Solutions for Homework 1

Problem 0.1: Comparing $f(n)$ and $g(n)$.

a. $f(n) = 100n + \log n, g(n) = n + \log^2 n$.

Note, $f(n) \leq 101g(n)$ and $g(n) \leq f(n)$. Hence, both $f(n) = O(g(n))$ and $g(n) = O(f(n))$ are true.

b. $f(n) = \log(2n), g(n) = \log_{10} 10000n$.

Note, we have that:

$$g(n) = \log_{10}(10000n) = \log_{10}(10000) + \log_{10} n = 4 + \frac{\log(n)}{\log_2(10)}.$$

$$f(n) = 1 + \log(n).$$

Therefore, we have that both $f(n) = O(g(n))$ and $g(n) = O(f(n))$ are true.

c. $f(n) = 10 \log n, g(n) = \log(n^5)$.

It is easy to see that $g(n) = 5 \log n$. So both $f(n) = O(g(n))$ and $g(n) = O(f(n))$ are true.

d. $f(n) = (\log n)^2, g(n) = 100 \log(n^{10})$.

Note that, $g(n) = 1000 \log n$. So there exists an n large enough, such that $\log n > 1000$ and hence $f(n) > g(n)$. Therefore, $g(n) = O(f(n))$, but it is not true that $f(n) = O(g(n))$.

e. $f(n) = (\log n)^{\log n}$ and $g(n) = n^{100}$.

First of all, we try to make $f(n)$ and $g(n)$ in the same format:

$$f(n) = (\log n)^{\log n} = \left(2^{\log \log n}\right)^{\log n} = 2^{(\log n)(\log \log n)} = \left(2^{\log n}\right)^{\log \log n} = n^{\log \log n}$$

Since $\log \log n > 100$ for n large enough. We can conclude that $g(n) = O(f(n))$.

f. $f(n) = n^2$ and $g(n) = 2^{3 \log n}$.

Rearranging $g(n)$ we have:

$$g(n) = 2^{3 \log n} = (2^{\log n})^3 = n^3.$$

Then it is easy to see that $f(n) = O(g(n))$, and that $g(n) \neq O(f(n))$.

g. $f(n) = 2^n$, and $g(n) = 3^n$.

For $n \geq 1$, $f(n) < g(n)$, therefore $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

h. $f(n) = 100\sqrt{n}$ and $g(n) = 5^{\log n}$.

Manipulating $g(n)$ we have:

$$g(n) = 5^{\log n} = \left(2^{\log 5}\right)^{\log n} = \left(2^{\log n}\right)^{\log 5} = n^{\log 5}$$

Then since $\log 5 > 1 > 1/2$, we have that $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.