Problem 1: DPV 7.7

Find necessary and sufficient conditions on the reals a and b for different cases for the following linear program, denoted as LP:

$$\begin{aligned} \max\ & x + y \\ ax + by &\leq 1 \\ x, y &\geq 0 \end{aligned}$$

(a) **LP is infeasible:**

Solution.

Note that this program will always be feasible at the origin (where $x = 0$ and $y = 0$) regardless of a and b, since $ax + by$ would be 0, which is less than 1. Therefore, there is no combination of a and b that will make this program infeasible.

(b) **LP is unbounded:**

Solution.

The dual program (DP) of LP is:

$$\text{(DP)} \qquad \begin{aligned} \min\ & z \\ az &\geq 1 \\ bz &\geq 1 \\ z &\geq 0 \end{aligned}$$

We claim that DP is infeasible if and only if $a \leq 0$ or $b \leq 0$. To see this, consider 2 cases:

(1) $a \leq 0$ or $b \leq 0$: we may assume $a \leq 0$ without loss of generality. It follows from $z \geq 0$ that $az \leq 0$. Hence, the program is infeasible since no z can satisfy both $az \leq 0$ and $az \geq 1$.

(2) $a > 0$ and $b > 0$: any $z \geq \max(1/a, 1/b)$ is a feasible solution.

If DP is infeasible, then LP can be either infeasible or unbounded. However, from part (a) we know that LP is always feasible. Therefore, infeasibility of DP implies unboundedness of LP. Note that the other direction is always true.

Hence, LP is unbounded if and only if DP is infeasible, which in turn happens if and only if $a \leq 0$ or $b \leq 0$.

(c) **LP has a finite and unique optimal solution.**

Solution.

From part (a) and (b), LP has a finite optimal value if and only if both a and b are positive. However, the optimal solution might not be unique. We consider 2 cases:

(1) $a = b$: the optimal value is $1/a$ and any (x, y) satisfying $x + y = 1/a$ is an optimal solution.

(2) $a \neq b$: without loss of generality we may assume that $a > b$. The optimal value $1/b$ is obtained at $(x, y) = (0, 1/b)$.

Therefore, LP has a unique solution if and only if $a, b > 0$ and $a \neq b$.

Problem 2: ILP

Show how to reduce Vertex Cover to integer programming. An integer program is a linear program with the additional constraint that the variables must take only integer values.

Solution.

We formulate Vertex Cover as an integer programming (IP) problem. Recall that in Vertex Cover, we have a graph $G = (V, E)$ and want to find $C \subseteq V$ of minimum cardinality such that each edge in E is incident with at least one vertex of the set.

In our integer program (IP), we have a variable x_v for each $v \in V$, and

$$x_v = \begin{cases} 1 & \text{if } v \in C \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for each edge $e = (u, v) \in E$, it must be incident with at least one vertex in C:

$$x_u + x_v \geq 1.$$

Therefore, the IP is:

$$\begin{aligned} \min \quad & \sum_{v \in V} x_v \quad \text{s.t.} \\ \forall e = (u, v) \in E, \quad & x_u + x_v \geq 1 \\ \forall v \in V, \quad & x_v \in \{0, 1\} \end{aligned}$$