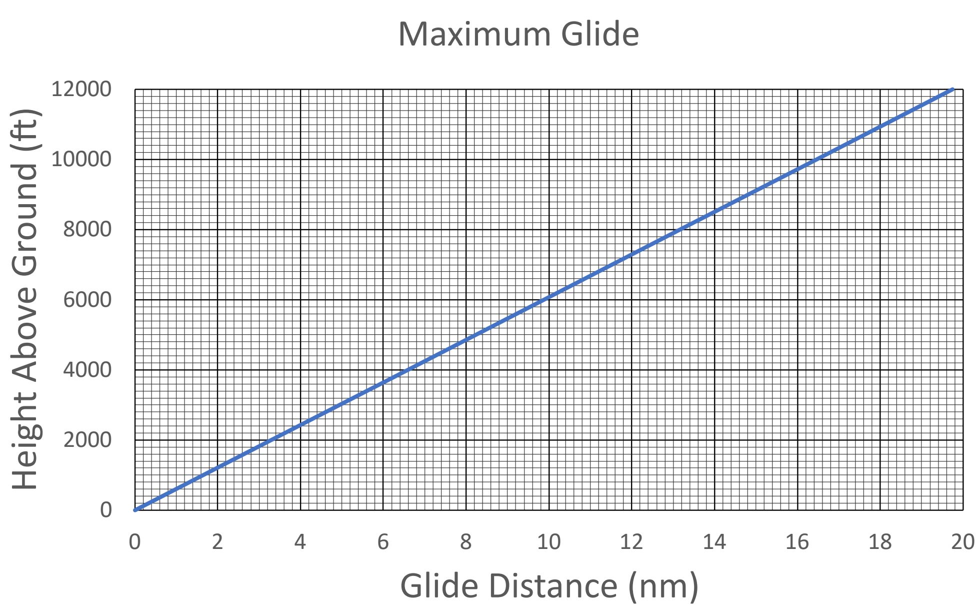
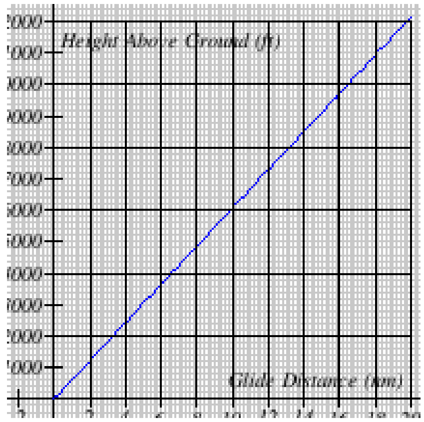


Figure 3.2.12 Graph Representing Maximum Engine Out Glide

Checkpoint 3.2.13



Based on the chart (graph) how far can the plane glide if it is 600 ft above the ground? ____

You may round to the nearest 0.2 of a nautical mile.

When we look at any table, graph, or figure, we should ask ourselves why various choices were made in the construction of the table, graph, or figure. We may need to ask someone with related knowledge for explanations of those decisions.

Example 3.2.14 Consider Figure 3.2.12. The input we use is “Height Above Ground (ft)”. Frequently we place the inputs on the x-axis. Why was the y-axis chosen for the inputs here?

Consider that the inputs are *heights*. This is a y-axis concept, so it matches our expectations. Reading the graph is not affected by this choice.

Why do the inputs begin at 0 and end at 12000?

They begin at 0, because we are talking about a plane gliding to the ground. A plane must be above the ground (above 0) to glide.

They end at 12000 in this case, because this aircraft cannot fly higher than that altitude. We do not need data for cases that cannot occur.

Why are the inputs labeled every 2000 and the inputs every 2?

This is purely space available. If we put in more labels they would overlap each other. □

Graphs can be from raw data which may appear random. We still read these graphs the same way.

Example 3.2.15 Figure 3.2.16 has the temperature and dewpoint read by a radiosonde (instruments on weather balloon) as it rose in the atmosphere. Note the vertical axis is the pressure reading. This is not the same as altitude, but it does correspond mostly to altitude. Dew point is the temperature at which water will condense, so it is also a temperature.

What are the temperature and dewpoint at the 700 millibar level?

We follow the 700 mb line over to the dewpoint (green, dashed) line. It is a little less than halfway between -20° C and 0° (closer to 0°). We estimate -7° C. Continuing across the 700 mb line to the temperature (red, solid) line we find it a little closer to 0°. We estimate the temperature is 5° C. □

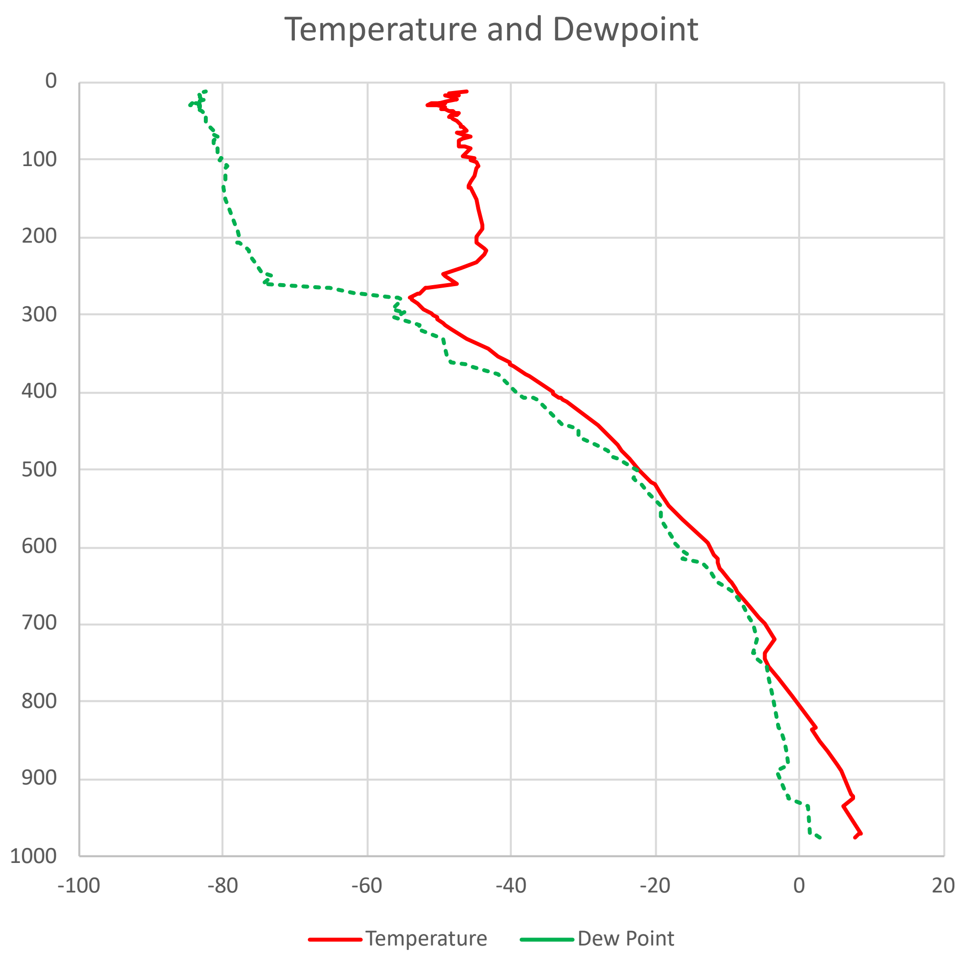


Figure 3.2.16 Graph of Temperature and Dewpoint

Note some charts like Figure 3.2.16 are not meant to convey specific numbers but rather to show trends.

Example 3.2.17 Notice that while the temperature (red, solid line) wiggles around, it trends down as the pressure decreases. That is, it shows temperature generally decreasing as altitude increases. We expect this, because it is farther from the ground which heats the air. The increase above 300 millibars level is the result of other factors which you can research at `noaa.gov/jetstream/atmosphere/layers-of-atmosphere`.

Clouds form when the temperature reaches the dewpoint and the air is saturated (has enough moisture). We see in Figure 3.2.16 three places where temperature and dewpoint are the same. The lowest is between 800 and 700 millibars (we estimate 750 millibars). The second is between 700 and 600 millibars (we estimate 650 millibars). The third is at about 500 millibars. We would expect clouds to form at these altitudes. □

Example 3.2.18 Consider Figure 3.2.16. The input we use is “Pressure (millibars)”. Why was the y-axis chosen for the inputs here?

The pressure readings correspond to altitudes (height) which we tend to think of as up. Putting this on the y-axis matches this expectation.

Why do the y-axis labels decrease as they go up?

Atmospheric pressure decreases with altitude, so low pressure means higher altitude. The pressure readings are arranged to be low altitude at the bottom and high altitude at the top.

Why do the inputs begin at 1000?

Because the pressure readings correspond to altitude the highest pressure should be on the ground. It turns out 1013 is a typical pressure at ground level, so much higher pressure readings are not expected.

Why are the output labels from -100 to 40?

These are based on commonly experienced temperatures. Temperatures lower than -100° C are not expected. Temperatures above 40° C do occur, but not in the location where this sounding was taken.

Use Example 3.1.7 to convert 50° C to Fahrenheit to see why this temperature is uncommon in most locations. □

The input for the glide ratio questions is altitude. Altitudes are **continuous** that is it makes sense to refer to an altitude of 2453.27 feet (fractional feet). Similarly the pressure levels are continuous, that is it makes sense to refer to 501.7 millibar level. However, there is data where a fraction does not make sense. This **discrete** data is often graphed differently. The next examples illustrate a way of presenting discrete data.

Example 3.2.19 Increasing Income. When Vasya was hired in 2017, she was paid an annual salary of $62,347.23. Her work has been good, so each year she has received raises of $5000.00.

To represent this data we first need to calculate her salary for each year. We do this by starting with her initial salary, then for each year adding the $5000 raise to the previous year's salary. This is an **iterative** process. Table 3.2.20 contains the results. A table is an effective means to represent discrete data like this.

We will represent her salary over time using the bar graph in Figure 3.2.21. Notice the horizontal axis is labeled with years and the vertical axis is labeled in dollars. There is one bar for each year, because her salary was changed only once each year. Bar graphs are a good option for discrete data.

Consider the bar graph (ignore the table). Can you tell that Vasya's salary is increasing? Can you tell how much? How might the graph be changed to make information easier to find? □

Table 3.2.20 Vasya's Salary

2017	$62,347.23
2018	$67,347.23
2019	$72,347.23
2020	$77,347.23
2021	$82,347.23
2022	$87,347.23
2023	$92,347.23
2024	$97,347.23

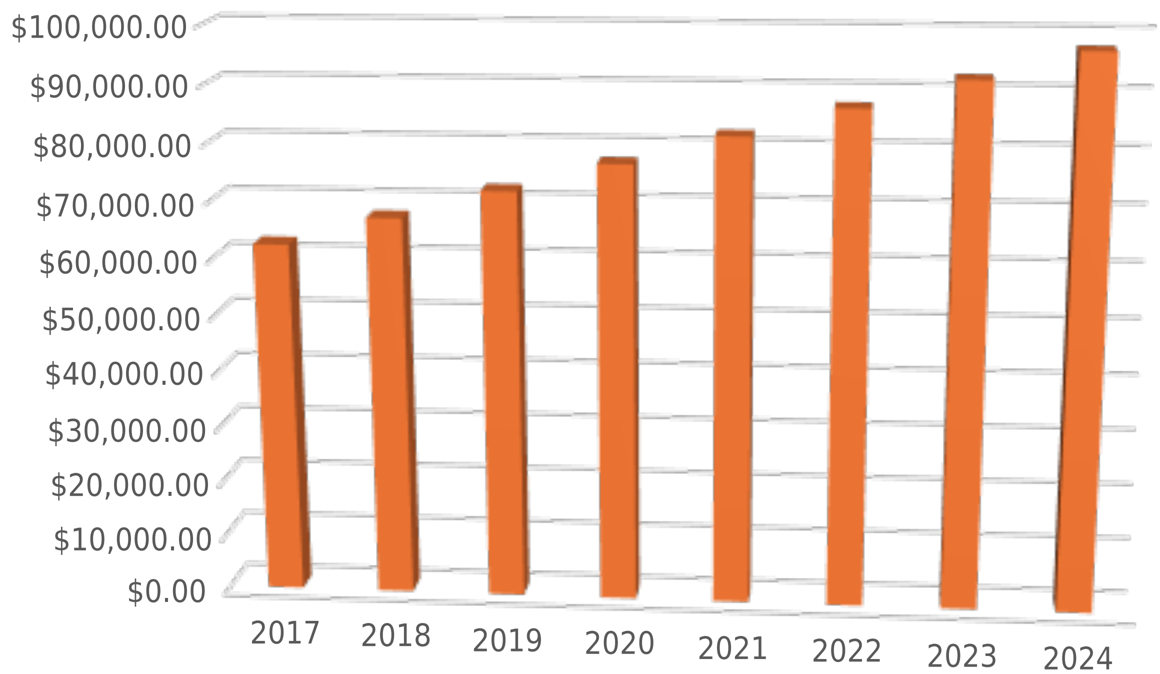


Figure 3.2.21 Vasya's Salary

Example 3.2.22 Vasya wishes to know how her raises are helping her keep up with increasing costs. Figure 3.2.23 shows her raises as a percent of her previous year's salary and the inflation rate (`usinflationcalculator.com/inflation/current-inflation-rates/`) If her raises are at least as large as inflation, then her spending power is not diminished (keeping up)

(a) Using Table 3.2.20 confirm that the graph shows the correct percent increase for 2021. Recall her raise is $5000 each year. The graph shows the raise as a percent of the previous year.

(b) For these years is she keeping up with inflation?

Solution. Her raise is a larger percent each year except for two. In those two years it is close. She has been more than keeping up with inflation.

(c) What trend do you notice in the percent increase of salary? Why is this happening?

Solution. Her percent drops from about 8% to a little over 5%. This results from her raise being the same amount but her previous year's salary is bigger each year. In the percent (part/whole) the part remains fixed while the whole increases.

Unless there is a change this will lead to her raises eventually not keeping up with inflation.

□

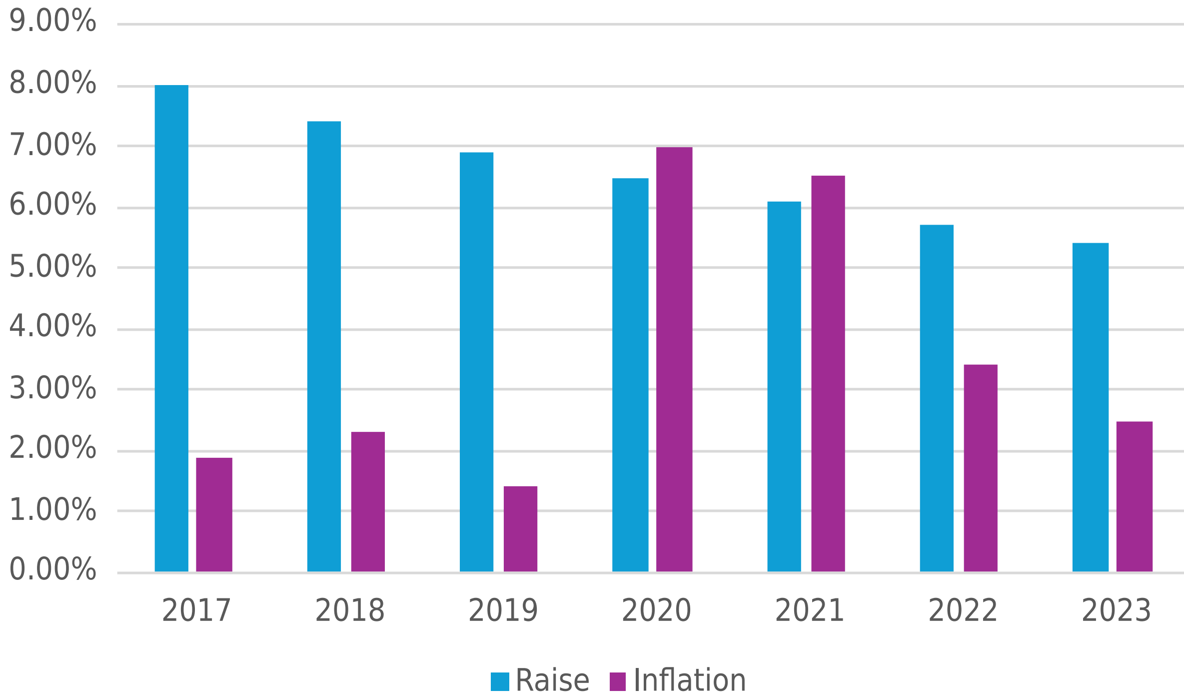


Figure 3.2.23 Vasya's Salary

Checkpoint 3.2.24 Consider Figure 3.2.25. It contains Guido's annual salary for each year listed. The second bar is the first year's salary increased each year to match inflation. That is it shows what Guido's salary would have been if his raises had exactly matched inflation.

(a) In which years did Guido receive a raise?

(b) In which years did Guido's salary appear to grow at least as much as inflation?

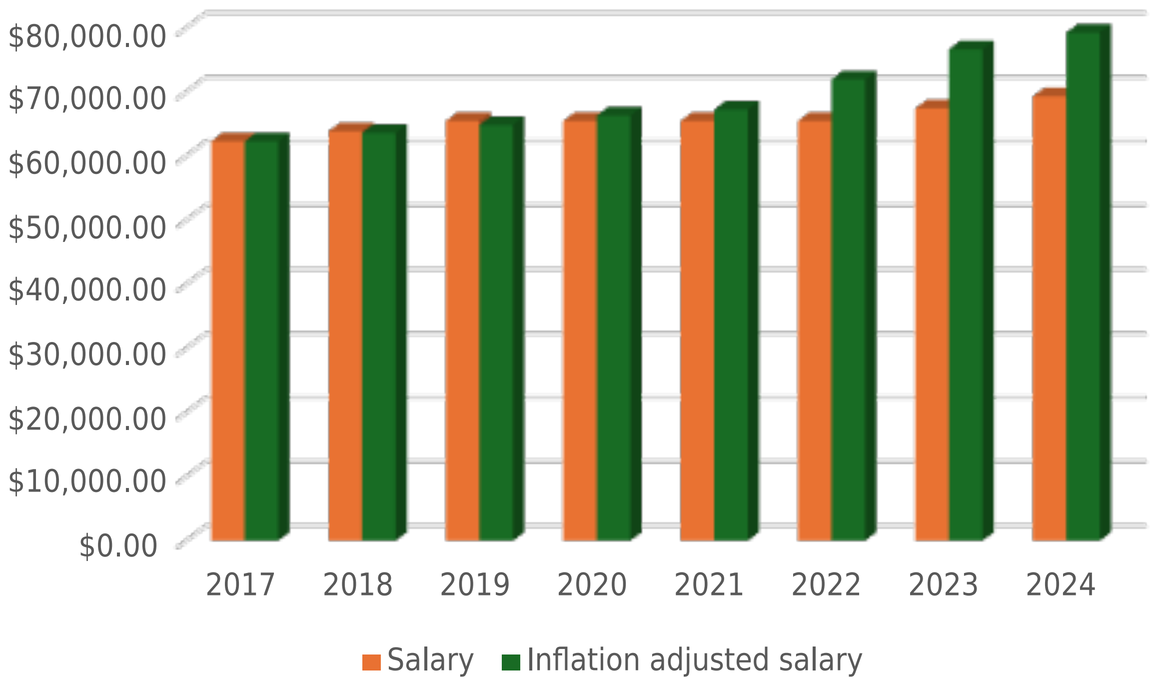


Figure 3.2.25 Salary vs Inflation

3.2.3 Using Graphs to Analyze Models

Above we practiced interpreting graphs provided for us. This section presents generating graphs to analyze and interpret models. While we will rely on technology to produce graphs, these examples begin with manual generation of graphs because that process helps us understand a model and it helps us understand what the graphs mean.

Example 3.2.26 Scale Model. A model of a space shuttle is labeled as 1:144. This means one inch on the model represents 144 inches on the actual shuttle, that is, there is a ratio between the size of objects on the model and the size of the objects on the actual shuttle. If a portion of the model is 1.72 inches then means the part on the actual shuttle is $1.72 \cdot 144 = 247.68$ inches. In general $A = 144M$ where M is the size on the model and A is the size on the actual shuttle.

To represent this scale conversion as a graph we will generate a table like Table 3.2.20 then we will use that to plot the graph.

Model	Full Size
1.0	144
1.5	216
2.0	288
2.5	360
3.0	432

We sketch a graph by plotting the points first. Notice the five points based on the table above. Through the points we draw a curve: in this case it is a line. The graph is in Figure 3.2.28.

For the curious, software uses this same process to produce a graph. It usually plots a much larger number of points and then connects the dots with short line segments. □

Example 3.2.27 Why does the graph start at 0? The inputs are lengths on the model; negative lengths do not make sense.

Why does it end at 4? If we wanted all sizes from zero to the largest dimension of the shuttle, we would need a bigger graph. However, because this is a line, we have a good idea what the rest of the graph looks like.

This graph is a line. We knew it would be because $A = 144M$ is in the form of a line (as shown in Subsection 3.1.1). □

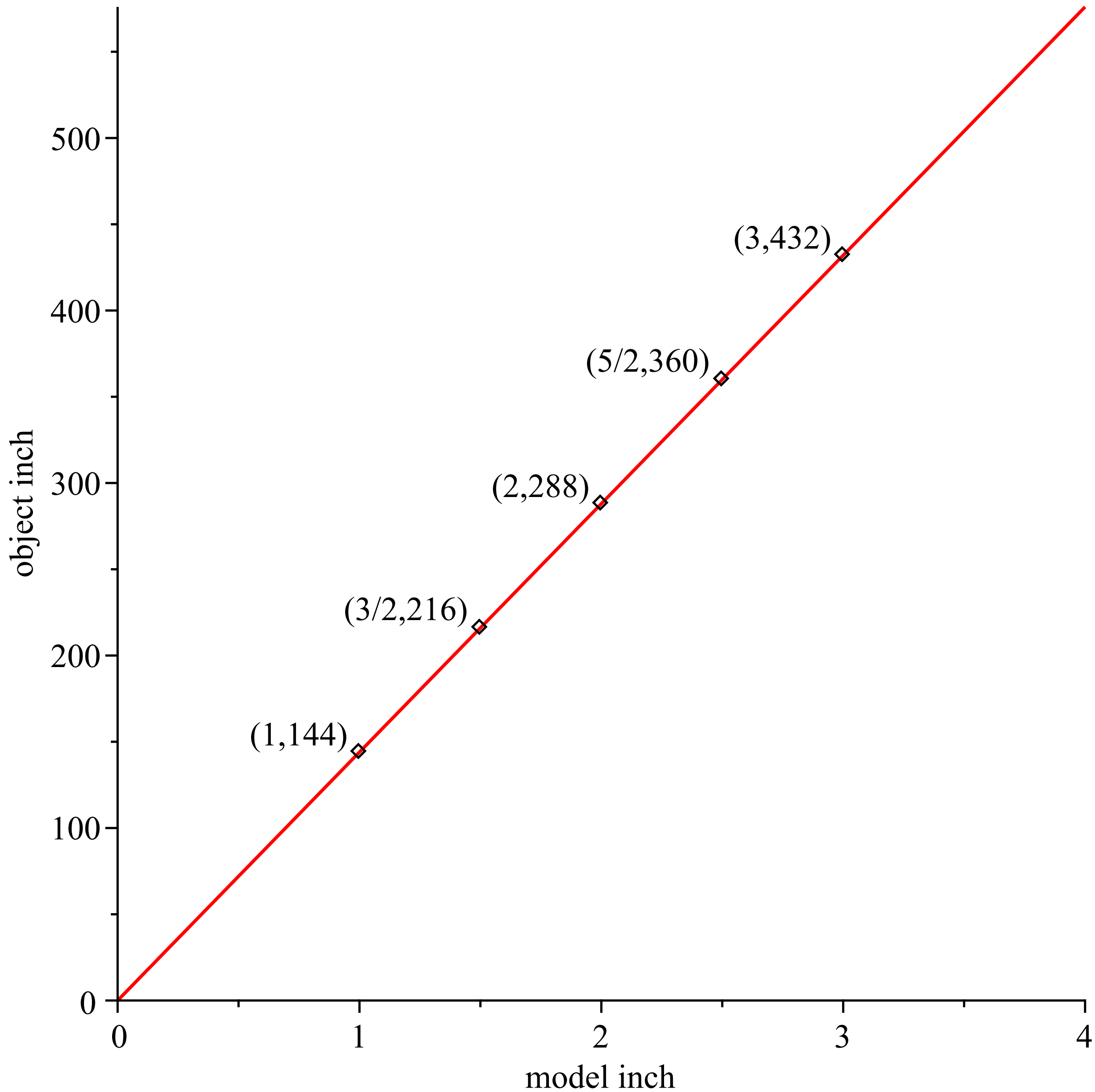


Figure 3.2.28 Graph of Scale

The next example is a shape we have not yet encountered in this text.

Example 3.2.29 Ohm's Law. Recall Ohm's Law $V = IR$ from Model 1.3.1. We will explore the relationship between current (I) and resistance (R).

To begin the exploration and to enable graphing we will complete a table. First it will be convenient to solve Ohm's law for current (I).

$$V = IR.$$

$$V \cdot \frac{1}{R} = IR \cdot \frac{1}{R}.$$
$$\frac{V}{R} = I.$$

Because we are interested in the effect of resistance on current we will pick a fixed voltage: $V = 12$ V. Thus our equation is $I = \frac{12}{R}$.

Resistance	Current
1.0	12
4.0	3.0
8.0	1.5
12.0	1.0
16.0	0.75

We can plot these points and sketch a curve through them. This graph is in Figure 3.2.30.

The apparent relationship between current and resistance for a fixed voltage is that current decreases as resistance increase.

The graph starts with 1 Ohm. Why does it not start at 0? If resistance were 0, then the equation becomes $I = \frac{12}{0}$. Division by zero does not make arithmetic sense. 0 ohm resistance means no resistance and this is not physically possible (nothing is perfect). Thus the math model fits the physical reality. □

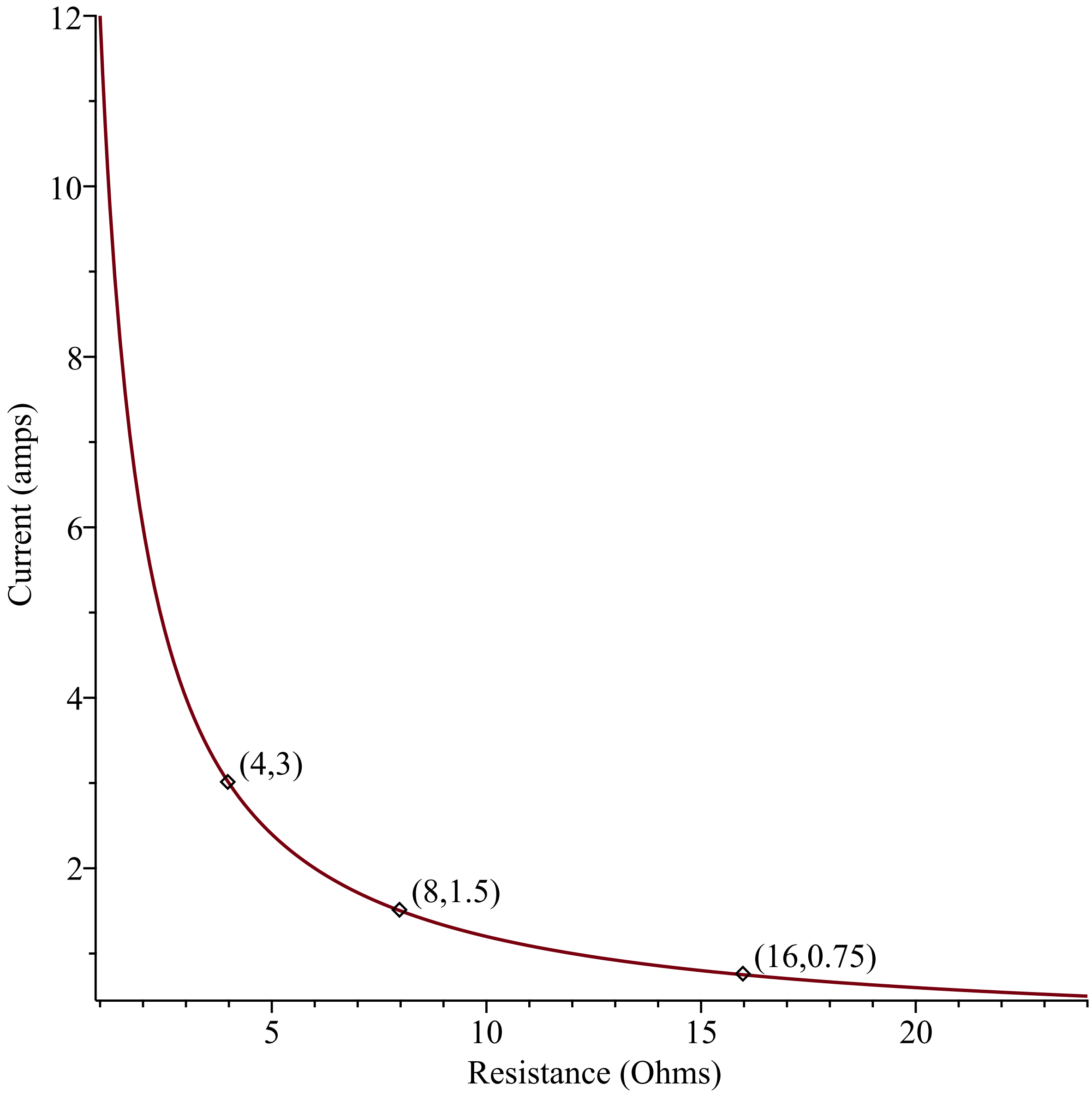


Figure 3.2.30 Graph of Ohms Law

Checkpoint 3.2.31 The ideal gas law expresses a relationship between pressure, volume, and temperature of a gas. It is given by

$$P \cdot V = k \cdot T$$

where P is the pressure, V is the volume, T is the temperature, and k is a constant dependent on the specific gas.

(a) Draw a graph for the equation $P = \frac{8.3145T}{2.0000}$. Note the units are Kelvin (Celsius + 273.15) for temperature and Jules/litre for pressure. These do not need to be labeled here.

(b) Draw a graph for the equation $P = \frac{8.3145 \cdot 293.15}{V}$.

3.2.4 Graphing Lines

We have seen what linear data looks like in data tables, discrete graphs (e.g., bar graph), and continuous graphs. This section presents how to graph lines if we have the equation and presents analyzing linear models based on their graphs.

As described in Subsection 3.1.1 a linear equation (model) has two parts: the ratio (slope, rate) m and the shift $+b$. First, we address the role of the ratio in the graph.

Definition 3.2.32 Slope. The rate of change of a line (graph) is called its **slope**. The numerator is the change in y and the denominator is the change in x. Slope can be calculated as

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Rise and run are terms to help us remember the formula. ◊

In Example 3.2.26 we graphed a linear equation with the ratio 144 real inches to 1 model inch. In the linear model we have $m = \frac{a}{b} = \frac{144}{1}$ that is 144 is the change in y and and 1 is the change in x.

Because the rate of change is the fixed, the slope can be calculated from any two points. We can calculate the slope from points in a table or points from a graph using

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

The next example illustrates calculating the slope from two points and that the slope is the same regardless of points selected.

Example 3.2.33 Calculate Slope. The graph in Figure 3.2.34 is linear. We will calculate the slope twice.

Solution 1. First, we will use the points $(4000, 25.92)$ and $(2000, 27.92)$.

$$\begin{aligned}\frac{y_2 - y_1}{x_2 - x_1} &= \frac{25.92 - 27.92 \text{ inHg}}{4000 - 2000 \text{ ft}}\\ &= \frac{-2.00 \text{ inHg}}{2000 \text{ ft}}\\ &= -\frac{1.00 \text{ inHg}}{1000 \text{ ft}}.\end{aligned}$$

This means the slope is a decrease of 1.00 inHg (inch of mercury) per increase of $100\bar{0}$ feet above mean sea level.

Solution 2. We will use the points $(8000, 21.92)$ and $(4000, 25.92)$.

$$\begin{aligned}\frac{y_2 - y_1}{x_2 - x_1} &= \frac{21.92 - 25.92 \text{ inHg}}{8000 - 4000 \text{ ft}}\\ &= \frac{-4.00 \text{ inHg}}{4000 \text{ ft}}\\ &= -\frac{1.00 \text{ inHg}}{1000 \text{ ft}}.\end{aligned}$$

As expected this is the same slope, because on a line the rate of change (slope) is constant. □

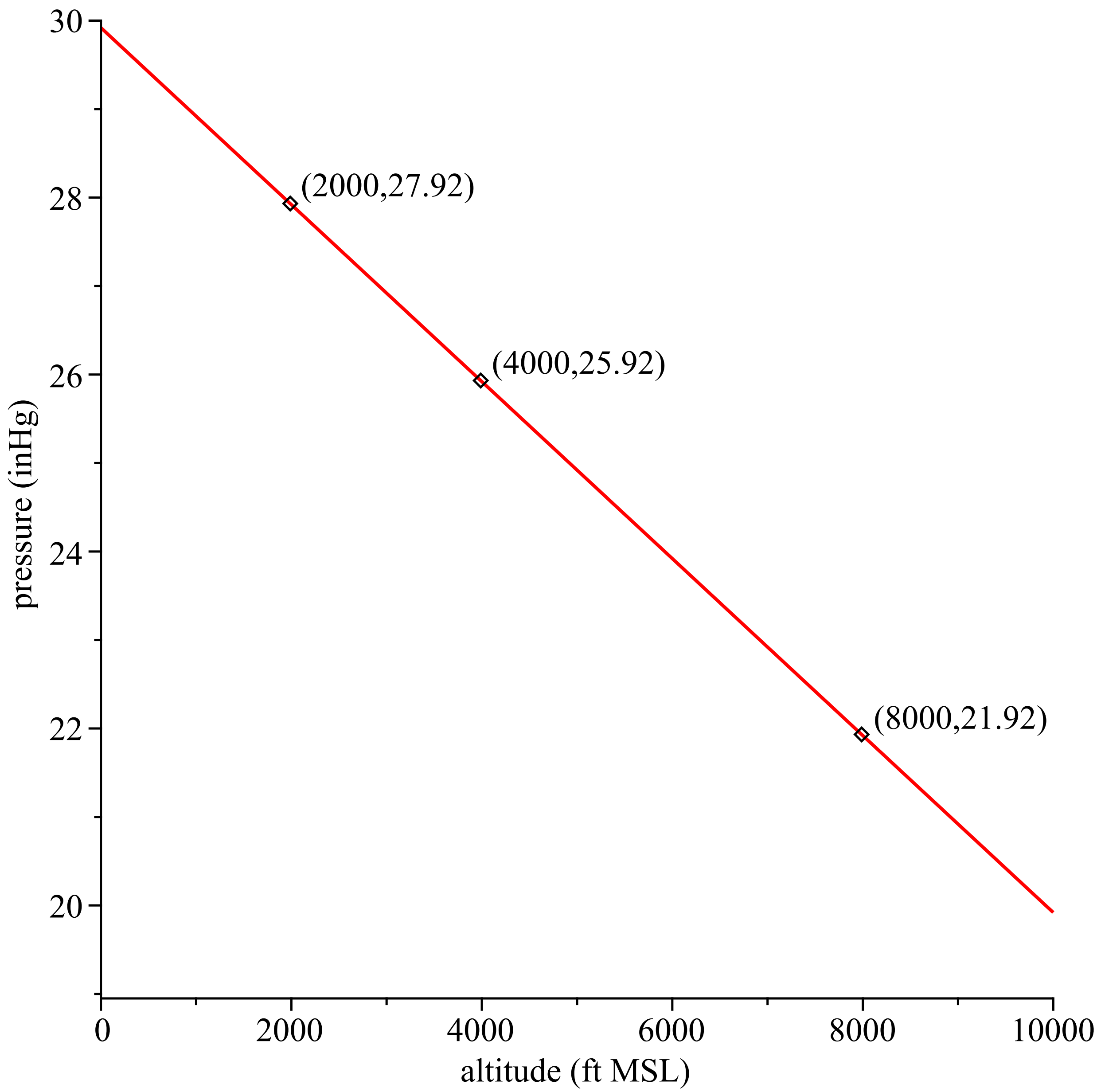


Figure 3.2.34 Calculating Slope

To write the equation for a line we need the shift $+b$ as well as the slope. This can also be read from the graph. The slope determines how tilted the line is. After this that line can be moved up or down. b controls this shift. It is typically easiest to read this shift at $x = 0$ because in the linear form $y = mx + b$ we have $y = m \cdot 0 + b = b$, so the y coordinate at $x = 0$ is the shift which is why it is often called the y-intercept.

Example 3.2.35 Calculate Shift. The graph in Figure 3.2.34 is linear. We will calculate the shift.

Solution. The shift can be read when $x = 0$. That point is not labeled on the graph. However, we can calculate it using one of the points and the ratio.

We will use $(2000 \text{ ft}, 27.92 \text{ inHg})$ to find the pressure using a proportion. We want the point 2000 feet below this point, and pressure increases as we go down so we set up

$$\frac{1.00 \text{ inHg}}{1000 \text{ ft}} = \frac{d \text{ inHg}}{2000 \text{ ft}}.$$

$$\frac{1.00 \text{ inHg}}{1000 \text{ ft}} \cdot 200\bar{0} \text{ ft} = \frac{d \text{ inHg}}{2000 \text{ ft}} \cdot 2000 \text{ ft}.$$
$$2.00 \text{ inHg} = d.$$

Thus the pressure should increase by 2.00 inHg giving us $P = 27.92 \text{ inHg} + 2.00 \text{ inHg} = 29.92 \text{ inHg}$. Thus the point is $(0.00 \text{ ft}, 29.92 \text{ inHg})$ and the shift is $b = 29.92$ inHg.

Combining this shift with the slope from the example above the model is

$$P = 29.92 \text{ inHg} - \frac{1 \text{ inHg}}{1000 \text{ ft}} A.$$

- P is the pressure at altitude A.
- 29.92 is the initial pressure.
- The rate is $\frac{1 \text{ inHg}}{1000 \text{ ft}}$.
- A is the altitude above ground level.

If we replaced 29.92 with a parameter P_G we could generalize the model to $P_A = P_G - \frac{1}{1000}A$. □

Checkpoint 3.2.36 Suppose that as dry air rose its temperature dropped in a linear fashion. The temperature was measured at 1000 ft MSL as 20° C and at 4000 ft MSL as 25.4° C.

What is the rate of change of temperature with respect to altitude? ____________________

Your response may be as a fraction such as 2/1000 or decimal such as 0.002

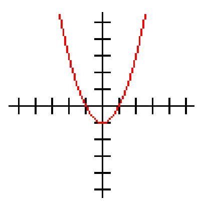
What does this data imply about the temperature at 0 ft MSL? ____________________

3.2.5 Exercises

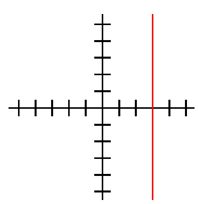
Exercise Group. Answer these questions about interpreting data.

1. Determine Linear. Identify what each graph below represents?

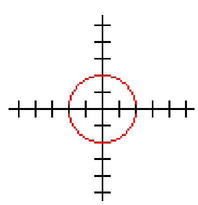
(a)



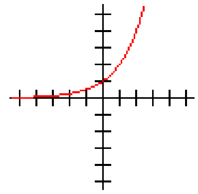
(b)



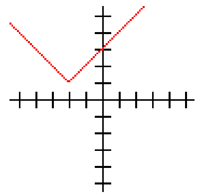
(c)



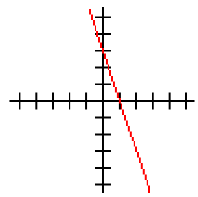
(d)



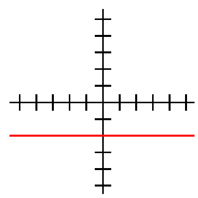
(e)



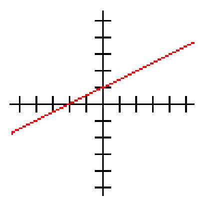
(f)



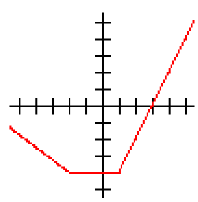
(g)



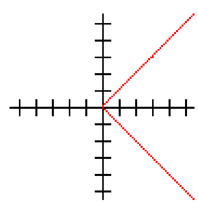
(h)



(i)



(j)



(a) Linear function

(b) Nonlinear function

(c) Not a function

2. Graph and Table. Determine if the following table represents a linear relation.

Table 3.2.37

x	0	1	2	3	4	5
y	0.3	1.8	3.3	4.8	6.3	7.8

(a) Yes

(b) No

3. Graph and Table. Determine if the following table represents a linear relation.

Table 3.2.38

x	0	1	2	3	4	5
y	2.3	4.6	11.5	23	39.1	59.8

(a) Yes

(b) No

4. Graph and Table. Determine if the following table represents a linear relation.

Table 3.2.39

x	0	1	2	3	4	5
y	3.9	5.9	11.9	29.9	83.9	245.9

(a) Yes

(b) No

5. Graph and Table. Determine if the following table represents a linear relation.

Table 3.2.40

x	1	2	3	4	5	6
y	936	234	104	58.5	37.44	26

(a) Yes

(b) No

6. Graph and Table. Calculator[1]

Examine the linear function below.

$y = 3x - 5$

Which table represents the same function?

(a)

Table 3.2.41

x	y
0	-5
5	25
10	55
15	85

(b)

Table 3.2.42

x	y
-4	-7
-2	-1
0	5
2	11

Table 3.2.43

(c)

x	y
-2	-11
0	-5
2	1
4	7

Table 3.2.44

(d)

x	y
0	3
3	12
6	21
9	30

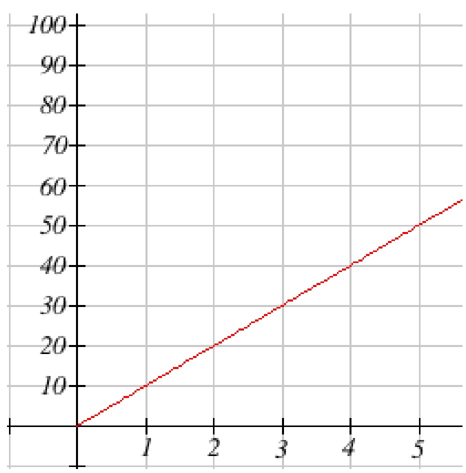
7. **Compare Linear Functions.** Put the the people in order from lowest pay rate to greatest pay rate.
(Note variables: x = time in hours, y = dollars earned)

- *Person A*$y = 12.5x$
- *Person B*Draeden earned $42 after 4 hours of work.
- *Person C*

Table 3.2.45

Time (hours)	Total (dollars)
0.5	6.75
2	27
6	81

- *Person D*



Lowest to Greatest

(a) A

(b) B

(c) C

[1] #

(d) D

(a) A

(b) B

(c) C

(d) D

(a) A

(b) B

(c) C

(d) D

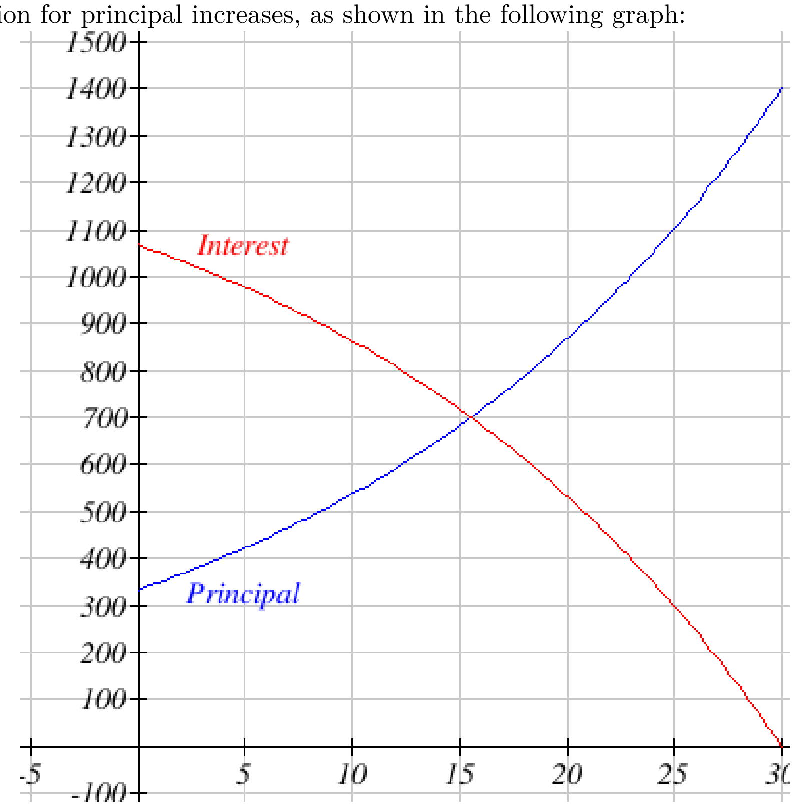
(a) A

(b) B

(c) C

(d) D

8. Interpret Graph. Becky takes out a 30-year mortagage for which her monthly payment is $1400. During the early years of the mortgage, most of each payment is for interest and the rather small remainder for principal. As time goes on, the portion of each payment that goes for interest decreases while the portion for principal increases, as shown in the following graph:

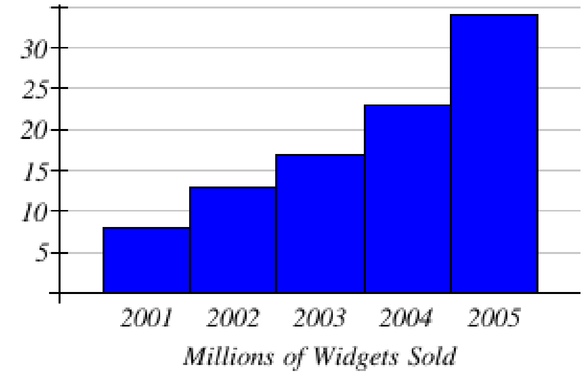


a) Approximately how much of the $1400 monthly payment goes for interest in year 15? $____________________

b) In what year will the monthly payment be equally divided between interest and principal?

Year ____________________

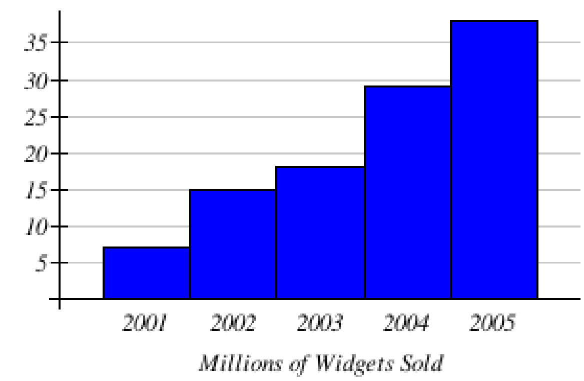
9. Interpret Graph. Our company's new widget has been growing in sales. The histogram below shows sales in millions for the years shown.



Approximately how many millions of widgets were sold in year 2004?

____

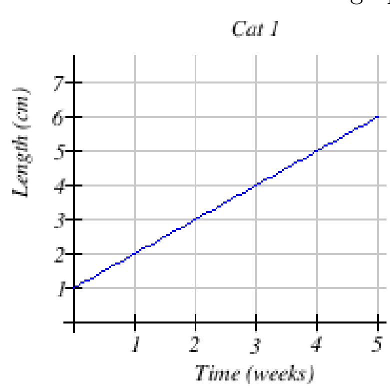
10. Interpret Graph. Our company's new widget has been growing in sales. The histogram below shows sales in millions for the years shown.

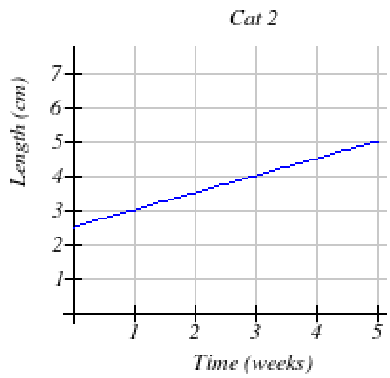


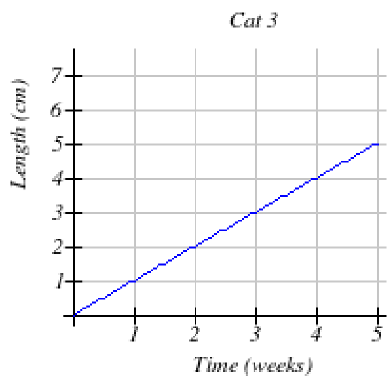
In which year did sales first exceed 11 million widgets?

____

11. Interpret Graph. An animal shelter once tracked the length of the whiskers of three cats every week starting from the time they received the cats. The three graphs are shown below.







Which cat was born in the animal shelter?

(a) Cat 1

(b) Cat 2

(c) Cat 3

At what rate are the whiskers of Cat 3 growing? ____

(a) weeks

(b) cm

(c) cm per week

(d) weeks per cm

Which cat's whiskers are growing more slowly than the other two?

(a) Cat 1

(b) Cat 2

(c) Cat 3

How long were the whiskers of Cat 1 when it was received by the animal shelter? ____

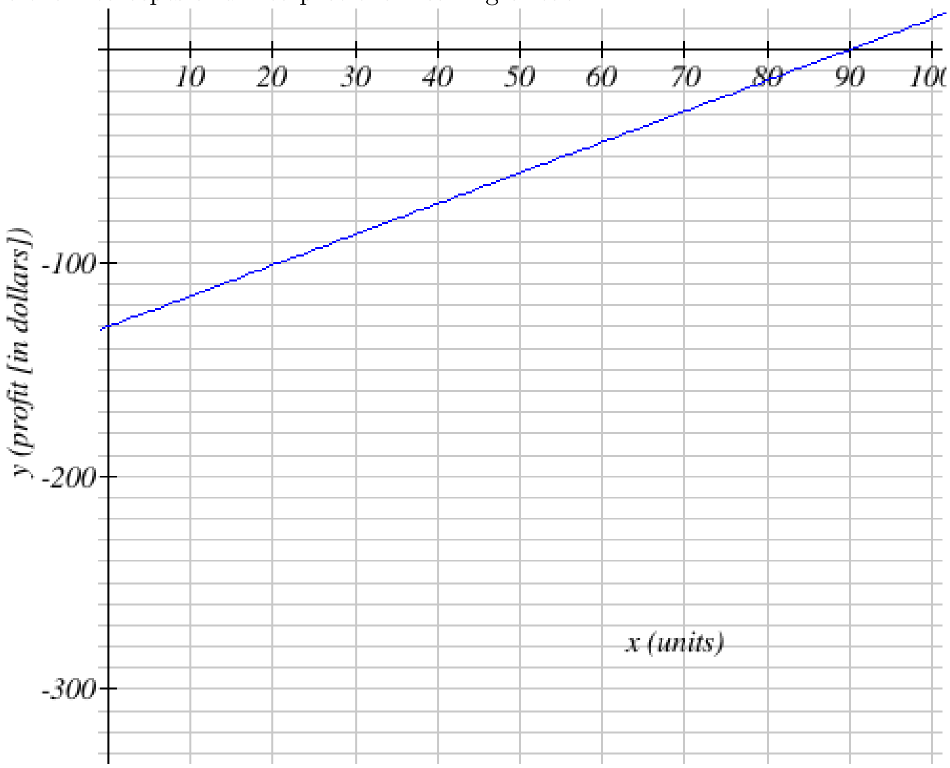
(a) weeks per cm

(b) cm

(c) weeks

(d) cm per week

12. Interpret Graph. In the graph below the input x is the number of units produced by a machine in a factory. The output y is the profit made by the sale of these units when they are produced. Determine the intercepts and interpret the meaning of each.



A. Give the coordinates of the x-intercept: ____________________

B. Interpret the meaning of the x-intercept:

(a) The profit is \$90 when -130 display units are produced and sold.

(b) The profit is \$-130 when 90 display units are produced and sold

(c) The profit is zero when -130 units are produced and sold.

(d) The profit is zero when 90 units are produced and sold.

(e) There are zero units produced and sold when the profit is \$90.

(f) There are zero units produced and sold when the profit is \$-130

C. Give the coordinates of the y-intercept: ____________________

D. Interpret the meaning of the y-intercept:

(a) The profit is zero when -130 units are produced and sold.

(b) The profit is \$-130 when 90 display units are produced and sold

(c) The profit is zero when 90 units are produced and sold.

(d) The profit is \$90 when -130 display units are produced and sold.

(e) There are zero units produced and sold when the profit is \$90.

(f) The profit is \$-130 when zero units are produced.

E. Determine the slope of the line: ____________________ [Make sure your slope is written as a reduced fraction.]

F. Interpret the meaning of the slope. For each 7-unit increase in units sold there is a(n)

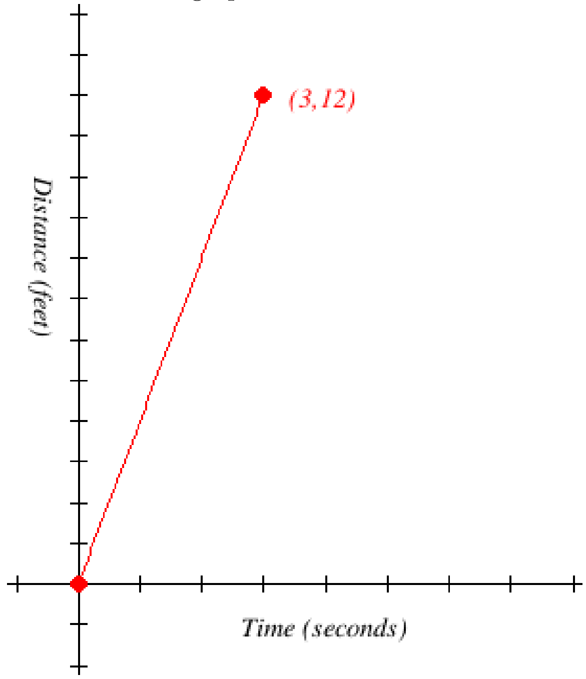
(a) increase

(b) decrease

____________________ *G.* Write the equation of the line in $y = mx + b$ form: ____________________

H. Use your equation to determine what quantity of units will yield a profit of \$26: ____________________

13. Interpret Graph. The graph below shows the distance a cat is from his bowl of milk over time. Which sentence is a good match for the graph?



(a) The cat was 12 feet away from the milk and ran toward it reaching it after 3 seconds.

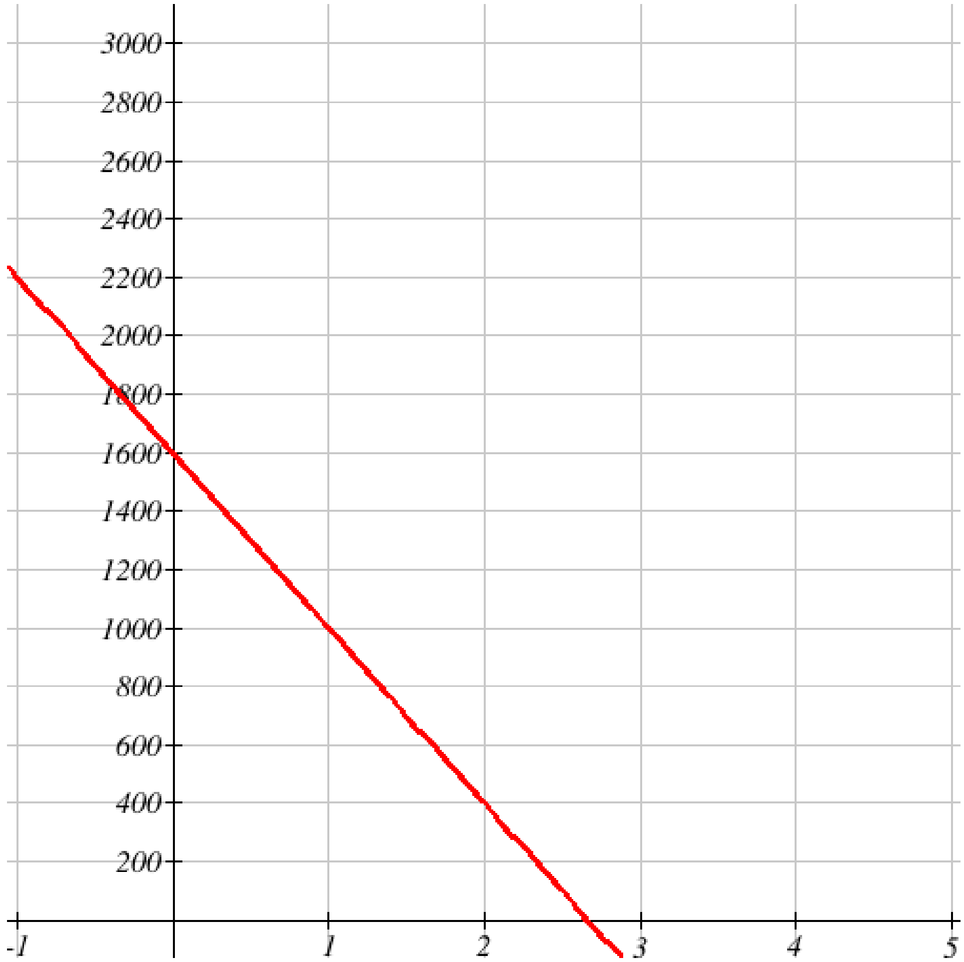
(b) The cat was 3 feet away from the milk and ran toward it reaching it after 12 seconds.

(c) The cat ran away from the milk at a rate of 3 feet per second.

(d) The cat was 12 feet away from the milk and ran away from it at a rate of 3 feet per second.

(e) The cat ran away from the milk at a rate of 4 feet per second.

14. Interpret Graph. The graph below shows the value of Bob's Beanie Baby collection over several years.



The collection's value decreased at a rate of ____________________...

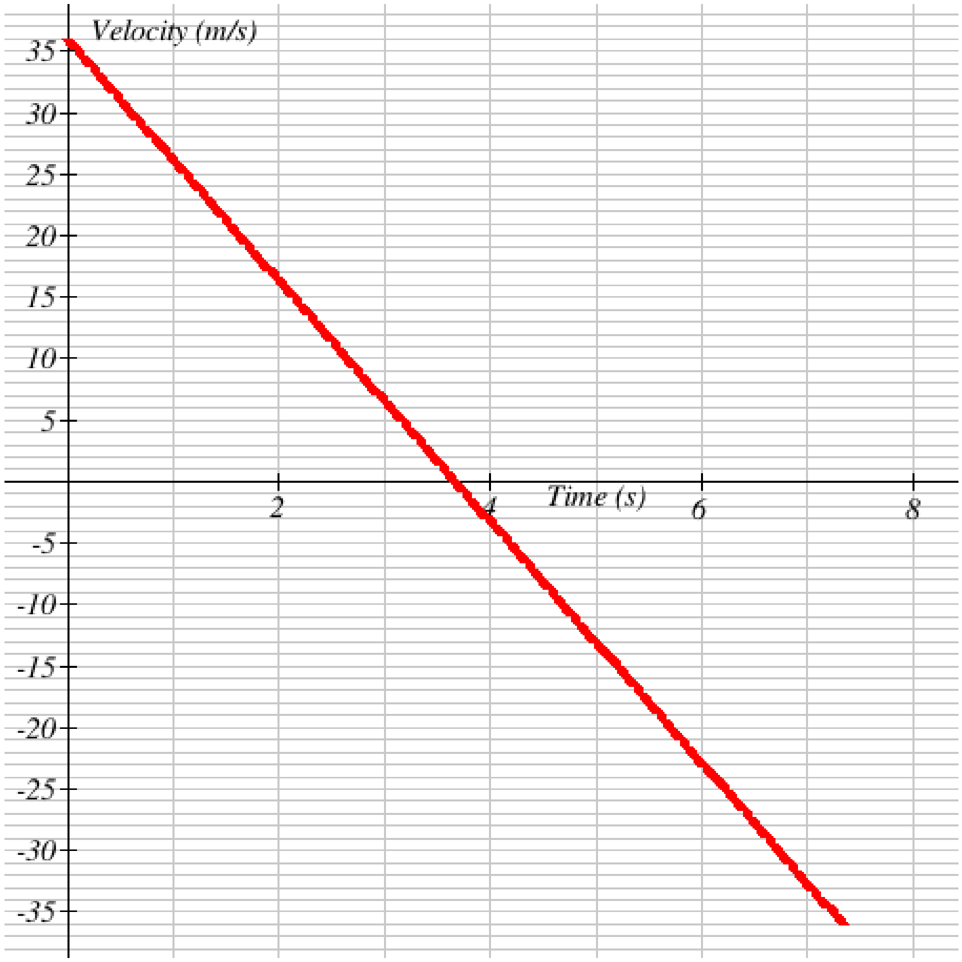
(a) dollars per Beanie Baby

(b) years per dollar

(c) dollars per year

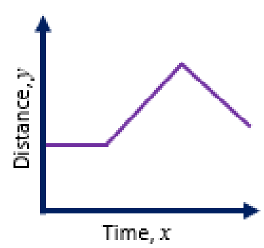
(d) Beanie Babies per dollar

15. Interpret Graph. The graph shows the velocity of a ball that is thrown upwards and remains in the air for 7.35 seconds.



What is the final velocity of the ball?
v_f = ____________________ unit____________________
At approximately what time does the ball change direction?
t_f = ____________________ unit____________________

16. Interpret Graph. The graph shows the distance traveled by a vehicle over time. Describe the change in distance.



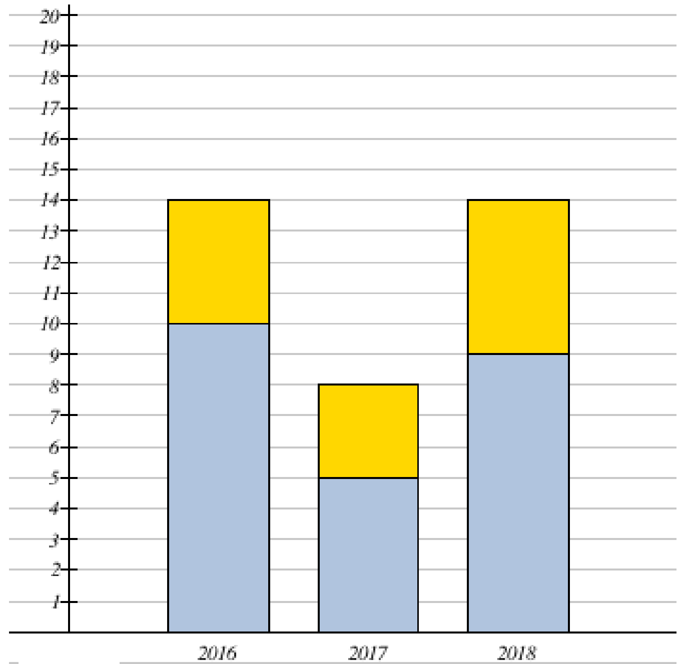
(a) First the distance *increases* at a constant rate. Then the distance *stays the same* for a while before *increasing* at a constant rate.

(b) The distance *stays the same* for a while. Then the distance *increases* at a constant rate before *decreasing* at a constant rate.

(c) First the distance *decreases* at a constant rate. Then the distance *increases* at a constant rate before staying the *same* for a while.

(d) First the distance *increases* at a constant rate. Then the distance *decreases* at a constant rate before staying the *same* for a while.

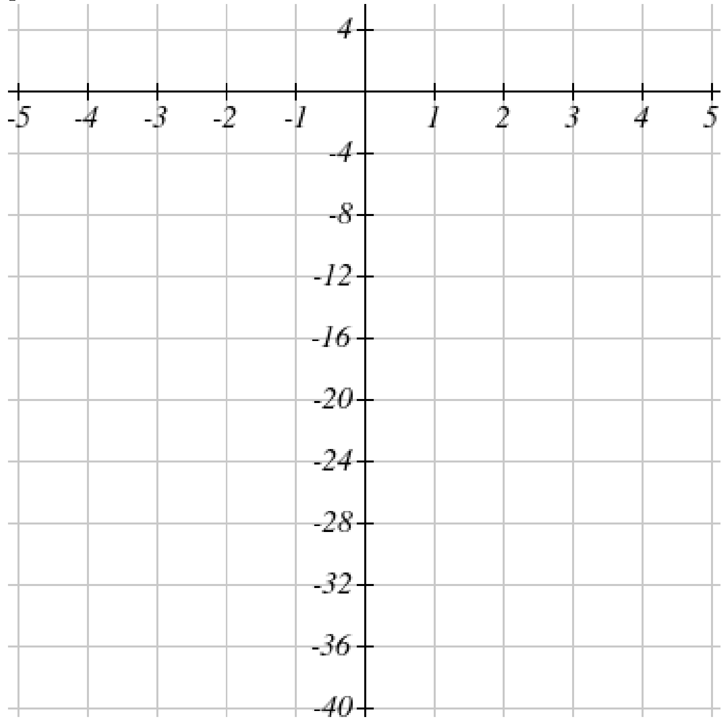
17. Interpret Graph. The park service records the number of reported rattle snake sightings on Bald Mountain Trail (light blue) and Camel Back Trail (gold). The data gives rise to the following stacked bar graph.



(a) How many rattle snake sightings were there on Bald Mountain Trail in 2016? _____
(b) How many rattle snake sightings were there on Camel Back Trail in 2016? _____
(c) How many total rattle snake sightings were there in 2017? _____
(d) What percentage of rattlesnake sightings in 2018 were from Bald Mountain trail? _____%
Report your answer to at least two decimal places.

18. Interpret Graph. In Biology class, students store their specimen's for labs at -22 $^\circ$C. To thaw, a specimen is brought to a refrigeration unit. After 1 hour, the specimen's temperature in the refrigeration unit is -14 $^\circ$C.

a. Sketch a graph to model the situation.



b. Calculate the rate of change. $m =$ _____ degrees per hour
c. The y-intercept is _____. What does it represent in this situation?

(a) The temperature goes up 22 degrees per hour.

(b) The specimen starts at 22 degrees below zero.

(c) The slope is 22.

(d) It takes 22 minutes to warm up.

d. The slope is _____. What does it represent in this situation?

(a) The specimen starts at 8 degrees below zero.

(b) The temperature goes up 8 degrees per hour.

(c) It takes 8 minutes to warm up.

(d) The y-intercept is 8.

19. Interpret Graph. The cost to take a taxi from the airport is a function of the distance driven. A 5 mile taxi ride from the airport costs $12. The cost is $21 for a 10 mile ride.
y is the cost and x is number of miles.
a. Write an equation to model the situation. $y =$ ____________________
b. What is the y-intercept? $b =$ ____ What does it represent?

(a) The cost for getting into the cab is $3.

(b) It takes 5 miles to get home.

(c) You start at $12

(d) The meter drop is $1.80.

c. What is the slope? $m =$ ____ What does it represent?

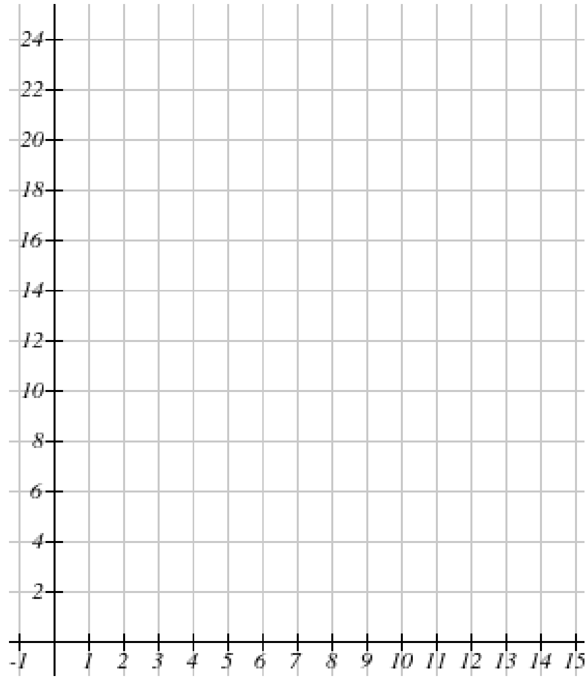
(a) It takes 5 miles to get home.

(b) The meter drop is \$3.

(c) You start at \$3

(d) It costs \$1.80 per mile.

d. Graph the equation.



20. Interpret Graph. In 1991, the moose population in a park was measured to be 3980. By 1997, the population was measured again to be 3680. If the population continues to change linearly:

Find a formula for the moose population, P, in terms of t, the years since 1990.

$P(t) =$ ____________________

What does your model predict the moose population to be in 2004?

____

21. Determine if Linear. Select all of the following tables which could represent a linear function.

(a) **Table 3.2.46**

x	$f(x)$
0	-4
5	6
10	16
15	26

(b) **Table 3.2.47**

x	$g(x)$
0	8
5	-7
10	-22
15	-37

(c)

Table 3.2.48

x	$h(x)$
0	8
5	33
10	108
15	233

(d)

Table 3.2.49

x	$k(x)$
5	13
10	28
20	58
25	73

22. Find Equation. Is the function $g(x) = 6x - 2$ a linear function?

(a) Yes

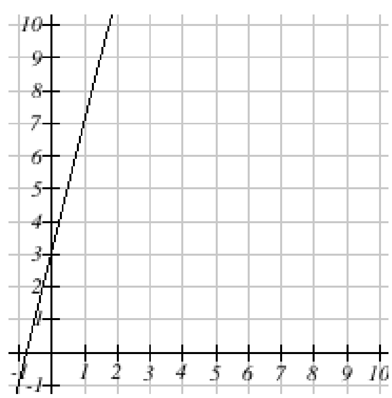
(b) No

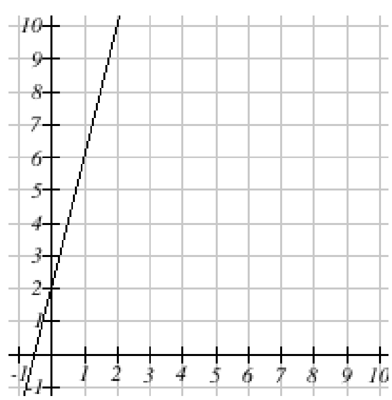
(c) Cannot Be Determined

23. Interpret Graph. Below, one description, one graph, and one equation are equivalent. Choose the proper set.

(a) You start with 3 Xbox games and each month you buy 1 new games.

(b) You start with 2 Xbox games and each month you buy 4 new games.





(a) $y = x + 3$

(b) $y = 4x + 2$

Exercise Group. Answer these questions about lines.

24. Rate from Data. Superman needs to save Lois from the clutches of Lex Luthor. It takes Superman 11 seconds to get to Lois who is 462 feet away. What is Superman's rate?

_____

(a) ft/s or feet per second

(b) s/ft or seconds per foot

25. Rate from Data. You are on an oceanographic research expedition that began in San Juan, Puerto Rico on September 14.

The ship left port at 0630 hr on 14 September and covered a distance of 1612 km to the first drill location (Site 1) where you are going retrieve a drill core of seafloor sediments. The ship arrived at the first drill site at 1800 hr on 16 September.

Calculate the rate of travel (i.e., speed) of the ship during its transit to the first drill site. *Round your answer to the nearest tenth.*

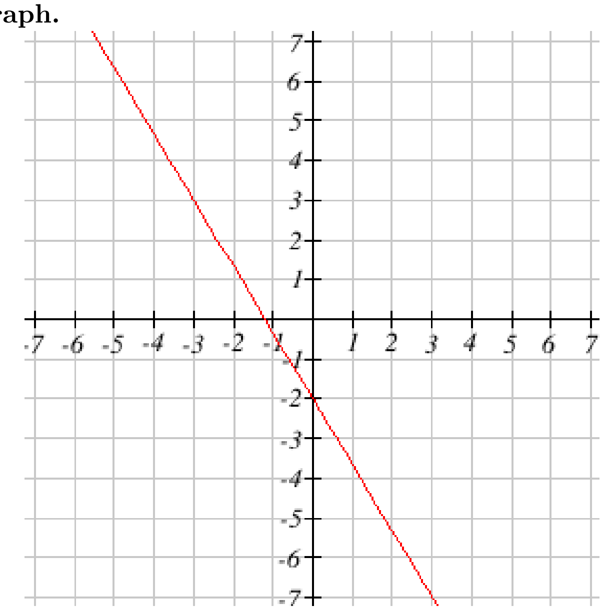
_____ km/hr

26. Find Slope from Points. Find the slope of the line that goes through the points (14,-5) and (13,2).

Slope, $m =$ ____________________

Enter your answer as an integer or a reduced fraction in the form A/B

27. Find Slope from Graph.



Find the slope of the line.

Slope = $m =$ __________

Enter your answer as an integer or as a reduced fraction in the form A/B.

28. Population. A city's population in the year x =1980 was y =3,485,600. In 1957 the population was 3,482,150.

Compute the slope of the population growth (or decline) and choose the most accurate statement from the following:

(a) The population is decreasing at a rate of 300 people per year.

(b) The population is decreasing at a rate of 50 people per year.

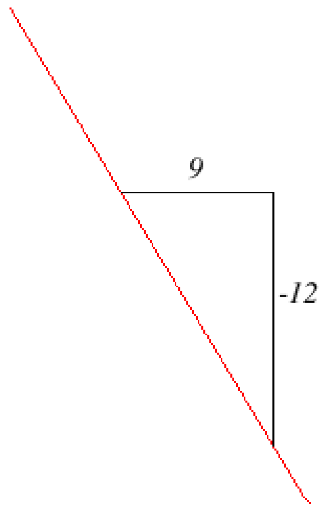
(c) The population is increasing at a rate of 150 people per year.

(d) The population is decreasing at a rate of 150 people per year.

(e) The population is increasing at a rate of 50 people per year.

(f) The population is increasing at a rate of 300 people per year.

29. Identify Slope.



State the run, rise, and slope of the line above.
run = _
rise = _
m = _

30. Equation from Table. Given the table of Celsius and corresponding Fahrenheit temperatures, find the linear relationship where Celsius, c, is the input and Fahrenheit, f, is the output.

Table 3.2.50

Celsius	Fahrenheit
-10	14
0	32
10	50
20	68
30	86
40	104
50	122
60	140

____

31. Equation from Table. Find the constant rate of change described from the table.

Table 3.2.51

hours	dollars
6	64
7	55
8	46
9	37

Rate/Slope: _____

(a) dollars

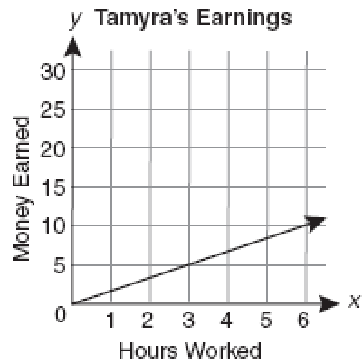
(b) hours per dollar

(c) dollars per hour

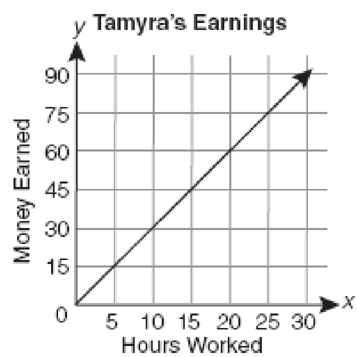
(d) hours

32. Find Slope from Graph. Tamrya is babysitting to earn money to visit her aunt. She earns $5.00 for each hour of babysitting. Which graph represents her earnings from babysitting?

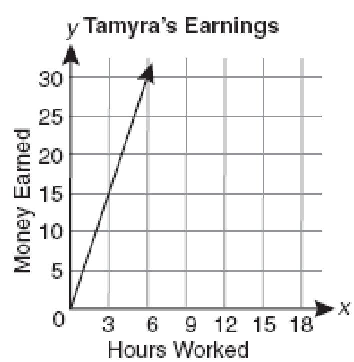
(a)



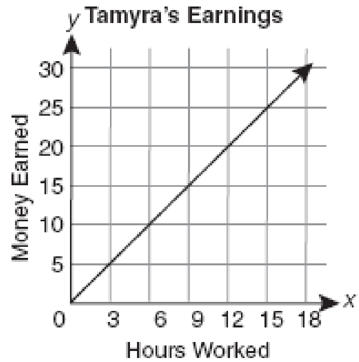
(b)



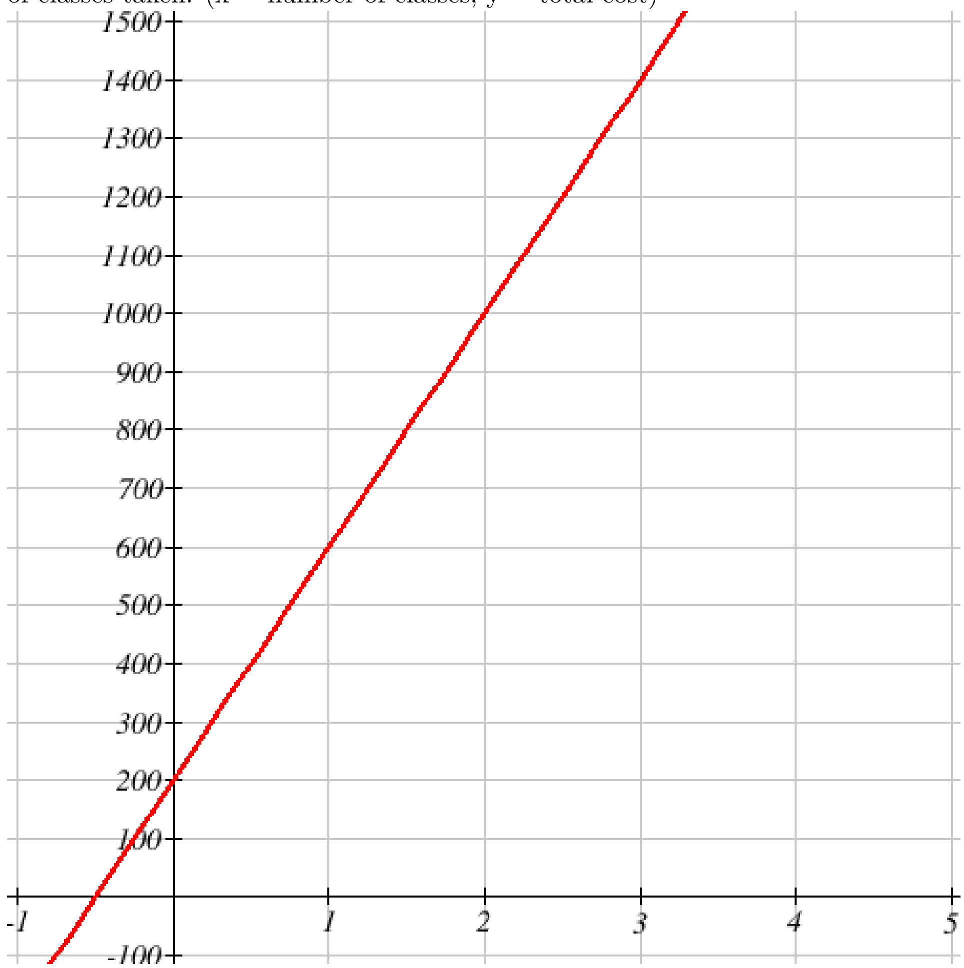
(c)



(d)



33. Tuition. The graph below shows the total cost of attending a particular college based on the number of classes taken. (x = number of classes, y = total cost)



The slope of this line tells us that the cost of attending this college increases by ________________

(a) dollars per year

(b) dollars

(c) dollars per class

(d) classes

$ ____________________

34. Check if Linear. Identify the rate of change and initial value for the linear situation modeled below.

Table 3.2.52

x	y
0	8
2	20
4	32
6	44
8	56

Rate of change: _______
Initial Value: _______

35. Check if Linear. Identify the rate of change and initial value for the linear situation modeled below.

$y = 3x + 7$
Rate of change: _______
Initial Value: _______

36. Write Equation. Find the equation (in terms of x) of the line through the points (-1,3) and (1,1)

$y =$ ____________________

37. Write Eqation. Write the equation in slope-intercept form of the line that has slope -2 and y-intercept $(0, -3)$.

$y =$ ____________________

38. Write Eqation. Suppose you start with a full tank of gas (17 gallons) in your truck. After driving 5 hours, you now have 2 gallons left.

If x is the number of hours you have been driving, then y is the number of gallons left in the tank.

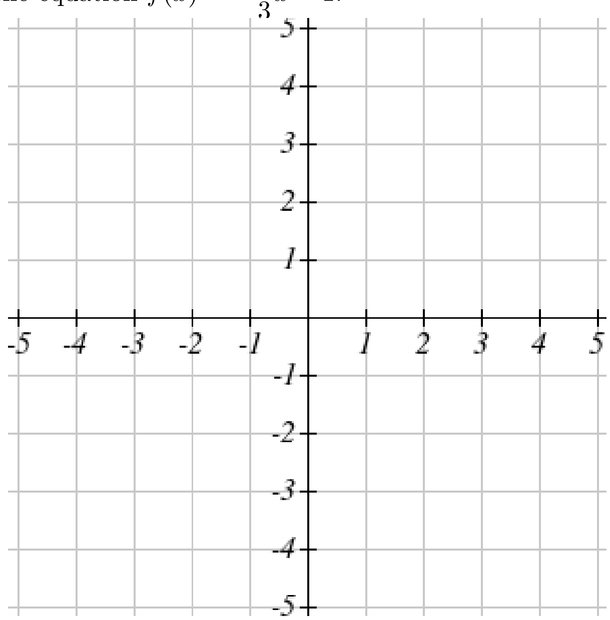
At what rate is the gas left in the tank changing? State your answer as a reduced fraction.

_ gallons per hour

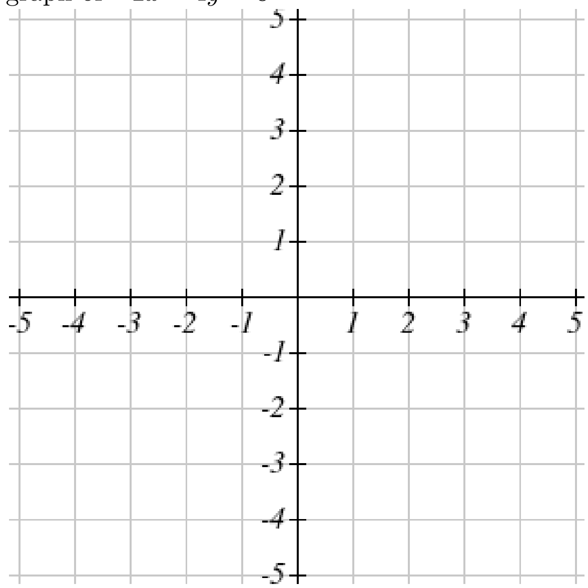
Find an equation of a line in the form y = mx + b that describes the amount of gas in your tank.

$y =$ _____

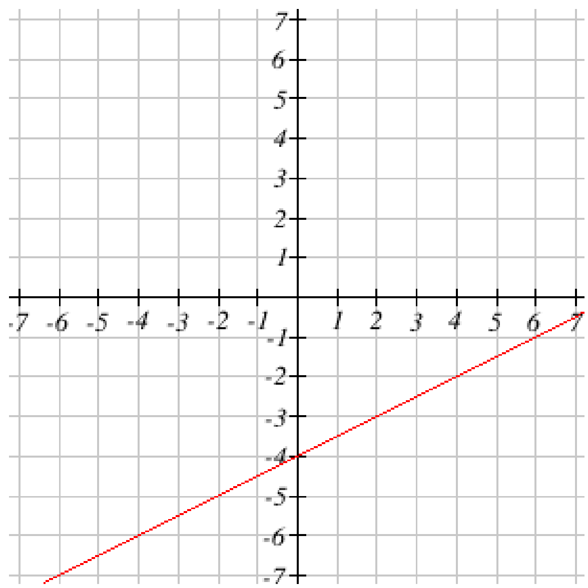
39. Graph Line. Graph the equation $f(x) = -\frac{4}{3}x - 1$.



40. Graph Line. Sketch a graph of $-2x - 4y = 8$



41. Graph Line.



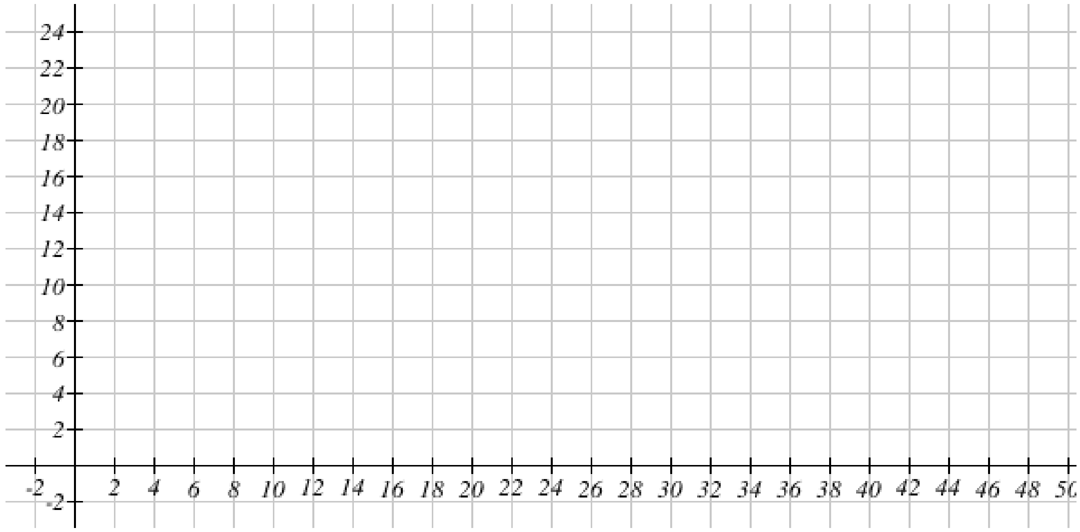
What is the slope of the graph? Leave your answer in simplest form.
Slope = ___
Identify the y-intercept.
y-intercept = (0, __)
Write an equation in slope-intercept form for the graph above.
y = ______

42. Write Equation and Graph Line. Prince John has 17 bags of gold. While he is sleeping, Robin Hood is stealing one bag every two minutes.

(a) Write an equation in slope-intercept form to model the situation, where B is the number gold bags and t is the time in minutes.

B = ____________________

(b) Graph your equation below.



3.3 Identifying Rates

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models

This section covers the following mathematical concepts.

- Identify rates as linear, quadratic, exponential, or other (critical thinking)

So far we have looked at linear models. We will add quadratic, exponential, and some variations in later sections. One of the ways we distinguish between models is by the rate at which they grow. Often the rate at which something is happening is more important than how much there currently is. This section presents two methods for identifying rates from tables of data. How to identify each type by graph is presented in the appropriate chapter and section.

3.3.1 Differences

One way to measure rates is to look at the differences between data points. Calculating these differences is illustrated in the table below.

Table 3.3.1 Tien's Salary

Year	Salary	Difference
2017	\$52,429.33	
2018	\$55,050.80	\$55,050.80-\$52,429.33=\$2,621.47
2019	\$57,803.34	\$57,803.34-\$55,050.80=\$2,752.54
2020	\$60,693.50	\$60,693.50-\$57,803.34=\$2,890.17
2021	\$63,728.18	\$63,728.18-\$60,693.50=\$3,034.68
2022	\$66,914.59	\$66,914.59-\$63,728.18=\$3,186.41
2023	\$70,260.32	\$70,260.32-\$66,914.59=\$3,345.73
2024	\$73,773.33	\$73,773.33-\$70,260.32=\$3,513.02

In order to see how these differences can help us distinguish between linear and other models, consider Vasya's salary in Example 3.2.19. We know that the difference between each year's salary is \$5000.00, because we are told that was the raise each year. This is linear model. In contrast Tien's raises are different each year (they grow year to year). This means his salary does not grow linearly.

Example 3.3.2 Differences for Atmospheric Pressure Model. Consider the model in Example 3.2.33. Table 3.3.3 calculates the differences every 2000 ft. Notice that the differences are all the same namely -2.00. This model is linear.

Table 3.3.4 calculates the differences every 1000 ft. In this table the differences are all -1.00. This still indicates that the model is linear. It does not matter what interval we choose. If the differences over evenly spaced intervals are the same, then the model is linear.

The differences we obtained do match. Consider $\frac{-2.00}{2000} = \frac{-1.00}{1000}$. When written as a ratio the differences are the same number. □

Table 3.3.3 Atmospheric Pressure Differences (2000 ft intervals)

Altitude (ft MSL)	Expected Pressure (inHg)	Difference
0	29.92	
2000	27.92	27.92-29.92=-2.00
4000	25.92	25.92-27.92=-2.00
6000	23.92	23.92-25.92=-2.00
8000	21.92	21.92-23.92=-2.00

Table 3.3.4 Atmospheric Pressure Differences (1000 ft intervals)

Altitude (ft MSL)	Expected Pressure (inHg)	Difference
0	29.92	
1000	28.92	28.92-29.92=-1.00
2000	27.92	27.92-28.92=-1.00
3000	26.92	26.92-27.92=-1.00
4000	25.92	25.92-26.92=-1.00

Definition 3.3.5 Linear Relation. A relation is **linear** if and only if the rate of change is constant. ◊

This states that a linear model grows by the same amount from one step to the next (rate of change or difference). This equal growth results from the ratio m in the form $y = mx + b$. Consider the specific case $y = 3x + 2$. In the table below notice that the differences are all the same and that the difference is 3. 3 is the ratio from the equation. This is always the case. The slope is how fast the line grows.

Table 3.3.6 Differences for Lines

x	y	Difference
0	2	
1	5	5-2=3
2	8	8-5=3
3	11	11-8=3
4	14	14-11=3

Note we used this constant addition property when working with ratio problems like Example 2.3.4. Relations defined by fixed ratios like these are linear.

For linear data consecutive differences are always the same. The next examples (Example 3.3.7 to Example 3.3.10) illustrate known, non-linear data and how the differences for those look.

Example 3.3.7 Quadratic Data. Consider Table 3.3.8. The first differences (what we calculated above) are not the same. Thus this data is not linear.

However, the first differences increase in a suspiciously simple pattern. Checking the second differences (the differences of the 1st differences) we see a linear pattern. This turns out to be the pattern for all **quadratic** data. □

Table 3.3.8 Quadratic Data

n	n^2	1st difference	2nd difference
1	1		
2	4	4-1=3	
3	9	9-4=5	5-3=2
4	16	16-9=7	7-5=2
5	25	25-16=9	9-7=2
6	36	36-25=11	11-9=2

Definition 3.3.9 Quadratic Relation. A relation is **quadratic** if and only if the second differences are constant. ◊

Example 3.3.10 Exponential Data. Consider Table 3.3.11. The differences are not the same nor do they show the same pattern of quadratics. However, there is a pattern in the differences. Notice that the differences are exactly equal to the original data. This means that the rate of increase is determined by the current scale. In other words, the bigger it is, the faster it grows. This is the pattern of data that varies exponentially. □

Table 3.3.11 Exponential Data

n	2^n	Difference
1	2	
2	4	4-2=2
3	8	8-4=4
4	16	16-8=8
5	32	32-16=16
6	64	64-32=32

The next example illustrates that the differences for exponential data are not always exactly equal to the data.

Example 3.3.12 Exponential Data Differences. Consider Table 3.3.13. The differences are not exactly equal to the original numbers. However, notice that $6 = 2 \cdot 3$ $18 = 2 \cdot 9$, and $54 = 2 \cdot 27$. The differences are double the original numbers. In general for exponential data the differences will be the original data scaled by some number.

Happily there is an easier way to determine that data is exponential shown in the next section. □

Table 3.3.13 Exponential Data with Scale

n	3^n	Difference
1	3	
2	9	9-3=6
3	27	27-9=18
4	81	81-27=54
5	243	243-81=162
6	729	729-243=486

Definition 3.3.14 Exponential Relation. A relation is **exponential** if and only if the differences are a multiple of the original values, that is the rate is proportional to the value. ◊

3.3.2 Quotients

The previous section analyzed change as the difference (subtraction) of consecutive numbers (salaries in these examples). This section analyzes change using the percent increase for each pair of consecutive numbers.

Example 3.3.15 Salary Percent Incease. We will first calculate the percent increase of salary each year for Tien and Vasya. Because salary numbers are exact, we will not use significant digits. Rather we will round to the nearest percent. This is in Table 3.3.16 and Table 3.3.17.

Notice that for Tien the percent increase is the same each year. It is 5%. For Vasya, the percent increase is not the same each year. How does the percent increase change for her? □

Table 3.3.16 Percent Increase for Tien

Year	Ratio		Increase
2018	\$55,050.80/\$52,429.33	=1.05	5%
2019	\$57,803.34/\$55,050.80	=1.05	5%
2020	\$60,693.50/\$57,803.34	=1.05	5%
2021	\$63,728.18/\$60,693.50	=1.05	5%

Table 3.3.17 Percent Increase for Vasya

Year	Ratio		Increase
2018	\$67,347.23/\$62,347.23	=1.08	8%
2019	\$72,347.23/\$67,347.23	=1.07	7%
2020	\$77,347.23/\$72,347.23	=1.07	7%
2021	\$82,347.23/\$77,347.23	=1.06	6%

Definition 3.3.18 Exponential. A relation is **exponential** if and only if the percent increase is constant. ◊

Example 3.3.19 The table below gives an amount of caffeine in the blood stream. This data is exponential with a ratio of 0.87. This ratio means there is a 13% decrease per hour of caffeine in the blood stream per hour in this example. This is in contrast to the previous example which was an increasing exponential.

Hour	Caffeine	Ratio
0	95 mg	
1	83 mg	$83/95 \approx 0.87$
2	72 mg	$72/83 \approx 0.87$
3	63 mg	$63/72 \approx 0.87$
4	55 mg	$55/63 \approx 0.87$
5	48 mg	$48/55 \approx 0.87$
6	41 mg	$41/48 \approx 0.87$
7	36 mg	$36/41 \approx 0.87$
8	31 mg	$31/41 \approx 0.87$

□

Although Definition 3.3.14 and Definition 3.3.18 are phrased differently they both accurately describe exponential relations. Generally it is easier to test if data is exponential by testing the ratios of terms rather than the differences. Table 3.3.20 shows an example of analyzing data using both differences and ratios. Notice that the differences are a scaled version of the original data (scaled by 1/3). The ratio from the quotient is 4/3 which gives about a 33% increase. For the curious the data was generate by $5\left(\frac{4}{3}\right)^n$.

Table 3.3.20 Exponential Data 2 Ways

Data	Difference	Ratio
$\frac{20}{3}$		
$\frac{80}{9}$	$\frac{20}{9} = \frac{1}{3} \cdot \frac{20}{3}$	$\frac{4}{3} \approx 1.33$
$\frac{320}{27}$	$\frac{80}{27} = \frac{1}{3} \cdot \frac{80}{9}$	$\frac{4}{3} \approx 1.33$
$\frac{1280}{81}$	$\frac{320}{81} = \frac{1}{3} \cdot \frac{320}{27}$	$\frac{4}{3} \approx 1.33$
$\frac{5120}{243}$	$\frac{1280}{243} = \frac{1}{3} \cdot \frac{1280}{81}$	$\frac{4}{3} \approx 1.33$
$\frac{20480}{729}$	$\frac{5120}{729} = \frac{1}{3} \cdot \frac{5120}{243}$	$\frac{4}{3} \approx 1.33$

Checkpoint 3.3.21 Determine whether each person's salaries followed a linear or an exponential growth pattern.

Table 3.3.22

Year	Hillary	Marco	Bautista
2019	44105	42395.83	66051
2020	44987.1	43630.66	67372.02
2021	45886.84	44865.49	68719.46
2022	46804.58	46100.32	70093.85
2023	47740.67	47335.15	71495.73
2024	48695.48	48569.98	72925.64

Hillary:

1. linear
2. exponential

Marco:

1. linear
2. exponential

Bautista:

1. linear
2. exponential

3.3.3 Extrapolation

In Example 3.2.7 we found a value between two entries in a table. That is interpolation. In other cases we want to find a value past the end of a table. This is called **extrapolation**.

Example 3.3.23 Extrapolation from a Table. Based on Table 3.3.1 what do we expect Tien's salary to be in 2022, 2025?

From Example 3.3.15 we know each entry is 1.05 times the previous year's salary. Because that was the pattern every year, we might safely suppose it will occur again. Thus we expect his 2022 salary to be $1.05 \cdot \$73,773.33 \approx \$77,462.00$.

To extrapolate to 2025 we repeat this process for 2023, 2024, and 2025.

$$1.05 \cdot \$77,462.00 \approx \$81,335.10.$$
$$1.05 \cdot \$81,335.10 \approx \$85,401.85.$$
$$1.05 \cdot \$85,401.85 \approx \$89,671.94.$$

If his raises continue at the same rate he will have a salary of \$89,671.94 in 2025. □

Example 3.3.24 Based on Table 3.2.20 what do we expect Vasya's salary to be in 2025, 2028?

From Example 3.2.19 we know that Vasya has received a \$5,000 raise each year. Because that has been the pattern, we might safely suppose it will continue. Thus we expect her 2025 salary to be

$$\$97,347.23 + \$5,000.00 = \$102,347.23.$$

To calculate her expected 2028 salary we note that is 4 years after 2024, so she should have four raises of \$5,000 each. Her expected salary will be

$$\$97,347.23 + 4(\$5,000.00) = \$117,347.23.$$

□

Notice that in both of these examples we needed to know the growth rate (i.e., exponential or linear respectively) before we could extrapolate.

Checkpoint 3.3.25 Determine whether Jeffrey's salary follows a linear or an exponential growth pattern. Then extrapolate to determine the expected salary in 2028.

Table 3.3.26

Year	Jeffrey
2019	69106.04
2020	69107.08
2021	69108.12
2022	69109.16
2023	69110.2
2024	69111.24

The salary growth is

1. linear
2. exponential

The salary in 2028 is expected to be ____________________

3.3.4 Exercises

1. Determine Rate. For each table below, could the table represent a function that is linear, exponential, or neither?

Table 3.3.27

x	1	2	3	4
$f(x)$	70	40	10	-20

f(x) is

(a) Exponential

(b) Linear

(c) Neither

Table 3.3.28

x	1	2	3	4
$g(x)$	70	49	34.3	24.01

g(x) is

(a) Exponential

(b) Linear

(c) Neither

Table 3.3.29

x	1	2	3	4
$h(x)$	40	-11	-55.7	-95.99

h(x) is

(a) Exponential

(b) Linear

(c) Neither

2. Check if Linear. Is the following equation linear and consequently produce a graph that is a straight line?

$y = x^2 + 2x + 6$

(a) Yes

(b) No

3. Check if Linear. Is the following equation linear and consequently produce a graph that is a straight line?

$4x + 5y = 11$

(a) Yes

(b) No

4. Check if Linear. Is the following equation linear and consequently produce a graph that is a straight line?

$7x = 4 - 9y$

(a) Yes

(b) No

5. Check if Linear. Is the following equation linear and consequently produce a graph that is a straight line?

$16y = \dfrac{5}{x - 21}$

(a) Yes

(b) No

6. Find Slope for Linear. Determine Linearity and Slope from a Table

For each of the following functions, determine if the function is linear.

If it is linear, give the slope. If it is not linear, enter "DNE" for the slope.

Table 3.3.30

x	1	13	15	17	19
$f(x)$	-22	-46	-50	-54	-58

Behavior:

(a) Not Linear

(b) Linear

Slope: ____________________

Table 3.3.31

x	4	6	8	15	19
$f(x)$	1	-4	-7.5	-26.5	-36.5

Behavior:

(a) Not Linear

(b) Linear

Slope: ____________________

Table 3.3.32

x	4	5	7	9	18
$f(x)$	18	24.5	37.5	50.5	109

Behavior:

(a) Linear

(b) Not Linear

Slope: ____________________

Table 3.3.33

x	3	7	8	9	13
$f(x)$	3	27	38	45	73

Behavior:

(a) Linear

(b) Not Linear

Slope: ____________________

7. Determine Rate. One of the patterns in these tables is a quadratic relationship. Which table is quadratic?

(a) **Table 3.3.34**

x	y
0	3
1	9
2	15
3	21
4	27
5	33

(b) **Table 3.3.35**

x	y
0	5
1	10
2	20
3	40
4	80
5	160

(c) **Table 3.3.36**

x	y
0	3
1	7
2	15
3	27
4	43
5	63

8. Determine Rate.

Table 3.3.37

x	y
0	4
1	6
2	10
3	16
4	24
5	34

is:

(a) Linear

(b) Exponential

(c) Quadratic

Table 3.3.38

x	y
0	4
1	8
2	16
3	32
4	64
5	128

is:

(a) Linear

(b) Exponential

(c) Quadratic

Table 3.3.39

x	y
0	7
1	12
2	17
3	22
4	27
5	32

is:

(a) Linear

(b) Exponential

(c) Quadratic

9. Determine Rate. Identify which type of pattern each table is. Continue each pattern, filling in the missing rows.

Table 3.3.40

x	y
0	3
1	4
2	6
3	9
4	__
5	__
6	24

is:

(a) Linear

(b) Exponential

(c) Quadratic

Table 3.3.41

x	y
0	9
1	18
2	36
3	72
4	__
5	__
6	576

is:

(a) Linear

(b) Exponential

(c) Quadratic

Table 3.3.42

x	y
0	7
1	9
2	11
3	13
4	__
5	__
6	19

is:

(a) Linear

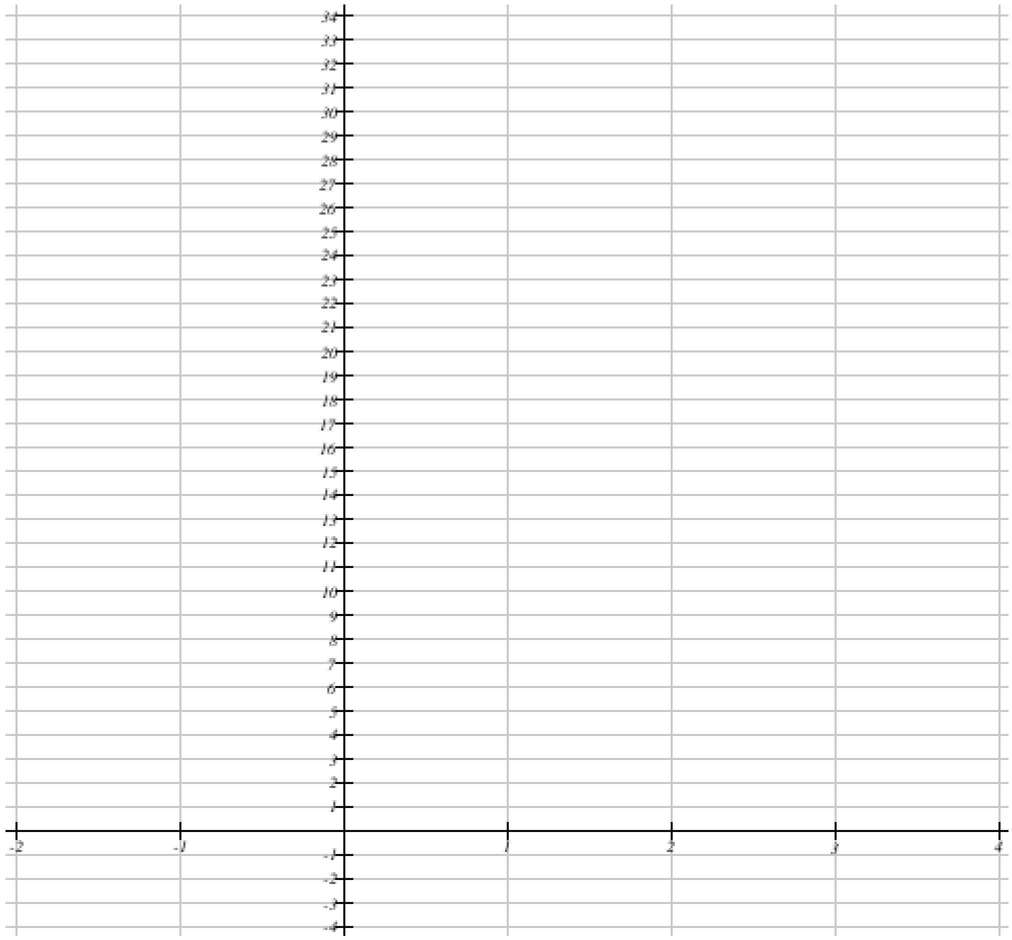
(b) Exponential

(c) Quadratic

10. Determine Rate. Directions: Graph each set of values (just the points) and determine whether the function is linear, quadratic, or exponential.

Table 3.3.43

x	-1	0	1	2	3
y	1	-3	1	13	33



The function type is:

(a) Linear

(b) Quadratic

(c) Exponential

I know because the graph/table shows that

(a) the rate of change is constant

(b) the change of the change is constant

(c) each subseqent value is multiplied by a constant number

11. Determine Rate. For each scenario, identify the appropriate growth model that describes how it's changing.

(a) The number of new polio cases has been cut in half each year due to vaccination efforts

(b) The amount of pollutants in the lake has been increasing by 4 milligrams per Liter each year

(c) Tuition is currently $2,000 a quarter has been growing by 7% a year

(d) The number of arrests grew for several years, but now has been decreasing

(a) Linear

(b) Exponential

(c) Neither

12. Validity of Model. Your friend Pat says to you, "I'm on a new diet plan and I've been able to lose about a pound a week." Come up with a linear equation that models this situation and use it to answer the following questions:
If Pat currently weighs 225 pounds, how much would Pat weigh in a year (52 weeks)?
____________________ pounds
Assuming Pat stays on this plan and the equation is still valid, how much would Pat weigh in 4 years?
____________________ pounds
Does this equation still seem like it would be valid after 4 years? Why or why not?

13. Write next terms. For the following sequence, state the next three terms.
5, 9, 13, 17, ___, ___, ___
State a formula for the nth term: ______
What type of sequence is this?

(a) arithmetic (linear)

(b) quadratic

(c) geometric (exponential)

(d) other

14. Write next terms. For the following sequence, state the next three terms.
4, 8, 16, 32, ___, ___, ___
What type of sequence is this?

(a) arithmetic (linear)

(b) quadratic

(c) geometric (exponential)

(d) other

15. Write next terms. For the following sequence, state the next three terms.
3, 5, 8, 12, ___, ___, ___
What type of sequence is this?

(a) arithmetic (linear)

(b) quadratic

(c) geometric (exponential)

(d) other

3.4 Variation

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Identify rates as linear, quadratic, exponential, or other (critical thinking)
- Identify data varrying directly or indirectly (critical thinking)
- Solve linear, rational, quadratic, and exponential equations and formulas (skill)

Section 3.3 presents how to determine the rate at which one variable grows with respect to another. In all of those cases we considered only one variable changing. However, many models have more than one variable or parameter, and we wish to analyze the impact each one has on the result. This section will also present one more type of rate.

3.4.1 Analyzing Models with Multiple Variables

This first example illustrates how a result can be linear with respect to one or more variables and quadratic (or other relationships) with another.

Example 3.4.1 Review Model 1.3.3. Consider how lift changes with respect to each of the parameters.

Consider ρ the density of air. If the other parameters remain constant (only density changes), then the model looks like

$$\begin{aligned} L &= \frac{1}{2}\rho s C_L v^2 \\ &= \left(\frac{1}{2} s C_L v^2\right)\rho \\ &= k\rho. \end{aligned}$$

All we used is commutativity of multiplication. We can see that lift (L) changes linearly with respect to air density (ρ).

Consider s the surface area of the airfoil (wing or propeller). If the other parameters remain constant (only surface area changes), then the model looks like

$$\begin{aligned} L &= \frac{1}{2}\rho s C_L v^2 \\ &= \left(\frac{1}{2}\rho C_L v^2\right) s \\ &= ks. \end{aligned}$$

We can see that lift (L) changes linearly with respect to surface area (s).

Consider C_L the coefficient of lift of the airfoil. If the other parameters remain constant, then the model looks like

$$\begin{aligned} L &= \frac{1}{2}\rho s C_L v^2 \\ &= \left(\frac{1}{2}\rho s v^2\right) C_L \\ &= kC_L. \end{aligned}$$

We can see that lift (L) changes linearly with respect to the coefficient of lift (C_L).

Consider v the velocity. If the other parameters remain constant, then the model looks like

$$\begin{aligned} L &= \frac{1}{2}\rho s C_L v^2 \\ &= \left(\frac{1}{2}\rho s C_L\right) v^2 \\ &= kv^2. \end{aligned}$$

We can see that lift (L) changes quadratically with respect to velocity (v^2).

How can we apply this knowledge? Lift changes linearly with respect to all parameters except for velocity. If greater lift (to handle greater weight) is needed, velocity provides bigger bang for our buck than any other change. □

This second example has us look at the impact on more than just one variable. It also introduces a new relationship.

Example 3.4.2 Review Model 1.3.3.

First consider the impact temperature (T) has on pressure (P). If the other parameters remain constant, then the model looks like

$$\begin{aligned}
PV &= nRT. \\
P &= \frac{nRT}{V}. && \text{Divide to isolate } P \\
P &= \frac{nR}{V}T. && \text{Use commutativity} \\
P &= kT.
\end{aligned}$$

Pressure (P) grows linearly with respect to temperature.

Notice that we can write that last equation at $\frac{1}{k}P = T$ which means we can also make the reverse statement: temperature increases linearly with pressure.

Next consider the impact temperature (T) has on volume (V). If the other parameters remain constant, then the model looks like

$$\begin{aligned}
PV &= nRT. \\
V &= \frac{nRT}{P}. \\
V &= \frac{nR}{P}T. \\
V &= kT.
\end{aligned}$$

Volume (V) also grows linearly with respect to temperature.

Finally consider the impact volume (V) has on pressure (P). If the other parameters remain constant, then the model looks like

$$\begin{aligned}
PV &= nRT. \\
P &= \frac{nRT}{V}. \\
P &= (nRt)\frac{1}{V}. \\
P &= k \cdot \frac{1}{V}.
\end{aligned}$$

This pattern is not linear (nor quadratic, nor exponential). If volume is increased the right hand side will decrease (dividing by a larger number). This means that pressure will decrease. Conversely if volume is decreased the right hand side will increase (divide by a smaller number). This means that pressure will increase. □

Definition 3.4.3 Vary Directly. For two quantities a and b, if increasing b increases a, then a is said to **vary directly** with b. ◊

Definition 3.4.4 Vary Indirectly. For two quantities a and b, if increasing b decreases a, then a is said to **vary indirectly** with b. ◊

Example 3.2.29 illustrated a model where the increase in one variable caused a decrease in the other. We used an experiment (plugging in numbers) to discover the inverse relationship.

If two quantities vary directly, the model may be linear or quadratic. If two quantities vary inversely, the model may be neither linear nor quadratic. In contrast the next example illustrates two quantities can vary inversely, and the model is still exponential.

Example 3.4.5 Decreasing Exponential. Review the data in the table below. First, notice that a decreases as n increases which means a varies inversely with n. Second, note that the data is exponential with a ratio of 1/2. All exponential models that are decreasing have this property.

n	$a = \frac{1}{2^n}$	Ratio
1	1/2	
2	1/4	1/2
3	1/8	1/2
4	1/16	1/2
5	1/32	1/2
6	1/64	1/2

□

3.4.2 More Model Usage

In the section above and in Section 3.3 we determined growth rates (e.g., linear, quadratic exponential, and direct and inverse variation) by generating data and checking the consecutive differences and/or the consecutive ratios. In this section we learn a method to directly calculate the relationship.

Model 3.4.6 *Pressure is a measure of the amount of force spread over an area.*

$$P = \frac{F}{A}$$

where

- *P is pressure,*
- *F is the force in units of pounds (lbs) or Newtons (N),*
- *A is the area in units of square inches, or meters, or similar.*

Example 3.4.7 Walking on snow (or mud) is difficult because our feet tend to puncture the snow. We can use snowshoes to avoid this problem. The reduction in pressure is the reason snowshoes work. This is a science model with measurements, we will round using significant digits.

Suppose Guido weighs 172 lbs. Each foot has an area of $22\,\text{in}^2$. The pressure he exerts on the snow is

$$\begin{aligned} P &= \frac{172 \text{ lbs}}{2 \cdot 22 \text{ in}^2} \\ &= \frac{172 \text{ lbs}}{44 \text{ in}^2} \\ &\approx 3.909090 \text{ lbs/in}^2 \\ &\approx 3.9 \text{ psi.} \end{aligned}$$

If he wears snowshoes that have a surface area of $144\,\text{in}^2$, what is the pressure?

$$\begin{aligned} P &= \frac{172 \text{ lbs}}{2 \cdot 144 \text{ in}^2} \\ &= \frac{172 \text{ lbs}}{288 \text{ in}^2} \\ &\approx 0.597222 \text{ lbs/in}^2 \\ &\approx 0.60 \text{ psi.} \end{aligned}$$

□

The next example illustrates how we can construct another model when we are interested in only some of the variables.

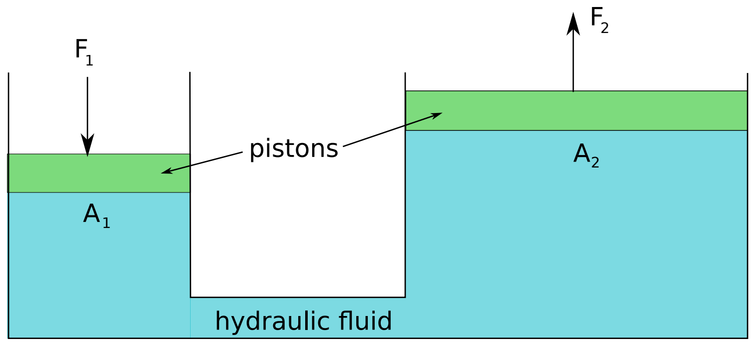


Figure 3.4.8 Hydraulic Press

Model 3.4.9 Hydraulics. *Consider the situation in Figure 3.4.8. The pressure exerted on a fluid by a piston is the ratio of the force exerted and the area of the piston. On the left that is* $P = \frac{F_1}{A_1}$*. The same is true on the right* $P = \frac{F_2}{A_2}$*. Because the hydraulic fluid is contiguous the pressure is the same on both sides. Thus*

$$\frac{F_1}{A_1} = \frac{F_2}{A_2}.$$

Example 3.4.10 If the left piston has area 16 cm^2, and 5.0 N of force is exerted, what is the force exerted on the second piston if it has area 25 cm^2? This is a science model with measurements; we will round using significant digits.

Notice this is a proportion problem. We can use the relationship

$$\begin{aligned}\frac{5.0\text{ N}}{16\text{ cm}^2} &= \frac{F_2}{25\text{ cm}^2}.\\ \frac{5.0\text{ N}}{16\text{ cm}^2}\cdot(25\text{ cm}^2) &= F_2.\\ 7.8125 &= F_2.\\ 7.8\text{ N} &\approx F_2.\end{aligned}$$

□

The next example illustrates how we can determine how one variable varies with respect to another by direct calculation. Specifically, consider how changing the size of the second piston affects the resulting force.

Example 3.4.11 If the second piston's area is changed from 25 cm^2 to 36 cm^2, what is the increase or decrease in force?

Solution. First we setup the same problem as in Example 3.4.10

$$\begin{aligned}\frac{5.0\text{ N}}{16\text{ cm}^2} &= \frac{F_2}{36\text{ cm}^2}.\\ \frac{5.0\text{ N}}{16\text{ cm}^2}\cdot(36\text{ cm}^2) &= F_2.\\ 11.25 &= F_2.\\ 11\text{ N} &\approx F_2.\end{aligned}$$

The piston is larger and the force is also larger. Thus we know that force and area vary directly. Note the equation also shows us that they increase linearly, because all terms are linear. In particular $F_2 = A_2\left(\frac{F_1}{A_1}\right)$ is in the form of a line with ratio F_1/A_1 and no shift. □

Model 1.3.5 gives two forms of the ideal gas law. The second is produced from the first in the same way as the hydraulics model is produced from the pressure definition. To illustrate consider the case of the air/water mixture in a pressure cooker. First recall the ideal gas law $PV = nRT$. The pressure cooker changes the temperature of the gas, but the volume, number of molecules and gas constant do not change. Thus we have $P_1V = nRT_1$ and $P_2V = nRT_2$. In both cases we can solve for the shared variables. Note

$$\begin{aligned} P_1V &= nRT_1 \\ P_1V \cdot \frac{1}{VT_1} &= nRT_1 \cdot \frac{1}{VT_1} \\ \frac{P_1}{T_1} &= \frac{nR}{V} \end{aligned}$$

Similarly $\frac{P_2}{T_2} = \frac{nR}{V}$. Because both expressions equal nR/V we can set them equal giving $\frac{P_1}{T_1} = \frac{P_2}{T_2}$ which is the other model.

The example above produces a second form of the ideal gas law. This example produces a second form of Ohm's Law. In many applications the voltage is fixed. For example it might be 12V from a car battery. Thus if we are considering current and resistance in two places we have $V = I_1R_1$ and $V = I_2R_2$. Because voltage is the same (same source of electricity) we have $I_1R_1 = I_2R_2$.

Example 3.4.12 If the current is 3 amps when the resistance is 8 Ohms, what will the current be when the resistance is 6 Ohms? 4 Ohms?

Solution. We can write

$$\begin{aligned} I_1R_1 &= I_2R_2 \\ (3\text{ amps})(8\Omega) &= I_2(6\Omega) \\ \frac{(3\text{ amps})(8\Omega)}{6\Omega} &= I_2 \\ 4\text{ amps} &= I_2. \end{aligned}$$

The second case can be written

$$\begin{aligned} (3\text{ amps})(8\Omega) &= I_2(4\Omega) \\ \frac{(3\text{ amps})(8\Omega)}{4\Omega} &= I_2 \\ 6\text{ amps} &= I_2. \end{aligned}$$

Notice as the resistance decreased, the current increased. Current varies inversely with resistance. □

3.4.3 Limitations of Models

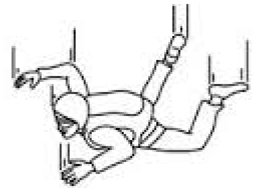
In Example 3.4.2 we solved the model equation to see how one value changes with respect to another. We considered how temperature impacts pressure if the other properties do not change. We also considered how temperature impacts volume if the other properties do not change. We might ask how how volume impacts temperature if the other properties do not change.

We will not however, because this is impossible. If volume is reduced (e.g., we press down on the handle of a tire pump and do not let the air out), then the pressure will increase. There is no way around that. The increase of pressure will then result in an increase of temperature. This still matches the model we have.

When using a model, mathematics is used to understand what is meant, but we must understand some of the background so we do not draw false conclusions.

3.4.4 Exercises

1. Distinguish Direct and Indirect Variation.



$$v = \left(\frac{2D}{\rho CA}\right)^{1/2}$$

- D is the skydiver's weight
- ρ is the density of the air
- C is the skydiver's coefficient of drag
- A is the skydiver's ground-facing surface area

(a) Increasing C will

(a) increase velocity

(b) decrease velocity

(b) Increasing ρ will

(a) decrease velocity

(b) increase velocity

(c) Decreasing D will

(a) increase velocity

(b) decrease velocity

(d) Decreasing A will

(a) increase velocity

(b) decrease velocity

2. **Describe Relation.** For the following exercise, assume the constant k is positive.
C varies directly with L. Describe what happens to the value of C as L increases.

(a) undeterminable

(b) no change

(c) decreases

(d) increases

3. **Describe Relation.** For the following exercise, assume the constant k is positive.
C varies inversely with L. Describe what happens to the value of C as L decreases.

(a) no change

(b) increases

(c) decreases

(d) undeterminable

4. Application. The force F (in pounds) needed on a wrench handle to loosen a certain bolt varies inversely with the length L (in inches) of the handle. A force of 50. pounds is needed when the handle is 8.0 inches long. If a person needs 15 pounds of force to loosen the bolt, estimate the length of the wrench handle. Calculate using significant digits.

____________________ inches

5. Application. The electrical current, in amperes, in a circuit varies directly as the voltage. When 24 volts are applied, the current is 8 amperes. What is the current when 39 volts are applied?

____________________ amperes

6. Application. The number of hours required to build a fence is inversely proportional to the number of people working on the fence. If it takes 3 people 50 hours to complete the fence, then how long will it take 11 people to build the fence?

(Round the answer to 2 decimal places if needed)

____ hours.

7. Application. The capacitive reactance, X, in a circuit varies inversely as the frequency, f, of the applied voltage. If the reactance is 772 ohms when the frequency is 65.2 hertz, find the reactance when the frequency is 51.2.

_____ Ω

8. Application. The loudness, L, of a sound (measured in decibels, dB) is inversely proportional to the square of the distance, d, from the source of the sound.

When a person 12 feet from a jetski, it is 70.0 decibels loud. How loud is the jetski when the person is 41 feet away?

Round using the rules of significant figures.

____________________ dB

9. Contextless Practice. S varies directly as p and q. If $p = 4$ and $q = 6$ then $S = 151.2$. Find the constant of proportionality.

$k =$ ____________________

10. Contextless Practice. Write the equation representing the relationship, use k for the constant of variation.

a varies directly as q

(a) a q = k

(b) a / q = k

11. Contextless Practice. Write the equation representing the relationship, use k for the constant of variation.

a is inversely proportional to u

(a) a u = k

(b) a / u = k

12. Application. Hooke's law states that the distance that a spring is stretched by hanging object varies directly as the mass of the object. If the distance is 160.0 cm when the mass is 24.0 kg, what is the distance when the mass is 12.0 kg?

Round using the rules of significant figures.

____________________ cm

13. Application. The volume of a gas varies inversely as the pressure upon it. The volume of a gas is 400 cm^3 under a pressure of 16 kg/^2. What will be its volume under a pressure of 160 kg/cm^2?

Round your answer to two significant figures.

____________________ cm^3

14. Application. The wavelength of a radio wave varies inversely as its frequency. A wave with a frequency of 1800 kilohertz has a length of 200 meters. What is the length of a wave with a frequency of 400 kilohertz?

____________________ meters

15. Contextless Practice. Write a function describing the relationship of the given variables.
h varies directly with the square of d and when $d = 7$, $h = 637$
$h =$____________________

16. Contextless Practice. Write a function describing the relationship of the given variables.
V varies inversely with the square of t and when $t = 2$, $V = 10$
$V =$____________________

17. Contextless Practice. Write a function describing the relationship of the given variables.
W varies directly with the square root of n and when $n = 36$, $W = 84$
$W =$____________________

18. Application. The velocity v of a falling object varies directly with the time t of the fall. If after 3.00 seconds, the velocity of an object is 96.0 feet per second, what is the velocity after 8.0 seconds?
Your answer should have 3 significant figures.
_____ feet per second

19. Application. The weight of an object above the surface of Earth varies inversely with the square of distance from the center of Earth. If an object weighs 90.00 pounds when it is 3960 miles from Earth's center, what would the same object weigh when it is 3,990.0 miles from Earth's center?
Your answer should have 4 significant figures.
______ pounds

20. Application. Newton's Law of Gravitation says that two objects with masses m_1 and m_2 attract each other with a force F that is jointly proportional to their masses and inversly proportional to the square of the distance r between the objects. Newton discovered the constant of proportionality is 6.67×10^{-11}.
In a small laboratory experiment, two 700 kg masses are separated by 0.6 meters. What would the gravitational force between the objects be?
Force = ____________________ Newtons

3.5 Rational Expressions

This section addresses the following topics.

- Read and use mathematical models in a technical document
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Solve linear, rational, quadratic, and exponential equations and formulas (skill)

This section presents algebra needed to work with models involving more complex, rational (fractional) expressions; and it presents how to answer questions requiring adding rates.

3.5.1 Re-arranging Rational Expressions

In previous sections, many of the models involving rational (fractional) expressions could be set up so we solved for a variable in the numerator. In this section we look at multiple examples in which we must solve for a variable in the denominator.

Model 3.5.1 Gear Design. *For ANSI standard gears, there is a relationship between the number of teeth, and diameters of the gear.*

$$D_p = \frac{D_o N}{N+2}$$

where

- *N is the total number of teeth*

- D_o *the diameter of the outside of the gear, and*
- D_p *is the pitch diameter.*

?? Does not make sense

Note pitch diameter is the diameter of a circle such that this where this gear meets the other gear.

Example 3.5.2 Suppose we know the outer diameter (17/8") and pitch diameter (2") needed for a gear. What are the steps to solve for the number of teeth? The number of teeth must end up an integer (half a gear tooth is just a broken gear). Thus there will be no rounding. If the calculation gives a non-integer result, we must change something in the design.

Converting $17/8 = 2.125$ and using the model we obtain

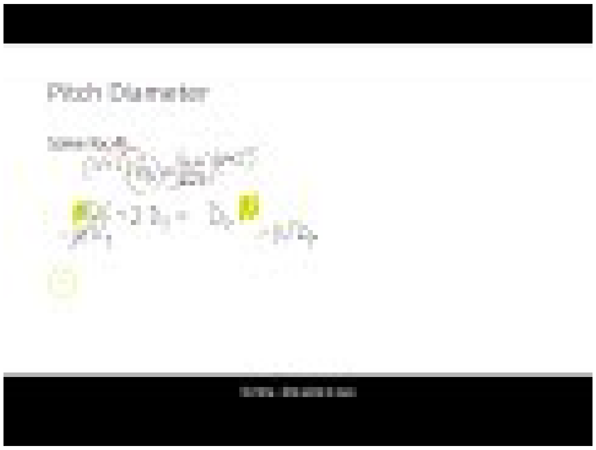
$$\begin{aligned}
D_p &= \frac{D_o N}{N+2}. \\
2 \text{ in} &= \frac{(2.125 \text{ in})N}{N+2}. \\
(2 \text{ in}) \cdot (N+2) &= \frac{(2.125 \text{ in})N}{N+2} \cdot (N+2). \\
(2 \text{ in})N + (4 \text{ in}) &= (2.125 \text{ in})N \\
-(2 \text{ in})N + (2 \text{ in})N + 4 \text{ in} &= -(2 \text{ in})N + (2.125 \text{ in})N. \\
4 \text{ in} &= (0.125 \text{ in})N \\
\frac{4 \text{ in}}{0.125 \text{ in}} &= \frac{(0.125 \text{ in})N}{0.125 \text{ in}}. \\
32 &= N.
\end{aligned}$$

Thus this gear will have 32 teeth (total). □

Example 3.5.3 Formula for Number of Teeth. If we are going to perform this calculation regularly, we can solve the equation for N.

$$\begin{aligned}
D_p &= \frac{D_o N}{N+2}. \\
D_p \cdot (N+2) &= \frac{D_o N}{N+2} \cdot (N+2). \\
D_p(N+2) &= D_o N. \\
D_p N + 2D_p &= D_o N. \\
D_p N - D_o N &= -2D_p. \\
(D_p - D_o)N &= -2D_p. \\
\frac{(D_p - D_o)N}{D_p - D_o} &= \frac{-2D_p}{D_p - D_o}. \\
N &= \frac{-2D_p}{D_p - D_o}.
\end{aligned}$$

Notice we needed to collect the terms with N. This required distributing (third line), collecting on one side, then factoring. □



Standalone

Figure 3.5.4 Pitch Diamter

A problem with similar algebra is the Model 1.3.5 when we are solving for temperature. The next example illustrates the necessary algebra.

Example 3.5.5 Suppose the conditions for a tire are $P_1 = 3\bar{0}$ psi at a temperature of $T = 52°$ F. At what temperature will the pressure drop below the safe value of 28 psi? Because our gauges are not more accurate we will round to units.

so $V_1 = V_2$

We can use the ideal gas law version below. Note that the volume does not change (the tire size does not change).

Have students used subscript notation before this problem?

$$\begin{aligned}
\frac{P_1V_1}{T_1+460°} &= \frac{P_2V_2}{T_2+460°}\\
\frac{(30\text{ psi})V_1}{52°+460°} &= \frac{(28\text{ psi})V_1}{T_2+460°} && \text{Because the volume does not change}\\
\frac{(30\text{ psi})V_1}{512°}\cdot\frac{1}{V_1} &= \frac{(28\text{ psi})V_1}{T_2+460°}\cdot\frac{1}{V_1}\\
\frac{30\text{ psi}}{512°} &= \frac{28\text{ psi}}{T_2+460°}\\
\frac{512°}{30\text{ psi}} &= \frac{T_2+460°}{28\text{ psi}}\\
\frac{512°}{30\text{ psi}}\cdot(28\text{ psi}) &= T_2+460°\\
477.866667° &\approx T_2+460°\\
477.866667°-460° &= T_2\\
17.8666666° &\approx T_2.\\
18° &\approx T_2.
\end{aligned}$$

I'm assuming this step has already been explained on a previous example

□

The final example in this section presents a type of problem involving adding multiple fractions. This is needed for the resistors in parallel problem and is connected to the next section on additive rates.

Model 3.5.6 Parallel Resistors. *When two resistors are in parallel as shown in Figure 3.5.7 then the resulting resistance is given by*

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

If there are more than two resistors, the model is expanded by adding an additional $1/R$ *term for each resistor.*

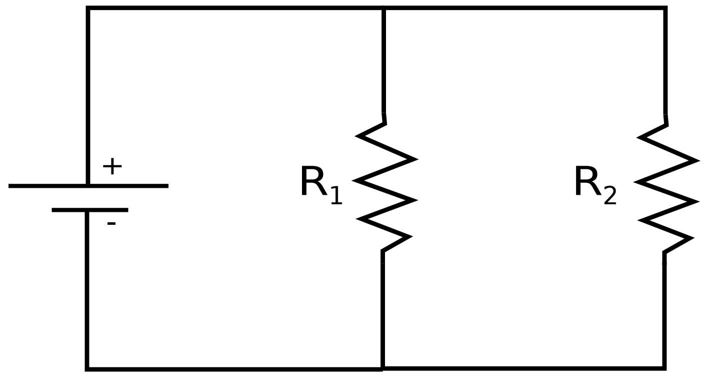


Figure 3.5.7 Resistors in Parallel

The primary algebra technique required is to obtain common denominators.

Example 3.5.8 Parallel Resistance. Calculate the resulting resistance when one resistor is 4 Ohms ($R_1 = 4$) and the other is 12 Ohms ($R_2 = 12$).

$$\begin{aligned}
\frac{1}{R} &= \frac{1}{R_1} + \frac{1}{R_2}.\\
\frac{1}{R} &= \frac{1}{4} + \frac{1}{12}.\\
\frac{1}{R} &= \frac{3}{3}\cdot\frac{1}{4} + \frac{1}{12}. \text{ Common denominator}\\
\frac{1}{R} &= \frac{3}{12} + \frac{1}{12}.\\
\frac{1}{R} &= \frac{4}{12}\\
&= \frac{1}{3}.\\
\frac{1}{R}\cdot 3R &= \frac{1}{3}\cdot 3R \text{ Clear both denominators}\\
3 &= R.
\end{aligned}$$

← Do you ever clear fractions at this level?

Note the need for a common denominator in the third line. The final step is our now frequently used clearing of denominators (i.e., 'cross multiplication'). □

Example 3.5.9 Parallel Resistance Solving. In the previous example we knew the two resistors and calculated the resulting resistance. In other cases we know how much resistance we need and one of the resistors. We must calculate the resistance for the other resistor.

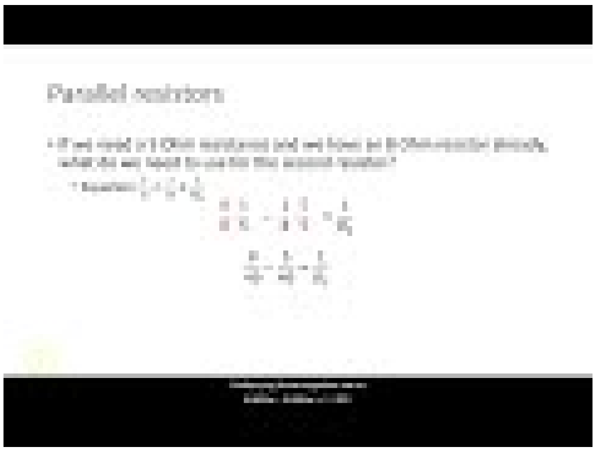
If we need a 5 Ohm resistance and have an 8 Ohm resistor already, what do we add as the second resistor? Resistor measurements are commonly accurate to $\pm 5\%$. For this problem that allows us to round to units. This is because 5% of 5 is much less than one.

$$\begin{aligned}
\frac{1}{R} &= \frac{1}{R_1} + \frac{1}{R_2}.\\
\frac{1}{5} &= \frac{1}{8} + \frac{1}{R_2}.
\end{aligned}$$

$$
\begin{aligned}
\frac{1}{5}-\frac{1}{8} &= \frac{1}{R_2}.\\
0.2-0.125 &= \frac{1}{R_2}\\
0.075 &= \frac{1}{R_2}\\
0.075R_2 &= 1\\
R_2 &= \frac{1}{0.075}\\
&\approx 13.333333\\
&\approx 13.
\end{aligned}
$$

I like that you used decimals in this example.

□



Standalone

Figure 3.5.10 Parallel Resistance Solving

Checkpoint 3.5.11 If two resistors are arranged in parallel one is 9.0Ω and the other is 6.0Ω, what is the resulting resistance? ____ Ω

Checkpoint 3.5.12 We need a resulting resistance of 1.0Ω If one resistor in parallel is 9.0Ω, what should the other resistance be? ____ Ω

Checkpoint 3.5.13 If we need 5 Ohm resistance and one of our resistors is a 4 Ohm resistor, can we find a second resistor to make this work? Explain.

3.5.2 Rates

There are many times when we need to calculate the rate at which something can be accomplished when more than one person/thing is working on it. This section illustrates how to obtain a resulting rate from the individual rates. This process requires algebra similar to that above.

Example 3.5.14 Joint Work: Draining Basement. A company has two pumps available for draining flooded basements. One pump can drain a basement in 4 hours, whereas the other pump can do the job in only 3 hours. How long would it take to drain the basement if both pumps are used simulataneously?

The question is how to find the speed of the combined pumps. Adding a pump would increase the speed; we want to find a way to add the speeds. We can start by writing down the rates to see what that suggests. The first pump operates at a rate of $\frac{1 \text{ basement}}{4 \text{ hours}}$ and the second pump operates at a rate of $\frac{1 \text{ basement}}{3 \text{ hours}}$. Because rates are ratios (fractions), and the units match, we know how to add them. The combined rate is

So we

$$\frac{1 \text{ basement}}{4 \text{ hours}}+\frac{1 \text{ basement}}{3 \text{ hours}} = (\text{Need a common denominator})$$

I don't like using equal here-
personal preference

$$\begin{aligned}
\frac{3}{3}\cdot\frac{1 \text{ basement}}{4 \text{ hours}} + \frac{4}{4}\cdot\frac{1 \text{ basement}}{3 \text{ hours}} &= (\text{Scaling accomplishes this})\\
\frac{3 \text{ basement}}{12 \text{ hours}} + \frac{4 \text{ basement}}{12 \text{ hours}} &= \frac{7 \text{ basement}}{12 \text{ hours}}.
\end{aligned}$$

We did not convert to decimal notation, because we will be setting up a proportion, so we want the rate in fraction form.

The next step is to use this rate to determine how long it takes to empty one basement. This is setting up a proportion.

$$\begin{aligned}
\frac{7 \text{ basement}}{12 \text{ hours}} &= \frac{1 \text{ basement}}{N \text{ hours}}.\\
\frac{12 \text{ hours}}{7 \text{ basement}} &= \frac{N \text{ hours}}{1 \text{ basement}}.\\
\frac{12 \text{ hours}}{7 \text{ basement}}\cdot(1 \text{ basement}) &= \frac{N \text{ hours}}{1 \text{ basement}}\cdot(1 \text{ basement}).\\
1.714285714 \text{ hours} &\approx N.\\
1.72 \text{ hours} &\approx N.
\end{aligned}$$

Given this is in hours, it makes sense to round to the ~~hundreds~~ hundredths (a little less than a minute). ~~Because it~~ is ~~how long it will take~~ ~~it~~ It also makes sense to round up (better to expect the worst and be happy it was better). We expect the two pumps to complete the work in 1.72 hours (a little less than 1 hour and 45 minutes).

Because the denominator was 1 basement, when we cleared the denominator (cross multiplication) all we did was adjust the units (basement/basement divides out). This suggests we could have simply scaled the combined rate to obtain the final result.

$$\frac{1/7}{1/7}\cdot\frac{7 \text{ basement}}{12 \text{ hours}} = \frac{1 \text{ basement}}{1.72 \text{ hours}}.$$

Thus if both pumps are working it will take 1.72 hours drain the basement.

The increase in speed, results in less time required to complete the job. This is why the faster rate (1 basement per 1.72 hours) has a smaller denominator (1.72 vs 3 or 4). □

Example 3.5.15 How to Use an Example: Joint Work. If one housekeeper can clean a hotel room in 11 minutes, and another can clean a room in 13 minutes. How long will it take them combined to clean 27 rooms?

Because the question asks us to determine combined speed, we recognize this as a "joint work" problem. Looking at Example 3.5.14 we see that the first step was to write down the two rates. $\frac{1 \text{ room}}{11 \text{ minutes}}$ and $\frac{1 \text{ room}}{13 \text{ minutes}}$.

After writing the rates, we realize we need to add the rates.

$$\begin{aligned}
\frac{1 \text{ room}}{11 \text{ minutes}} + \frac{1 \text{ room}}{13 \text{ minutes}} &= (\text{Need a common denominator})\\
\frac{1 \text{ room}}{11 \text{ minutes}}\cdot\frac{13}{13} + \frac{1 \text{ room}}{13 \text{ minutes}}\cdot\frac{11}{11} &= (\text{this scales the rates})\\
\frac{13 \text{ rooms}}{143 \text{ minutes}} + \frac{11 \text{ rooms}}{143 \text{ minutes}} &= \frac{24 \text{ rooms}}{143 \text{ minutes}}.
\end{aligned}$$

After calculating the rate, the next step ~~was~~ is setting up the proportion. In this case we want to find the time to clean 27 rooms.

$$\begin{aligned}
\frac{24 \text{ rooms}}{143 \text{ minutes}} &= \frac{27 \text{ rooms}}{N \text{ minutes}}.\\
\frac{143 \text{ minutes}}{24 \text{room}} &= \frac{N \text{ minutes}}{27 \text{ rooms}}.\\
\frac{143 \text{ minutes}}{24 \text{ rooms}}\cdot(27 \text{ rooms}) &= \frac{N \text{ minutes}}{27 \text{ rooms}}\cdot(27 \text{ rooms}).\\
140.875 \text{ minutes} &= N.
\end{aligned}$$

$$141 \text{ minutes} \approx N.$$

Because this is about the time required to complete a task it makes sense to round up. □

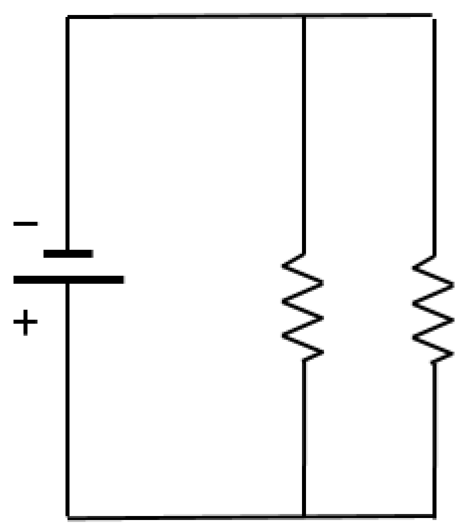
Checkpoint 3.5.16 If one shop can do 7 float changes in two days, and a second shop can do 13 float changes in three days, how long will it take the pair of shops to do 52 float changes? ____

If they start on Monday and work only weekdays, on what day of the week will they finish?

1. Monday
2. Tuesday
3. Wednesday
4. Thursday
5. Friday

3.5.3 Exercises

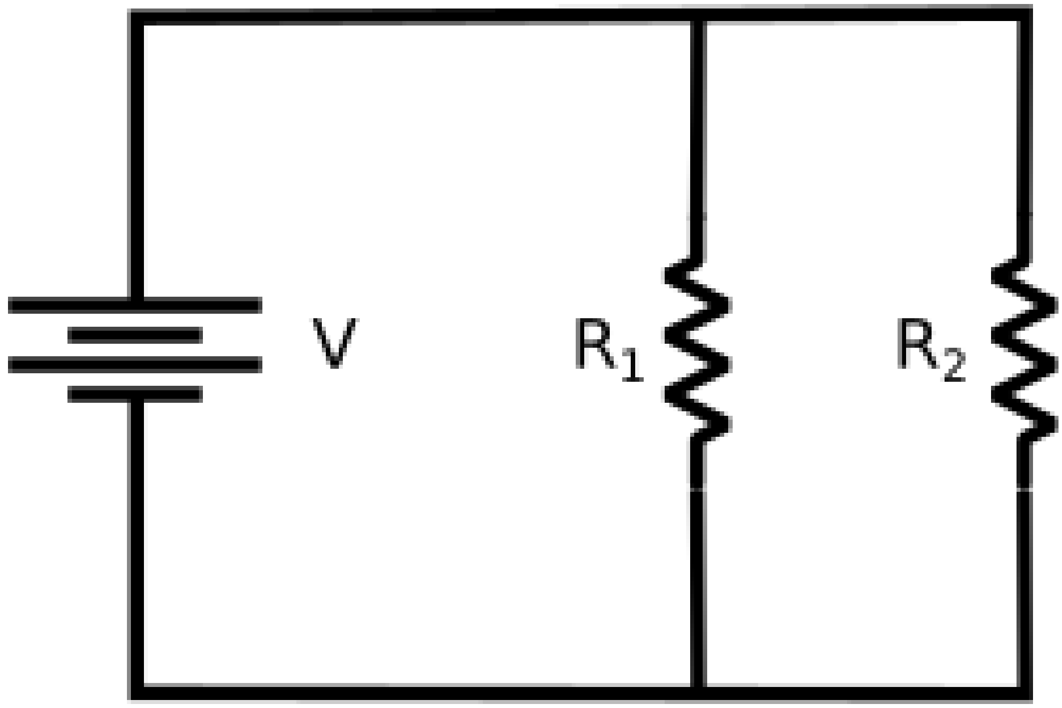
1. What is the total resistance of the circuit shown below? Use the formula $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ to answer this question, where R is the total overall resistance and R_1 and R_2 are individual resistors in parallel.



Round your answer using the rules of working with measurements.
$R_T =$ _____ unit_____
To enter the unit, in the pop-up box click on the heading ”vars”.

2. What is the total resistance of the circuit shown below?

3. Consider the following diagram of a parallel circuit.



V = 45.0 Volts
R_1 = 450.Ω
R_2 = 18$\bar{0}$0Ω
Using the information above, determine the total (equivalent) resistance, and the current.
R_{eq} = ____________________ Ω
I = ____________________

4. A known volume of ideal gas is contained in a cylinder fitted with a piston and held at constant pressure. If the temperature changes from 25^{o}C to 200. ^oC, by what factor does the volume change? Round your answer using the rules for working with significant figures.

For this problem you should know that when pressure is held constant, gasses follow Charles' Law, which states $\frac{V_1}{T_1 + 273} = \frac{V_2}{T_2 + 273}$, where V_1 and T_1 is the initial volume and temperature and V_2 and T_2 is the new volume and temperature.

The volume

(a) decreases

(b) increases

______

5. The ideal gas law can be expressed as $PV = nRT$ where P is pressure in atm (atmospheres), V is volume in liters, n is the number of moles (a count of molecules), R is a constant from physics, and T is the temperature (from absolute zero).

What is the pressure, in atmospheres, of 14.0 moles of a gas at a temperature of 278.5 K in a 470 liter container when R = 0.0821 (atm∗L)/(mol∗K)?

____________________ atm

6. A relationship between the outer D_o and pitch D_p diameters of a gear and the number of teeth N is given by $D_p = \dfrac{D_o N}{N+2}$.

If a gear needs $N = 26$ teeth and pitch diameter $D_p = 35$ mm, what should the outer diameter be? Note that both measurement are given to two significant digits.

$D_o =$ ____

7. A contractor finds that Crew A takes 6.50 hours to construct a retaining wall and Crew B can do the same job in 7.50 hours. If Crew A and Crew B work together, how long will it take them to construct the retaining wall?

Round using the rules of significant figures.

You may find it useful to know the work formula is $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{1}{t}$.

Working together, the two crews will construct the wall in ____________________ hours.

8. Angela can shovel the snow from her driveway in 2.50 hours. When Franklin joins her, the driveway can be finished in just 78 minutes.

Working alone, Franklin can shovel the driveway in ____________________ minutes. Round your answer to the nearest minute.

9. You know it takes 12.0 hours to fill your pool with your hose. Last year, you used both your hose and a hose borrowed from a neighbor, and it took 7.50 hours. This year, your faucet is broken and you can only use your neighbor's hose.

Round appropriately using significant figures.

Just using the neighbor's hose, it will take ____________________ hours.

10. Three hoses are used to fill a swimming pool. The first hose can fill the pool in 29 minutes, the second can fill the pool in 55 minutes, and the third can fill the pool in 44 minutes. All numbers have two significant digits.

Using all three hoses, the pool will be filled in ____________________ minutes.

11. A gear with 35 teeth directly drives another gear with 25 teeth. If the larger gear spins at 370 rpm, at what speed will the smaller gear spin? ____________________

12. A gear with 30 teeth is connected via a belt to another gear with 15 teeth. If the larger gear spins at 570 rpm, at what speed will the smaller gear spin? ____________________

13. A gear with 35 teeth directly drives a smaller gear. If the larger gear spins at 380 rpm, and the smaller gear must spins at 665, how many teeth should the smaller gear have? ____________________

14. On a bicycle the pedals move a chain ring with 34 teeth. The chain moves a gear on the cassette which has 24 teeth. The chain also goes over a chain pulley with 11 teeth.

If the cyclist is pedalling at 96 rpm, at what rpm are the other gears spinning? Round to the nearest integer

gear on cassette ____________________

pulley ____________________

15. Solve: $I = \dfrac{V}{R+r}$ for R.

$R =$ ____________

3.6 Linear Systems

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Solve a system of linear equations (skill)

In Example 2.2.1 we solved a problem with two different mixtures. In that case we knew how much of each we were adding. In other cases we know what result we want and need to figure out how much of each substance. This section presents a method for answering questions involving two or more constraints (equations) at the same time.

3.6.1 Motivation

This section provides a first example of a problem involving two linear equations. It illustrates how we would recognize this type of problem.

In previous dilution problems we diluted a mixture using just diluent (water in that case). In other situations we will have two mixtures with different percents that we will combine to obtain a new mixture.

Example 3.6.1 Combining Mixtures. Suppose we have 16 oz of 91% isopropyl alcohol and 12 oz of 75% isopropyl alcohol. How much of each do we need to mix to produce 10.0 oz of 85% alcohol?

A common technique in mathematics is to start by writing the answer. We will declare that we will use A oz of 91% alcohol and B oz of 75% alcohol. Next we will express our dual constraints using these answers (variables).

The first constraint is that we end up with 10 oz of solution. Thus

$$A + B = 10.0.$$

The second contraint is the percent alcohol. Because our variables are in terms of amount of solution, we need to express the percent alcohol constraint in terms of ounces (not percents). We can obtain amount from percent using the definition of percent. Because the resulting solution will be 85% alcohol there will be

$$(0.85)10.0 = 8.5 \text{ oz.}$$

This will be the result of adding the amounts from each solution (just as in previous mixture problems). Because A oz of the first solution will be added and it is 91% alcohol, it will contribute $(0.91)A$ oz of alcohol. Similarly the second solution will contribute $(0.75)B$ oz of alcohol. Combined we will obtain

$$(0.91)A + (0.75)B = 8.5 \text{ oz.}$$

Now we just need a way to solve this pair of equations. □

3.6.2 Crossing Lines

This section connects solving a system of two linear equations to their graphs. Graphs will help us understand why (and when) ~~their~~ there should be a solution. The next section provides the method for solving.

Our goal here is to consider what causes lines to cross. We will do this by looking at a pair of lines and seeing where they cross.

Recall that a line is a relation (set of points) such that the change between any two, equally spaced points is the same. Often you have heard this described as rise over run or slope. Slope is a geometric interpretation referring to how **steep** the line is.

Checkpoint 3.6.2 In Figure 3.6.3 there are two lines. One goes through point $A = (1, 3)$. It rises at a slope of 2/1 (two up for each one over). The other line goes through the point $B = (1, 2)$ which is below A.

(a) Use the slider to set the slope of the second line to 3. Does the line cross the one through A? Where (left or right of point B)?

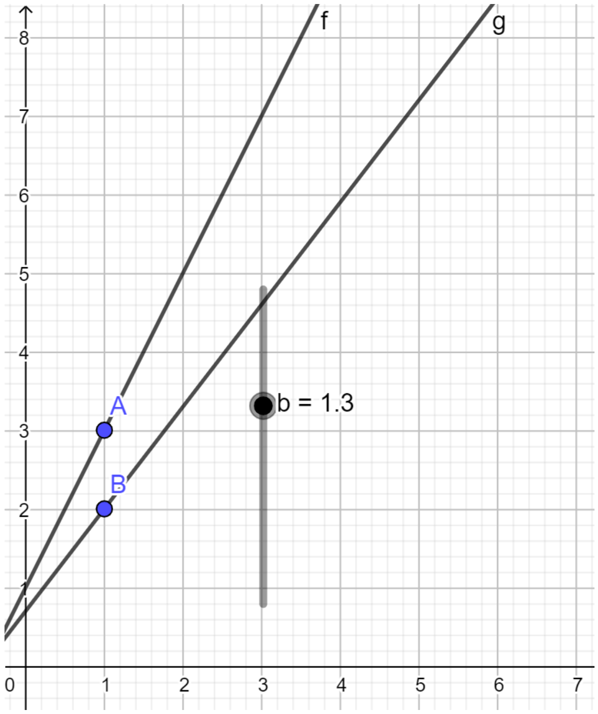
← 3/1, to stay consistent with 2/1 ?

(b) Use the slider to set the slope to something bigger than 3. Does the line cross the first one? Where (left or right of point B)?

(c) Use the slider to set the slope to 1. Does the line cross the first one? Where (left or right of point B)?

(d) In general if either slope is steeper than the other slope will the two lines cross?

(e) Can you select a slope for the second line so that these two lines do not cross?



Standalone
Embed

Figure 3.6.3 Crossing Lines

This next example applies the idea of a line starting lower but rising faster to answer a financial question.

Example 3.6.4 Vasya's initial pay was \$62,347.23. She received \$5,000 raises each year. Pyotr's initial pay was \$67,242.33. He receives \$3,500 raises each year. If they were both hired in 2012 in what year does Vasya first have a higher salary?

We could make a table.

Year	Vasya	Pyotr
2012	\$62,347.23	\$67,242.33
2013	\$67,347.23	\$70,742.33
2014	\$72,347.23	\$74,242.33
2015	\$77,347.23	\$77,742.33
2016	\$82,347.23	\$81,242.33

We see that is in 2016 that Vasya is first paid more.

Alternatively we could note that Vasya's raises are \$1,500 more each year than Pyotr's raises. This means she closes the gap by \$1,500 each year. The difference in their initial salaries is $67242.33 - 62347.23 = 4895.10$. Because she gains by 1500 each year it will take $4895.10/1500 = 3.2634$ years. Because they receive raises once a year this result must be rounded up to 4 years. Thus she is first paid more in 2016. □

3.6.3 Solving Linear Systems

This section presents two ways of solving linear systems of this type. The second method is very important for larger systems.

Example 3.6.5 We will solve the system from Example 3.6.1. The two equations are

$$A + B = 10.0.$$
$$(0.91)A + (0.75)B = 8.5.$$

Notice we can solve the first equation for B, then substitute it into the second. The result is a single equation with only one variable. We already know how to solve that one.

$$\begin{aligned} A+B &= 10.0. \\ B &= 10.0 - A. \end{aligned}$$

$$\begin{aligned} (0.91)A + (0.75)B &= 8.5. && \\ (0.91)A + (0.75)(10.0 - A) &= 8.5. && \text{Substitute B from above} \\ (0.91)A + 7.5 - (0.75)A &= 8.5. && \text{Distribute} \\ (0.91)A - (0.75)A &= 8.5 - 7.5. && \\ (0.91 - 0.75)A &= 1.0. && \\ (0.16)A &= 1.0. && \\ A &= \frac{1.0}{0.16} && \\ &= 6.25. && \end{aligned}$$

Now that we know that $A = 6.25$ we can substitute that into $A + B = 10.0$. This gives us

$$\begin{aligned} A+B &= 10.0. \\ 6.25 + B &= 10.0. \\ B &= 3.75. \end{aligned}$$

We can check that this works in the other equation (about percent alcohol).

$$\begin{aligned} (0.91)A + (0.75)B &= \\ (0.91)(6.25) + (0.75)(3.75) &= \\ 5.6875 + 2.8125 &= 8.5. \end{aligned}$$

□

If we had 7 variables instead of two, substituting would take a while. Instead we can use the following method which is more like solving as we know it, that is isolating a variable. This method is called **elimination**.

Example 3.6.6 We will solve the system

$$\begin{aligned} A+B &= 10.0. \\ (0.91)A + (0.75)B &= 8.5. \end{aligned}$$

In the second line below notice how we modify the first equation to partially match the second one by scaling it.

$$\begin{aligned} A+B &= 10.0. && \text{Original equation 1} \\ -(0.91)(A+B) &= -(0.91)10.0. && \text{Scale to match 2nd equation} \\ -(0.91)A - (0.91)B &= -9.1. && \text{Distribute} \end{aligned}$$

Now add the two equations.

$$\begin{aligned} -(0.91)A - (0.91)B &= -9.1. && \text{Scaled equation 1} \\ (0.91)A + (0.75)B &= 8.5. && \text{Original equation 2} \\ -(0.16)B &= -0.6. && \text{Result of adding previous 2 lines} \\ B &= \frac{-0.6}{-0.16} && \end{aligned}$$

$$= 3.75.$$

In the fifth line we added the two equations. Because they had opposite coefficients for A, that variable was eliminated, leaving us with just B. This can always be done with systems of linear equations.

We finish solving this system the same way as the previous example, by substituting the value of B back into the first equation.

$$\begin{aligned} A + B &= 10.0. \\ A + 3.75 &\approx 10.0. \\ A &\approx 6.25. \end{aligned}$$

why ≈ here?

A=6.25 B=3.75

A = 6.25 B = 3.75

??

□

Notice that we ended up with slightly different solutions in Example 3.6.5 and Example 3.6.6. This is not the result of differences in the methods. Rather it is the result of rounding and it is a result of our choice of variable to solve first. These slightly different results are a reminder to be careful when rounding is involved. If the difference between these results makes a difference in our lives, then we need to measure more precisely.

Checkpoint 3.6.7 Solve this system of equations using substitution.

$-x + 2y = -7$
$8x - 7y = 38$

———

Checkpoint 3.6.8 Solve this system of equations using elimination.

$9x + 8y = -54$
$-6x - 9y = 69$

———

3.6.4 Other Cases

In Figure 3.6.3 we found a slope that resulted in no intersection. If we were solving a pair of linear equations that represented lines like this we would find no solution. These are known as **inconsistent** systems. This section provides two examples of such systems and demonstrates how we identify them. For this book, identifying these cases and correctly describing them is all you are expected to do.

Example 3.6.9 Inconsistent Linear System. Find all solutions to the system

$$\begin{aligned} 2x + 3y &= 5. \\ 4x + 6y &= 7. \end{aligned}$$

We will use elimination. If we multiply -2 by the first equation we will obtain -4 (opposite of x in the second equation).

$$\begin{aligned} 2x + 3y &= 5. \\ -2(2x + 3y) &= -2(5). \\ -4x - 6y &= -10. \\ 4x + 6y &= 7. \\ 0 &= -3. \end{aligned}$$

Our work is correct, but the conclusion is clearly false. You can think of this as saying, for a solution to exist 0 must equal -3. Because this is a contradiction, we call the system **inconsistent**. This means there are no solutions. □

There is a third case.

Example 3.6.10 Dependent System. Find all solutions to the system

$$2x + 3y = 5.$$
$$4x + 6y = 10.$$

We will use elimination. If we multiply -2 by the first equation we will obtain -4 (opposite of x in the second equation).

$$2x + 3y = 5.$$
$$-2(2x + 3y) = -2(5).$$
$$-4x - 6y = -10.$$
$$4x + 6y = 10.$$
$$0 = 0.$$

This time we have a true, but rather uninformative statement. We notice that after scaling (multiplying by -2) the two equations were identical. Essentially we had only one equation. Because one can be obtained from the other we call them **dependent**. □

Checkpoint 3.6.11 Determine what type of linear system this is.
$-5x + 9y = 7$
$6x + 6y = -42$

1. Consistent
2. Inconsistent
3. Dependent

3.6.5 Exercises

1. Two Equations. Use substitution to solve this system of linear equations. $-5x + 5y = -15$ $x = 3y + 7$ Select the correct choice below and, if necessary, enter an ordered pair (x, y) to complete your answer.

- One or more solutions: ____________________
- No solution
- Infinite number of solutions

2. Two Equations. Use substitution to solve this system of linear equations. $-2x + 5y = -28$ $-5x + y = -1$ Select the correct choice below and, if necessary, enter an ordered pair (x, y) to complete your answer.

- One solution: ____________________
- No solution
- Infinite number of solutions

3. Two Equations. Use substitution to solve this system of linear equations. $x + 2y = -4$ $8y = -16 - 4x$ Select the correct choice below and, if necessary, enter an ordered pair (x, y) to complete your answer.

- One solution: ____________________
- No solution
- Infinite number of solutions

4. Two Equations. Use substitution to solve this system of linear equations. $y = -3x + 1$ $2y = -6x + 1$ Select the correct choice below and, if necessary, enter an ordered pair (x, y) to complete your answer.

- One solution: ____________________
- No solution
- Infinite number of solutions

5. **Two Equations.** Use elimination to solve this system of linear equations. $\begin{array}{rcrcr} x & + & 5y & = & 18 \\ -x & + & 2y & = & 3 \end{array}$
Select the correct choice below and, if necessary, enter an ordered pair (x, y) to complete your answer.

- One solution: ____________________
- No solution
- Infinite number of solutions

6. **Two Equations.** Use elimination to solve this system of linear equations. $\begin{array}{rcrcr} 6x & - & 2y & = & 16 \\ 7x & + & 8y & = & -2 \end{array}$
Select the correct choice below and, if necessary, enter an ordered pair (x, y) to complete your answer.

- One solution: ____________________
- No solution
- Infinite number of solutions

7. **Two Equations.** Use elimination to solve this system of linear equations. $\begin{array}{rcrcr} x & + & 3y & = & -1 \\ -4x & + & 4y & = & -12 \end{array}$
Select the correct choice below and, if necessary, enter an ordered pair (x, y) to complete your answer.

- One solution: ____________________
- No solution
- Infinite number of solutions

8. **Two Equations.** Use elimination to solve this system of linear equations. $\begin{array}{rcrcr} 6x & - & 3y & = & 36 \\ -18x & + & 9y & = & -108 \end{array}$
Select the correct choice below and, if necessary, enter an ordered pair (x, y) to complete your answer.

- One solution: ____________________
- No solution
- Infinite number of solutions

9. **Two Equations.** Use elimination to solve this system of linear equations. $\begin{array}{rcrcr} 4x & - & 2y & = & -2 \\ 6x & - & 3y & = & 1 \end{array}$
Select the correct choice below and, if necessary, enter an ordered pair (x, y) to complete your answer.

- One solution: ____________________
- No solution
- Infinite number of solutions

10. **Linear System Application.** We have a jar of coins, all pennies and quarters. All together, we have 282 coins, and the total value of all coins in the jar is \$ 31.38. How many pennies are there in the jar?
Answer: ____ pennies

11. **Linear System Application.** A hoverboard manufacturer has just announced the *Glide 5* hoverboard. The accounting department has determined that the cost to manufacturer the *Glide 5* hoverboard is $y = 43.53x + 24771$. The revenue equation is $y = 79.43x$. What is the break even point for the *Glide 5* hoverboard?

The break even point for the *Glide 5* hoverboard is __________

12. Linear System Application. A store owner wants to mix chocolate and nuts to make a new candy. How many pounds of chocolate costing \$18.40 per pound should be mixed with 19 pounds of nuts that cost \$3.30 per pound to create a mixture worth \$9.96 per pound?

The owner needs to mix __________ pounds of chocolate.

(round to the nearest whole pound)

13. Linear System Application. A coffee distributor plans to mix some Queen City coffee that sells for \$9.90 per pound with some House coffee that sells for \$12.60 per pound to create 10 pounds of a new coffee blend that will sell for \$10.44 per pound.

How many pounds of each kind of coffee should they mix? Round to the nearest pound.

____ pounds of Queen City coffee.

____ pounds of House coffee.

14. Linear System Ap

*2360*The return trip takes *3* hours flying against the wind.

What is the speed of the airplane in still air and how fast is the wind blowing?

Answer:

The speed of the airplane in still air is _____ miles per hour.

The wind speed is _____ miles per hour.

Round your values to the nearest whole number.

15. Linear System Appli

A certain bread recipe asks you to combine yeast and flour with *1* cups of warm *120* °F water. If the water is hotter or colder than that, then the bread won't rise.

All you have available are boiling water that is *220* °F and cold tap water that is *60* °F.

How much boiling water and cold tap water should you mix together to get *1* cups of *120* °F water?

Answer:

____ cups of boiling water.

____ cups of cold tap water.
Round your answers to 2 decimal places.

16. Linear System Application. A 3.00 % solution of pesticide and a 6.00 % solution of pesticide must be combined to produce 135 mL of a 4.33 % solution. How much of each type should be mixed? Round using the rules of working with significant figures.
____________________ mL of 3 %
____________________ mL of 6 %

17. Linear System Application. At a farmers' market, Frederick buys 4 pounds of apples and 4 pounds of cherries for $12.32. At the same farmers' market, Wilhelmina buys 12 pounds of apples and 8 pounds of cherries for $28.12. Determine the price per pound of apples and cherries at the farmers' market.
Apples cost $_______ per pound.
Cherries cost $_______ per pound.

3.7 Project: Biking in Kansas and Alaska

Project 4 Project: Biking in Kansas and Alaska. In this project, we're going to think about what makes a relationship linear or not linear. Each question is worth two points.

(a) This is a graph of a linear relationship. Looking at it, what about it tells you that it is linear?

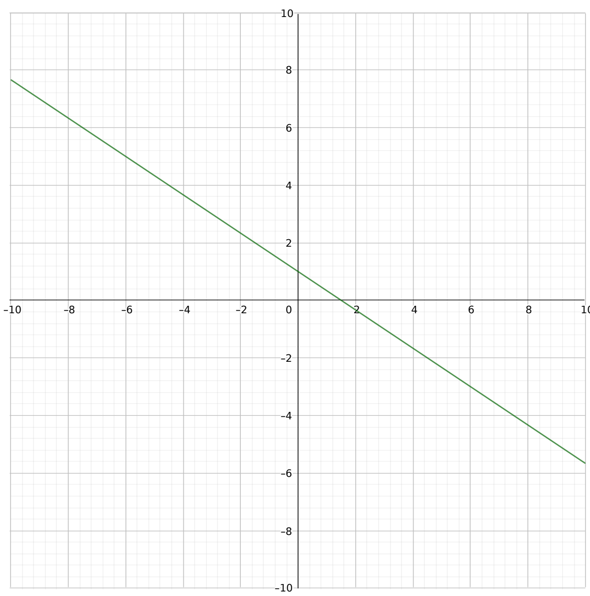


Figure 3.7.1 Graph of Line

(b) Here is a table of some of the points represented on the above graph. This data also represents a linear relationship. Without graphing, how can you tell that this relationship is linear?

Table 3.7.2 Table of Points

x	y
-6	5
-3	3
0	1
3	-1
6	-3

(c) Friends Jacob and Mike like to bike. For a math conference, the two traveled to Kansas and decided to go on a bike ride one evening. Mike enjoys tracking his data and so took note of his distance traveled at regular intervals. Here is a table of Mike's time and mileage:

Table 3.7.3 Time and Distance

Time biked (in minutes)	Distance traveled (in miles)
10	2
20	4
30	6
40	8
50	10

Does this table represent a linear relationship? Give some supporting computations OR write a sentence to support your answer.

(d) Jacob is more absent minded in tracking his mileage over time, and so took note of his distance traveled sporadically. Here is a table of Jacob's time and mileage:

Table 3.7.4 Time and Distance

Time biked (in minutes)	Distance traveled (in miles)
7	1.4
12	2.4
20	4
35	7
54	10.8

Does this table represent a linear relationship? Give some supporting computations OR write a sentence to support your answer.

(e) After returning home to Alaska, the friends decide to go on another ride. This bike ride was on a trail in the foothills of the Chugach Mountains. Again, Mike took note of his distance traveled at regular intervals Here is a table of Mike's time and mileage:

Table 3.7.5 Time and Distance

Time biked (in minutes)	Distance traveled (in miles)
10	2.3
20	4
30	5.2
40	6
50	8

Does this table represent a linear relationship? Give some supporting computations OR write a sentence to support your answer.

(f) Again, Jacob is absent minded in tracking his mileage over time, and so took note of his distance traveled sporadically. Here is a table of Jacob's time and mileage:

Table 3.7.6 Time and Distance

Time biked (in minutes)	Distance traveled (in miles)
11	2.5
20	4
25	4.8
43	7
65	10.5

Does this table represent a linear relationship? Give some supporting computations OR write a sentence to support your answer.

(g) Slope is $\frac{\text{change in y values}}{\text{change in x values}}$. If the first columns of the four tables above represent x values and the second columns represent y values, find the unit of the slope. Your answer should be a unit, like $\frac{\text{ft}}{\text{s}}$ or in^2, *not a number*.

(h) (1 point extra credit): Consider your answer to the previous question. What does this unit represent? Your answer can be one word.

3.8 Project: Constraints on Dilution Problems

Project 5 Project: Constraints on Dilution. Curiosity is an important mathematical virtue. We have seen limitations on the results we can obtain when diluting a mixture. This project guides us through conclusions on those constraints and asks at what rate it grows. Each question is worth I DONT KNOW points.

In Section 2.2 we learned to calculate percents for mixtures and how to dilute a mixture to a specified percent.

By adding water we can of course not increase the percent alcohol, so if we start with 91% alcohol 91% is the highest we can achieve. On the other side we can reduce the percent alcohol to as little as we want (not quite to 0%) if we dilute it enough. This dilution requires not restricting the final volume. This pair of restrictions should make us wonder about a relationship between the desired volume and the minimum/maximum amount of alchol.

For all of these questions start with 16 oz of 91.0% alcohol solution.

(a) This first question is the same as Example 2.2.6. Use it to review the basic dilution calculation.

Suppose you have 16 oz of 91.0% alcohol solution. How much water must we add to obtain at least 20.0 oz of 70.0% alcohol solution?

How many ounces is the resulting solution?

(b) Next we will illustrate that for a percent alcohol such as 70%, there is a maximum volume we can achieve. Specifically this is the number we already calculated.

(i) If we dilute to a 70.0% alcohol solution what is the resulting amount of solution? You calculate this above.

Add one ounce of water. Calculate the resulting percent alcohol.

(ii) Did the percent alcohol decrease, stay the same, or increase?

(iii) As a result can we produce more 70.0% alcohol solution starting with 16.0 oz of 91.0% alcohol?

Note, if we added less water we would have less solution, so that is a decrease (not the maximum).

(c) Next, we ask at what rate does the maximum percent alcohol increase or decrease as we increase the desired amount of solution. To figure this out we will calculate the percent for multiple amounts and analyze the data as we did in Section 3.3 and Section 3.4

(i) How much water must be added for 20 oz of solution?

What is the resulting percent alcohol?

(ii) How much water must be added for 22 oz of solution?

What is the resulting percent alcohol?

(iii) How much water must be added for 24 oz of solution?

What is the resulting percent alcohol?

(iv) Does the maximum percent alcohol increase or decrease with the increase in the number of ounces?

(v) Does it grow linearly, quadratically, exponentially, or otherwise?

(d) We can ask this question in reverse as well. If we want a percent alcohol, what is the resulting maximum amount of resulting solution? Then we measure the growth of the amount.

Once again we start with 16.0 oz of 91.0% alcohol solution.

- **(i)** If we want exactly 80.0% alcohol, what is the resulting volume of solution?
- **(ii)** If we want exactly 70.0% alcohol, what is the resulting volume of solution?
- **(iii)** If we want exactly 60.0% alcohol, what is the resulting volume of solution?
- **(iv)** Does the volume of solution increase or decrease as the percent concentration decreases?
- **(v)** Does the volume of solution grow/shrink linearly, quadratically, exponentially, or otherwise?

Note if in some model a variable a varies directly with respect to a variable b, then b must vary directly with respect to a. It cannot be direct in one direction and inverse in the other.

3.9 Project: Radiation Dosage

Project 6 Calculating Effects of Radiation. The purpose of this project is to build a mathematical model for a situation. We will focus on the structure of the equations, and what they tell us about the mathematical relationships of the data. The emphasis is not on actual numbers.

For an unrelated example of a model, consider dropping a ball off a cliff. Ignoring air resistance, the ball's position can be modeled by the equation $h = kt^2$, where t is the time and h is the height of the ball. There are some other numbers that go in there, but what is changing is the time and position (height). That equation framework is the mathematical model.

You may find this to be a challenging project. Do the best you can and use your own common sense. Math should make sense! After you finish this, please read back over your work and make sure your answers are logically consistent.

Remember that you can ask questions and meet up with a tutor, but you should *not* be looking up answers or just writing down what someone else says. Do not let someone else copy your answers. That is academic dishonesty and you should not allow it. Your work should be your own.

For this project, imagine that you are working with radioactive material. Since we do not want you harmed by radiation, we should understand how time and distance impacts the radioactive dose. (You should also be behind material that shields you from the radiation, but that math is more complicated, so we'll focus on time and distance.)

(a) As your time near a radioactive source increases, does the radiation level in your body increase, decrease, or stay the same?

(b) Do you think the relationship between time and dosage is linear or not linear?

(c) Let E stand for radiation exposure, t stand for length of time of exposure, and k be the constant of variation. Write an equation representing the relationship between E, t, and k.

(d) Examine the above equation you just wrote.

- (a) Is it linear or non-linear? This should match your answer to Task 6.b.
- (b) What in the equation indicates it is linear or non-linear? That is, how did you know the answer to part Item 1?

(e) We have determined the relation between time and radiation exposure. Next we determine the relation between distance to radiation source and radiation exposure. These can be different.

Radiation radiates outwards from a source evenly in all directions (like light radiates out evenly from a lightbulb) unless it is obstructed by something (like a lead shield). Imagine a radiation source floating in the center of a sphere. All parts of the sphere would be getting hit with an equal amount of radiation. We are going to figure out the radiation for a given patch of area on this sphere.

It will be helpful to know the following formula: $A = 4\pi r^2$, where A is the surface area of the sphere and r is the radius of the sphere.

If a sphere had a radius of 2 m, what is the surface area of the sphere? Remember to include units. Leave your answer in terms of π (meaning it should look like __________π m^2).

(f) A very rough approximation of the surface area of the front of a person is $1\,\mathrm{m}^2$. We will consider what percent of the surface area of a sphere this is.

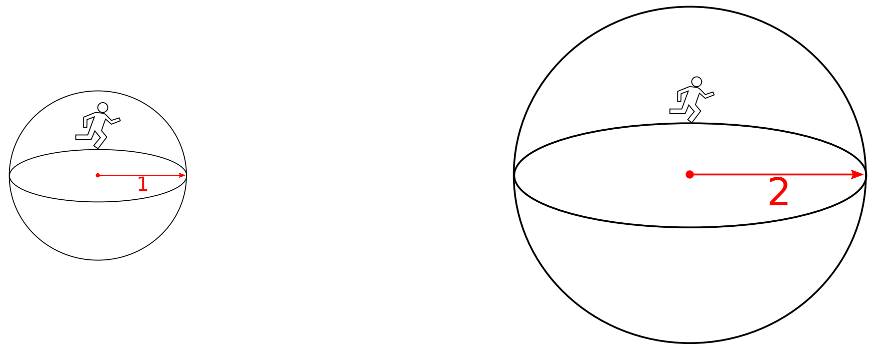


Figure 3.9.1 Surface Area Ratio

(i) What percent of the surface area of a sphere of radius 1 is $1\,\mathrm{m}^2$?

(ii) What percent of the surface area of a sphere of radius 2 is $1\,\mathrm{m}^2$?

(iii) Why does the change in percent make sense? See Figure 3.9.1.

(iv) In general as the radius increases what will the percent of the surface that is $1\,\mathrm{m}^2$ do? Increase/ remain the same/decrease?

(g) Note that the amount of radiation (energy) remains the same regardless of the radius of the sphere. That is a sphere with radius 1 and surface area 4π has the same energy as a sphere with radius 2 and surface area 16π.

(i) If the radiation is being emitted at an intensity of 5 Sieverts per hour ($\frac{\mathrm{Sv}}{\mathrm{h}}$), what amount of radiation will be hitting our $1\,\mathrm{m}^2$ person who is at a distance of $1\,\mathrm{m}$ from the source?

(ii) If the radiation is being emitted at an intensity of 5 Sieverts per hour ($\frac{\mathrm{Sv}}{\mathrm{h}}$), what amount of radiation will be hitting our $1\,\mathrm{m}^2$ person who is at a distance of $2\,\mathrm{m}$ from the source?

(iii) In general as the distance (radius) increases what happens to the amount of radiation absorbed by the person do? Increase/remain the same/decrease?

(h) If the radiation is being emitted at an intensity of x Sieverts per hour ($\frac{\mathrm{Sv}}{\mathrm{h}}$), what amount of radiation will be hitting our human-sized cutout on the surface of the sphere? Your answer should be in terms of x.

(i) Complete the following table. Notice that you already found the values for the first row.

Table 3.9.2 Ratio of Surface Areas

Radius of circle	Ratio of $1\,\mathrm{m}^2$ to surface area
2 m	
3 m	
4 m	
5 m	
r m	

(j) Graph the points in Table 3.9.2.

Hint. Your horizontal axis (radius) should be from 0 to 5. Because the output numbers are small, we need a scale that matches. Make the units on the vertical scale 0.002.

(k) Does the data represented in the table above represent a linear relationship or a non-linear relationship? Give a reason to justify your answer.

(l) Is the relationship between distance and the potential amount of radiation hitting the person better modeled by direct variation or inverse variation?

(m) Consider the numbers 16, 36, 64, 100. These numbers are significant in mathematics. What is the pattern or significance of these numbers? (Note: you only see these numbers if you completed the table in terms of π (do not multiply and round). Go back and fix your table if you do not see these numbers.)

(n) As the radius of the sphere increases, does the level of radiation hitting our person-sized cutout increase or decrease? Does this increase or decrease change linearly (at a constant rate) or non-linearly (at a changing rate)? Circle the appropriate answer.

As the radius of the sphere increases, the radiation intensity is *increasing / decreasing* (circle one) in a *linear / non-linear* (circle one) fashion.

(o) The sphere with a floating radiation source is a good model for us to use when thinking through how distance impacts radiation levels because we can disregard complicating factors like the walls, floor, and ceiling of the room as long as a person is still directly exposed to the radiation source. Still, the relationship you found between radius and radiation holds true. With that in mind, answer the following questions.

Let I stand for radiation experienced by the person, r stand for distance, and k be the constant of variation. Write an equation of variation representing the relationship between I, r, and k.

Hint. Look back at your recent answers and the table you built. Is the formula you wrote logically consistent with these answers? That is, if you plugged in the r with some constant value k, would you get the right answer for I?

(p) Let R stand for radiation exposure, t stand for length of time of exposure, r stand for distance, and k be the constant of variation (this k may be different from your k in Task 6.c or Task 6.n). Write an equation of joint variation representing the relationship between R, r, t, and k.

Hint. Is your answer consistent with your answers to #3 and #13?

(q) Use the equation you just built. If you are 5 meters away for 3 hours, how long would you stay at 2 meters away to receive the same radiation dose?

(r) At the end of it all, if you find yourself next to a radioactive source, what do you do? Full points will be awarded for all reasonable answers that address both time and distance.

Chapter 4

Geometry

4.1 Geometric Reasoning 2D

This section addresses the following topics.

- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Identify shapes and apply their properties (skill)

This section presents geometric properties, illustrates identifying shapes in applications, and illustrates breaking down complex shapes into simple ones.

4.1.1 Formulae

This section defines the two properties of interest and provides the formulae for some common shapes. Memorizing all of the formulae is not likely useful: in a job you will be able to look them up. However, anything you use a lot (e.g., triangles) is worth memorizing.

Two of the properties of shapes we will consider are **perimeter** and **area**. The **perimeter** of a shape is a measure of the size of its border (edges). The **area** of a shape is a measure of what it takes to fill the shape. In the figure below showing the outline of Kapi'olani Regional Park the perimeter is the distance someone would travel walking around the edge of the park. The area determines how much grass seed it would take to spread over the park.

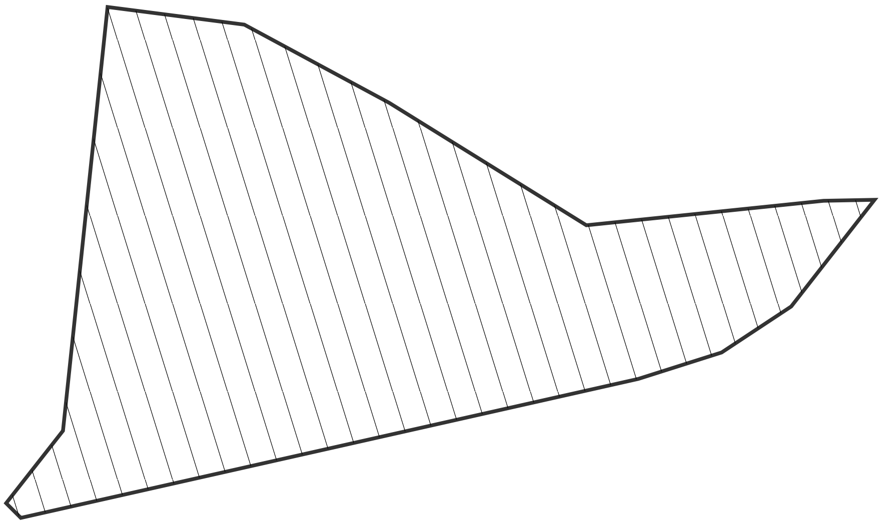


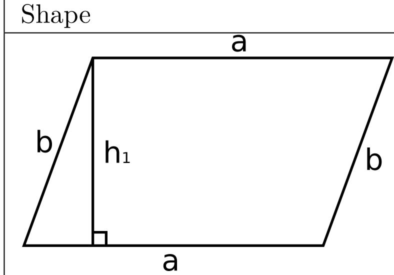
Figure 4.1.1 Complex Shape with Perimeter and Area

Rounding. How we round on geometric problems will depend as always on the application. For example a carpenter will round to units that can be measured with a tape measure like 1/16 of an inch. Contextless examples in this section will be rounded using significant digits. When using significant digits treat all numbers in the formulae as exact numbers.

Definition 4.1.2 Parallelogram. A **parallelogram** is a four sided shape for which opposing pairs of sides are parallel. ◊

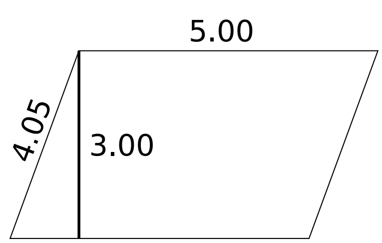
Parallelograms include **rectangles**, which are parallelograms with four right angles, and **rhombi**, which are parallelograms with four equal length sides. Notice that a square is a rectangle and a rhombus.

Table 4.1.3 Parallelograms

Shape	Perimeter	Area
	$2(a+b)$	$h_1 a$

The perimeter formula is the sum of the four sides. Because the sides come in two pairs which each have the same length we end up with $a+b+a+b = a+a+b+b = 2a+2b = 2(a+b)$.

Example 4.1.4 What are the perimeter and area of this parallelogram?



The perimeter is the sum of the side lengths which in this case is

$$2(4.05+5.00) = 18.1.$$

The area of a parallelogram, given in Table 4.1.3 is $h_1 a$. For this parallelogram that is

$$\text{Area} = 3.00 \cdot 5.00 = 15.0.$$

□

Example 4.1.5 What are the perimeter and area of this parallelogram?

The perimeter is the sum of the side lengths which in this case is

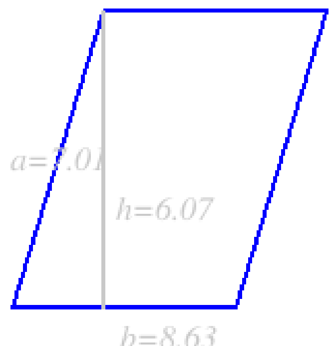
$$2(18.1 + 28.2) = 92.8.$$

The area of a parallelogram, given in Table 4.1.3 is $h_1 a$. Because this is a rectangle a side length can be used as the height.

$$\text{Area} = 18.1 \cdot 28.2 = 510.42 \approx 510.$$

□

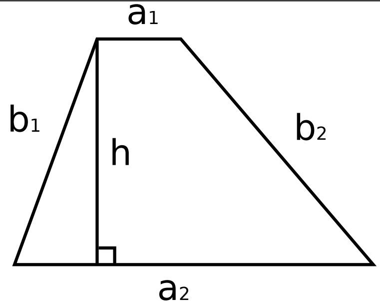
Checkpoint 4.1.6



Find the perimeter and area of the parallelogram above. Side lengths are measurements.
perimeter = ______
area = ______

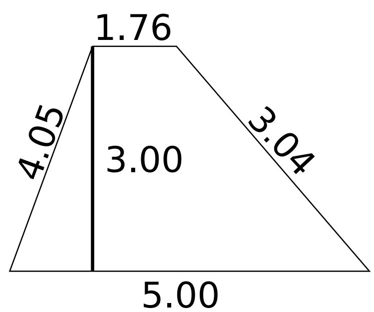
Definition 4.1.7 Trapezoid. A **trapezoid** is a four sided shape for which one pair of opposing sides are parallel. ◊

Table 4.1.8 Trapezoid

Shape	Perimeter	Area
	$a_1 + b_1 + a_2 + b_2$	$\frac{h}{2}(a_1 + a_2)$

Because none of the side lengths need be the same, the perimeter "formula" is just the sum of the four sides. The height must be between the two, parallel sides. The formula does not work if a line is connected between the other two sides.

Example 4.1.9 What are the perimeter and area of this trapezoid?



The perimeter of this trapezoid is the sum of the four side lengths

$$4.05 + 1.76 + 3.04 + 5.00 = 13.85.$$

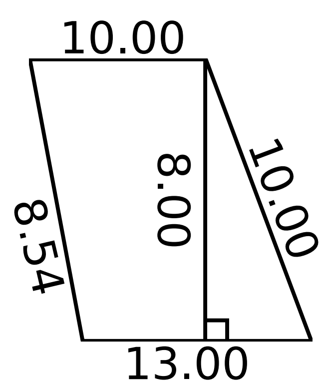
The area of a trapezoid, given in Table 4.1.8 is $\frac{h}{2}(a + b)$. For this trapezoid that is

$$\text{Area} = \frac{3.00}{2}(1.76 + 5.00) = 10.14 \approx 10.1.$$

□

Note the following is also a trapezoid and the formulae still apply.

Example 4.1.10 What are the perimeter and area of this trapezoid?



The perimeter of this trapezoid is the sum of the four side lengths

$$10.00 + 10.00 + 8.54 + 13.00 = 41.54.$$

The area of a trapezoid, given in Table 4.1.8 is $\frac{h}{2}(a+b)$. For this trapezoid that is

$$\text{Area} = \frac{8.00}{2}(10.00 + 13.00) = 92.0.$$

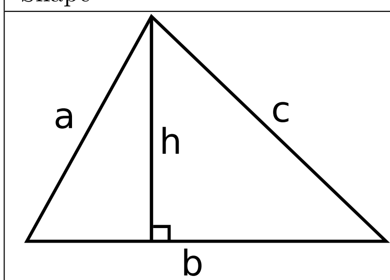
□

Checkpoint 4.1.11

Find the perimeter and area of the trapezoid above. Side lengths are measurements.
perimeter = ______
area = ______

Definition 4.1.12 Triangle. A **triangle** is a three sided shape. ◊

Table 4.1.13 Triangle

Shape	Perimeter	Area
	$a+b+c$	$\frac{1}{2}bh$

Because the three sides can all be different, the perimeter formula is just the sum of the lengths of each side. The height is a segment from a vertex down (perpendicularly) to the opposite side. Any vertex/side pair can be used. Be aware that the height may not intersect the opposite side; consider Figure 4.1.15. The vertical, dashed line segments are heights for those two triangles. The one on the left is from the top vertex down to the bottom side. The one on the right is from the top vertex down to the extension (to the left) of the bottom side.

Example 4.1.14 What are the perimeter and area of the triangles in Figure 4.1.15?
The perimeter of the triangle on the left is

$$5.98 + 10.9 + 8.19 = 25.07 \approx 25.1.$$

The perimeter of the triangle on the right is

$$5.98 + 2.87 + 8.19 = 17.04 \approx 17.0.$$

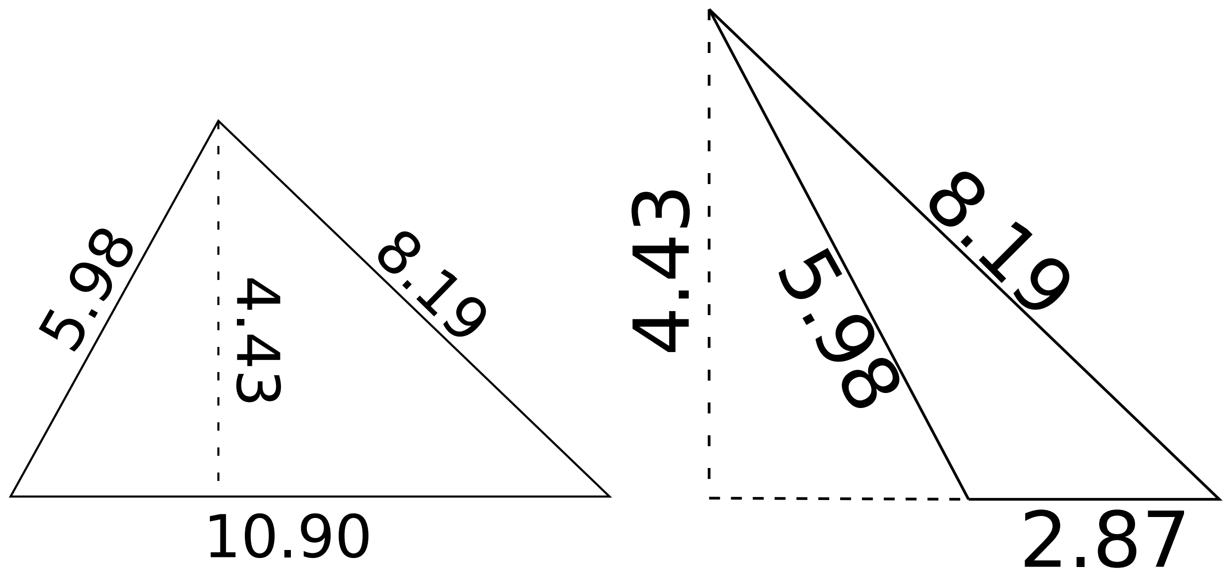
The area of a triangle, given in Table 4.1.13 is $\frac{1}{2}bh$. For the triangle on the left that is

$$\text{Area} = \frac{1}{2}10.90 \cdot 4.43 = 24.\underline{1}435 \approx 24.1.$$

For the triangle on the right the area is

$$\text{Area} = \frac{1}{2}2.87 \cdot 4.43 = 6.3\underline{5}705 \approx 6.36$$

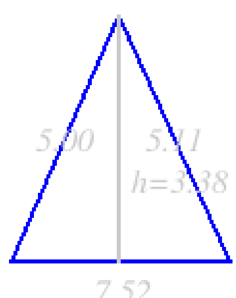
□



(a) Obtuse Triangle **(b)** Acute Triangle

Figure 4.1.15 Two Triangles

Checkpoint 4.1.16



Find the perimeter and area of the triangle above. Side lengths are measurements.
perimeter = ______
area = ______

Checkpoint 4.1.17 In Table 4.1.13 the height is labeled h. Draw the other two heights.

Theorem 4.1.18 Pythagorean Theorem. *For a triangle containing a right angle*

$$a^2 + b^2 = c^2$$

where a and b are the lengths of the sides adjacent to the right angle and c is the third side.

Example 4.1.19 Consider the triangle in Figure 4.1.15(b). Consider the segments of length 4.43, 5.98, and the horizontal dashed segment. If we want to know the length of the dashed segment, we can use this theorem. 5.98 is the length of the side not adjacent to (opposite from) the right angle (c in the formula).

$$\begin{aligned} 4.43^2 + b^2 &= 5.98^2. \\ 19.\underline{6}249 + b^2 &= 35.\underline{7}604. \\ b^2 &= 16.\underline{1}355. \\ \sqrt{b^2} &= \sqrt{16.\underline{1}355}. \\ b &\approx 4.0\underline{1}690179 \\ &\approx 4.02. \end{aligned}$$

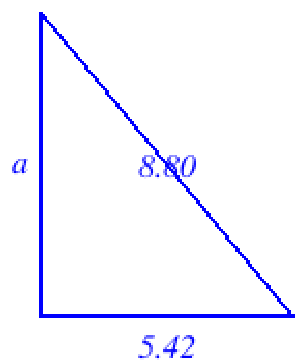
□

Example 4.1.20 Another use of this theorem is determining if a triangle is a right triangle. Notice that the triangle in Figure 4.1.15(a) appears to have a right angle at the top (opposite the side of length 10.90). If this is a right triangle, then $a^2 + b^2$ will equal c^2.

$$\begin{aligned} a^2 + b^2 &= \\ 5.98^2 + 8.19^2 &= \\ 35.7604 + 67.0761 &= 102.8365. \\ c^2 &= 102.8365. \\ c &= \sqrt{102.8365} \\ &\approx 10.14083330 \\ &\approx 10.1. \end{aligned}$$

Because $10.1 \neq 10.9$ this is not a right triangle, but it appears to be close. □

Checkpoint 4.1.21



What is the length of the unmeasured side of the triangle? ____________________

In Section 7.3 we will develop a version of this statement for triangles without a right angle.

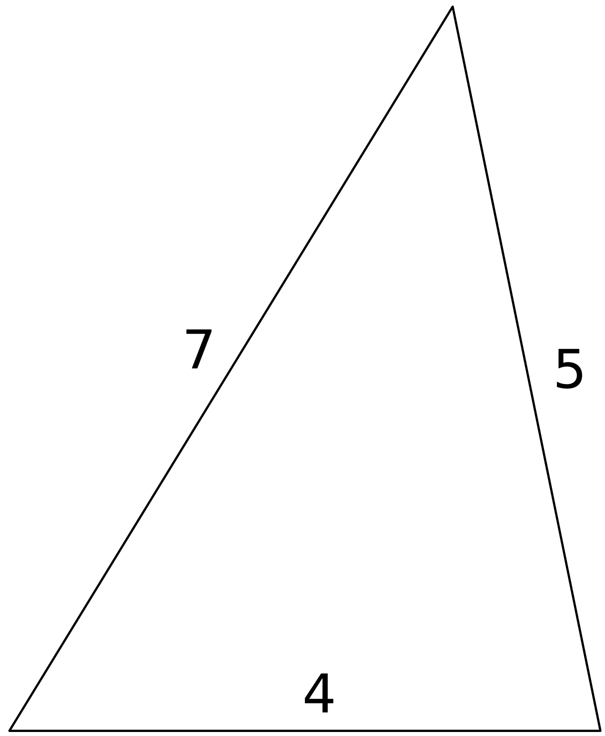
The formula for area of a triangle above requires that we be able to calculate the height. In Section 7.1 we will learn to calculate this height if we know an angle. This next formula enables us to calculate the area of a triangle without knowing any angles or the height.

Theorem 4.1.22 Heron's Formula. *The area of a triangle can be calculated using the three sides.*

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a + b + c)$.

Example 4.1.23 Calculate the area of a triangle with side lengths 4, 5, and 7.



According to Heron's formula

$$\begin{aligned} s &= \frac{1}{2}(4+5+7) \\ &= 8. \\ \text{Area} &= \sqrt{8(8-4)(8-5)(8-7)} \\ &= \sqrt{96} \\ &\approx 9.79795897 \\ &\approx 9.8. \end{aligned}$$

Rounding to one decimal place here is an arbitrary choice for this example. □

The next example illustrates tracking significant digits while using Heron's formula to calculate area.

Example 4.1.24 Calculate the area of the triangles in Figure 4.1.15.

According to Heron's formula for the triangle in Figure 4.1.15(a)

$$\begin{aligned} s &= \frac{1}{2}(5.98 + 8.19 + 10.90) \text{ Each term is precise to the 100ths (2 places)} \\ &= 12.5\underline{3}5. \text{ Addition maintains to the 100ths (4 sigfigs)} \\ \text{Area} &\approx \sqrt{12.5\underline{3}5(12.5\underline{3}5 - 5.98)(12.5\underline{3}5 - 8.19)(12.5\underline{3}5 - 10.90)} \end{aligned}$$

Terms precise to 100ths, 3 sigfigs

$$\begin{aligned} &\approx \sqrt{12.5\underline{3}5(6.5\underline{5}5)(4.3\underline{4}5)(1.6\underline{3}5)} \text{ Product maintains 3 sigfigs} \\ &\approx \sqrt{58\underline{3}.7199977} \text{ Root maintains 3 sigfigs} \\ &\approx 24.\underline{1}6 \\ &\approx 24.2. \end{aligned}$$

For the triangle in Figure 4.1.15(b)

$$\begin{aligned} s &= \frac{1}{2}(5.98 + 8.19 + 2.87) \text{ Each term precise to 100ths} \\ &\approx 8.52. \text{ Sum maintains to the 100ths} \\ \text{Area} &\approx \sqrt{8.52(8.52 - 5.98)(8.52 - 8.19)(8.52 - 2.87)} \end{aligned}$$

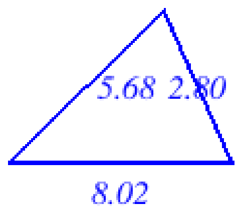
Sums maintain to the 100ths

$$\begin{aligned} &\approx \sqrt{8.52(2.54)(0.33)(5.65)} \text{ Smallest factor has 2 sigfigs} \\ &\approx \sqrt{4\underline{0}.34927160} \text{ Root maintains 2 sigfigs} \\ &\approx 6.\underline{3}5210765 \\ &\approx 6.4. \end{aligned}$$

□

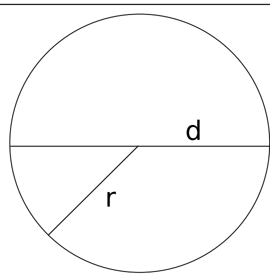
Notice that the results from Heron's formula are slightly different from the results using the $\frac{1}{2}bh$ formula. This reminds us that different calculations can result in different rounding. If the difference does not impact the application we do not care. If it does have a negative impact, then we must measure more precisely.

Checkpoint 4.1.25



What is the area of the triangle above? ______
Side lengths are measurements.

Table 4.1.26 Circle

Shape	Perimeter	Area
	$2\pi r$ πd	πr^2 $\pi \frac{d^2}{4}$

When performing calculations with π we will need to select an appropriate approximation. Because it is possible to obtain an approximation with arbitrary precision (as many decimal places as we want), we will select the approximation so that rounding is not affected.

Example 4.1.27 For a circle with radius 7.31 what are the perimeter and area?

The perimeter, given in Table 4.1.26, is $2\pi r$. For radius 7.31

$$\begin{aligned}\text{perimeter} &= 2\pi(7.31)\\ &\approx 2(3.14159)(7.31)\\ &= 45.\underline{9}300458.\\ &\approx 45.9.\end{aligned}$$

We need to approximate π to at least as many significant digits as the others numbers to avoid reducing precision. We could use more.

The area, given in Table 4.1.26, is πr^2. For radius 7.31

$$\begin{aligned}\text{area} &= \pi(7.31)^2\\ &= (3.14159)(7.31)^2\\ &\approx 16\underline{7}.8743174.\\ &\approx 168.\end{aligned}$$

□

Example 4.1.28 What are the perimeter and area of a semi-circle with diameter 11.7?

The perimeter includes half the usual perimeter plus the length of the diameter. We use the diameter version of the perimeter formula: πd.

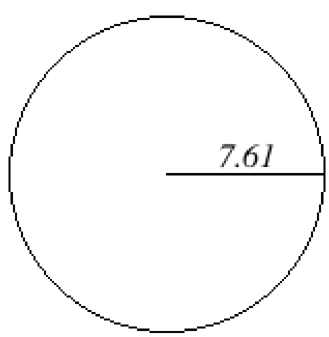
$$\begin{aligned}\text{perimeter} &= \frac{1}{2}\pi(11.7) + 11.7\\ &= \frac{1}{2}(3.14159)(11.7) + 11.7\\ &= 18.\underline{3}783015 + 11.7\\ &= 30.\underline{0}783015\\ &\approx 30.1.\end{aligned}$$

The area is simply half of the usual area.

$$\begin{aligned}\text{area} &= \frac{1}{2}\pi(11.7)^2\\ &= \frac{1}{2}(3.14159)(11.7)^2\\ &\approx 21\underline{5}.0261276.\\ &\approx 215.\end{aligned}$$

□

Checkpoint 4.1.29



Find the perimeter and area of the circle above. The radius is a measurement.
perimeter = _____
area = _____

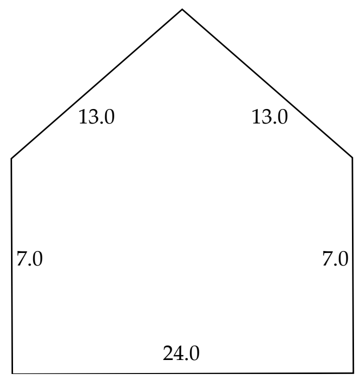
4.1.2 Applying Geometry

When we encounter geometric questions, they will often involve shapes consisting of more than one basic shape such as the park boundry above. Our first task then is to break down these shapes into pieces that are basic shapes (ones we already know). Then we can use the geometric properties to calculate results.

Note that often we can break down a complex shape in more than one way. Which way we choose will depend on what we are trying to calculate and what information we have available.

Example 4.1.30

(a) Find the area of this wall given the dimensions given in feet.



First we notice that we can describe the wall as a rectangle with a triangle on top of it.

The sides of the rectangular area are 7.0 ft (height) and 24.0 ft (width). This means that area is 7.0 ft × 24.0 ft = 168 ft^2.

The top is a triangle with two sides of length 13.0 and one of length 24.0. We don't know the height of the triangle so it will be easier to use Heron's formula for area.

$$\begin{aligned} s &= \frac{1}{2}(13.0 + 13.0 + 24.0) \\ &= 25.0. \\ \text{area} &= \sqrt{25.0(25.0 - 13.0)(25.0 - 13.0)(25.0 - 24.0)} \\ &= \sqrt{25.0(12.0)(12.0)(1.0)} \\ &= \sqrt{3600} \\ &= 6\bar{0}. \end{aligned}$$

The total area then is $168 + 6\bar{0} = 228 \approx 230$ square feet.

(b) Find the perimeter of this wall given the dimensions given in feet.

There are five (5) edges. There sum is $7.0 + 24.0 + 7.0 + 13.0 + 13.0 = 64.0$ feet.

(c) What is the (tallest) height of the wall?

The height comes from the center of the wall (peak of the triangle). The height of the wall is the height of the rectangle (7.0 ft) plus the height of the triangle. We must calculate the height of the triangle. At this time we know the area and two area formulae. Height is part of the first area formula, and from Heron's formula we already know the area. Putting these two together gives us the following.

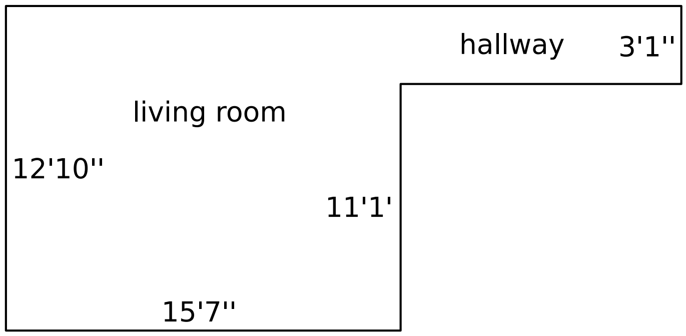
$$6\bar{0} = \frac{1}{2}24.0h.$$

$$\begin{aligned} 6\bar{0} &= 12.0h. \\ \frac{6\bar{0}}{12.0} &= h. \\ 5.0 &= h. \end{aligned}$$

The total height is then $7.0 + 5.0 = 12.0$.

□

Example 4.1.31 Calculate the square footage of the combined living room and hallway.



This layout is rectangular. The living room is a rectange with height 12 ft 10 in and width 15 ft 7 in. The hallway adds an additional rectangle of height 3 ft 1 in and width 11 ft 1 in.

To calculate square footage we need to convert everything to the same unit. To avoid rounding issues we will convert to inches then back to feet in the end.

The living room height is $12 \text{ ft} \cdot \frac{1 \text{ ft}}{12 \text{ in}} + 10 \text{ in} = 154 \text{ in}$. The width is $15 \text{ ft} \cdot \frac{1 \text{ ft}}{12 \text{ in}} + 7 \text{ in} = 187 \text{ in}$. The area is $(154 \text{ in})(187 \text{ in}) = 28798 \text{ in}^2$.

The hallway height is $3 \text{ ft} \cdot \frac{1 \text{ ft}}{12 \text{ in}} + 1 \text{ in} = 37 \text{ in}$. The width is $11 \text{ ft} \cdot \frac{1 \text{ ft}}{12 \text{ in}} + 1 \text{ in} = 122 \text{ in}$. The area is $(37 \text{ in})(122 \text{ in}) = 4514 \text{ in}^2$.

The total floor surface is $28798 \text{ in}^2 + 4514 \text{ in}^2 = 33312 \text{ in}^2$. We can now convert that back to feet.

$$33312 \text{ in}^2 \cdot \left(\frac{1 \text{ ft}}{12 \text{ in}}\right)^2 = \frac{33312}{144} \text{ ft}^2 \approx 231.333 \text{ ft}^2 \approx 231 \text{ ft}^2.$$

We are using significant digits rounding. All measurements were accurate to one inch. If we converted to decimal feet we would end up with measurements like 3 ft 1in ≈ 3.0833 ft. The repeated decimal would have to be rounded whereas the inch measurement did not need to be rounded. Avoiding rounding avoids introducing error. □

The following example requires geometric thinking, but does not require any formula. Especially important is the type of rounding which is determined by physical constraints.

Example 4.1.32 Katie is building a large scale abacus for a park. Her plan is to build it from treated 2x4 lumber. Her plan is shown in Figure 4.1.33 Note the depth of each piece of wood is 3.5". If you are wondering why a 2x4 is 1.5 in x 3.5 in, note that the nominal size (2x4 in this case) is based on the initial cut. The lumber shrinks as it cures and again when it is planed smooth.

Because we must have enough wood, we will round up all approximations. Measurements are accurate to an 8th of an inch.

(a) What is the total number of feet of lumber (2x4) needed?

Solution. There are two boards of length 60 inches and three boards of length 27 inches. The total length is

$$2(60 \text{ in}) + 3(27.0 \text{ in}) = 120 \text{ in}.$$

We convert this to feet using a ratio.

$$120 \text{ in} \cdot \frac{1 \text{ ft}}{12 \text{ in}} = 10 \text{ ft}$$

(b) If a standard 2x4 is 96.0 inches long, what is the smallest number of boards Katie can purchase to have enough lumber?

Solution. If a 60 inch section is cut from a 96 inch board, we have 96 in − 60 in = 36 in left. This is long enough for one of the 27 inch segments but not more. Thus two boards will cover all but the last 27 inch segment. We need three (3), 96 inch boards.

(c) If the boards are painted before they are assembled, what is the total surface area of the boards to be painted? Paint cans are rated for the number of square feet they can cover. We will want the solution in units of square feet.

Solution. Each board has six surface. Each surface size appears twice (e.g., top and bottom). For the long segments these areas are

$$\begin{aligned} 60 \text{ in} \cdot 3.5 \text{ in} &= 210 \text{ in}^2, \\ 60 \text{ in} \cdot 1.5 \text{ in} &= 90 \text{ in}^2, \\ 1.5 \text{ in} \cdot 3.5 \text{ in} &= 5.25 \text{ in}^2. \end{aligned}$$

For the short segments these are

$$\begin{aligned} 27 \text{ in} \cdot 3.5 \text{ in} &= 94.5 \text{ in}^2, \\ 27 \text{ in} \cdot 1.5 \text{ in} &= 40.5 \text{ in}^2, \\ 1.5 \text{ in} \cdot 3.5 \text{ in} &= 5.25 \text{ in}^2. \end{aligned}$$

Thus the total area is

$$\begin{aligned} &2(2)(210 \text{ in}^2) + 2(2)(90 \text{ in}^2) + 2(2)(5.25 \text{ in}^2) \\ &+3(2)(94.5 \text{ in}^2) + 3(2)(40.5 \text{ in}^2) + 3(2)(5.25 \text{ in}^2) = \\ &\qquad 810 + 360 + 21 + 567 + 246 + 31.5 \text{ in}^2 = 2035.5 \text{ in}^2 \end{aligned}$$

Finally we convert this to square feet using the method of Example 1.1.23.

$$\begin{aligned} 2035.5 \text{ in}^2 \cdot \left(\frac{1 \text{ ft}}{12 \text{ in}}\right)^2 &= \\ \frac{2035.5 \text{ in}^2}{12^2 \text{ in}^2} \text{ ft}^2 &= \\ \frac{2035.5}{144} \text{ ft}^2 &\approx 14.13541667 \text{ ft}^2. \\ &\approx 15. \end{aligned}$$

We round up because we want to be safe and because we likely cannot order an amount to cover exactly 14.1 square feet.

(d) What is the area that is hidden, that is, cannot be seen after assembling?

Solution. This would be where the three short boards touch the long boards. There are six places where this happens which are all the same shape $3.5 \text{ in} \cdot 1.5 \text{ in} = 5.25 \text{ in}^2$. The covered surface is on both the short boards and the matching spot on the long boards, so there are 12 of these surfaces for $12 \cdot 5.25 \text{ in}^2 \approx 63 \text{ in}^2$.

□

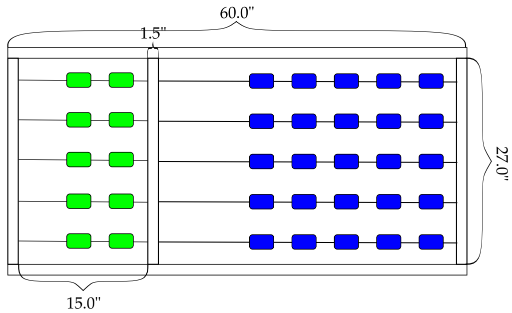
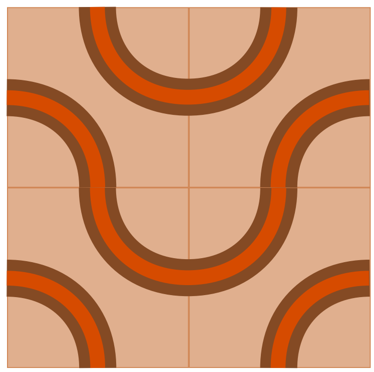


Figure 4.1.33 Abacus

The following example also illustrates rounding according to physical constraints.

Example 4.1.34 We are purchasing carpet squares to carpet a rectangular room that is 15.5 ft by 19.0 ft. These carpet squares are 18 in by 18 in. A box contains 10 squares. How many boxes of carpet squares do we need purchase? This image shows four carpet squares together.



We need to know how many square across and down the room. For the number of squares across we can divide the width of the room (15.5 ft) by the width of a carpet square (18 in). This requires matching units, so $15.5\text{ ft} \cdot \frac{12\text{ in}}{\text{ft}} = 186\text{ in}$ Thus the number of squares across is $\frac{186}{18} = 10.333$. This means the final square will be a third of a square in width. While it might seem efficient to reuse the rest of that carpet square in the rows above, we cannot because the pattern would not continue correctly. Thus we simply round up to 11 squares across.

Next we calculate the number of squares down. $19.0\text{ ft} \cdot \frac{12\text{ in}}{\text{ft}} = 228\text{ in}$. Thus the number of squares down is $\frac{228}{18} = 12.666$. Again to have enough squares we must round up to 13 squares down.

Thus the total number of carpet squares needed is $11 \cdot 13 = 143$. Because each box has 10 squares each, we need $\frac{143}{10} = 14.3$ boxes. To have enough we need 15 boxes. In practice we would likely buy at least one extra box in case there are problems. □

Example 4.1.35 At first the following might seem to be a reasonable way to solve this problem. First, calculate the total square inches of floor. Using the conversion to inches from above $186 \cdot 228 = 42408$ square inches of floor.

Next we can divide by the number of square inches each carpet squares covers to count the number of squares. Each square covers $18^2 = 324$ square inches. Thus we calculate that we need $\frac{42408}{324} = 130.888$ carpet squares.

This is substantially fewer than the 143 we calculated previously. The reason is this calculation assumes we can use each of the partial squares somewhere else in the room. This is true when the pattern does not need to match, but not in this case.

Now you may understand why such repeating patterns are avoided in most carpets. □

Checkpoint 4.1.36

What is the area of this floor in square feet? ____

4.1.3 Exercises

1. Area Application. Chris wishes to carpet a rectangular room. He will not carpet the floor inside the closet.

Which is the number of square yards, to the nearest square yard, of carpet needed to carpet the floor of the room?

HINT: The diagram gives dimensions in FEET but you are asked for square YARDS. How many square feet are there in a square yard? Draw a picture if you need to.

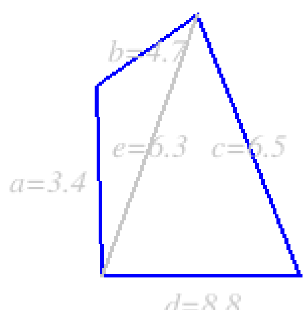
(a) 17

(b) 13

(c) 114

(d) 156

2. Contextless Area.

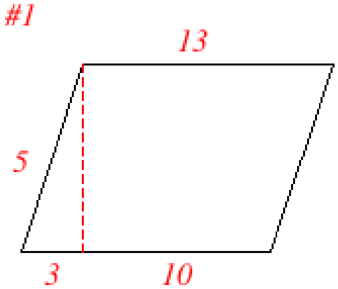


Find the area of the quadrilateral shown. The lengths of the four sides (counterclockwise) are $a = 3.4$, $b = 4.7$, $c = 6.5$, and $d = 8.8$, with diagonal from first to third point $e = 6.3$. Measurements are in yards. What is the area of this figure?

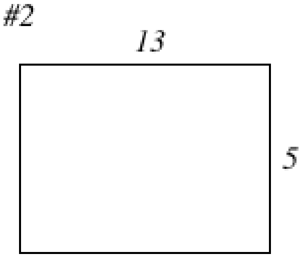
area = ____________________ yd^2

3. Perimeter and Area Theory. Below are three quadrilaterals

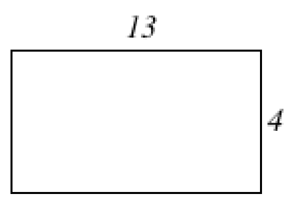
#1



#2

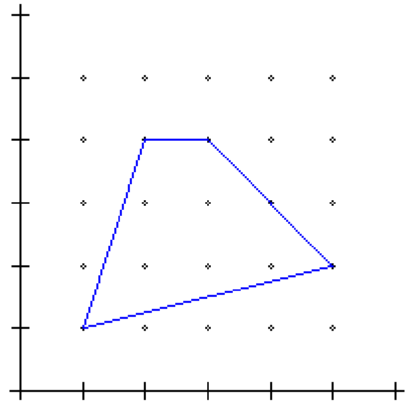


#3



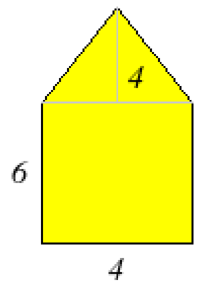
What is the perimeter and area of the 1st quadrilateral?
Perimeter =__________ Area =__________
What is the perimeter and area of the 2nd quadrilateral?
Perimeter =__________ Area =__________
What is the perimeter and area of the 3rd quadrilateral?
Perimeter =__________ Area =__________

4. **Contextless Area.** What is the area of the quadrilateral on the Geoboard below? The distance between points vertically and horizontally is 1 cm.



The area is ___ cm^2.

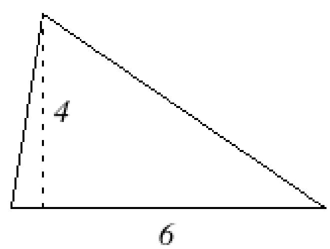
5. Contextless Composite Area.



Find the area of the shaded region above.

___

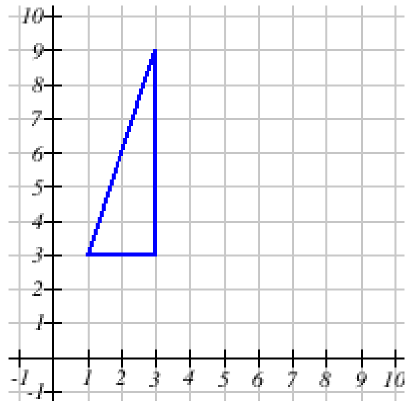
6. Contextless Area.



Find the area of the triangle pictured above.
Round your answer to the nearest tenth

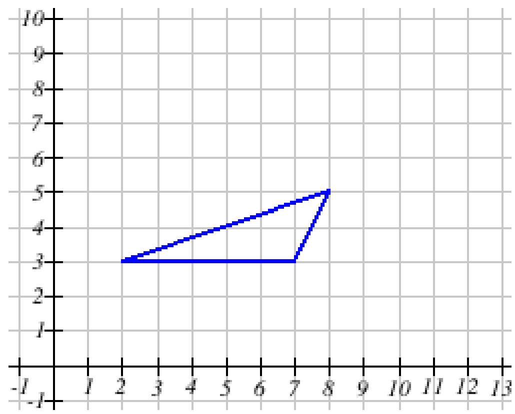
___

7. Contextless Area. Find the area of the triangle shown below



Area = ____________________

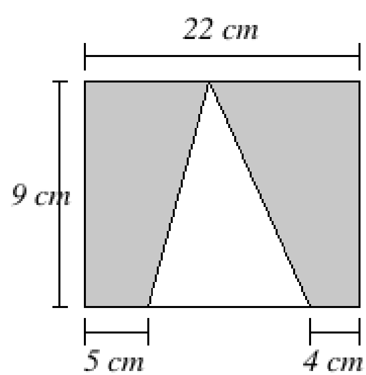
8. Contextless Area. Find the area of the triangle shown below



Area = ____________________

9. Contextless Composite Area. Find the area of the shaded region.

Hint: Area of a rectangle: $A = lw$. Area of a triangle: $A = \frac{1}{2}bh$.



Area = ____________________

10. Area Application. A triangular parcel of land has sides of lengths 110. feet, 330. feet and 334 feet. Notice all three of these measurements are precise to the ones digit.

(a) What is the area of the parcel of land? Round your answer to appropriate significant digits.

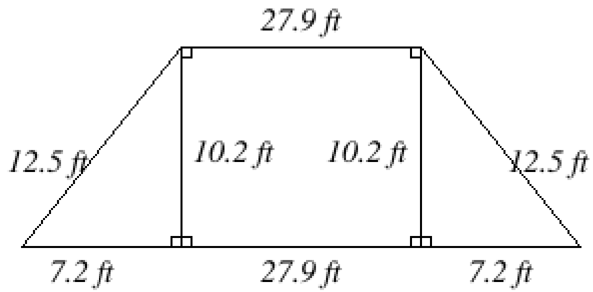
Area = ____________________ ft^2

(b) If land is valued at \$2300 per acre (1 acre is 43,560 ft^2), what is the value of the parcel of land?

Base on rounded area. Round your value to the nearest dollar.

value =$ ____________________

11. Contextless Perimeter and Area. Find the perimeter and area of the following composite figure:



Perimeter = _____ feet (Follow the rules for working with precision.)
Area = _____ square feet (Follow the rules for working with accuracy.)
NOTE: Figures are NOT to scale.

12. Contextless Area. Find the area of a triangle with sides *26*, *43*, and *11*

____

Your answer should be accurate to the tenths place.
Enter DNE if the triangle cannot exist.

13. Contextless Area. The diagram below shows a triangle inscribed in a rectangle.

3.00 7.00 5.00 4.24

What is the area of the shaded inscribed triangle? ____ $units^2$
What is the area of the rectangle? ____ $units^2$
What is the area of the unshaded parts of the rectangle? ____ $units^2$

14. Area and Perimeter Application. Valerie needs enough mulch to cover the triangular garden shown and enough paving stones to border it. If one bag of mulch covers 13 square feet and one paving stone provides a 6-inch border, how many bags of mulch and how many stones does she need to buy?

Figure is not to scale

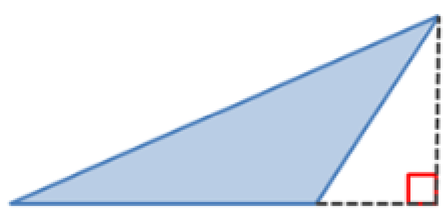
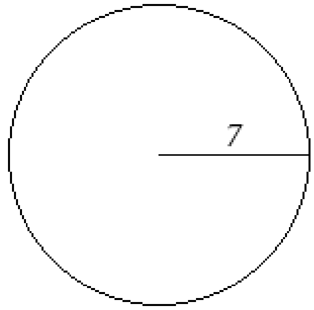


Table 4.1.37

Area: _____ ft^2	_____ bags of mulch
Perimeter: _____ ft	_____ paving stones

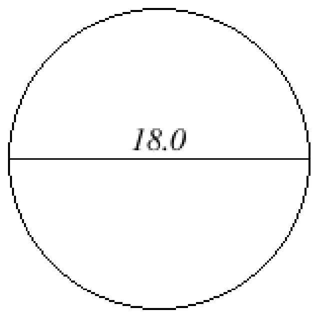
15. Contextless Area.



Find the area of the circle pictured above.
Round your answer to the nearest tenth

____

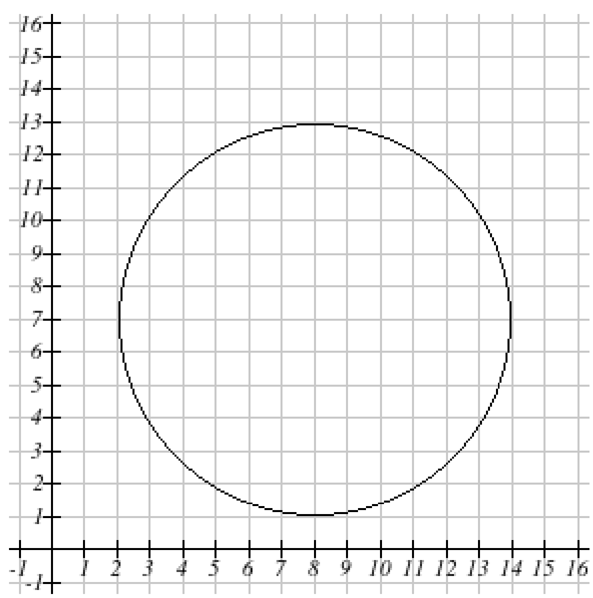
16. Contextless Area.



Find the area of the circle pictured above.
Round your answer using the rules of working with significant figures. Note using the approximation of 3.14 for π may result in an incorrect answer.

________________ $units^2$

17. Contextless Area.

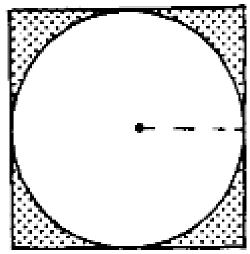


Find the area of the circle pictured above.

Round your answer to the nearest tenth

____

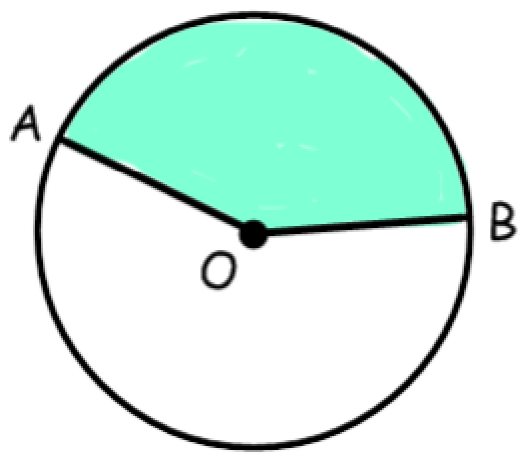
18. Contextless Composite Area. Here is a figure made of a circle inscribed in a square.



If the radius of the circle is 8.00 ft:
(a) What is the area of the square? ____________________ ft^2
(b) What is the area of the circle? Round to the nearest tenth. ____________________ ft^2
(c) What is the area of the shaded section? ____________________ ft^2

19. Area Application. Kimani has a sprinkler than covers a circular area with diameter 34 feet. What is the area of lawn that the sprinkler covers? (Use 3.14 for π)
_______ square feet

20. Contextless Area. In the diagram below, the circle has a radius length of$r = 11$. If the measure of arc $AB = 161$ °, then what is the area of the unshaded part of the circle O?



[***not drawn to scale***]
Area of unshaded part of circle $O =$ _______ $units^2$

21. A grain silo is shown below. Measurements are in feet.

If 25 ft^3 of grain is stored in the silo, what percent of the silo is full? _____

4.2 Geometric Reasoning 3D

This section addresses the following topics.

- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Identify shapes and apply their properties (skill)

This section presents geometric properties, illustrates identifying shapes in applications, and illustrates breaking down complex shapes into simple ones.

4.2.1 Formulae

This section defines the two properties of interest and provides the formulae for some common shapes. Memorizing all of the formulae is not likely useful: in a job you will be able to look them up. However, anything you use a lot (e.g., prisms and spheres) is worth memorizing.

Use this section to learn to identify which measurement is used in each part of a formula and how to find those from diagrams or descriptions of shapes. The next section will illustrate using these properties in applications.

The **surface area** of a 3D shape is the cummulative area of all the 2D areas of the shape. You can think of it as the amount of paint needed to cover the object. The **volume** of a 3D shape is a measure of what it takes to fill a 3D shape. In the figure below showing a paper cone for holding popcorn, the surface area is the amount of wax needed to coat the inside of the cone so the popcorn (butter) does not soak through the paper. The volume is how much popcorn can be held.

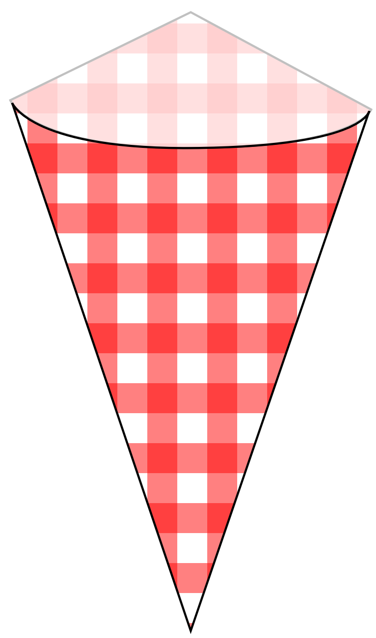


Figure 4.2.1 Complex Shape with Surface and Volume

Rounding. How we round on geometric problems will depend as always on the application. For example a carpenter will round to units that can be measured with a tape measure like 1/16 of an inch. Contextless examples in this section will be rounded using significant digits. When using significant digits treat all numbers in the formulae as exact numbers.

Definition 4.2.2 Prism. A **prism** is a solid consisting of two identical polygons connected by parallelograms. ◊

These look like a polygon has been extruded. If the sides are rectangles, then it is called a **right prism**. Prisms are named for their base shape. For example, there are triangular prisms and pentagonal prisms. The figure below shows multiple prisms. Those on the left are right prisms (the angles on the bottom are right angles) and those on the right are skewed (some angle other than a right angle). The top two are triangular prisms and the bottom two are trapezoidal prisms.

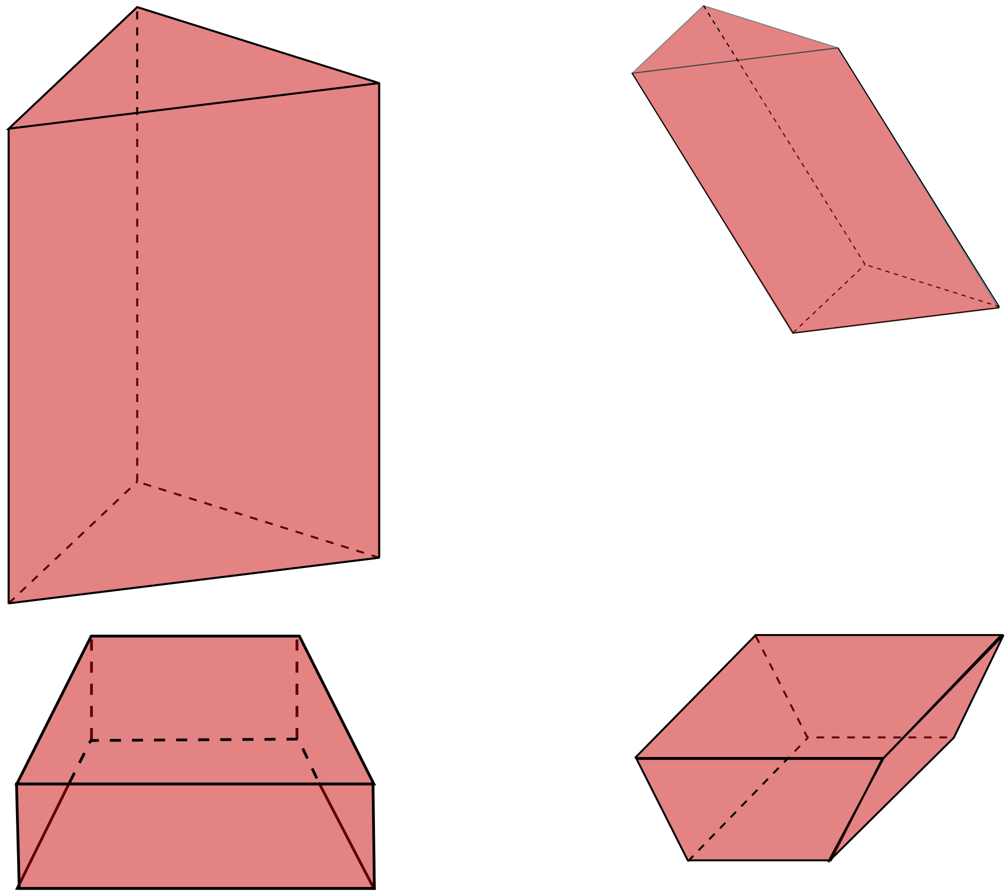
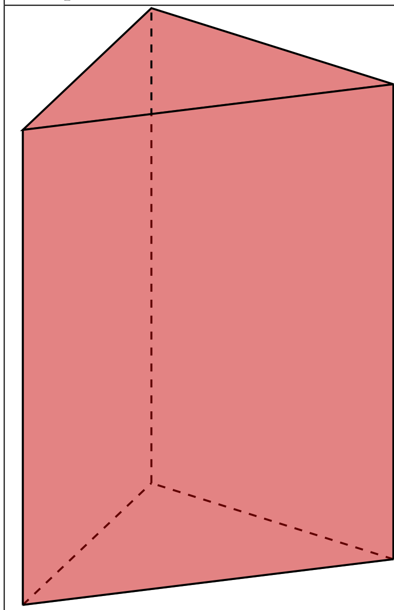


Figure 4.2.3 Various Prisms

Table 4.2.4 Prisms

Shape	Surface Area	Volume
	sum of area of all sides	$V = Bh$

For surface area the sides are always parallelograms (rectangles for right prisms). For the volume B is the area of the base (the triangle or trapezoid in the four examples). h is the height or distance from the bottom to the top (or front to back if we tip the prism over). Note this is not along an edge, but perpendicular to the base.

Example 4.2.5

(a) What is the surface area of the right triangular prism in Figure 4.2.6?

The surface area consists of the areas of the two right triangles and the three rectangles. Because the triangles are right triangles, they both have area $\frac{1}{2} \cdot 2.12\underline{5} \cdot 3.12\underline{5} = 3.3\underline{2}03125$. The rectangle areas are $2.12\underline{5} \cdot 10.25\underline{0} = 21.7\underline{8}125$ and $3.12\underline{5} \cdot 10.25\underline{0} = 32.0\underline{3}125$. For the third rectangle we need to calculate the side length. Because this is a right triangle we can use the Pythagorean theorem.

$$\begin{aligned}
(2.12\underline{5})^2 + (3.12\underline{5})^2 &= c^2. & \\
4.51\underline{5}625 + 9.76\underline{5}625 &= c^2. & &\text{Squaring maintains 4 sigfigs}\\
14.28\underline{1}25 &= c^2. & &\text{Adding precise to 1000ths}\\
\sqrt{14.28\underline{1}25} &= c. & &\text{Root maintains 5 sigfigs}\\
3.779\underline{0}5 &\approx c. & \\
3.7791 &\approx c. &
\end{aligned}$$

The third rectangle has area $3.779\underline{0}5 \cdot 10.25\underline{0} \approx 38.73\underline{5}2625$. The total area is $2(3.3\underline{2}03125) + 21.7\underline{8}125 + 32.0\underline{3}125 + 38.73\underline{5}2625 \approx 99.1\underline{8}83875 \approx 99.19$.

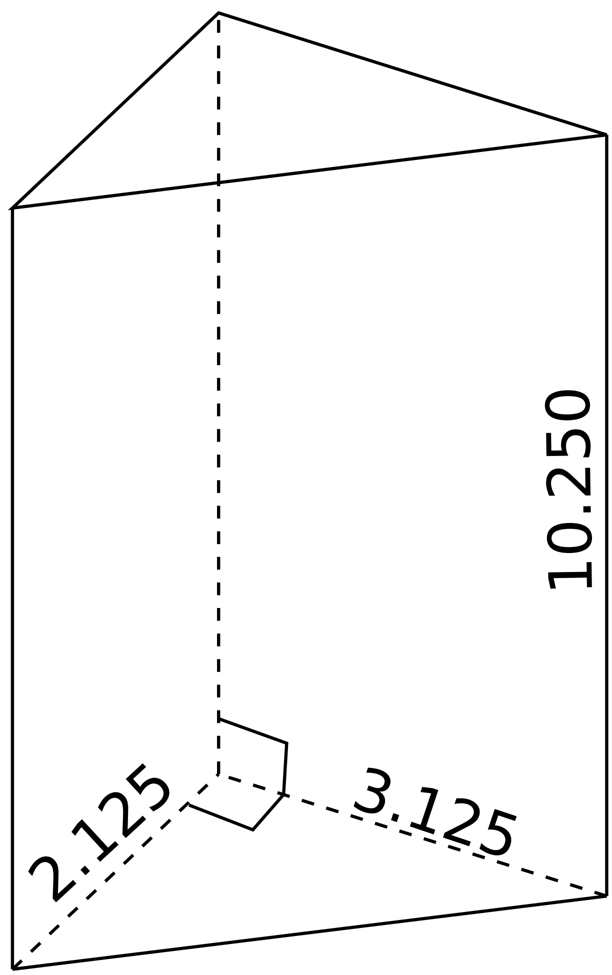
(b) What is the volume of the triangular prism in Figure 4.2.6(a)?

The volume is the area of the triangle, 3.320, times the height of the prism, 10.250. Thus the area is $3.320 \cdot 10.250 = 34.03$.

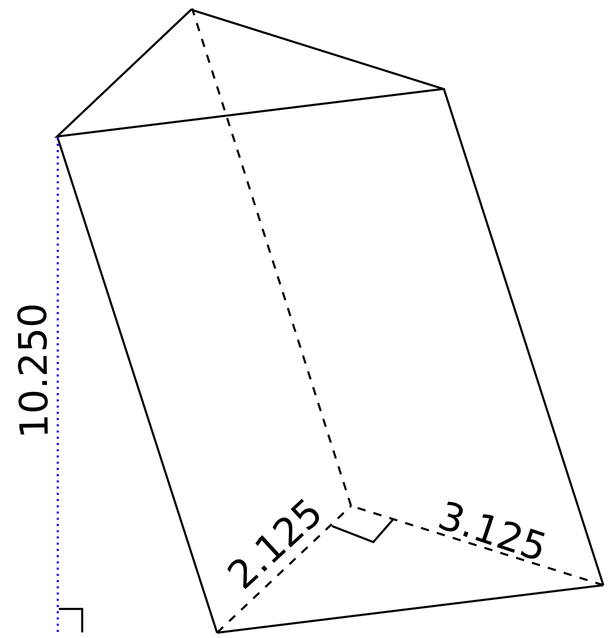
(c) What is the volume of the triangular prism in Figure 4.2.6(b)?

The volume is the area of the triangle, 3.320, times the height of the prism, 10.250. Thus the area is $3.320 \cdot 10.250 = 34.03$. Notice that the volume is not affected by the tilt of the prism. This is not true of the surface area which we do not have sufficient information to calculate here.

□



(a) Right Triangular Prism

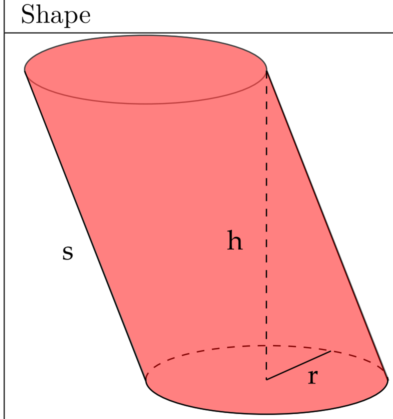


(b) Non-Right Triangular Prism

Figure 4.2.6 Calculate the surface area and volume

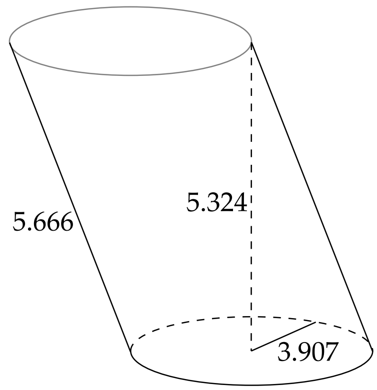
Definition 4.2.7 Cylinder. A **cylinder** is a circular prism. ◊

Table 4.2.8 Cylinder

Shape	Lateral Surface Area	Volume
	$A = 2\pi r \cdot s$	$V = \pi r^2 h$

If the cylinder is not slanted then $s = h$. Notice the volume is not dependent on the angle of slant but the lateral surface area is. To obtain the complete surface area add the area of the top and bottom circles. The surface area of the side a cylinder can be imagined to be the result of peeling off the surface which results in a rectangle.

Example 4.2.9 What are the total surface area and volume of this slanted cylinder?



The lateral surface area is

$$2\pi(3.90\underline{7})(5.66\underline{6}) \approx$$
$$2(3.1415\underline{9})(3.90\underline{7})(5.66\underline{6}) \approx 139.\underline{0}911452.$$

We must approximate π precise to at least as many positions as the rest of the numbers. Otherwise we will decrease the overall precision. We do not need to use more positions: the extra disappears in the rounding at the end. Here we use the portion most commonly memorized or listed in a table.

The area of the top and bottom are both

$$\pi(3.90\underline{7})^2 \approx$$
$$(3.1415\underline{9})(3.90\underline{7})^2 \approx 47.9\underline{5}526865.$$

Thus the total surface area is

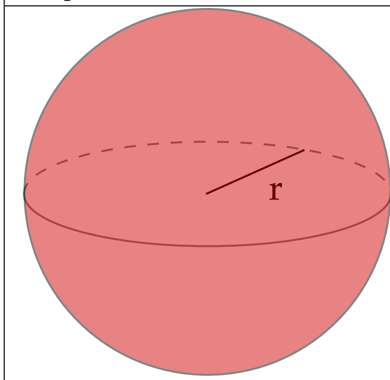
$$2(47.9\underline{5}526865) + 139.\underline{0}911452 \approx 235.\underline{0}016825 \approx 235.0.$$

The volume is

$$\begin{aligned} V &= \pi(3.90\underline{7})^2(5.32\underline{4}) \\ &\approx (3.1415\underline{9})(3.90\underline{7})^2(5.32\underline{4}) \\ &\approx 255.\underline{3}138503 \\ &\approx 255.3 \end{aligned}$$

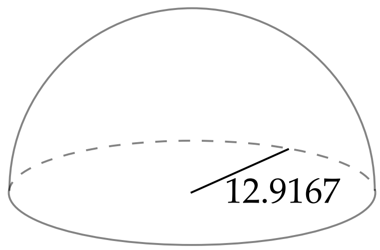
□

Table 4.2.10 Sphere

Shape	Surface Area	Volume
	$A = 4\pi r^2$	$V = \frac{4}{3}\pi r^3$

The 4 and 4/3 are exact numbers.

Example 4.2.11 What are the surface area and volume of this spherical dome? The dome is sitting on the ground so the only surface area of importance is the top surface.



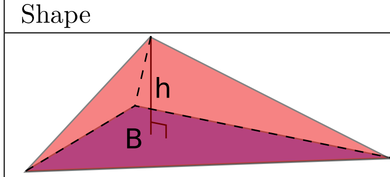
Because this is a dome (half sphere) we divide the formula by 2.

$$\begin{aligned} A &= \frac{1}{2}(4\pi(12.9167)^2) \text{ Squaring mainatains 6 sigfigs} \\ &\approx \frac{1}{2}(4(3.14159)(12.9167)^2) \text{ Approximate to 6 sigfigs} \\ &\approx 1048.2\underline{9}2907 \text{ Product maintains 6 sigfigs} \\ &\approx 1048.29 \end{aligned}$$

$$\begin{aligned} V &= \frac{1}{2}\left(\frac{4}{3}\pi(12.9167)^3\right) \\ &\approx \frac{1}{2}\left(\frac{4}{3}(3.14159)(12.9167)^3\right) \\ &\approx 4513.4\underline{9}4997 \\ &\approx 4513.49 \end{aligned}$$

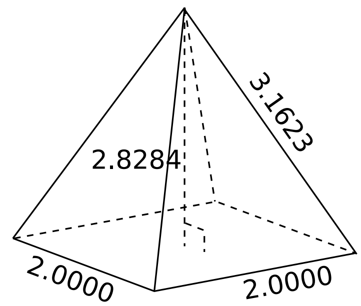
□

Table 4.2.12 Pyramid

Shape	Surface Area	Volume
	sum of surfaces	$V = \frac{1}{3}Bh$

Example 4.2.13

(a) What is the surface area of this pyramid?



This pyramid consists of four, identical triangular sides and a square base. Note the side lengths are given as exact numbers.

The area of one of the triangles can be found using Heron's formula.

$$\begin{aligned} s &= \frac{1}{2}(2.0000 + 3.1623 + 3.1623) \\ &= 4.1623. \text{ All numbers precise to 4 decimal places.} \\ A &= \sqrt{s(s - 3.1623)(s - 3.1623)(s - 2.0000)} \end{aligned}$$

Subtraction maintains 4 decimal places

$$\begin{aligned} &= \sqrt{(4.1623)(1.0000)(1.0000)(2.1623)} \text{ Product will have 5 sigfigs} \\ &= \sqrt{9.000\underline{1}4129} \text{ Root will maintain 5 sigfigs} \\ &= 3.000\underline{0}23548 \\ &= 3.0000. \end{aligned}$$

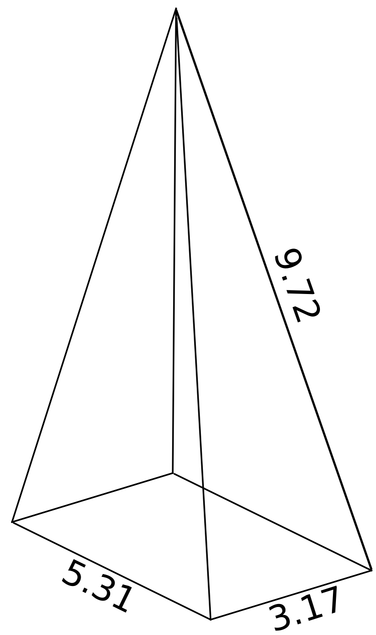
The area of the base is $2.0000 \cdot 2.0000 = 4.0000$. Thus the surface area is $4.0000 + 4(3.0000) = 16.0000$.

(b) What is the volume of this pyramid?

The volume is $V = \frac{1}{3}(2.0000^2)(2.8284) = 3.7712$.

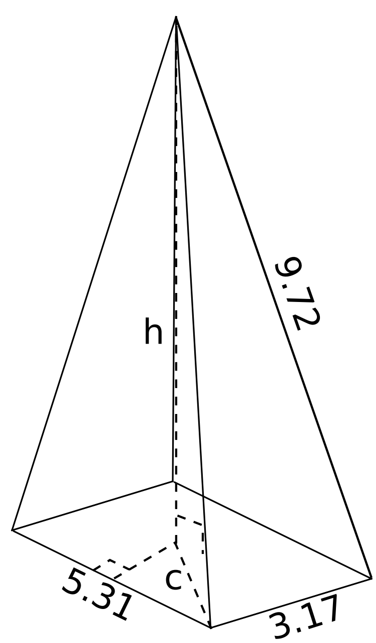
□

Example 4.2.14 Find the volume of this rectangular pyramid.



To calculate the volume we need the height which is not labeled. This is probably because we cannot put a measuring tape along the height (interior). However, we can calculate the height based on the right triangle with the height as one leg, the segment on the bottom (center of base to corner drawn below) and the edge length of the pyramid (labeled).

We do not have the length of the segment on the base (center to corner). We do know that the highest point of this pyramid is above the middle of the base. Thus this segment goes from the middle (both directions) to the corner. The length we need can be caluclated using the Pythagorean theorem. Because the point is in the middle the lengths are half of the measurements labeled.



$$\left(\frac{5.31}{2}\right)^2 + \left(\frac{3.17}{2}\right)^2 = c^2.$$

$$7.0\underline{4}9025 + 2.5\underline{1}2225 = c^2. \qquad \text{Square maintains 3 sigfigs}$$

$$\begin{aligned}
9.5\underline{6}125 &= c^2. && \text{Addition maintains precision to 2 places}\\
\sqrt{9.5\underline{6}125} &= c. && \text{Root maintains 3 sigfigs}\\
3.0\underline{9}21271 &\approx c.\\
3.09 &\approx c.
\end{aligned}$$

Next, we can use this length and the given side length to calculate the height.

$$\begin{aligned}
3.0\underline{9}21271^2 + h^2 &= 9.72^2.\\
9.5\underline{6}125 + h^2 &= 94.\underline{4}784. && \text{Square maintains 3 sigfigs}\\
h^2 &= 94.\underline{4}784 - 9.5\underline{6}125.\\
h^2 &= 84.\underline{9}1715.
\end{aligned}$$

Subtraction maintains precision to only one place

$$\begin{aligned}
h &= \sqrt{84.\underline{9}1715}. && \text{Root maintains 3 sigfigs}\\
h &\approx 9.2\underline{1}505019\\
h &\approx 9.22.
\end{aligned}$$

Finally we can calculate the volume. The base has area $5.31 \cdot 3.17 = 16.\underline{8}327 \approx 16.8$. This pyramid has volume

$$V = \frac{1}{3}(16.\underline{8}327)(9.2\underline{1}505019) \approx 51.\underline{7}0472510 \approx 51.7.$$

□

Checkpoint 4.2.15 3.66 7.21 6.00 4.44

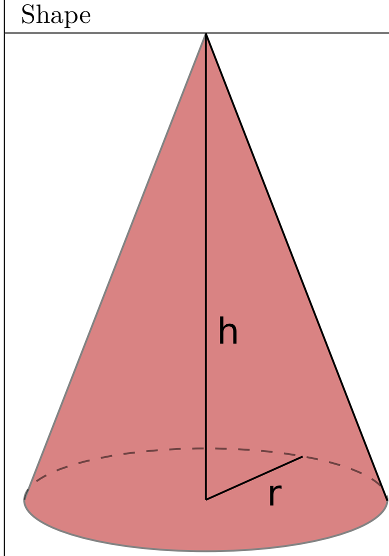
For the pryamid above (not to scale) determine the

Surface area (all): ______

Volume: ______

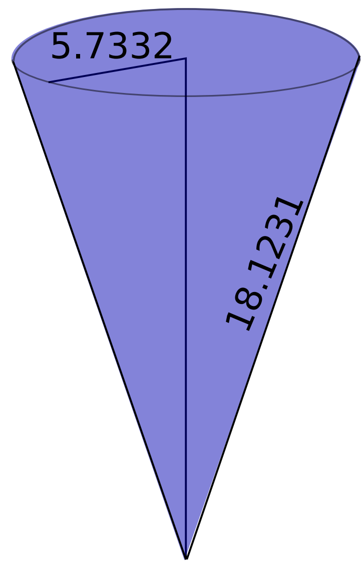
Checkpoint 4.2.16 What is the relationship between the volume of a pyramid to the volume of a prism with the same base? “same base” means the same shape and size. The pyramid could then be placed inside the prism.

Table 4.2.17 Cone

Shape	Lateral Surface Area	Volume
	$A = \pi r\sqrt{h^2+r^2}$ or $A = \pi r s$	$V = \frac{\pi}{3}r^2h$

The surface area formulas apply only when it is a right circular cone (tip of the cone directly above the center of the circle). The volume formula works for all cones. For the full surface area we add the area of the circle at the bottom of the cone.

Example 4.2.18 Calculate the total surface area and the volume of this cone.



The surface area can be calculated directly. The lateral surface area is

$$\begin{aligned} LA &= \pi r s \\ &\approx (3.14159)(5.7332)(18.1231) \text{ Minimum 5 sigfigs} \\ &\approx 326.4\underline{2}17471 \\ &\approx 326.42 \end{aligned}$$

The area of the top circle is

$$\begin{aligned} TA &= \pi r^2 \\ &\approx (3.14159)(5.7332)^2 \text{ Minimum 5 sigfigs} \\ &\approx 103.2\underline{6}27509 \\ &\approx 103.26 \end{aligned}$$

The total area therefore is $A = 326.4\underline{2}17471 + 103.2\underline{6}27509 \approx 429.6\underline{8}44980 \approx 429.68$.

To calculate the volume we need to first calculate the height. Looking at the image we see that the labeled radius and side length are part of a right triangle with the height. This means

$$\begin{aligned} h^2 + 5.7332^2 &= 18.1231^2 \text{ Squaring maintains sigfigs} \\ h^2 + 32.86\underline{9}5224 &\approx 328.44\underline{6}7536 \text{ Precise to 3 places} \\ h^2 &\approx 328.44\underline{6}7536 - 32.86\underline{9}5224 \\ h^2 &\approx 295.57\underline{7}1714 \\ h &\approx 17.192\underline{3}5794 \text{ Root preserves 6 sigfigs} \\ &\approx 17.1924. \end{aligned}$$

Having calculated the height we can calculate the volume

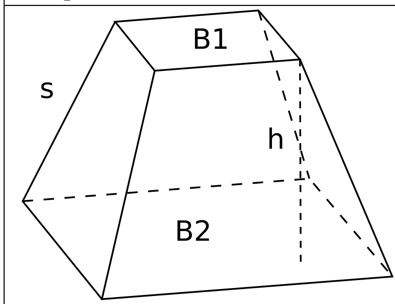
$$\begin{aligned} V &= \frac{\pi}{3} r^2 h \\ &\approx \frac{3.14159}{3}(5.7332)^2(17.192\underline{3}5794) \\ &\approx 591.7\underline{7}67250 \end{aligned}$$

$$\approx 591.78$$

□

The frustrums are compound shapes: they are obtained by subtracting one shape from itself. The latin word **frustrum** means "cut off." This is the etymology of "frustrated" which refers to a cut off hope. For these formulae the bases must be parallel. Note P_i below refers to the perimeter of the bases.

Table 4.2.19 Frustrum of a Pyramid

Shape	Lateral Surface Area	Volume
	$\frac{1}{2}s(P_1+P_2)$	$V=\frac{1}{3}h(B_1+B_2+\sqrt{B_1B_2})$

P_1 is the perimeter of the lower base (bottom polygon). It will be the sum of the lengths of each side. P_2 is calculated similarly. B_1 is the area of the lower base (bottom polygon). It will be calculated based on the type of polygon (e.g., area of a triangle, parallelogram, etc.). B_2 is the area of the upper base.

Example 4.2.20

(a) What is the surface area of the frustrum of the pyramid in Figure 4.2.21?

The perimeter of the lower base (triangle) is $3(3.500) = 10.500$ (3 is an exact number here so precision is not changed). The perimeter of the upper base (triangle) is $3(2.200) = 6.600$. To calculate the area of the two bases we will need Heron's formula.

$$\begin{aligned} s_1 &= 10.500. \\ A_1 &= \sqrt{10.500(10.500-3.500)^3} \\ &= \sqrt{10.500(7.000)^3} \\ &= \sqrt{360\underline{1}.5} \\ &\approx 60.0\underline{1}24987 \\ &\approx 60.01. \\ s_2 &= 6.600. \\ A_2 &= \sqrt{6.600(6.600-2.200)^3} \\ &= \sqrt{6.600(4.400)^3} \\ &= \sqrt{562.\underline{2}144} \\ &\approx 23.7\underline{1}106071 \\ &\approx 23.71. \end{aligned}$$

The three sides are rhombi. The area of one of them is

$$\begin{aligned} A &= \frac{1}{2}(2.422)(3.500+2.200) \\ &= \frac{1}{2}(2.422)(5.700) \text{ Remains precise to 3 places} \\ &= 6.90\underline{2}7 \text{ Maintains 4 sigfigs} \\ &\approx 6.903. \end{aligned}$$

Thus the surface area of the entire shape is $60.0\underline{1}24987 + 23.7\underline{1}106071 + 3(6.90\underline{2}7) \approx 104.4\underline{3}16594 \approx 104.43$.

(b) What is the volume of the frustrum of a pyramid in Figure 4.2.21?

We need the height (labeled) and the area of the two bases which were calculated in the previous step. Using these values

$$\begin{aligned} V &= \frac{1}{3}(2.333)(60.01 + 23.71 + \sqrt{(60.01)(23.71)}) \text{ Product maintains 4 sigfigs} \\ &= \frac{1}{3}(2.333)(60.01 + 23.71 + \sqrt{142\underline{2}.8371}) \text{ Root maintains 4 sigfigs} \\ &= \frac{1}{3}(2.333)(60.01 + 23.71 + 37.7\underline{2}051299) \text{ All terms precise to 2 places} \\ &= \frac{1}{3}(2.333)(121.4\underline{4}0513) \text{ Product maintains 4 sigfigs} \\ &= 94.4\underline{4}023895 \\ &\approx 94.44. \end{aligned}$$

□

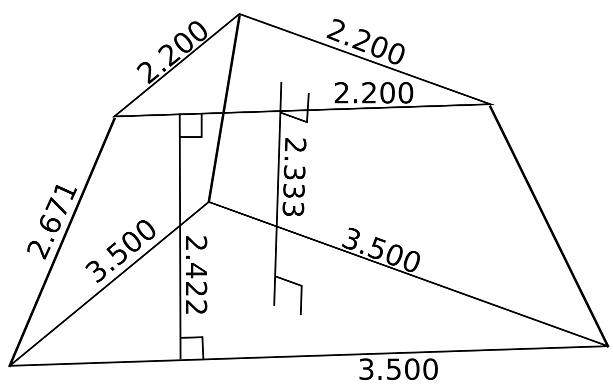
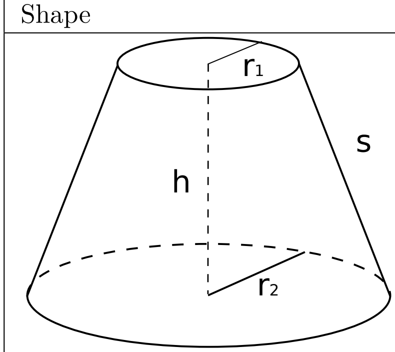


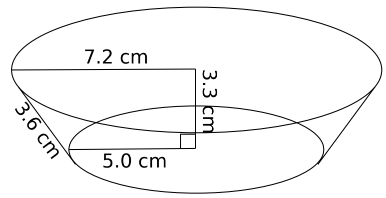
Figure 4.2.21 Calculate Area and Volume of this Frustrum

Table 4.2.22 Frustrum of a Cone

Shape	Lateral Surface Area	Volume
	$\pi s(r_1 + r_2)$	$V = \frac{1}{3}h(B_1 + B_2 + \sqrt{B_1 B_2})$

The total surface area adds the area of the two discs (bottom and top) to the lateral surface area. The volume is the same formula as for a frustrum of a pyramid because these are pyramids built on a circle.

Example 4.2.23 What is the surface area an volume of one part of a muffin tin? The surface area would be the area of a paper liner (assuming it is flat and not corrugated) and the volume is the amount of batter that can be held (if fully filled).



The surface area will be the lateral surface area plus the area of the bottom base (muffin tin is open on top).

$$\begin{aligned}
LA &= \pi s(r_1 + r_2)\\
&\approx (3.14159)(3.6)(5.0 + 7.2) \text{ Sum maintains 1 place precision}\\
&\approx (3.14159)(3.6)(12.2) \text{ Product maintains 2 sigfigs}\\
&\approx 1\underline{3}7.9786328\\
&\approx 140.\\
BA &= \pi r^2\\
&\approx (3.14159)(5.0)^2\\
&\approx 7\underline{8}.53975\\
&\approx 79.
\end{aligned}$$

The total surface area is $1\underline{3}7.9786328 + 7\underline{8}.53975 \approx 2\underline{1}6.5183828 \approx 220$ cm squared.

The volume requires calculating the area of the top surface as well.

$$\begin{aligned}
TA &= \pi r^2\\
&\approx (3.14159)(7.2)^2\\
&\approx 1\underline{6}2.8600256\\
&\approx 160.
\end{aligned}$$

Now we can calculate the volume of the muffin tin.

$$\begin{aligned}
V &= \frac{1}{3}\left(3.3)(78 + 160 + \sqrt{(78)(160)}\right) \text{ Product maintains 2 sigfigs}\\
&= \frac{1}{3}(3.3)\left(78 + 160 + \sqrt{1\underline{2}480}\right) \text{ Root maintains 2 sigfigs}\\
&\approx \frac{1}{3}(3.3)(78 + 160 + 1\underline{1}1.7139) \text{ Root maintains 2 sigfigs}\\
&\approx \frac{1}{3}(3.3)(3\underline{4}9.7139) \text{ Product maintains 2 sigfigs}\\
&= 3\underline{8}4.6853\\
&= 380
\end{aligned}$$

The volume is 380 square centimeters. □

If we have the slant height (length of the side of a frustrum of a cone) we can calculate the height. This is similar to the calculation of height in Example 4.2.14.

Example 4.2.24 Suppose the larger radius is 7.32, the smaller radius is 5.44, and the slant height is 6.09. The slant height is the hypotenuse of a right triangle with the height being one other side. The third side is on the larger base and has length $7.32 - 5.44 = 1.88$. We can use the Pythagorean theorem to calculate the height.

$$\begin{aligned}
h^2 + 1.88^2 &= 6.09^2. \text{ Square maintains 3 sigfigs}\\
h^2 + 3.5\underline{3}44 &= 37.\underline{0}881.
\end{aligned}$$

$$\begin{aligned}
h^2 &= 37.\underline{0}881 - 3.5\underline{3}44. \text{ Subtraction precise to 1 place}\\
h^2 &= 33.\underline{5}537.\\
h &= \sqrt{33.\underline{5}537} \text{ Root maintains 3 sigfigs}\\
&\approx 5.7\underline{9}255567\\
&\approx 5.79.
\end{aligned}$$

□

Checkpoint 4.2.25 For the surface area of a frustrum of a cone we use the slant height: the distance from the edge of the bottom base to the edge of the top base. Here we consider limitations on that length.

For these questions suppose the bottom base has radius 5. The top base will vary depending on the slant height.

(a) If the height is 3, can the slant height be 2?

(b) What is the bottom of the range of possible slant heights for this frustrum?

(c) If the height is 3 and the top base has radius 1, what is the slant height?

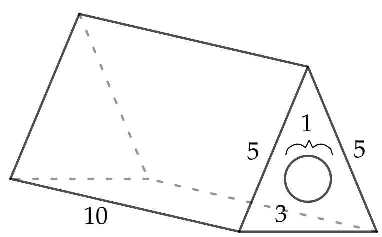
(d) If the height is 3 and the top base has radius 0.5, what is the slant height?

(e) What is the top of the range of possible slant heights for this frustrum? Note this is for bottom base radius 5, height 3, and top base radius unrestricted (but smaller than bottom base).

4.2.2 Applying Geometry

Effectively using knowledge of geometric properties requires recognizing the shapes we know in problems we encounter and figuring out missing parameters from the parameters we do know. The latter we did in Example 4.2.14 and Example 4.2.24. We will extend our ability to calculate missing parameters in Chapter 7. In the following problems we identify shapes and apply the appropriate formulae.

Example 4.2.26 What is the total, external area of this birdhouse? All measurments are in inches and accurate to 1/8 inch.



First, we identify the shapes. The front and back are triangles. The front triangle has a circular hole in it. The bottom and two sides are rectangles. Although this is a 3D object surface area uses 2D formulae.

Because we have edge lengths and not the height (it is easier to measure the edges than the middle), we will use Heron's formula for area.

$$\begin{aligned}
s &= \frac{1}{2}(5+5+3)\\
&= \frac{13}{2}\\
&= 6.5\\
TA &= \sqrt{(6.5)\,(6.5-3)\,(6.5-5)^2}\\
&= \sqrt{(6.5)\,(3.5)\,(1.5)^2}\\
&= \sqrt{51.1875}
\end{aligned}$$

$$\approx 7.1545$$

For the hole we know the diameter so we use the area formula $\frac{\pi}{4}d^2$. This hole has area

$$\frac{\pi}{4}(1)^2 \approx \frac{3.14159}{4} \approx 0.7853975$$

The area of the front triangle is $7.1545 - 0.7853975 \approx 6.3691025$.

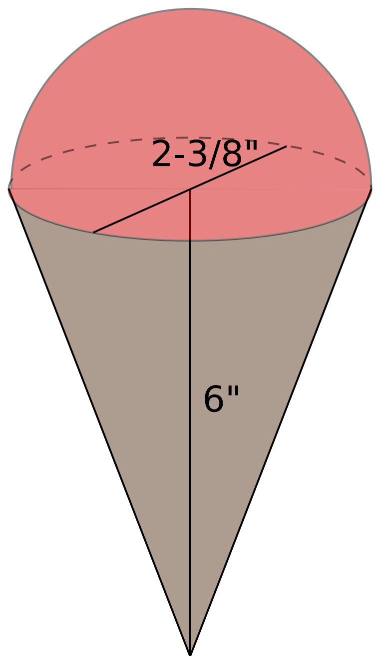
The bottom rectangle has area $3 \cdot 10 = 30$ square inches. The two sides both have area $5 \cdot 10 = 50$ square inches.

Thus the total area is

$$7.1545 + 6.3396025 + 2(50) + 30 \approx 143.5236025 \approx 143.6.$$

We round up because this is the safer estimate (better to have a little more than not enough for construction). □

Example 4.2.27 An ice cream cone has the dimensions shown below.



(a) What is the volume of the ice cream? No person can scoop with sufficient accuracy for a fraction of a square or cubic inch, so we will round to units.

Solution. First, we identify the shapes. This is a right circular cone with a half sphere on top. The diameter of the cone and sphere is $2\,\frac{3}{8} = 2.375$ inches. The radius then is half of that or $1\,\frac{3}{16} = 1.1875$. We convert to decimal inches, because in the end we would want to convert cubic inches (volume) to cups or pints or similar volume measures. This is easier with decimal inches.

The volume of the cone is

$$\begin{aligned} V_c &= \frac{\pi}{3}(1.1875)^2(7) \\ &\approx \frac{3.14159}{3}(1.1875)^2(7) \\ &\approx 10.33697647. \end{aligned}$$

The volume of the half sphere is

$$V_s = \frac{1}{2} \cdot \frac{4}{3}\pi(1.1875)^3$$

$$\approx \frac{2}{3}(3.14159)(1.1875)^3$$
$$\approx 3.507188445.$$

Thus the total volume of the ice cream is $10.33697647 + 3.507188445 \approx 13.84416492 \approx 14$ cubic inches.

(b) What is the surface area of the ice cream? Note the greater the area the faster it melts.

Solution. This is a right circular cone with a half sphere on top. The surface area of the cone is

$$\begin{aligned} A_c &= \pi(1.1875)\sqrt{6^2 + (1.1875)^2} \\ &\approx (3.14159)(1.1875)\sqrt{6^2 + (1.1875)^2} \\ &\approx (3.14159)(1.1875)\sqrt{37.410156} \\ &\approx (3.14159)(1.1875)(6.116384) \\ &\approx 22.81801626. \end{aligned}$$

The surface area of the half sphere is $\frac{1}{2} \cdot 4\pi(1.1875)^2 \approx 8.860265547$. Thus the total surface area of the ice cream is $22.81801626 + 8.860265547 \approx 31.67828181 \approx 32$ square inches.

□

4.2.3 Exercises

1. **Volume Application.** How many loads of gravel will be needed to cover 2.4 miles of roadbed, 34 feet wide, to a depth of 3.0 inches, if one truckload contains 8.0 yd^3 of gravel?

Remember that 1 mi = 5280 ft.

Find the volume of gravel needed, rounded to the nearest cubic foot.

____________________ ft^3

Convert that volume to cubic yards.

____________________ yd^3

Finally, how many truckloads? Think about appropriate rounding. Recall that one truckload contains 8.0 yd^3 of gravel.

____________________ truckloads

2. **Volume Application.** A box contains 9 identical glass spheres that are used to make snow globes. The spheres are tightly packed, as shown below.

6.0 in 6.0 in

What is the total volume, in cubic inches, of all 9 spheres? ______ cubic inches

If a styrofoam inset is made to fill the area between the box and the spheres, what is the volume of foam needed? ______

3. A grain silo is shown below. Measurements are in feet.

6.0 5.0

If 94 ft^3 of grain is stored in the silo, what percent of the silo is full? _____

4. Part 1 of 2

Use the rules for working with measurements to give your answer to the appropriate level of accuracy or precision.

Gravel is piled in the shape of a cone. The circumference of the base is 69.1 feet. The slant height is 61.0 feet. Find the volume of gravel.

____________________ ft^3

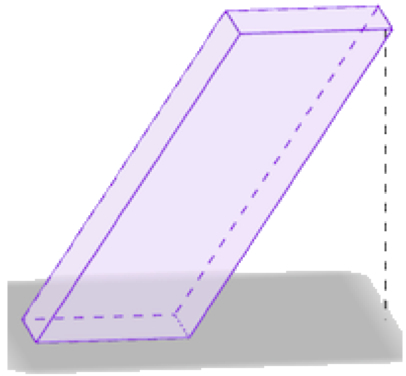
Part 2 of 2

If the gravel weighs $3,200\frac{lb}{(yd)^3}$, how many 18-ton truckloads are needed to transport the gravel? Think about what the appropriate rounding would be for the number of truckloads.

____________________ truckloads.

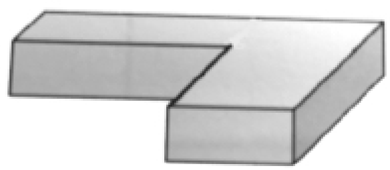
It may be useful to know the following:
3 feet = 1 yard
2000 lb = 1 ton

5. Contextless Volume.



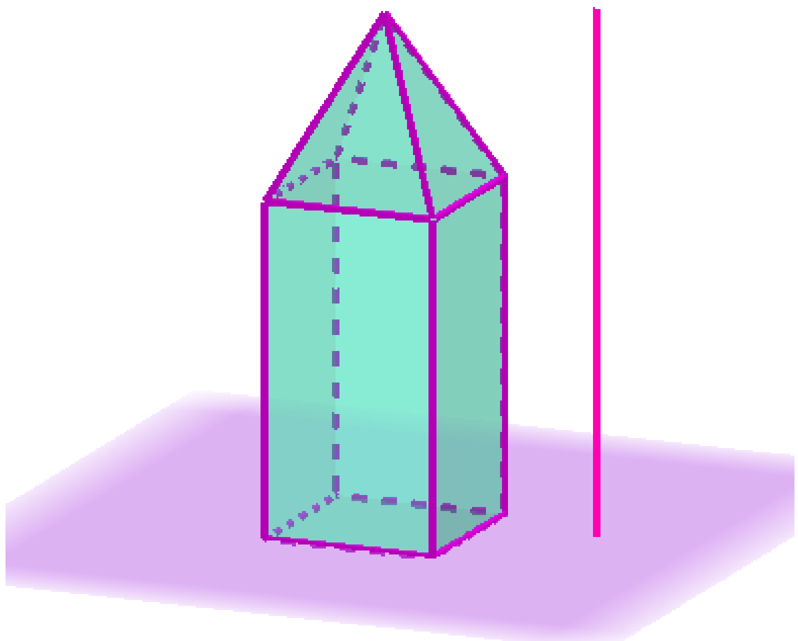
ft^2Calculate the volume.
$V =$ ____ ft^3

6. Contextless Composite Volume. Find the volume of the figure below. Round using significant digits.



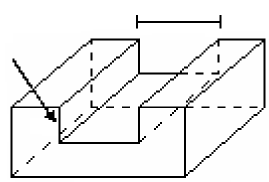
____________________ $feet^3$

7. Contextless Composite Volume. Find the volume of the composite figure below.



19.08.008.0025.0All measurements are in centimeters.
Calculate the volume of just the square prism.
$V_{\text{prism}} =$ ________ cm^3
Calculate the volume of just the pyramid.
$V_{\text{pyramid}} =$ ________ cm^3
Calculate the composite volume.
$V_{\text{composite}} =$ ________ cm^3

8. Contextless Composite Volume. Figure is not drawn to scale. All units are in feet. No rounding required.



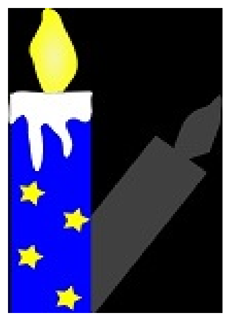
What is the volume of the box? __________ cubic feet

9. Contextless Volume. 18.00 8.00
All measurements are in feet.
What is the volume of this cylinder?
________ ft^3
Round your answer using the rules of working with measurements.

10. Volume Application. A cylindrical soup can has a diameter of 3.0 in and is 6.1 in tall. Find the volume of the can.

Round your answer using the rules for working with measurements.
____________________ in.^3

11. Volume Application. Tamika is making candles in the shape of a cylinder. She needs to determine how many cubic centimeters of wax she needs.



Suppose the radius of the candle is 4.0 cm and the height of the candle is 6.0 cm. Round using significant figures.

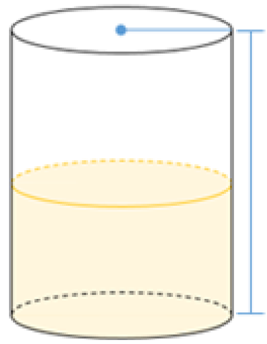
(a) How much wax is needed for one candle? _____ cm^3

(b) How much wax is needed for 20 candles? _____ cm^3

(c) If wax is sold in cubic meters, how many complete candles can be made with 1 cubic meter of wax? _____ candles

Hint: 1 cubic m = 1,000,000 cubic cm

12. Volume Application. A grain silo is shown below. The silo is only filled halfway full with grain.
Figure is not drawn to scale.



$ydyd$_____

(a) yards

(b) meters

(c) cubic meters

(d) cubic feet

(e) cubic yards

(f) feet

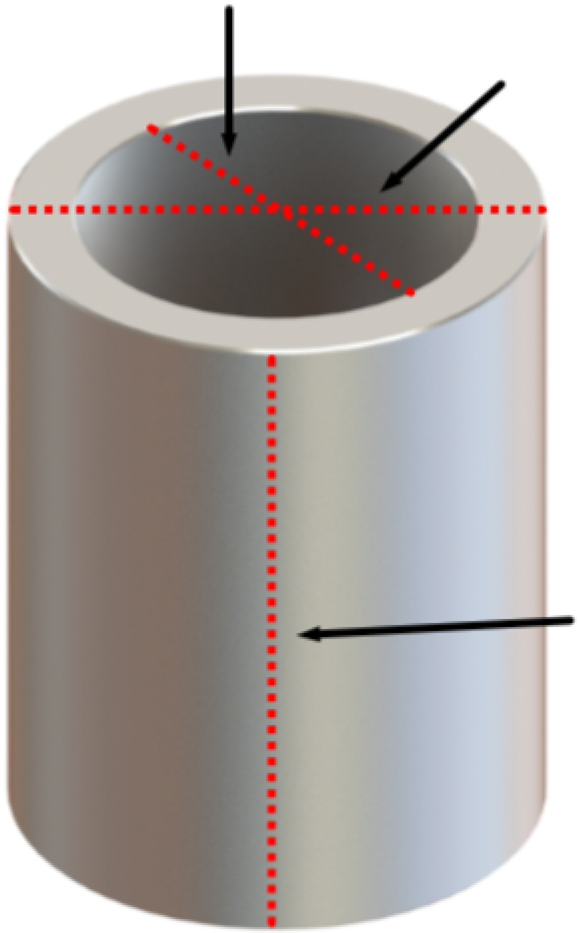
13. Contextless Volume. 30 8

All units are in feet. Round answers to the nearest whole.

Lateral Surface Area: ____________________ ft^2

Volume: ____________________ ft^3

14. Volume Application. The diameter of the pipe shown is 26.0 yd. The diameter of the hole in the pipe is 24.0 yd. Note the image is not to scale. How much metal is in this pipe? Round your answer to appropriately using the rules of significant figures.



The pipe is made of ____________________ yd^3 of metal.

4.3 Project: Unicorn Cake

Project 7 Building a Unicorn Cake. The purpose of this project is to use knowledge of geometric properties to solve a scaling problem.

You have been asked to bake the birthday cake of a little girl who is about to turn three. This little girl has been begging for weeks for a unicorn cake, and you decide to indulge her. When looking up suggestions online on how to bake this cake, you found that most everyone suggests baking the cake in smaller circular cake pans that have a diameter of 6 inches. That's the best way to get the height for the unicorn head. A normal circular cake pan (and the only kind you have) has a diameter of 9 inches. You need to buy cake pans. As an experienced baker, you know that the recipe you are planning on using usually fills two regular (9-inch) cake pans, with a little room to spare. How many 6-inch pans do you need to buy? We're going to assume all cake pans are the same height.

In this project, we will figure out how many 6 inch circular cake pans we're going to need, assuming we're sticking with the recipe we planned.

Instead of a traditional assignment where the questions guide you from step to step, this project has you practice your math solving skills from start to finish. You will demonstrate your ability to answer the question without any leading prompts. This will likely take you about a page or less.

You will be graded on the following.

1. Is your thought process clear? Are the steps laid out in a logical manner? If you used any formulas, did you write them down and label them? Don't just race to the answer; set this up as if you are explaining it to someone else.

2. You need to show some calculations to justify your answer.

3. Some kind of written justification or explanation as to how you solved the problem. (Yes, use words!)

4. A clearly stated (and correct) answer.

(a) How many 6-inch cake pans will you need to use to bake a unicorn cake, assuming you're sticking with the original recipe?

(b) We "assumed all cake pans are the same height." Why is this important and how does it impact your solution?

Chapter 5

Quadratics

5.1 Quadratics

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Identify rates as linear, quadratic, exponential, or other (critical thinking)
- Read and interpret models (critical thinking)

In Section 3.4 we learned to identify data that has a quadratic relation. This section presents algebraic notation for quadratics with emphasis on the forms we will use in this book. You should be able to identify a model as quadratic by looking at the equation.

One of comparing different rates is to enable us to enable us to provide more specific statements than "slow" or "fast." Quadratic is faster than linear. In the next chapter we will learn that exponential is faster than quadratic (or any polynomial). Frequently descriptions of rates in casual conversation and also in the media are lacking in detail or are even inaccurate. This is part of the concept "Identify rates as linear, quadratic, exponential, or other."

5.1.1 Algebraic Forms of Quadratics

Quadratic refers to any expression or equation that has a non-zero squared term. The three most common forms are below. All three rows have the same quadratic.

Table 5.1.1 Algebraic Notation for Quadratics

Example	Form	Description
$y = 5x^2 - 17x - 12$	$y = ax^2 + bx + c$, $a \neq 0$	Standard
$y = (5x+3)(x-4)$	$y = (a_1x - b_1)(a_2x - b_2)$, $a_i \neq 0$	Factored
$y = 5\left(x - \frac{17}{10}\right)^2 - \frac{1489}{100}$	$y = a(x-h)^2 + k$, $a \neq 0$	Convenient

For this class the form we will see the most is

$$y = kx^2$$

where k has some meaning in each model.

When reading models quadratics may look a little different. Table 5.1.2 shows examples of equations that are quadratic and some that are not (but might look like it).

Table 5.1.2 Quadratic and Non-quadratic

Quadratic	Non-quadratic
$11x^2 + 32x - 3$	$5x + 3$
$2(x-3)^2 + 7$	$y = x^3 + 7x^2 - 5x + 3$
$y = 23 - 3x^2$	$y = \frac{17}{x^2}$
$x(6x-5) = 21$	$x^2(x-5) = 7$
$y = (x+3)(x-5)$	$y = 0x^2 + 3x + 2$

5.1.2 Quadratic Models

Now that we know how to recognize models (equations) as quadratic, we will present a few.

Model 5.1.3 Load Factor. *The maximum load factor on an aircraft is*

$$n_{max} = \left(\frac{v}{v_s}\right)^2$$

where

- *n is the load factor,*
- *v is the velocity, and*
- *v_s is the stall speed.*

Load factor is a measure of the force exerted on the aircraft by a maneuver. Maximum load factor is the greatest load factor an aircraft would experience if at the given speed, the aircraft executed a maximum performance maneuver.

Load factor is measured as a multiple of the force of gravity. Thus a load factor of 2 means the object is subject to a force twice as strong as earth's gravity. The expression "pulling g's" refers to experiencing a load factor greater than one.

Model 5.1.4 Kinetic Energy. *The kinetic energy of an object in motion is given by*

$$E = \frac{1}{2}mv^2$$

where

- *E is the energy (in foot pounds or Joules)*
- *m is the mass (think weight) of the object (in pound mass or grams)*
- *v is the velocity of the object (in feet or meters per second).*

Example 5.1.5 What is the maximum load factor if the stall speed is 54 kias, and the current speed is 95 kias, 105 kias? How many g's does it increase between those two speeds? Round to one decimal place.

We use the formula

$$\begin{aligned} n_{max} &= \left(\frac{95}{54}\right)^2 \\ &\approx (1.759259259)^2 \\ &\approx 3.094993140 \\ &\approx 3.1. \end{aligned}$$

$$n_{max} = \left(\frac{105}{54}\right)^2$$

$$\approx (1.9444444449)^2$$
$$\approx 3.780864196$$
$$\approx 3.8.$$

The increase of 10 kias increased the g's by $3.8 - 3.1 = 0.7$. □

Example 5.1.6 What is the kinetic energy of a glider weighing 1323 lbs and flying at 34 kias, at 54 kias? How much did it increase? Use significant digits.

First, we need to convert the speeds to feet per second. Table 1.1.2 suggests we can multiply.

$$\frac{34 \text{ nm}}{\text{hr}} \cdot \frac{6076 \text{ ft}}{\text{nm}} \cdot \frac{\text{hr}}{3600 \text{ sec}} \approx 5\underline{7}.38444444.$$
$$\frac{54 \text{ nm}}{\text{hr}} \cdot \frac{6076 \text{ ft}}{\text{nm}} \cdot \frac{\text{hr}}{3600 \text{ sec}} \approx 9\underline{1}.14.$$

Now, we can calculate the kinetic energy.

$$\begin{aligned}
E_{34} &= \frac{1}{2}(1323 \text{ lbs}) \left(\frac{5\underline{7}.38444444 \text{ ft}}{\text{sec}}\right)^2 \\
&\approx 2.\underline{1}78302608 \times 10^6 \\
&\approx 2.2 \times 10^6. \\
E_{54} &= \frac{1}{2}(1323 \text{ lbs}) \left(\frac{9\underline{1}.14 \text{ ft}}{\text{sec}}\right)^2 \\
&\approx 5.\underline{4}94749485 \times 10^6 \\
&\approx 5.5 \times 10^6.
\end{aligned}$$

The increase was $5.5 \times 10^6 - 2.2 \times 10^3 = 3.3 \times 10^3$ foot pounds. □

5.1.3 Inversely Quadratic Relations

Table 5.1.2 has $y = 17/x^2$ as an example that is not quadratic. Instead y varies inversely with the square of x. While not quadratic, as illustrated below by the tables of differences, they can be solved using the same techniques. Solving quadratics will be demonstrated in the next section.

Table 5.1.7 Quadratic Relation

x	$3x^2$	1st Difference	2nd Difference
1	3		
2	12	9	
3	27	15	6
4	48	21	6
5	75	27	6
6	108	33	6

Table 5.1.8 Inversely Quadratic Relation

x	$\frac{3}{x^2}$	1st Difference	2nd Difference
1	3		
2	3/4	-9/4	
3	1/3	-5/12	11/6
4	3/16	-7/48	13/48
5	3/25	-27/400	47/600
6	1/12	-11/300	37/1200

In Table 5.1.7 the second differences are all 6, so that is quadratic. In contrast in Table 5.1.8 the second differences are all different.

Model 5.1.9 Gravitational Attraction. *Two objects exert a gravitational pull on each other related by their masses and the distance between them. The relationship is*

$$F = G \cdot \frac{m_1 m_2}{r^2}$$

where

- *F is the resulting gravitational force in Newtons*
- *G is the gravitational constant*
- *m_1, m_2 are the masses of the two objects in kilograms*
- *r is the distance between the two objects in meters*

The gravitational constant is $G = 6.6743 \times 10^{-11} \frac{m^3}{kg \cdot s^2}$. The constant has not been determined to any greater precision.

Example 5.1.10 What force does the earth exert on the moon? The mass of the earth is approximately 5.97219×10^{24} kg, and the mass of the moon is approximately 7.34767×10^{22} kg. The average distance between the earth and moon is 382,500 km.

We need to convert the kilometers to meters so units match (gravitational constant). Because it is kilo the conversion is $382,500 \text{ km} \cdot \frac{1000 \text{ m}}{\text{km}} = 382,500,000$ m. Using the model

$$\begin{aligned} F &= (6.6743 \times 10^{-11}) \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2} \left(\frac{(5.97219 \times 10^{24} \text{ kg})(7.34767 \times 10^{22} \text{ kg})}{(382500000 \text{ m})^2} \right) \\ &\approx 2.001824977 \times 10^{20} \frac{\text{m kg}}{\text{s}^2} \\ &\approx 2.002 \times 10^{20} \text{N} \end{aligned}$$

□

5.2 Solving Quadratics

This section addresses the following topics.

- Read and use mathematical models in a technical document

This section covers the following mathematical concepts.

- Solve linear, rational, quadratic, and exponential equations and formulas (skill)

This section presents how to solve equations that contain quadratics. For this text we present two methods of solving quadratics. Both should be memorized and practiced.

5.2.1 Solving Quadratics with Inversion

The first method is for very simple quadratics of the form $ax^2 + c = 0$. Notice that there is no linear term in x. This is what makes solving these simpler. The majority of quadratic models in this text can be solved with this technique.

Example 5.2.1 Find all solutions to $10 - 21x^2 = 0$. Round solutions to 2 decimal places.

We can solve this by undoing each operation.

$$10 - 21x^2 = 0. \qquad \text{Subtract to isolate} x$$

$$
\begin{aligned}
-21x^2 &= -10. && \text{Divide to isolate } x\\
x^2 &= \frac{-10}{-21}.\\
x^2 &= \frac{10}{21}. && \text{Square root to undo square.}\\
\sqrt{x^2} &= \sqrt{10/21}.\\
x &= \pm\sqrt{10/21}.\\
x &\approx \pm 0.6900655593.\\
x &\approx \pm 0.69.
\end{aligned}
$$

Notice that we end up with two results. The $\pm$ results from squaring eliminating a negative. That is, $(2)^2 = 4$ and $(-2)^2 = 4$. So $\sqrt{4}$ could be either 2 or -2. □

Example 5.2.2 Find all solutions to $7(x-3)^2 - 4 = 0$. Round solutions to 2 decimal places.

Solution. We can solve this by undoing each operation.

$$
\begin{aligned}
7(x-3)^2 - 4 &= 0.\\
7(x-3)^2 &= 4.\\
(x-3)^2 &= \frac{4}{7}.\\
\sqrt{(x-3)^2} &= \sqrt{\frac{4}{7}}.\\
x - 3 &= \pm\sqrt{\frac{4}{7}}.\\
x &= \pm\sqrt{\frac{4}{7}} + 3.\\
x &\approx 3.755928946, 2.244071054.\\
x &\approx 3.76, 2.24.
\end{aligned}
$$

□

Checkpoint 5.2.3 What are the solutions to $-5(x+1)^2 + 20 = 0$? ____________

Round solutions to two decimal places.

Significant Digits beyond Arithmetic. There are significant digits rules for addition/subtraction and multiplication/division. We can handle signficant digits with exponents, such as $(0.0230)^2$, because this can be interpreted as multiplication. In the case of $(0.023\underline{0})^2 = (0.023\underline{0})(0.023\underline{0}) = 0.00052\underline{9}$.

For square roots we rely on mathematics we do not need to present here that indicates we can calculate the square root so that the number of significant digits is maintained. For example $\sqrt{12\underline{3}\underline{0}} \approx= 35.07135583 \approx 35.1$

Example 5.2.4 Recall the lift equation in Model 1.3.3. If $\rho = 0.002309$ slugs/ft^3, $S = 174.0$ ft^2, and $C_L = 0.5001$, what velocity in nautical miles per hour is needed to produce $L = 3500.$ lbs?

We start by filling in the information we know in the equation.

$$
\begin{aligned}
3500. &= \frac{1}{2}(0.002309)(174.0)(0.5001)v^2.\\
3500. &\approx 0.100\underline{4}615883v^2.\\
\frac{3500.}{0.100\underline{4}615883} &\approx v^2.\\
348\underline{3}9.18639 &\approx v^2.\\
\sqrt{348\underline{3}9.18639} &\approx \sqrt{v^2}.
\end{aligned}
$$

$$186.\underline{6}525821 \approx v.$$
$$186.7 \approx v.$$

We do not consider the negative square root, because a negative velocity (flying backwards) does not work. Remember to consider reality constraints.

Note the units for velocity are feet per second. Now we need to convert units (like Example 1.1.20).

$$\frac{186.\underline{6}525821 \text{ ft}}{\text{s}} \cdot \frac{\text{mi}}{6076 \text{ ft}} \cdot \frac{3600 \text{ s}}{\text{hr}} \approx 110.\underline{5}907333\frac{\text{mi}}{\text{hr}} \approx 110.6\frac{\text{mi}}{\text{hr}}.$$

□

Example 5.2.5 Consider Model 5.1.3 What velocity will produce a maximum load factor of $n_{max} = 2$ if the stall speed is $v_s = 54$?

Note that 2 is an exact number, and 54 is a measurement.

Solution.

$$\begin{aligned}
2 &= \left(\frac{v}{v_s}\right)^2. \\
2 &= \left(\frac{v}{54}\right)^2. \\
\sqrt{2} &= \sqrt{\left(\frac{v}{54}\right)^2}. \\
\sqrt{2} &= \frac{v}{54}. \\
54\sqrt{2} &= 54\frac{v}{54}. \\
54\sqrt{2} &= v. \\
7\underline{6}.36753235 &\approx v. \\
76 &\approx v.
\end{aligned}$$

The aircraft would have to be travelling at 76 nm/hour to be able to experience 2 G's in a maximum performance maneuver.

Again reality constraints eliminate the possibility of the negative square root. Negative velocity (backwards motion) does not produce flight. □

5.2.2 Solving Quadratics with the Formula

When the quadratic has more than a square term, e.g., $11x^2 + 32x - 3 = 0$ we cannot solve by undoing each operation. In this text we will solve all of these quadratics using the quadratic formula. For $ax^2 + bx + c = 0$ the solutions are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 5.2.6 Find all solutions to $11x^2 + 32x - 3 = 0$.

We note that $a = 11$, $b = 32$, $c = -3$.

$$\begin{aligned}
x &= \frac{-32 \pm \sqrt{(32)^2 - 4(11)(-3)}}{2(11)} \\
&= \frac{-32 \pm \sqrt{1024 + 132}}{22} \\
&= \frac{-32 \pm \sqrt{1156}}{22} \\
&= \frac{-32 \pm 34}{22}
\end{aligned}$$

$$= \frac{1}{11}, -3$$

□

Checkpoint 5.2.7 What are the solutions to $-5x^2 + 30x - 6 = 0$? ____________
Round results to 2 decimal places.

5.2.3 Exercises

Exercise Group. Use these contextless problems to practice using the quadratic formula.

1. Solve by the quadratic formula. List the solutions, separated by commas.
$4x^2 - 13x + 10 = 0$
$x =$ ____________________

2. Solve using the quadratic formula. List your answers, separated by commas.
$2x^2 - 14 = 0$
____

3. Solve using the quadratic formula. List your answers, separated by commas.
$2x^2 - 14 = 0$
____

4. Solve by the quadratic formula. List the solutions, separated by commas.
$4x^2 - 13x + 10 = 0$
$x =$ ____________________

Exercise Group. Answer these questions which involve a quadratic model. Select rounding that is appropriate or as directed.

5. A rocket launch occurs at $t = 0$ seconds. Its height, in meters above sea-level, as a function of time is given by $h = -4.9t^2 + 172t + 185$.
Assuming that the rocket will splash down into the ocean, at what time does splashdown occur? __________ seconds.
At what time will the rocket become level with the take-off location? __________ seconds

6. An aircraft in flight has total energy given by $\frac{1}{2}mv^2 + mgh$ where
m is the mass of the aircraft in pounds
v is the velocity in feet/second
$g = 32.1740$ is earth's standard gravity
h is the height above sea level in feet
Suppose a glider with mass 1242 lbs and initial velocity 83 ft/s is at an altitude of 2,700. ft MSL. If the glider forces a descent to 2,457 ft MSL, what must the new velocity be? _____ ft/s
Altitudes have four significant digits. Hint: The total energy will be the same before and after the altitude change.

7. An aircraft in flight has total energy given by $\frac{1}{2}mv^2 + mgh$ where
m is the mass of the aircraft in pounds
v is the velocity in feet/second
$g = 32.1740$ is earth's standard gravity
h is the height above sea level in feet
Suppose a glider with mass 1177 lbs and initial velocity 83 ft/s is at an altitude of 1,800. ft MSL. If the pilot wants to increase the velocity to 117 ft/s to what altitude must it descend? _____ ft MSL
The initial altitude has four significant digits. Hint: The total energy will be the same before and after the altitude change.

8. Load factor refers to the amount of force imposed on an aircrafts body and contents (e.g., pilot). It is typically measured in g-forces. The maximum load factor is calculated by $n_{\max} = \left(\frac{V}{V_s}\right)^2$ where

$n_{\max}$ is the maximum load factor
V_s is the stall speed, and
V is the current speed.

The stall speed (V_s) for a Cessna 206 is 54 kias. Maximum structural cruising speed (V_{NO}) is 147 kcas. Maneuvering speed (V_A) at this weight is about 122 kcas.

Calculate the V speed for a maximum load factor of 3. Give a numeric approximation rounded to the nearest unit. ____________________

9. The lift equation is $L = \frac{1}{2}\rho S C_L v^2$ where ρ is the density of air, S is the surface area of the airfoil (wing), C_L is called the coefficient of lift, and L is the lift (measured in pounds).

For this question $\rho = 0.001988$, $S = 174$, $C_L = 3.8$, and $L = 3200$. What is the velocity v necessary? ____________________

10. A door 2.7 feet wide by 7 feet high. If both dimensions are increased by 3 inches, what will the new area be? ____________________

Give the answer in feet as the base unit. Round your answer to two decimal places.

11. A door is 2.7 feet wide by 6 feet high. If both width and height are increased by the same number of inches, what must be added to make the total area 20.2 ft^2? By how many inches must the height and width be increased? ____________________

Round your answer to two decimal places.

5.3 Roots

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models

This section covers the following mathematical concepts.

- Identify rates as linear, quadratic, exponential, or other (critical thinking)

This section is for those who are curious for more details about roots, like the square root. These extensions do not show up in any models in this text. The rate question is an important general concept however.

5.3.1 Definition of Roots

As implied by their use in solving equations with a square term, square roots are a sort of opposite to squares. For example $3^2 = 9$ means that 9 is the result of multiplying 3 by itself. $\sqrt{9} = 3$ means that 3 is a number that multiplied by itself is 9. This means in general

$$\sqrt{n^2} = n.$$

There is one detail that we will not use, but should be acknowledged. Consider that $(3)^2 = 9$ and $(-3)^2 = 9$. Thus $\sqrt{9}$ might be considered to have two solutions. We saw this in Subsection 5.2.2. However when solving using inversion as in Subsection 5.2.1 reality constraints often meant the negative root was not a solution to the application even if it fits the equation.

5.3.2 Generalized Roots

Just as we can multiply a number by itself (e.g., $3^2 = 3 \cdot 3 = 9$) we can multiply a number by itself more than once (e.g., $3^3 = 3 \cdot 3 \cdot 3 = 27$). In general

$$3^m = \overbrace{3 \cdot 3 \cdot \ldots 3}^{m}.$$

Checkpoint 5.3.1 Evaluate each of these by multiplying enough times.

$2^5 =$ ______

$7^3 =$ ______

$2.7^4 =$ ______

As noted above square roots perform the reverse action of squaring. To solve problems involving other powers, there are matching roots. These are denoted with a small number to show which root. For example this is a third root:

$$\sqrt[3]{8} = 2.$$

This root means that because $2^3 = 8$, $\sqrt[3]{8} = \sqrt[3]{2^3} = 2$. In general

$$\sqrt[n]{x^n} = x.$$

There are restrictions (n is an integer; if n is even, x must be non-negative). For models in this text none of these play a role.

Just as with square roots we can use a device to calculate the values. The device may have a key like $\sqrt[m]{x}$.

Another notation for roots is a type of exponent. For example,

$$\sqrt{9} = 9^{1/2} = 3.$$

Likewise,

$$\sqrt[3]{8} = 8^{1/3} = 2.$$

This notation can be used when solving problems with powers.

Checkpoint 5.3.2 Evaluate each of these. You may need a computational device.

$\sqrt[3]{343} =$ ______

$\sqrt[3]{729} =$ ______

$\sqrt[5]{243} =$ ______

$\sqrt[4]{18} =$ ______

5.3.3 Rates of Roots

This section illustrates where roots fall with respect to the question, "How fast is it growing?" It ends by addressing an important general question: if growth is slowing is it stopping?

Figure 5.3.3 shows a linear equation and a square root equation. The square root eventually grows much more slowly than the linear grows. Because a linear grows more slowly than a quadratic or exponential, the root also grows more slowly than quadratics and exponentials.

The first important idea here is that the square root grows more quickly than the linear at first, but then slows down and is passed up. When we are comparing rates, we typically look at what happens eventually. It does not matter so much what happens at first. For example for salaries, exponential growth might be slower for the first few years, but in our careers it will be substantially more.

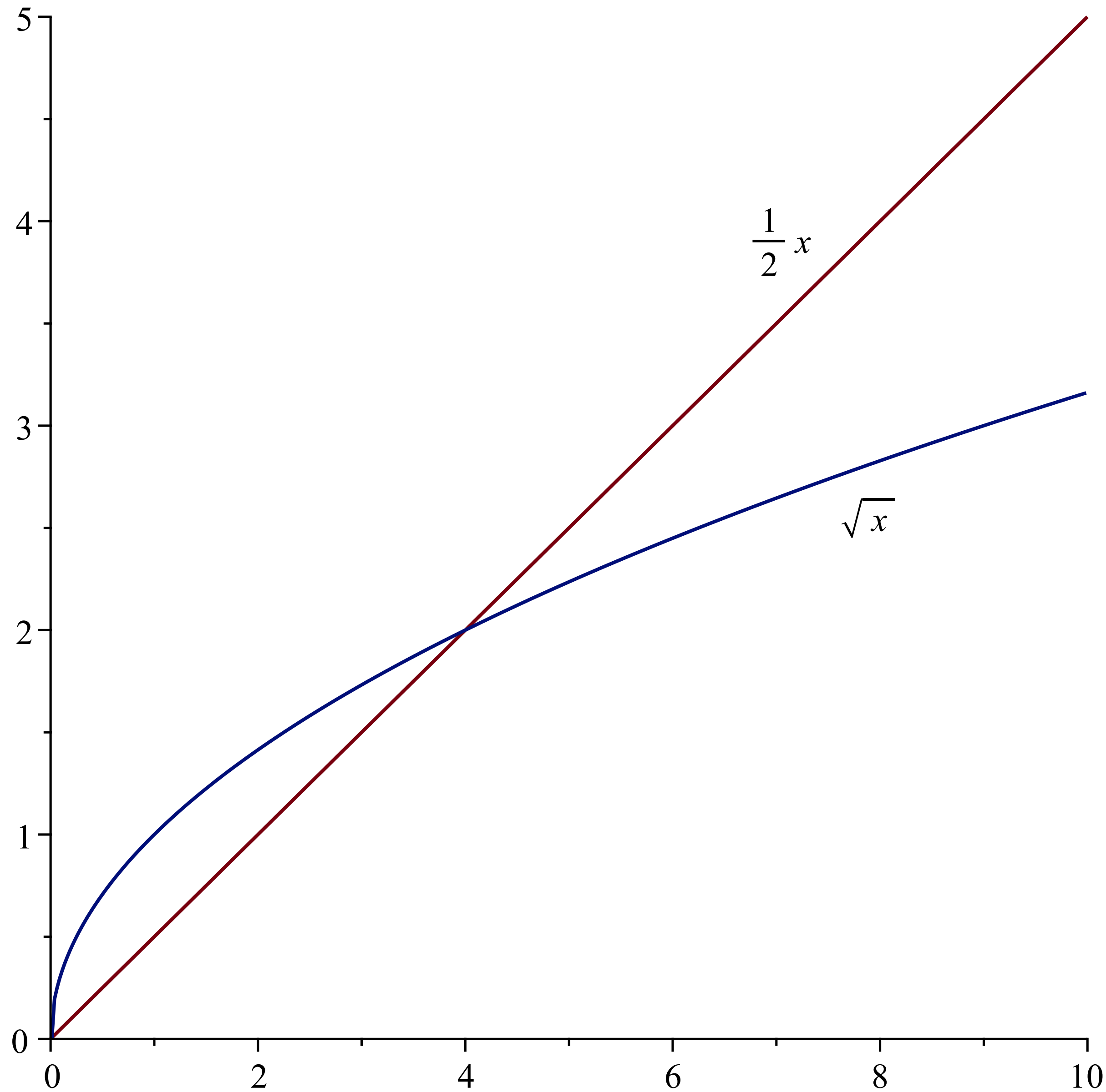


Figure 5.3.3 Rate Comparison: Linear and Square Root

With rates it is important to test if a slow growth rate is leveling off or not. By level off we mean that even though it is constantly growing it never passes some fixed amount. For example, the numbers 0.5, 0.75, 0.875, 0.9375, 0.96875 is growing, but never passes 1.0. We will show that a square root (and other roots) do not level off.

Consider Table 5.3.4. Clearly this grows larger than 1, because we have a 1 in the first entry. It also grows larger than 2, because we can obtain 2 from $n = 4$. In general if we ask, can the square root grow beyond n, the answer is: consider that $\sqrt{(}n^2) = n$, so if we choose $n^2 + 1$ that will be larger than n. While slow, the square root continues to grow without bound (infinite growth).

Table 5.3.4 Rate of a Square Root

n	$\sqrt{n}$
$1^2 = 1$	1
$2^2 = 4$	2
$3^2 = 9$	3
$10^2 = 100$	10
n^2	n

5.4 Graphs of Quadratics

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models

This section covers the following mathematical concepts.

- Solve linear, rational, quadratic, and exponential equations and formulas (skill)
- Use models including linear, quadratic, exponential/logarithmic, and trigonometric (skill)

This section will introduce the graph (shape) of quadratic equations, illustrate all the variations that are possible, and demonstrate how these can be used to answer questions.

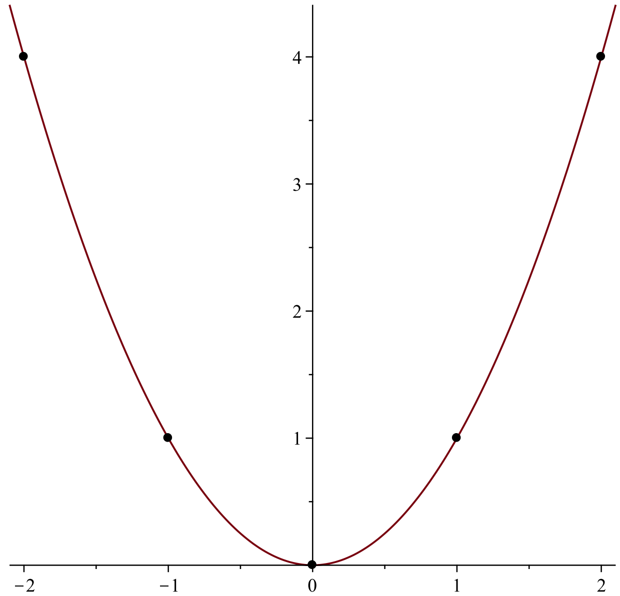
5.4.1 Properties of Quadratics

First we will graph the most basic quadratic equation $y = x^2$. This will show us the basic shape and properties. Later examples will show us the variations and how to control them.

Example 5.4.1 To discover the shape we will graph following the steps shown first in Subsection 3.2.3. First we complete this table of values. We must choose the x values randomly for now.

x	x^2	
-2	$(-2)^2$	$= 4$
-1	$(-1)^2$	$= 1$
0	0^2	$= 0$
1	1^2	$= 1$
2	2^2	$= 4$

Next we graph these points and sketch the graph through them.



□

The graph of a quadratic is known as a **parabola**. Notice that there is a single point at the bottom from which the parabola grows upward to the left and upward to the right. This is known as the **vertex**. It is at (0,0) for this parabola.

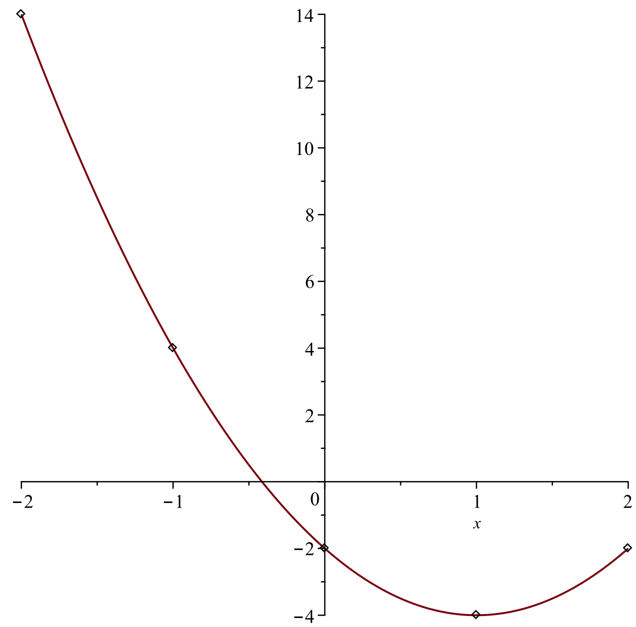
Notice as well that the left and right sides are mirrors of each other. Specifically they are mirrored over the line through the vertex known as the **line of symmetry**. In this case that is the vertical line $x = 0$.

The following example combines three modifications of the basic quadratic. We will graph it, notice the differences, and then use an activity to understand how each part of the equation produces a particular change in the graph.

Example 5.4.2 We will graph $y = 2(x-1)^2 - 4$. First we complete this table

x	$2(x-1)^2-4$	
-2	$2(-2-1)^2-4$	$=14$
-1	$2(-1-1)^2-4$	$=4$
0	$2(0-1)^2-4$	$=-2$
1	$2(1-1)^2-4$	$=-4$
2	$2(2-1)^2-4$	$=-2$

Next we graph these points and sketch the graph through them.



Notice that this time the vertex is at $(1, -4)$. The line of symmetry is $x = 1$. □

The following activity is a set of experiments to understand how each part of the parabola equation works. It is designed to help you understand by seeing the results in calculations. Do these exercises by hand. While you can look up the meaning of each equation part, memorizing them will be harder (and your ability to use them in later chapters reduced) if you do not see how each transformation is caused by the arithmetic.

Activity 8 Discovering Parabola Parameters.

(a) In this exercise we will check on the effect of multiplying a quadratic by a scalar (number).

(i) Complete the table of points below. Compare the results to the table in Example 5.4.1

x	$2x^2$
-2	$2(-2)^2 = 8$
-1	
0	
1	
2	

(ii) Graph these points.

(iii) Compare the entries in the $2x^2$ column to the entries in the x^2 in Example 5.4.1. Describe how they changed.

(iv) Compare this graph to the graph in Example 5.4.1. Describe how the graph changed.

(b) In this exercise we will continue to check on the effect of multiplying a quadratic by a scalar (number).

(i) Complete the table below. Compare the results to the table in Example 5.4.1

x	$\frac{1}{2}x^2$
-2	$\frac{1}{2}(-2)^2 = 2$
-1	
0	
1	
2	

(ii) Graph these points.

(iii) Compare the entries in the $\frac{1}{2}x^2$ column to the entries in the x^2 in Example 5.4.1. Describe how they changed.

(iv) Compare this graph to the graph in Example 5.4.1. Describe how the graph changed.

(c) In this exercise we will check on the effect of adding inside the square.

(i) Complete the table below. Compare the results to the table in Example 5.4.1

x	$(x-1)^2$
-2	$(-2-1)^2 = 9$
-1	
0	
1	
2	

(ii) Graph these points.

(iii) Compare the entries in the $(x-1)^2$ column to the entries in the x^2 in Example 5.4.1. Describe how they changed.

(iv) Compare this graph to the graph in Example 5.4.1. Describe how the graph changed.

(d) In this exercise we will continue to check on the effect of adding inside the square.

(i) Complete the table below. Compare the results to the table in Example 5.4.1

x	$(x+1)^2$
-2	$(-2+1)^2 = 1$
-1	
0	
1	
2	

(ii) Graph these points.

(iii) Compare the entries in the $(x+1)^2$ column to the entries in the x^2 in Example 5.4.1. Describe how they changed.

(iv) Compare this graph to the graph in Example 5.4.1. Describe how the graph changed.

(e) In this exercise we will check on the effect of adding outside the square.

(i) Complete the table below. Compare the results to the table in Example 5.4.1

x	x^2-1
-2	$(-2)^2-1 = 3$
-1	
0	
1	
2	

(ii) Graph these points.

(iii) Compare the entries in the x^2-1 column to the entries in the x^2 in Example 5.4.1. Describe how they changed.

(iv) Compare this graph to the graph in Example 5.4.1. Describe how the graph changed.

(f) In this exercise we will continue to check on the effect of adding outside the square.

(i) Complete the table below. Compare the results to the table in Example 5.4.1

x	x^2+1
-2	$(-2)^2+1=5$
-1	
0	
1	
2	

(ii) Graph these points.

(iii) Compare the entries in the x^2+1 column to the entries in the x^2 in Example 5.4.1. Describe how they changed.

(iv) Compare this graph to the graph in Example 5.4.1. Describe how the graph changed.

(g) In this exercise we will check on the effect of multiplying a quadratic by a negative.

(i) Complete the table below. Compare the results to the table in Example 5.4.1

x	$-x^2$
-2	$-(-2)^2=-4$
-1	
0	
1	
2	

(ii) Graph these points.

(iii) Compare the entries in the $-x^2$ column to the entries in the x^2 in Example 5.4.1. Describe how they changed.

(iv) Compare this graph to the graph in Example 5.4.1. Describe how the graph changed.

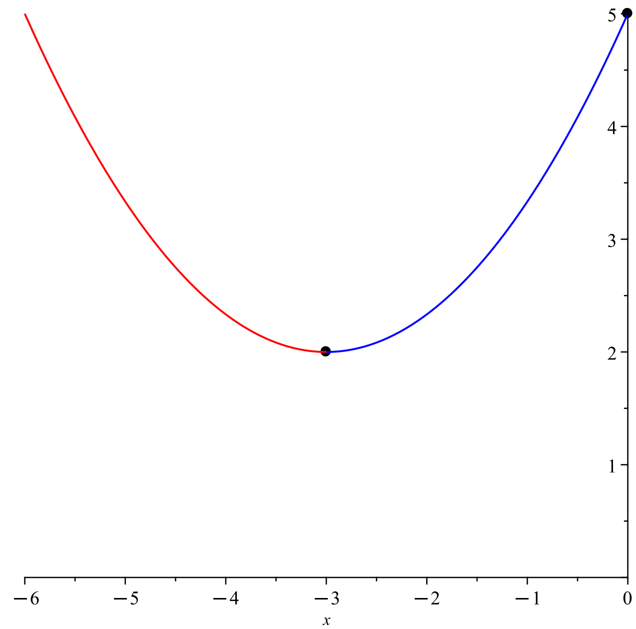
Example 5.4.3 Graph the parabola $y=\frac{1}{3}(x+3)^2+2$ by identifying the vertex and one other point.

From the activity we know that the vertex location is determined by the numbers added inside and outside the square. In this case the vertex is at (-3,2), because this is $(x-[-3])^2+[2]$.

The second point can be any point. $x=0$ is convenient. $y=\frac{1}{3}(0+3)^2+2=\frac{1}{3}\cdot 9+2=3+2=5$.

Because the center (axis of symmetry) is $x=-3$, we want to plot a portion of the curve left and right of that value. x from -6 to 0 (3 left to 3 right) will be convenient.

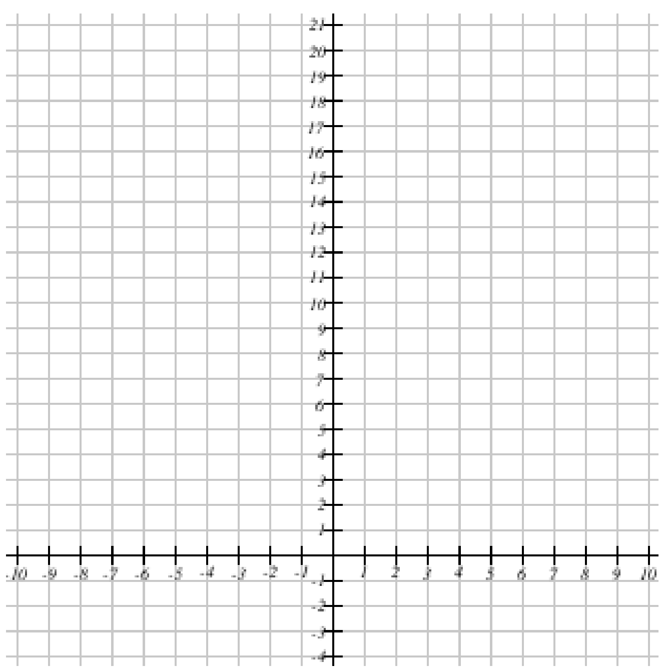
We can plot the two points we calculate and then sketch the portion of the curve to the right. The portion to the left we then sketch to be symmetrical.



□

Checkpoint 5.4.4 Graph the parabola $y = 6(x + 2)^2 - 3$

The vertex is ________



Recall that lines can all be expressed in the form $y = mx + b$ where m, the slope, defines steepness, and b represents a shift. Based on the work above we now know that all quadratics (parabolas) can be written as

$$y = a(x - h)^2 + k$$

where a indicates how steeply the parabola rises, and h and k indicate horizontal and vertical shifts of the parabola. The shifts are also the location of the vertex which we used in the examples above. In applications these parameters can have meaning.

5.4.2 Applications

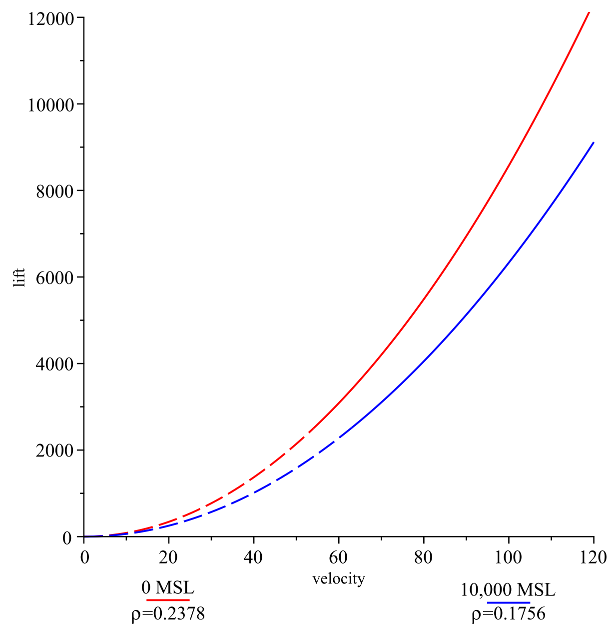
Now that we understand what each part of the equation of a parabola tells us, we can use that to interpret quadratic models and answer some questions using a quadratic model.

Example 5.4.5 The lift equation is a parabola. Note, only the right side of the parabola is part of the model, because negative speeds do not apply.

The lift equation has $h = 0$ and $k = 0$. This makes sense because there should be zero lift when there is zero speed. If $k > 0$ then there would be lift at zero speed. If $h > 0$ there would be no lift until that speed.

The coefficients, $\frac{1}{2}\rho SC_L$, make the parabola steeper. This means than increased air density, wing surface area, or coefficient of lift make the increase in lift occur faster with respect to increased speed.

For example, for a specific aircraft the increase in lift at ground level is steeper than at 10,000 MSL (air density given below).



□

Example 5.4.6 The maximum load factor can be written as

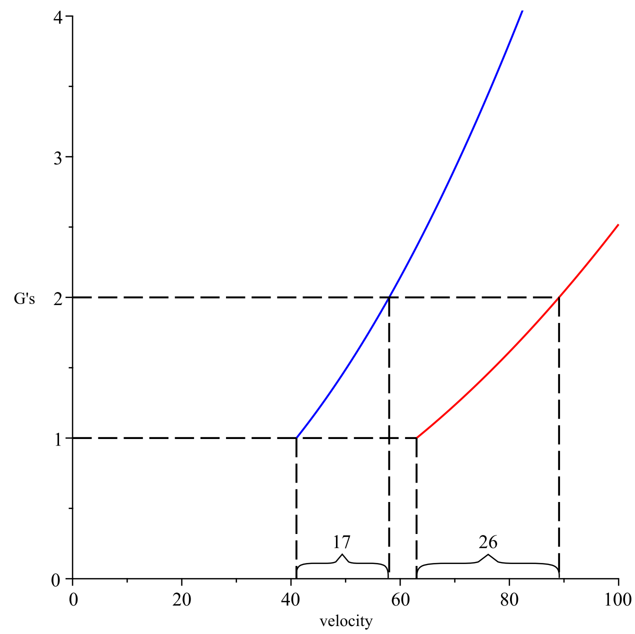
$$n_{max} = \left(\frac{V}{V_S}\right)^2 = \frac{1}{V_S^2}V^2.$$

This is a parabola.

This equation has $h = 0$ and $k = 0$. Note the stall speed of a fixed wing aircraft is the slowest speed at which it can fly. Thus the smallest number we will input into the model is $V = V_S$ which gives us 1 G. That is the normal state of flight is 1 G (the one produced by gravity). If $k > 0$ the model would indicate more than 1 G produced by gravity. $h \neq 0$ would indicate this occuring at some other speed.

The coefficient $(1/V_S^2)$ indicates that a higher stall speed (divide by a bigger number) will make the increase in G's slower (graph is less steep). This means it takes a greater increase of speed to move from 1 G to 2 G's.

For example if the stall speed is 41 nm/hr, then 1 G occurs at 41 nm/hr and 2 G's occurs at $V = 41\sqrt{2} \approx 58$ nm/hr. To increase the maximum load factor requires an increase of $58 - 41 = 17$ nm/hr. However if the stall speed were 63 nm/hr then 1 G occurs at 63 nm/hr and 2 G's occurs at $V = 63\sqrt{2} \approx 89$ nm/hr. To increase the maximum load factor required an increase of $89 - 63 = 26$ nm/hr. The higher stall speed implies a greater $(22 > 16)$ increase in speed.



□

The next example shows how we can determine the vertex and axis of symmetry even when the equation is not in the easy to read form. Here we use a method that does not require additional algebraic techniques. If you know how to complete the square, that will also work.

Example 5.4.7 Find the vertex of $y = 2x^2 - 11x + 12$. All numbers are exact.

Because of the symmetry of a parabola we know that the vertex lies half way between any pair of points with the same height. For example it is half way between the solutions to $2x^2 - 11x + 12 = 0$. We can solve this using the quadratic formula.

$$\begin{aligned} x &= \frac{-(-11) \pm \sqrt{(-11)^2 - 4(2)(12)}}{2(2)} \\ &= \frac{11 \pm \sqrt{121 - 96}}{4} \\ &= \frac{11 \pm \sqrt{25}}{4} \\ &= \frac{11 \pm 5}{4} \\ &= 4, 1.5 \end{aligned}$$

To find what is half way in between we average these two values.

$$\begin{aligned} x &= \frac{4 + 1.5}{2} \\ &= \frac{5.5}{2} \\ &= 2.75. \end{aligned}$$

Thus the x-coordinate of the vertex is 2.75. To find the y-coordinate we substitute the x value into the quadratic.

$$2(2.75)^2 - 11(2.75) + 12 = -3.125.$$

□

Simplified Effect of Gravity. A model of motion due to gravity is beyond the scope of this course, but we can understand a connection to quadratics. Gravity is a force that accelerates objects toward each other. The strength of earth's gravity at the surface is approximately 9.80665 $\frac{\mathrm{m}}{\mathrm{s}^2}$. Be aware that this value actually varies a little by location and decreases with altitude, but those effects are not needed for any problems in this book.

Notice that gravity is an acceleration (i.e., m/s is a velocity and this is (m/s)/s meaning a change in velocity) and as a result the time portion is squared. This means accleration due to gravity is quadratic with respect to time.

For objects that are dropped, we can figure out both their speed and their height can be determined.

Example 5.4.8 Gravity's Effect on Velocity. If we drop a ball from 1.0 meters above the surface, it will start with a velocity of 0 m/s. Acceleration due to gravity is (9.8 m/s)/s which means after one second the ball will be travelling down at 9.8 m/s. After another second (2 seconds total) it would increase (accelerate) another 9.8 m/s so its velocity would be 18.6 m/s.

We can write an equation for the velocity after t seconds. Notice that the change in velocity is a constant, so the change in velocity is a linear equation. The rate (slope) is 9.80665 $\frac{\mathrm{m}}{\mathrm{s}^2}$. The shift is the initial velocity (0 m/s in this case).

$$v = 0\ \frac{\mathrm{m}}{\mathrm{s}} + \frac{9/8\ \mathrm{m/s}}{\mathrm{s}}(t\ \mathrm{s})$$

□

Calculating the exact height of an object falling due to gravity requires mathematics beyond this textbook. A little more information can be found in Section 5.5. The following example illustrates calculations using a provided simplification.

Example 5.4.9 Falling Due to Gravity. Suppose the height of an object falling due to gravity is given by

$$h = -4.9t^2 + 1.0.$$

(a) Determine when the object strikes the ground. The ground is at height 0.

We can find this by solving the following equation.

$$\begin{aligned} -4.9t^2 + 1.0 &= 0. \\ -4.9t^2 &= -1.0. \\ t^2 &= \frac{-1.0}{-4.9}. \\ t^2 &\approx 0.2\underline{0}40816327. \\ t &\approx \sqrt{0.2\underline{0}40816327}. \\ t &\approx 0.4\underline{5}17539515 \\ t &\approx 0.45 \end{aligned}$$

It will take 0.45 seconds for the object to reach the ground.

(b) Determine the height from which it was dropped.

Because the object is dropped, the heighest point is when it is dropped which is time $t = 0$. Thus the height from which it was dropped is $-4.9(0) + 1.0 = 1.0$ meters.

□

Example 5.4.10 Gravity's Effect on a Thrown Object. Consider an object that is thrown upward. Gravity will first slow down the object as it climbs, then it will cause the object to fall (as in the previous example). Suppose the height of this object is given by

$$h = -4.9t^2 + 5.1t + 1.2.$$

(a) Determine when the object strikes the ground. The ground is at height 0.

We can find this by solving the $-4.9t^2 + 5.1t + 1.2 = 0$. To do this we can use the quadratic formula. All numbers have two significant digits and are precise to the 10ths.

$$\begin{aligned}
t &= \frac{-(5.1) \pm \sqrt{(5.1)^2 - 4(-4.9)(1.2)}}{2(-4.9)} \\
&= \frac{-5.1 \pm \sqrt{2\underline{6}.01 + 2\underline{3}.52}}{-9.8} \text{ Products maintains 2 sigfigs} \\
&= \frac{-5.1 \pm \sqrt{4\underline{9}.53}}{-9.8} \text{ Sum maintains to units} \\
&\approx \frac{-5.1 \pm 7.\underline{0}37755324}{-9.8} \text{ Root maintains 2 sigfigs} \\
&\approx \frac{1.\underline{9}37755324}{-9.8}, \frac{-12.\underline{1}37755324}{-9.8} \text{ Sum maintains to 10ths} \\
&\approx -0.1\underline{9}77301351, 1.\underline{2}38546461 \text{ Division maintains 2 sigfigs} \\
&\approx -0.20, 1.2
\end{aligned}$$

The negative time does not make sense, so $t \approx 1.2$ seconds is when the object will reach the ground.

(b) Determine the maximum height the object reached.

This will occur at the vertex of the parabola. Because the equation is not in the easy form we will use the technique demonstrated in Example 5.4.7.

We use any two points. In the previous step we found the two times when the height is zero. It does not matter that one is negative: this will still work to find the vertex. The time at which the maximum height occurs is

$$\frac{-0.1\underline{9}77301351 + 1.\underline{2}38546461}{2} \approx 0.5\underline{2}0408163 \approx 0.52.$$

The height at that time is

$$\begin{aligned}
-4.9(0.5\underline{2}0408163)^2 + 5.1(0.5\underline{2}0408163) + 1.2 \approx& \text{ Products maintain 2 sigfigs} \\
-1.\underline{3}27040815 + 2.\underline{6}54081631 + 1.2 \approx& \text{ Sum maintains to 10ths} \\
2.\underline{5}27040816 \approx& \ 2.5
\end{aligned}$$

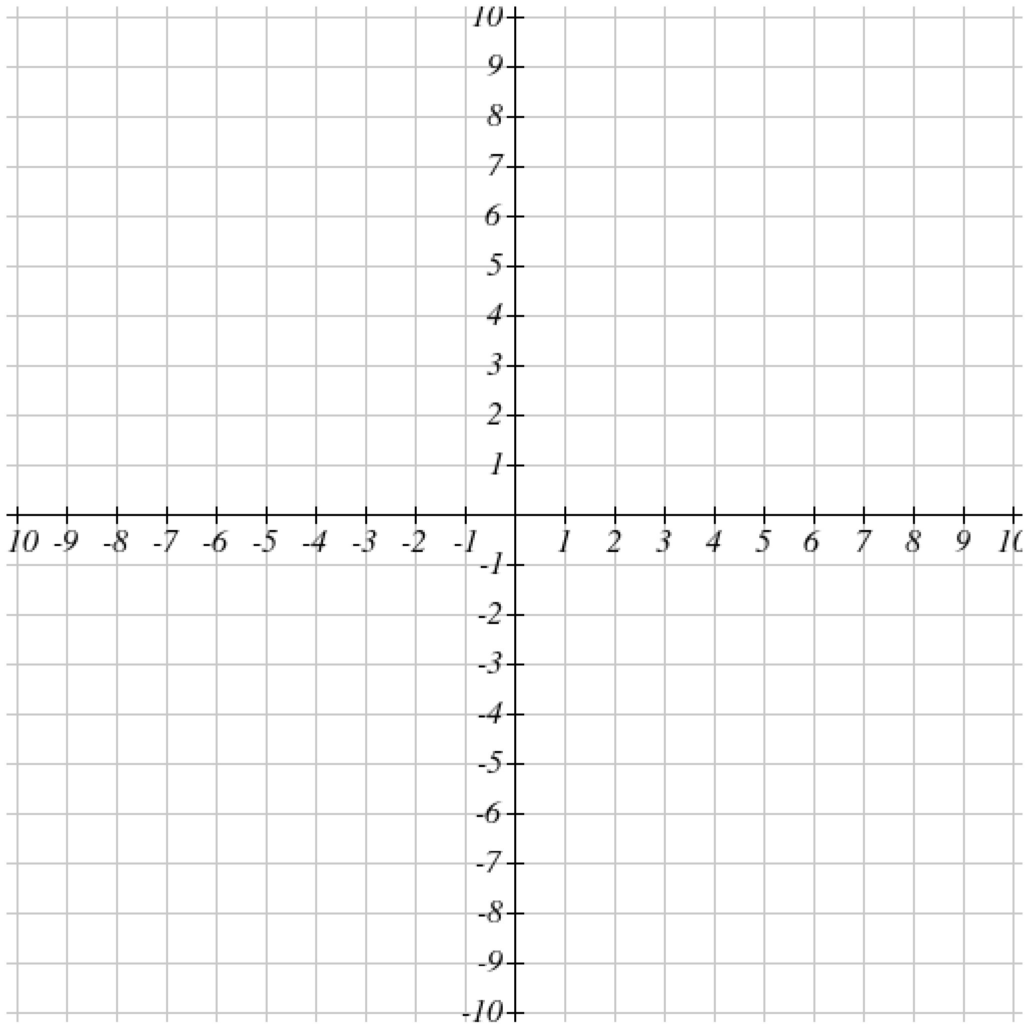
The object will reach a maximum height of 2.5 meters at 0.52 seconds after it is thrown.

□

5.4.3 Exercises

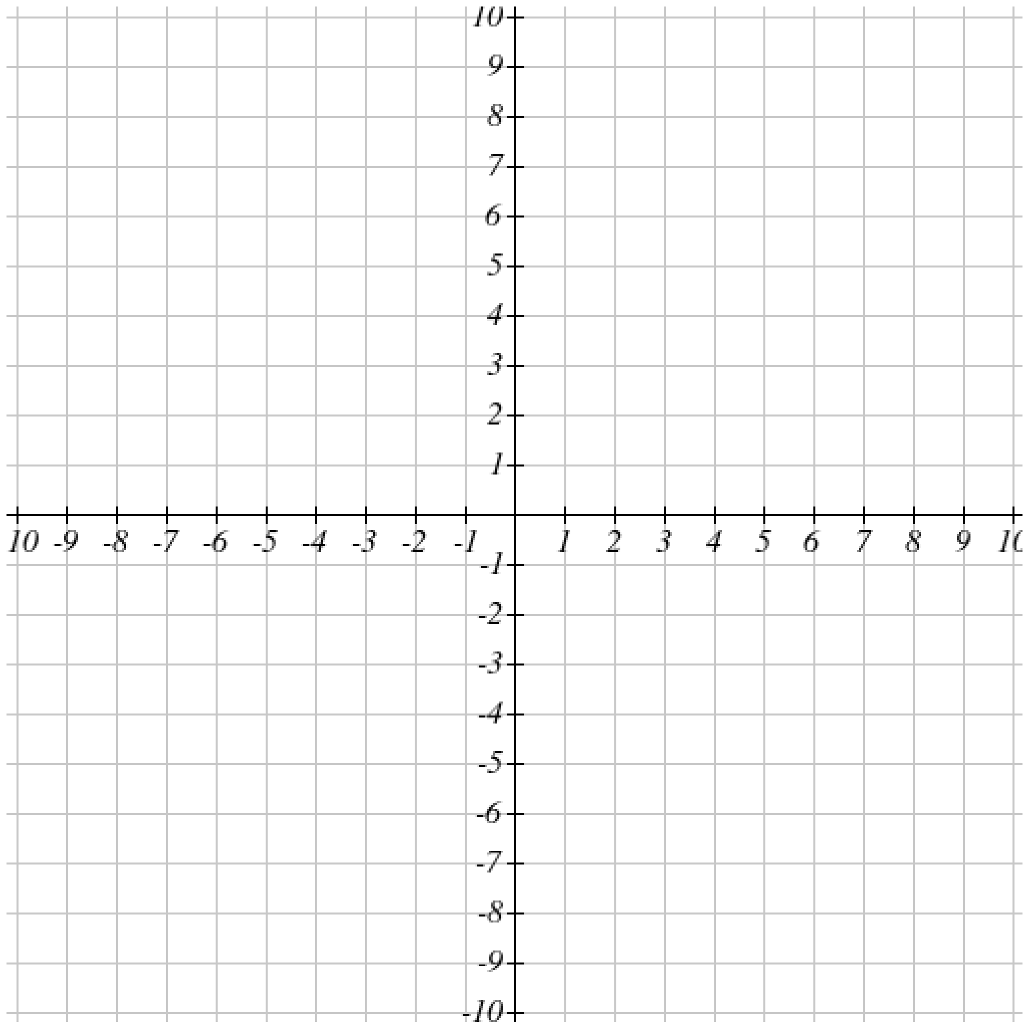
1. Contextless Graphing. Graph the parabola: $f(x) = (x-1)^2$

The vertex is _______

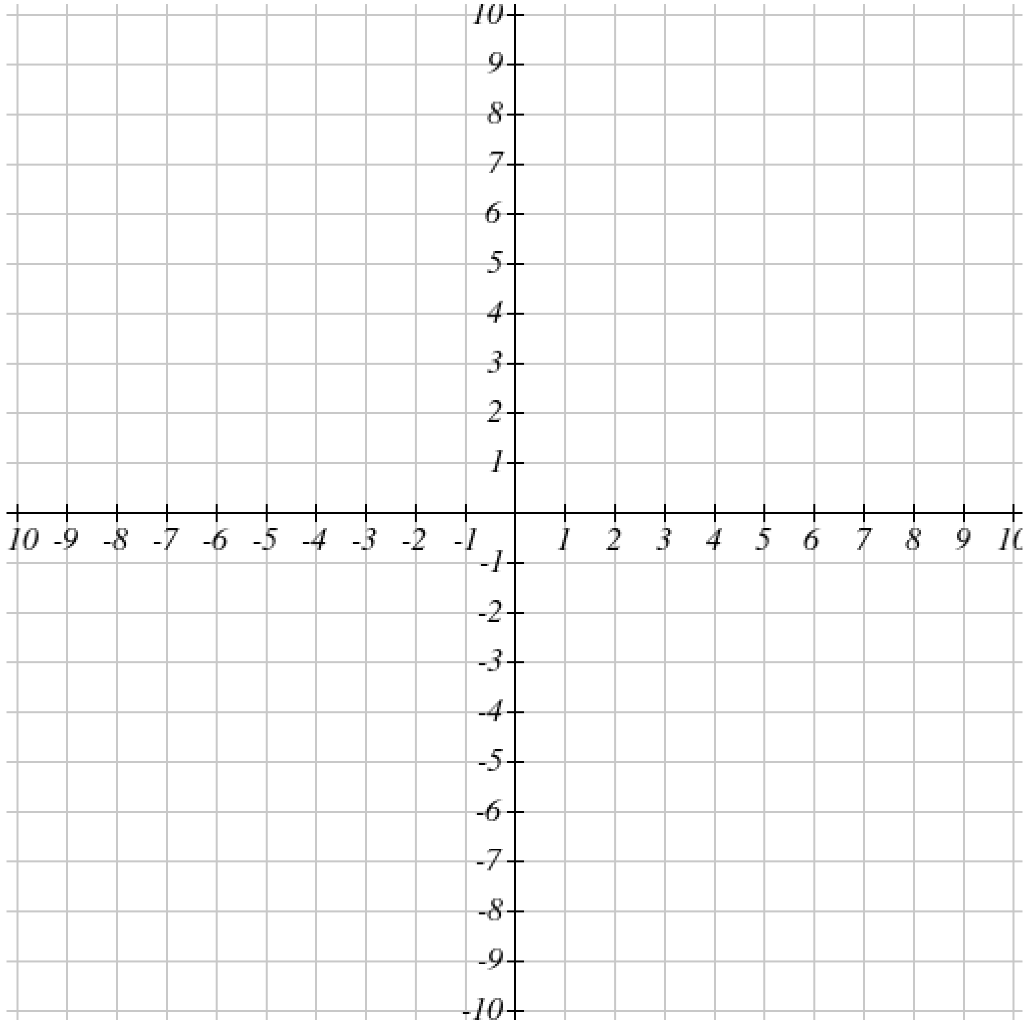


2. **Contextless Graphing.** Graph the parabola: $f(x) = (x-2)^2 + 3$

The vertex is _______

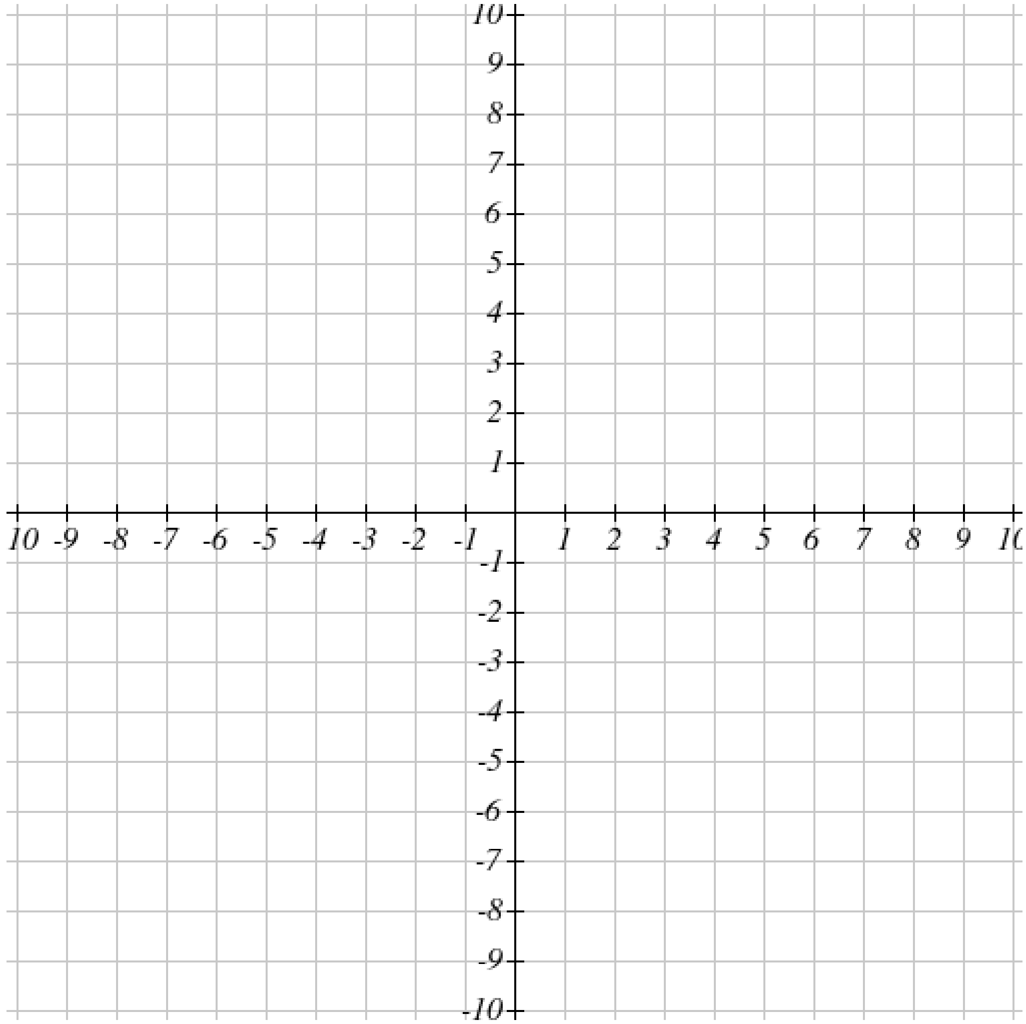


3. Contextless Graphing. Graph the parabola: $f(x) = 4x^2$
The vertex is _______



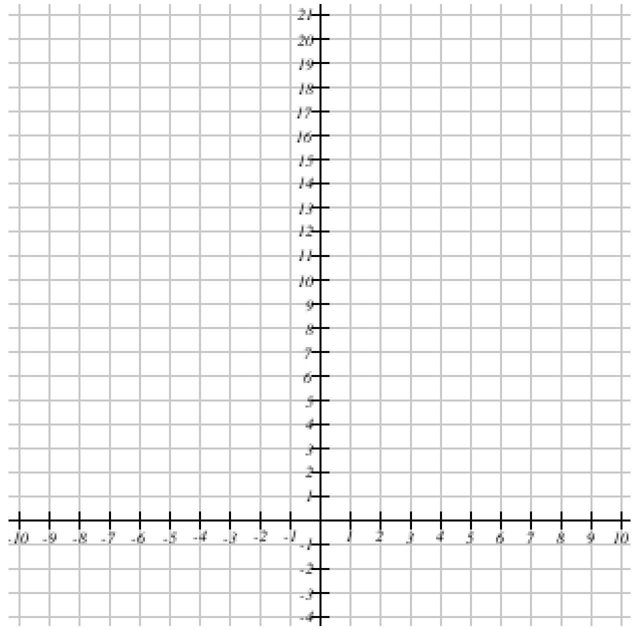
4. Contextless Graphing. Graph the parabola: $f(x) = \frac{1}{3}x^2$

The vertex is _______

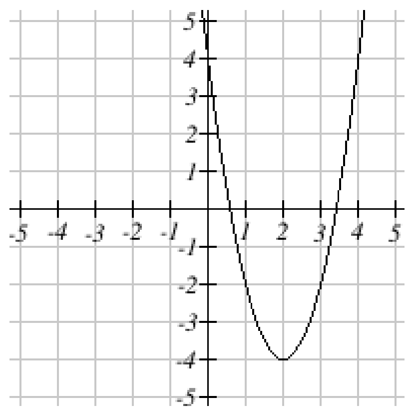


5. Contextless Graphing. Graph the parabola $y = 6(x+2)^2 - 3$

The vertex is _______

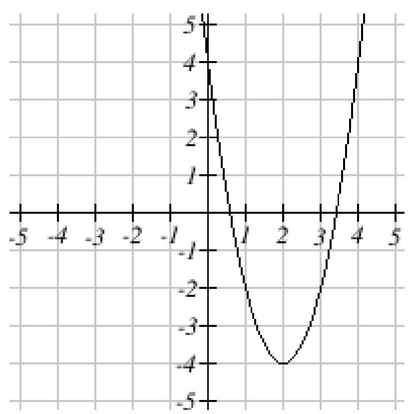


6. **Read Graph.** Identify the vertex of the parabola $y = 2x^2 - 8x + 4$ graphed below



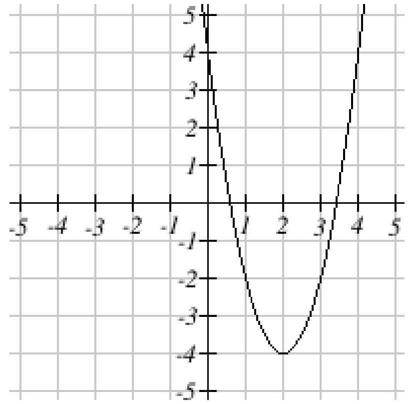
Vertex = (__, __)

7. **Read Graph.** Identify the vertex of the parabola $y = 2x^2 - 8x + 4$ graphed below



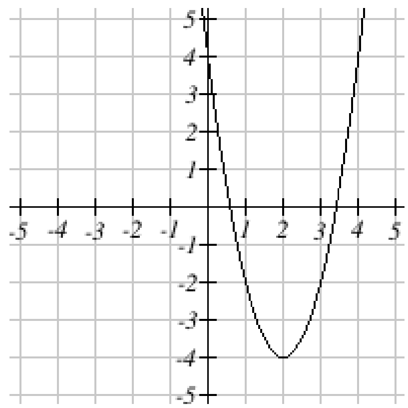
Vertex = (__, __)

8. **Read Graph.** Identify the vertex of the parabola $y = 2x^2 - 8x + 4$ graphed below



Vertex = (__, __)

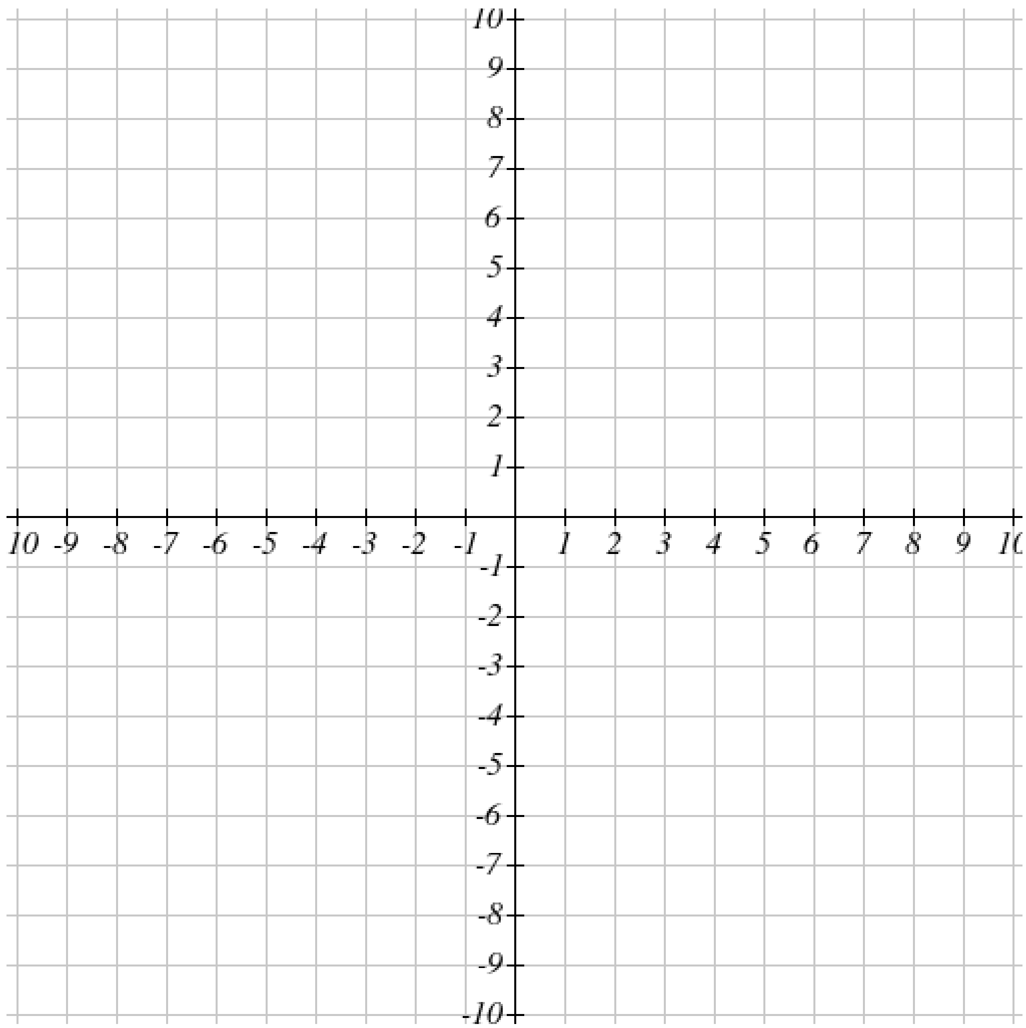
9. **Read Graph.** Identify the vertex of the parabola $y = 2x^2 - 8x + 4$ graphed below



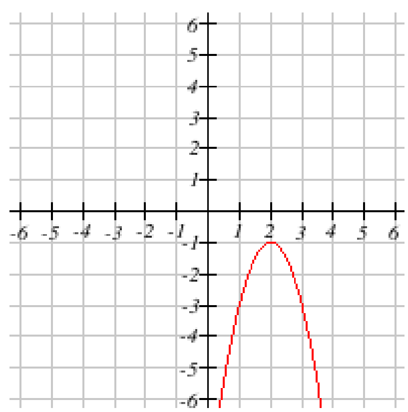
Vertex = (__, __)

10. **Read Graph.** Graph the parabola: $f(x) = 2x^2 + 12x + 16$

The vertex is _______

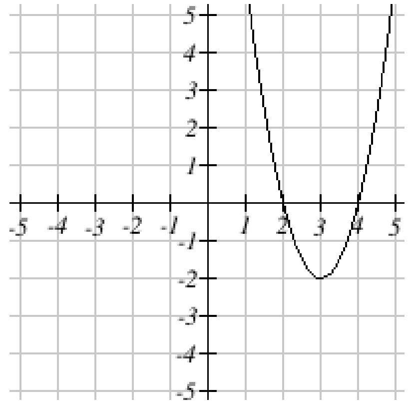


11. Write Equation. Find an equation for the graph shown below. (*Hint*: use the form $y = a(x-h)^2 + k$)



$y =$ __________________________

12. Write Equation. Write an equation (any form) for the quadratic graphed below



y = ____________________

13. Write Equation. A quadratic function has its vertex at the point $(-3, 3)$. The function passes through the point $(8, -8)$. Find the expanded form of the function.

The coefficient a is __________ .

The coefficient b is __________ .

The constant term c is __________ .

14. In Context. NASA launches a rocket at $t = 0$ seconds. Its height, in meters above sea-level, as a function of time is given by $h(t) = -4.9t^2 + 301t + 94$.

(A) Assuming that the rocket will splash down into the ocean, at what time does splashdown occur? (Round answer to 2 decimal places)

The rocket splashes down after __________ seconds.

(B) How high above sea-level does the rocket get at its peak? (Round answer to 2 decimal places)

The rocket peaks at __________ meters above sea-level.

5.5 Project: Approximation of Sums

Project 9 Approximate Height of a Dropped Object. Using the acceleration due to gravity, $9.80665\ \frac{\text{m}}{\text{s}^2}$, we can calculate the velocity at any time. However, we can only approximate its position, because the velocity is constantly changing. This activity demonstrates a method of approximation that is used in many problems including calculating areas (e.g., calculating the amount of asphalt needed for a driveway with curved sides). This activity also illustrates how approximations depend on choices we make.

(a) Review the process shown here.

Suppose a ball is dropped from a height of 60.0 meters. We will approximate its height above ground by calculating its approximate position each second. For example, after 1 second the ball will have accelerated from 0 m/s to 9.8 m/s. If we pretend that it was travelling 9.8 m/s the whole time, then we estimate it will be 9.8 m lower than it began. This is a height of $60.0 - 9.8 = 50.2$. A second later the velocity will have increased 9.8 m/s so it will be $9.8 + 9.8 = 19.6$ m/s. If we pretend that it was travelling 19.6 m/s the whole time, then we estimate it will be 19.6 m lower than it began. This is a height of $50.2 - 19.6 = 30.6$. Another second later (third second) the velocity will have increased by 9.8 m/s so it will be $19.6 + 9.8 = 29.4$. If we pretend that it was travelling 29.4 m/s the whole time, then we estimate it will be 29.4 m lower than it began. This is a height of $30.6 - 29.4 = 1.2$. With only 1.2 meters left and a velocity much greater than 1.2, we know it will reach the ground in less than one more second.

The table below contains these results.

Time	Velocity	Distance Traveled	Height
0	0	0	60.0
1	9.8	9.8	50.2
2	19.6	19.6	30.6
3	29.4	29.4	1.2

(b) Repeat this exercise using half second intervals instead of one second intervals. Remember that in a half second the acceleration will be $9.8/2 = 4.9$ m/s. In the first half second the ball will fall $0.5 \text{ s} \cdot 4.9 \, \frac{\text{m}}{\text{s}} = 2.45$ m. In the second half second the velocity would be 9.8 m/s so it would drop $0.5 \text{ s} \cdot 9.8 \, \frac{\text{m}}{\text{s}} = 2.45$ m. The first two entries are completed for you. Add more rows if you need to.

Time	Velocity	Distance Traveled	Height
0	0	0	60.0
0.5	4.9	2.45	57.55
1.0	9.8	4.9	52.65
1.5			
2.0			
2.5			
3.0			

(c) The end result of this task will be a third table calculated in 1/4 second intervals.

(i) What is the acceleration in 1/4 second? Recall it is 9.8 m/s which was 4.9 m/half second.

(ii) Fill out a table in 1/4 second intervals.

(d) Notice that in the first table (1 second intervals), we estimated after one second the height would be 50.2 m. In the second table (1/2 second intervals), we estimated after one second the height would be 52.65 m.

(i) Find other times in which the 3 tables have different estimates.

(ii) Are the estimates in one table always less than or greater than those in the other tables?

(iii) Explain why these estimates are different.

(iv) Which estimate do you think is most accurate?

Chapter 6

Exponential

6.1 Exponential Relations

This section addresses the following topics.

- Read and use mathematical models in a technical document
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Read and interpret models (critical thinking)
- Use models including linear, quadratic, exponential/logarithmic, and trigonometric (skill)

Section 3.3 presented different rates at which data changes including linear, quadratic, and exponential. This section presents models that require exponentials and notation so we can perform calculations. The ability to solve equations with exponentials comes in Section 6.4.

6.1.1 Comparing Growth Rates

In colloquial speech **exponential** is used to mean “very fast”. In this text we are using a more detailed definition. This section presents examples that illustrate that an exponential relation is very fast in that it grows faster than linear and quadratic relations. To complete the presentation of comparative rates we will also show another relation that is even faster.

Consider the linear, quadratic, and exponential relations in Table 6.1.1, Table 6.1.2, and Table 6.1.3 respectively. Linear data has the same differences (3 in the example below). In contrast the differences for quadratic and exponential data grow as the data grows. Their growth is not the same however. Quadratic differences grow at the same rate (2nd differences are linear, 4 in the example below). In contrast exponential differences grow faster as the data grows faster (indeed at the same rate).

Another way to look at this, is to note that for linears and quadratics eventually the data is bigger than the differences. For example 5,8,11 are all bigger than 3 for the linear below, and 6,10,14 are less than 8,18, and 32 respectively for the quadratic below. For exponential relations, however, the rate of growth of the values is a multiple of the values, that is the rate of growth increases as quickly as the values do (e.g., the values are doubling and so are the differences). Because of this exponential growth will always outpace linear and quadratic growth eventually.

Table 6.1.1 Linear Relation

n	0	1	2	3	4	5
$3n+2$	2	5	8	11	14	17
Differences		3	3	3	3	3

Table 6.1.2 Quadratic Relation

n	0	1	2	3	4	5
$2n^2$	0	2	8	18	32	50
Differences		2	6	10	14	18

Table 6.1.3 Exponential Relation

n	0	1	2	3	4	5
$3(2^n)$	3	6	12	24	48	96
Differences		3	6	12	24	48

We can also compare the ratios of consecutive terms (which can be thought of as percent increase) of the relations. Table 6.1.4 and Table 6.1.5 have the ratios for a linear and a quadratic relation respectively. From the decimal approximation rows we can see that the ratios are decreasing. A closer look shows they are both trending toward one. In contrast in Table 6.1.6 we are reminded that the ratios for an exponential relation are constant. This also implies that exponential relations grow faster than linear or quadrataic relations.

Table 6.1.4 Linear Relation

n	0	1	2	3	4	5
$3n+2$	2	5	8	11	14	17
Ratios		$5/2$	$8/5$	$11/8$	$14/11$	$17/14$
Decimal		2.5	1.6	1.375	1.2727	1.2143

Table 6.1.5 Quadratic Relation

n	1	2	3	4	5	6
$2n^2$	2	8	18	32	50	72
Ratios		$8/2$	$18/8$	$32/18$	$50/32$	$72/50$
Decimal		4	2.25	1.7778	1.5625	1.44

Table 6.1.6 Exponential Relation

n	0	1	2	3	4	5
$3(2^n)$	3	6	12	24	48	96
Ratios		2	2	2	2	2

The comparisons to linear and quadratic might suggest that exponential is the fastest growing relation. It is not. Table 6.1.7 shows the ratios for a relation known as factorial. Interestingly, the ratios grow linearly. This is vaguely like quadratic relations which grow faster than a linear relation because their differences grow linearly. Factorial relations are growing faster than exponential relations because their ratios are growing linearly rather than remaining constant.

Table 6.1.7 Factorial Relation

n	0	1	2	3	4	5
$n!$	1	1	2	6	24	120
Ratios		1	2	3	4	5

Checkpoint 6.1.8 Rank the following based on how fast they grow.

Table 6.1.9

a	16	40	100	250	625	1562.5

Table 6.1.10

b	0	1	16	81	256	625

Table 6.1.11

c	-7	-2	3	8	13	18

1. a
2. b
3. c

1. a
2. b
3. c

1. a
2. b
3. c

6.1.2 Applications

This section presents some simple applications that introduce us to the nature of exponential relations and introduce us to how to write equations for exponential models. The next few examples are used to present the mathematics, they are not a complete presentation of the science.

Bacteria (any cells) grow by each cell dividing into two, completely functioning cells. The new cells eventually reproduce by dividing in half as well. This means that the population is doubling. Because each cell takes roughly the same amount of time to grow enough to be able to divide, the population will double again when that much time has elapsed. Because we are interested in the idea, we will not round in the next few examples.

Example 6.1.12 The bacteria **lactobacillus acidophilus** is part of turning milk into yogurt. Based on experiments a new generation of bacteria are formed every 70 minutes, that is, the population doubles every 70 minutes.

Suppose when we start tracking the data there are 3000 cells. After 70 minutes, all of these will have divided into two, so there will be $4000(2) = 8000$ cells. After another 70 minutes, all of these 8000 cells will have divided into two, so there will be $8000(2) = 16000$ cells. These results are shown in Table 6.1.13.

If the population grows by the same ratio every 70 minutes, we might ask ourselves if it grows by the same ratio every 35 minutes. Using math from later in this chapter we can construct Table 6.1.14 which shows populations every 35 minutes. We discover that the population does grow by a fixed percent every 35 minutes. If we look a little deeper we realize that we cut the time in half and switched from doubling (times 2) to times $\sqrt{2} \approx 1.4142$.

We could produce a table for any amount of time (e.g., every hour) and we would find that the population grows by the same multiple every time. ☐

Table 6.1.13 Growth of Lactobacillus Acidophilus

Minutes	0	70	140	210
Population	4000	8000	16000	32000
Increased by		4000	8000	16000

Table 6.1.14 Growth of Lactobacillus Acidophilus

Minutes	0	35	70	105	140
Population	4000	5657	8000	11314	16000
Ratio		1.4142	1.4142	1.4142	1.4142

Having emphasized that exponential data grows by a fixed ratio (multiple) every time unit, we can now work toward mathematical notation.

Example 6.1.15 First Exponential Model. To obtain a model (equation) we will review our calculations from the previous example and note a pattern in those calculations.

The initial amount (information we are given) is $P_0 = 4000$. After 70 minutes all of these split into two so the population is $P_{70} = 4000(2) = 8000$. After 140 minutes they have all split into two a second time so the population is $P_{140} = 4000(2)(2) = 16000$. After 210 minutes they have split a third time increasing the population to $P_{210} = 4000(2)(2)(2) = 32000$. We repeatedly mutiply the initial amount by 2. This means we will have a power of 2 times the initial amount.

The number of 2's by which we multiply is determined by how many multiples of 70 minutes have expired. $140/70 = 2$ so we multiplied by 2^2. $210/70 = 3$ so we multiplied by 2^3. In general we multiply by two $t/70$ times.

Putting these together implies we want to multiply the initial amount (4000 in this case) by two (double the population) for each of multiple of 70. This gives us

$$P = 4000 \cdot 2^{t/70}.$$

□

Example 6.1.16 First Exponential Model Redux. Of course we cannot actually count the number of cells in a colony of bacteria. It is easier to measure by mass (units of grams). If the number of cells has doubled then the mass will have about doubled as well.

Suppose 3 grams of lactobacillus acidophilus is placed in milk. What is a model for the mass of bacteria if it doubles every 70 minutes?

From the previous example we know we can multiply the initial amount (3 g) by 2 for each 70 minutes. This gives us

$$A = 3(2^{t/70}).$$

□

Some exponential relations show the amount decreasing. That is the ratio that is multiplied is between 0 and 1. One example is radioactive decay. Radioactive substances are not stable. The radiation they give off is the result of the atoms breaking down into other substances. This means that over time the amount of the radioactive substance decreases. It has been shown that this decrease is exponential. The rate of decay is expressed in **half-life**, the amount of time for the substance to be reduced by half.

Example 6.1.17 Barium-133 has a half-life of 10.551 years. This means that afeter 10.551 years only half of the original amount will remain. This is the result of the radioactive isotope breaking down into other substances.

Suppose that we obtain 7.000 grams of Barium-133. After 10.551 years we will have $7.000 \cdot \frac{1}{2} = 3.500$ grams. Note the 1/2 is an exact number (the measurement error is in the number of years). After 21.102 years there will be $7.000 \cdot \left(\frac{1}{2}\right)^2 = 1.750$. In general the amount left after t years will be

$$A = 7.000 \left(\frac{1}{2}\right)^{t/10.551}.$$

This could also be written as $A = 7.000 \cdot 2^{-t/10.551}$. □

Exponential relations are not all doubling or cutting in half. The ratio can be anything. Going viral on the internet could be (but is not always) an exponential growth of views.

Example 6.1.18 A new cat video is posted and 12 people view it the first day. Every 4 days afterward the number of people who see it triples. Write an equation to model the total number of people who have viewed this video.

To help ourselves figure this out we can calculate the first few days results. The 4 days after the video is posted, there will be $12(3) = 36$ views. The eighth day, there will be $12(3)(3) = 108$ views. Each additional four days we multiply the result by 3.

Thus we need to divide the number of days by 4 to determine how many times it has tripled. For example after 24 days we expect it to triple $24/4 = 6$ times.

To calculate the total number of views we multiply the original twelve by 3 for each time it tripled. Thus the number of people viewing the video is

$$v = 12(3^{d/4})$$

□

Checkpoint 6.1.19 At a particular temperature and acidity bacillus megaterium will doube every 0.62 hours. Suppose we begin with 1.7 grams of the bacteris. Write an equation for the amount of bacteria after t hours.

$b =$ ____________________

Use t as the variable.

We learned in Subsection 3.3.2 that salaries increased by a fixed percent each year are exponential in nature. Now we can write a model for this and calculate results.

Example 6.1.20 Tien's initial salary was \$52,429.33. He received a 5% raise each year. What should Tien's salary be entering the sixth year?

Because the raise is a 5% increase, the percent is $p = 1.05$. Thus after one year his salary will be $\$52,429.33(1.05)$. After two years his salary will be $\$52,429.33(1.05)(1.05)$. This pattern will continue.

The model then is

$$S = \$52,429.33(1.05)^t$$

where t is the number of years since he was hired. Entering the sixth year would mean he has just received his fifth raise. His salary would be

$$S = \$52,429.33(1.05)^5 = \$66,914.59.$$

□

Example 6.1.21 If Moses' salary after six raises was \$72,311.54, and he received a 4% raise each year. What was his initial salary?

Because the raise is a 4% increase, the percent is $p = 1.04$. The model then is

$$S = S_0(1.04)^t$$

where t is the number of years since he was hired, and S_0 is the initial salary. We can now solve for the initial salary.

$$\begin{aligned}
\$72,311.54 &= S_0(1.04)^6. \\
\$72,311.54 &\approx S_0(1.2653). \\
\frac{\$72,311.54}{1.2653} &\approx \frac{S_0(1.2653)}{1.2653}. \\
\$57,149.72 &\approx S_0.
\end{aligned}$$

Because the salary would be rounded each year this might be off by a small amount, but not enough to matter for our curiosity. □

In the previous example we rounded in the second step to 4 decimal places. Was this the right choice? We can test by trying the calculation with fewer decimal places. If we had rounded to 5 places, the last step would have been $\frac{\$72,311.54}{1.26532} \approx \$57,148.82$. If we had rounded to 6 places, the last step would have been $\frac{\$72,311.54}{1.265319} \approx \$57,148.86$. If we had rounded to 7 places, the last step would have been $\frac{\$72,311.54}{1.265320} \approx \$57,148.86$. Likewise testing 8 places would not change the amount. Thus any more than 6 places is not enough to change the result. However, was the 4 places result bad? The amounts between the roundings are different from each other but not by enough to impact anyone's life. Remember rounding is often about being practical: if the variation we round away has no impact, why bother with all the extra work?

Because we just calculated an initial salary from a current salary and the annual percent increase, we might wonder if we can calculate the percent increase from a past and current salary. We can, but it requires a technique from Section 5.3.

Example 6.1.22 We will calculate the percent increase given initial and final salaries. If Raven's initial salary was \$53,242.17, and her salary at the end of 7 years was \$67,368.33, what was her annual percentage increase?

Solution. The end of seven years means there have been six raises. From the two data points we know

$$\begin{aligned} 67368.33 &= 53242.17(r^6) \\ \frac{67368.33}{53242.17} &= r^6 \\ \sqrt[6]{\frac{67368.33}{53242.17}} &= \sqrt[6]{r^6} \\ 1.04 &= r \end{aligned}$$

Thus her annual percentage increase was 4%. The full model is

$$S = 53242.17(1.04)^t$$

where t is time in years. □

6.1.3 Two Exponential Models

Above we worked examples in which some initial amount (e.g., mass of bacteria or salary) was repeatedly multiplied by a number. These were written in slightly different ways giving us the following two models.

Model 6.1.23 Exponential Growth (rate). *An amount that is growing exponentially with respect to time is given by*

$$A = A_0\left(r^{t/d}\right)$$

where

- *A is the final amount (units vary)*
- *A_0 is the initial amount (same units as A)*
- *r is the doubling ($r = 2$), trebling ($r = 3$), or other increase. It is unitless.*
- *t is the amount of time (units typically days/hours/minutes)*
- *d is how long until the increase occurs (same units as t)*

Model 6.1.24 Exponential Growth (percent). *An amount that is growing exponentially with respect to time is given by*

$$A = A_0\left(1+p\right)^t$$

where

- *A is the final amount (units vary)*
- *A_0 is the initial amount (same units as A)*
- *p is the percent increase or decrease. It is unitless.*
- *t is the amount of time (units typically days/hours/minutes)*

It is possible to convert from one form to the other. The following examples demonstrate using these models, determining the percent increase/decrease, and converting between forms.

Example 6.1.25 Suppose $P = 100(1.032)^t$ where t is the number of years. What is the annual percent increase?

We know that

$$\begin{aligned} 1.032 &= 1 + p \\ 1.032 - 1 &= p \\ 0.032 &= p \end{aligned}$$

so this is a 3.2% increase. □

Before we can change that form into one that shows us how long until it doubles, we will need to learn another function. We will do this in Section 6.4.

Example 6.1.26 Suppose $P = 100(0.88)^t$ where t is the number of years. What is the annual percent decrease?

We know that

$$\begin{aligned} 0.88 &= 1 + p \\ 0.88 - 1 &= p \\ -0.12 &= p \end{aligned}$$

so this is a 12% decrease. □

Generally if the value $1 + p$ is greater than one this is an increasing rate and if it is less than one it is a decreasing rate.

Example 6.1.27 Suppose $P = 230(1.03)^t$. Is this increase or decrease and what is the percent increase/decrease?

Solution. Because $1.03 > 1.00$ this is a percent increase.

$$\begin{aligned} 1.03 &= 1 + p \\ 1.03 - 1 &= p \\ 0.03 &= p \end{aligned}$$

Thus this is a 3% increase. □

Even when the exponential model is expressed in terms of doubling or similar, we can determine the percent increase or decrease. This requires knowing that $2^{t/3} = (2^3)^t = 8^t$ or in general

$$B^{t/r} = (B^r)^t.$$

Significant Digits beyond Arithmetic. In Significant Digits beyond Arithmetic we noted that a square root maintains the same number of significant digits. The same mathematics applies to exponential. For example, $2^{3.51} \approx 11.39240156 \approx 11.4$.

Example 6.1.28 In the exponential model $P = 342\,(2)^{t/2.06}$ what is the percent increase or decrease? Note these numbers are measurements.

First, we use the algebra above to convert the form

$$\begin{aligned} (2)^{t/2.06} &\approx \\ \left(2^{\frac{1}{2.06}}\right)^t &\approx 1.400009768^t \\ &\approx 1.40^t \end{aligned}$$

Because 1.4 is bigger than one it is a percent increase. Note $1.40 = 1 + 0.40$, so this is a 40% increase.

For rounding we used that 2 and 1 (in the division) are exact numbers. Calculating the power of 2 maintains the number of significant digits (3). When we subtract precision is maintained to the hundredths position (2 significant digits). □

Example 6.1.29 In the exponential model $P = 342\,(2)^{-t/27}$ what is the percent increase or decrease?

Solution. First, we use the algebra above to convert the form

$$\begin{aligned}(2)^{-t/27} &= \\ \left(2^{\frac{-1}{27}}\right)^t &= 0.9746546091^t \\ &\approx (0.97)^t\end{aligned}$$

Because 0.97 is less than one it is a percent decrease. Note $1 - 0.97 = 0.03$, so this is a 3% decrease.

We round to two decimal places because 27 has 2 significant digits. The subtraction retains precision to the hundredths position which is only one significant digit. □

Checkpoint 6.1.30 In the model $A = 1730 \cdot \left(2^{\frac{t}{28}}\right)$ what is the percent increase or decrease? ______

Note all numbers are scientific measurements.

A common base for exponentials in many scientific models is e. We can work with this base using calculation devices which will have an e^x button or an $\exp(x)$ function.

Example 6.1.31 What is the percent increase or decrease for the exponential model $P = 27e^{0.0200t}$?

Just as with the fractions we convert to decimal. Note that we can also write the model $P = 27(e^{0.0200})^t$. Do you see the difference? $e^{0.0200} \approx 1.02$ which we found using a device. Thus this is a percent increase of 2%. □

Checkpoint 6.1.32 In the model $A = 174.6e^{0.0357t}$ what is the percent increase or decrease? ______

Note all numbers are scientific measurements.

6.1.4 Exercises

Exercise Group. Calculate results for exponential relations.

1. **Contextless Practice.** Evaluate the expression $y = 4^x$ for the values given. Use two (non-zero) decimal places for fractional values.
   For $x = 5$, $y =$ ____________________
   For $x = -9$, $y =$ ____________________
2. **Contextless Practice.** Evaluate the expression $y = 4^x$ for the values given. Use two (non-zero) decimal places for fractional values.
   For $x = 5$, $y =$ ____________________
   For $x = -9$, $y =$ ____________________
3. **Contextless Practice.** Evaluate the expression $y = 4^x$ for the values given. Use two (non-zero) decimal places for fractional values.
   For $x = 5$, $y =$ ____________________
   For $x = -9$, $y =$ ____________________
4. **Contextless Practice.** Evaluate the expression $y = 3.96^x$ for the values given. Use two decimal places.
   For $x = 4$, $y =$ ____________________
   For $x = -3$, $y =$ ____________________
5. **Contextless Practice.** Evaluate the expression $y = 3.96^x$ for the values given. Use two decimal places.
   For $x = 4$, $y =$ ____________________
   For $x = -3$, $y =$ ____________________
6. **Contextless Practice.** Evaluate the expression $y = 3.96^x$ for the values given. Use two decimal places.
   For $x = 4$, $y =$ ____________________
   For $x = -3$, $y =$ ____________________

7. Contextless Practice. Approximate e^2 to 4 decimal places. ____________________

Approximate $e^{-1.87}$ to 4 decimal places. ____________________

8. Faux Application. A population numbers 14,000 organisms initially and grows by 18.3% each year.

Suppose P represents population, and t the number of years of growth. An exponential model for the population can be written in the form $P = a \cdot b^t$ where

$a =$ _____ and $b =$ _____.

9. Application. The number of bacteria in a culture is given by the function

$n(t) = 915e^{0.3t}$

where t is measured in hours.

(a) What is the initial population of the culture (at t=0)?

Your answer is _______________

(b) How many bacteria will the culture contain at time t=5?

Your answer is _______________

10. Application. Find the final hourly wage if a \$6.50 starting wage is increased by 4.5% each year for 7 years.

The final wage is \$ ____________________. *Round answer to 2 decimal places*

11. Application. The population of the world in 1987 was 5 billion and the annual growth rate was estimated at 2 percent per year. Assuming that the world population follows an exponential growth model, find the projected world population in 1999.

Round your answer to two decimal places.

Your answer is ____________________ billion

12. Application. A vehicle purchased for \$25000 depreciates at a constant rate of 4%. Determine the approximate value of the vehicle 15 years after purchase.

Round to the nearest whole number.

____

13. Application. A worker's contract states that the hourly wage will start at \$9.50 and will increase by $r = 6\%$ annually, with the raise given every 12 months.

The hourly wage can be modeled by the exponential formula $S = P\left(1 + \frac{r}{n}\right)^{nt}$, where S is the future value, P is the present value, r is the (nominal) yearly rate of increase, n is the number of times each year that the wage is increased, and t is the time in years.

(A) What values should be used for P, r, and n?

$P =$ _____, $r =$ _____, $n =$ _____

(B) The final hourly wage in 9 years will equal which dollar amount?

Answer = \$ _____.

Round answer to the nearest penny.

14. Application. A worker's contract states that the hourly wage will start at \$8.00 and will increase by $r = 6\%$ annually, with half the annual raise given every 6 months.

The hourly wage can be modeled by the exponential formula $S = P\left(1 + \frac{r}{n}\right)^{nt}$, where S is the future value, P is the present value, r is the (nominal or stated) annual rate of increase, n is the number of times each year that the wage is increased, and t is the time in years.

(A) What values should be used for P, r, and n?

$P =$ _____, $r =$ _____, $n =$ _____

(B) The final hourly wage in 10 years will equal which dollar amount?

Answer = \$ _____.

Round answer to the nearest penny.

(C) The actual annual percent increase is _____%.

Round answer to 3 decimal places.

15. Application. In April 1986, a flawed reactor design played a part in the Chernobyl nuclear meltdown. Approximately 14252 becqurels (Bqs), units of radioactivity, were initially released into the environment. Only areas with less than 800 Bqs are considered safe for human habitation. The function $f(x) = 14252(0.5)^{\frac{x}{32}}$ describes the amount, $f(x)$, in becqurels, of a radioactive element remaining in the area x years after 1986.

Find $f(150)$, to one decimal place, in order to determine the amount of becqurels in 2136.

_____

Determine if the area is safe for human habitation in the year 2136.

(a) No, because by 2136, the radioactive element remaining in the area is greater than 800 Bqs.

(b) Yes, because by 2136, the radioactive element remaining in the area is greater than 800 Bqs.

(c) No, because by 2136, the radioactive element remaining in the area is less than 800 Bqs.

(d) Yes, because by 2136, the radioactive element remaining in the area is less than 800 Bqs.

16. Application. A newly hatched channel catfish typically weighs about 0.6 grams. During the first 6 weeks of life, its weight increases about 9% each day.

Identify the initial amount:

____

Identify the growth/decay factor (the b value):

_____

Write an exponential equation to model the situation in the form $f(x) = a(b)^x$

f(x) = __________

How much does the catfish weigh after 5 weeks?

__________ (Round to 2 decimals)

17. Application. Coral reefs throughout the world are dying at a rate of about 3% per year. Write an equation that can be used to determine the future area of a reef that now has an area of 250 km^2. Let c be the area of the coral reef in t years from now.

(a) The equation is: ____________________

(b) The equation is in the family:

i. linear

ii. exponential

iii. quadratic

(c) Use the equation to determine how many years until the reef decays to an area of 128 km^2? Round to the nearest year.

____

Source[1]

18. Application. Initially, there were 810 flies in a population, and it increases by 8% every 3 days.

- What will be the population at $t = 10$ days? [Round your answer to the nearest whole number.]

  Answer: _____

- Write a formula for the population of flies as a function of t.

  Answer: $P(t) =$____________________

[1] www.theworldcounts.com/counters/ocean_ecosystem_facts/coral_reef_destruction_facts

19. Atenolol has a half life of 6-7 hours depending on patient factors. For this problem use a half life of 6.3 hours. Suppose Victor is prescribed 25 mg/day. If Victor takes his first 25 mg dose on Monday morning, how much is expected in his blood Tuesday morning at the same time? ____

6.2 Graphs of Exponential Functions

This section addresses the following topics.

- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Identify rates as linear, quadratic, exponential, or other (critical thinking)
- Identify data varrying directly or indirectly (critical thinking)

We have learned how to identify exponential data (Section 3.4) and have learned to work with some exponential applications (Section 6.1). This section presents the graph (shape) of exponentials and emphasizes two traits of exponential models.

Because devices can quickly produce accurate graphs for us, it is not the goal of this section to help you become proficiant at sketching graphs. Rather, it is the goal for you to be able to read a model, and recognize what different parts of the model imply, because you know what it would look like.

6.2.1 Shape of Exponentials

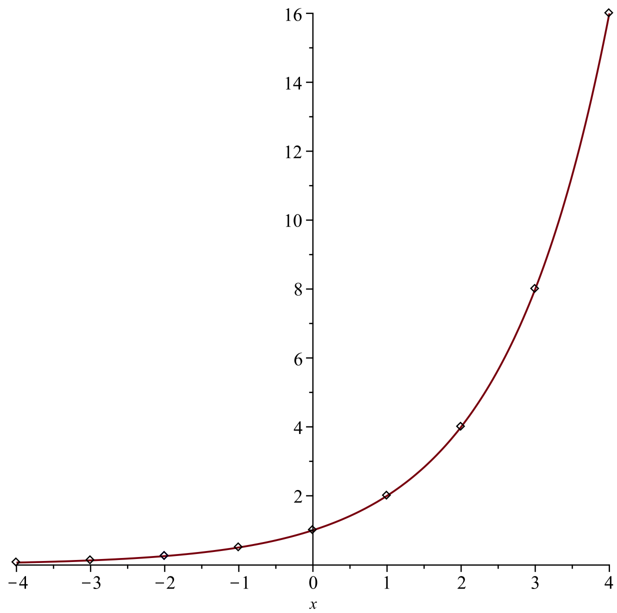
In order to understand the graph of an exponential function we are going to graph one that is not limited by interpretation or rounding.

Example 6.2.1 Graph $y = 2^x$.

First we will generate a table of points with which to start.

x	2^x
-4	$2^{-4} = \frac{1}{2^4} \quad = \frac{1}{16}$
-3	$2^{-3} = \frac{1}{2^3} \quad = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{2^2} \quad = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2^1} \quad = \frac{1}{2}$
0	$2^0 \quad = 1$
1	$2^1 \quad = 2$
2	$2^2 \quad = 4$
3	$2^3 \quad = 8$
4	$2^4 \quad = 16$

We can plot these points then sketch a curve smoothly through the points.



Starting at $x = 0$ consider the curve as it extends to the right. We can use the analogy of walking up hill to understand this curve. At $x = 0$ we are walking up hill so we are a little slower than on a flat surface. At $x = 1$ the height has doubled but so also has the slope (remember the differences are a multiple of the height). Walking the next segment will be more strenuous. At $x = 2$ the height has doubled again, so also has the steepness. Now it is too steep to walk. We will need to scramble or use climbing equipment. Generally an exponential curve becomes increasingly steep.

Starting back at $x = 0$ again consider the curve as it extends to the left. We can see that we are walking downhill; we might pick up a little speed. At $x = -1$ it continues downhill to the left but it is not enough to help us maintain a faster speed. At $x = -2$ we can no longer see the downhill (too gentle) though we would notice it if we were on a bike or skateboard. Just as the curve becomes twice as high and also twice as steep as we move the to right, it becomes half as high and half as steep as we go to the left. The inability to "see" the downhill nature illustrates that the exponential to the left begins to look like a horizontal line.

Let's clarify that last statement. We notice that all values (left and right) are positive. So to the left the curve is going down, but it never crosses zero. It must therefore become increasingly close to zero. This is why it begins to look like a line namely $x = 0$. When one curve becomes increasingly like another we call that second curve (the line $x = 0$ in this case) an **asymptote**. □

This decrease to the left can be understood in contexts as well.

Example 6.2.2 In Checkpoint 6.1.19 we determined that the amount of bacteria was modeled by

$$b = 30(2^{t/25}).$$

Initially ($t = 0$) there are 30 g of bacteria. Now we will calculate how much bacteria there was before this initial measurement.

Time	Mass of Bacteria		
0	$30(2^{0/25})$	$= 30 \cdot 1$	$= 30$
-25	$30(2^{-25/25})$	$= 30 \cdot \frac{1}{2}$	$= 15$
-50	$30(2^{-50/25})$	$= 30 \cdot \frac{1}{4}$	$= 15$
-75	$30(2^{-75/25})$	$= 30 \cdot \frac{1}{8}$	$= 7.5$
-100	$30(2^{-100/25})$	$= 30 \cdot \frac{1}{16}$	$= 3.75$

We notice that the amount of bacteria decreases. More specifically we see that it is divided in half each time. This is just the reverse of it doubling each time as it grows.

We also notice that because each previous entry is half of the next, there is always some left. No matter how far back in time we go, there will always be some. This is part of all exponentials: rapid growth in one direction and an increasingly gentle decrease toward some amount (zero in this case) in the other direction.

The model measures the bacteria by weight in grams. This can always be cut in half. What reality constraint should we put on this model? □

Not all exponential curves increase to the right and decrease to the left. Some reverse (mirror) this pattern. The following example uses radioactive decay as an example of this.

Example 6.2.3 Uranium-242 has a half-life of 16.8 minutes. Suppose we begin with $13\bar{0}$ g of uranium-242.

(a) How much is left at times 16.8, 33.6, and 50.4 minutes?

At time 0 we have $13\bar{0}$ g. At time 16.8 it will be half of the original which is 65.0 g. The 1/2 is an exact number so 3 significant digits is maintained. At time 33.6 it will be reduced by half again ($33.6 = 18.8 \cdot 2$, 2nd half-life). Thus there will be $\frac{65.0}{2} = 32.5$ g. At time 50.4 it will be reduced by half a third time ($50.4 = 16.8 \cdot 3$), so the amount remaining will be $\frac{32.5}{2} = 16.25 \approx 16.3$ g.

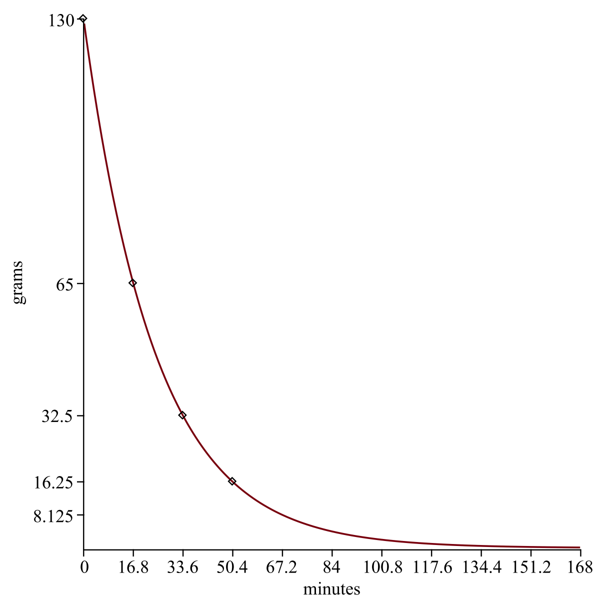
(b) Write a model for this decay.

We can use the process from Subsection 6.1.2. The time in minutes is divided by the half-life 16.8 minutes. Because this is reducing by half the scale is $\frac{1}{2}$.

$$A = 13\bar{0}\left(\frac{1}{2}\right)^{t/16.8}$$

(c) Graph this model.

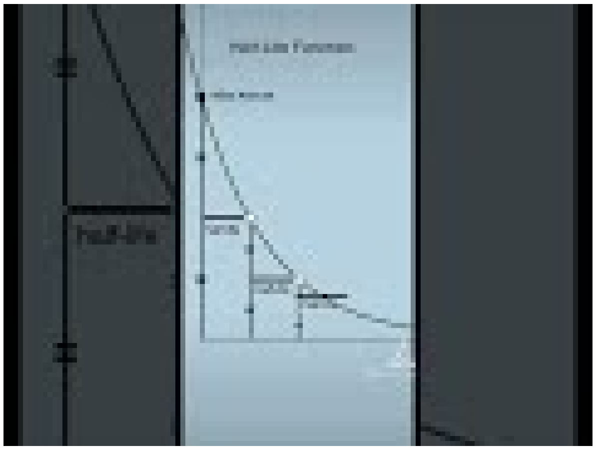
We can use the points from the first task and the method from Example 6.2.1 to produce a graph.



Notice the gentle decrease toward zero is to the right this time as opposed to the left in Example 6.2.1. That is because we are cutting in half each time instead of doubling.

□

The following video illustrates the connection between a fixed time (half-life) and decreasing by a ratio (1/2).



Standalone

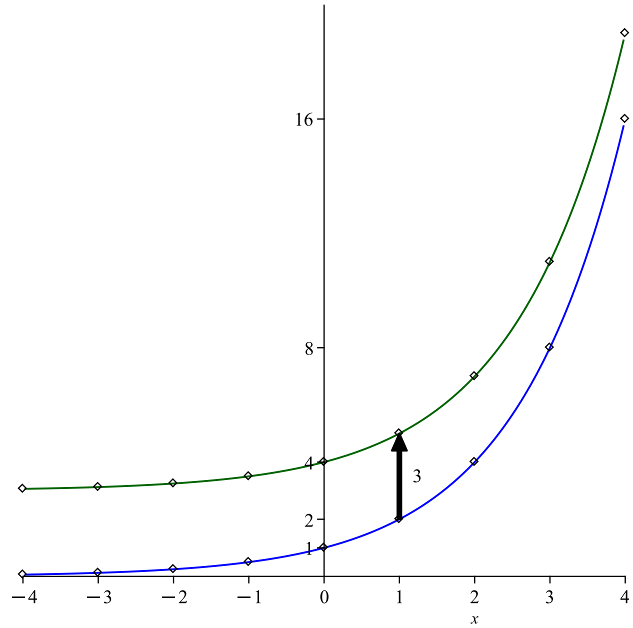
Figure 6.2.4 Geometric Illustration of Half-Life

6.2.2 Transforming Exponential Functions

In Section 5.4 we learned how to shift and stretch parabolas (graphs of quadratics). Those same transformations work on exponentials (and any other function). The following examples illustrate that the effects are the same.

Example 6.2.5 Translate an Exponential. Graph $y = 2^x + 3$ using the graph and table in Example 6.2.1. We can extend the table to include the $+3$.

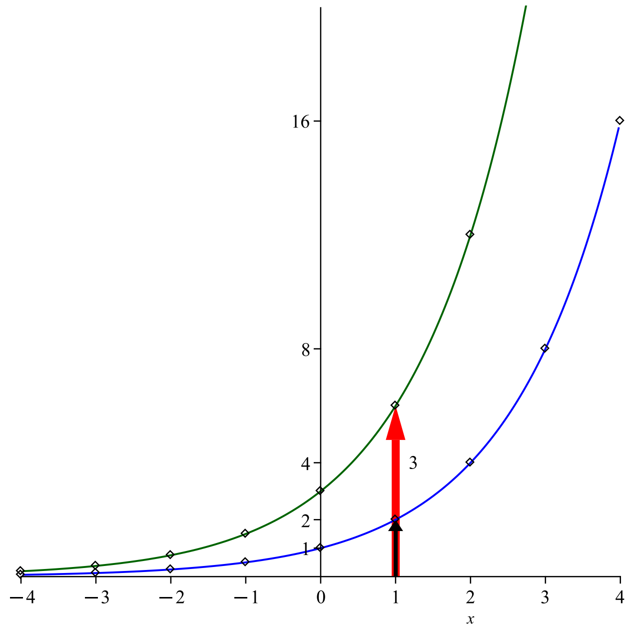
x	2^x	$2^x + 3$
-4	$\frac{1}{16}$	$3 + \frac{1}{16}$
-3	$\frac{1}{8}$	$3 + \frac{1}{8}$
-2	$\frac{1}{4}$	$3 + \frac{1}{4}$
-1	$\frac{1}{2}$	$3 + \frac{1}{2}$
0	1	$1 + 3 = 4$
1	2	$2 + 3 = 5$
2	4	$4 + 3 = 7$
3	8	$8 + 3 = 11$
4	16	$16 + 3 = 19$



Notice that every point is 3 units higher than the point in the original graph. We have moved the graph 3 units up. □

Example 6.2.6 Stretch an Exponential. Graph $y = 3(2^x)$ using the graph and table in Example 6.2.1. We can extend the table to include the 3·.

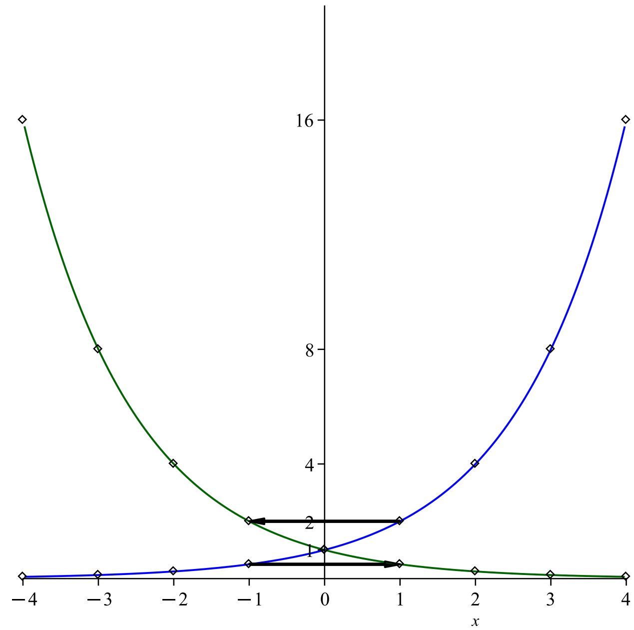
x	2^x	$3(2^x)$	
-4	$\frac{1}{16}$	$3 \cdot \frac{1}{16} =$	$\frac{3}{16}$
-3	$\frac{1}{8}$	$3 \cdot \frac{1}{8} =$	$\frac{3}{8}$
-2	$\frac{1}{4}$	$3 \cdot \frac{1}{4} =$	$\frac{3}{4}$
-1	$\frac{1}{2}$	$3 \cdot \frac{1}{2} =$	$\frac{3}{2}$
0	1	$3 \cdot 1 =$	3
1	2	$3 \cdot 2 =$	6
2	4	$3 \cdot 4 =$	12
3	8	$3 \cdot 8 =$	24
4	16	$3 \cdot 16 =$	48



Notice that every point is 3 times farther from the x-axis. This makes it 3 times steeper. □

Example 6.2.7 Reflect an Exponential. Graph $y = 2^{-x}$ using the graph and table in Example 6.2.1. We can extend the table to include the $-x$.

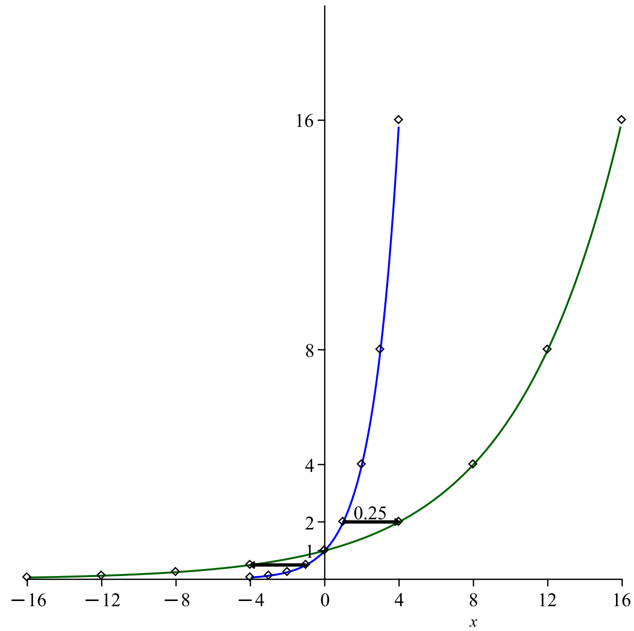
x	2^x	$-x$	2^{-x}
-4	$2^{-4} = \frac{1}{2^4} = \frac{1}{16}$	$-(-4) = 4$	$2^{-(-4)} = 2^4 = 16$
-3	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$	$-(-3) = 3$	$2^{-(-3)} = 2^3 = 8$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$-(-2) = 2$	$2^{-(-2)} = 2^2 = 4$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$-(-1) = 1$	$2^{-(-1)} = 2^1 = 2$
0	$2^0 = 1$	$-(0) = 0$	$2^{-(0)} = 2^0 = 1$
1	$2^1 = 2$	$-(1) = -1$	$2^{-(1)} = 2^{-1} = \frac{1}{2}$
2	$2^2 = 4$	$-(2) = -2$	$2^{-(2)} = 2^{-2} = \frac{1}{4}$
3	$2^3 = 8$	$-(3) = -3$	$2^{-(3)} = 2^{-3} = \frac{1}{8}$
4	$2^4 = 16$	$-(4) = -4$	$2^{-(4)} = 2^{-4} = \frac{1}{16}$



Notice that every point end up on the other side of the y-axis. This is a reflection of the original curve. □

Example 6.2.8 Reflect an Exponential. Graph $y = 2^{0.25x}$ using the graph and table in Example 6.2.1. We can extend the table to include the $0.25x$. Because this changes the x value, we will need a different set of inputs. Look at the result to understand why we selected them.

x	$0.25x$	$2^{0.25x}$
-16	$0.25(-16) = \quad -4$	$2^{-4} = \frac{1}{2^4} = \quad \frac{1}{16}$
-12	$0.25(-12) = \quad -3$	$2^{-3} = \frac{1}{2^3} = \quad \frac{1}{8}$
-8	$0.25(-8) = \quad -2$	$2^{-2} = \frac{1}{2^2} = \quad \frac{1}{4}$
-4	$0.25(-4) = \quad -1$	$2^{-1} = \frac{1}{2^1} = \quad \frac{1}{2}$
0	$0.25(0) = \quad 0$	$2^0 = \quad 1$
4	$0.25(4) = \quad 1$	$2^1 = \quad 2$
8	$0.25(8) = \quad 2$	$2^2 = \quad 2$
12	$0.25(12) = \quad 3$	$2^3 = \quad 8$
16	$0.25(16) = \quad 4$	$2^4 = \quad 16$



We ended up with the same y values, but they occurred spread out over a wider area. Multiplying stretched the curve horizontally. Because the number is between 0 and 1 (small fraction) it stretched it out (we could squeeze it in). □

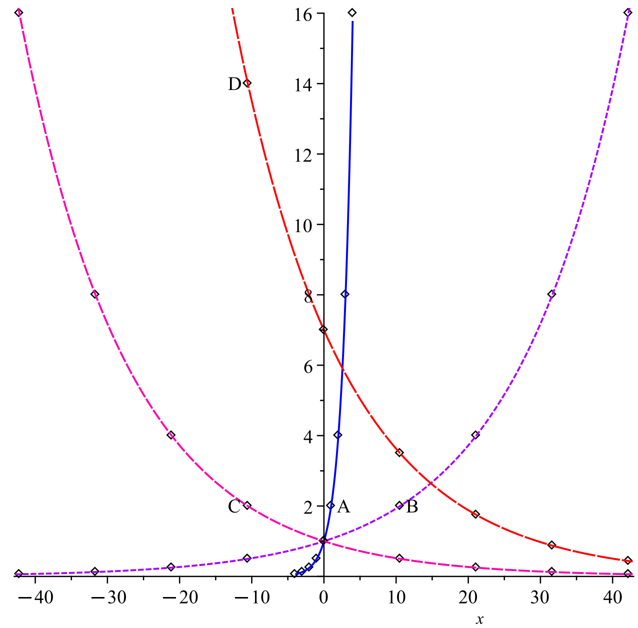
Now that we know how each piece affects the result we can put them all together. Our general equation for a base of 2 is

$$y = A \cdot 2^{B(x-h)} + k$$

where h and k represent the horizontal and vertical shifts respectively. A and B represent the vertical and horizontal stretches respectively. The first two examples below start by using each of these transformations individually. The third example illustrates using a shorter method that we typically use.

Example 6.2.9 Graph the model $7.000 \cdot 2^{-t/10.551}$ from Example 6.1.17. In order to match our examples above we can express the exponent as $7.000 \cdot 2^{-0.094778t}$. We simply calculated 1/10.551 and maintained 5 significant digits.

First, we identify the changes from $y = 2^x$. The 7.000 in front is a vertical stretch. The negative in the exponent is a reflection over the y-axis. The coefficient of the exponent is a horizontal stretch. We can make these changes in order of operation: horizontal stretch, reflection, vertical stretch. These are shown in the graph below. Graph A is the original. B shows the horizontal stretch. C shows the reflection. D adds the vertical stretch. The labels are located to show how one point is transformed by each step.

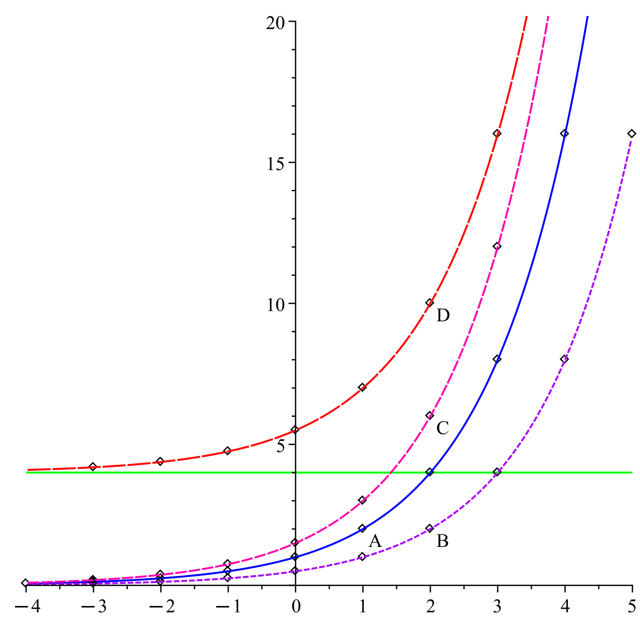


Looking at the final graph we can note that we really needed only three details to graph it. These are the asymptote (which remained unchanged from $y = 0$ here) and two points. One point tells us where to start and the other indicates how steep (because of the change between the two points). □

Before demonstrating graphing using the asymptote and two points, we graph an exponential with shifts as well.

Example 6.2.10 Graph the equation $y = 3\left(2^{x-1}\right) + 4$.

First, we identify that this is a modified version of $y = 2^x$. Next we identify the changes. The -1 is a horizontal shift. The 3 is a vertical stretch. The +4 is a vertical shift. We perform the changes in this order (following order of operations).These are shown in the graph below. Graph A is the original. B shows the horizontal shift. C shows the vertical stretch. D adds the vertical shift. The labels are located to show how one point is transformed by each step. Notice that the asymptote was moved up 4 units by the vertical shift as well.



Again, we could have sketched the asymptote knowing only the vertical shift would change it. Then we could plot points at $x = 0, 1$. The curve is then sketched down to the left (to approach the asymptote) and up to the right with steepness determined by the second point. □

Now that we have practiced using the various transformations, we can look at how a graph can be produced from just two, particular points. First, we consider that $2^0 = 3^0 = e^0 = 1$; no matter the base the zeroth power is 1. We also know that the asymptote is one unit below this point. Third, we note that $2^1 = 2$, $3^1 = 3$, $e^1 = e$; the base can be determined by looking at the point one unit left/right from the one at 0. In the following example we use these three ideas to identify the shifts and reflection.

Example 6.2.11 Graph the equation $y = 3^{x+2} + 5$.

First, we find the point where $x + 2 = 0$. This occurs at $x = -2$. This implies that the curve is shifted to the left (we start graphing two left of zero instead of at zero). At $x = -2$ we calculate $y = 6$. We can plot this point

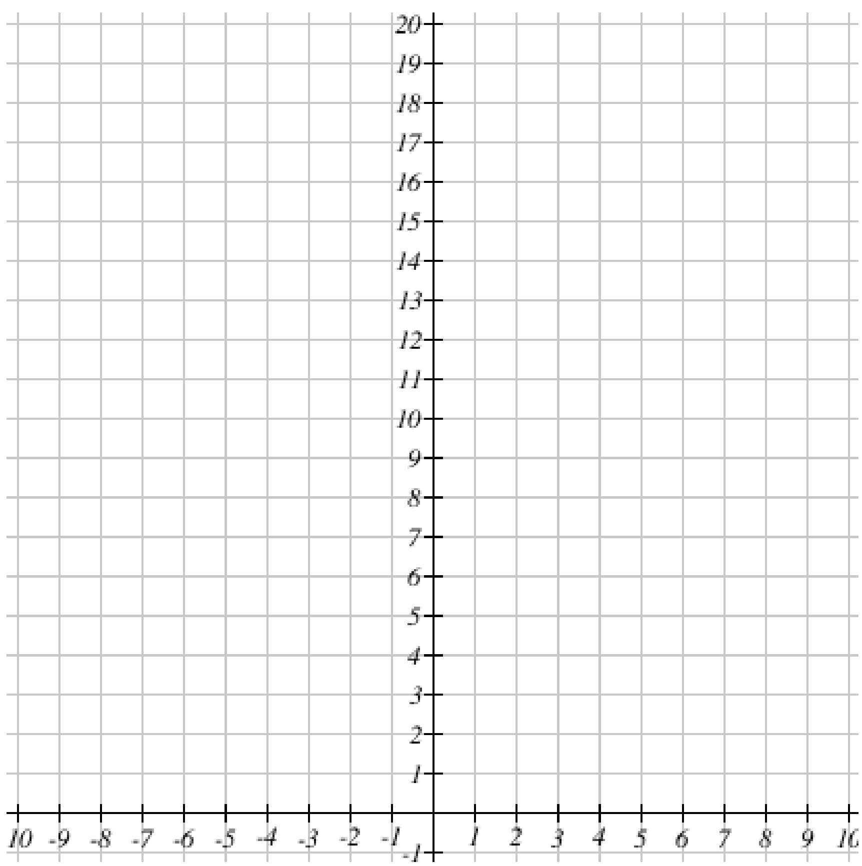
Because the asymptote is always one unit lower than this point, we now know the asymptote is $y = 5$.

The next point we want is at $x = -2 + 1 = -1$. At this x value we calculate $y = 8$. Because we know the asymptote is at $y = 5$, we know this point is $8 - 5 = 3$ units above the asymptote. Notice that that matches the base of 3 (i.e., the 3 in $3^{x+2} + 5$).

So from these two points, we can determine the asymptote, the base, and the vertical shift. This is enough for graphing. When you graph exponentials in exercises you will enter these two points, and the software will calculate the rest using these ideas. □

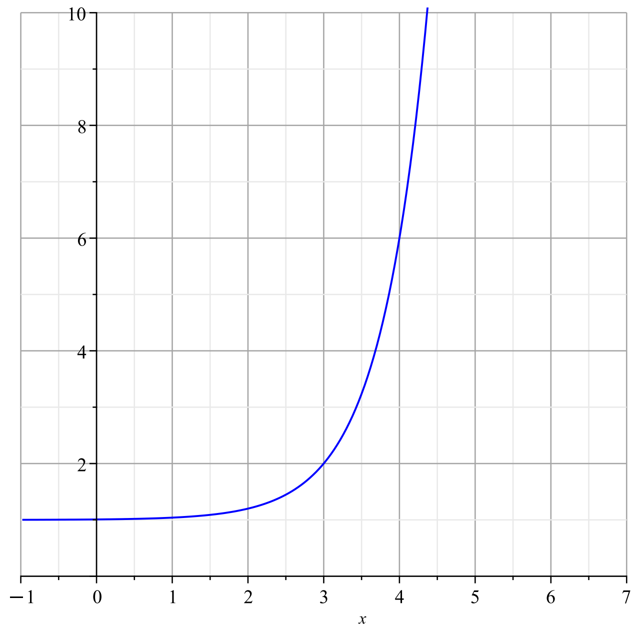
Similar ideas enable us to use just two points if we have horizontal or vertical stretches, but the explanation is more complicated than is useful in this text. We should be aware that a vertical stretch means the asymptote is not one unit beneath the first point ($x = -2$ in the example above), because the stretch, stretches that distance. For example for $3(2^{x-3})$ the asymptote is at $y = 0$, the first, convenient point is at $x = 3$ which gives $y = 3$. This is 3 above the asmyptote instead of just 1.

Checkpoint 6.2.12 Graph the equation $y = 3\left(4^{x-4}\right) + 5$. You will select a point on the asymptote (only y coordinate matters); then you select two points on the exponential.



The following examples use these ideas in reverse as you will be expected to do. We read the base, vertical shift, and horizontal shift from a graph.

Example 6.2.13 The graph below is of the equation $y = b^{x-h} + k$. Determine the values of b, h, and k.



First, we notice by looking at the left side of the graph that the asymptote is at $y = 1$. The asymptote is also the vertical shift, i.e., $k = 1$.

We know there will be a point one unit higher ($y = 2$). This occurs at $x = 3$. This is the horizontal shift $h = 3$.

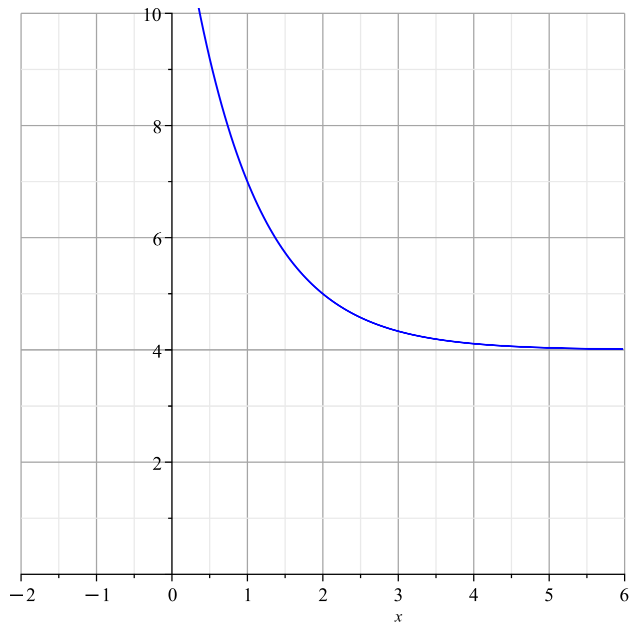
Finally, we can look one unit to the right of this point ($x = 4$). The y value there is $y = 6$. This is $6 - 1 = 5$ units above the asymptote. This is the base $b = 5$.

The equation is

$$y = 5^{x-3} + 1.$$

□

Example 6.2.14 The graph below is of the equation $y = b^{\pm(x-h)} + k$. Determine the values of b, the sign, h, and k.



First, we notice that the graph decreases to the right instead of the left. We know this is a reflection so it will be $-x$.

Second, we notice by looking at the right side of the graph that the asymptote is at $y = 4$. The asymptote is also the vertical shift, i.e., $k = 4$.

We know there will be a point one unit higher ($y = 5$). This occurs at $x = 2$. This is the horizontal shift $h = 2$.

Finally, we can look one unit to the left of this point ($x = 1$). The y value there is $y = 7$. This is $7 - 4 = 3$ units above the asymptote. This is the base $b = 3$.

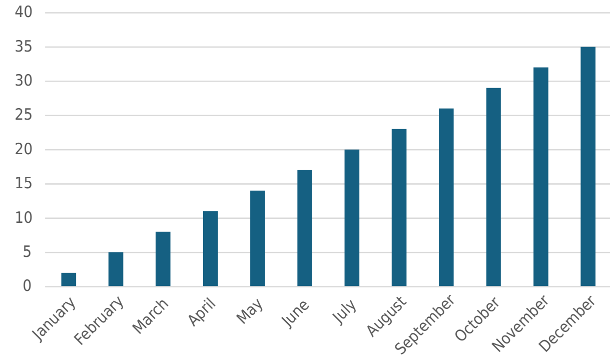
The equation is

$$y = 3^{-(x-2)} + 4.$$

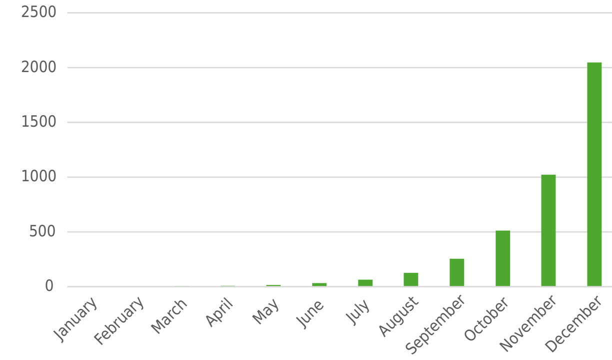
□

6.2.3 Identifying Exponential Graphs

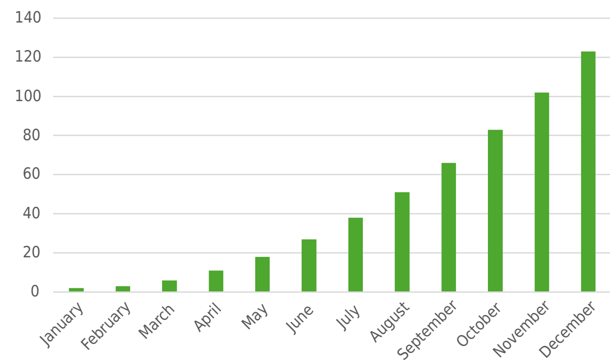
When we look at data, which may not be labeled as linear, quadratic, or exponential, how can we quickly determine which one it is? Look at the four examples below and try to determine which growth rate each has. Note one graph is a growth rate we have not covered.



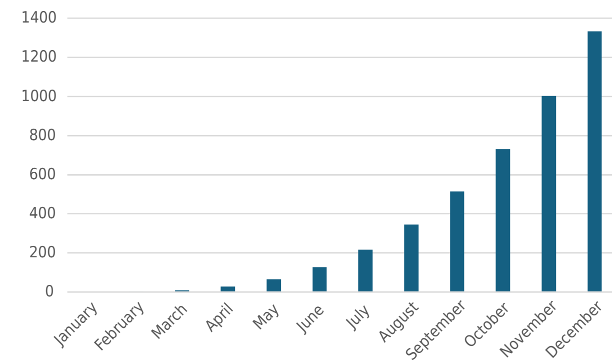
(a) Bar Chart 1



(b) Bar Chart 2



(c) Bar Chart 3



(d) Bar Chart 4

Figure 6.2.15 Identify the Rates of these Charts

A linear relationship is defined by the differences between pairs of data points being the same. On bar graphs this means the difference in height between consecutive bars must be the same. If we compare the change from January to February to the change from November to December we see that Figure 6.2.15(b)–6.2.15(d) they are vastly different (very small early and much larger later). For Figure 6.2.15(a) however, the increase between bar heights looks the same. If we wished, we could use a ruler to measure and convince ourselves. This appears to be linear.

An exponential relationship can be recognized by the ratio between pairs of data points being the same. This is hard to use on a bar graph, because it means the height of each bar is a multiple of the previous. For example each bar would be 2 or 3 times as high. This would be hard to identify if each bar is 2.3 times as high however. Rather we will use two, alternative traits.

We know that the graph of an exponential is nearly flat in one direction and very steep in the other. Figure 6.2.15(c) does not appear to flatten heading left. The bottom two graphs do have this flat appearance to the left. The other trait is that the differences of consecutive terms in an exponential relationship are the same scale as the entries. This means the change in height of consecutive bars should be on scale with the size of a bar. Consider September and October on Figure 6.2.15(b). The change in height from September to October looks like the height of the September bar. In contrast the change in height from September to October in Figure 6.2.15(d) is much less than the height of the September bar. Thus we can be convinced that the data in Figure 6.2.15(b) is exponential and none of the others are.

We identify quadratic data by looking at the second differences. To do this on a bar graph would mean constructing a bar graph where each new bar's height is the change in height of consecutive bars on the original graph. Then we would look if this new bar graph was linear. This is not a reasonable task. We would be better off asking for the original data.

The two bar graphs on the right have different relationships: one is quadratic, the other is not. Both can be described in the same terms (e.g., change in heights is increasing). This is a warning to not make many assumptions about data from limited data or simple graphs.

6.2.4 Exercises

1. **Contextless Practice.** Find the exponential function $f(x) = Ca^x$ whose graph goes through the points $(0, 3)$ and $(3, 24)$.

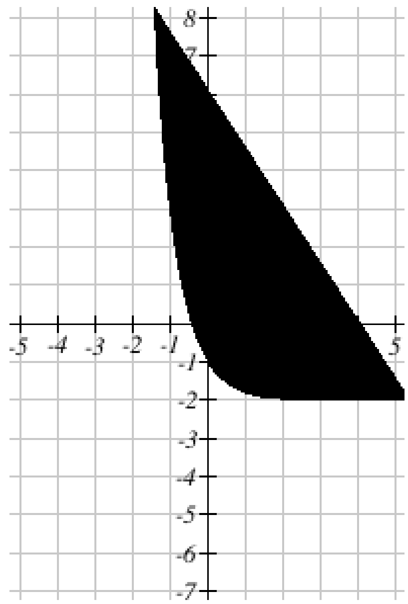
$a =$ ____________ ,

$C =$ ____________ .

2. **Contextless Practice.** Find a formula for the exponential function passing through the points (-1,2) and (3,32)

y = ________________________

3. **Contextless Practice.** The function below has the form $f(x) = b^x + k$.



Which of the following functions is shown on the graph?

(a) $f(x) = \left(\frac{1}{5}\right)^x - 2$

(b) $f(x) = \left(\frac{1}{5}\right)^x - 1$

(c) $f(x) = \left(\frac{1}{5}\right)^x + 2$

(d) $f(x) = 5^x + 2$

(e) $f(x) = 5^x - 2$

(f) $f(x) = \left(\frac{1}{5}\right)^x - 3$

(g) $f(x) = 5^x - 3$

(h) $f(x) = 5^x - 1$

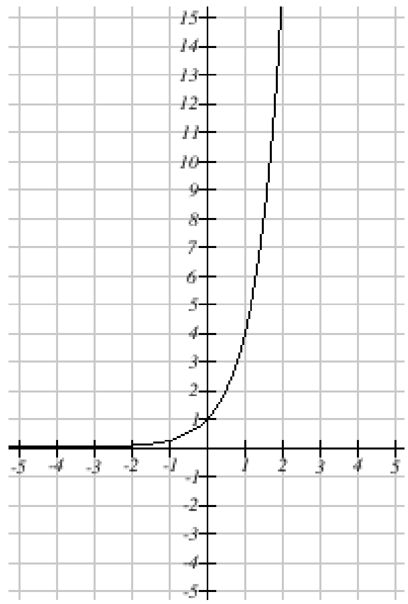
4. **Contextless Practice.** The table below shows values from the function $f(x) = b^x$. Identify b.

Table 6.2.16

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{100}$	$\frac{1}{10}$	1	10	100

$b =$ ____________________

5. Contextless Practice. What function is graphed below?



(a) $f(x) = \left(\frac{1}{4}\right)^x$

(b) $f(x) = 5^x$

(c) $f(x) = \left(\frac{1}{5}\right)^x$

(d) $f(x) = 2^x$

(e) $f(x) = 3^x$

(f) $f(x) = \left(\frac{1}{3}\right)^x$

(g) $f(x) = 4^x$

(h) $f(x) = \left(\frac{1}{2}\right)^x$

6. Contextless Practice. Determine whether the following equation represents an exponential growth or exponential decay.
$y = 266.5 \cdot (0.82)^x$

(a) exponential decay

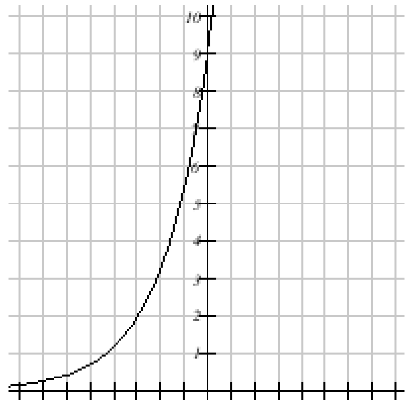
(b) exponential growth

7. Contextless Practice. Determine whether the following equation represents an exponential growth or exponential decay.
$y = 42.9 \cdot (1.83)^x$

(a) exponential decay

(b) exponential growth

8. Interpreting (Contextless). Does this graph show exponential growth, decay or neither?

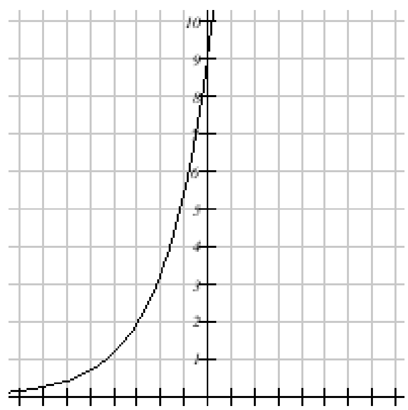


(a) Growth

(b) Decay

(c) Neither

9. Interpreting (Contextless). Does this graph show exponential growth, decay or neither?



(a) Growth

(b) Decay

(c) Neither

10. Interpreting (Contextless). Does this graph show exponential growth, decay or neither?
$7(3)^x$

(a) Growth

(b) Decay

(c) Neither

11. Interpreting (Contextless). Does this graph show exponential growth, decay or neither?
$7(3)^x$

(a) Growth

(b) Decay

(c) Neither

6.3 Logarithm Properties

This section addresses the following topics.

- Read and use mathematical models in a technical document
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Read and interpret models (critical thinking)

The previous two sections presented exponential models, and we answered some questions about applications using these models. In order to solve more problems involving exponential models we need the **logarithm** which is presented here. This section presents a definition of the logarithm as well as presenting how to graph them and one application. These latter will help us understand how it works.

Historically logarithms also have a connection to drunk pigeons and a Scotsman's bones.

6.3.1 Definition of Logarithm

The following definition is commonly used when we are using logarithms to solve equations.

Definition 6.3.1 Logarithm. $\log_a(x) = y$ if and only if $a^y = x$ ◊

This definition is simply that the logarithm is the opposite of an exponential in the same sort of way that a square root is the opposite of a square.

Example 6.3.2 Each of the following is a conversion between exponential and logarithmic notation.

(a) $\log_2(8) = 3$ is the same as $2^3 = 8$.

(b) $\log_5(1/25) = -2$ is the same as $5^{-2} = 1/25$.

(c) $\log_{10}(\sqrt{10}) = 1/2$ is the same as $10^{1/2} = \sqrt{10}$.

□

Checkpoint 6.3.3 Write $\log_5(625) = 4$ in exponential notation. ____________________

Use a calculator to confirm that the statement is true.

Checkpoint 6.3.4 Write $25^{\frac{5}{2}} = 3125$ in logarithmic notation. ____________________

6.3.2 Graphing Logarithms

The purposes of this section are to become proficient with the defition through practice, and to understand the logarithm through practice and seeing its graph.

Example 6.3.5 We will graph $y = \log_2(x)$. As before we will begin by completing a table. Before we can fill out a table we need to figure out how we can produce points.

While we would normally pick values like $x = 3$ plug that in, producing $y = \log_2(3)$, and calculating, that will not work with logs. Indeed we will never calculate this particular point. Because it is easier to evaluate exponentials than logs, we will fill out a table of related exponentials then converting the points. The related exponential here is $2^y = x$.

y	$x = 2^y$	
0	$2^0 =$	1
1	$2^1 =$	2
2	$2^2 =$	4
3	$2^3 =$	8
4	$2^4 =$	16

To produce points on the graph of this logarithm, we need only note that x and y reverse by our definition. The next table was constructed by swapping the x and y columns of the previous table.

x	$y = \log_2(x)$
1	0
2	1
4	2
8	3
16	4

Notice that as promised we do not have a point with $x = 3$. This is because our method works only on values of x that are powers of 2.

To obtain points left of $(1, 0)$ we will need to expand the table of points for the exponential. The previous table used positive y values. So we will select negative y values this time.

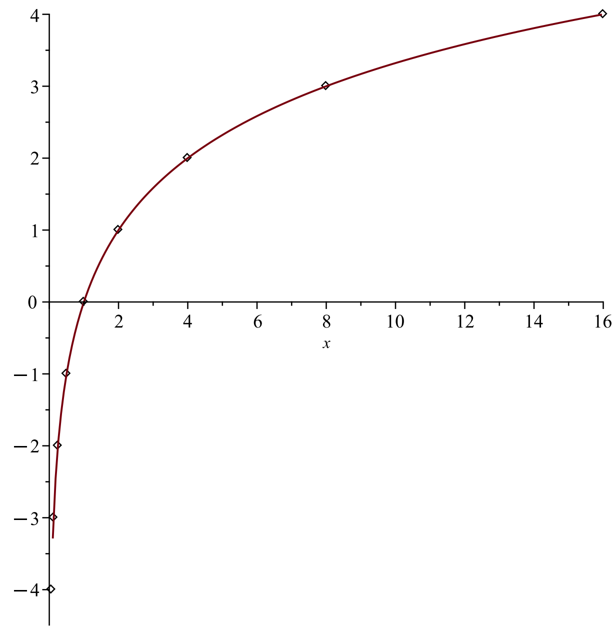
y	$x = 2^y$	
-1	$2^{-1} =$	$1/2$
-2	$2^{-2} =$	$1/4$
-3	$2^{-3} =$	$1/8$
-4	$2^{-4} =$	$1/16$

Again we produce points on the graph of the logarithm by swapping the x and y coordinates.

x	$y = \log_2(x)$
$1/2$	-1
$1/4$	-2
$1/8$	-3
$1/16$	-4

These are points to the left of the previous ones. However, there are none with negative x values. To understand why consider what a negative x value implies. Consider $\log_2(-4) = y$ means $2^y = -4$. However, now matter how many times we multiply positive 2 it will never be negative. Therefore negative x values do not make sense in a logarithm.

The graph of $y = \log_2(x)$ based on these points is below.



□

Checkpoint 6.3.6 Following the previous example construct a table for $x = 3^y$, then use it to construct a table for $y = \log_3(x)$. Finally, graph this logarithm.

Next we consider the shape of the graph and what it tells us about logarithms. The first question is how high the logarithm curve grows. To explain this question consider Figure 6.3.7 below. As we look from left to right both curves are always increasing. Both curves increase by less as we go to the right. However, the lower (blue, dashed) curve levels off. That is it never increases above $y = 1$. We want to know if the logarithm curve ever levels off.

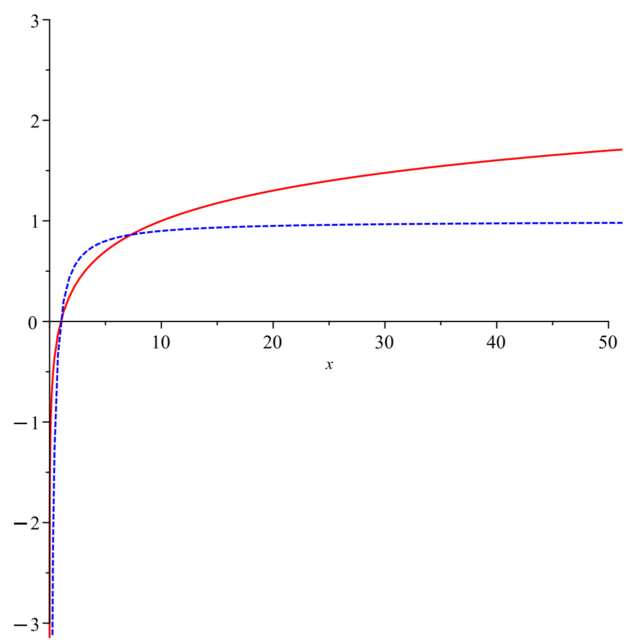


Figure 6.3.7 Graph of two always increasing curves

Example 6.3.8 Consider the example $y = \log_2(x)$. We know from the table above that it grows above $y = 1$. Consider whether it grows above $y = 100$. This would mean $100 < 2^x$. We know it is easier to calculate this logarithm for powers of 2 so instead we consider

$$100 < 128 = 2^7.$$

Does it grow above $y = 1000$. We can use

$$1000 < 1024 = 2^{10}.$$

By now we can see a pattern. For any y value we select there is a power of 2 that is greater (we can double forever). The logarithm evaluated at that power is bigger. Thus we know the logarithm increases forever. This is not dependent on the base 2. The same statement is true of any number (e.g., 3^x and 29^x increase forever as well).

In practice this means we should draw the graph of any logarithm with the curve pointing slightly up at the end. □

We claimed above that the two graphs in Figure 6.3.7 increase more slowly as they continue to the right. The next example explains why this is true of the graph of a logarithm.

Example 6.3.9 Looking at the table of points for $y = \log_2(x)$ in Example 6.3.5 we note that the curve increased by 1 in height from $x = 1$ to $x = 2$. The next increase of 1 in height is at $x = 4$. The next one is at $x = 8$. This is a jump of 1 then 2 then 4. Each time the distance we have to travel to see an increase in height of 1 unit doubles. This means the rate of increase was $1, 1/2, 1/4$. This is a decreasing sequences, so while the logarithm increases forever, it does so increasingly slowly. □

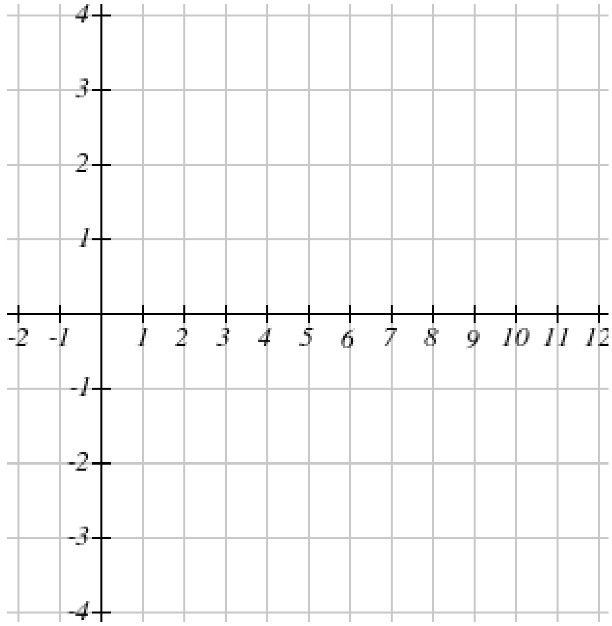
The final trait of the graph of a logarithm to consider is what happens as the graph goes to the left. We have already established that it is not defined for negative x values. However, the graph does not have a starting point (first point on the left). The next example illustrates the asymptotic nature of the left part of the graph.

Example 6.3.10 First consider the question, is there an x value such that $0 = 2^x$? We know that $2^0 = 1$. If we plug in negative values we obtain fractions, e.g., $2^{-3} = 1/2^3 = 1/8$. If we use a bigger negative number (e.g., -29) we end up with a smaller fraction ($2^{-29} = 1/2^{29}$). No matter how far we go, 0 is never the result. Because of this $\log_2(0)$ is not defined.

The calculations above also illustrated that we can get as close to zero as we want (just pick a big enough negative number). This means that just as the exponential curve has an asymptote to the left (approaches $y = 0$), so the logarithm curve has an asymptote to the left, namely $x = 0$. □

Putting these traits together we realize the graph of any logarithm can be determined by the vertical asymptote and two points. This matches exponential graphs. From the left most point we draw asymptotically toward the vertical asymptote. To the right we draw with a slight increase.

Checkpoint 6.3.11 Graph $y = \log_4(x)$ by first placing the vertical asymptote.



6.3.3 Special Logarithms

Some logarithms occur sufficiently frequently in applications that they have their own notation.

Logarithms were developed to manage large numbers base 10. Thus we call base 10 logarithms: **common logs**. This is written without the base. For example $\log(100) = 2$ is the same as $\log_{10}(100) = 2$.

In science and from mathematics we need another log called the **natural logarithm**. This is written as $\ln(x)$ (for logarithm natural). Natural logarithms are paired with the base e. So $\ln(5) = y$ is the same as $e^y = 5$. e is a naturally occuring constant. You do not need to memorize an approximation (your calculator can handle that for you). For the curious $e \approx 2.1718281828$.

Checkpoint 6.3.12 Evaluate the following logarithms using a device. Round to 3 decimal places.

$\log\left(1.90 \times 10^{36}\right) =$ ____________________

$\ln(2.54) =$ ____________________

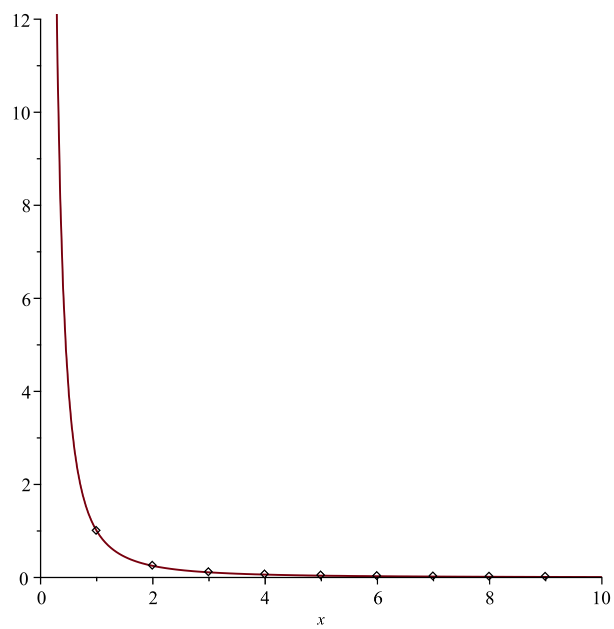
6.3.4 Logarithmic Scale

This section demonstrates how logarithms can be used to better analyze data. The technique is to modify how we graph, specifically how we label one axis.

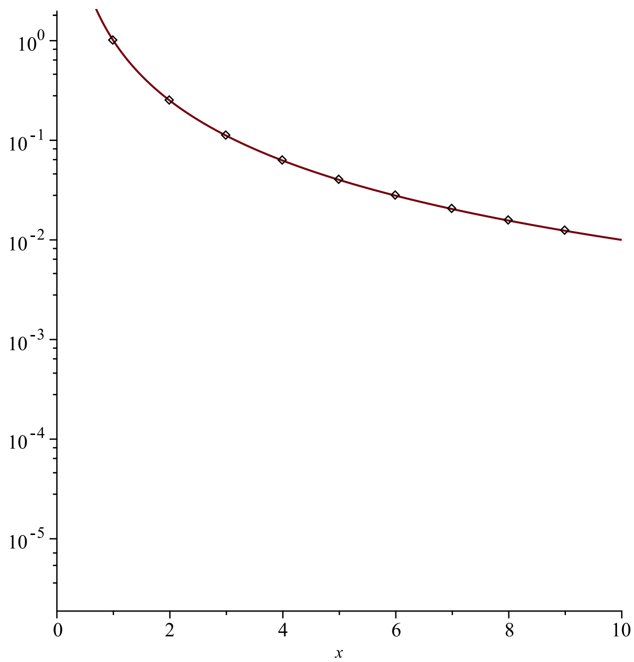
Often the scales on the two axes are different. Consider the graph of $y = \log_2(x)$ in Example 6.3.5 above. The x-axis is labeled 0 to 16 and the y-axis is labeled - 4 to 4. That means the x-axis, though it takes up the same length of area as the y-axis represents 16 units to the y-axis' 8 units. We do this because many graphs would otherwise be too tall or too wide to fit.

What these graphs do require is that each tick mark represents the same number of units. For example on the graph of this log, every tick mark on the x-axis represents one unit (only half are labeled). On the y-axis every tick mark represents 1/2 unit (only half are labeled). Sometimes this requirement means it is hard to read a graph.

Consider the two graphs in Figure 6.3.13. Note they are the same data drawn with different scales. Write down coordinates for the first three points. Is it easier to be precise using the version on the left (normal scale) or the version on the right (log scale)?



(a) Normal Scale



(b) Log Scale

Figure 6.3.13 Comparison of Scales

The graph on the left uses the normal (linear) scale. Specifically on both the x and y axes each tick represents the same difference (add 1 unit for each tick or 2 units for each label). The graph on the right is the same data using a logarithmic scale. The x axis is still linear (same as the left), but the y axes is labeled such that each tick is the same product (multiply by 10 for each label). Basing a scale on multiplying rather than adding makes this match exponentials. It is called a log scale because it looks like graphing $(x, \log(y))$.

When is this scale useful? Any time data is growing by a multiple rather. This includes anything exponential. Recall that exponential growth means growing by a fixed percent. Signal strength in electronics is an example. For example consider how much of the signal is lost as it passes through a long cable. The resistance of the cable will cause a fixed percent to be lost rather than a fixed amount. A stronger signal will lose more (but still be stronger).

Consider an electronic amplifier: a device to increase intensity of sound by increasing the power of the signal. The mechanism used multiplies the incoming signal power by some factor. This enables your media devices to send out a very weak audio signal, but the speakers to receive a lot more power which they need to move enough air to be heard. Based on this design if more than one amplifier is connected in series, each one multiplies the signal strength from the previous one.

Studies of the human brain indicate that our brains interpret sensory data using a logarithmic scale. That is, we perceive things based on how much of a multiple they are of current condition rather than how much is added. For example, a 10 degree temperature change feels like more of change if the initial temperature is 20° F (when it is half of the current value) than if it is 50° F (now only one fifth of the current value).

6.3.5 Exercises

Exercise Group. Calculate using logarithms

1. **Notation.** Write the following logarithmic equation in exponential form

$$\log_3\left(\frac{1}{81}\right) = -4$$

____

2. **Notation.** Write the following logarithmic equation in exponential form

$\log_3\left(\frac{1}{81}\right) = -4$

____

3. Notation. Write the following exponential equation in logarithmic form

$8^{-1} = \frac{1}{8}$

____

4. Notation. Write the following exponential equation in logarithmic form

$8^{-1} = \frac{1}{8}$

____

5. Notation. Evaluate the following expressions.

(a) $\log_4 4^2 =$ __________

(b) $\log_3 81 =$ __________

(c) $\log_5 125 =$ __________

(d) $\log_7 7^{14} =$ __________

6. Evaluate. Evaluate using your calculator, giving at least 3 decimal places:

$\log(390) =$ ____________________

7. Evaluate. Evaluate $\ln(0.068)$. Give your answer to 4 decimal places.

$\ln(0.068) =$ ____________________

8. Notation. Express each equation in logarithmic form.

(a) $e^x = 5$ is equivalent to the logarithmic equation: ____________________

(b) $e^4 = x$ is equivalent to the logarithmic equation: ____________________

9. Notation. Find the logarithm.

$\log_5\left(\frac{1}{5}\right) =$ ____________________

10. Solve. Solve: $\log_4(m) = 3$

Give an exact answer. Simplify as much as possible.

$m =$ ____________________

11. Solve. Solve: $\log_5(y) = 7$

$y =$ ____________________

12. Solve. If $\log_2(4x + 3) = 4$, then $x =$ ____________________ .

13. Solve. Solve each equation for x. If needed, first convert from exponential to logarithmic form. *Round answers to 3 decimal places. If there is no solution enter* DNE.

(a) $10^x = 7548$

$x =$ _____

(b) $10^x = 0.002$

$x =$ _____

(c) $10^x = -2$

$x =$ _____

(d) $10^x = 0$

$x =$ _____

14. Solve. Solve for x. Round answers to four decimal places.

$\ln(x) = 6$

____

15. Solve. Solve by taking the logarithm of each side. *Round answer to 3 decimal places.*

$6^w = 1730$

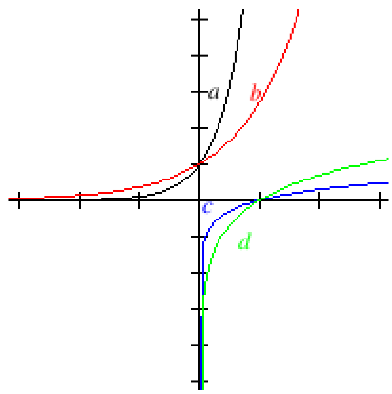
$w =$ ________

16. Solve. Solve by taking the logarithm of each side. *Round answer to 3 decimal places.*
$600(1.3)^x = 52,620$
$x =$ ________

17. Application. The Richter Scale reading, R, of an earthquake is based on a logarithmic equation. Suppose:
$R = \log\left(\frac{A}{A_0}\right)$
where
A - is the measure of the amplitude of the earthquake wave, and
A_0 - is the amplitude of the smallest detectable wave (or standard wave).
Suppose an earthquake is measured with a wave amplitude $A = 0.2011$ while the smallest detectable wave A_0 is measured at 0.0001 cm. What is the magnitude of this earthquake using the Richter scale, to the nearest tenth?
The earthquake registered ____________________ on the Richter scale.

Exercise Group. Graph Logarithms

18. Graphs.



Match each equation with a graph above

(a) $\log(x)$

(b) e^x

(c) $\ln(x)$

(d) 10^x

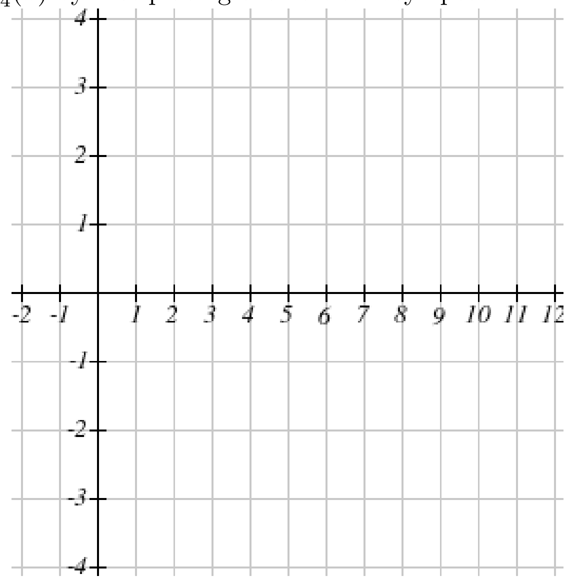
(a) black

(b) red

(c) blue

(d) green

19. Graphs. Graph $y = \log_4(x)$ by first placing the vertical asymptote.

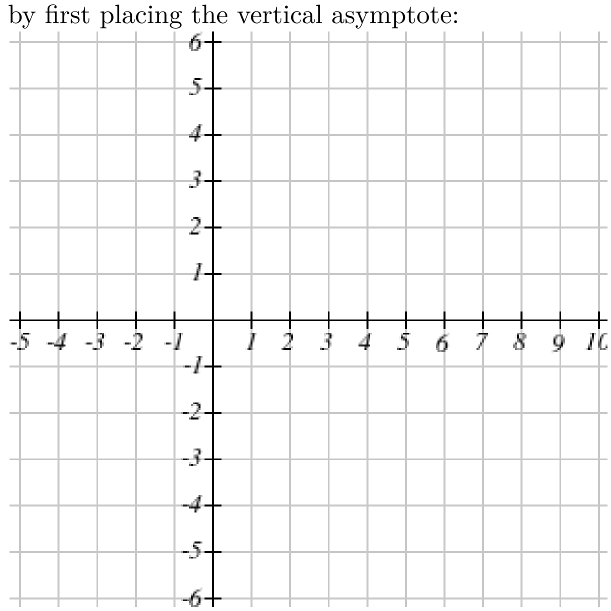


20. Graphs. Fill in the table of values for the function $f(x) = \log_4(x+2)$. Enter *DNE* if the answer does not exist.

Table 6.3.14

x	$f(x)$
-6	____
-1	____
2	____

Graph the function by first placing the vertical asymptote:

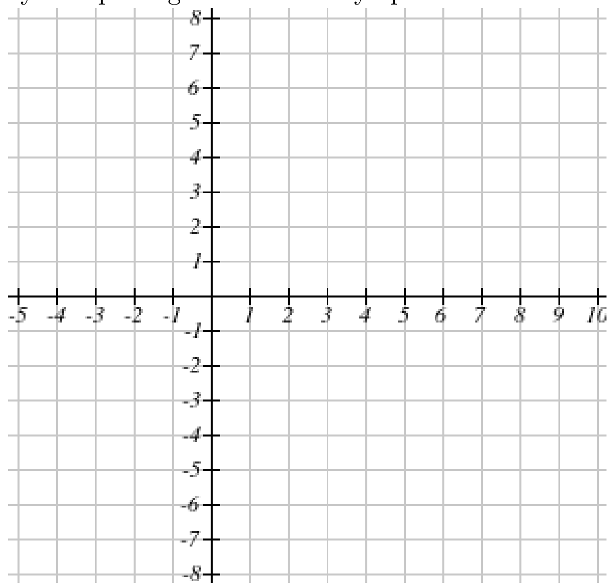


21. Graphs. Fill in the table of values for the function $f(x) = \log_2(x) - 2$. Enter *DNE* if the answer does not exist.

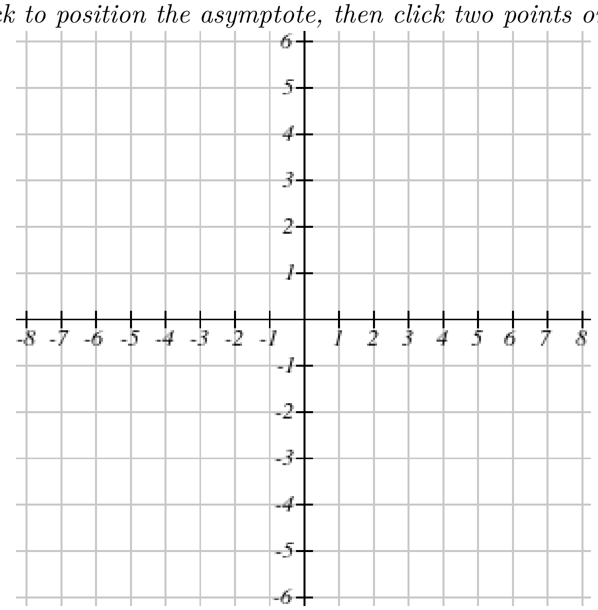
Table 6.3.15

x	$f(x)$
0	____
1	____
2	____

Graph the function by first placing the vertical asymptote:



22. Graphs. Graph the function $y = \log_2(x)$
Tool help: First click to position the asymptote, then click two points on the graph.



6.4 Solving Equations Using Logarithm

This section addresses the following topics.

- Read and use mathematical models in a technical document

- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Solve linear, rational, quadratic, and exponential equations and formulas (skill)
- Read and interpret models (critical thinking)
- Use models including linear, quadratic, exponential/logarithmic, and trigonometric (skill)

This section presents methods for solving equations that have an exponential by using the relationship with the logarithm, and also how to solve some equations with logarithms.

6.4.1 Solving Equations with Exponentials

The first two examples demonstrate how we can use the inverse relationship between exponentials and logarithms to solve an equation with an exponential.

Example 6.4.1 Solve $6.0 = 10^{\frac{3}{2}(x+10.7)}$.

Order of operations dictates the following

$$6.0 = 10^{\frac{3}{2}(x+10.7)}.$$

Use a log to undo an exponential.

$$\begin{aligned} \log(6.0) &= \log\left(10^{\frac{3}{2}(x+10.7)}\right). \\ \log(6.0) &= \frac{3}{2}(x+10.7). \\ \frac{2}{3}\log(6.0) &= x+10.7. \\ 0.51876750 &\approx x+10.7. \\ 0.51876750 - 10.7 &= x. \\ -10.18123250 &\approx x. \\ -10.181 &\approx x. \end{aligned}$$

Rounding here was arbitrarily chosen to be 3 decimal places, because there is no context. □

Example 6.4.2 Solve $13 = 2 + 3e^{x-1}$.

Order of operations dictates the following

$$\begin{aligned} 13 &= 2+3e^{x-1}. \\ 11 &= 3e^{x-1}. \\ \frac{11}{3} &= e^{x-1} \end{aligned}$$

Use a ln to undo an exponential.

$$\begin{aligned} \ln\left(\frac{11}{3}\right) &= \ln\left(e^{x-1}\right). \\ \ln\left(\frac{11}{3}\right) &= x-1. \\ 1.29928298 &= x-1. \\ 1.29928298+1 &= x. \\ 2.29928298 &\approx x. \\ 2.299 &\approx x. \end{aligned}$$

Rounding here was arbitrarily chosen to be 3 decimal places, because there is no context. □

Checkpoint 6.4.3 Solve the equation: $14\left(10^{2x}\right) - 1 = 0$.

$x =$______

Round to 3 decimal places.

The next two examples show us how to solve equations involving exponentials other than the common (base 10) or natural (base e).

Example 6.4.4 Solve $5.0 = 3.0\left(2^{\frac{t}{70}}\right)$.

Because the exponential is base 2, we will use a log base 2.

$$\begin{aligned} 5.0 &= 3.0\left(2^{\frac{t}{70}}\right). \\ 5.0/3.0 &= 2^{\frac{t}{70}}. \\ \log_2(5.0/3.0) &= \log_2\left(2^{\frac{t}{70}}\right). \\ \log_2(5.0/3.0) &= \frac{t}{7\bar{0}}. \\ 7\bar{0}\log_2(5.0/3.0) &= t. \end{aligned}$$

Standalone

□

Your calculator likely does not have a button for calculating $\log_2(x)$. We can use a property of all logarithms to solve this equation with the natural log. In general

$$\log_b(a^p) = p\log_b(a).$$

Example 6.4.5 Solve $5.0 = 3.0\left(2^{\frac{t}{70}}\right)$. Round to units.

$$\begin{aligned} 5.0 &= 3.0\left(2^{\frac{t}{70}}\right). \\ 5.0/3.0 &= 2^{\frac{t}{70}}. \\ \ln(5.0/3.0) &= \ln\left(2^{\frac{t}{70}}\right). \\ \ln(5.0/3.0) &= \frac{t}{7\bar{0}}\cdot\ln(2). \\ \frac{7\bar{0}\ln(5.0/3.0)}{\ln(2)} &= t. \\ 51.58759159 &\approx t \\ 52 &\approx t \end{aligned}$$

Your calculator does have a button for $\ln(x)$. Note this implies that $\log_2(5.0/3.0) = \frac{\ln(5.0/3.0)}{\ln(2)}$. This relationship works for logs of any base. □

Checkpoint 6.4.6 nobelium-252 has a half-life of 2.27 seconds, and its decay is modeled by $A = A_0\left(\frac{1}{2}\right)^{\frac{t}{2.27}}$. If the initial amount is $A_0 = 5.5$, how long will it take until only one third, approximately $A \approx 1.8$ is left? ____

6.4.2 Solving Equations with Logarithms

As with equations involving exponentials, we can solve equations involving logarithms using the inverse relationship between exponentials and logarithms. The first two examples demonstrate using the definition of logarithm.

Example 6.4.7 Solve

$$\log_3(x) = 2.$$

We can re-write this as

$$3^2 = x$$

which tells us that $x = 9$. □

Example 6.4.8 Solve $14 = 12 + \log(5x)$.

First we combine the terms outside the logarithm, the we re-write that as an exponential.

$$\begin{aligned} 14 &= 12 + \log(5x) \\ 2 &= \log(5x) \end{aligned}$$

Re-write the logarithm as an exponential.

$$\begin{aligned} 10^2 &= 5x \\ 100 &= 5x \\ 20 &= x \end{aligned}$$

□

Example 6.4.9 Solve $7 = \ln(3x + 2)$. Round to units.

Solution. We re-write this as an exponential.

$$\begin{aligned} 7 &= \ln(3x+2) \\ e^7 &= 3x + 2 \\ e^7 - 2 &= 3x \\ \frac{1}{3}(e^7 - 2) &= x \\ 365 &\approx x \end{aligned}$$

□

Checkpoint 6.4.10 Solve $6.8 = \left(\frac{2}{3}\right)\log(M_0) - 10.7$.

$M_0 =$ ____________________

6.4.3 Applications with Exponentials and Logarithms

Example 6.4.11 In Example 6.1.16 we produced the model $P = 3.0 \cdot 2^{t/7\bar{0}}$. Here we will redo this problem using the model $P = P_0e^{kt}$.

The bacteria **lactobacilus acidophilus** doubles in population every 70 minutes. If the initial amount was 3 grams, what how much would there be after 24 hours?

To use this model we must first calculate the constant k. We know the amount doubles every 70 minutes, so we can start with the following.

$$\begin{aligned}
6.0 &= 3.0e^{7\bar{0}k}.\\
\frac{6.0}{3.0} &= e^{7\bar{0}k}.\\
2.0 &= e^{7\bar{0}k}.\\
\ln(2.0) &= \ln\left(e^{7\bar{0}k}\right).\\
\ln(2.0) &= 7\bar{0}k.\\
\frac{\ln(2.0)}{7\bar{0}} &= k.\\
k &\approx 0.009902.\\
&\approx 0.0099.
\end{aligned}$$

Using this value we can now calculate the value after 24 hours. Note 24 hours is $24 \cdot 60 = 1440$ minutes.

$$\begin{aligned}
P &= 3.0e^{(144\bar{0})(0.0099)}\\
&\approx 4.674533843 \times 10^6\\
&\approx 4.7 \times 10^6
\end{aligned}$$

This is $4.7 \times 10^3 \approx 47,000$ kilograms. Naturally, this is an unreasonable prediction. This tells us there must be other factors in bacteria growth. □

Example 6.4.12 Plutonium-241 has a half-life of 14.4 years. This means if you start with 10.0 g of Pu-241 in 14.4 years there will be only 5.0 g of Pu-241. Generally, this can also be modeled by

$$P = P_0e^{kt}.$$

P_0 is the initial amount of material. P is the amount left after t units of time. k is a constant that is derives from the speed how fast the material decays.

(a) Write the model for Plutonium-241.

Solution. We must first find k. We can use Example 6.4.1 as an example.

$$\begin{aligned}
5.0 &= 10.0e^{k(14.4)}\\
\frac{1}{2.0} &= e^{k(14.4)}\\
\ln\left(\frac{1}{2.0}\right) &= \ln\left(e^{k(14.4)}\right)\\
\ln\left(\frac{1}{2.0}\right) &= k(14.4).\\
\frac{1}{14.4}\ln\left(\frac{1}{2.0}\right) &= k\\
-0.04813522 &\approx k.\\
-0.048 &\approx k.
\end{aligned}$$

Thus the model is

$$P = P_0e^{-0.048t}.$$

(b) If a lab has 12.0g of Pu-241, how much will be left in 6 years?

Solution. We use the model from the previous step.

$$P = 12.0e^{-0.048(6)}$$

$$\approx 8.997139$$
$$\approx 9.0.$$

$\square$

Note, that the example in Example 6.1.16 and Example 6.4.11 imply that $3.0 \cdot 2^{t/70} = 3.0e^{(0.0099)t}$. In particular we can convert the power of 2 to a power of e. More generally, we can write $2^x = e^{kx}$ or $3^x = e^{kx}$ or similar for some value of k. The next example shows how we can perform this conversion.

Example 6.4.13 Write 2^x as e^{kx}. Numbers are exact.

$$\begin{aligned}
2^x &= e^{kx}.\\
\ln(2^x) &= \ln(e^{kx}).\\
x\ln(2) &= (kx)\ln(e).\\
x\ln(2) &= kx.\\
\frac{x\ln(2)}{x} &= \frac{kx}{x}.\\
\ln(2) &= k
\end{aligned}$$

Thus $2^x = e^{x\ln(2)}$. $\square$

Definition 6.4.14 pH (percent hydrogen). Acidity is measured in pH (percent hydrogen). The calculation is

$$\text{pH} = -\log(H_3O^+)$$

where H_3O^+ is the concentration of hydronium ions per mole. This is obtained experimentally. $\lozenge$

Example 6.4.15

(a) A solution of hydrochloric acid has a concentration of 0.0025. Find its pH.

$$\begin{aligned}
\text{pH} &= -\log(0.0025)\\
&\approx 2.6.
\end{aligned}$$

(b) Sweat has a pH between 4.5 and 7. Suppose sweat is measured to have a pH of 5.3. Determine the concentration of ions.

We setup the pH calculation and solve using Example 6.4.9.

$$\begin{aligned}
5.3 &= -\log(c)\\
-5.3 &= \log(c)\\
10^{-5.3} &= 10^{\log(c)}\\
10^{-5.3} &= c\\
5.0 \times 10^{-6} &\approx c
\end{aligned}$$

$\square$

What is the purpose behind defining pH using a log? What does it do, that simply giving the concentration of H_3O^+ does not? The next example illustrates what the use of a log adds.

Example 6.4.16

(a) Calculate the concentration of H_3O^+ for pH levels of 5, 6, 7 (all within the range of human sweat).

Solution.

$$\begin{aligned}5 &= -\log(c_5).\\10^{-5} &= c_5.\\6 &= -\log(c_6).\\10^{-6} &= c_6.\\7 &= -\log(c_7).\\10^{-7} &= c_7.\end{aligned}$$

(b) What is the ratio of the concentration from pH of 5 to 6? of 6 to 7? of 5 to 7? This means calculate the ratios c_5/c_6, c_6/c_7, and c_5/c_7.

Solution.

$$\begin{aligned}\frac{c_5}{c_6} &= \frac{10^{-5}}{10^{-6}} = 10.\\\frac{c_6}{c_7} &= \frac{10^{-6}}{10^{-7}} = 10.\\\frac{c_5}{c_7} &= \frac{10^{-5}}{10^{-7}} = 100.\end{aligned}$$

□

We see that a change of one pH means the substance is 10 times as strong. A change of 2 pH means the substance is $10^2 = 100$ times as strong. The log scale gives us growth as a ratio.

Definition 6.4.17 Moment Magnitude. Larger earthquakes today are measured and reported using the **moment magnitude** scale. This is calculated via

$$M_w = \frac{2}{3}\log(M_0) - 10.7$$

where M_w is the moment magnitude, and M_0 is the seismic moment in Newtons per meter (a measure of energy). ◊

Example 6.4.18

(a) Based on seismic readings $M_0 = 7.2 \times 10^{22}$. What was the moment magnitude? These are rounded to one decimal place.

Solution. Using the formula we obtain

$$\begin{aligned}M_w &= \frac{2}{3}\log(7.2 \times 10^{22}) - 10.7\\&\approx 4.53822167\\&\approx 4.5\end{aligned}$$

(b) What was the seismic moment for an earthquake with magnitude 7.1?

Solution. We setup the calculation and solve using Example 6.4.9.

$$\begin{aligned}7.1 &= \frac{2}{3}\log(M_0) - 10.7\\17.8 &= \frac{2}{3}\log(M_0)\\26.7 &= \log(M_0)\\10^{26.7} &= M_0\end{aligned}$$

$$5.011872336 \times 10^{26} \approx M_0$$
$$5.01 \times 10^{26} \approx M_0$$

□

Just as with pH, the moment magnitude seismic scale enables us to compare how much stronger one earthquake is than another. The 2/3 means that a change of 1 is not 10 times as strong but a different ratio. Can you figure out what that ratio is?

6.4.4 Exercises

Exercise Group. Solve equations with logarithms and exponentials

1. **Contextless.** Solve for x: $5^x = 44$
   $x =$ ____________________

2. **Contextless.** Find the solution of the exponential equation
   $1000(1.02)^{2t} = 50,000$
   in terms of logarithms, or correct to four decimal places.
   $t =$ ____________________

3. **Contextless.** Find the solution of the exponential equation
   $2e^x = 15$
   in terms of logarithms, or correct to four decimal places.
   $x =$ ____________________

4. **Contextless.** Find the solution of the exponential equation
   $e^{1-4x} = 4$
   in terms of logarithms, or correct to four decimal places.
   $x =$ ____________________

5. **Contextless.** Solve for x:
   $10^{3x-4} = 2^{4x-5}$
   $x =$ __________ .

6. **Contextless.** Solve.
   $\log_4(t) = 6$
   $t =$____________________

7. **Contextless.** Solve *exactly*.
   $\ln(-3x) = 67$
   $x =$____________________

8. **Contextless.** Solve *exactly*.
   $11\log_6(x) = 1$
   $x =$____________________

9. **Contextless.** Solve for x
   $\log_4(4x + 2) = 2$
   ____

Exercise Group. Use logarithms and exponentials to work with applications.

10. The pH scale for acidity is defined by $\mathrm{pH} = -\log_{10}\left[\mathrm{H}^+\right]$ where $\left[\mathrm{H}^+\right]$is the concentration of hydrogen ions measured in moles per liter (M).
    A solution has a pH of 2.2.
    Calculate the concentration of hydrogen ions in moles per liter (M).
    The concentration of hydrogen ions is __________ moles per liter.

11. The number of bacteria in a culture is given by the function
    $n(t) = 925e^{0.2t}$

where t is measured in hours.
(a) What is the relative rate of growth of this bacterium population?
Your answer is __________ percent
(b) What is the initial population of the culture (at t=0)?
Your answer is _______________
(c) How many bacteria will the culture contain at time t=5?
Your answer is _______________

12. A population of bacteria is growing according to the equation $P(t) = 1600e^{0.16t}$. Estimate when the population will exceed 2545.
t= ____________________
Give your answer accurate to one decimal place.

13. An unknown radioactive element decays into non-radioactive substances. In 240 days the radioactivity of a sample decreases by 70 percent.
(a) What is the half-life of the element?
half-life: ______________________________ (days)
(b) How long will it take for a sample of 100 mg to decay to 61 mg?
time needed: ____ (days)

14. A cell of some bacteria divides into two cells every 20 minutes.The initial population is 5 bacteria.
(a) Find the size of the population after t hours
$y(t) =$ ____
(function of t)
(b) Find the size of the population after 3 hours.
$y(3) =$ ____
(c) When will the population reach 15?
$T =$ ____

15. Diseases tend to spread according to the exponential growth model. In the early days of AIDS, the growth factor (i.e. common ratio; growth multiplier) was around 2.2. In 1983, about 1900 people in the U.S. died of AIDS. If the trend had continued unchecked, how many people would have died from AIDS in 2005?
____________________ people
(Note: once diseases become widespread, they start to behave more like logistic growth, but don't worry about that for the purpose of this exercise)

16. A native wolf species has been reintroduced into a national forest. Originally 250 wolves were transplanted, and after 3 years the population had grown to 710 wolves. If the population grows exponentially according to the formula $P_t = P_0(1 + r)^t$
(a) Find the growth rate. Round your answer to the nearest tenth of a percent.
r = ____________________%
(b) If this trend continues, how many wolves will there be in ten years?
____________________ wolves
(c) If this trend continues, how long will it take for the population to grow to 1000 wolves? Round your answer to the nearest tenth of a year.
____________________ years

17. A wooden artifact from an ancient tomb contains 45 percent of the carbon-14 that is present in living trees. How long ago, to the nearest year, was the artifact made? (The half-life of carbon-14 is 5730 years.)
_______________ years.

18. The half-life of strontium-90 is 28 years. How long will it take a 44 mg sample to decay to a mass of 11 mg?
Your answer is _______________ years.

19. The half-life of Palladium-100 is 4 days. After 20 days a sample of Palladium-100 has been reduced to a mass of 6 mg.

What was the initial mass (in mg) of the sample? ____________________
What is the mass 8 weeks after the start? ____________________

20. The half-life of caffeine in the human body is about 6.1 hours. A cup of coffee has about 115 mg of caffeine.

(a) Write an equation for the amount of caffeine in a person's body after drinking a cup of coffee? Let C be the milligrams of caffeine in the body after t hours. ____________________

(b) How much caffeine will remain after 10 hours? ___ mg. State your answer to the nearest hundredth of a mg.

(c) How long until there are only 20 mg remaining? ___ hours. State your answer to the nearest hundredth of an hour.

21. Many substances are metabolized by our body so that the amount of the substance in our system decreases exponentially. Sometimes studies state this in terms of "half-life", and sometimes as an hourly rate of decrease.

$$A \cdot \left(\frac{1}{2}\right)^{\frac{t}{HL}} = A \cdot (1-r)^t$$

Where A is the initial amount, HL is the half-life in hours, t is the time in hours, and r is the hourly rate of decrease.

Suppose a substance has a half-life of 4 hours.
What is the hourly decay rate, to the nearest tenth of a percent? _____ %

6.5 Project: Time of Death

Project 10 Estimating Time of Death. The purpose of this project is to practice reading a mathematical model, using it to calculate a result, and interpreting its features. This model involves an exponential relation.

If a person is believed to have died within a day or so of the body's discovery, it's possible to estimate the time of death using body temperatures. Isaac Newton's idea was that since hot things cool much faster than cool things, the rate of cooling is more or less proportional to the temperature of the object, resulting in an exponential decay model.

Theorem 6.5.1 Newton's Law of Cooling. *The temperature T at a time x of a cooling object follows the function*

$$T = A + Be^{kx}.$$

A is the ambient (or room) temperature. B and k are constants that depend on the object.

Suppose a forensics technician arrived at a murder scene and recorded the temperature of the surroundings as well as the body. The technician decides it is fair to assume that the room temperature has been holding steady at about 68 °F. A thermometer was placed in the liver of the corpse and the following table of values was recorded.

Table 6.5.2 Time and Temperature

Actual Time	Minutes Elapsed (x)	Temperature, T, of the Body (°F)
2:00 pm	0	85.90
2:20 pm	20	85.17
2:40 pm	40	84.47
3:00 pm	60	83.78

The key to estimating time of death is to estimate A, B, k, and x in Theorem 6.5.1.

(a) Recall that the technician thinks the room temp was 68 °F. By substituting this number and the first recording (0, 85.9) into the Cooling Equation, find $A =$ ______ and $B =$ ______.

(b) Once you know A and B, substitute some *other* data point into the equation so that k is the only variable. Solve the resulting equation and round k to 6 decimal places. Show your work. Remember the exponential must be isolated before you take the natural log of both sides.

$$k \approx \quad$$

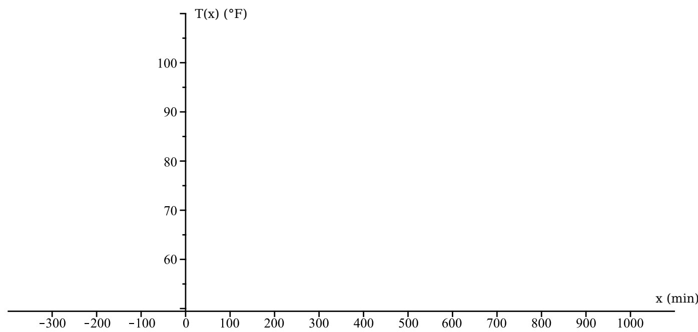
(c) Note: the number k is called the cooling (or warming) constant. If an object cools, k should be negative. Mathematically, looking at the equation, why should k be negative?

(d) Using the numbers you found for A, B, and k, write the equation for the temperature at any time x.

$T(x) =$

(e) Draw a graph of the temperature function $T(x)$. Completing the table of values may help you graph.

x	y
-300	
-100	
100	
300	
500	
700	
900	



(f) The graph of $T(x)$ has a horizontal asymptote. What is the height of this asymptote and what does it tell you about the way corpses cool?

(g) Notice that this equation deviates from reality if the x-value goes too far negative. Generally speaking, (no numbers required), at what point does the model no longer work? What in reality gives us an indication that we've taken it too far?

(h) Assuming that the temperature of the person at the time of death (TOD) was 98.6 °F, set up a TOD equation using the values of A, B, and k you've calculated. Then, solve the equation using the same logarithm method you used to solve for k.

Write your answer as a time, not just as x minutes. Recall that when $x = 0$, the time is 2:00 PM.

(i) When a forensic expert determines time of death, they often have additional information besides body temperature. Suppose a coroner finds that the person who was murdered had an infection that probably raised the core body temperature to around 102 °F. Using the same cooling constant, ambient room temperature, and temperature data as in Task 10.a and Task 10.b, make a new estimate for the time of death.

Again, write your answer as a time, not just as x minutes.

Chapter 7

Trigonometry

7.1 Trigonometric Ratios

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Read and use mathematical models in a technical document
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Use models including linear, quadratic, exponential/logarithmic, and trigonometric (skill)
- Analyze right triangles (skill)
- Identify properties of sine and cosine functions (skill)

Section 4.1 presented areas of triangles and a relationship between the three sides of a right triangle. This section and the following ones present relationships between angles of triangles and the lengths of their sides.

7.1.1 Angle Relationships in Triangles

The Pythagorean Theorem states a relationship between the side lengths of all right triangles. There is also a relationship between the three angles of all (not just right) triangles.

Theorem 7.1.1 Triangle Angle Sum. *The sum of the angles of any triangle is* $180°$.

Example 7.1.2 If two angles of a triangle are $40°$ and $70°$ what is the other angle?

The third angle must satisfy

$$\begin{aligned} 40° + 70° + \theta &= 180°. \\ \theta &= 70°. \end{aligned}$$

□

Example 7.1.3 If one angle of a right triangle is $55°$ what is the other angle?

Solution. We know the measure of one angle is $90°$ (right angle) and the measure of another angle is $55°$. Thus the third angle must satisfy

$$\begin{aligned} 90° + 55° + \theta &= 180°. \\ \theta &= 35°. \end{aligned}$$

□

Checkpoint 7.1.4 If two sides of a triangle have measures $65°$ and $56°$, what is the measure of the third angle? ____

For a right triangle each of the other two angles have measure less than a right angle. This is a result of the Triangle Angle Sum theorem. $180° - 90° = 90°$ so the remaining two angles have a sum that adds to $90°$. This implies both non-right angles are smaller than a right angle.

7.1.2 Defining Trig Functions

This section presents the trigonometric functions and demonstrates that they make sense. First, we need terminology with which to describe right triangles.

For right triangles we have names for the sides. Consider the labels in Figure 7.1.5. These names are relative to the particular non-right angles with which we are working. In this case it is the one labeled α. The **adjacent** is the side that connects the angle α and the right angle. The **opposite** is the side touching the right angle but not touching the angle (α). Both the adjacent and opposite sides are known as **legs** of the right triangle. The **hypotenuse** is opposite the right angle (the one side not touching it).

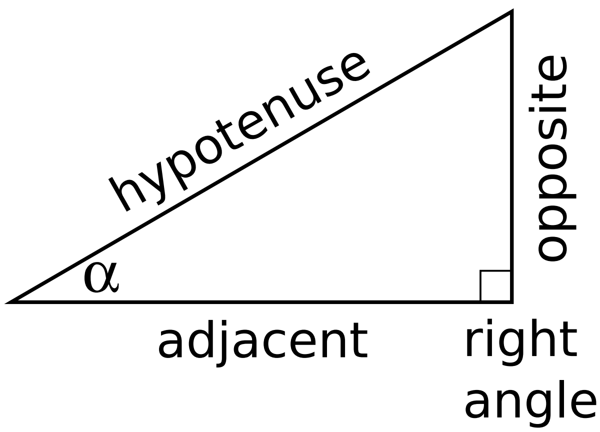


Figure 7.1.5 Right Triangle Terminology

Example 7.1.6 Consider the triangle in Figure 7.1.8. With respect to the angle α the adjacent side is the one with length 5, and the opposite side is the one with length 3.

With respect to the angle θ the adjacent side is the one with length 3, and the opposite side is the one with length 5. □

The trigonometric functions are defined below as ratios of side lengths. We will use only the first three in this text.

Table 7.1.7 Trig Functions as Ratios

sine	$\sin(\alpha) = \frac{\text{opposite}}{\text{hypotenuse}}$
cosine	$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}}$
tangent	$\tan(\alpha) = \frac{\text{opposite}}{\text{adjacent}}$
secant	$\sec(\alpha) = \frac{\text{hypotenuse}}{\text{adjacent}}$
cosecant	$\csc(\alpha) = \frac{\text{hypotenuse}}{\text{opposite}}$
cotangent	$\cot(\alpha) = \frac{\text{adjacent}}{\text{opposite}}$

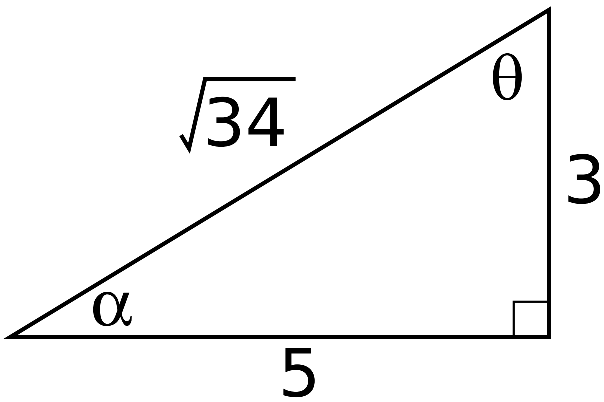


Figure 7.1.8 Right Triangle with Side Lengths

Example 7.1.9 Given the side lengths in Figure 7.1.8 what are each of the following trig ratios?

(a) $\sin(\alpha) =$

From the perspective of α, the opposite side has length 3. The hypotenuse has length $\sqrt{34}$. Thus

$$\sin(\alpha) = \frac{3}{\sqrt{34}}$$

(b) $\cos(\alpha) =$

From the perspective of α, the adjacent side has length 5. The hypotenuse has length $\sqrt{34}$. Thus

$$\cos(\alpha) = \frac{5}{\sqrt{34}}$$

(c) $\sec(\alpha) =$

$\sec(\alpha)$ flips the ratio of $\cos(\alpha) = \frac{5}{\sqrt{34}}$ (from the previous problem). Thus

$$\sec(\alpha) = \frac{\sqrt{34}}{3}$$

(d) $\sin(\theta) =$

From the perspective of θ, the opposite side has length 5. The hypotenuse has length $\sqrt{34}$. Thus

$$\sin(\theta) = \frac{5}{\sqrt{34}}$$

□

7.1.3 Solving Triangles

Now that we have defined trigonometric functions, we can use them to analyze triangles. The goal is to calculate all of the side lengths and/or angles given only some of them.

Example 7.1.10 For a right triangle with angle $\alpha = 50°$ and corresponding opposite side of length 7, what are the other side lengths and angles?

First, we know that two of the angles are 90° and 50°, so the third angle has measure $180° - 90° - 50° = 40°$.

To calculate the length of the hypotenuse recall that $sin(\alpha) = \frac{\text{opposite}}{\text{hypotenuse}}$. We know the angle and the length of the opposite.

$$\sin(50°) \approx 0.76604444. \qquad \text{Use a device to approximate.}$$

$$\begin{aligned}
\frac{7}{h} &\approx 0.76604444. && \text{Write the ratio.}\\
7 &\approx h \cdot 0.76604444. && \text{Clear the denominator.}\\
\frac{7}{0.76604444} &\approx h.\\
h &\approx 9.1378510.
\end{aligned}$$

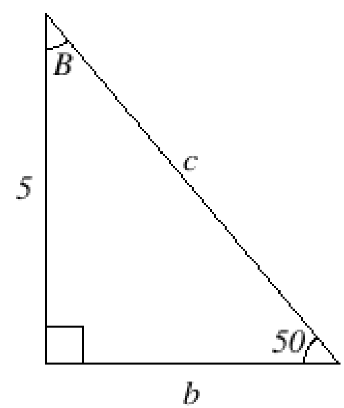
The hypotenuse has length 9.13. Now that we know two sides we can use the Pythagorean Theorem to calculate the length of the adjacent.

$$\begin{aligned}
7^2 + b^2 &= 9.1378510^2.\\
b^2 &= 9.1378510^2 - 7^2.\\
b^2 &= 34.500321.\\
\sqrt{b^2} &= \sqrt{34.500321}.\\
b &\approx 5.8736974.
\end{aligned}$$

The adjacent has length 5.87.

Rounding was arbitrarily chosen to be two (2) decimal places, because we have no context. □

Checkpoint 7.1.11



Suppose a = 5 and A = 50 degrees. Calculate:
b = ____________________
c = ____________________
B = ____________________ degrees
Round all answers to one decimal place. Give angles in *degrees*

7.1.4 Calculating lengths using trig functions

This section demonstrates using trigonometric functions to calculate lengths in an application when we know an angle and a side length.

First, we define terminology we need to describe the applications.

Definition 7.1.12 Angle of Elevation. The **angle of elevation** of an object or observation is the angle measured from level (often the ground) up to the object (or line of sight). ◊

Definition 7.1.13 Angle of Depression. The **angle of depression** of an object or observation is the angle measured from level down to the object (or line of sight). ◊

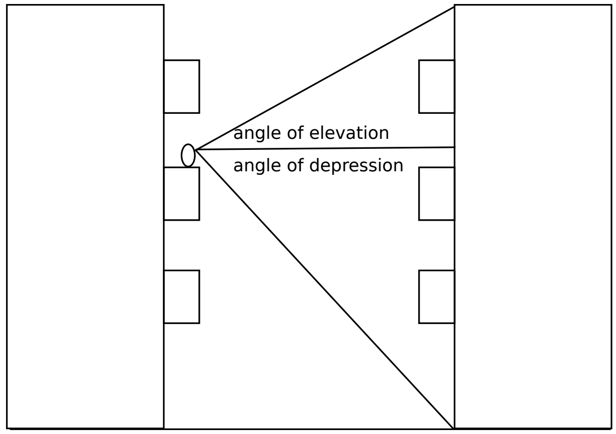
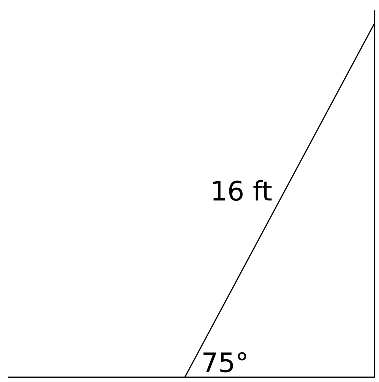


Figure 7.1.14 Illustrations of Angles of Elevation and Depression

For all of these applications our first task is to recognize a right triangle in the problem. We must also identify what the two legs and/or the hypotenuse are in the application. Then we can set up an equation using a trigonometric function, and use the equation to calculate something.

Example 7.1.15 For safety reasons the optimal angle of elevation of a ladder is 75°. If the ladder is 16 ft long, at what height will the top of the ladder be resting against a wall? We can measure a tenth of a foot but not very easily measure a hundredth of a foot, especially for placing a ladder.

First, it is often useful to sketch an image. This makes it easier to identify triangles or other shapes. Note the sketch does not need to be artistic.



We notice that the ladder forms the hypotenuse of a right triangle with the ground and the wall. Next we identify details. We know an angle (75° angle of elevation from the ground) and the length of the hypotenuse (length of the ladder). We want the length side opposite the angle (height along the wall). From this information (opposite, hypotenuse, angle) we can recognize the need for the sine function.

$$\begin{aligned}\sin(75^\circ) &= \frac{B}{16}\\ 16\sin(75^\circ) &= B\\ 15.45 &\approx B\end{aligned}$$

Thus the top of the ladder is 15.5 feet up the wall. □

Example 7.1.16 We may also wish to know how far from the wall to place the bottom of the ladder. That is calculating the length of the side adjacent to the angle of elevation, so we use the cosine function.

$$\begin{aligned}\cos(75^\circ) &= \frac{A}{16}\\ 16\cos(75^\circ) &= A\\ 4.14 &\approx A\end{aligned}$$

Thus we place the ladder a little more than 4 feet from the wall.

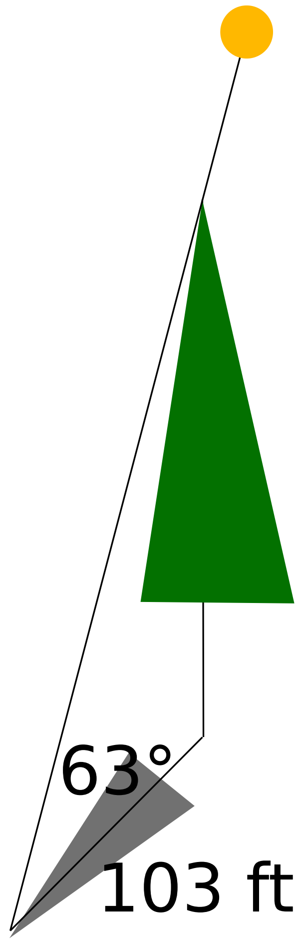
If we had already calculated the height up the wall (previous example) we could also use the Pythagorean Theorem.

$$\begin{aligned}A^2 + 15.45^2 &= 16^2\\ A^2 &= 16^2 - 15.45^2\\ A^2 &= 256 - 238.7025\\ A^2 &= 17.2975\\ A &= \sqrt{17.2975}\\ &\approx 4.1590\\ &\approx 4.16\end{aligned}$$

This is quite close to the previous solution, specifically the difference is too small to effect ladder placement. The difference is the result of using the 15.45 length which was rounded. □

Example 7.1.17 Measuring the heights of tall objects is a use of trigonometry that has been around for millenia.

We want to determine the height of a tree, but dropping a measuring tape from the top is impractical. Instead we can use its shadow, which being on the ground, is easier to access.



The shadow of a tree is measured to be 103 ft (measured from the base of the tree to the end of the shadow). From the end of the shadow the angle of elevation to the sun is measured to be 63°. How tall is the tree? Round using significant digits because this is based on measurements. We do not want to claim a precision about the height which is not valid.

This forms a right triangle with angle 63°, and an adjacent side length of 103 ft. We want the length of the opposite leg. Because we know the adjacent and want the opposite we use the tangent function.

$$\begin{aligned} \tan(63°) &= \frac{H}{103} \\ 103\tan(63°) &= H \\ 2\underline{0}2.15 &\approx H \\ 2\bar{0}0 &\approx H \end{aligned}$$

The tree is approximately 200 feet high. □

Just as in Significant Digits beyond Arithmetic and Significant Digits beyond Arithmetic trigonometric functions and their inverses can be calculated to preserve the same number of significant digits.

Example 7.1.18 Aircraft typically fly a 3° angle of depression to a point 1020 ft past the start of the runway. How high would the plane be when it crosses the runway threshhold? Round to the nearest foot, because aircraft cannot be controlled sufficiently precisely for greater precision to matter here.

Solution. This is a right triangle with adjacent leg length 1020 ft and angle 3°. The length of the opposite is the height at the threshold.

$$\tan(3°) = \frac{T}{1020}$$

$$\begin{aligned} 1020\tan(3°) &= T \\ 53.4559 &\approx T \\ 53 &\approx T \end{aligned}$$

□

Checkpoint 7.1.19 At a particular airport the angle of depression flown by aircraft following the visual descent angle (VDA) is 3.51°. This leads to a point 1020 ft from the threshold of the runway. How high is the aircraft at the threshold? ____

Round to units. This height is called the Threshhold Crossing Height (TCH).

7.1.5 Limitations on Triangle Side Lengths

We should always check that results of calculations make sense. This section presents limitations on triangles we can use for these reality checks.

It is possible to define the trigonometric functions as ratios of sides, because there is a connection between how big an angle is, and how big the side across from it must be. The following activity illustrates how the three main trigonometric functions change as the angle increases or decreases because of this connection between angle measure and length of sides.

Activity 11 This activity has two steps. First notice the relationship between the how big an angle is and how long the side opposite that angle is. Second notice how the trig functions change as a result of this first fact. Use the activity in Figure 7.1.20

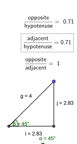
(a) Use the slider for α to increase the angle from 0° to 90°. As the angle increases what does the length of the opposite side (j) do?

(b) Use the slider for α to increase the angle from 0° to 90°. As the angle increases what does the length of the adjacent side (j) do?

(c) Note that the hypotenuse does not change in this example. Based on your result in Task 11.a, as the angle α increases what will the ratio of opposite to hypotenuse (j/g) do?

(d) Note that the hypotenuse does not change in this example. Based on your result in Task 11.b, as the angle α increases what will the ratio of adjacent to hypotenuse (i/g) do?

(e) Based on your result in Task 11.a and Task 11.b, as the angle α increases what will the ratio of opposite to adjacent (j/i) do?



Standalone
Embed

Figure 7.1.20 Sides vs Angles

While angle size and triangle side length is connected, we can always scale a triangle (e.g., make it twice as large) without changing the angles and hence not changing the trig function values. This is why the trig functions are defined as ratios: the scale is divided out.

Another way to look at this is to recall similar triangles (Subsection 2.4.3). The ratios between corresponding sides of two, similar triangles is fixed (all three ratios are the same value). This means ratios of sides of one triangle will be the same as ratios of sides of the other triangle: one will be expressed in non-reduced form. The next examples illustrate this idea.

Example 7.1.21 Consider a right triangle with side lengths 8, 15, and 17. If A is the angle opposite from the side of length 8, then $\sin(A) = \frac{8}{17}$.

Next consider a right triangle with side lengths 16, 30, and 34. If A' is the angle across from 16, then $\sin(A') = \frac{16}{34} = \frac{8}{17}$. This is the same ratio as the previous triangle although the triangle is larger (double in each side length). □

Example 7.1.22 Consider the right triangle with side lengths 8, 15, and 17. If we scale this triangle until the side of length 8 is now length 40, what are the other side lengths?

If A is the angle opposite from the side of length 8 in the original triangle, then $\sin(A) = \frac{8}{17}$. Scaling the triangle does not change the angles, so the new triangle has an angle with the same angle measure as A, call it A'. Putting these together gives us the following.

$$\begin{aligned}
\sin(A) &= \sin(A'). \\
\frac{8}{17} &= \frac{40}{h}. \\
h &= \frac{40 \cdot 17}{8} \\
&= 85 \\
&= 17 \cdot 5.
\end{aligned}$$

The last line is included to show how the hypotenuse is scaled the same way the opposite leg is. We could use tangent to show the other leg is also scaled by 5, that is the adjacent leg will be length $15 \cdot 5 = 75$. □

The Pythagorean Theorem tells us that if we know two sides of a right triangle, the length of the third side is already determined. This means there are restrictions on the side lengths from which a right triangle can be assembled.

Activity 12 Considering the Shortest Hypotenuse. By calculating the sides of a triangle, we will recognize a limitation on how small a hypotenuse can be relative to either leg. Treat all numbers as exact. Do not round.

(a) For a right triangle with a leg of length 4 and hypotenuse of length 5, what is the length of the other leg?

(b) For a right triangle with a leg of length 4 and hypotenuse of length 4.5, what is the length of the other leg?

(c) For a right triangle with a leg of length 4 and hypotenuse of length 4.1, what is the length of the other leg?

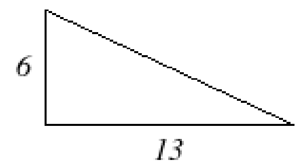
(d) For a right triangle with a leg of length 4 and hypotenuse of length 3.5, what is the length of the other leg?

(e) For a right triangle with the longest leg of length 4, how small can the hypotenuse become?

7.1.6 Exercises

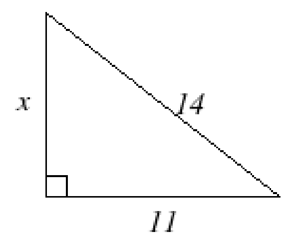
Exercise Group. Use the Pythagorean Theorem and angle sum fact to calculate side lengths and angles.

1. Triangle Side Length. Find the length of the hypotenuse of the triangle pictured below. Give your answer accurate to at least 2 decimal places.



hypotenuse = ______

2. Triangle Side Length. Find the length of the leg x, as an exact value or to at least 2 decimal places.



$x =$ ____________________

3. Triangle Angles. The measures of two angles of a triangle are 48° and 84°. Find the measure of the third angle.

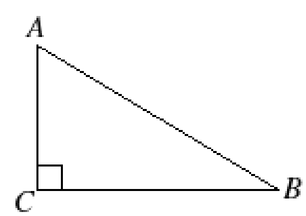
$\angle C =$ ____°

4. Triangle Angles. The measures of two angles of a triangle are 48° and 84°. Find the measure of the third angle.

$\angle C =$ ____°

Exercise Group. Use the ratio definitions of trigonometric functions to answer these.

5. Right Triangle Side Names.



Match each side as hypotenuse, opposite, or adjacent of angle A.

(a) Opposite

(b) Hypotenuse

(c) Adjacent

(a) AC

(b) BC

(c) AB

6. Trig Function Definitions. Match each trig function with its ratio.

(a) Cosine

(b) Sine

(c) Tangent

(a) $\frac{adj}{hyp}$

(b) $\frac{hyp}{adj}$

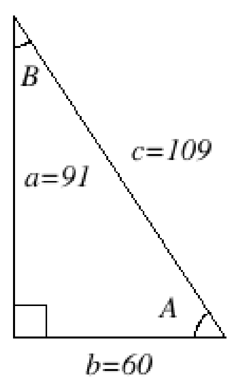
(c) $\frac{opp}{hyp}$

(d) $\frac{adj}{opp}$

(e) $\frac{hyp}{opp}$

(f) $\frac{opp}{adj}$

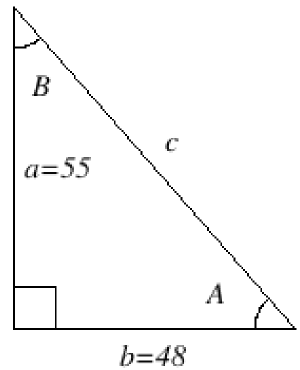
7. Trig Function Value.



Suppose the legs have lengths $a = 91$ and $b = 60$ and the hypotenuse has length $c = 109$. Answers must be exact results (fractions).
$\sin(A) =$ ____________________
$\cos(A) =$ ____________________
$\tan(A) =$ ____________________
$\sec(A) =$ ____________________
$\csc(A) =$ ____________________

$\cot(A) =$ ____________________

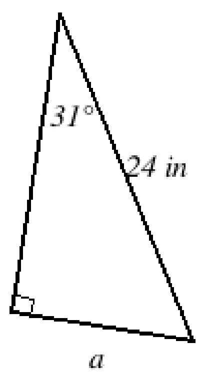
8. Trig Function Value.



Suppose the legs have lengths $a = 55$ and $b = 48$.
Answers must be exact results (fractions).
$\sin(B) =$ ____________________
$\cos(B) =$ ____________________
$\tan(B) =$ ____________________
$\sec(B) =$ ____________________
$\csc(B) =$ ____________________
$\cot(B) =$ ____________________

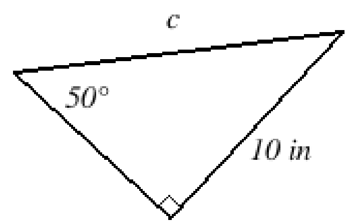
Exercise Group. Calculate side lengths and angles using trigonometric functions.

9. Find a side length. For the right triangle below, find the length of a.



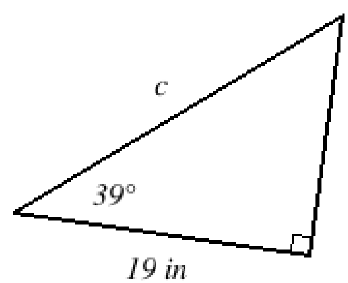
Enter the value for a (accurate to at least two decimal places) and include the units of measure.
$a =$ ______

10. Find a side length. For the right triangle below, find the length of c.



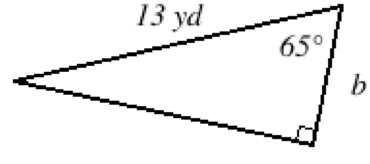
Enter the value for c (accurate to at least two decimal places) and include the units of measure.
$c =$ ______

11. Find a side length. For the right triangle below, find the length of c.



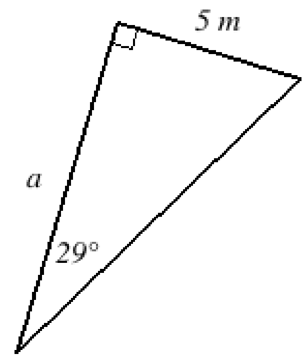
Enter the value for c (accurate to at least two decimal places) and include the units of measure.
$c =$ ______

12. Find a side length. For the right triangle below, find the length of b.



Enter the value for b (accurate to at least two decimal places) and include the units of measure.
$b =$ ______

13. Find a side length. For the right triangle below, find the length of a.

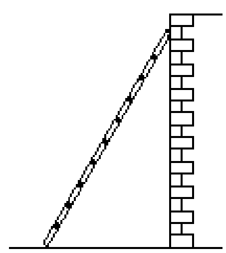


Enter the value for a (accurate to at least two decimal places) and include the units of measure.
$a =$ ______

14. Find a side length. What is the height of a right triangle with an angle that measures 57 degrees and an adjacent side of length 8. Enter your answer accurate to 2 decimal places.
Opposite = ____________________

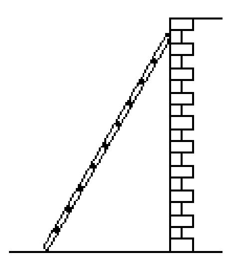
Exercise Group. Identify a triangle in each problem then select an appropriate trigonometric function to answer these application questions.

15. The proper angle for a ladder is about 75° from the ground. Suppose you have a 20 foot ladder. How far from the house should you place the base of the ladder? Round to the nearest 10th of a foot



____________________ feet

16. The proper angle for a ladder is about 75° from the ground. Suppose you have a 14 foot ladder. How high can it reach?



_____ feet
Make your answer accurate to at least 2 decimal places.

17. A smokestack is 190 feet high. A guy wire must be fastened to the stack 10 feet from the top. The guy wire makes an angle of 40° with the ground. Find the length of the guy wire rounded to the nearest foot.

_____ feet

18. The angle of elevation to the top of a Building in New York is found to be 3 degrees from the ground at a distance of 1.25 miles from the base of the building. Using this information, find the height of the building. Round to the nearest whole number.

Your answer is _____ feet.

19. A radio tower is located 300 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 33° and that the angle of depression to the bottom of the tower is 26°. How tall is the tower?

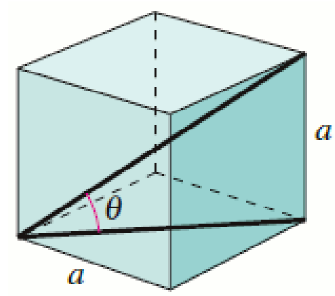
____________________ feet

Give your answer rounded to the nearest foot.

20. From the top of a 219-ft lighthouse, the angle of depression to a ship in the ocean is 18°. How far is the ship from the base of the lighthouse? Give your answer to the nearest foot.

The boat is ____________________ feet from the base of the light house.

21. What is the angle , to the nearest tenth of a degree, between a diagonal of a cube and a diagonal of a face of that cube.



angle = ____________________ °

Report answer accurate to 1 decimal places.

22. 2000 ft h a b

A plane is flying 2000 feet above sea level toward a mountain as shown. The pilot observes the top of the mountain to be $\alpha = 17°$ above the horizontal, then immediately flies the plane at an angle of $\beta = 20°$ above horizontal. The airspeed of the plane is 100 mph. After 5 minutes, the plane is directly above the top of the mountain. How high is the plane above the top of the mountain (when it passes over)? What is the height of the mountain? Round to the nearest 10 feet.

The height of plane above mountain is ______ feet

The height of the mountain is ______ feet

23. From the observation deck of the lighthouse at Sasquatch Point 52 feet above the surface of Lake Ippizuti, a lifeguard spots a boat out on the lake sailing directly toward the light house. The first sighting had a angle of depression of 8.5º and the second sighting had an angle of depression of 25.9º. How far had the boat traveled between the sightings?

_______ ft

24. Below is a picture of a lean-to greenhouse. The angle of elevation of the roof is 21°. The width is 5 feet and the height from the ground to the low part of the roof is 4 feet. What is the height from the ground to the top of the roof?

21°The height from the ground to the top of the roof is _______ feet.
Round to appropriate significant figures.

25. A boat is 1500 meters from a cliff. If the angle of depression from the top of the cliff to the boat is 17°, how tall is the cliff? Round your answer to the nearest tenth.

The cliff is _____ meters tall.
Figure is not to scale.

26. From a window 28 feet above the ground, the angle of elevation to the top of another building is 31°. The distance between the buildings is 55 feet. Find the height of the building to the nearest tenth of a foot.
The height of the building is ____ feet.

7.2 Inverse Trigonometric Relationships

This section addresses the following topics.

- Interpret data in various formats and analyze mathematical models
- Read and use mathematical models in a technical document
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Use models including linear, quadratic, exponential/logarithmic, and trigonometric (skill)
- Analyze right triangles (skill)
- Identify properties of sine and cosine functions (skill)

The previous section presented relationships between side lengths and angles and presented solving problems where we know an angle. This section presents examples where we have side lengths and desire to know an angle.

7.2.1 Inverse Trigonometric Functions

The trigonometric functions presented above provide a side ratio given an angle. It is also possible to find the angle given a side ratio. We use the so called inverse trigonometric functions for this. There are two common notations for them which are shown in Table 7.2.1.

Table 7.2.1 Inverse Trigonometric Functions

Function	Inverse Function	
$\sin\alpha = r$	$\arcsin r = \alpha$	$\sin^{-1} r = \alpha$
$\cos\alpha = r$	$\arccos r = \alpha$	$\cos^{-1} r = \alpha$
$\tan\alpha = r$	$\arctan r = \alpha$	$\tan^{-1} r = \alpha$

Note the notation $\sin^{-1} x$ shows up on calculator keys and in many books. It is unfortunately easy to confuse with $(\sin x)^{-1} = \frac{1}{\sin x} = \csc x$. As a result that notation will not be used in this text.

Example 7.2.2 What is the measure of both non-right angles in Figure 7.1.8? Use technology to calculate.
We can use the arcsine function.

$$\alpha = \arcsin(3/\sqrt{34}) \approx 31^\circ.$$

$$\theta = \arcsin(5/\sqrt{34}) \approx 59^\circ$$

We can also use the arccosine function.

$$\alpha = \arccos(5/\sqrt{34}) \approx 31^\circ.$$

$$\theta = \arccos(3/\sqrt{34}) \approx 59^\circ$$

□

Example 7.2.3 A right triangle has legs of lengths 4 and 8. What are the measures of the non-right angles?

Solution 1. Because we have the two legs, we can use the arctangent function to calculate the angles. The roles of adjacent and opposite switch for the two angles.

$$\arctan\left(\frac{4}{8}\right) \approx 26.57.$$
$$\arctan\left(\frac{8}{4}\right) \approx 63.43.$$

Solution 2. Because we have two legs, we can use the Pythagorean Theorem to calculate the third side length, then use arcsine.

$$\begin{aligned} 4^2 + 8^2 &= c^2. \\ 80 &= c^2. \\ 8.944 &\approx c. \end{aligned}$$

$$\arcsin\left(\frac{4}{8.944}\right) \approx 26.57.$$
$$\arcsin\left(\frac{8}{8.944}\right) \approx 63.44.$$

Notice that the larger angle is slightly different from the first solution. This is the result of using the approximate hypotenuse. □

Checkpoint 7.2.4 If a right triangle has legs of length $a = 19$ and $b = 22$, what are the angles
opposite from side of length a: $A =$ ___ $^\circ$
opposite from side of length b: $B =$ ___ $^\circ$

7.2.2 Solving Triangles

With inverse trigonometric functions we can calculate all side lengths and all angles starting with just side lengths.

Example 7.2.5 A right triangle has a leg of length 12 and the hypotenuse has length 13, what is the length of the other leg? What are the measures of the angles?

We know lengths of two sides so we can apply the Pythagorean Theorem to calculate the length of the third side.

$$\begin{aligned}12^2 + b^2 &= 13^2.\\ b^2 &= 13^2 - 12^2.\\ b^2 &= 25.\\ b &= 5.\end{aligned}$$

For the angle opposite the side of length 12, we can use the inverse sine to calculate the angle.

$$\arcsin\left(\frac{12}{13}\right) \approx 67^\circ.$$

For the angle opposite the side of length 5, we can now use the angle sum theorem.

$$\begin{aligned}90^\circ + 67^\circ + B &\approx 180^\circ.\\ B &\approx 180^\circ - 90^\circ - 67^\circ.\\ B &\approx 23^\circ.\end{aligned}$$

The angle measure is approximate, because we rounded the result of the inverse sine calculation. Rounding to units here was arbitrarily chosen because we do not have a context. □

Example 7.2.6 A right triangle has legs of length 14 and 48.

(a) What is the length of the other side?

Solution. The other side is the hypotenuse. We can calculate it using the Pythagorean theorem.

$$\begin{aligned}c^2 &= 14^2 + 48^2\\ &= 2500.\\ c &= \sqrt{2500}\\ &= 50.\end{aligned}$$

(b) What are the measures of the angles?

Solution. For the angle opposite the side of length 14 because we know the lengths of both legs, we can use

$$\arctan\left(\frac{14}{48}\right) \approx 16^\circ.$$

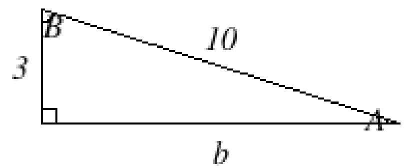
We could have used arcsine and the length of the hypotenuse, however, that length has rounding error which could affect this calculation.

For the other angle we can use the angle sum theorem.

$$\begin{aligned}90^\circ + 16^\circ + B &\approx 180^\circ.\\ B &\approx 180^\circ - 90^\circ - 16^\circ.\\ B &\approx 74^\circ.\end{aligned}$$

□

Checkpoint 7.2.7

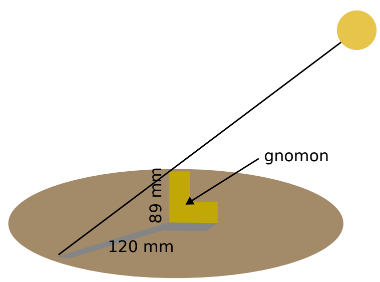


Suppose a = 3 and c = 10. Calculate:
b = ____________________
A = ____________________ degrees
B = ____________________ degrees
Round all answers to one decimal place. Give angles in *degrees*.

7.2.3 Calculating angles using trig functions

This section demonstrates how to calculate the angles when we know the side lengths of a right triangle.

Example 7.2.8 If the gnomon of a sundial is 89 mm tall and the shadow the sun casts on the sundial is $12\bar{0}$ mm long, what is the angle of elevation of the sun? These are measurements, so use significant digits.



The gnomon and surface form the legs of a right triangle. This means we can use the inverse tangent to calculate the angle. With respect to the angle of the sun, the surface is the adjacent and the gnomon is the opposite.

$$\begin{aligned}
\tan(\theta) &= \frac{89}{12\bar{0}}. \\
\theta &= \arctan\left(\frac{89}{12\bar{0}}\right) \\
&\approx \arctan(0.7\underline{4}166667) \\
&\approx 3\underline{6}.563096 \\
&\approx 37^\circ.
\end{aligned}$$

□

Example 7.2.9 An airliner is at 13,000 feet MSL and is cleared to descend to 9,000 feet MSL. This descent will be accomplished over 22 nm. What is the angle of descent? Round to units, because aircraft instruments are not more precise.

First, we will need to convert 22 nm to feet. The unit conversions in Table 1.1.2 suggest we can multiply $\frac{6076 \text{ ft}}{1 \text{ nm}} \cdot 22 \text{ nm} = 133672 \text{ ft}$.

The 133,672 ft is the length of the adjacent side. The length of the opposite side is the change in altitude which is 13,000−9,000 = 4,000 ft. Because the horizontal and vertical components are legs of a right triangle, we can use the inverse tangent function to calculate the angle. The horizontal change is the adjacent side, and the vertical change is the opposite side.

$$\begin{aligned}\tan(\theta) &= \frac{4000}{133672}.\\ \theta &= \arctan\left(\frac{4000}{133672}\right)\\ &\approx \arctan(0.02992399)\\ &\approx 1.71400703\\ &\approx 2^\circ.\end{aligned}$$

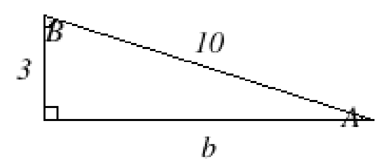
□

Checkpoint 7.2.10 An aircraft is at 17,000 feet MSL and is cleared to descend to 3,000 feet MSL. This descent will be accomplished over 23 nm. What is the angle of descent? ___
Round to the nearest unit.

7.2.4 Exercises

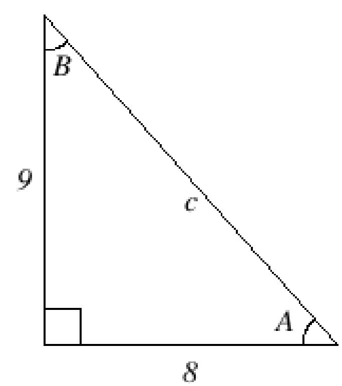
Exercise Group. Calculate side lengths and angles using trigonometric functions.

1. **Find angles and side lengths.**



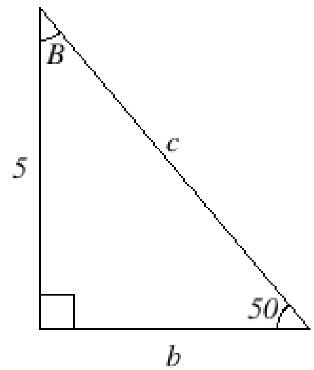
Suppose a = 3 and c = 10. Calculate:
b = ____________________
A = ____________________ degrees
B = ____________________ degrees
Round all answers to one decimal place. Give angles in *degrees*.

2. **Find angles and side lengths.**



Suppose a = 9 and b = 8. Calculate:
c = ____________________
A = ____________________ degrees
B = ____________________ degrees
Round all answers to one decimal place. Give angles in *degrees*.

3. Find angles and side lengths.



Suppose a = 5 and A = 50 degrees. Calculate:
b = ____________________
c = ____________________
B = ____________________ degrees
Round all answers to one decimal place. Give angles in *degrees*

Exercise Group. Use inverse trigonometric functions to answer these questions.

4. Trig Application. You are skiing down a mountain with a vertical height of 1291 feet. The distance from the top of the mountain to the base of the mountain is 3227.5 feet. What is the angle of elevation from the base to the top of the mountain?
Express your answer as a whole angle.
____________________ degrees

5. Trig Application. A 37-foot tree casts a 15-foot shadow. Find the measure of the angle of elevation to the sun to the nearest degree.
Angle of elevation is ____ degrees.

6. Trig Application. A plane flying at an altitude of 17,000 feet begins descending when the end of the runway is 55,000 feet from a point on the ground directly below the plane. Find the measure of the angle of descent (depression) to the nearest degree.
The angle of descent is ____ degrees.

7.3 Non-Right Triangles

This section addresses the following topics.

- Read and use mathematical models in a technical document
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Use models including linear, quadratic, exponential/logarithmic, and trigonometric (skill)
- Analyze non-right triangles (skill)

In Section 7.1 we learned about relationships between angles of the triangles and their sides. However, most of our work was restricted to right triangles. This section demonstrates how to make similar calculations on non-right triangles.

The one relationship that did not require a right angle is the Triangle Angle Sum Theorem.

7.3.1 Law of Sines

In Section 7.1 we saw that there was a relationship between angles of a triangle and the side ratios. More generally there is a relationship between the magnitude of an angle of a triangle and the length of the side opposite it. The following theorem expresses this relationship.

Theorem 7.3.1 Law of Sines. *For a triangle with angles and sides as labeled in Figure 7.3.2,*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

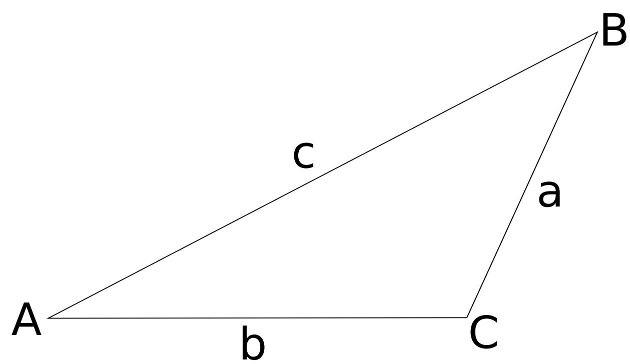


Figure 7.3.2 Labeled Triangle

Example 7.3.3 A triangle has an angle with measure 50° which is opposite a side of length 6. The triangle has another angle with measure 45°.

Rounding in this example is arbitrarily chosen to be two decimal places.

(a) What is the length of the side opposite the 45° angle?

Because we know two angles and a side opposite one of them we can use the law of sines.

$$\begin{aligned} \frac{\sin(50°)}{6} &= \frac{\sin(45°)}{b}. && \text{Set up Law of Sines} \\ b\sin(50°) &= 6\sin(45°). && \text{Clear the denominators} \\ b &= 6\frac{\sin(45°)}{\sin(50°)}. && \text{Divide to solve for } b \\ &\approx 5.54. \end{aligned}$$

(b) What is the third angle and the length of the side opposite it?

We can use the angle sum fact to calculate the third angle. $50° + 45° + \alpha = 180°$ so $\alpha = 85°$. As in the previous step we know an angle (50°), the side opposite it (6), and another angle (85°) so we can use the Law of Sines.

$$\begin{aligned} \frac{\sin(50°)}{6} &= \frac{\sin(85°)}{c}. && \text{Set up the Law of Sines} \\ c\sin(50°) &= 6\sin(85°). && \text{Clear the denominators} \\ c &= 6\frac{\sin(85°)}{\sin(50°)} && \text{Divide to solve for } c \\ &\approx 7.80. \end{aligned}$$

For triangle congruency this was known as Angle-Angle-Side. □

Example 7.3.4 A triangle has two angles with measure 40° and 60.20°. The side between these two angles has length 7.66.

What is the measure of the other angle, and what are the other side lengths? Everything will be rounded to two decimal places.

The third angle measure is the easiest to calculate, because we can use the angle sum theorem. $40° + 60.20° + \alpha = 180°$. Thus $\alpha = 79.80°$.

Now we know an angle (α) and the length of the side opposite it (7.66). This enables us to use the Law of Sines to calculate the other two side lengths.

$$\begin{aligned} \frac{\sin(40°)}{a} &= \frac{\sin(79.80°)}{7.66}. && \text{Set up Law of Sines} \\ \frac{a}{\sin(40°)} &= \frac{7.66}{\sin(79.80°)}. && \text{Proportion true both ways} \\ a &= \frac{7.66\sin(40°)}{\sin(79.80°)} && \text{Multiply to solve} \\ &\approx 5.00281961 \\ &\approx 5.00. \end{aligned}$$

We calculate the third side the same way.

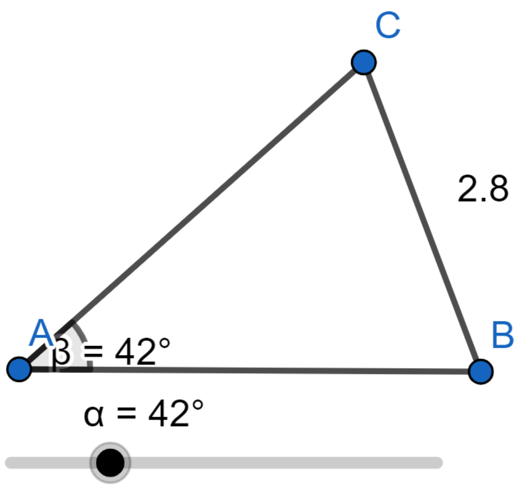
$$\begin{aligned} \frac{\sin(60.20°)}{a} &= \frac{\sin(79.80°)}{7.66}. \\ \frac{a}{\sin(60.20°)} &= \frac{7.66}{\sin(79.80°)}. \\ a &= \frac{7.66\sin(60.20°)}{\sin(79.80°)} \\ &\approx 6.75382345 \\ &\approx 6.75. \end{aligned}$$

If you recall triangle geometry this was known as Angle-Side-Angle. □

Checkpoint 7.3.5 A triangle has angles of measure $A = 45°$ and $B = 75°$. The side opposite angle A has length 6.0. What is the length of the side opposite angle B? _____

Round to the nearest tenth.

The following activity demonstrates the relationship between the magnitude of an angle and the length of the side opposite that angle.



Standalone
Embed

Figure 7.3.6 Experiment to Demonstrate the Angle/Opposite Side Relationship

Activity 13 Use the illustration above to answer the questions below.

(a) Use the slider to set the angle to about 10°.

What happens to the length of the opposite side as you increase the angle to 170°?

(b) Recall the Triangle Angle Sum relationship. As the angle at A increases what must be happening to the sum of the measures of the other two angles?

What must be happening to each of the other two angles?

(c) In this illustration the side connecting A and B is remaining the same length. The angle opposite that side is changing as you noted in the previous step.

Combining the results of the previous two steps, what do you think is true about the following. The angle opposite a larger side is bigger/smaller/unrelated to the angle opposite a shorter side.

7.3.2 Ambiguous Triangles

Above we calculated angles and side lengths given partial information about a triangle (two angles and a side). This section presents a case (two sides and an angle) that we cannot resolve without additional information.

Example 7.3.7 A triangle has an angle of measure 45° and the side opposite it is length 4. Another side has length 5. What are the other angles and side lengths?

We can try to use the Law of Sines.

$$\begin{aligned}
\frac{\sin(45^\circ)}{4} &= \frac{\sin(\theta)}{5}.\\
5\cdot\frac{\sin(45^\circ)}{4} &= \sin(\theta).\\
\arcsin\left(\frac{5}{4}\sin(45^\circ)\right) &= \theta.\\
62.11^\circ &\approx \theta.
\end{aligned}$$

Using the triangle angle sum theorem we learn the other angle measure. $45^\circ + 62.11^\circ + \alpha = 180^\circ$ or $\alpha \approx 72.89^\circ$. We use the Law of Sines again to find the length of the final side.

$$\frac{\sin(45^\circ)}{4} = \frac{\sin(72.89^\circ)}{c}.$$

$$\begin{aligned}\frac{4}{\sin(45°)} &= \frac{c}{\sin(72.89°)}.\\ c &= 4\frac{\sin(72.89°)}{\sin(45°)}.\\ c &\approx 5.41.\end{aligned}$$

This gives us a triangle with angles: 45°, 62.11°, and 72.89°; and with side lengths: 4, 5, and 5.41.

However, $\sin(117.89°) = \sin(62.11°)$, that is, $\arcsin\left(\frac{5}{4}\sin(45°)\right)$ had multiple possible angles. We will repeat the calulcations above using 117.89° as the second angle. The third angle is $45° + 117.89° + \alpha = 180°$ or $\alpha \approx 17.11°$.

$$\begin{aligned}\frac{\sin(45°)}{4} &= \frac{\sin(17.11°)}{c}.\\ c\sin(45°) &= 4\sin(17.11°).\\ c &= 4\frac{\sin(17.11°)}{\sin(45°)}.\\ c &\approx 1.66.\end{aligned}$$

Notice we have two, distinct triangles that match the initial angle and side information. They can be seen in Figure 7.3.8. This indicates an ambiguity if what we know is this particular information.

If you recall previous geometry this was the case Side-Side-Angle which does not prove congruent triangles. □

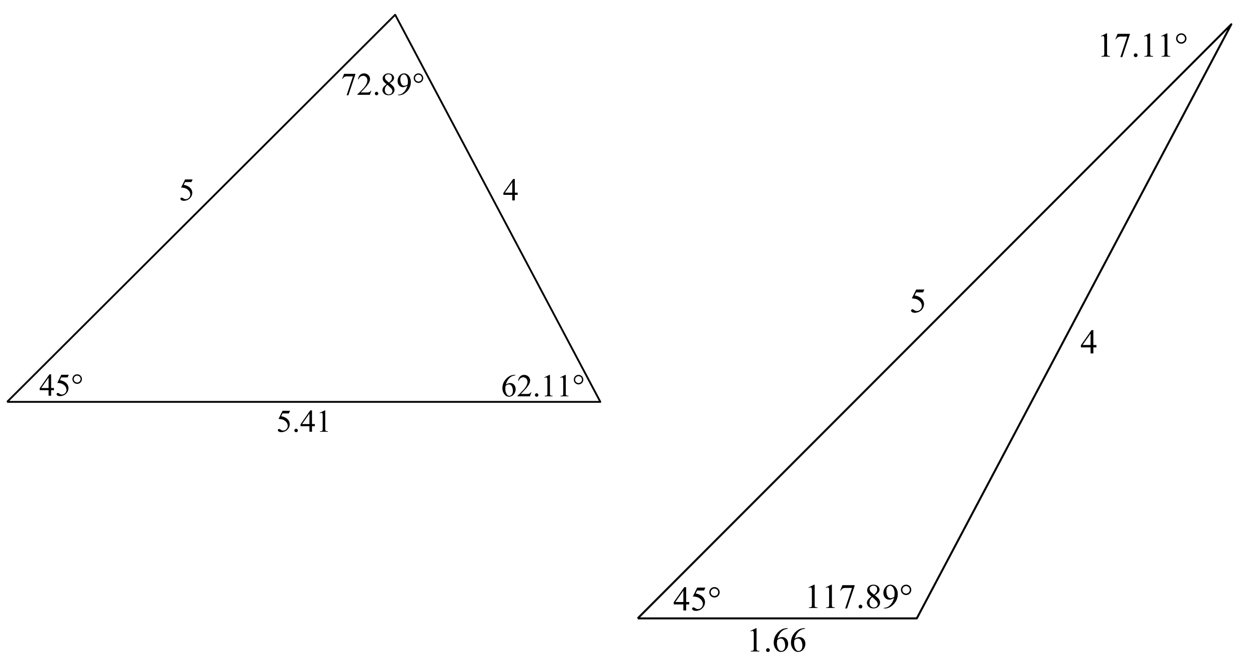


Figure 7.3.8 Two Possible Triangles

When using the Law of Sines we will need to restrict ourselves to the two cases for which it works: Angle-Angle-Side and Angle-Side-Angle.

7.3.3 Law of Cosines

For right triangles we know the Pythagorean theorem is a relationship between the sides of those triangles. For triangles without a right angle that relationship must be slightly modified. The more general statement is below.

Theorem 7.3.9 Law of Cosines. *For any triangle with side lengths a, b, c and angle C which is opposite the side with length c*

$$c^2 = a^2 + b^2 - 2ab\cos(C).$$

Example 7.3.10 A triangle has sides of lengths 4.00, 5.39, and 6.13. What are the angles?

We can use the Law of Cosines.

$$\begin{aligned}
4^2 &= 5.39^2 + 6.13^2 - 2(5.39)(6.13)\cos(A).\\
16 &= 29.0521 + 37.5769 - 66.0814\cos(A).\\
-50.629 &= -66.0814\cos(A).\\
\frac{-50.629}{-66.0814} &= \cos(A).\\
0.76616113 &\approx \cos(A).\\
\arccos(0.76616113) &\approx A.\\
39.98959796^\circ &\approx A.\\
40.0^\circ &\approx A.
\end{aligned}$$

With an angle, we could now use the Law of Sines, but for practice we will use the Law of Cosines again.

$$\begin{aligned}
5.39^2 &= 4^2 + 6.13^2 - 2(4)(6.13)\cos(B).\\
29.0521 &= 16 + 37.5769 - 49.04\cos(B).\\
-24.5248 &= -49.04\cos(B).\\
0.50009788 &\approx \cos(B).\\
\arccos(0.50009788) &\approx B.\\
59.99352413^\circ &\approx B.\\
60.0^\circ &\approx B.
\end{aligned}$$

Knowing two of the angles we can use the triangle angle sum theorem to calculate the third angle measure. $40.0^\circ + 60.0^\circ + C = 180^\circ$ so $C = 80.0^\circ$.

For triangle congruency this was known as Side-Side-Side. □

Example 7.3.11 A triangle has sides with lengths 5 and 7 and the angle between them is 40°. What are the length of the other side and the measures of the other angles?

Because we have two sides (a, b) and the angle between them (C) we can use the Law of Cosines.

$$\begin{aligned}
c^2 &= 5^2 + 7^2 - 2(5)(7)\cos(40^\circ)\\
c^2 &\approx 20.37688900.\\
\sqrt{c^2} &\approx \sqrt{20.37688900}.\\
c &\approx 4.514076761\\
&\approx 4.51.
\end{aligned}$$

Now that we know a side and the angle opposite it, we can use the Law of Sines to find the remaining two angles.

$$\begin{aligned}
\frac{\sin(40^\circ)}{4.514076761} &= \frac{\sin(A)}{5}.\\
0.71198126 &= \sin(A).\\
\arcsin(0.71198126) &= A.\\
45.39634537^\circ &\approx A.\\
45.4^\circ &\approx A.
\end{aligned}$$

Finally we can use the triangle angle sum theorem to calculate the final angle. $40^\circ + 45.4^\circ + B = 180^\circ$ so $B = 94.6^\circ$.

For triangle congruency this was known as Side-Angle-Side. □

Checkpoint 7.3.12 A triangle has sides with lengths $a = 6.00$, $b = 13.98$, and $c = 13.71$. Calculate all of the angles of this triangle.

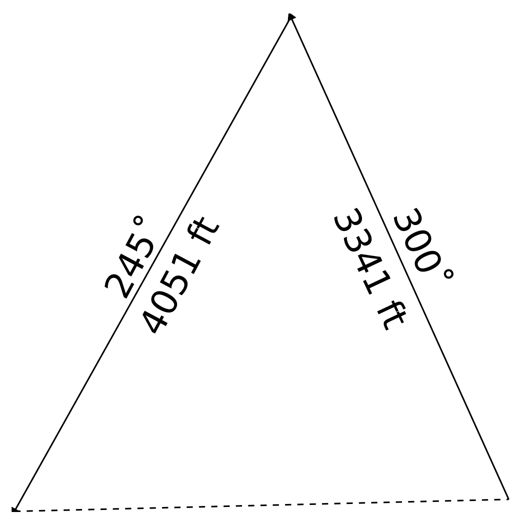
A=____ (across from side of length $a = 6.00$)
B=____ (across from side of length $b = 13.98$)
C=____ (across from side of length $c = 13.71$)
Round to two decimal places.

7.3.4 Using Trigonometric Laws in Applications

When we encounter non-right triangles in applications, we will need to check if we have the information needed to use the Law of Sines or the Law of Cosines.

Example 7.3.13 A sailboat sails 3341 ft at heading 300° then turns to heading 245° and sails 4051 ft. How far is the sailbot from its starting position? Round to units, because fractions of a foot are not meaningful in the motion of a vehicle.

First, it helps to sketch an image.



In our sketch we see a non-right triangle. We know the lengths of two sides and can calculate the angle between them. This is side-angle-side which allows us to use the Law of Cosines to calculate the desired distance (length of dashed segment).

We are given the headings for two segments but not an angle. The angle at the top is the change in headings. If we were at 300° and ended up at 245° then we turned left $300° - 245° = 55°$.

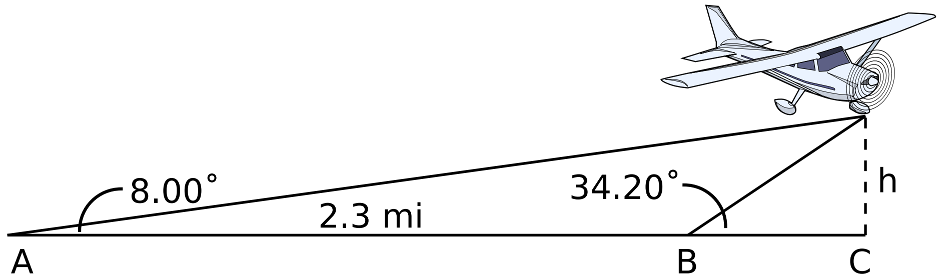
Now we can apply the Law of Cosines.

$$\begin{aligned}
c^2 &= 3341^2 + 4051^2 - 2(3341)(4051)\cos(55°)\\
&= 27572882 - 27068782\cos(55°)\\
&\approx 27572882 - 15526015.51\\
&\approx 12046866.49.\\
c &\approx \sqrt{12046866.49}\\
&\approx 3470.859618\\
&\approx 3471.
\end{aligned}$$

Thus the sailboat ended up 3471 ft from its starting point. □

Example 7.3.14 Two observers are 2.3 miles apart on the ground at the same altitude. At the same time they record the angle at which they saw an aircraft pass. The observer at point A recorded an angle of 8.00° from horizontal. The observer at point B recorded an angle of 34.20° from horizontal. How high above ground was the aircraft?

First, it helps to sketch an image.



How do we know that the larger angle is closer to the aircraft? If we think about an aircraft flying toward us, we realize that our head tips up (bigger angle) as it becomes closer. Thus the closer observer will have the larger angle.

In our sketch we see multiple triangles including some right triangles. We want to calculate the height of the aircraft above ground which is part of the small, right triangle on the right of the diagram. Initially we know only the bottom angles (34.20° and the right angle) and none of the lengths. If we can calculate the length of the side on the left, then we could use the angles and that side length with the Law of Sines to calculate the height.

That left side is also part of a non-right triangle with vertices at points A, B, and the aircraft. The side from A to B has length 2.9 mi, and we know the angles on either side. Because we have angle-side-angle information, we can apply the Law of Sines to calculate the length of the hypotenuse we need. In particular

$$\frac{\sin(P)}{2.3} = \frac{\sin(8.00°)}{c}$$

To calculate the angle at the plane (labeled P above) we will need to use the Triangle Angle Sum theorem. However, the angle we are given is the other side of the non-right triangle. The angle we need now is on the other side of that line, so its measure is $180° - 34.20° = 145.80° = B$. Now, b $8.00° + 145.80° + P = 180°$ So, the angle at the aircraft is $P = 26.20°$.

Now we can apply the Law of Sines.

$$\begin{aligned}
\frac{\sin(8.00°)}{c} &= \frac{\sin(26.20°)}{2.3 \text{ mi}}. \\
\frac{c}{\sin(8.00°)} &= \frac{2.3 \text{ mi}}{\sin(26.20°)}. \\
c &= \frac{(2.3 \text{ mi})\sin(8.00°)}{\sin(26.20°)} \\
&\approx 0.7\underline{2}501447 \text{ mi} \\
&\approx 0.73 \text{ mi}.
\end{aligned}$$

We now know the length of the hypotenuse of the right triangle whose leg is the desired aircraft altitude. This gives us angle-angle-side (first angle is the right angle). This means we can use the Law of Sines to calculate the height.

$$\begin{aligned}
\frac{\sin(34.20°)}{h} &= \frac{\sin(90°)}{0.7\underline{2}501447 \text{ mi}}. \\
\frac{h}{\sin(34.20°)} &= \frac{0.7\underline{2}501447 \text{ mi}}{\sin(90°)}. \\
h &= \frac{(0.7\underline{2}501447 \text{ mi})\sin(34.20°)}{\sin(90°)} \\
&\approx 0.4\underline{0}75185828 \text{ mi} \\
&\approx 0.41 \text{ mi}.
\end{aligned}$$

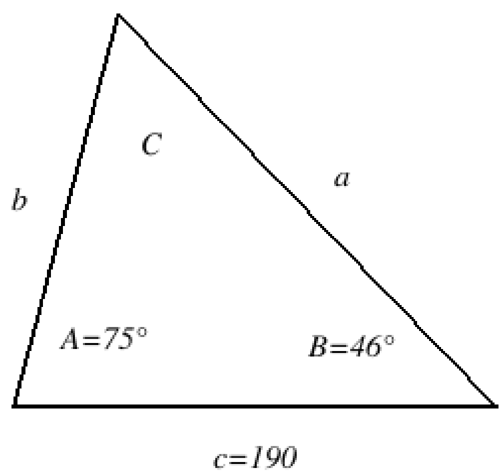
This is in miles. It will be easier to interpret in feet. The conversion ratio in Table 1.1.2 suggests we can multiply

$$(0.4\underline{0}751862 \text{ mi}) \cdot \frac{5280 \text{ ft}}{1 \text{ mi}} \approx 2\underline{1}51.6983 \text{ ft} \approx 2200 \text{ ft}.$$

□

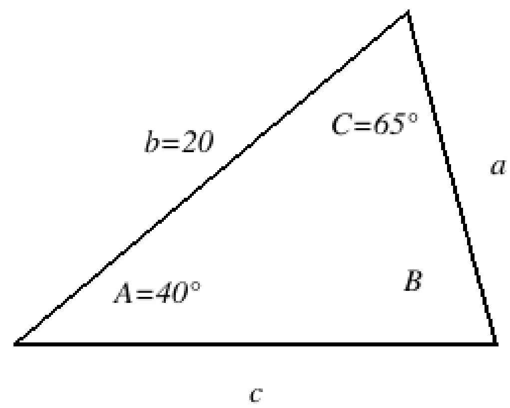
7.3.5 Exercises

1. Contextless.



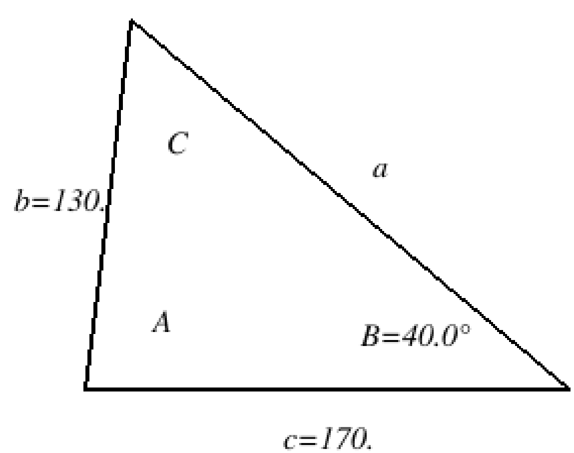
$b =$ ________
Round to 2 decimal places.

2. Contextless. Solve the triangle



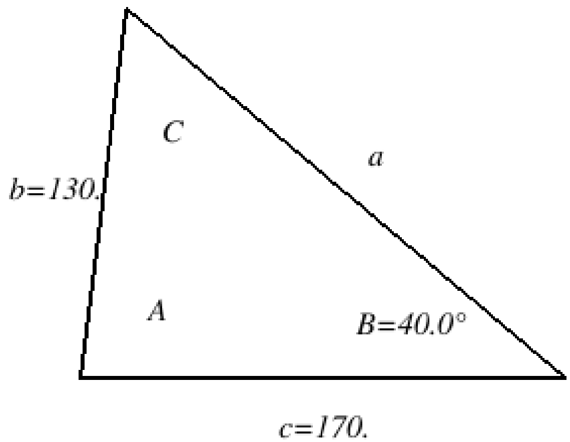
$\angle B$ is _____ degrees
$a =$ _____
$c =$ _____
Round to 2 decimal places.

3. Contextless.



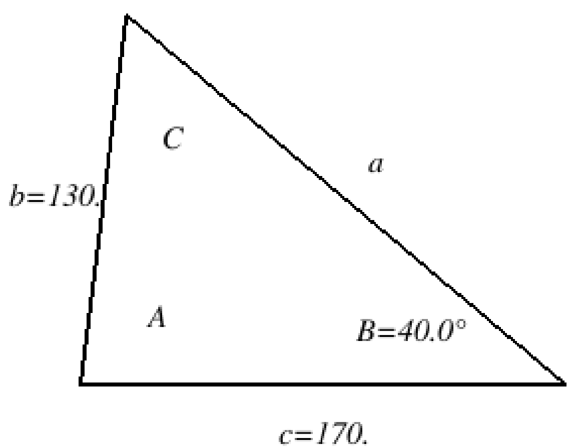
$a =$ _____
$\angle A =$ _____ degrees
$\angle C =$ _____ degrees
Use significant figures.

4. Contextless.



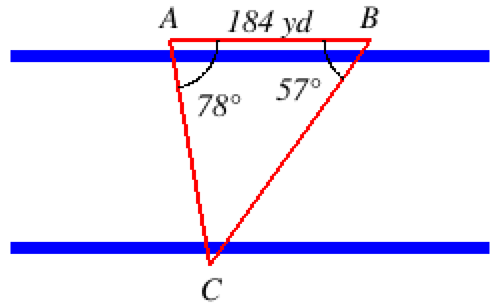
$a =$ _____
$\angle A =$ _____ degrees
$\angle C =$ _____ degrees
Use significant figures.

5. Contextless.



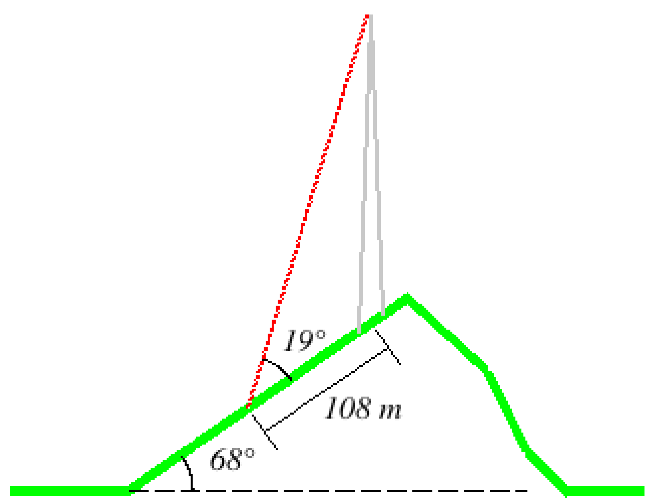
$a =$ _____
$\angle A =$ _____ degrees
$\angle C =$ _____ degrees
Use significant figures.

6. **Application.** To find the distance across a river, a surveyor choose points A and B, which are 184 yd apart on one side of the river. She then chooses a reference point C on the opposite side of the river and finds that $\angle BAC \approx 78°$ and $\angle ABC \approx 57°$.



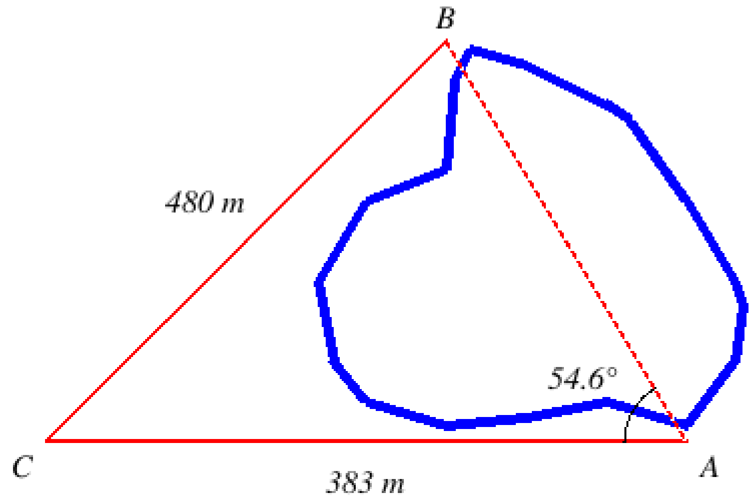
*NOTE: The picture is NOT drawn to scale.*Approximate the distance from point A to point C.
distance = ________ yd
Find the distance across the river.
height = ________ yd
Enter your answer as a number; your answer should be accurate to 2 decimal places.

7. **Application.** A communications tower is located at the top of a steep hill, as shown. The angle of inclination of the hill is 68°. A guy wire is to be attached to the top of the tower and to the ground, 108 m downhill from the base of the tower. The angle formed by the guy wire is 19°. Find the length of the cable required for the guy wire.



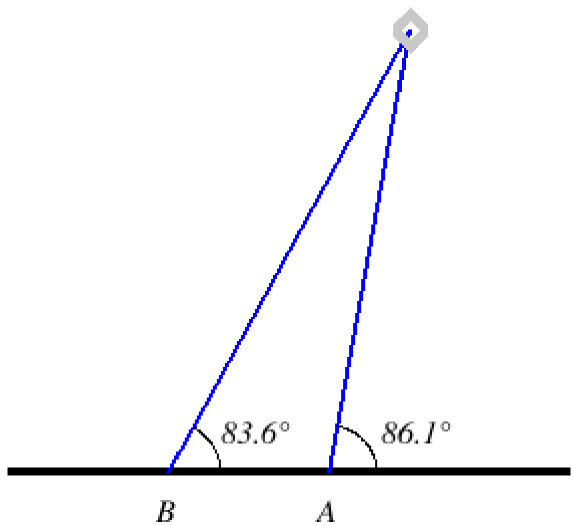
*NOTE: The picture is NOT drawn to scale.*length of guy-wire = ____________________ m
Enter your answer as a number; your answer should be accurate to 2 decimal places.

8. **Application.** Points A and B are separated by a lake. To find the distance between them, a surveyor locates a point C on land such than $\angle CAB = 54.6°$. Find the distance across the lake from A to B.



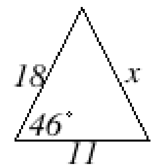
*NOTE: The triangle is NOT drawn to scale.*distance = ____________________ m
Enter your answer as a number; your answer should be accurate to 2 decimal places.

9. Application. The path of a satellite orbiting the earth causes it to pass directly over two tracking stations A and B, which are 47 km apart. When the satellite is on one side of the two stations, the angles of elevation at A and B are measured to be 86.1° and 83.6°, respectively.



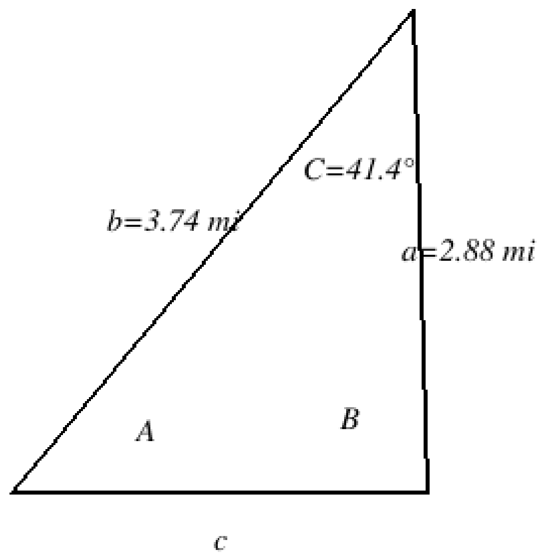
*NOTE: The picture is NOT drawn to scale.*How far is the satellite from station A?
distance from A = ________ km
How high is the satellite above the ground?
height = ________ km
Enter your answer as a number; your answer should be accurate to 2 decimal places.

10. Contextless. Given the triangle



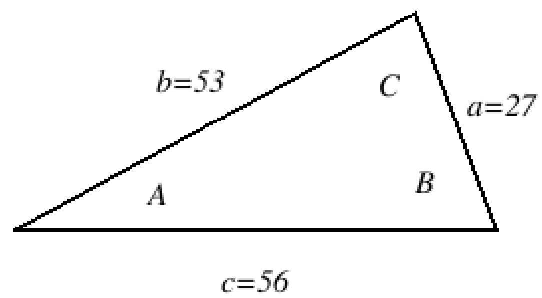
$xx =$ ____________________

11. Contextless.



$c =$ _______ mi
Round to 2 decimal places.

12. Contextless.

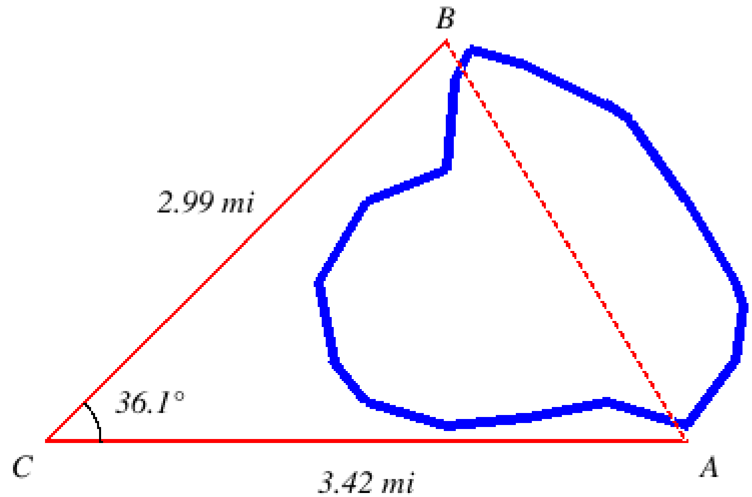


Solve the triangle
$A =$ _______ °
$B =$ _______ °
$C =$ _______ °
Round to 2 decimal places.

13. Application. A pilot flies in a straight path for 1 h 30 min. She then makes a course correction, heading 10 degrees to the right of her original course, and flies 2 h in the new direction. If she maintains a constant speed of 700 mi/h, how far is she from her starting position?
Your answer is _______________ mi;

14. Application. A steep mountain is inclined 74 degree to the horizontal and rises to a height of 3400 ft above the surrounding plain. A cable car is to be installed running to the top of the mountain from a point 930 ft out in the plain from the base of the mountain. Find the shortest length of cable needed.
Your answer is _______________ ft;

15. Application. To find the distance across a small lake, a surveyor has taken the measurements shown. Find the distance across the lake using this information.



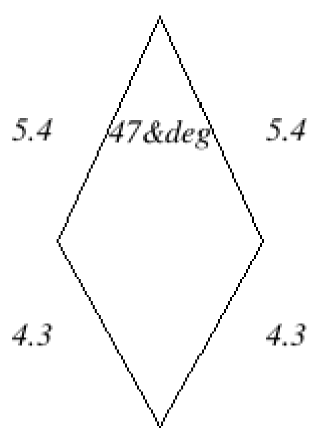
*NOTE: The triangle is NOT drawn to scale.*distance = ____________________
Enter your answer as a number; your answer should be accurate to 2 decimal places.

16. Application. The four sequential sides of a quadrilateral have lengths $a = 3.9$, $b = 6.3$, $c = 7.2$, and $d = 9.9$ (all measured in yards). The angle between the two smallest sides is $\alpha = 105°$.

What is the area of this quadrilateral?
area = ____________________ yd²
Round to 2 decimal places.

17. Application. A *kite* is a quadrilateral with two pairs of adjacents sides of equal length (think of simple ones you can fly).

A kite has two sides of length $a = b = 5.4$ and two sides of length $c = d = 4.3$ (all measured in yards). The angle between the largest sides is $\alpha = 47°$.



What is the area of this kite? ____________________ yd²
Round to using measurement rules.

18. Application. A surveyor starting from a point A moves N. 31°4 E. a distance of 670 5.75 to point B. Next, she moves N. 58°29 E. a distance of 509 10.25 to point C. Next, she walks S. 3°27 W. a distance of 1714 7.75 to point D. Finally, she returns to the starting point.

What distance must she walk to return to the starting point? (Answer accurate to the nearest quarter inch.)

________ feet &; ________ inches

What heading does she walk from the fourth point to return to the starting point? (Answer accurate to the nearest minute.)

N. ________° ________ W.

What is the acreage of this plot of land?

area = ________ acre

Your answer should be accurate to 3 decimal places.

Notes:

∗ the prime symbol in an angle represents minutes: 60 = 1°

∗ the prime symbol in a length represents feet; double prime = inches;

∗ 1 acre = 4840 yd^2

7.4 Sine Wave Properties

This section addresses the following topics.

- Read and use mathematical models in a technical document
- Communicate results in mathematical notation and in language appropriate to the technical field

This section covers the following mathematical concepts.

- Use models including linear, quadratic, exponential/logarithmic, and trigonometric (skill)
- Identify properties of sine and cosine functions (skill)

We began by looking at trigonometric functions in the context of triangles where they represent the ratio of side lengths. Here we will consider trigonometric functions in the context of properties of their graphs. The graphs have direct applications.

7.4.1 Beyond Triangles

This section illustrates how trigonometric functions can be defined on angles greater than 180°. In triangles every angle had to be less than 180° because the sum of the angles of a triangle are only 180°. However, in many applications rather than measuring angles on objects (like triangles), we are measuring how far or how many times around something has moved. For example a wheel on a car moves more than 180° when we are driving. Use Activity 14 to explore this idea.

Activity 14 Use the the illustration in Figure 7.4.1 to see how angles, including those bigger than 180°, are measured and how the trig functions act on these angles.

(a) Angles of measure 30° and 210°

- **(i)** What is the sine value for both angles?
- **(ii)** Compare the x coordinates of these two points on the sine graph.
- **(iii)** Where is the triangle created by the angle 210°?

(b) Angles of measure 45° and 315°

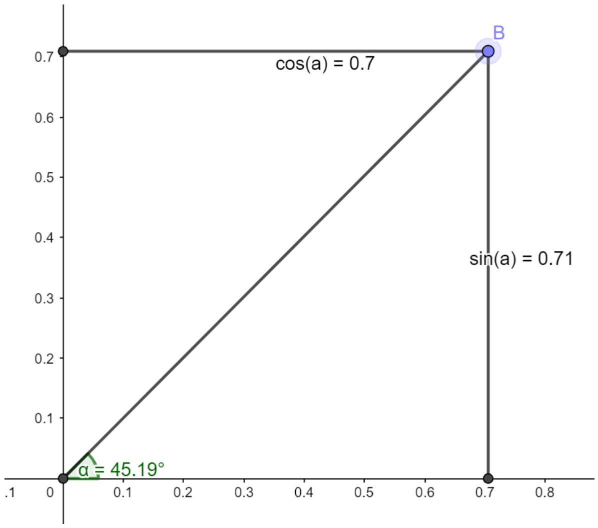
- **(i)** Compare the sine values for these two points.
- **(ii)** Where is the triangle created by the angle 315°?

(c) Angles of measure 45° and 405°

- **(i)** Compare the sine values for these two angles.
- **(ii)** Where is the triangle created by the angle 405°?
- **(iii)** Note the angle displayed at the origin for 405°: why does it not match the slider angle?

(d) Angles of measure -45° and 315°

- **(i)** Where are the triangles for these two points?
- **(ii)** Move the slider from 0° to -45°. Which direction does the point move?
- **(iii)** Note the angle displayed at the origin: explain why it is reasonable.



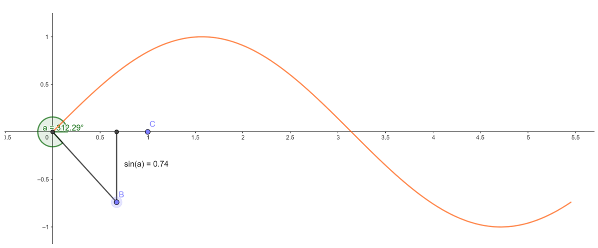
Standalone
Embed

Figure 7.4.1 Bigger Angles

Checkpoint 7.4.2 The angle $-1068°$ has the same sine value as which of these angles?

1. 12
2. 168
3. 192
4. 348

We can use the definition of sine as a ratio and this understanding of angles to produce a graph. In Figure 7.4.3 drag the slider until you have the full graph. A graph that extends over a longer range (and labeled in degrees) is in Figure 7.4.4.



Standalone
Embed

Figure 7.4.3 Produce Graph of Sine

Now we know what the graph of the sine function looks like and why it looks that way.

7.4.2 Properties of Sine Waves

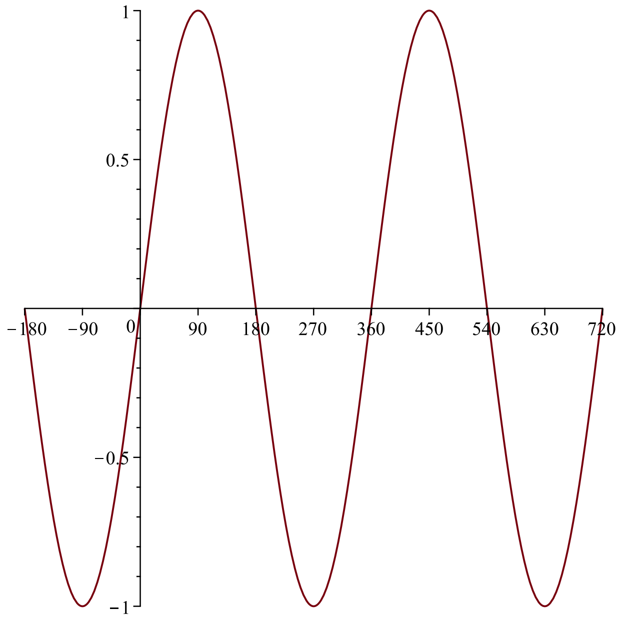


Figure 7.4.4 Graph of Sine

From the first section we know that the graph of sine is a wave that repeats. The piece that is repeated is called a **cycle**. In the default graph this is from 0° to 360° as shown in Figure 7.4.3.

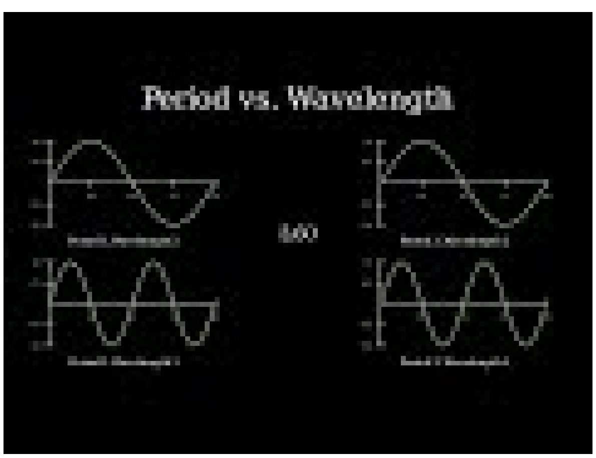
The length of the cycle can be modified. Depending on the application we interpret and measure the length of the cycle differently. This section defines two traits of the cycle, illustrates the traits, and gives an alternate definition for one trait.

Definition 7.4.5 Period. The length of a cycle measured in time is called the **period**. ◊

Definition 7.4.6 Wavelength. The length of a cycle measured in distance is called the **wavelength**. ◊

The video below illustrates the difference between wavelength and period of a sine wave. The top two sine curves have the same wavelength (2), and the bottom two sine curves have the same wavelength (1). We can see this because the top two curves have one cycle from 0 to 2, and the bottom two curves have one cycle from 0 to 1. This implies the bottom curves have a wavelength that is half that of those on top.

The left two sine curves have the same period. They complete one cycle in one (1) seconds. The right two sine curves have the same period. They complete one cycle in two (2) seconds.



Standalone

Sometimes instead of measuring how long a single cycle is in units of time, we measure how many cycles occur in a fixed unit of time. This is called **frequency**.

Definition 7.4.7 Frequency. The number of cycles that occur per second is called the **frequency**. This is typically measured in Hertz (Hz). 1 Hz is one cycle per second. ◊

Note that frequency is the inverse of the period as shown below. In the video the sine waves above, the sine waves on the right have a period of two (2) and a frequency of 1/2.

Table 7.4.8 Period and Frequency are Inverses

Period	Frequency
$\frac{n\text{ seconds}}{1\text{ cycle}}$	$\frac{n\text{ cycles}}{1\text{ second}}$

Example 7.4.9 If a wave has a period of 1/3 seconds, what is its frequency?

We can see how many 1/3 of a second there are in one second. That is

$$\begin{aligned} f \cdot \frac{1}{3} &= 1 \\ f &= 3 \end{aligned}$$

The frequency is 3.

We can consider this a conversion of units. If the period is

$$\frac{1/3\text{ seconds}}{\text{cycle}}$$

and frequency is in cycles per second what we want is to remove the 1/3 from the denominator (turn it into 1).

$$\begin{aligned} \frac{\text{cycles}}{1/3\text{ seconds}} &= \\ \frac{\text{cycles}}{1/3\text{ seconds}} \cdot \frac{3}{3} &= \frac{3\text{ cycles}}{\text{second}} \end{aligned}$$

□

Generally, if the period is T then the frequency is

$$f = \frac{1}{T}.$$

Example 7.4.10 What are the period and wavelength of middle C which has a frequency of 261.63 Hz? This is a measurement in a science model so we will use significant digits.

Because we know the frequency we can directly calculate the period.

$$\begin{aligned} 261.63 &= \frac{1}{T}. \\ 261.63 \cdot T &= \frac{1}{T} \cdot T. \\ 261.63T &= 1. \\ \frac{T \cdot 261.63}{261.33} &= \frac{1}{261.33}. \\ T &= \frac{1}{261.63} \\ &\approx 0.003822\underline{1}916 \\ &\approx 0.0038222. \end{aligned}$$

For the wavelength we need to know that the speed of sound is 1116 feet/second. Now we can use the fact that Hz is cycles per second to convert frequency (cycles per second) to wavelength (feet per cycle). The units suggest that we multiply as follows.

$$\frac{\text{second}}{261.63\text{ cycles}} \cdot \frac{1116\text{ feet}}{\text{second}} \approx 4.265\underline{5}65875 \approx 4.266\frac{\text{feet}}{\text{cycle}}.$$

□

Example 7.4.11 A local AM radio station broadcasts at 750.0 Hz. Note radio waves move at the speed of light which is approximately 2.9979×10^8 meters per second. What are the period and wavelength of this radio signal?

Solution. Because we know the frequency we can directly calculate the period.

$$\begin{aligned} T &= \frac{1}{750.0} \\ &\approx 0.00133\underline{3}333333 \\ &\approx 0.00133 \end{aligned}$$

For the wavelength we need to convert units from seconds per cycle to meters per cycle

$$\frac{1 \text{ s}}{750.0 \text{ cycles}} \cdot \frac{2.9979 \times 10^8 \text{ m}}{\text{s}} \approx 399\underline{7}20 \frac{\text{feet}}{\text{cycle}} \approx 399700 \frac{\text{m}}{\text{cycle}}.$$

□

Checkpoint 7.4.12 Consider the note G#4 which has frequency 415.305 Hz.

What is the period for G#4? ______

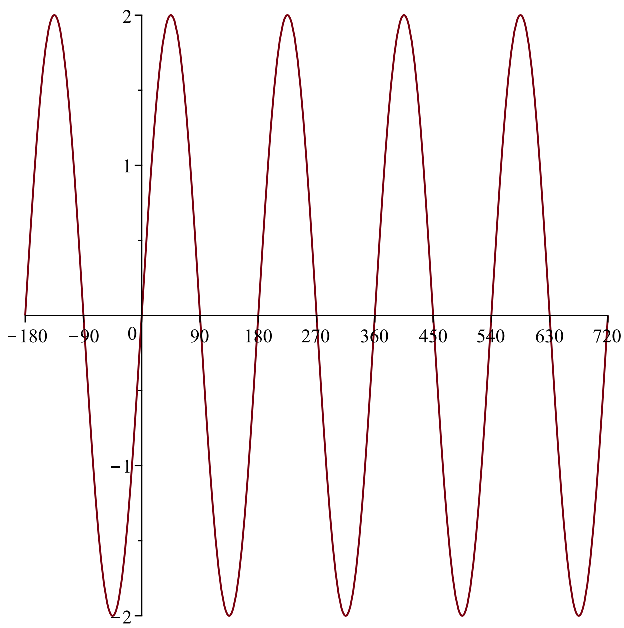
What is the wavelength for G#4? ______

Recall that the speed of sound is 1116 feet per second.

Frequency, period, and wavelength are all about how fast a sine wave moves. Amplitude is about how strong it is.

Definition 7.4.13 Amplitude. The height of the wave (from center to top) is called the **amplitude**. ◊

Example 7.4.14

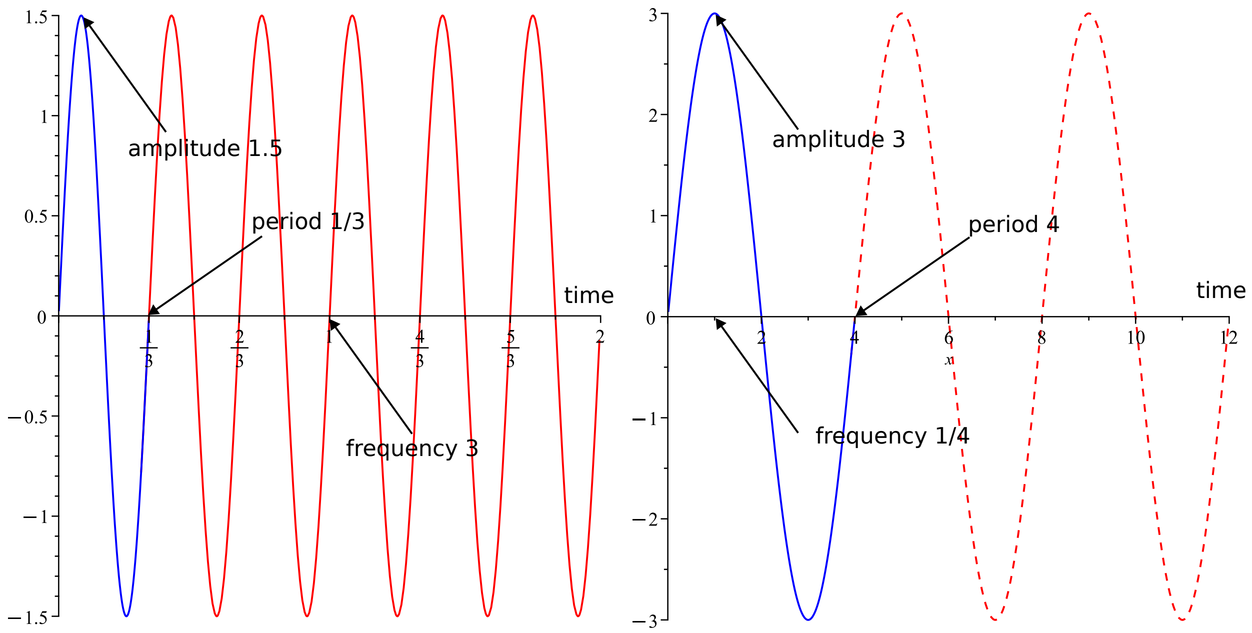


The amplitude of this sine wave is 2. The period is 180. Note without a context period and wavelength are the same. The frequency is

$$f = \frac{1}{180} \approx 0.0056$$

□

In applications the wavelength, period/frequency, and amplitude are determined experimentally: we rarely see the wave. However, in order to practice distinguishing between these properties, we will use visual examples and exercises.



(a) Sine Wave **(b)** Another Sine Wave

Figure 7.4.15 Identifying Sine Wave Properties

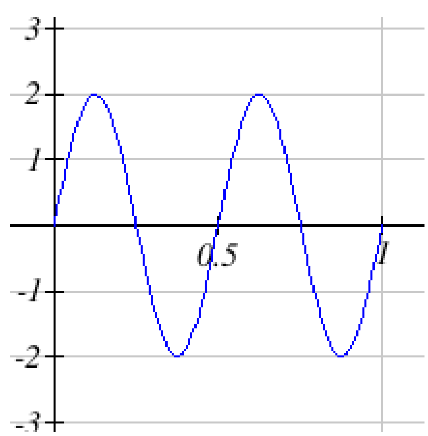
Example 7.4.16 Consider the sine wave in Figure 7.4.15(a). The amplitude is 3 because the wave reaches a max of 1.5 and a min of -3.

The period is 1/3, because a full cycle ends at 1/3. The frequency is 3, because there are three copies of the cycle from 0 to 1.

Switch to the sine wave in Figure 7.4.15(b). The amplitude is 3 because the wave reaches a max of 3 and a min of -3.

The period is 4, because a full cycle ends at 4. The frequency is 1/4, because the cycle is only 1/4 complete by 1. □

Checkpoint 7.4.17



Note the graph shows one second.
What is the frequency of this wave? ____
What is the period of this wave? ____
What is the amplitude of this wave? ____

7.4.3 Transformations of Sine

In Subsection 5.4.1 and Subsection 6.2.2 we learned how to transform a graph by shifting it and reflecting it. Those apply to trigonometric graphs as well. This section illustrates vertical and horizontal stretches and connects them to amplitude and wavelength.

Activity 15 This activity demonstrates how the sine function is modified to change amplitude. Use Figure 7.4.18 to answer the following. Note the amplitude of the unmodified graph is 1.

(a) If you set $a = 2$, that is graph $2\sin(\pi x)$ what is the amplitude?

(b) If you set $a = 3$, that is graph $3\sin(\pi x)$ what is the amplitude?

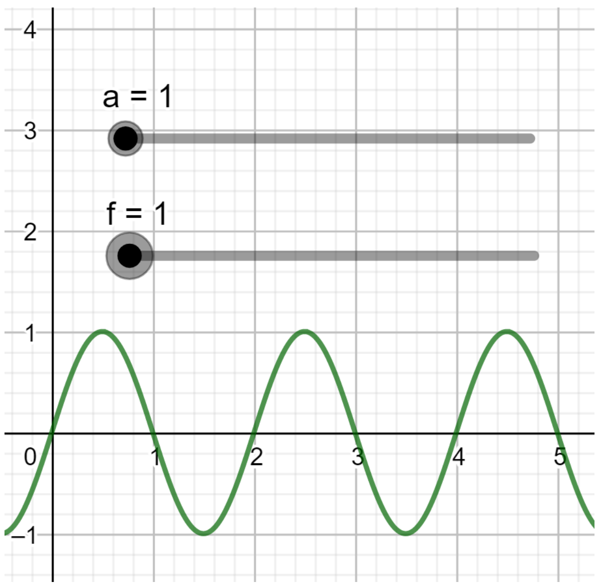
(c) How could you obtain an amplitude of $1/2$?

Activity 16 This activity demonstrates how the sine function is modified to change wavelength. Use Figure 7.4.18 to answer the following. Note the wavelength of the unmodified graph is 2.

(a) If you set $f = 2$, that is graph $\sin(2\pi x)$ what is the wavelength?

(b) If you set $f = 3$, that is graph $\sin(3\pi x)$ what is the wavelength?

(c) How could you obtain a wavelength of 4?

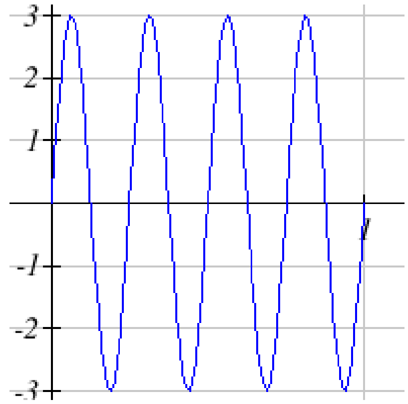


Standalone
Embed

Figure 7.4.18 Amplitude and Wavelength

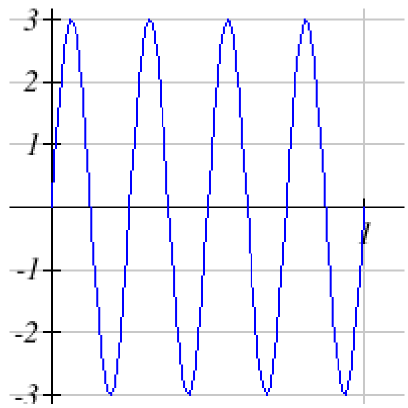
7.4.4 Exercises

1. Contextless.



What is the frequency of this wave? ____________________
Note the graph shows one second.
What is the amplitude of this wave? ____________________

2. Contextless.



What is the frequency of this wave? ____________________
Note the graph shows one second.
What is the amplitude of this wave? ____________________

3. Contextless. Find the period (in seconds) of a wave whose frequency is 5 KHz. ____________________
Write to two, non-zero decimal places.

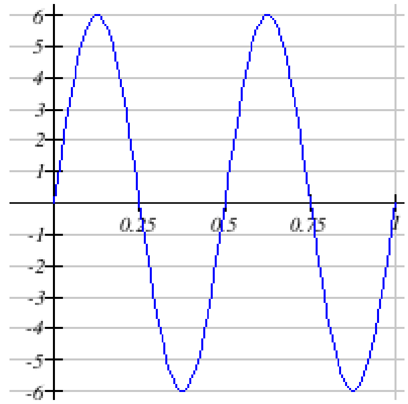
4. Contextless. What is the frequency in Hz of a wave with period 0.14? ____________________
Write accurate to 2 decimal places.

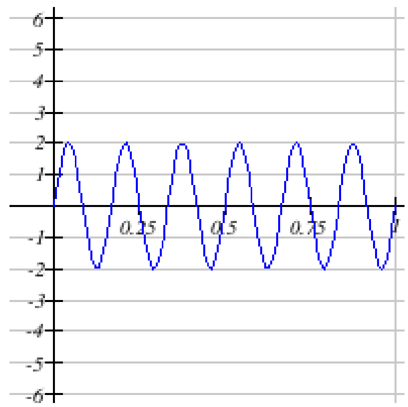
5. Application. A local AM radio station broadcasts at 750 Hz. Radio waves travel at the speed of light which is approximately 3.0×10^8 meters/second. What is the wavelength? ____________________
Round to the units (1's).

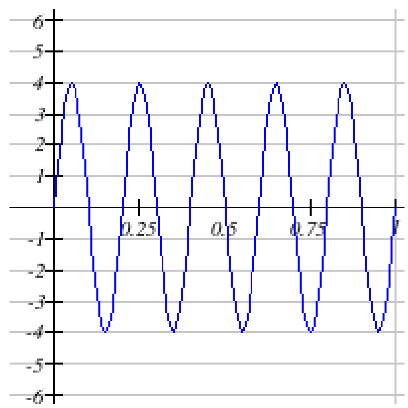
6. Application. Recall that light is a wave and so we can use the relationship that the speed of light is the wavelength of a photon multiplied by the frequency of a photon. $c = \lambda f$. A famous photon in astronomy is the photon emitted by a hydrogen atom with a frequency of $f = 1440$ MHz. What is the wavelength of this photon in centimeters? ____________________ centimeters

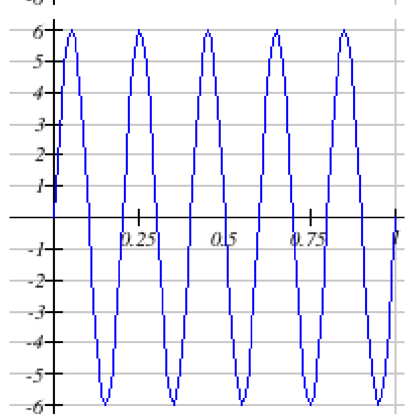
7. Application. Near ultraviolet light has a wavelength of 300 nanometers. Note a nanometer is 1×10^{-9} of a meter. The speed of light is 3×10^8 meters/second. What is the frequency in Hertz?
Enter your answer in scientific notation with three significant figures. For example: 3.29E13 or 4.29E-12.
____________________ Hz

8. Contextless. Which of the following graphs is the correct plot of $y = 6\sin(2x)$?





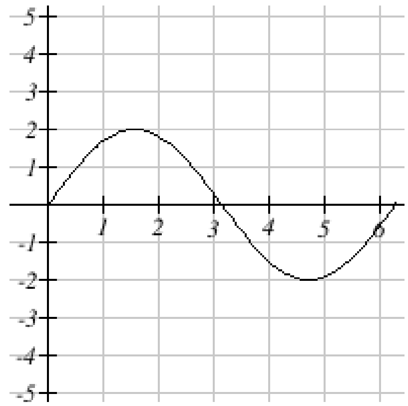


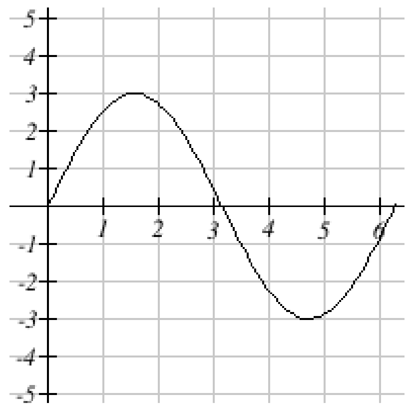


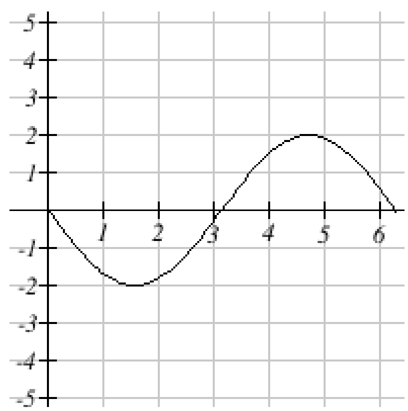
9. Application. The frequency of a vibrating object doubles. By what factor does the period change? *You can enter a decimal or fraction.*

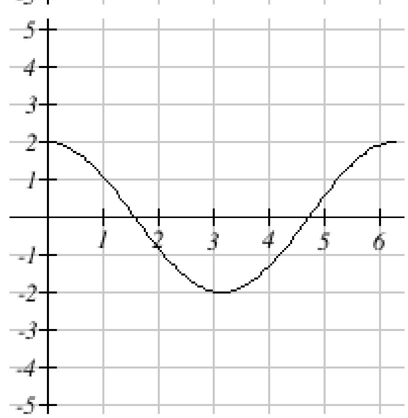
The period is ____________________ times what it had been.

10. Contextless. Which of the following graphs is the correct plot of $y = 2\sin(x)$?









7.5 Project: Effects of Scale on Error

Project 17 Select the Best Time of Day. In Example 7.1.17 we learned to indirectly measure height of an object from the length of its shadow and angle of elevation. Here we consider the effects of the angle of elevation on the precision of our result.

(a) Suppose the shadow of a tree is measured to be 73 ft. If the the angle of elevation of the sun from the end of the shadow is 45°, what is the height of the tree.

(b) We will consider the impact of the sun's angle of elevation on the calculation. For this section suppose the tree is exactly 73 ft tall.

- **(i)** Suppose the angle of elevation of the sun is 45°. What is the length of the shadow?
- **(ii)** Suppose the angle of elevation of the sun is 55°. What is the length of the shadow?
- **(iii)** Suppose the angle of elevation of the sun is 65°. What is the length of the shadow?
- **(iv)** Suppose the angle of elevation of the sun is 75°. What is the length of the shadow?
- **(v)** Suppose the angle of elevation of the sun is 85°. What is the length of the shadow?
- **(vi)** As the angle of elevation grows from 0° toward 90° does the length of the shadow increase directly or inversely? Is it linear?
- **(vii)** Note the angle of elevation of the sun grows from morning until (high) noon and then decreases again. At what time of day would it be easiest to measure the length of the shadow in order to estimate the height of the tree?

(c) We will consider the effect of error in measurement of the angle of elevation of the sun on our calculation of the height of the tree.

- **(i)** Suppose the shadow's length is 73 ft. What is the estimated height of the tree if the angle of elevation is measured to be 46°? 44°?
- **(ii)** Suppose the shadow's length is 51 ft. What is the estimated height of the tree if the angle of elevation is measured to be 56°? 54°?
- **(iii)** Suppose the shadow's length is 34 ft. What is the estimated height of the tree if the angle of elevation is measured to be 66°? 64°?
- **(iv)** Suppose the shadow's length is 20 ft. What is the estimated height of the tree if the angle of elevation is measured to be 76°? 74°?
- **(v)** Suppose the shadow's length is 6 ft. What is the estimated height of the tree if the angle of elevation is measured to be 86°? 84°?
- **(vi)** How much effect on the estimated height of the tree can error in measurement of the angle have?
- **(vii)** Does the error change as the angle of elevation increases?

(d) We will consider the effect of error in measurement of the length of the shadow on our calculation of the height of the tree.

- **(i)** Suppose the angle of elevation of the sun from the end of the shadow is 45°. What is the estimated height of the tree if the length of the shadow is measured to be 72 ft? 74 ft?
- **(ii)** Suppose the angle of elevation of the sun from the end of the shadow is 55°. What is the estimated height of the tree if the length of the shadow is measured to be 50 ft? 52 ft?
- **(iii)** Suppose the angle of elevation of the sun from the end of the shadow is 65°. What is the estimated height of the tree if the length of the shadow is measured to be 33 ft? 35 ft?
- **(iv)** Suppose the angle of elevation of the sun from the end of the shadow is 75°. What is the estimated height of the tree if the length of the shadow is measured to be 19 ft? 21 ft?
- **(v)** Suppose the angle of elevation of the sun from the end of the shadow is 85°. What is the estimated height of the tree if the length of the shadow is measured to be 5 ft? 7 ft?
- **(vi)** At what angle of elevation does the difference in shadow length make the greatest difference?
- **(vii)** What does this suggest about when we should measure the shadow?