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**NOTE ON THE JACKIW-REBBI POTENTIAL —**

# 1 Model

To the non-interacting Hamiltonian of the Luttinger liquid that we already know

$$H_0 = -iv_F \int_{L/2}^{L/2} dx \left(\psi_R^\dagger(x)\partial_x\psi_R(x) - \psi_L^\dagger(x)\partial_x\psi_L(x)\right) \tag{1.1}$$

we add a Jackiw-Rebbi like potential term

$$H_{JR} = \int_{-L/2}^{L/2} \Delta \tanh(x)\psi^\dagger(x)\psi(x) \tag{1.2}$$

where $\Delta$ is a constant parameter. Around the barrier of potential $x = 0$ the main effect is the backscattering, so when we decompose $\psi(x) = \psi_R(x) + \psi_L(x)$ we can keep only the $\psi_L^\dagger(x)\psi_R(x) + h.c.$ terms in the J-R term, and from the Klein factors in the bosonization method $\eta_L\eta_L$ we have to add an $i$ in the front, which gives

$$H_{JR} = \Delta \int_{-L/2}^{L/2} \tanh(x) i\psi_L^\dagger(x)\psi_R(x) + h.c. \tag{1.3}$$

Now we can write everything in terms simply of x, introducing $\psi(x) = \psi_R(x)$ and $\psi(x) = \psi_L(-x)$ if $x > 0$ and the opposite if $x < 0$ : $H_0$ becomes

$$H_0 = -2iv_F \int_{L/2}^{L/2} dx\psi^\dagger(x)\partial_x\psi(x) \tag{1.4}$$

and $H_{JR}$ becomes

$$H_{JR} = i\Delta \int_{-L/2}^{L/2} (\tanh(x) - \tanh(-x))\psi^\dagger(x)\psi(-x) = 2i\Delta \int_{-L/2}^{L/2} \tanh(x)\psi^\dagger(x)\psi(-x) \tag{1.5}$$

# 2 Bound states

To show the presence of bound states and clarify the amplitude and length of localization, we look for energy eigenstates resolved in energy space

$$c_\varepsilon^\dagger = \int_{-L/2}^{L/2} dx\chi_\varepsilon(x)\psi^\dagger(x) \tag{2.1}$$

Such that

$$H = H_0 + H_{JR} = \sum_\varepsilon \varepsilon c_\varepsilon^\dagger c_\varepsilon \tag{2.2}$$

Using the correspondance between (anti-)commutation relations for fermions : $[c_\varepsilon^\dagger c_\varepsilon, c_{\varepsilon'}^\dagger] = c_\varepsilon^\dagger\{c_\varepsilon, c_{\varepsilon'}^\dagger\} = c_\varepsilon^\dagger \delta_{\varepsilon,\varepsilon}$ and $[\psi^\dagger(x)\partial_x\psi(x), \chi_\varepsilon(y)\psi^\dagger(y)] = \psi^\dagger(x)\partial_x\delta(x-y)\chi_\varepsilon(y) = -\psi^\dagger(x)\partial_x\chi_\varepsilon(x)$ and $[\psi^\dagger(x)\psi(-x), \chi_\varepsilon(y)\psi^\dagger(y)] = \psi^\dagger(x)\delta(x+y)\chi_\varepsilon(y) = \psi^\dagger(x)\chi_\varepsilon(-x)$ we verify

$$[H, c_\varepsilon^\dagger] = \varepsilon c_\varepsilon^\dagger = \int_{-L/2}^{L/2} \varepsilon\chi_\varepsilon(x)\psi^\dagger(x) \tag{2.3}$$

$$[H_0, c_\varepsilon^\dagger] = +2iv_F \int_{-L/2}^{L/2} \psi^\dagger(x)\partial_x\chi_\varepsilon(x) \tag{2.4}$$

$$[H_{JR}, c_\varepsilon^\dagger] = 2i\Delta \int_{-L/2}^{L/2} \tanh(x)\psi^\dagger(x)\chi_\varepsilon(-x) \tag{2.5}$$

Putting together these equations from $H = H_0 + H_{JR}$ we find

$$\varepsilon\chi_\varepsilon(x) = 2iv_F\partial_x\chi_\varepsilon(x) + 2i\Delta\tanh(x)\chi_\varepsilon(-x) \tag{2.6}$$

Which simplifies at $\pm\infty$ to

$$\varepsilon\chi_\varepsilon(x) = 2iv_F\partial_x\chi_\varepsilon(x) \pm 2i\Delta\chi_\varepsilon(-x) \tag{2.7}$$

We can verify that solutions at zero energy $\varepsilon = 0$ do exist and are of the form

$$\chi_0(x) = \chi_0 e^{-\Delta|x|/v_F} \tag{2.8}$$

# 3 Localization

From the integral $\int_{-\infty}^{\infty} e^{-a|x|}dx = 2/a$ assuming $L$ sufficiently large we deduce that the wave-function of the bound state at the interface is

$$\chi_0(x) = \sqrt{\frac{\Delta}{v_F}} e^{-\Delta|x|/v_F} \tag{3.1}$$

So that the norm of the wavefunction is $|\chi_0(x)|^2 = \frac{\Delta}{v_F}e^{-2\Delta|x|/v_F}$. The correlation length of the spreading of the wavefunction is $\xi = \frac{v_F}{2\Delta}$ and its amplitude $A$ is $|\chi_0|^2 = \frac{\Delta}{v_F}$. At half-filling for $v_F = 2t$ this gives $\xi = \frac{t}{\Delta}$ and $A = \frac{\Delta}{2t}$.

# 4 Numerical observation

From DMRG results with a wire of 300 sites, OBC, 35 sweeps, when we plot $|\psi(x)|^2(\Delta/t)$ we observe (see Figure 1)

Renormalizing with $|\psi(x)|^2(\Delta/t = 0)$ to highlight the localization we see (see Figure 2)

Plotting the amplitude $A$ and localization length $\xi$ as a function of $\Delta/t$ (for $\Delta/t > 0.5$ for better numerical fitting results) we see good agreement with the theory (see Figures 3 and 4)

# 5 Suite

RG ??

Pinning of phi, friedel oscillations (for phi, not theta), theta fluctuates more

*Submitted by Magali Korolev under the supervision of Karyn Le Hur on February 18, 2025.*

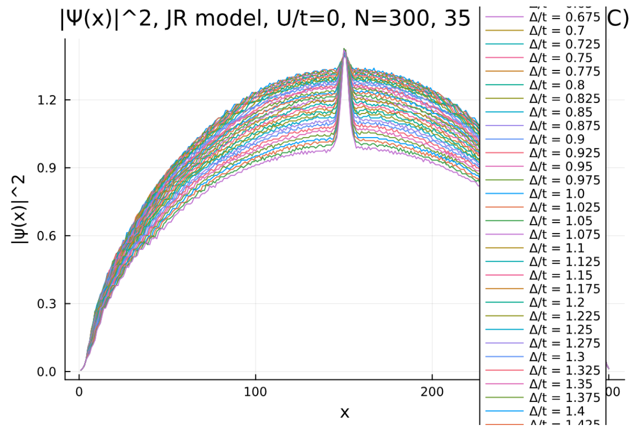


Figure 1: We see localization around the center of the wire (x=150)

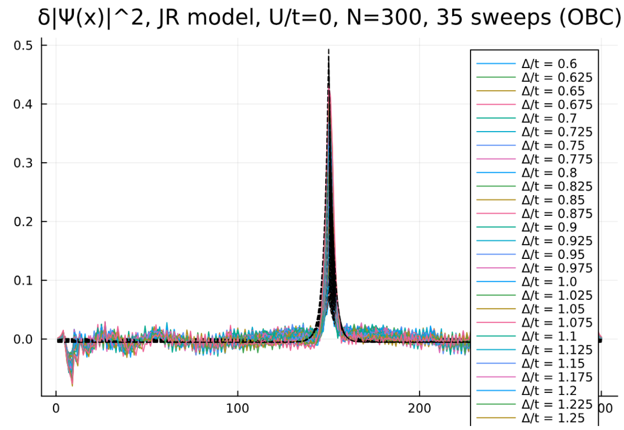


Figure 2: Plotting and fitting (dashed black lines) of the localization

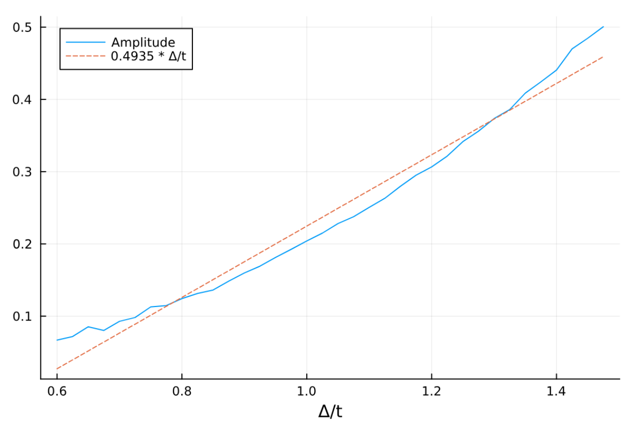


Figure 3: $A(\Delta/t)$ the theoretical value is $0.5\Delta/t$

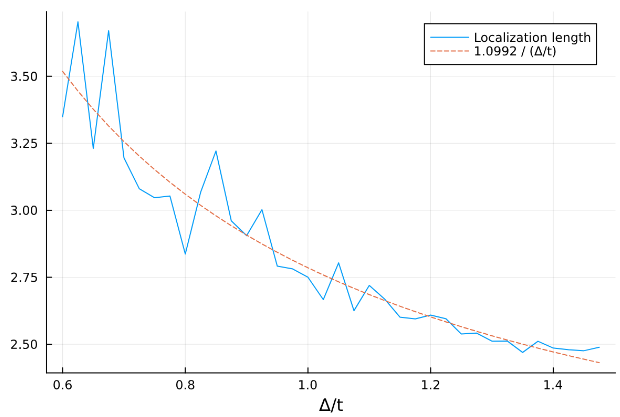


Figure 4: $\xi(\Delta/t)$ the theoretical value is $1 * (t/\Delta)$