note 175 views

Solving Polynomial Multiplication using the matrix approach

(Spoiler alert - DPV 2.9 Part a).

The thread @280 prompted some questions regarding the matrix approach that I mentioned as the second method to solve Polynomial Multiplication. I decided to solve the exercise 2.9 Part (a) using this approach to help everyone else and have the instructors correct me if I'm doing something wrong.

DPV 2.9 Part (a): Multiply $x+1$ and x^2+1

Let's have $A(x) = x+1$ and $B(x) = x^2+1$

A is a polynomial of degree 1, and B is a polynomial of degree 2. Multiplying $A \times B$ will give us a polynomial of degree 3 (degree of A + degree of B), so we need to make n the next power of 2, in this case, $n = 4$.

Now let's define the matrixes $M_n(\omega)$ and $M_n(\omega^{-1})$. For $n=4$, we have $\omega = i$, so here are our matrixes:

$$M_n(\omega) = M_4(i) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

$$M_n(\omega^{-1}) = M_4(i^{-1}) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i^{-1} & i^{-2} & i^{-3} \\ 1 & i^{-2} & i^{-4} & i^{-6} \\ 1 & i^{-3} & i^{-6} & i^{-9} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

The vector a represents the coefficients of the polynomial A, and $a = (1,1,0,0)$ We can evaluate $A(x)$ by rotating the vector a using $M_n(\omega)$:

$$FFT(a,\omega) = FFT((1,1,0,0),i) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = (2, 1+i, 0, 1-i)$$

The vector b represents the coefficients of the polynomial B, and $b = (1,0,1,0)$ We can evaluate $B(x)$ by rotating the vector b using $M_n(\omega)$:

$$FFT(b,\omega) = FFT((1,0,1,0),i) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = (2,0,2,0)$$

Now we can compute $C(x) = A(x) \times B(x)$:

$c = (2, 1+i, 0, 1-i) \times (2,0,2,0) = (4,0,0,0)$

Finally, we can interpolate our vector c to obtain the coefficients of our resultant polynomial using the following formula:

$$FFT^{-1}(c) = \frac{1}{n} FFT(c, \omega^{-1})$$

Substituting:

$$FFT^{-1}((4,0,0,0)) = \frac{1}{4} FFT((4,0,0,0), i^{-1})$$

$$FFT^{-1}((4,0,0,0)) = \frac{1}{4} \times \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \times \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{4} \times (4,4,4,4) = (1,1,1,1)$$

So, our solution is the polynomial $x^3 + x^2 + x + 1$

hw3

~ An instructor (Murali Raghu Babu B) thinks this is a good note ~

Updated 1 month ago by Santiago L. Valdarrama

followup discussions *for lingering questions and comments*

Resolved Unresolved

Collin Lee 1 month ago
So is the Matrix approach here considered a divide and conquer approach or is that debatable?

Santiago L. Valdarrama 1 month ago DPV page 67 explains a divide and conquer approach using these matrixes.

Obviously, the solution explained here is not using divide and conquer. Turns out that computers are really good solving recurrences, but humans are really bad at them. There's really no point in manually running a recursion to solve FFT when we can easily do it as explained above.

The professor confirmed in @280 that this approach is valid for both the homework and the exam.

Shihgian Lee 1 month ago Thanks for posting the example, Santiago. I will take a look.

You are correct that human is not good at doing machine work. After a lot of pencil pushing and mistakes, I managed to multiply the polynomials using FFT and IFFT presented in the lectures. I don't think I can go another around of multiplying polynomials using FFT by hand.

EDIT: Our TA, Tiancheng Gong provided an example of the Matrix approach too. @277.

Resolved Unresolved

Balaji Sundaresan 1 month ago
Does i (for w^n when n is odd) always cancel out when we multiply two polynomials?

Santiago L. Valdarrama 1 month ago Do you refer to the step when we determine c by multiplying a with b? The answer is no: you might end up with some i's here.

Balaji Sundaresan 1 month ago Yes and thank you

Resolved Unresolved

Nikita Nerkar 1 month ago
Thank you for the post Santiago.
I got stuck in choosing value of n while solving similar problems. How do we choose value of n?
For example, if we have two polynomials $A(x) = x^2 + 1$ and $B(x) = x^2 + 1$, then as per the logic mentioned above, the output polynomial will be of degree 4. In this case, what will be the value of n?

Santiago L. Valdarrama 1 month ago n will be 4 as well.

The idea is that you will choose a value of n such as n is greater or equal to the degree of the resultant multiplication and n is a power of 2.

Nikita Nerkar 1 month ago But in that case, FFT(a,ω) = (2, 0, 2, 0) and FFT(b,ω) = (2, 0, 2, 0) and c = (4, 0, 4, 0)

So for computing $FFT^{-1}(c) = n^{-1}FFT(c, \omega^{-1})$

$$\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & i \end{bmatrix} \text{ X } \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix} = (2, 0, 2, 0)$$

Then the solution will be $2x^3 + 2x$, which is not correct.

Am I missing here something ?

Santiago L. Valdarrama 1 month ago I think you are absolutely right. To select n we need to add 1 to the degree of the resultant polynomial and then select n as the next power of two.

In this case, the degree of the polynomial will be 4, so we need n to be greater than 5, so we will select 8.

That should solve the problem.

What do you think?

Nikita Nerkar 1 month ago Yes I think you are right. But if we select n=8 then the computation starts becoming tricky as in this case,

$w^k = e^{\frac{i2\pi k}{n}}$ and n = 8
$w^0 = 1$
$w^1 = e^{\frac{i2\pi}{8}} = \cos\frac{\pi}{4} + i.\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$
$w^2 = i$
and so on..
Don't you think, this computation will not be as simple as it was for n=4?

Santiago L. Valdarrama 1 month ago The professor already confirmed that they are not going to ask us to run FFT manually. For anything that's greater than n = 4, it will be really tricky to do it manually.

Nikita Nerkar 1 month ago Ok. One last question. I just saw that in post @277, professor mentioned that when the degree of polynomial is d, then we determine d + 1 points. So do you think for polynomial of degree 4, n will be 5 instead of 8 ?

Santiago L. Valdarrama 1 month ago I think that's exactly what I was missing above. d + 1 is the degree we should consider to compute n, but n should always be a power of 2. It won't work if it isn't.

So, in your example, we need to consider d + 1 = 5, so we need to use $n = 8$.

Nikita Nerkar 1 month ago Ok great. This answers my doubt. Thank you so much Santiago for your help.

Resolved Unresolved

Estelle Yeh 1 month ago
Thanks for this post, Santiago. It couldn't be better explained.

Resolved Unresolved

Jim Bullington 1 month ago
+1 Great post Santiago - I could not have done the HW problem without this!