Problem 1: DPV 7.18 (c) and (d) Max-flow Variants
Reduce to linear programming.

(c) Each edge has not only a capacity, but also a lower bound on the flow it must carry.

Solution.

Recall that c_e is the capacity of edge e, and f_e is the flow across edge e. Let $d_e \geq 0$ be the lower bound of flow on edge e. The LP can be written as:

$$\begin{array}{lll} \max & \sum_{sv \in E} f_{sv} & \\ \text{s.t.} & d_e \leq f_e \leq c_e & \forall e \in E \\ & \sum_{wv \in E} f_{wv} = \sum_{vz \in E} f_{vz} & \forall v \in V \backslash \{s, t\} \end{array}$$

The first constraint ensures the flow across each edge is within bounds, and the second constraint ensures flow preservation.

(d) The outgoing flow from each node u is not the same as the incoming flow, but is smaller by a factor of $(1 - \epsilon_u)$, where ϵ_u is a loss coefficient associated with node u.

Solution.

The LP for this problem is:

$$\begin{array}{lll} \max & \sum_{vt \in E} f_{vt} & \\ \text{s.t.} & 0 \leq f_e \leq c_e & \forall e \in E \\ & (1 - \epsilon_v) \sum_{wv \in E} f_{wv} = \sum_{vz \in E} f_{vz} & \forall v \in V \backslash \{s, t\} \end{array}$$

The first constraint ensures the flow across each edge is at most its capacity, and the second constraint ensures the modified flow preservation.

Problem 2: DPV 7.8 Best Fit Line
You are given the following points in the plane:

$$(1,3),(2,5),(3,7),(5,11),(7,14),(8,15),(10,19)$$

You want to find a line $ax+by=c$ that approximately passes through these points (no line is a perfect fit). Write a linear program (you don't need to solve it) to find a line that maximizes the maximum absolute error,

$$\max_{1\leq i\leq 7}|ax_i+by_i-c|$$

Solution.

Our LP for the problem is:

$$\begin{array}{rll} \min & e & \\ \text{s.t.} & e \geq ax_i+by_i-c & \forall i: 1\leq i\leq 7 \\ & e \geq -(ax_i+by_i-c) & \forall i: 1\leq i\leq 7 \\ & \text{b} = 1 & \end{array}$$

The constraints guarantee that

$$e \geq \max_{1\leq i\leq 7}|ax_i+by_i-c|,$$

and since we minimize e,

$$e = \max_{1\leq i\leq 7}|ax_i+by_i-c|.$$

Therefore, e is the maximum absolute error, and the LP correctly finds the line that minimizes it. We added in the constraint $b=1$ to avoid the trivial solution $a=b=0$ (we will not penalize you if you didn't include it).

Problem 3: Infeasible
For an infeasible LP, the dual is always feasible:

TRUE or FALSE

If TRUE explain why it's true, and if FALSE give a counterexample.

Solution.

FALSE

Primal LP:

$$\begin{aligned} \max 2x - y & \\ x - y &\leq 1 \\ -x + y &\leq -2 \\ x, y &\geq 0 \end{aligned}$$

The corresponding dual LP:

$$\begin{aligned} \min w - 2z & \\ w - z &\geq 2 \\ -w + z &\geq -1 \\ w, z &\geq 0 \end{aligned}$$

Both problems are infeasible. For the primal LP, the first constraint simplifies to $x \leq 1 + y$; and the second constraint simplifies to $x \geq y + 2$. Clearly, $1 + y < 2 + y$ so there is no valid solution for the primal LP. For the dual LP, the first constraint simplifies to $w \geq 2 + z$, and the second constraint simplifies to $w \leq z + 1$. As before $z + 1 < z + 2$ so there is no valid solution for the dual LP.

*For those wondering why duality does not prove that the claim is true, note that weak duality only implies that if the dual LP is unbounded then the primal LP is infeasible. Similarly, if the primal LP is unbounded then the dual LP is infeasible. Unbounded means that the optimal value is unbounded (it may be that the feasible region is unbounded but there is a bounded value for the optimal solution). What we've shown in this problem is that the converse does not hold: the primal LP is infeasible but the dual LP may not be unbounded. If the primal LP is infeasible then the dual LP cannot be bounded (because of strong duality, the primal LP is bounded iff the dual LP is bounded).