Charge fractionalization in Quantum Wires

KOROLEV Magali, under the supervision of **LE HUR Karyn**
Centre de Physique Théorique, École Polytechnique
Email: magali.korolev@ens.fr

Abstract

In one-dimensional (1D) quantum systems, electronic interactions give rise to unique and paradigmatic phenomena not observed in higher-dimensional systems. Unlike two- (2D) or three-dimensional (3D) systems, the electron-electron interactions are enhanced in 1D, the low-energy properties of 1D systems are dominated by collective rather than individual excitations, and the concept of quasiparticles break down which makes them indescribable by the celebrated *Fermi liquid theory* and a new theory, so-called *Luttinger liquid* (LL) *theory* has to be used. This internship report explores in particular the special phenomenon of *charge fractionalization*, where an electron in a 1D quantum system (a quantum wire described by LLT) splits into separate, fractional, charge excitations. This phenomenon is a hallmark of 1D systems and challenges the conventional understanding of charge carriers. Charge fractionalization is particularly compelling because it defies the intuitive notion of electron indivisibility and highlights the richness of quantum behaviors in reduced dimensions. In this report we explain a new way of measuring these irrationnal fractional charges $f_{\pm} = (1 \pm K)/2$ characterized by the Luttinger parameter K through the fluctuation of chiral charge-current fields $\theta_{R/L}(x,t) = \theta(x,t) \mp \phi(x,t)$ in one wire for the first time, and apply this method numerically via the *Density Matrix Renormalization Group* (DMRG).

Contents

1 Introduction

1.1 Three... Two...

Since the late 1950's, the complete description of conduction electrons in conventional metallic systems in two and three dimensions has been successfully achieved by the beautiful Landau's Fermi liquid theory (FLT) [1–4]. The remarkable result of FLT is that the low energy excitations can be effectively described the same way as the free electrons gas, where the electrons are *dressed* by the density fluctuations around them, composed of quasiparticles (particle-hole pairs) which obey Fermi statistics with momenta just above (particle) / below (hole) the Fermi surface. In three and two dimensions this is characterized by a spectral function $A(k,\omega)$ which is a δ-function peak with a spectral weight $Z = 1$ for free electrons and a (sharp) Lorentzian with a spectral weight $Z < 1$ for interacting electrons. In the interacting case, provided Z is non-zero, there exist well defined low-energy quasiparticle excitations, which are in a one-to-one correspondence with the bare electron excitations of the reference free electron gas which, we have to note, are *individual* excitations. It is then possible to build a transport theory of Fermi liquids in direct analogy with Drude theory of the free electron gas, by focussing on the scattering of the low energy quasiparticle excitations. For a weakly-interacting electron gas, deviations in the spectral weight Z from one can be computed perturbatively in the interaction strength, provided the spatial dimensionality, d, is greater than one.

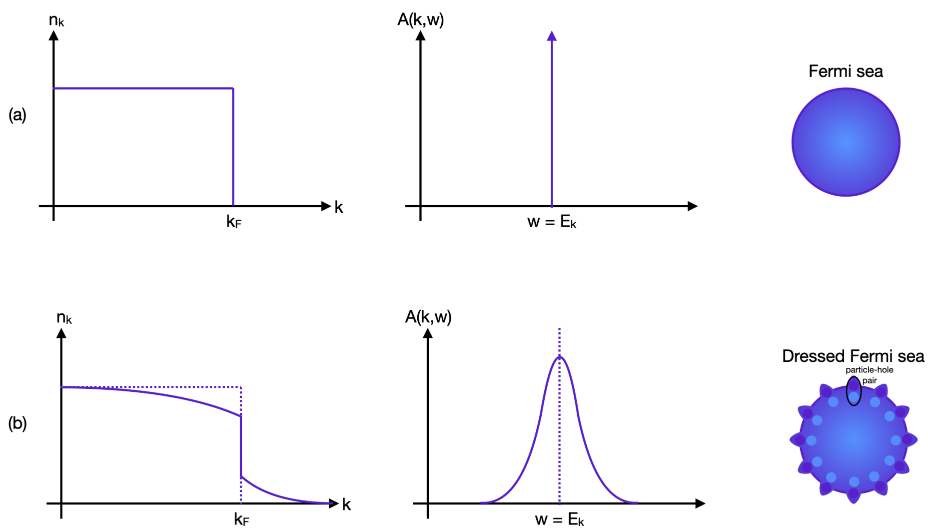

Figure 1: Illustration of the Fermi Liquid Theory: (a) free electrons in 2D and 3D are described by an occupation number that is like a Heaviside distribution, a spectral function that is a δ-peak and individual excitations that form a Fermi sea. (b) Interacting electrons in 2D and 3D are described by quantities that are in one-to-one correspondance with that of the free electrons: the occupation number slightly deviates from a Heaviside function, but still has a discontinuity only with a jump Z smaller than one. The spectral function is no longer a δ-peak but a sharp Lorentzian, and the Fermi sea becomes "dressed" with particle-hole pair excitations.

1.2 One... Go !

Now, what remains of this beautiful theory, and its perturbative expansion in one dimension ? First of all, without any calculations it is easy to see that interactions have drastic effects compared to higher dimensions. If we look at Figure 2 we see that in high dimension ($d = 2, 3$), electrons can nearly propagate freely as individual particles which lead to the existence of individual excitations whereas in one dimension, an electron that tries to propagate has to push its neighbors because of electron-electron interactions, individual motions are rendered impossible and therefore any individual excitation has to become a collective one. This fact is also reflected in the low-order terms of the perturbation theory which become divergent in 1d, leading to the breakdown of Landau's FLT.

This signals that the physical properties of one-dimensional electron systems are drastically different from the ones of the free electron gas, and therefore of FLT. For electrons with spins it's even more drastic: because only collective excitations can exist, excitations fall under the so-called process of *electron fractionalization* where the excitation is split between a collective excitation carrying charge like a sound wave, and a collective excitation carrying

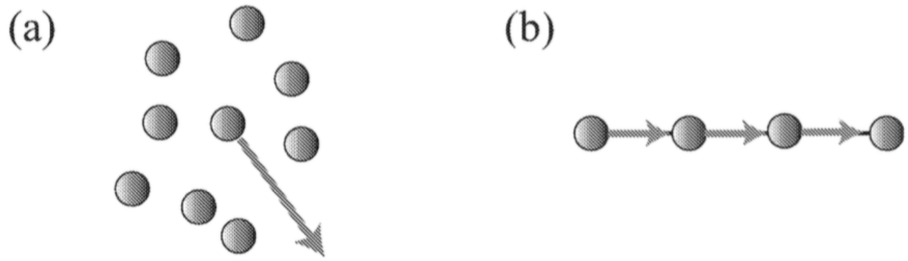

Figure 2: (a) In high ($d = 2,\ d = 3$) dimensions, individual movements and excitaions are possible. (b) In a $d = 1$ interacting system, an electron cannot move without pushing all the other electrons, individual excitations are transformed into collective ones. Figure taken from [5].

spin like a spin wave, and both of them generally have different velocities. This is called *spin-charge separation* and along with the other properties we have mentioned just above which mark the separation from FLT are the essence of the (Tomaga-) Luttinger liquid theory (LLT) [6–9] which is needed in order to describe interacting 1D quantum systems. Surprisingly, LLT can actually describe the divergence previously mentioned in terms of a perturbation theory which accounts in particular for an anomalous mass and an electron lifetime $\tau_F \propto T^{-1}$ different from the FLT electron lifetime $\propto T^{-2}$ where T is the temperature, which both result from an exact expression of the 1D electronic Green's function being characterized by so-called *fractional charges* [10, 11]. *Charge fractionalization* is the phenomenon where the extra charge produced by the tunnelling of an electron into the middle of a uniform Luttinger liquid will break up into pieces, moving in opposite directions, which carry definite fractions $f_+ = \frac{1+K}{2}$ and $f_- = \frac{1-K}{2}$ of the electron charge e, determined by the so-called *Luttinger parameter* K which measures the strength of the interactions in the quantum wire [10–16]. The goal of this internship is to study this phenomenon, and characterize new ways to observe it.

In this report we start by presenting a detailed introduction to Luttinger liquids and their transport properties, and then review known methods for characterizing charge fractionalization. We stress in this first part that measuring and observing charge fractionalization is particularly challenging, motivating thus why in a second time, we introduce a novel method to probe this phenomenon through the fluctuations of chiral fields in a single wire, and propose an experiment that should easily be able to observe these fractional charges. We finish by providing a thorough numerical analysis with a *Density Matrix Renormalization Group* (DMRG) algorithm, to show these fractional charges in the above-mentioned fluctuations. In addition, we draw connections between charge fractionalization and the fractional quantum Hall effect, which is a 2D phenomenon, highlighting the broader significance of our work within the context of condensed matter physics and as a prospect towards building a bridge between 1D and 2D quantum physics.

2 Luttinger Liquids

2.1 Building the model

In this first part, we introduce the Luttinger liquid model as a Gaussian model, such that calculations are easy to carry out.

2.1.1 Chiral basis: left- and right- movers

We start from the free Hamiltonian around the Fermi energy:

$$\begin{aligned} H_0 &= \sum_k \varepsilon(k) c_k^\dagger c_k \\ &\simeq \sum_{k=-k_F-\Lambda}^{k_F+\Lambda} \varepsilon(k+k_F) c_{k+k_F}^\dagger c_{k+k_F} + \varepsilon(k-k_F) c_{k-k_F}^\dagger c_{k-k_F} \end{aligned} \tag{2.1}$$

Using a linear spectrum around k_F: $\varepsilon(k \pm k_F) = \pm v_F k$ and splitting into right (R) and left (L) movers $c^{(\dagger)}_{k+k_F} = c^{(\dagger)}_R(k)$, $c^{(\dagger)}_{k-k_F} = c^{(\dagger)}_L(k)$:

$$\begin{aligned} H_0 &= \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} v_F k (c^\dagger_R(k) c_R(k) - c^\dagger_L(k) c_L(k)) \\ &= \int dx\; v_F [\psi^\dagger_R(x)(i\partial_x)\psi_R(x) - \psi^\dagger_L(x)(i\partial_x)\psi_L(x)] \end{aligned} \tag{2.2}$$

after Fourier transform and introduce now the chiral densties

$$\rho_{R/L}(x) = \psi^\dagger_{R/L}(x)\psi_{R/L}(x) \tag{2.3a}$$

$$\rho_\pm(x) = \rho_R(x) \pm \rho_L(x) \tag{2.3b}$$

with the following properties

$$\rho_{R/L}(x) = \frac{1}{L}\sum_q \rho_{R/L}(q) e^{iqx} \text{ with } \rho_{R/L}(q) = \sum_k c^\dagger_{R/L}(k+q) c_{R/L}(k) \tag{2.4a}$$

$$\rho^\dagger_{R/L}(q) = \rho_{R/L}(-q) \Leftrightarrow \rho^\dagger_{R/L}(x) = \rho_{R/L}(x) \tag{2.4b}$$

2.1.2 Bosonisation

The next step is to define some fields $\phi(x)$ and $\theta(x)$ that satisfy the following:

$$\boxed{\rho_+(x) = -\frac{1}{\pi}\partial_x\phi(x)} \tag{2.5a}$$

$$\boxed{\rho_-(x) = \frac{1}{\pi}\partial_x\theta(x)} \tag{2.5b}$$

$$\boxed{\rho_{R/L}(x) = \frac{-1}{2\pi}\left(\partial_x\phi(x) \mp \partial_x\theta(x)\right)} \tag{2.5c}$$

And the creation and annihilation operators (*bosonization formula*):

$$\psi_{R/L}(x) = \frac{\eta_R}{\sqrt{2\pi\alpha}} e^{i\theta_{R/L}(x)} = \frac{\eta_{R/L}}{\sqrt{2\pi\alpha}} e^{i[\theta(x) \mp \phi(x)]} \tag{2.6}$$

These stepes are refered to as *bosonozation* because they consist in grouping two fermionic creation/annihilation operators together in density operators, which then can be rewritten as bosonic operators:

$$b^\dagger_q = \sqrt{\frac{2\pi}{L|q|}}[\Theta(q)\rho_R(q) + \Theta(-q)\rho_L(q)], \quad b_q = \sqrt{\frac{2\pi}{L|q|}}[\Theta(q)\rho_R(-q) + \Theta(-q)\rho_L(-q)] \tag{2.7}$$

where $q \neq 0$ and with $\Theta(q)$ the Heaviside function. They verify $[b_q, b^\dagger_p] = \delta_{p,q}$, $[b_q, H_0] = v_F|q|b_q$, $[H_0, b^\dagger_q] = [b_q, H_0]^\dagger = v_F|q|b^\dagger_q$ which is the algebra of the harmonic oscillator. Now, we can rewrite our free Hamiltonian in a nice harmonic form

$$\boxed{H_0 = \sum_q v_F|q| b^\dagger_q b_q} \tag{2.8}$$

We now define the "Fourier transform" of these operators, the fields

$$\phi(x) = \phi_0(x) - i\sum_q \mathrm{sgn}(q)\sqrt{\frac{\pi}{2L|q|}}(b_q + b^\dagger_{-q}) e^{iqx} e^{-\alpha|q|/2} \tag{2.9a}$$

$$\theta(x) = \theta_0(x) + i\sum_q \sqrt{\frac{\pi}{2L|q|}}(b_q - b^\dagger_{-q}) e^{iqx} e^{-\alpha|q|/2} \tag{2.9b}$$

equivalent to

$$\phi(x) = \phi_0(x) - i\frac{\pi}{L}\sum_q \frac{1}{q}e^{iqx}\rho_+(q)e^{-\alpha|q|/2} = \phi_0(x) + \frac{1}{\sqrt{L}}\sum_{n\neq 0}\phi_n e^{i\frac{2\pi n}{L}x} \tag{2.10a}$$

$$\theta(x) = \theta_0(x) + i\frac{\pi}{L}\sum_q \frac{1}{q}e^{iqx}\rho_-(q)e^{-\alpha|q|/2} = \theta_0(x) + \frac{1}{\sqrt{L}}\sum_{n\neq 0}\theta_n e^{i\frac{2\pi n}{L}x} \tag{2.10b}$$

where the exponential terms with α are just to ensure non-divergence of some quantities we will derive later, but can now be forgotten, or thought as when taken the limit $\alpha \to 0$ which then indeed verify equations (2.5). The terms ϕ_0 and θ_0 are the zero-modes of the fields and the interpretation of these modes and of the fields is not yet straight-forward, but will be addressed in Section 2.2 about transport.

Now, we have introduced all of these new quantities so that we can re-express the Hamiltonian (2.2) with the densities operators, to find a nice harmonic bosonic form (2.8), which we can rewrite again in terms of the fields to end up with a convenient string model (2.12). However unlike what the literature usually states, reaching this final form is far from trivial. The detailed steps are left to refs [5] (Chapter 2) and [17] (eqs. (54) to (84)) and the interested reader may find a sketch of these steps in Appendix B. In the end, we find that we can rewrite the Hamiltonian (2.2) in terms of density operators

$$H_0 = v_F\pi\int dx[\rho_R^2(x) + \rho_L^2(x)] = \frac{v_F\pi}{2}\int dx[\rho_+^2(x) + \rho_-^2(x)] \tag{2.11}$$

and then, with the field operators

$$H_0 = \frac{v_F}{2\pi}\int_0^L dx\ (\partial_x\theta(x))^2 + (\partial_x\phi(x))^2 \tag{2.12}$$

which is a standard string model with $[\phi(x), \nabla\theta(y)] = i\pi\delta(x-y)$ (see Appendix A). To the non-interacting Hamiltonian we can now add the interactions to obtain the full Hamiltonian $H = H_0 + H_I$ with

$$\begin{aligned} H_I &= \frac{1}{L}\sum_q g_2[\rho_R(q)\rho_L(-q) + \rho_L(q)\rho_R(-q)] + g_4[\rho_R(q)\rho_R(-q) + \rho_L(q)\rho_L(-q)] \\ &= \int dx\ 2g_2(\rho_R(x)\rho_L(x)) + g_4(\rho_R^2(x) + \rho_L^2(x)) \\ &= \int dx\ \left(\frac{g_4+g_2}{2}\rho_+^2(x) + \frac{g_4-g_2}{2}\rho_-^2(x)\right) \end{aligned} \tag{2.13}$$

So that the full hamiltonian is

$$H = \int_0^L dx\ \left\{\left(\frac{v_F + \pi(g_4+g_2)}{2\pi}\right)\rho_+^2(x) + \left(\frac{v_F + \pi(g_4-g_2)}{2\pi}\right)\rho_-^2(x)\right\} = \int_0^L dx\ \frac{u}{2\pi K}(\partial_x\phi(x))^2 + \frac{uK}{2\pi}(\partial_x\theta(x))^2 \tag{2.14}$$

It is in the same form as the non-interacting Hamiltonian H_0 simply with different coefficients, we have introduced u the *Luttinger velocity* and K the *Luttinger parameter*

$$u = \sqrt{(v_F + \pi g_4)^2 - (\pi g_2)^2} \tag{2.15a}$$

$$K = \sqrt{\frac{v_F + \pi(g_4 - g_2)}{v_F + \pi(g_4+g_2)}} \tag{2.15b}$$

Finally, using the variable conjugate $\Pi(x) = \frac{1}{\pi}\partial_x\theta(x)$ we can rewrite

$$\boxed{H = \frac{u}{2\pi}\int dx\ \frac{1}{K}(\partial_x\phi(x))^2 + K(\partial_x\theta(x))^2 = \frac{u}{2}\int dx\ \frac{1}{(\pi K)}(\partial_x\phi(x))^2 + (\pi K)\Pi(x)^2} \tag{2.16}$$

This Hamiltonian is the Hamiltonian of the so-called *Luttinger liquid model*. By a rescaling $\tilde{\phi} \to \sqrt{K}\phi(x)$, $\tilde{\theta} \to \theta(x)/\sqrt{K}$, we reabsorb the K's in the interacting Hamiltonian, and we can then write $\tilde{H}$ with $K = 1$ in terms of bosonic operators $b_q^{(\dagger)}$:

$$\tilde{H} = \sum_q u|q| b_q^\dagger b_q \tag{2.17}$$

just like the free Hamiltonian. Note that if $K \neq 1$ we would have annoying quadratic bb and $b^\dagger b^\dagger$ terms in the Hamiltonian. Those can be treated by means of a Bogoliubov transformation, see [4]. In the end the rescaling is equivalent, so to preserve ourselves from tedious calculations we will just consider $K = 1$ for now. We insist now that the whole point of these first few pages was to introduce the model of the **interacting** 1D quantum wires (LL) as a **quadratic** model, just like the free Hamiltonian, so that derivations are much easier to carry out, which we have finally achieved in (2.16). Lastly, we can write the action of this model. We use the fact that ϕ and $\Pi = \frac{1}{\pi}\partial_x\theta$ on which the Hamiltonian depends quadratically are conjugate variables: $[\phi(x), \Pi(y)] = i\delta(x - y)$ and then we can write the traditional partition function

$$Z = Tr\left(e^{-\beta H(\phi(x),\Pi(x))}\right) \tag{2.18}$$

which can be expressed via a functional integral:

$$Z = \int \mathcal{D}\phi(x,\tau)\mathcal{D}\Pi(x,\tau)e^{-S} \tag{2.19}$$

with S the action (*i.e.* the integral of the Lagrangian $\mathcal{L}$):

$$\begin{aligned} S &= -\int_0^\beta d\tau \int dx \left\{i\Pi(x,\tau)\partial_\tau\phi(x,\tau) - H(\phi(x,\tau),\Pi(x,\tau))\right\} \\ &= -\frac{1}{\pi}\int_0^\beta d\tau \int dx \left\{i\nabla\theta(x,\tau)\partial_\tau\phi(x,\tau) - \frac{u}{2}\left(K(\nabla\theta(x,\tau))^2 + K^{-1}(\nabla\phi(x,\tau))^2\right)\right\} \end{aligned} \tag{2.20}$$

which rewrites in Fourier space

$$S = -\frac{1}{\beta\Omega\pi}\sum_{k,\omega_n}\left\{ik\omega_n\theta(-k,-\omega_n)\phi(k,\omega_n) + \frac{uk^2}{2}\left(K\theta(k,\omega_n)\theta(-k,-\omega_n) + K^{-1}\phi(k,\omega_n)\phi(-k,-\omega_n)\right)\right\} \tag{2.21}$$

We can write it in a more compact form using the spinors $\Psi_\mathbf{q} = \begin{pmatrix}\theta_\mathbf{q} \\ \phi_\mathbf{q}\end{pmatrix}$, $\Psi_\mathbf{q}^\dagger = \begin{pmatrix}\theta_\mathbf{q}^* & \phi_\mathbf{q}^*\end{pmatrix}$ with $\mathbf{q} = \{k, \omega_n\}$ and $\theta(\mathbf{q})^* = \theta(-\mathbf{q})$, $\phi(\mathbf{q})^* = \phi(-\mathbf{q})$:

$$S = \frac{1}{2\pi}\frac{1}{\beta\Omega}\sum_\mathbf{q}\Psi_\mathbf{q}^\dagger M(k,\omega_n)^{-1}\Psi_\mathbf{q} = \frac{1}{2}\frac{1}{\beta\Omega}\sum_\mathbf{q}\Psi_\mathbf{q}^\dagger\begin{pmatrix}k^2uK & ik\omega_n \\ ik\omega_n & k^2u/K\end{pmatrix}\Psi_\mathbf{q} \tag{2.22}$$

The action being quadratic, standard Gaussian integration will allow for straightforward calculation of average values of thermodynamical quantities.

2.2 Transport

As mentioned in the introduction, due to the importance of electronic interactions in one dimension which render individual excitations impossible, transport in 1D quantum systems is highly different from its 2D and 3D counterpart. And beyond theoretical curiosity, as technology evolved starting from the 1970's and carbon nanotubes and other quasi-1d organic conductors gained a lot of interest, the Luttinger liquid approach to transport in quantum wires became very popular, and recent advances in semiconductor, nano technology, and quantum experiments (*e.g.* Quantum Point Contacts) renewed this interest[10, 18–24]. Additionnally, current understanding of topological effects such as the Fractional Quantum Hall Effect (FQHE) lead to believe that the edge modes are isomorphic to a 1D interacting Luttinger liquid [25–27], which again raises interest in the understanding of its transport properties, which is what we address in this section.

2.2.1 Charge and current

Here we discuss the essential quantities of transport: the charge and current densities $\rho(x,t)$ and $j(x,t)$, and the *chiral* fields $\vartheta_{R/L}(x,t)$. First, we recall equations (2.5) and see that we can easily relate ρ_+ and ρ_- respectively to the total charge (per unit volume)[1] and the current to the right (per unit volume) and define

$$\rho(x,t) \equiv q(x,t) \propto \rho_+(x,t) = -\frac{1}{\pi}\partial_x\phi(x,t) \; ; \; Q(t) = \int_0^L dx\, q(x,t) \tag{2.23a}$$

$$j(x,t) \propto \rho_-(x,t) = \frac{1}{\pi}\partial_x\theta(x,t) \; ; \; J(t) = \int_0^L dx\, j(x,t) \tag{2.23b}$$

where Q and J operators are integer numbers: respectively the total number of charges: $N_R + N_L$ and the conserved current quantum number $N_R - N_L$ with $N_{R/L}$ the (integer) number of fermions going to the right/left (i.e. with momenta close to $\pm k_F$). They are also the zero-modes of the bosonized fields ϕ and θ:

$$\phi_0(x,t) = \phi_0 - \frac{\pi Q(t)}{L}x \tag{2.24a}$$

$$\theta_0(x,t) = \theta_0 + \frac{\pi J(t)}{L}x \tag{2.24b}$$

The densities ρ and j are linked by the continuity equation

$$\partial_t\rho(x,t) + \partial_x j(x,t) = 0 \tag{2.25}$$

and where $\phi(x,t)$ and $\theta(x,t)$ are the time-dependant fields obeying the equations of motions (where H is given by (2.16), and using results from Appendix A)

$$\partial_t\phi = i[H,\phi] = i\frac{u}{2\pi}(-i)K2\pi\partial_x\theta = uK\partial_x\theta \tag{2.26a}$$

$$\partial_t\theta = i[H,\theta] = i\frac{u}{2\pi}i\frac{1}{K}2\pi\partial_x\phi = \frac{-u}{K}\partial_x\phi \tag{2.26b}$$

which leads to the following wave equations:

$$\left(\partial_t^2 - u^2\partial_x^2\right)\phi(x,t) = 0 \tag{2.27a}$$

$$\left(\partial_t^2 - u^2\partial_x^2\right)\theta(x,t) = 0 \tag{2.27b}$$

where u is the Luttinger velocity. The proportionality constants in equations (2.23) are ambiguous but from the equations of motion (2.26) we can see that in order to satisfy the continuity equation (2.25) we need to define

$$\rho(x,t) = \rho_+(x,t) = -\frac{1}{\pi}\partial_x\phi(x,t) \tag{2.28a}$$

$$j(x,t) = uK\rho_-(x,t) = \frac{uK}{\pi}\partial_x\theta(x,t) \tag{2.28b}$$

We can now rewrite equations (2.28) in terms of new densities $\rho(x,t) = \tilde{\rho}_+(x,t)$ and $j(x,t) = u\tilde{\rho}_-(x,t)$ that verify the propagation of chiral fields $\vartheta_{R/L}$ at speed u:

$$\tilde{\rho}_+ = \rho_+ \tag{2.29a}$$

$$\tilde{\rho}_- = K\rho_- \tag{2.29b}$$

$$\tilde{\rho}_R = \frac{\tilde{\rho}_+ + \tilde{\rho}_-}{2} = -\frac{1}{2\pi}(\partial_x\phi - K\partial_x\theta) = \frac{1}{2\pi}K\partial_x\vartheta_R = \rho_R\frac{1+K}{2} + \rho_L\frac{1-K}{2} \tag{2.30a}$$

$$\tilde{\rho}_L = \frac{\tilde{\rho}_+ - \tilde{\rho}_-}{2} = -\frac{1}{2\pi}(\partial_x\phi + K\partial_x\theta) = -\frac{1}{2\pi}K\partial_x\vartheta_L = \rho_R\frac{1-K}{2} + \rho_L\frac{1+K}{2} \tag{2.30b}$$

[1] Technically, the total charge $\rho_{tot}(x)$ should be defined as $\rho_+(x) + (e^{-2ik_Fx}\psi_R^\dagger\psi_L + h.c.)$ but we can neglect the left-right processes because of the factor $2k_F$ in the exponential these terms oscillate rapidly and will vanish after integration.

$$\vartheta_{R/L}(x,t) = \theta(x,t) \mp \frac{\phi(x,t)}{K} \tag{2.31a}$$

$$\vartheta_{+}(x,t) = \vartheta_R(x,t) + \vartheta_L(x,t) = 2\theta(x,t) \tag{2.31b}$$

$$\vartheta_{-}(x,t) = \vartheta_R(x,t) - \vartheta_L(x,t) = -2\frac{\phi(x,t)}{K} \tag{2.31c}$$

and the Hamiltonian (2.16) becomes

$$H = \frac{uK}{4\pi}\int dx\ (\partial_x\vartheta_R)^2 + (\partial_x\vartheta_L)^2 = \frac{u\pi}{K}\int dx\ \tilde{\rho}_R^2 + \tilde{\rho}_L^2 \tag{2.32}$$

Now we can check (see Appendix A) that $[\tilde{\rho}_{R/L}(x), \tilde{\rho}_{R/L}(y)] = \mp\frac{iK}{2\pi}\partial_x\delta(x-y)$ and $\partial_t\tilde{\rho}_{R/L} = i[H, \tilde{\rho}_{R/L}]$ yield $(\partial_t \pm u\partial_x)\tilde{\rho}_{R/L} = 0$ which ensures $\tilde{\rho}_{R/L} = F(x \pm ut)$ (the condition to satisfy for the densities to be "chiral"). Now, from a completely different yet equally beautiful point of view, it makes perfect sense that the chiral fields are of the form of eq. (2.31a) because they are the fields which diagonalize the action (2.22), which becomes

$$S = \frac{1}{2}\frac{1}{\beta\Omega}\sum_{\mathbf{q}} \Psi^\dagger_{\mathbf{q}} \begin{pmatrix} k^2 uK & ik\omega_n \\ ik\omega_n & k^2 u/K \end{pmatrix} \Psi_{\mathbf{q}} = \frac{K}{4}\frac{1}{\beta\Omega}\sum_{\mathbf{q}} \tilde{\Psi}^\dagger_{\mathbf{q}} \begin{pmatrix} k(uk - i\omega_n) & 0 \\ 0 & k(uk + i\omega_n) \end{pmatrix} \tilde{\Psi}_{\mathbf{q}} \tag{2.33}$$

where $\tilde{\Psi}_{\mathbf{q}} = \begin{pmatrix} \vartheta_{R,\mathbf{q}} \\ \vartheta_{L,\mathbf{q}} \end{pmatrix}$ and $\tilde{\Psi}^\dagger_{\mathbf{q}} = \begin{pmatrix} \vartheta^*_{R,\mathbf{q}} & \vartheta^*_{L,\mathbf{q}} \end{pmatrix}$

Finally, we can introduce the "other" chiral field $\varphi_{R/L}(x,t)$, as well as the chiral current densities $j_{R/L}$ and the chiral total charge and current operators $Q_{R/L}$ and $J_{R/L}$

$$\varphi_{R/L}(x,t) = -\phi(x,t) \pm K\theta(x,t) \tag{2.34a}$$

$$j_{R/L}(x,t) = \pm u\tilde{\rho}_{R/L}(x,t) \tag{2.34b}$$

$$\tilde{\rho}_{R/L}(x,t) = \frac{1}{2\pi}\partial_x\varphi_{R/L}(x,t) = \frac{q(x,t) \pm Kj(x,t)}{2} \tag{2.34c}$$

$$J_{R/L} = \int dx\ j_{R/L}(x,t) = u\frac{KJ \pm Q}{2} \tag{2.34d}$$

$$Q_{R/L} = \int dx\ \tilde{\rho}_{R/L}(x,t) = \frac{Q \pm KJ}{2} \tag{2.34e}$$

and for charged fermions of elementary charge e: $j_{e,R/L} = \pm eu\tilde{\rho}_{R/L}$. The $\pm$ in (2.34b) is to ensure a satisfaction of the continuity equation on both sides: $\partial_t\tilde{\rho}_{R/L} + \partial_x j_{R/L} = \partial_t\tilde{\rho}_{R/L} \pm u\partial_x\tilde{\rho}_{R/L} = 0$. We see that those quantities are in fact all linked to each other and that[2]

$$J_{R/L}\varphi_{R/L}(x,t) = uKQ_{R/L}\vartheta_{R/L}(x,t) = v_F Q_{R/L}\vartheta_{R/L}(x,t) \tag{2.35}$$

which is still consistent. Now that we have defined the chiral charges and currents, we can also define chiral creation and annihilation operators, like (2.6) which carry the chiral charges $Q_{R/L} = (Q \pm KJ)/2$ which are in general nonintegral , and are written as

$$\tilde{\psi}^\dagger_R(x,t) = \frac{\eta_R}{\sqrt{2\pi\alpha}} e^{-iQ_R\vartheta_R(x,t)} = e^{-\frac{i}{v_F}J_R\varphi_R(x,t)} \tag{2.36a}$$

$$\tilde{\psi}^\dagger_L(x,t) = \frac{\eta_L}{\sqrt{2\pi\alpha}} e^{-iQ_L\vartheta_L(x,t)} = e^{-\frac{i}{v_F}J_L\varphi_L(x,t)} \tag{2.36b}$$

Using formulae from Appendix A, we can check that these operators verify the following commutation relations:

$$[\rho(x), \tilde{\psi}^\dagger_{R/L}(y)] = [\tilde{\rho}_+(x), \tilde{\psi}^\dagger_{R/L}(y)] = Q_{R/L}\tilde{\psi}^\dagger_{R/L}(x)\delta(x-y) \tag{2.37a}$$

$$[Q, \tilde{\psi}^\dagger_{R/L}(y)] = Q_{R/L}\tilde{\psi}^\dagger_{R/L}(x) \tag{2.37b}$$

$$[J, \tilde{\psi}^\dagger_{R/L}(y)] = \frac{Q_{R/L}}{K}\tilde{\psi}^\dagger_{R/L}(x) \tag{2.37c}$$

[2] if $g_4 = g_2$ in eqs. (2.15)

Therefore, for the creation of one electron exactly at the right Fermi point (for example $+k_F$) this is creation of a $(Q = 1, J = 1)$ excitation, and we see the propagation of a charge $Q_R = f_+ = (1 + K)/2$ to the right at velocity u and a charge $Q_L = f_- = (1 - K)/2$ to the left at velocity $-u$ (see Section 2.3).

This is otherwise seen from all of the equations that define the transport properties of the LL, especially the highlighted ones eqs. (2.30) which is separated in fractional charges $f_\pm = (1 \pm K)/2$ and eq. (2.34e) or (2.28) which give the ratio

$$\frac{Q_R - Q_L}{J} = \frac{\rho_R - \rho_L}{j} = K \Leftrightarrow \frac{Q_{R/L}}{J} = \frac{\rho_{R/L}}{j} = f_\pm = \frac{1 \pm K}{2} \tag{2.38}$$

Therefore, we see that charge fractionalization is *deeply* rooted in the theory of Luttinger liquids and its transport properties.

2.2.2 Conductance, conductivity

Finally, we address the conductance and conductivity, as it is usually what is examined in experiments. The first method is very standard material based on the Kubo formula

$$\langle j(x,t)\rangle = \langle j\rangle_0 + \frac{i}{\hbar}\int_0^L dx' \int_{-\infty}^{\infty} dt' \mathcal{T}_t\langle[j(x,t), j(x',t')]\rangle A(x',t') \tag{2.39}$$

with

$$j(x,t) = \sigma(x,t)E(x,t) \tag{2.40}$$

and usually well-addressed in the literature [4, 5, 18]. Skipping all the steps that are detailed in Appendix D, this first method leads to the conductance

$$G = \frac{e^2}{h}K \tag{2.41}$$

with K the Luttinger parameter. This result reminds us of the famous Landauer result for the conductance of non-interacting fermions $G = e^2/h \times \mathcal{T}$ where $\mathcal{T}$ is the transmission coefficient, $\mathcal{T} = 1$ at $T = 0$ and $\mathcal{T} < 1$ otherwise, in analogy with $K = 1$ for non-interacting fermions and $K < 1$ otherwise in the Luttinger model. We see therefore that K can be interpreted as a dimensionless conductance in the sense of Thouless.

Another method for evaluating the conductivity which is a lot simpler is using the classical Ohm's law

$$U = RI \Leftrightarrow G = \frac{1}{R} = \frac{I}{U} \tag{2.42}$$

using the Hamiltonian with the chiral fields (2.32) we can define the chiral chemical potentials

$$\frac{\partial H}{\partial \tilde{\rho}_{R/L}} = \mu_{R/L} = eU_{R/L} \tag{2.43}$$

with e the charge of an electron and $U_{R/L}$ the chiral potentials. From this is it easy to deduce (using the Hamiltonian with the $\hbar$ made explicit, see Appendix G)

$$\mu_{R/L} = \frac{2u\pi\hbar}{K}\tilde{\rho}_{R/L} = \frac{uh}{K}\tilde{\rho}_{R/L} = eU_{R/L} \tag{2.44}$$

which plugged into the formula for the chiral currents $I_{R/L} = j_{e,R/L} = eu\tilde{\rho}_{R/L}$ (2.34b) gives $I_{R/L} = e^2uU_{R/L}K/h$, which yields after using Ohm's law:

$$G = \frac{e^2K}{h} \tag{2.45}$$

which agrees with (2.41). However now, since we have mentionned the analogy with the Landauer-Büttiker formula, we have to wonder if that really makes sense. Here we have considered an isolated quantum wire not connected to anything, and in that context our formula for the conductance with a factor K is correct. However when conducting an experiment in which one tries to measure charge fractionalization, and the asymmetrical ratio of eq. (2.38) for example and the factors $f_\pm$, or when attempting to derive a Landauer formula rigorously, one has to face the problem of modeling the reservoirs properly, a point which has lead to different points of view in the literature. So our calculation of the conductance shouldn't be made with a single isolated LL (Luttinger liquid) but rather with one connected to leads as it is formulated for the FL (Fermi liquid). But then, the theory doesn't fit the experiments anymore, where we find a conductance $G = G_0 = e^2/h$ without the factor K. This subtle point has been addressed in the literature for example in [21, 28] using a Fabry-Perot analysis and here we give a simple interpretation.

We consider the setup of Figure 3 where the quantum wire (interacting LL, $K \neq 1$, with $K < 1$ in the context of repulsive interactions) is connected to long non-interacting leads, *i.e.* $K = 1$ in the leads, with the injection of 1 electron in the middle.

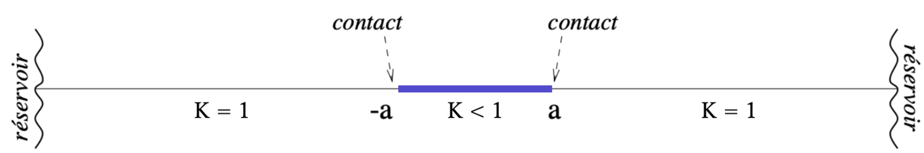

Figure 3: An interacting one-channel wire connected perfectly to very long leads. We suppose the leads are non-interacting: $K = 1$. Figure readapted from [21].

At the contact points ($\pm a$ on Figure 3), we use the continuity of the charge density to find that

$$\rho_R + \rho_L(x \lesssim a) = \rho_R + \rho_L(x \gtrsim a) = 1 \tag{2.46}$$

Respectively with $-a$. Now, we use the equations of Section 2.2.1 to find

$$\begin{aligned} j &= u(\tilde{\rho}_R - \tilde{\rho}_L)(x \lesssim a) = u(\tilde{\rho}_R - \tilde{\rho}_L)(x \gtrsim a) = uK = v_F \\ &\Rightarrow (\tilde{\rho}_R - \tilde{\rho}_L)(x \lesssim a) = K \Rightarrow \rho_R = 1,\ \rho_L = 0,\ \tilde{\rho}_{R/L} = \frac{1 \pm K}{2} \end{aligned} \tag{2.47}$$

in the region of the wire. Therefore we can calculate the current $j_e = e \times j$ and the conductance with the chiral chemical potentials of the setup with leads by interpreting the Fermi velocity as a group velocity:

$$v_F \equiv \frac{1}{\hbar}\frac{\partial E}{\partial k} \tag{2.48}$$

$$I = \int \frac{dk}{2\pi} j_e(k) = \frac{e}{\hbar} \int \frac{dk}{2\pi}\frac{dE}{dk} = \frac{e}{h}\int_{\mu_L}^{\mu_R} dE = \frac{e}{h}(\mu_R - \mu_L) = \frac{e^2}{h}V = GV \tag{2.49}$$

Leading indeed to $G = G_0 = e^2/h$ in an interacting quantum wire in contact with perfect non-interacting leads and reservoirs. We do not see the Luttinger parameter in this formula, this means that conducting an experiment where we try to measure charge fractionalization in an interacting quantum wire connected to leads and reservoirs is actually very difficult, hence why it has been a challenge for the past decades and justifies the look for new ways to observe these fractional charges and this research project.

2.3 Charge fractionalization

Strongly correlated systems in condensed-matter physics are particularly challenging since the interactions completely defy our intuitions and unexpected phenomena can arise. One of the most astonishing properties of some of these systems is *fractionalization* which is the emergence of elementary excitations carrying only part of the quantum number or of the constituent particles of the system. A prime example of this phenomenon is the charge 1/3 Laughlin quasiparticle excitation of the fractional quantum Hall effect (FQHE) at filling $\nu = 1/3$ [25, 29, 30]. This 2-dimensional topological effect was thoroughly studied and awarded the 1998 Nobel prize, and its experimental confirmation consisted in a series of shot noise experiments. The earliest example however of fractionalization is found in one dimension: the exact solution of the Hubbard model by the Bethe Ansatz revealed that the charge and spin of the electron split into two excitations with independant dynamics, known as the *holon* and *spinon* [5, 31]. The evidence of this so-called *spin-charge separation* in Luttinger systems has been obtained via tunneling conductance between parallel wires in a transverse magnetic field [10–16].

Another remarkable effect predicted by LLT is *charge fractionalization* where the extra charge produced by an electron tunneling into the middle of a uniform Luttinger liquid splits up into pieces, moving in opposite directions, which will carry definite fractions $f_+ = \frac{1+K}{2}$ and $f_- = 1 - f_+ = \frac{1-K}{2}$ of the electron charge, determined by the *Luttinger parameter* K that measures the strength of charge interactions in the wire.

As previously mentioned, this phenomenon is deeply rooted in the transport properties of Luttinger liquids, the fractional excitations are carried by the fractional states created by the chiral creation operators defined in (2.36) and (2.6) and they have been predicted and measured via momentum-resolved tunnelling and shot-noise experiments in the early 2000's [10–16]. In particular, the asymmetric ratio between the right- and left-going charges highlighted in (2.38) has been similarily theoretically proposed by [32] and measured in [16] for the asymettry of the chiral currents measured in the wire as fractions of the tunnelling current injected

$$I_L(x,t) = f_+ I(x,t) \tag{2.50a}$$

$$I_R(x,t) = f_- I(x,t) \tag{2.50b}$$

This allows to measure the fractional charges as the fraction of the tunnelling current that goes to left/right in the lower wire, but also directly in the noise in a three-terminal geometry using the Keldish formalism [32] from eqs. (2.50) we find the autocorrelation noise at zero frequency $S(x,x,\omega=0)$

$$S(0,0,\omega=0) = f_+ e \langle I(x=0) \rangle \tag{2.51a}$$

$$S(L,L,\omega=0) = f_- e \langle I(x=L) \rangle \tag{2.51b}$$

which shows that charges found are not just a quantum average and give rise to two counterpropagating pieces of charge f_+e and f_-e where e is the elementary charge of the electron. We should insist that the measurement of these fractional charges done in [16] was tuly a feat because as explained in the previous section, measuring the ratio K between chiral observables as in eq. (2.38) is non-trivial in a regular geometry where a quantum wire is connected to leads.

The most important property of these fractional states is that they are exact eigenstates of the Gaussian Hamiltonian. The proof is done in [12] and requires a proper definition of their Fourier transform because they are anyons [13], which results from the Baker-Campbell-Haussdorf formula (see Appendix A): if $A(x) = e^{-i(a\theta(x) - b\phi(x))}$ then $A(x)A(y) = A(y)A(x)e^{-i\,ab\,\mathrm{sgn}(y-x)}$. Here, the commutation relations of the creation operators defined in (2.36) are anyonic $\tilde{\psi}^\dagger_{R/L}(x)\tilde{\psi}^\dagger_{R/L}(y) = \tilde{\psi}^\dagger_{R/L}(y)\tilde{\psi}^\dagger_{R/L}(x)e^{i\Phi \mathrm{sgn}(x-y)}$ with an anyonic phase

$$\Phi = \pm \frac{Q^2_{R/L}}{K} \tag{2.52}$$

This proves that the creation operators $\tilde{\psi}^\dagger_{R/L}$ do not obey periodic boundary conditions. From the Fourier decomposition of the fields eqs. (2.10) and the expression of the zero modes eqs. (2.24) we find

$$\tilde{\psi}^\dagger_{R/L}(x+L) = e^{\mp i2\pi \frac{Q^2_{R/L}}{K}} \tilde{\psi}^\dagger_{R/L}(x) \tag{2.53}$$

The Fourier transform is defined by

$$\tilde{\psi}^\dagger_{R/L}(q_n) = \frac{1}{\sqrt{L}} \int_0^L dx\, e^{-iq_n x} \tilde{\psi}^\dagger_{R/L}(x) \tag{2.54}$$

where

$$q_n = \frac{2\pi}{L} n \mp \frac{2\pi}{L} \frac{Q^2_{R/L}}{K} = \overline{q_n} \mp \frac{2\pi}{L}\frac{Q^2_{R/L}}{K} \tag{2.55}$$

is a quantized pseudomomentum, with $\overline{q_n}$ the phonon part. The operators $\tilde{\psi}^\dagger_{R/L}(q_n)$ are such that

1. $\tilde{\psi}^\dagger_{R/L}(q_n)\,|\psi_0\rangle$ is an exact eigenstate of the chiral Hamiltonian $H_{R/L}$ with energy

$$E(Q_{R/L}, q_n) = u|\overline{q_n}| + \frac{\pi u}{2L}\frac{Q^2_{R/L}}{K} \tag{2.56}$$

where $|\psi_0\rangle$ is the interacting Ground State. It has a linear dispersion.

2. The states created by the $\tilde{\psi}^\dagger_{R/L}(q_n)$ to which one adds the phonon excitations form a complete set.

It is also important to note that the wave-function of these fractional excitations are exactly the Laughlin many-body wave-function, with exponent $1/K$ [12]. The ground-state is given by

$$\psi_0(\{z_k\}) = \prod_{i<j}|z_i - z_j|^{1/K} \tag{2.57}$$

and the excited states

$$\tilde{\psi}^\dagger_{R/L}(z)\psi_0(\{z_k\}) = \prod_i (z_i^* - z^*)^{Q_{R/L}/K}\prod_{i<j}|z_i - z_j|^{1/K}e^{\pm ik_F Q_{R/L}/K} \tag{2.58}$$

Finding an exact many-body wave-function for an interacting system is something highly non-trivial, and it being a Laughlin wave-function makes a direct analogy with the FQHE that is interesting to mention. We also see in this form that the fractional charges are directly carried by the wave-function in the exponential of the last-term, but that the eigenenergy depends on $Q^2_{R/L}/K$ with a prefactor $1/L$ where L is the system size, which makes the fractional charges again very difficult to measure directly in the energy. In conclusion to the previous sections, we understand that while charge fractionalization is at the heart of interacting Luttinger systems, is it genuinely difficult to measure. In the next section, we provide a new idea to that end.

3 A new measure of fractional charges via the fluctuations of chiral fields

In the Fractional Quantum Hall Effect (FQHE) which we have just mentioned is the epitome of fractionalization phenomena, the fractional charges are revealed with the fluctuations of the current [33]. In the FQHE however, the edge states that are propagating are *chiral* Luttinger liquids which means they move only in one direction whereas in this report, we are considering *non-chiral* LL, in which the fractional charges can move in *both* directions. The goal of this section is to reveal the ratio of (2.38) and the fractional charges observed in transport from equilibrium theory, using the fluctuations of the chiral charge and current. The advantage is that it is numerically attractive and experimentally feasible, and that fluctuations (or generally, Green's functions) are also well admitted as a good probe of many-body physics and phases transitions [12–14], in link with quantum Information theory and entanglement entropies [34, 35], and reveal the unusual properties of LL systems (electron lifetime, conductivity ...) [11, 36, 37] and it has previously been shown they reveal some aspects of quantum Hall systems [38].

3.1 Correlations and fluctuations

As we have stressed before, our Hamiltonian (2.16) is Gaussian in terms of the fields $\phi(x,t)$ and $\theta(x,t)$ which makes the calculation of the Green's functions of these fields very easy to calculate. In this first section, we outline the steps for the calculations of the correlations $\langle\phi(x)\phi(0)\rangle$ and fluctuations $\langle(\phi(x)-\phi(0))^2\rangle$. We will calculate the correlators by two methods: the first one is rather classical, we just take the expressions of the fields as given by eqs. 2.9 and perform a straightforward integration, the other method comes from QFT and involves calculating the path integral with the action. Some steps are skipped in this main body, but detailed in Appendix C.

3.1.1 Classical integral for the time-independant fields

Let's start by calculating explicitely the $\phi\phi$ correlators, the other ones ($\phi\theta$ and $\theta\theta$) follow the same logic.

From eq. (2.9a) (and by forgetting about the zero-mode terms ϕ_0 and θ_0 because they average to 0) we can rewrite

$$\phi(x)\phi(0) = -\sum_{q,p}\mathrm{sgn}(q)\mathrm{sgn}(p)\frac{\pi}{2L}\frac{1}{\sqrt{|q||p|}}(b_q + b^\dagger_{-q})(b_p + b^\dagger_{-p})e^{-\alpha(|q|+|p|)/2}e^{iqx} \tag{3.1a}$$

$$\phi(x)^2 = -\sum_{q,p} \text{sgn}(q)\text{sgn}(p)\frac{\pi}{2L}\frac{1}{\sqrt{|q||p|}}(b_q + b^\dagger_{-q})(b_p + b^\dagger_{-p})e^{-\alpha(|q|+|p|)/2}e^{i(q+p)x} \tag{3.1b}$$

$$\phi(0)^2 = -\sum_{q,p} \text{sgn}(q)\text{sgn}(p)\frac{\pi}{2L}\frac{1}{\sqrt{|q||p|}}(b_q + b^\dagger_{-q})(b_p + b^\dagger_{-p})e^{-\alpha(|q|+|p|)/2} \tag{3.1c}$$

$\theta\theta$ are similarily expressed thanks to eq. (2.9b) and $\phi\theta$ is given by

$$\phi(x)\theta(0) = \sum_{q,p} \text{sgn}(q)\frac{\pi}{2L}\frac{1}{\sqrt{|q||p|}}(b_q + b^\dagger_{-q})(b_p - b^\dagger_{-p})e^{-\alpha(|q|+|p|)/2}e^{iqx} \tag{3.2}$$

Now taking the average value only amounts to calculating the average values of the bosonic creation and annihilation terms: $\langle b_q b_p\rangle$, $\langle b_q b^\dagger_{-p}\rangle$, $\langle b^\dagger_{-q} b_p\rangle$ and $\langle b^\dagger_{-q} b^\dagger_{-p}\rangle$. We already know they are bosonic operators obeying $[b_q, b^\dagger_p] = \delta_{q,p} \Leftrightarrow b_q b^\dagger_p = \delta_{q,p} + b^\dagger_p b_q$ and at zero temperature, the number operator $N = b^\dagger_q b_q$ averages to 0: $\langle b^\dagger_q b_q\rangle = 0$, and the operators bb and $b^\dagger b^\dagger$ also average to zero. Therefore the only non-zero term left out is $\langle b_q b^\dagger_{-p}\rangle = \delta_{q,-p}$:

$$\begin{aligned}\langle\phi(x)\phi(0)\rangle = \langle\theta(x)\theta(0)\rangle &= \frac{\pi}{2L}\sum_q \frac{e^{-\alpha|q|}}{|q|}e^{iqx} \\ &= \frac{1}{2}\int_0^\infty dq\frac{e^{-\alpha q}}{q}\cos(qx) = \mathcal{F}_0(x)\end{aligned} \tag{3.3a}$$

$$\begin{aligned}\langle(\phi(x)-\phi(0))^2\rangle = \langle(\theta(x)-\theta(0))^2\rangle &= \frac{\pi}{2L}\sum_q \frac{e^{-\alpha|q|}}{|q|}\left(2-(e^{iqx}+e^{-iqx})\right) \\ &= \int_0^\infty dq\frac{e^{-\alpha q}}{q}(1-\cos(qx)) = \mathcal{F}_1(x)\end{aligned} \tag{3.3b}$$

$$\begin{aligned}\langle\phi(x)\theta(0)\rangle &= -\frac{\pi}{2L}\sum_q \text{sgn}(q)\frac{e^{-\alpha|q|}}{|q|}e^{iqx} \\ &= -\frac{1}{4}\int_{-\infty}^\infty dq\frac{e^{-\alpha|q|}}{q}e^{iqx} \\ &= -i\int_0^\infty dq\frac{e^{-\alpha q}}{q}\sin(qx) = \mathcal{F}_2(x)\end{aligned} \tag{3.3c}$$

Integrals $\mathcal{F}(x)$ can be calculated thanks to the cutoff $\alpha > 0$ and give

$$\mathcal{F}_0(x) = -\frac{1}{2}\log\left(\alpha^2 + x^2\right) \tag{3.4a}$$

$$\mathcal{F}_1(x) = \frac{1}{2}\log\left(\frac{\alpha^2 + x^2}{\alpha^2}\right) \tag{3.4b}$$

$$\mathcal{F}_2(x) = -i\arctan\left(\frac{x}{\alpha}\right) \tag{3.4c}$$

Now, the reasoning above was actually done with $\tilde{\phi}$ and $\tilde{\theta}$ which are rescaled by $\sqrt{K}^{\pm 1}$ that make the total K's disappear, but let's put them back for consistency:

$$\boxed{\langle(\phi(x)-\phi(0))^2\rangle = K\mathcal{F}_1(x)} \tag{3.5a}$$

$$\boxed{\langle(\theta(x)-\theta(0))^2\rangle = \frac{1}{K}\mathcal{F}_1(x)} \tag{3.5b}$$

$\langle\phi\theta\rangle$ rests unchanged because ϕ has a $\sqrt{K}$ which cancels with the $1/\sqrt{K}$ of θ.

3.1.2 Time-dependant correlations

Using now the standard interaction picture with $\tau = it$ the imaginary time, we can define the time-dependant fields

$$\phi(x,\tau) = e^{\tau H}\phi(x)e^{-\tau H} \tag{3.6a}$$

$$\theta(x,\tau) = e^{\tau H}\theta(x)e^{-\tau H} \tag{3.6b}$$

and use the Baker-Campbell-Haussdorf formula (see Appendix A) to calculate the previous correlation functions for the time-dependant fields $\phi(x,\tau)$, $\theta(x,\tau)$. The *time-ordered* correlation functions are then calculated: $G_{\phi\phi}(x,\tau) = \langle T_\tau(\phi(x,\tau)-\phi(0,0))^2\rangle = \langle \phi(x,\tau)^2 + \phi(0,0)^2 - \Theta(\tau)\phi(x,\tau)\phi(0,0) - \Theta(-\tau)\phi(0,0)\phi(x,\tau)\rangle$ with $\Theta(\tau)$ the Heaviside function. Similarily, $G_{\theta\theta}(x,\tau) = \langle T_\tau(\theta(x,\tau)-\theta(0,0))^2\rangle$ and $G_{\phi\theta}(x,\tau) = \langle T_\tau \phi(x,\tau)\theta(0,0)\rangle$. By doing the same derivation as the previous section we find (see Appendix C.2)

$$\boxed{G_{\phi\phi}(x,\tau) = K\mathcal{F}_1(x,\tau)} \tag{3.7a}$$

$$\boxed{G_{\theta\theta}(x,\tau) = \frac{1}{K}\mathcal{F}_1(x,\tau)} \tag{3.7b}$$

$$G_{\phi\theta}(x,\tau) = \mathcal{F}_2(x,\tau) \tag{3.7c}$$

with

$$\mathcal{F}_1(x,\tau) = \int_0^\infty dq \frac{e^{-\alpha q}}{q}\left(1 - e^{-u|\tau|q}\cos(qx)\right) = \frac{1}{2}\log\left(\frac{x^2 + u|\tau|\alpha^2}{\alpha^2}\right) \tag{3.8a}$$

$$\mathcal{F}_2(x,\tau) = -i\,\mathrm{sgn}(\tau)\int_0^\infty dq\frac{e^{-\alpha q}}{q}e^{-u|\tau|q}\sin(qx) = -i\,\mathrm{sgn}(\tau)\arctan\left(\frac{x}{u|\tau|+\alpha}\right) \tag{3.8b}$$

3.1.3 Path integral

The second method is quite elegant and involves the path integrals. We use the expression of the action (2.22) and using standard results of Gaussian integration, one can show that the matrix M is associated with the average values of the fields

$$M = \frac{1}{k^2(u^2k^2+\omega_n^2)}\begin{pmatrix} k^2u/K & -ik\omega_n \\ -ik\omega_n & k^2uK\end{pmatrix} = \beta\Omega\begin{pmatrix}\langle\theta^*(\mathbf{q})\theta(\mathbf{q})\rangle & \langle\theta^*(\mathbf{q})\phi(\mathbf{q})\rangle \\ \langle\phi^*(\mathbf{q})\theta(\mathbf{q})\rangle & \langle\phi^*(\mathbf{q})\phi(\mathbf{q})\rangle\end{pmatrix} \tag{3.9}$$

Therefore, the correlators $\phi\phi$, $\theta\theta$ and $\phi\theta$ in Fourier space are given by

$$\langle\phi(\mathbf{q_1})\phi(\mathbf{q_2})\rangle = \langle\phi^*(-\mathbf{q_1})\phi(\mathbf{q_2})\rangle = \frac{\beta\Omega\pi uK}{u^2k^2+\omega_n^2}\delta_{-\mathbf{q_1},\mathbf{q_2}} \tag{3.10a}$$

$$\langle\theta(\mathbf{q_1})\theta(\mathbf{q_2})\rangle = \langle\theta^*(-\mathbf{q_1})\theta(\mathbf{q_2})\rangle = \frac{\beta\Omega u\pi}{K(u^2k^2+\omega_n^2)}\delta_{-\mathbf{q_1},\mathbf{q_2}} \tag{3.10b}$$

$$\langle\phi(\mathbf{q_1})\theta(\mathbf{q_2})\rangle = \langle\phi^*(-\mathbf{q_1})\theta(\mathbf{q_2})\rangle = \frac{-i\omega_n\beta\Omega\pi}{k(u^2k^2+\omega_n^2)}\delta_{-\mathbf{q_1},\mathbf{q_2}} \tag{3.10c}$$

Fourier transforming back to real space using $\mathbf{q}r = (kx - \omega_n\tau)$ we find

$$G_{\phi\phi}(x,\tau) = \langle T_\tau(\phi(x,\tau)-\phi(0,0))^2\rangle = \frac{1}{\beta\Omega}\sum_{\mathbf{q}=(k,\omega_n)}\frac{2uK\pi}{u^2k^2+\omega_n^2}(1-\cos(kx-\omega_n\tau)) = K\tilde{\mathcal{F}}_1(x,\tau) \tag{3.11a}$$

$$G_{\theta\theta}(x,\tau) = \langle T_\tau(\theta(x,\tau)-\theta(0,0))^2\rangle = \frac{1}{\beta\Omega}\sum_{\mathbf{q}=(k,\omega_n)}\frac{2u\pi}{K(u^2k^2+\omega_n^2)}(1-\cos(kx-\omega_n\tau)) = \frac{1}{K}\tilde{\mathcal{F}}_1(x,\tau) \tag{3.11b}$$

$$G_{\phi\theta}(x,\tau) = \langle T_\tau\phi(x,\tau)\theta(0,0)\rangle = \frac{1}{\beta\Omega}\sum_{\mathbf{q}=(k,\omega_n)}\frac{-i\omega_n\pi}{k(u^2k^2+\omega_n^2)}e^{i(kx-\omega_n\tau)} = \tilde{\mathcal{F}}_2(x,\tau) \tag{3.11c}$$

we can verify, with ω_n which are bosonic Matsubara frequencies that at zero temperature $\beta\to\infty$ (see [4]):

$$\tilde{\mathcal{F}}_1(x,\tau) = \mathcal{F}_1(x,\tau) = \int_0^\infty dq \frac{e^{-\alpha q}}{q}\left(1 - e^{-u|\tau|q}\cos(qx)\right) = \frac{1}{2}\log\left(\frac{x^2 + u|\tau|\alpha^2}{\alpha^2}\right) \tag{3.12a}$$

$$\tilde{\mathcal{F}}_2(x,\tau) = \mathcal{F}_2(x,\tau) = -i\,\mathrm{sgn}(\tau)\int_0^\infty dq \frac{e^{-\alpha q}}{q} e^{-u|\tau|q}\sin(qx) = -i\,\mathrm{sgn}(\tau)\arctan\left(\frac{x}{u|\tau| + \alpha}\right) \tag{3.12b}$$

which agrees with the previous result eqs. (3.7).

3.2 Interpretation of the fields and mapping to real observables

What do the fields ϕ and θ and their fluctuations mean, and how do we measure them ? The goal in this section is to give a physical interpretation of these fields to better understand and measure them, and to map our Luttinger liquid model to a fermionic chain model and then to a spin model such that we can easily perform numerical calculations on these operators.

From Section 2.2 it is quite obvious that ϕ and θ correspond respectively to the charge and current : we recall eqs. (2.23) : $\rho(x,t) = -\frac{1}{\pi}\partial_x\phi(x,t)$ and $j(x,t) = \frac{1}{\pi}\partial_x\theta(x,t)$ and see that if we consider the integral operators N (or Q) as the total number of charge operator, and J as the total current operator, the fluctuations of ϕ and θ correspond to the fluctuations of charge, and current

$$G_{\phi\phi} = \pi^2 G_{NN} = \pi^2\langle(N(x,t) - N(0,0))^2\rangle \tag{3.13a}$$

$$G_{\theta\theta} = \pi^2 G_{JJ} == \pi^2\langle(J(x,t) - J(0,0))^2\rangle \tag{3.13b}$$

Now that we know the physical meaning of these fluctuations without any calculations, let's express them in the language of second quantization and of spins to mathematically prove eqs. (3.13). First we start from the free Hamiltonian (2.12) which is obtained from a free fermion model on a 1D lattice: the standard tight-binding model

$$\boxed{H_0 = -t\sum_j c_j^\dagger c_{j+1} + h.c.} = \sum_k \varepsilon(k) c_k^\dagger c_k \tag{3.14}$$

where $\varepsilon(k) = -2t\cos(k)$. The Hamiltonian H_0 can be transformed into an XX spin-1/2 chain thanks to the famous Jordan-Wigner transformation (see Appendix E) and yields

$$H_0 = -t\sum_j S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+ = -2t\sum_j S_j^x S_{j+1}^x + S_j^y S_{j+1}^y \tag{3.15}$$

In the same manner we can map the interacting fermions on a 1D lattice model $H = H_0 + H_I$ with

$$\boxed{H_I = U\sum_j n_j n_{j+1} = \sum_{k,k',q} U(q) c_{k+q}^\dagger c_k c_{k'-q}^\dagger c_{k'} = \sum_q U(q)\rho(q)\rho(-q)} \tag{3.16}$$

where $U(q)$ corresponds to the g_2 and g_4 terms as $\rho(q)$ is split into left- and right-movers as $\rho(q) = \rho_R(q) + \rho_L(q)$ (see Section 2.1.2), to a spin-1/2 XXZ chain absorbing some terms in the chemical potential (see Appendix E):

$$H_I = U\sum_j S_j^z S_{j+1}^z \tag{3.17}$$

If we suppose $g_2 = g_4 = g$ for simplicity, the final Hamiltonian reads

$$\boxed{H = -2t\sum_j S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + U\sum_j S_j^z S_{j+1}^z} \tag{3.18}$$

Where the link between t and U in the chain model and u and K in the continuous model is given by

$$uK = v_F = 2t \text{ and } \frac{u}{K} = v_F + \pi U = 2t + \pi U \tag{3.19a}$$

$$\Leftrightarrow u = \sqrt{2t(2t + \pi U)} \text{ and } K = \sqrt{\frac{2t}{2t + \pi U}} \tag{3.19b}$$

In a perturbative approximation [5]. But since the spin-chain model is exactly solvable by the Bethe Ansatz, we can get exact values of the LL parameter [6, 31, 34, 35, 39, 40]:

$$u = \frac{t\sqrt{1-(U/2t)^2}}{1-\arccos(-U/2t)/\pi} \text{ and } K = \frac{\pi}{2\arccos(-U/2t)} \tag{3.20}$$

From the expression of $S^z(x)$ in terms of fermionic operators thanks to the Jordan-Wigner transformation,

$$S^z(x)\left(+\frac{1}{2}\right) = c^\dagger c(x) = \rho(x) = \underbrace{\rho_+(x)}_{-\frac{1}{\pi}\nabla\phi} + (e^{-2ik_F x}\psi_R^\dagger(x)\psi_L(x) + h.c.) \tag{3.21}$$

it is clear that we can interpret S_j^z as $\nabla\phi$ and

$$\nabla\phi = \pi(S_i^z + 1/2) \tag{3.22a}$$

$$\sum_{i=0}^{L} S_i^z = -\frac{1}{\pi}(\phi(L)-\phi(0))\left(-\frac{L}{2}\right) \tag{3.22b}$$

Therefore, calculating the fluctuations of $\sum_i S_i^z$

$$G_{S^zS^z}(x) = \langle(\sum_{i=0}^{x} S_i^z)^2\rangle - \langle\sum_{i=0}^{x} S_i^z\rangle^2 = \sum_{i,j=0}^{x}\left(\langle S_i^z S_j^z\rangle - \langle S_i^z\rangle\langle S_j^z\rangle\right) \tag{3.23}$$

or similarily of the total density/number operator $Ni = c_i^\dagger c_i$

$$G_{NN}(x) = \langle(\sum_{i=0}^{x} N_i)^2\rangle - \langle\sum_{i=0}^{x} N_i\rangle^2 = \sum_{i,j=0}^{x}\left(\langle N_i N_j\rangle - \langle N_i\rangle\langle N_j\rangle\right) \tag{3.24}$$

should lead to the same result as the calculations for the fluctuations of ϕ done in eqs. (3.7) (with a factor $1/\pi^2$), as predicted from eqs. (3.13). These calculations are done in [35] and we have redone them Appendix F, and indeed we find $G_{S^zS^z}(x) \sim \log(x)$ as expected.

For the field θ it's not as obvious and most importantly, it has never been done before. To interpret theta we will use the local continuity equation $\nabla j_i + \partial_t\rho_i = 0$ with $\partial_t\rho_i = i[H,\rho_i] = i[H, c_i^\dagger c_i] = -it(c_{i+1}^\dagger c_i - c_i^\dagger c_{i-1} + c_{i-1}^\dagger c_i - c_i^\dagger c_{i+1})$. Therefore we can write

$$\begin{aligned}
\nabla j_i &= it(\underbrace{c_{i+1}^\dagger c_i - c_i^\dagger c_{i-1}}_{\equiv\nabla c_i^\dagger c_{i-1}} + \underbrace{c_{i-1}^\dagger c_i - c_i^\dagger c_{i+1}}_{\equiv -\nabla c_{i-1}^\dagger c_i}) \\
&= it\nabla(c_i^\dagger c_{i-1} - c_{i-1}^\dagger c_i) \\
\Leftrightarrow j_i &= it(c_i^\dagger c_{i-1} - c_{i-1}^\dagger c_i)
\end{aligned} \tag{3.25}$$

where writing $c_i = \psi_R e^{ik_F x} + \psi_L e^{-ik_F x}$, $c_{i\pm1} = \psi_R e^{ik_F(x\pm1)} + \psi_L e^{-ik_F(x\pm1)}$ with $k_F = \frac{\pi}{2}$ leads to $c_i^\dagger c_{i-1} - c_{i-1}^\dagger c_i = -2i(\rho_R(x) - \rho_L(x)) = -\frac{2i}{\pi}\nabla\theta(x)$. Therefore we find $j(x) = \frac{2t}{\pi}\nabla\theta(x)$ which is consistent with eqs. (2.23) and (2.28). So our equations are physically consitent which is good news, and we can interprete θ in terms of fermionic operators:

$$\sum_{i=0}^{L}(c_i^\dagger c_{i-1} - c_{i-1}^\dagger c_i) = \frac{-2i}{\pi}(\theta(L)-\theta(0)) \tag{3.26}$$

We can now transform fermionic operators into spin operators, either directly by doing a JW transform on the $c_i^{(\dagger)}$'s or by using the local continuity equation with H written in terms of spins, and $\rho_i = S_i^z + \frac{1}{2}$. They both lead to the same result:

$$
\begin{aligned}
\nabla j_i &= 2t(\underbrace{S^x_{i+1}S^y_i - S^x_i S^y_{i-1}}_{\equiv \nabla S^x_i S^y_{i-1}} + \underbrace{S^y_i S^x_{i-1} - S^y_{i+1}S^x_i}_{\equiv -\nabla S^y_i S^x_{i-1}}) \\
&= 2t\nabla(S^x_i S^y_{i-1} - S^y_i S^x_{i-1}) \\
\Leftrightarrow j_i &= 2t(S^x_i S^y_{i-1} - S^y_i S^x_{i-1}) = \frac{2t}{\pi}\nabla\theta_i
\end{aligned}
\tag{3.27}
$$

Therefore, we see that indeed θ does not have a trivial expression in terms of spin operators, and reads

$$
\nabla\theta = \pi(S^x_i S^y_{i-1} - S^y_i S^x_{i-1}) \tag{3.28a}
$$

$$
\sum_{i=0}^{L}(S^x_i S^y_{i-1} - S^y_i S^x_{i-1}) = \frac{1}{\pi}\left(\theta(L) - \theta(0)\right) \tag{3.28b}
$$

We can then expect that the fluctuations of θ are recovered by the fluctuations of $\sum_i(S^x_i S^y_{i-1} - S^y_i S^x_{i-1})$, or equivalently of the $\sum_i(c^\dagger_i c_{i-1} - c^\dagger_{i-1}c_i)$. These calculations are again carried out very carefully (for the first time !) in Appendix F and we again recover $G_{S^xS^y}(x) \sim \log(x)$ in agreement with Section 3.1.

$$
G_{S^zS^z}(x) = G_{\phi\phi}(x) = K\mathcal{F}_1(x) \tag{3.29a}
$$

$$
G_{S^xS^y}(x) = G_{\theta\theta}(x) = \frac{1}{K}\mathcal{F}_1(x) \tag{3.29b}
$$

These fluctuations can easily be measured, as has already been done in [33–35], and for that we propose that the single quantum wire is subject to measurement from a grid that uses a capacity or an ammeter to measure the fluctuations of the charge and current in the wire, which correspond to the fluctuations of ϕ and θ. See Figure 4.

3.3 Fluctuations of the chiral fields

After having calculated the correlations and fluctuations of the non-chiral fields, and having interpreted them as the fluctuations of the charge and current and in the language of fermions and spins, we notice that these correlators yield the factor K in front, which is a first hint that it might be helpful to probe charge fractionalization. Now, if we go back to Section 2.3, we notice that the fractional charges are created by the creation operators carrying the *chiral* fields $\theta_{R/L} = \theta \mp \phi$ and $\vartheta_{R/L}$ or $\varphi_{R/L}$ defined in (2.31a) and (2.34a) as in eqs. (2.6) and (2.36). So by calculating and measuring the fluctuations of these fields, could we isolate and easily observe the fractional charges ? It turns out the answer is yes, but we have to be careful what we calculate the fluctuations of.

For the first case, indeed the fluctuations $G_{\theta_{R/L}\theta_{R/L}}(x,\tau) = \langle T_\tau(\theta_{R/L}(x,\tau) - \theta_{R/L}(0,0))^2\rangle$ will measure the fractional charges $f_\pm$ but for the second case, while we might be tempted to simply compute the fluctuations of $\vartheta_{R/L}$ or $\varphi_{R/L}$, we actually have to calculate the fluctuations of the excitations that are created with the corresponding carried charge: the fluctuations of $Q_{R/L}\vartheta_{R/L}$ or equivalently of $J_{R/L}\varphi_{R/L}/v_F$. Indeed, the operators creating the chiral states as defined in (2.36) carry in the exponential the chiral charge (operator) times the chiral fields. We write these fluctuations $G_{\vartheta\vartheta}$ and $G_{\varphi\varphi}$. For each case, the correlators are pretty straightforward to calculate them using results from Section 3.1:

$$
\begin{aligned}
G_{\theta_{R/L}\theta_{R/L}}(x,\tau) &= \langle T_\tau(\theta_{R/L}(x,\tau) - \theta_{R/L}(0,0))^2\rangle \\
&= \cdots = (G_{\theta\theta}(x,\tau) + G_{\phi\phi}(x,\tau) \mp (G_{\phi\theta}(x,\tau) + G_{\theta\phi}(x,\tau))) \\
&= \left(\frac{1}{K} + K\right)\mathcal{F}_1(x,\tau) \mp 2\mathcal{F}_2(x,\tau) \\
&\simeq \frac{2}{K}(f_+^2 + f_-^2)\mathcal{F}_1(x,\tau)
\end{aligned}
\tag{3.30a}
$$

$$
\begin{aligned}
G_{\vartheta\vartheta}(x,\tau) &= \langle T_\tau Q_{R/L}^2(\vartheta_{R/L}(x,\tau)-\vartheta_{R/L}(0,0))^2\rangle \\
&= \cdots = Q_{R/L}^2\left(G_{\theta\theta}(x,\tau)+\frac{1}{K^2}G_{\phi\phi}(x,\tau)\mp\frac{1}{K}(G_{\phi\theta}(x,\tau)+G_{\theta\phi}(x,\tau))\right) \\
&= \frac{2}{K}Q_{R/L}^2\{\mathcal{F}_1(x,\tau)\mp\mathcal{F}_2(x,\tau)\} \\
&\simeq \frac{2}{K}Q_{R/L}^2\mathcal{F}_1(x,\tau)
\end{aligned}
\tag{3.30b}
$$

$$
\begin{aligned}
G_{\varphi\varphi}(x,\tau) &= \langle T_\tau \frac{J_{R/L}^2}{v_F^2}(\varphi_{R/L}(x,\tau)-\varphi_{R/L}(0,0))^2\rangle \\
&= \cdots = \frac{J_{R/L}^2}{v_F^2}\left(K^2G_{\theta\theta}(x,\tau)+G_{\phi\phi}(x,\tau)\mp(G_{\phi\theta}(x,\tau)+G_{\theta\phi}(x,\tau))\right) \\
&= 2K\frac{J_{R/L}^2}{v_F^2}\{\mathcal{F}_1(x,\tau)\mp\mathcal{F}_2(x,\tau)\} \\
&\simeq \frac{2}{K}Q_{R/L}^2\mathcal{F}_1(x,\tau)
\end{aligned}
\tag{3.30c}
$$

where the $\mathcal{F}_2$ terms are subleading and neglected compared to the logarithmic $\mathcal{F}_1$ terms and essentially correspond to cross-term left-right correlations which decouple/become independant and vanish at large x. For the calculations with the chiral fields $\vartheta_{R/L}$ and $\varphi_{R/L}$, we see that these fluctuations are quantified by $\frac{Q_{R/L}^2}{K}$, which corresponds in fact to the anyonic phase Φ of the $\tilde{\psi}_{R/L}^\dagger$ operators and the eigenenergy "quantum" of the and $\tilde{\psi}_{R/L}^\dagger\,|\psi_0\rangle$ states as shown in Section 2.3. After injection of one additional electron in the system, this means that the fluctuations of these chiral states are characterized by a factor $\frac{2f_\pm^2}{K}$ where $f_\pm$ are the fractional charges.

For the fluctuations of $\theta_{R/L}$ however, the situation is very different: here the result is obtained at equilibrium, and the prefactor measures the addition of the fluctuations due to the fractional charges like $2/K(Q_R^2+Q_L^2)$, when the $1/K$ factor comes from the fact that $\theta_{R/L}$ has the same dimension as θ. Essentially, this result ensures that for one electron the addition of a fractional charge f_+ and a fractional charge f_-, we recover $f_+ + f_- = 1$. Indeed, we verify that if $K=1$ (no interactions), we are in the classical situation where $f_+=1$ and $f_-=0$. Additionnally, measuring the fluctuations of $\theta_{R/L}$ makes more sense: from eq. (3.30a) we conclude that $G_{\theta_{R/L}\theta_{R/L}}\simeq G_{\phi\phi}+G_{\theta\theta}$ especially at large x therefore, measuring the fractional charges $f_\pm$ only accounts to summing the measurements of the fluctuations of the charge and current, which is easily accessible via the apparatus proposed in Figure 4.

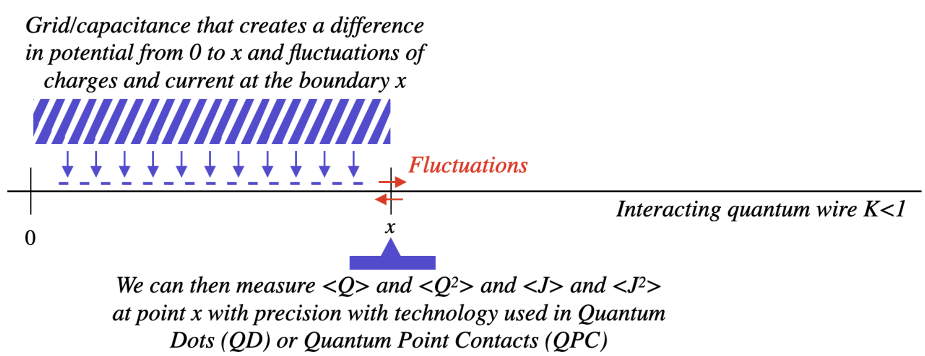

Figure 4: Proposed experimental setup to measure the fluctuations of charges and current, which correspond to the flucutations of ϕ and θ, which added should yield the fractional charges $f_+^2+f_-^2$ as a prefactor to the logarithmic dependance on distance x (see eq. (3.30a)). A metallic gate/grid controls the electron density in a part of the sample (for 1D quantum wires, it is usually a GaAs/AlGaAs sample) of size x by applying a voltage, and the fluctuations of charges and current at point x are precisely measured with modern technology that is able to measure the number of electrons, and their fluctuations at the individual, atomic level using a metallic scanning tip, see similar experiments in 1 and 2D [41, 42].

Lastly, we note that the logarithmic dependance on the distance x in the calculation of the fluctuations comes from everything being done in the low temperature limit, $\beta\to\infty$, otherwise a simple Statistial Physics argument would make us excpect a linear dependance on x, which is indeed the case when we place ourselves outside of this limit [34, 35].

4 Numerical analysis

Now that we have proposed a novel method for probing the fractional charges in a Luttinger liquid through the fluctuations of the chiral currents $G_{\theta_{R/L}\theta_{R/L}}$, we will numerically investigate these fluctuations, and compare them to $G_{\phi\phi} + G_{\theta\theta}$. To do so, we could think of several methods: the first that comes to mind is exact diagonalisation (ED) which would give *exact* results, but performing ED on big system sizes becomes quickly impossible, we can only have an efficiency on a system of a maximum of about 20 electrons, which is too little when we want to observe behaviors at $L \to \infty$ on a discrete chain model. Another mothod would be a random sampling like in Monte Carlo which is often used in statistical physics, but as soon as we work with fermions in the quantum world, commutation problems known as *the sign problem* appear and render calculations unfeasable. So lastly, we resort to variational methods with decimation and renormalization procedures which have shown to be very efficient, especially the so-called *Density Matrix Renormalization Group* (DMRG).

4.1 DMRG

DMRG is a powerful numerical technique used to study the ground state properties of strongly correlated one-dimensional quantum systems invented in the 1990's [43–45]. It works by iteratively optimizing the system's wavefunction within a truncated basis that captures the most significant degrees of freedom. Initially, the system is divided into blocks, and the density matrix of one block is constructed by tracing out the degrees of freedom of the other block. The eigenstates corresponding to the largest eigenvalues of the density matrix are retained to form a reduced basis. This process, known as renormalization, systematically increases the size of the blocks while maintaining computational efficiency. By sweeping through the system and repeatedly optimizing the basis, DMRG achieves high accuracy in capturing the ground state and low-energy excitations of 1D quantum systems. Once the ground state and eigenenergies of the system are known, calculating averages of quantum operators becomes easy.

Reinterpreting the fields ϕ and θ in terms of fermionic creation/annihilation and spin operators (which are easily expressed in terms of matrices) therefore maked the calculations easy to perform using DMRG. Here, we will verify that we can observe the fractional charges $f_\pm = (1 \pm K)/2$ in their fluctuations using DMRG. The first step is to calculate the fluctuations $G_{\phi\phi}(x)$ and $G_{\theta\theta}(x)$ independently for different values of U/t and check that they behave as $\log(x)$ with the prefactors K and K^{-1} that follow the expected result obtained from the Bethe-Ansatz solution eq. (3.20). In Figures 5 and 6 we show $G_{\phi\phi}(x)$ obtained from DMRG calculations (more details on the technical implementation in Appendix I) and how we extract its prefactor for different values of U/t, which are then all plotted on Figure 9. This numerical calculation and results (of how the prefactor of $G_{\phi\phi}(x)$ scales exactly like $K(U)\log(x)$ with $K(U)$ as in (3.20)) is already known [34, 35], but the same thing can be done for $G_{\theta\theta}(x)$ and we realize it here for the first time. We indeed verify the logarithmic dependance on the distance x and its prefactor as a function of U/t is extracted and shown in Figure 10.

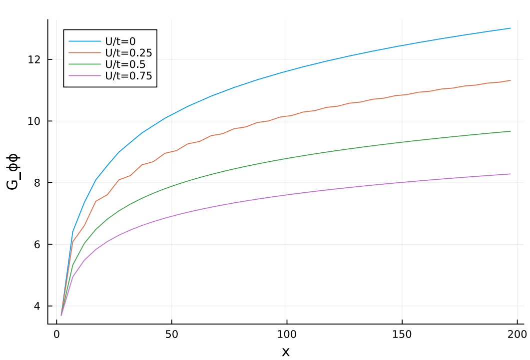

Figure 5: Plot of $G_{\phi\phi}$ for various values of U/t as a function of x : we clearly see the logarithmic dependance on x.

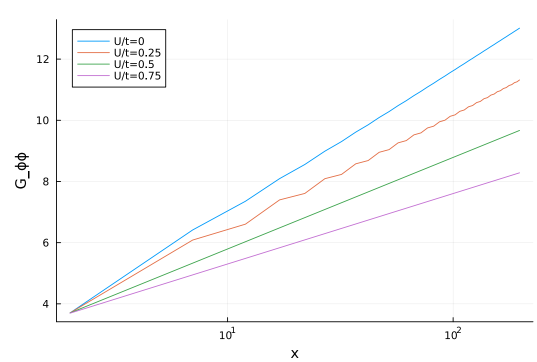

Figure 6: Same plot in x-log scale : from the slope of the "straight" line we can extract the prefactor K.

For the current fluctuations, we have to note that the fit is not exactly just $a * \log(x) + b$ but $a * \log(x) + b * x + c$ with b small but non-zero otherwise we don't have the right value for $a = K^{-1}$.

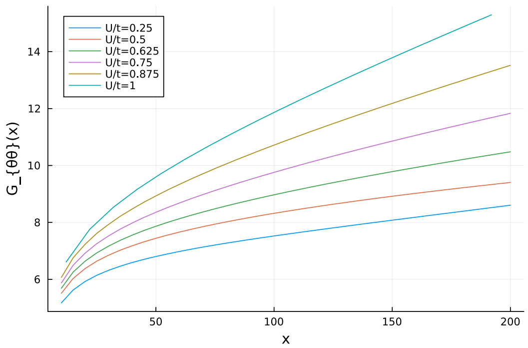

Figure 7: Plot of $G_{\theta\theta}(x)$ for various values of U/t as a function of x. We see a log dependance on x which seems to be dominated by a linear dependance on x at large U/t. Two fitting models are shown in the Figure on the right.

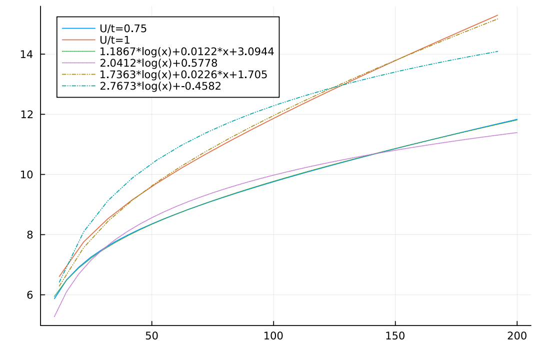

Figure 8: Focus on two values of U/t with a $a \log(x) + b$ fit and a $c * \log(x) + dx + e$ fit : we see the second fit is more accurate and the prefactor $c = K^{-1}$ is the correct one used to exctract the $K(U)$ dependance that is plotted in Figure 10.

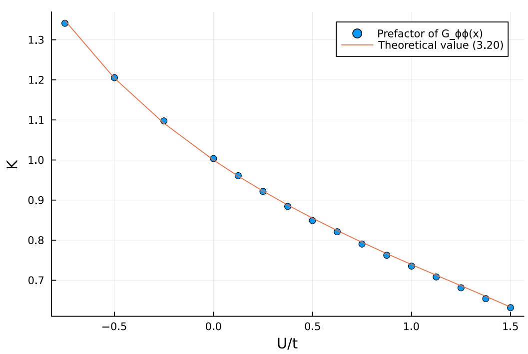

Figure 9: Fit of the prefactor of $G_{\phi\phi}(x)$ which should be K compared with eq (3.20). $U < 0$ corresponds to attractive interactions. We do not study this regime here but just use it for the fit.

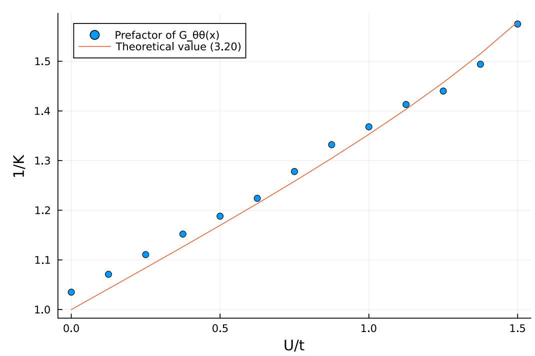

Figure 10: Fit of the prefactor of $G_{\theta\theta}(x)$ which should be K^{-1} compared with eq (3.20). Just as for the previous figure with the fit for $G_{\phi\phi}$, we see good agreement with theory.

We find a very good agreement with theory on how $G_{\phi\phi}(x)$ and $G_{\theta\theta}(x)$ scale with x and with U/t. From there, we can calculate directly $G_{\theta_{R/L}\theta_{R/L}}(x)$, look at the evolution of its prefactor (prefactor to the logarithmic dependance on x) as a function of U/t and check for some fixed values of this ratio, $G_{\theta_{R/L}\theta_{R/L}} - (G_{\phi\phi} + G_{\theta\theta})$ as a function of x. This is done in the following : Figures 11, 12 and 13.

5 Conclusion

In this report, we have introduced and studied the theory of Luttinger Liquids, focusing on their distinctive transport and conductance properties which give rise to the intriguing phenomenon of *charge fractionalization*, wherein an electron's charge appears to split into fractional components f_+ and f_-. By going through the derivation of the theory in the first part of this report, we have highlighted why this phenomenon is deeply rooted in the theory of Luttinger Liquids and is a unique feature of one-dimensional quantum systems.

Then, after reviewing this phenomenon, we have developed and presented a novel method for probing these fractional charges through the fluctuations of chiral currents, deriving our results both analytically and numerically,

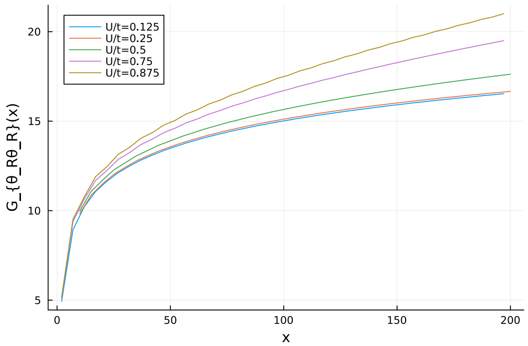

Figure 11: Plot of $G_{\theta_R\theta_R}(x)$ for different values of U/t.

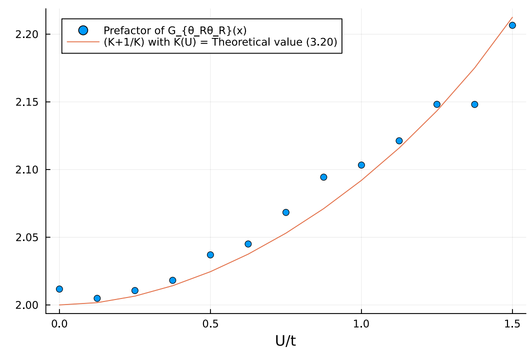

Figure 12: Plot of the pefactor of $G_{\theta_R\theta_R}$ with fitting curve from equation (3.30a). We see good agreement with theory.

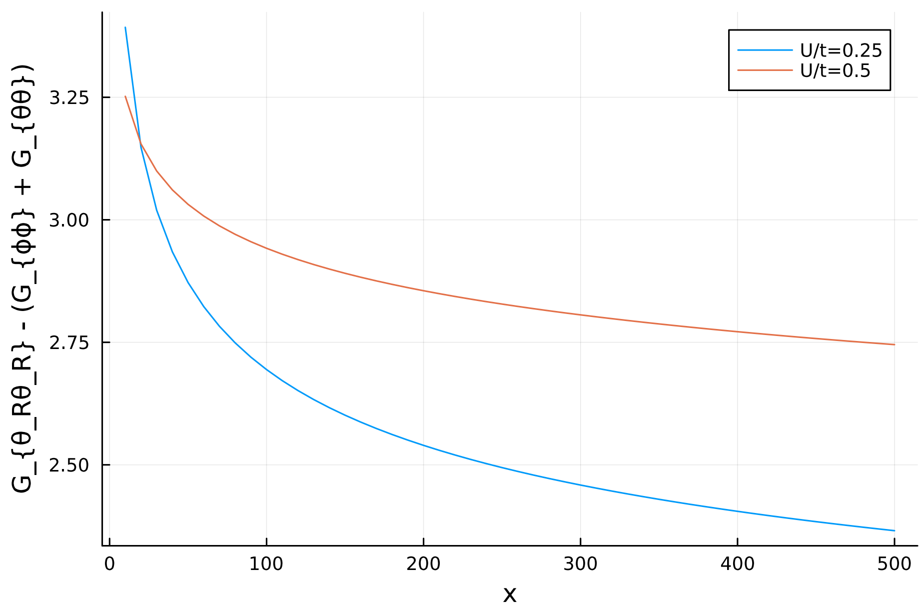

Figure 13: Difference between the fluctuations of the chiral current θ_R and the sum of the charge and current fluctuations, as a function of x for 2 fixed values of U/t.

and proposing an experimental setup to verify our hypothesis.

While the scope of this work was limited by the duration of the internship, numerous avenues for future research remain. As we already know there is a link between charge fluctuations and entanglement entropy [34, 35], one promising direction is to turn to quantum information theory and for example start looking at mapping between quantities of QIT and the current fluctuations. Additionally, further investigation into the relationship between charge fractionalization and the edge modes of the quantum Hall effect could provide valuable insights.

Applying our probing method to a system with two wires also presents an exciting opportunity to study a special kind of drag effect, where the interaction between the wires leads to novel transport phenomena, see eg. [38]. Moreover, examining the temperature dependence of charge fractionalization could reveal how thermal fluctuations influence this phenomenon and further elucidate the behavior of Luttinger liquids under different conditions.

In summary, our study of Luttinger liquids and charge fractionalization has opened up several potential research directions that promise to advance our understanding of one-dimensional quantum systems and their applications in quantum technology, which I am very excited to continue researching about during my PhD.

Appendix

A Commutators

A.1 Useful general formulae

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \text{ if } [A,[A,B]] = [B,[A,B]] = 0 \tag{A.1a}$$

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots} \tag{A.1b}$$

$$e^A B e^{-A} = B + \sum_{n>0} \frac{1}{n!}[A,[\cdots,[A,B]]]_n \tag{A.1c}$$

$$[A, e^{\lambda B}] = \lambda [A,B] e^{\lambda B} \text{ if } [A,[A,B]] = [B,[A,B]] = 0 \tag{A.1d}$$

A.2 Field commutators

$$[\phi(x), \theta(y)] = i\frac{\pi}{2}\mathrm{sgn}(y-x) = -i\frac{\pi}{2}\mathrm{sgn}(x-y) \tag{A.2a}$$

$$[\nabla\phi(x), \theta(y)] = -i\pi\delta(x-y) \tag{A.2b}$$

$$[\phi(x), \nabla\theta(y)] = i\pi\delta(x-y) \tag{A.2c}$$

$$[\vartheta_R(x), \vartheta_R(y)] = [\theta(x) - \frac{\phi(x)}{K}, \theta(y) - \frac{\phi(y)}{K}] = -\frac{1}{K}\big([\theta(x),\phi(y)] - [\phi(x),\theta(y)]\big) = \frac{1}{K} i\pi \mathrm{sgn}(x-y) \tag{A.2d}$$

$$[\vartheta_L(x), \vartheta_L(y)] = -[\vartheta_R(x), \vartheta_R(y)] = -\frac{1}{K} i\pi\mathrm{sgn}(x-y) \tag{A.2e}$$

$$[\vartheta_R(x), \nabla\vartheta_R(y)] = -\frac{1}{K}\big([\theta(x), \nabla\phi(y)] + [\phi(x), \nabla\theta(y)]\big) = -\frac{2i\pi}{K}\delta(x-y) \tag{A.2f}$$

$$[\vartheta_L(x), \nabla\vartheta_L(y)] = -[\vartheta_R(x), \nabla\vartheta_R(y)] = \frac{2i\pi}{K}\delta(x-y) \tag{A.2g}$$

A.3 Density commutators

$$[\tilde\rho_R(x), \vartheta_R(y)] = \frac{K}{2\pi}[\partial_x\vartheta_R(x), \vartheta_R(y)] = i\delta(x-y) \tag{A.3a}$$

$$[\tilde\rho_L(x), \vartheta_L(y)] = -\frac{K}{2\pi}[\partial_x\vartheta_L(x), \vartheta_L(y)] = i\delta(x-y) \tag{A.3b}$$

$$[\tilde\rho_R(x), \tilde\rho_R(y)] = \frac{K^2}{4\pi^2}[\partial_x\vartheta_R(x), \partial_y\vartheta_R(y)] = \frac{K^2}{4\pi^2}\partial_x[\vartheta_R(x), \partial_y\vartheta_R(y)] = -\frac{iK}{2\pi}\partial_x\delta(x-y) \tag{A.3c}$$

$$[\tilde\rho_L(x), \tilde\rho_L(y)] = \frac{K^2}{4\pi^2}[\partial_x\vartheta_L(x), \partial_y\vartheta_L(y)] = \frac{K^2}{4\pi^2}\partial_x[\vartheta_L(x), \partial_y\vartheta_L(y)] = \frac{iK}{2\pi}\partial_x\delta(x-y) \tag{A.3d}$$

A.4 Charge and current commutators

We use $Q = -\frac{1}{\pi}\int_0^L dx\, \partial_x\phi(x) = -\frac{1}{\pi}(\phi(L) - \phi(0))$ and $J = \frac{1}{\pi}\int_0^L dx\, \partial_x\theta(x) = \frac{1}{\pi}(\theta(L) - \theta(0))$ and calculate their commutators with the fields at point $0 < x < L$

$$[Q, J] = -\frac{1}{\pi^2}\big([\phi(L), \theta(L)] - [\phi(L), \theta(0)] - [\phi(0), \theta(L)] + [\phi(0), \theta(0)]\big) = 0 \tag{A.4a}$$

$$[Q, \theta(x)] = -\frac{1}{\pi}\big([\phi(L), \theta(x)] - [\phi(0), \theta(x)]\big) = i \tag{A.4b}$$

$$[J, \phi(x)] = \frac{1}{\pi}\big([\theta(L), \phi(x)] - [\theta(0), \phi(x)]\big) = i \tag{A.4c}$$

$$[Q, \partial_x\theta(x)] = [J, \partial_x\phi(x)] = 0 \tag{A.4d}$$

B Bosonization

Here we will sketch only the outline of the steps to rewrite the Hamiltonian (2.2) conviniently as (2.12). First and foremost, one must note that for this next step, if one wants to properly do it, the normal-ordering[3] should be taken (what we have not mentionned before) and therefore "naively" using the previous formulae will not give the right results. They serve only as to understand the physical idea but are mathematically not exact. The first step is to notice the commutators $[\phi(x), \theta(y)] = i\pi/2 \ \mathrm{sgn}(y-x)$, $[\phi(x), \nabla\theta(y)] = -[\nabla\phi(x), \theta(y)] = i\pi\delta(y-x)$ (see Appendix A). This last relation implies $\phi(x)$ and $\frac{1}{\pi}\nabla\theta(x) := \Pi(x)$ are *conjugate variables*. Then, we rewrite (kind of an Ansatz, details in the references) the creation and annihilation operators (*bosonization formulae*):

$$\psi_R(x) = \frac{\eta_R}{\sqrt{2\pi\alpha}} e^{i[\theta(x)-\phi(x)]} \tag{B.1a}$$

$$\psi_L(x) = \frac{\eta_L}{\sqrt{2\pi\alpha}} e^{i[\theta(x)+\phi(x)]} \tag{B.1b}$$

Where $\eta_{R/L}$ are the so-called "Klein factors" ensuring the anti-commutation properties of the fermionic fields $\psi_{R/L}$ (let's not care about them now, and just accept that $\eta_r^\dagger \eta_{r'} = cst \in \mathbb{C}$ if $r, r' \in \{R, L\}$). They are written this way to verify eqs (2.5) but taking into account everything that's not included in the "naive" expressions (2.3a) (that is to say, not simply writing $\rho = \psi^\dagger\psi$ but a slightly modified expression accounting for normal-ordering). First we will verify that this works and familiarize ourselves with how to use these operators, and then plug the expression of the ψ's in the Hamiltonian (2.2) to finally rewrite the free Hamiltonian in a more convenient way. This process is subtle, since normal ordering of these operators is requiblue (but often, implied and not explicitely mentionned). We use the so-called *point-splitting* technique and calculate the ρ's and H_0 using

$$\rho_{R/L}(x) = \lim_{\varepsilon \to 0} \left[\psi^\dagger_{R/L}(x+\varepsilon)\psi_{R/L}(x) - \langle \psi^\dagger_{R/L}(x+\varepsilon)\psi_{R/L}(x)\rangle \right] \tag{B.2}$$

The average term in the brackets of (B.2) comes from properly taking into account normal-ordering (see refs). So that, using the Baker-Campbell-Haussdorf formula and its normal-ordeblue equivalent[4],

$$\begin{aligned}
\rho_R(x) &= \lim_{\varepsilon\to 0}\left[\psi^\dagger_R(x+\varepsilon)\psi_R(x) - \underbrace{\langle\psi^\dagger_R(x+\varepsilon)\psi_R(x)\rangle}_{1/\varepsilon}\right] \\
&= \frac{i}{2\pi}\lim_{\varepsilon\to 0}\left[e^{-i[\theta(x+\varepsilon)-\phi(x+\varepsilon)]}e^{i[\theta(x)-\phi(x)]} - \frac{1}{\varepsilon}\right] \\
&= \frac{i}{2\pi}\lim_{\varepsilon\to 0}\left[e^{-i[(\theta(x+\varepsilon)-\theta(x))-(\phi(x+\varepsilon)-\phi(x))]} * \frac{1}{\varepsilon} - \frac{1}{\varepsilon}\right] \\
&= \frac{i}{2\pi}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[e^{-i\varepsilon[\partial_x\theta(x)-\partial_x\phi(x)]} - 1\right] \\
&= \frac{i}{2\pi}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[1 - i\varepsilon(\partial_x\theta(x) - \partial_x\phi(x)) - 1\right] \\
&= \frac{1}{2\pi}\left(\partial_x\theta(x) - \partial_x\phi(x)\right)
\end{aligned} \tag{B.3}$$

and similarily

$$\rho_L(x) = \frac{-1}{2\pi}\left(\partial_x\theta(x) + \partial_x\phi(x)\right) \tag{B.4}$$

which matches with (2.5). Now that we have verified how to calculate the ρ's and how to deal with the ψ's, we can plug them into eq. (2.2). For that, and just like before, we will use point-splitting and need to calculate

$$\lim_{\varepsilon\to 0}\left[\psi^\dagger_{R/L}(x+\varepsilon)(i\partial_x)\psi_{R/L}(x) - \langle\psi^\dagger_{R/L}(x+\varepsilon)(i\partial_x)\psi_{R/L}(x)\rangle\right]. \tag{B.5}$$

The $(i\partial_x)$ will just pull down a factor $(\partial_x\phi(x) \mp \partial_x\theta(x))$ (with an extra $\pm\sqrt{\pi}$) so we can easily see (or calculate in a very similar way as just before) it will give

[3] All that we need to know about normal ordering is that the normal ordering of two operators written in terms of creation and annihilation operators reads as $: AB := AB - \langle 0|AB|0\rangle$ and the normal ordering of creation and annihilation operators puts all the creation operators to the left, and the annihilation operators to the right (see Wick's theorem).

[4] $: e^A :: e^B :=: e^{A+B} : e^{\langle AB\rangle}$ if $[A, B] = cst$, which is the case here as our A and B are θ and ϕ.

$$\lim_{\varepsilon\to 0}\left[\psi^\dagger_{R/L}(x+\varepsilon)(i\partial_x)\psi_{R/L}(x)-\langle\psi^\dagger_{R/L}(x+\varepsilon)(i\partial_x)\psi_{R/L}(x)\rangle\right]=\frac{1}{4}\left(\partial_x\phi(x)\mp\partial_x\theta(x)\right)^2=\pi\rho^2_{R/L} \tag{B.6}$$

so that we can finally rewrite

$$\begin{aligned}H_0&=\int dx\, v_F\lim_{\varepsilon\to 0}\left[\psi^\dagger_R(x+\varepsilon)(i\partial_x)\psi_R(x)-\langle\psi^\dagger_R(x+\varepsilon)(i\partial_x)\psi_R(x)\rangle-\psi^\dagger_L(x+\varepsilon)(i\partial_x)\psi_L(x)+\langle\psi^\dagger_L(x+\varepsilon)(i\partial_x)\psi_L(x)\rangle\right]\\&=v_F\pi\int dx[\rho^2_R(x)+\rho^2_L(x)]\\&=\frac{v_F\pi}{2}\int dx[\rho^2_+(x)+\rho^2_-(x)]\end{aligned} \tag{B.7}$$

so it eventually reads

$$H_0=\frac{v_F}{2\pi}\int_0^L dx\,(\partial_x\theta(x))^2+(\partial_x\phi(x))^2 \tag{B.8}$$

which is indeed eq. 2.12

C Field correlations and fluctuations

We now want to study the correlations $\langle\phi(x)\phi(0)\rangle$ and fluctuations $\langle(\phi(x)-\phi(0))^2\rangle$. We will calculate the correlators by two methods: the first one is rather classical, we just take the expressions of the fields as given by eqs. 2.9 and perform a straightforward integration, the other method comes from QFT and involves calculating the path integral with the action.

C.1 Classical integral for the time-independant fields

Let's start by calculating explicitely the $\phi\phi$ correlators, the other ones ($\phi\theta$ and $\theta\theta$) follow the same logic.

From eq. (2.9a) (and by forgetting about the zero-mode terms ϕ_0 and θ_0 because they average to 0) we can rewrite

$$\phi(x)\phi(0)=-\sum_{q,p}\mathrm{sgn}(q)\mathrm{sgn}(p)\frac{\pi}{2L}\frac{1}{\sqrt{|q||p|}}(b_q+b^\dagger_{-q})(b_p+b^\dagger_{-p})e^{-\alpha(|q|+|p|)/2}e^{iqx} \tag{C.1a}$$

$$\phi(x)^2=-\sum_{q,p}\mathrm{sgn}(q)\mathrm{sgn}(p)\frac{\pi}{2L}\frac{1}{\sqrt{|q||p|}}(b_q+b^\dagger_{-q})(b_p+b^\dagger_{-p})e^{-\alpha(|q|+|p|)/2}e^{i(q+p)x} \tag{C.1b}$$

$$\phi(0)^2=-\sum_{q,p}\mathrm{sgn}(q)\mathrm{sgn}(p)\frac{\pi}{2L}\frac{1}{\sqrt{|q||p|}}(b_q+b^\dagger_{-q})(b_p+b^\dagger_{-p})e^{-\alpha(|q|+|p|)/2} \tag{C.1c}$$

$\theta\theta$ are similarily expressed thanks to eq. (2.9b) and $\phi\theta$ is given by

$$\phi(x)\theta(0)=\sum_{q,p}\mathrm{sgn}(q)\frac{\pi}{2L}\frac{1}{\sqrt{|q||p|}}(b_q+b^\dagger_{-q})(b_p-b^\dagger_{-p})e^{-\alpha(|q|+|p|)/2}e^{iqx} \tag{C.2}$$

Now taking the average value only amounts to calculating the average values of the bosonic creation and annihilation terms: $\langle b_qb_p\rangle$, $\langle b_qb^\dagger_{-p}\rangle$, $\langle b^\dagger_{-q}b_p\rangle$ and $\langle b^\dagger_{-q}b^\dagger_{-p}\rangle$. We already know they are bosonic operators obeying $[b_q,b^\dagger_p]=\delta_{q,p}\Leftrightarrow b_qb^\dagger_p=\delta_{q,p}+b^\dagger_pb_q$ and at zero temperature, the number operator $N=b^\dagger_qb_q$ averages to 0: $\langle b^\dagger_qb_q\rangle=0$, and the operators bb and $b^\dagger b^\dagger$ also average to zero. Therefore the only non-zero term left out is $\langle b_qb^\dagger_{-p}\rangle=\delta_{q,-p}$. Hence,

$$
\begin{aligned}
\langle\phi(x)\phi(0)\rangle = \langle\theta(x)\theta(0)\rangle &= \frac{\pi}{2L}\sum_q \frac{e^{-\alpha|q|}}{|q|}e^{iqx} \\
&= \frac{1}{4}\int_{-\infty}^{\infty} \frac{e^{-\alpha|q|}}{|q|}e^{iqx} \\
&= \frac{1}{4}\left(\int_{-\infty}^{0} dq\frac{e^{-\alpha|q|}}{|q|}e^{iqx} + \int_{0}^{\infty} dq\frac{e^{-\alpha|q|}}{|q|}e^{iqx}\right) \\
&= \frac{1}{4}\left(\int_{0}^{\infty} d\tilde{q}\frac{e^{-\alpha\tilde{q}}}{\tilde{q}}e^{-i\tilde{q}x} + \int_{0}^{\infty} dq\frac{e^{-\alpha q}}{q}e^{iqx}\right) \\
&= \frac{1}{2}\int_{0}^{\infty} dq\frac{e^{-\alpha q}}{q}\cos(qx) = \mathcal{F}_0(x)
\end{aligned}
\tag{C.3a}
$$

$$
\begin{aligned}
\langle(\phi(x)-\phi(0))^2\rangle = \langle(\theta(x)-\theta(0))^2\rangle &= \frac{\pi}{2L}\sum_q \frac{e^{-\alpha|q|}}{|q|}\left(2-(e^{iqx}+e^{-iqx})\right) \\
&= \frac{\pi}{L}\sum_q \frac{e^{-\alpha|q|}}{|q|}(1-\cos(qx)) \\
&= \frac{1}{2}\int_{-\infty}^{\infty} dq\frac{e^{-\alpha|q|}}{|q|}(1-\cos(qx)) \\
&= \int_{0}^{\infty} dq\frac{e^{-\alpha q}}{q}(1-\cos(qx)) = \mathcal{F}_1(x)
\end{aligned}
\tag{C.3b}
$$

$$
\begin{aligned}
\langle\phi(x)\theta(0)\rangle &= -\frac{\pi}{2L}\sum_q \mathrm{sgn}(q)\frac{e^{-\alpha|q|}}{|q|}e^{iqx} \\
&= -\frac{1}{4}\int_{-\infty}^{\infty} dq\frac{e^{-\alpha|q|}}{q}e^{iqx} \\
&= -\frac{1}{4}\left(\int_{-\infty}^{0} dq\frac{e^{\alpha q}}{q}e^{iqx} + \int_{0}^{\infty} dq\frac{e^{-\alpha q}}{q}e^{iqx}\right) \\
&= -\frac{1}{4}\left(-\int_{0}^{\infty} d\tilde{q}\frac{e^{-\alpha\tilde{q}}}{\tilde{q}}e^{-i\tilde{q}x} + \int_{0}^{\infty} dq\frac{e^{-\alpha q}}{q}e^{iqx}\right) \\
&= -i\int_{0}^{\infty} dq\frac{e^{-\alpha q}}{q}\sin(qx) = \mathcal{F}_2(x)
\end{aligned}
\tag{C.3c}
$$

Integrals $\mathcal{F}(x)$ can be calculated thanks to the cutoff $\alpha > 0$ and give

$$\mathcal{F}_0(x) = -\frac{1}{2}\log\left(\alpha^2+x^2\right) \tag{C.4a}$$

$$\mathcal{F}_1(x) = \frac{1}{2}\log\left(\frac{\alpha^2+x^2}{\alpha^2}\right) \tag{C.4b}$$

$$\mathcal{F}_2(x) = -i\arctan\left(\frac{x}{\alpha}\right) \tag{C.4c}$$

Now, the reasoning above was actually done with $\tilde{\phi}$ and $\tilde{\theta}$ which are rescaled by $\sqrt{K}^{\pm 1}$ that make the total K's disappear, but let's put them back for consistency:

$$\langle(\phi(x)-\phi(0))^2\rangle = K\mathcal{F}_1(x) \tag{C.5a}$$

$$\langle(\theta(x)-\theta(0))^2\rangle = \frac{1}{K}\mathcal{F}_1(x) \tag{C.5b}$$

$\langle\phi\theta\rangle$ rests unchanged because ϕ has a $\sqrt{K}$ which cancels with the $1/\sqrt{K}$ of θ.

C.2 Time-dependant correlations

Using now the standard interaction picture with $\tau = it$ the imaginary time, we can define the time-dependant fields

$$\phi(x,\tau) = e^{\tau H}\phi(x)e^{-\tau H} \tag{C.6a}$$

$$\theta(x,\tau) = e^{\tau H}\theta(x)e^{-\tau H} \tag{C.6b}$$

and use the Baker-Campbell-Haussdorf formula (see Appendix A) to calculate the previous correlation functions for the time-dependant fields $\phi(x,\tau)$, $\theta(x,\tau)$. The fields are expressed as a function of the bosonic operators $b_q^{(\dagger)}$ which obey the following commutation relation with the *rescaled*[5] Hamiltonian H (eq. (2.17)):

$$[b_q, b_p^\dagger] = \delta_{q,p} \tag{C.7a}$$

$$[b_q, H] = u|q|b_q \tag{C.7b}$$

$$[H, b_q^\dagger] = u|q|b_q^\dagger \tag{C.7c}$$

which allows us to write the time-dependant creation and annihilation operators

$$b_q(\tau) = e^{\tau H}b_q e^{-\tau H} = b_q + \sum_{n\geq 1}\frac{\tau^n}{n!}[H, ...[H, b_q]]_n = b_q\sum_n \frac{(-u|q|\tau)^n}{n!} = b_q e^{-u|q|\tau} \tag{C.8a}$$

$$b_q^\dagger(\tau) = e^{\tau H}b_q^\dagger e^{-\tau H} = b_q^\dagger + \sum_{n\geq 1}\frac{\tau^n}{n!}[H, ...[H, b_q^\dagger]]_n = b_q^\dagger\sum_n \frac{(u|q|\tau)^n}{n!} = b_q^\dagger e^{u|q|\tau} \tag{C.8b}$$

Finally, we can write the time-dependant fields:

$$\phi(x,\tau) = \phi_0(x,\tau) - i\sum_q \mathrm{sgn}(q)\sqrt{\frac{\pi}{2L|q|}}(b_q e^{-u|q|\tau} + b_{-q}^\dagger e^{u|q|\tau})e^{iqx}e^{-\alpha|q|/2} \tag{C.9a}$$

$$\theta(x,\tau) = \theta_0(x,\tau) + i\sum_q \sqrt{\frac{\pi}{2L|q|}}(b_q e^{-u|q|\tau} - b_{-q}^\dagger e^{u|q|\tau})e^{iqx}e^{-\alpha|q|/2} \tag{C.9b}$$

Then we need to calculate the *time-ordered* correlation function: $G_{\phi\phi}(x,\tau) = \langle T_\tau(\phi(x,\tau) - \phi(0,0))^2\rangle = \langle\phi(x,\tau)^2 + \phi(0,0)^2 - \Theta(\tau)\phi(x,\tau)\phi(0,0) - \Theta(-\tau)\phi(0,0)\phi(x,\tau)\rangle$ with $\Theta(\tau)$ the Heaviside function. Similarily, $G_{\theta\theta}(x,\tau) = \langle T_\tau(\theta(x,\tau) - \theta(0,0))^2\rangle$ and $G_{\phi\theta}(x,\tau) = \langle T_\tau\phi(x,\tau)\theta(0,0)\rangle$. By doing the same derivation as the previous section we find

$$\Theta(\tau)\langle\Phi(x,\tau)\Phi(0,0)\rangle + \Theta(-\tau)\langle\Phi(0,0)\Phi(x,\tau)\rangle = \sum_q \frac{2\pi}{L}\frac{e^{-\alpha|q|}}{|q|}e^{iqx}\underbrace{\left(\Theta(\tau)e^{-u|q|\tau} + \Theta(-\tau)e^{u|q|\tau}\right)}_{=e^{-u|q||\tau|}} \tag{C.10}$$

which gives finally, after doing a similar derivation to eqs. (C.3) and reintroducing the K dependance:

$$G_{\phi\phi}(x,\tau) = K\mathcal{F}_1(x,\tau) \tag{C.11a}$$

$$G_{\theta\theta}(x,\tau) = \frac{1}{K}\mathcal{F}_1(x,\tau) \tag{C.11b}$$

$$G_{\phi\theta}(x,\tau) = \mathcal{F}_2(x,\tau) \tag{C.11c}$$

with

$$\mathcal{F}_1(x,\tau) = \int_0^\infty dq\frac{e^{-\alpha q}}{q}\left(1 - e^{-u|\tau|q}\cos(qx)\right) = \frac{1}{2}\log\left(\frac{x^2 + u|\tau|\alpha^2}{\alpha^2}\right) \tag{C.12a}$$

$$\mathcal{F}_2(x,\tau) = -i\,\mathrm{sgn}(\tau)\int_0^\infty dq\frac{e^{-\alpha q}}{q}e^{-u|\tau|q}\sin(qx) = -i\,\mathrm{sgn}(\tau)\arctan\left(\frac{x}{u|\tau| + \alpha}\right) \tag{C.12b}$$

[5] It is important to note that we implicitely use the rescaled Hamiltonian $\tilde{H}$, which explains that the K's are not explicit in the expressions of the correlators until we reintroduce them "by hand", but in the following we still write H without the tilde for simplicity.

C.3 Path integral

The second method is quite elegant and involves the path integrals. We use the expression of the action (2.22) and using standard results of Gaussian integration, one can show that the matrix M is associated with the average values of the fields

$$M = \frac{1}{k^2(u^2k^2+\omega_n^2)}\begin{pmatrix} k^2u/K & -ik\omega_n \\ -ik\omega_n & k^2uK \end{pmatrix} = \beta\Omega \begin{pmatrix} \langle\theta^*(\mathbf{q})\theta(\mathbf{q})\rangle & \langle\theta^*(\mathbf{q})\phi(\mathbf{q})\rangle \\ \langle\phi^*(\mathbf{q})\theta(\mathbf{q})\rangle & \langle\phi^*(\mathbf{q})\phi(\mathbf{q})\rangle \end{pmatrix} \tag{C.13}$$

Therefore, the correlators $\phi\phi$, $\theta\theta$ and $\phi\theta$ in Fourier space are given by

$$\langle\phi(\mathbf{q_1})\phi(\mathbf{q_2})\rangle = \langle\phi^*(-\mathbf{q_1})\phi(\mathbf{q_2})\rangle = \frac{\beta\Omega\pi uK}{u^2k^2+\omega_n^2}\delta_{-\mathbf{q_1},\mathbf{q_2}} \tag{C.14a}$$

$$\langle\theta(\mathbf{q_1})\theta(\mathbf{q_2})\rangle = \langle\theta^*(-\mathbf{q_1})\theta(\mathbf{q_2})\rangle = \frac{\beta\Omega u\pi}{K(u^2k^2+\omega_n^2)}\delta_{-\mathbf{q_1},\mathbf{q_2}} \tag{C.14b}$$

$$\langle\phi(\mathbf{q_1})\theta(\mathbf{q_2})\rangle = \langle\phi^*(-\mathbf{q_1})\theta(\mathbf{q_2})\rangle = \frac{-i\omega_n\beta\Omega\pi}{k(u^2k^2+\omega_n^2)}\delta_{-\mathbf{q_1},\mathbf{q_2}} \tag{C.14c}$$

Now, our goal is to calcualte the correlators in real space. We write:

$$\langle(\phi(r_1)-\phi(r_2))^2\rangle = \frac{1}{(\beta\Omega)^2}\sum_{\mathbf{q_1},\mathbf{q_2}}\langle\phi(\mathbf{q_1})\phi(\mathbf{q_2})\rangle(e^{i\mathbf{q_1}r_1}-e^{i\mathbf{q_2}r_2})(e^{i\mathbf{q_2}r_1}-e^{i\mathbf{q_2}r_2}) \tag{C.15a}$$

$$\langle(\theta(r_1)-\theta(r_2))^2\rangle = \frac{1}{(\beta\Omega)^2}\sum_{\mathbf{q_1},\mathbf{q_2}}\langle\theta(\mathbf{q_1})\theta(\mathbf{q_2})\rangle(e^{i\mathbf{q_1}r_1}-e^{i\mathbf{q_2}r_2})(e^{i\mathbf{q_2}r_1}-e^{i\mathbf{q_2}r_2}) \tag{C.15b}$$

$$\langle\phi(r_1)\theta(r_2)\rangle = \frac{1}{(\beta\Omega)^2}\sum_{\mathbf{q_1},\mathbf{q_2}}\langle\phi(\mathbf{q_1})\theta(\mathbf{q_2})\rangle e^{i(\mathbf{q_1}r_1+\mathbf{q_2}r_2)} \tag{C.15c}$$

and therefore, after plugging expressions (C.14) into eqs. (C.15) and using $\mathbf{q}r = (kx-\omega_n\tau)$ we find

$$G_{\phi\phi}(x,\tau) = \langle T_\tau(\phi(x,\tau)-\phi(0,0))^2\rangle = \frac{1}{\beta\Omega}\sum_{\mathbf{q}=(k,\omega_n)}\frac{2uK\pi}{u^2k^2+\omega_n^2}(1-\cos(kx-\omega_n\tau)) = K\tilde{\mathcal{F}}_1(x,\tau) \tag{C.16a}$$

$$G_{\theta\theta}(x,\tau) = \langle T_\tau(\theta(x,\tau)-\theta(0,0))^2\rangle = \frac{1}{\beta\Omega}\sum_{\mathbf{q}=(k,\omega_n)}\frac{2u\pi}{K(u^2k^2+\omega_n^2)}(1-\cos(kx-\omega_n\tau)) = \frac{1}{K}\tilde{\mathcal{F}}_1(x,\tau) \tag{C.16b}$$

$$G_{\phi\theta}(x,\tau) = \langle T_\tau\phi(x,\tau)\theta(0,0)\rangle = \frac{1}{\beta\Omega}\sum_{\mathbf{q}=(k,\omega_n)}\frac{-i\omega_n\pi}{k(u^2k^2+\omega_n^2)}e^{i(kx-\omega_n\tau)} = \tilde{\mathcal{F}}_2(x,\tau) \tag{C.16c}$$

we can verify, with ω_n which are bosonic Matsubara frequencies that at zero temperature $\beta\to\infty$ (see [4]):

$$\tilde{\mathcal{F}}_1(x,\tau) = \mathcal{F}_1(x,\tau) = \int_0^\infty dq\frac{e^{-\alpha q}}{q}\left(1-e^{-u|\tau|q}\cos(qx)\right) = \frac{1}{2}\log\left(\frac{x^2+u|\tau|\alpha^2}{\alpha^2}\right) \tag{C.17a}$$

$$\tilde{\mathcal{F}}_2(x,\tau) = \mathcal{F}_2(x,\tau) = -i\,\mathrm{sgn}(\tau)\int_0^\infty dq\frac{e^{-\alpha q}}{q}e^{-u|\tau|q}\sin(qx) = -i\,\mathrm{sgn}(\tau)\arctan\left(\frac{x}{u|\tau|+\alpha}\right) \tag{C.17b}$$

which agrees with eqs C.11 and C.12.

D Kubo formula

In this section we derive the conductivity σ using the Kubo formula for two cases: (1) the Hamiltonian is perturbed by a vector potential, (2) the Hamiltonian is perturbed by a scalar potential. We will see that (obviously) both methods lead to the same result.

D.1 Vector potential

We start from the famous Kubo formula: assuming the Hamiltonian is perturbed $H \to H + \delta H$ with $\delta H = \sum_i \int d^d r A_i(r) h_i(r,t)$ where A is an observable and h a small field, the Kubo formula states

$$\langle A_i(r)\rangle(t) = \langle A_i(r)\rangle_0 + \sum_j \int dt' \int d^d r' \chi^R_{A_i,A_j}(r,t,r',t') h_j(r',t') \tag{D.1}$$

with

$$\chi^R_{A_i,A_j}(r,t,r',t') = \frac{i}{\hbar}\mathcal{T}_t \langle [A_i(r,t), A_j(r',t')]\rangle \tag{D.2}$$

In the case of a vector potential, the perturbation to the Hamiltonian is

$$\delta H = -\int dx j(x,t) A(x,t) \tag{D.3}$$

so that eq. (D.1) becomes

$$\langle j(x,t)\rangle = \cancelto{0}{\langle j\rangle_0} + \frac{i}{\hbar}\int_0^L dx' \int_{-\infty}^{\infty} dt' \mathcal{T}_t \langle [j(x,t), j(x',t')]\rangle A(x',t') \tag{D.4}$$

where we have put the constant term to zero for simplicity. Then we suppose E is uniform in space, and Fourier transform $A(x',t') = -\frac{i}{\omega}E(x',t') = -\frac{i}{\omega}E(x,t')$ to rewrite

$$\begin{aligned}\langle j(x,t)\rangle &= \frac{i}{\hbar}\int dx' \int dt' \mathcal{T}_t \langle [j(x,t), j(x',t')]\rangle \frac{-i}{\omega} E(x,t) e^{i\omega(t-t')} \\ &= \frac{E(x,t)}{\hbar\omega}\int dx' \int dt' e^{i\omega(t-t')} \mathcal{T}_t \langle [j(x,t), j(x',t')]\rangle\end{aligned} \tag{D.5}$$

Now, we identify $\langle j(x,t)\rangle$ with $j(x,t)$ from Ohm's law eq. (2.40) and replace the j's in the commutator by $j_e(x,t) = ej(x,t)$ where $j(x,t) = \frac{1}{\pi}\partial_t \phi$ (continuity equation eq. (2.25)) an go to imaginary time τ, and note.

$$\begin{aligned}\Rightarrow \sigma(x,\tau) &= \frac{i}{\hbar\omega}\int dx' \int d\tau' e^{\omega(\tau-\tau')} \mathcal{T}_\tau \langle [j_e(x,\tau), j_e(x',\tau')]\rangle \\ &= \frac{i}{\hbar\omega}\int dx' \int d\tau' e^{\omega(\tau-\tau')} \langle j_e(x,\tau), j_e(x',\tau')\rangle \\ &= \frac{-ie^2}{\pi^2\hbar\omega}\int dx' \int d\tau' e^{\omega(\tau-\tau')} \langle \partial_\tau\phi(x,\tau)\partial_\tau\phi(x',\tau')\rangle \\ \Rightarrow \sigma(q,\omega) &= \frac{-ie^2}{\pi^2\hbar}\omega\langle \phi^*(q,\omega)\phi(q,\omega)\rangle\end{aligned} \tag{D.6}$$

Where we have used $\mathcal{T}_\tau\langle [j_e(x,\tau), j_e(x',\tau')]\rangle = \Theta(\tau-\tau')\langle j_e(x,\tau)j_e(x',\tau')\rangle - \Theta(\tau'-\tau)\langle j_e(x',\tau')j_e(x,\tau)\rangle = \langle j_e(x,\tau)j_e(x',\tau')\rangle\,(\Theta(\tau-\tau') - \Theta(\tau'-\tau))$ because j commutes with itself $= \langle j_e(x,\tau)j_e(x',\tau')\rangle[\Theta(\tau-\tau') - (1-\Theta(\tau-\tau'))] = \langle j_e(x,\tau)j_e(x',\tau')\rangle$. In the limit $q \to 0$, we find using the results from Section 3.1, and using $\omega = \omega + i0^+$ and $i\omega_n \to \omega + i0^+$

$$\sigma(\omega) = \frac{ie^2}{\pi^2\hbar}\frac{\pi u K}{(\omega + i0^+)} = \frac{e^2}{\hbar}uK\left(\delta(\omega) + i\mathcal{P}\left(\frac{1}{\pi\omega}\right)\right) \tag{D.7}$$

This leads us to

$$G = \frac{e}{\pi\hbar}\lim_{\omega\to 0}\frac{1}{\omega + i0^+}\int \frac{dq}{2\pi}\mathrm{sinc}(qL/2)\frac{\omega_n^2 uK}{\omega_n^2 + u^2q^2} \tag{D.8}$$

which after integration gives the excpected conductance

$$G = \frac{e^2}{\hbar}K \tag{D.9}$$

D.2 Scalar potential

For the scalar potential it's essentially the same : we start from a perturbation

$$\delta H = \int dx \rho(x) V(x) \tag{D.10}$$

with $E(x) = -\nabla V(x)$ and following steps from [4], we replace $\langle [j(x,t), j(x',t')] \rangle A(x',t')$ by $\langle [j(x,t), \partial_{t'} \rho(x',t')] \rangle V(x')$ and using the continuity equation and doing an integration by part we find that it is equal to $\langle [j(x,t), j(x',t')] \rangle \nabla V(x') = \langle [j(x,t), j(x',t')] \rangle E(x')$ which is the same as before, and we fall back on our feet with all of the following steps which are exactly the same as before.

E Jordan-Wigner transformation

$$S^{x,y,z,\pm} = \frac{1(\hbar)}{2} \sigma^{x,y,z,\pm}, \quad S^{\pm} = S_i^x \pm i S_i^y \tag{E.1}$$

From fermions to spins

$$S_i^z = c_i^\dagger c_i - \frac{1}{2} \tag{E.2a}$$

$$S_i^+ = c_i^\dagger \prod_{j<i} (1 - 2c_j^\dagger c_j) = c_i^\dagger e^{i\pi \sum_{j<i} c_j^\dagger c_j} \tag{E.2b}$$

$$S_i^- = \prod_{j<i} (1 - 2c_j^\dagger c_j) c_i = e^{-i\pi \sum_{j<i} c_j^\dagger c_j} c_i \tag{E.2c}$$

From spins to fermions

$$c_i^\dagger = S_i^+ e^{-i\pi \sum_{j<1} S_j^+ S_j^-} \tag{E.3a}$$

$$c_i = e^{i\pi \sum_{j<1} S_j^+ S_j^-} S_i^- \tag{E.3b}$$

$$c_i^\dagger c_i = S_i^+ S_i^- \tag{E.3c}$$

F Spin fluctuations

Everything is calculated in PBC, and at low temperatures (this last assumption is very important !).

We know [16] that in the low temperature limit:

$$\begin{aligned} M_{i,j}(\beta) = \langle c_j^\dagger c_i \rangle &= \frac{1}{L} \sum e^{ik(i-j)} n_{FD}(\beta E_k) \\ &=_{\beta\to\infty} \frac{1}{L} \sum_k e^{ik(i-j)} \Theta(E_k) \\ &=_{L\to\infty} \delta_{i,j} - \frac{1}{2} \mathrm{sinc}\left(\frac{\pi}{2}(i-j)\right) = \delta_{i,j} - \frac{\sin\left(\frac{\pi}{2}(i-j)\right)}{\pi(i-j)} \end{aligned} \tag{F.1}$$

F.1 Spin z fluctuations

We want to calculate the S_i^z fluctuations and check that it corresponds to the fluctuations of the field ϕ as calculated in Appendix C:

$$G_{S^z S^z} = \langle (\sum_i S_i^z)^2 \rangle - \langle \sum_i S_i^z \rangle^2 = \sum_{i,j} \left(\langle S_i^z S_j^z \rangle - \langle S_i^z \rangle \langle S_j^z \rangle \right) \tag{F.2}$$

At half-filling we have $\langle S_i^z \rangle = \langle c_i^\dagger c_i - \frac{1}{2} \rangle = 0$ and $\langle (S_i^z)^2 \rangle = \frac{1}{4}$. Therefore $\langle S_i^z \rangle \langle S_j^z \rangle = 0$ and the other term is found using Wick's theorem:

$$
\begin{aligned}
\langle S_i^z S_j^z\rangle &= \langle (c_i^\dagger c_i - 1/2)(c_j^\dagger c_j - 1/2)\rangle = \langle c_i^\dagger c_i c_j^\dagger c_j\rangle - 1/2\langle c_i^\dagger c_i\rangle - 1/2\langle c_j^\dagger c_j\rangle + 1/4 \\
&= \frac{-1}{4} + \langle c_i^\dagger c_i\rangle\langle c_j^\dagger c_j\rangle + \langle c_i^\dagger c_j\rangle\langle c_i c_j^\dagger\rangle \\
&= \langle c_i^\dagger c_j\rangle(\delta_{i,j} - \langle c_j^\dagger c_i\rangle) \\
&= \delta_{i,j}\langle c_i^\dagger c_j\rangle - \langle c_i^\dagger c_j\rangle\langle c_j^\dagger c_i\rangle \\
&= \frac{1}{2}\delta_{i,j} - |M_{i,j}|^2 \\
&= \frac{1}{2}\delta_{i,j} - \frac{\sigma_{i-j}}{\pi^2(i-j)^2}
\end{aligned} \tag{F.3}
$$

where $\sigma_n = 0$ is n is even and 1 if n is odd, *i.e.* $\sigma_n = n \mod 2$ and $\frac{\sigma_n}{n^2} \to \frac{\pi^2}{4}$ as $n \to 0$. The next step is to separate the fluctuations as

$$
\begin{aligned}
G_{S^zS^z}(x) &= \sum_{i,j=1}^{x} \big(\langle S_i^z S_j^z\rangle - \langle S_i^z\rangle\langle S_j^z\rangle\big) = \sum_{i=1}^{x} \big(\langle S_i^{z2}\rangle - \langle S_i^z\rangle^2\big) + \sum_{i\neq j, i,j=1}^{x} \big(\langle S_i^z S_j^z\rangle - \langle S_i^z\rangle\langle S_j^z\rangle\big) \\
&= \frac{x}{4} + \sum_{i\neq j, i,j=1}^{x} \big(\langle S_i^z S_j^z\rangle - \langle S_i^z\rangle\langle S_j^z\rangle\big) \\
&= \frac{x}{4} - \sum_{i\neq j, i,j=1}^{x} \frac{\sigma_{i-j}}{\pi^2(i-j)^2}
\end{aligned} \tag{F.4}
$$

and to calculate the sum we notice it's a sum that depend only on the difference of the indices, so it can be converted into

$$
\sum_{i\neq j, i,j=1}^{x} \frac{\sigma_{i-j}}{\pi^2(i-j)^2} = \frac{2}{\pi^2}\sum_{k=1}^{x} \frac{(x-k)\sigma_k}{k^2} \tag{F.5}
$$

using

$$
\sum_{i,j=1}^{x} f(|i-j|) = f(0)x + 2\sum_{k=1}^{x}(x-k)f(k) \tag{F.6}
$$

The sum gives

$$
\sum_{k=1}^{x} \frac{(x-k)\sigma_k}{k^2} = \frac{\pi^2}{8}x - \frac{x}{4}\psi_1\left(\frac{x+1}{2}\right) - \frac{1}{2}\psi_0\left(\frac{x+1}{2}\right) + \frac{1}{2}\psi_0\left(\frac{3}{2}\right) - 1 \tag{F.7}
$$

Where ψ_n are digamma functions. Considering large x and $\psi_0(x\to\infty) \to \ln x - \frac{1}{2x} - \frac{1}{2x^2}$, we find

$$
\sum_{k=1}^{x} \frac{(x-k)\sigma_k}{k^2} = \frac{\pi^2}{8}x - \frac{1}{2}(\ln x + 1 + \gamma + \ln 2) + O(x^{-2}) \tag{F.8}
$$

Such that in the end,

$$
\pi^2 G_{S^zS^z}(x) = \ln x + f_1 \qquad f_1 = 1 + \gamma + \ln 2 \tag{F.9}
$$

This agrees with the non-interacting ($K = 1$) long-range ($x >> 1$) limit of the result obtained in Apendix C: $G_{\phi\phi}(x) = \frac{1}{2}\ln\big((\alpha^2 + x^2)/\alpha^2\big) \to_{x>>1} \ln x$.

F.2 Spins xy fluctuations

We want to calculate the $S_i^x S_{i\pm1}^y$ fluctuations and check that it corresponds to the fluctuations of the field θ as calculated in Appendix C.

$$\begin{aligned}
G_{S^xS^y} &= \langle(\sum_i (S_i^x S_{i-1}^y - S_i^y S_{i-1}^x))^2\rangle - \langle\sum_i (S_i^x S_{i-1}^y - S_i^y S_{i-1}^x)\rangle^2 \\
&= \sum_{i,j}\langle(S_i^x S_{i-1}^y - S_i^y S_{i-1}^x)(S_j^x S_{j-1}^y - S_j^y S_{j-1}^x)\rangle - \langle S_i^x S_{i-1}^y - S_i^y S_{i-1}^x\rangle\langle S_j^x S_{j-1}^y - S_j^y S_{j-1}^x\rangle \\
&= \frac{-1}{4}\sum_{i,j}\langle(c_i^\dagger c_{i-1} - c_{i-1}^\dagger c_i)(c_j^\dagger c_{j-1} - c_{j-1}^\dagger c_j)\rangle - \langle c_i^\dagger c_{i-1} - c_{i-1}^\dagger c_i\rangle\langle c_j^\dagger c_{j-1} - c_{j-1}^\dagger c_j\rangle
\end{aligned} \tag{F.10}$$

Using eq. (F.1) we find

$$\langle c_i^\dagger c_{i-1} - c_{i-1}^\dagger c_i\rangle = 0 \tag{F.11}$$

and, using Wick's theorem:

$$\begin{aligned}
\langle(c_i^\dagger c_{i-1} - c_{i-1}^\dagger c_i)(c_j^\dagger c_{j-1} - c_{j-1}^\dagger c_j)\rangle &= \langle c_i^\dagger c_{i-1} c_j^\dagger c_{j-1}\rangle - \langle c_i^\dagger c_{i-1} c_{j-1}^\dagger c_j\rangle - \langle c_{i-1}^\dagger c_i c_j^\dagger c_{j-1}\rangle + \langle c_{i-1}^\dagger c_i c_{j-1}^\dagger c_j\rangle \\
&= \underbrace{\langle c_i^\dagger c_{i-1}\rangle\langle c_j^\dagger c_{j-1}\rangle} + \underbrace{\langle c_i^\dagger c_{j-1}\rangle(\delta_{i,j+1} - \langle c_j^\dagger c_{i-1}\rangle)} - \underbrace{\langle c_i^\dagger c_{i-1}\rangle\langle c_{j-1}^\dagger c_j\rangle} \underbrace{- \frac{1}{2}\delta_{i,j} + \langle c_i^\dagger c_j\rangle\langle c_{j-1}^\dagger c_{i-1}\rangle} \\
&\quad - \underbrace{\langle c_{i-1}^\dagger c_i\rangle\langle c_j^\dagger c_{j-1}\rangle} \underbrace{- \frac{1}{2}\delta_{i,j} + \langle c_{i-1}^\dagger c_{j-1}\rangle\langle c_j^\dagger c_i\rangle} + \underbrace{\langle c_{i-1}^\dagger c_i\rangle\langle c_{j-1}^\dagger c_j\rangle} + \underbrace{\langle c_{i-1}^\dagger c_j\rangle(\delta_{i,j-1} - \langle c_{j-1}^\dagger c_i\rangle)}
\end{aligned} \tag{F.12}$$

Where the blue terms are all equal to $1/\pi^2$, the green terms to $\frac{1}{4}\mathrm{sinc}(\frac{\pi}{2}(i-j+1))\mathrm{sinc}(\frac{\pi}{2}(i-j-1))$ and the orange terms are exactly $-\frac{1}{2}\delta_{i,j} + |M_{i,j}|^2 = -\langle S_i^z S_j^z\rangle$, and we have calculated their sum in the previous paragraph. Therefore, the term simplifies to

$$\langle(c_i^\dagger c_{i-1} - c_{i-1}^\dagger c_i)(c_j^\dagger c_{j-1} - c_{j-1}^\dagger c_j)\rangle = -2\mathcal{F}_z(x) - \frac{1}{2}\mathrm{sinc}(\frac{\pi}{2}(i-j+1))\mathrm{sinc}(\frac{\pi}{2}(i-j-1)) \tag{F.13}$$

As before, we separate the sum into two terms: $i = j \pm 1$ and $i \neq j \pm 1$

$$\begin{aligned}
G_{S^xS^y}(x) &= \frac{1}{2}G_{S^zS^z}(x) + \frac{1}{8}\sum_{i=1}^{x}\mathrm{sinc}(\pi)\mathrm{sinc}(0) + \mathrm{sinc}(0)\mathrm{sinc}(-\pi) + \frac{1}{8}\sum_{i\neq j\pm1}^{x}\mathrm{sinc}(\frac{\pi}{2}(i-j+1))\mathrm{sinc}(\frac{\pi}{2}(i-j-1)) \\
&= \frac{1}{2}G_{S^zS^z}(x) + \frac{1}{8}\sum_{i\neq j\pm1}^{x}\mathrm{sinc}(\frac{\pi}{2}(i-j+1))\mathrm{sinc}(\frac{\pi}{2}(i-j-1)) \\
&= \frac{1}{2}G_{S^zS^z}(x) + \frac{1}{2\pi^2}\sum_{i\neq j\pm1}^{x}\frac{\sigma_{i-j}-1}{(i-j)^2-1} \\
&= \frac{1}{2}G_{S^zS^z}(x) + \frac{1}{2\pi^2}\left(\frac{(x-0)(\sigma_0-1)}{-1} + 2\sum_{k=2}^{x}\frac{(x-k)(\sigma_k-1)}{k^2-1}\right) \\
&= \frac{1}{2\pi^2}(\ln x + 1 + \gamma + \ln 2) + \frac{1}{2\pi^2}\left(x - x + \psi_0\left(\frac{x+1}{2}\right) - \psi_0\left(\frac{1}{2}\right)\right) \\
&= \frac{1}{2\pi^2}(\ln x + 1 + \gamma + \ln 2) + \frac{1}{2\pi^2}\left(\ln x - \ln 2 + \gamma + 2\ln 2\right) \\
&= \frac{1}{\pi^2}(\ln x + \gamma + \ln 2 + 1/2)
\end{aligned} \tag{F.14}$$

Such that in the end

$$\pi^2 G_{S^x S^y}(x) = \ln x + f_2 \qquad f_2 = \frac{1}{2} + \gamma + \ln 2 \tag{F.15}$$

We find $\pi^2 G_{S^z S^z} = \pi^2 G_{S^x S^y} + 1/2 = \mathcal{F}_1(x) = G_{\phi\phi} = G_{\theta\theta}$ in the non-interacting $K = 1$ case. In the interacting case $K \neq 1$ we recover

$$G_{S^z S^z}(x) = G_{\phi\phi}(x) = K\mathcal{F}_1(x) \tag{F.16a}$$

$$G_{S^x S^y}(x) = G_{\theta\theta}(x) = \frac{1}{K}\mathcal{F}_1(x) \tag{F.16b}$$

G Reintroducing $\hbar$

If we reintroduce the $\hbar$ such that $\phi(x)$ and $\Pi(x)$ are conjugate variables (i.e. $[\phi(x), \Pi(y)] = i\hbar\delta(x-y)$), $\Pi(x)$ is actually equal to $\Pi(x) = \frac{1}{\pi\hbar}\partial_x\theta(x)$ and the Hamiltonian reads

$$H = \frac{\hbar u}{2\pi}\int dx\ K(\partial_x\theta(x))^2 + \frac{1}{K}(\partial_x\phi(x))^2 = \frac{\hbar u}{2\pi}\int dx\ K(\pi\hbar\Pi(x))^2 + \frac{1}{K}(\partial_x\phi(x))^2 \tag{G.1}$$

this is taken care of, everytime that we have an equation of motion or use the Kubo formula because then we have a $i/\hbar$ in front of the commutator with the Hamiltonian so the $\hbar$'s cancel. Same with the action, it's $S/\hbar$ and the $\hbar$'s cancel.

H Variables index

There are lots of variables like ϕ, $\tilde{\phi}$, φ, Φ , θ, $\tilde{\theta}$, Θ, ϑ, ρ, $\rho_{R/L}$, $\tilde{\rho}_{R/L}$, $\rho_\pm$, $\tilde{\rho}_\pm$, σ, $\sigma^\pm$, σ_n etc. ... Here are all of them listed precisely with what they mean, to make sure not to be confused.

H_0: Free Hamiltonian.
$c^{(\dagger)}$: creation/annihilation operators.
$c^{(\dagger)}_{R/L}$: creation/annihilation operators around the Fermi points $\pm k_F$.
v_F: Fermi velocity.
ε: energy/dispersion relation.
$\rho_{R/L}$, $\rho_\pm$: charge densities.
ϕ and θ: bosonization fields, they can be interpreted respectively as a displacement field for phonons, and a superfluid phase. They are linked to the charge and current densities.
Π: partial derivative of θ in space: $\Pi = \nabla\theta$.
$b^{(\dagger)}$: bosonization operators.
$\psi^{(\dagger)}_{R/L}$: creation and annihilation operators.
H_I: Interacting Hamiltonian.
u: Luttinger velocity, K: Luttinger parameter.
$\tilde{\phi}$, $\tilde{\theta}$: Like ϕ and θ but in the interacting case, rescaled by $\sqrt{K}$. Used but implied, under the simpler notation ϕ and θ.
t: time, τ: imaginary time. In subscript, t refers to tunnelling in Section 2.3, $t_\perp$: tunnelling amplitude.
Θ: Heaviside distribution.
$G_{\phi\phi}$: Green's function of the $\phi\phi$ fields (can be replaced by other quantities).
T_τ: (imaginary-)time ordering operator.
$\Psi^{(\dagger)}_{\mathbf{q}}$: spinors $\Psi^\dagger_{\mathbf{q}} = (\theta^*_{\mathbf{q}}\ \phi^*_{\mathbf{q}})$.
ρ or q and j: charge and current densities.
Q and J: integer/integral/total charge and current.
$\tilde{\rho}_{R/L}$, $\tilde{\rho}_\pm$: chiral charge densities.
$j_{R/L}$: chiral current density.
$Q_{R/L}$ and $J_{R/L}$: chiral (total) charge and current.
$\vartheta_{R/L}$ and $\varphi_{R/L}$: chiral fields (chiral version of θ and ϕ).
$\psi^{(\dagger)}_{R/L}(x)$: creation/annihilation fields expressed in terms of ϕ and θ
$\tilde{\psi}^{(\dagger)}_{R/L}(x)$: chiral creation/annihilation fields expressed in terms of φ and ϑ, and carrying a fractional charge $J_{R/L}$ or $Q_{R/L}$
σ: conductivity, G: conductance, U: tension, R: resistance, I: macroscopic current (equivalent to J in 1D).
$\kappa = (1+K)/2$: fraction of charge.
Φ: anyonic phase.
$\sigma^{x,y,z,\pm}$, $S^{x,y,z,\pm}$: spin operators.
σ_n: 1 if n odd, 0 if n even.
$\phi_\pm$, $\theta_\pm$: sum/difference of the fields in two wires $\phi_1 \pm \phi_2$, $\theta_1 \pm \theta_2$.
$\tilde{\phi}_\pm/\tilde{\theta}_\pm$: rescaled $\phi_\pm/\theta_\pm$.
$\psi_n(x)$: digamma functions.

I DMRG

In this section we provide the codes that calculate the fluctuations $G_{\phi\phi}(x)$, $G_{\theta\theta}(x)$ and $G_{\theta_{R/L}\theta_{R/L}}(x)$.

I.1 Charge fluctuations

```julia
using ITensors, ITensorMPS

arg1 = 0.5 # Use parse(Float64, ARGS[1]) on the cluster

# Parameters of the system
t = 1
U = arg1
N = 2^10

# Get Hamiltonian and sites
function tight_binding_chain(L)
    sites = siteinds("S=1/2", L)
    h = OpSum()
    for i in 1:L-1
        add!(h, -2*t, "Sx", i, "Sx", i+1)
        add!(h, -2*t, "Sy", i, "Sy", i+1)
        add!(h, U, "Sz", i, "Sz", i+1)
    end
    add!(h, -2*t, "Sx", L, "Sx", 1) #PBC
    add!(h, -2*t, "Sy", L, "Sy", 1) #PBC
    add!(h, U, "Sz", L, "Sz", 1) #PBC
    return MPO(h, sites), sites
end

# Get operator O = phi
function operator_O(L, Sites)
    opO = OpSum()
    for i in 2:L
        add!(opO, "Sz", i)
    end
    return MPO(opO,Sites)
end

# Get operator O2 = phi^2
function operator_O2(L, Sites)
    opO2 = OpSum()
    for i in 2:L
        for j in 2:L
            add!(opO2, "Sz", i, "Sz", j)
        end
    end
    return MPO(opO2,Sites)
end

# Perform DMRG to find the ground state and calculate fluctuations

# Create the Hamiltonian and the sites and get the GS
H, sites = tight_binding_chain(N)
psi = randomMPS(sites)
sweeps = Sweeps(10)
maxdim!(sweeps, 10, 20, 100, 200)
cutoff!(sweeps, 1e-10)
energy, psi0 = dmrg(H, psi, sweeps)

function calculate_fluctuations(L)
```

```
    # Calculate <O> = <phi>
    O1 = operator_O(L, sites)
    avgO = inner(psi0, O1, psi0)

    # Calculate <O^2> = <phi^2>
    O2 = operator_O2(L, sites)
    avgO2 = inner(psi0, O2, psi0)

    # Calculate fluctuations
    fluctuations = avgO2 - avgO^2

    println("system size =", L)

    return fluctuations
end

# Calculate the average value of O for different system sizes L

System_sizes = 5:5:200  # Change this range to explore more values of L
fluct = [calculate_fluctuations(x) for x in System_sizes]
fluct_re = pi^2*real(fluct)

open("flc_phi_U_$arg1.txt", "w") do file
    for i in 1:length(System_sizes)
        println(file, "$(System_sizes[i]),$(fluct_re[i])")
    end
end
```

For the charge fluctuations, we have to note that finite size effects lead us to fit the fluctuations of a subsystem of size x for a spin chain of total size L with $\log\left(\frac{L}{\pi}\sin\left(\frac{\pi x}{L}\right)\right)$ instead of $\log(x)$. The fit with $\log(x)$ is accurate for big system sizes $L \gtrsim 500$ for $x < L/2$ ($G_{\phi\phi}(L/2 \leq x \leq L)$ is the symmetric of $G_{\phi\phi}(x < L/2)$ and vanishes). See Figures 14 and 15.

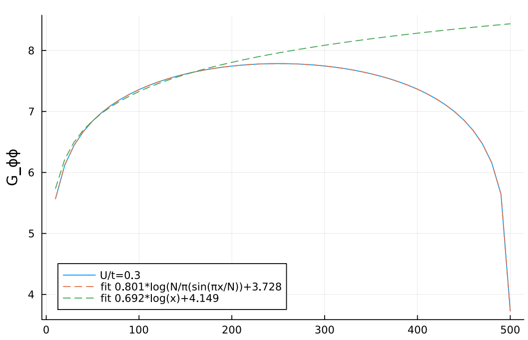

Figure 14: Here we see the fluctuations $G_{\phi\phi}(x)$ for a system of total size 500. We have plotted both fits : $\log(x)$ and $\log(N/\pi(sin(\pi x/N)))$. A zoom on the first half of this plot if displayed in Figure 15.

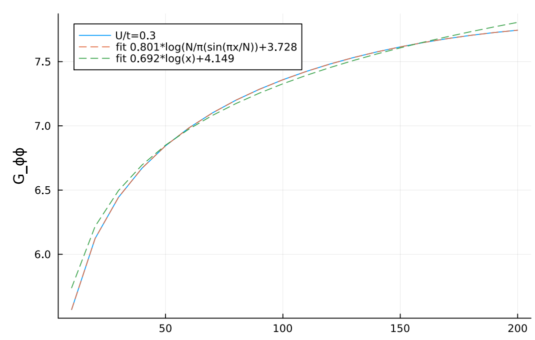

Figure 15: While we see that if we look at the whole picture we clearly see the difference, once we zoom in we don't really see the difference anymore but the prefactor is still quite different. For that reason we must be very careful when evaluating the prefator K.

I.2 Current fluctuations

```julia
    using ITensors, ITensorMPS

arg1 = 0.5 #parse(Float64, ARGS[1])

# Parameters of the system
t = 1
U = arg1
N = 2^10

# Get Hamiltonian and sites
function tight_binding_chain(L)
    sites = siteinds("S=1/2", L)
    h = OpSum()
    for i in 1:L-1
        add!(h, -2*t, "Sx", i, "Sx", i+1)
        add!(h, -2*t, "Sy", i, "Sy", i+1)
        add!(h, U, "Sz", i, "Sz", i+1)
    end
    add!(h, -2*t, "Sx", L, "Sx", 1) #PBC
    add!(h, -2*t, "Sy", L, "Sy", 1) #PBC
    add!(h, U, "Sz", L, "Sz", 1) #PBC
    return MPO(h, sites), sites
end

# Get operator O = theta
function operator_O(L, sites)
    opO = OpSum()
    for i in 2:L
        add!(opO, 1, "Sx", i, "Sy", i-1)
        add!(opO, -1, "Sy", i, "Sx", i-1)
    end
    return MPO(opO,sites)
end

# Get operator O2 = theta^2
function operator_O2(L, sites)
    opO2 = OpSum()
    for i in 2:L
        for j in 2:L
            add!(opO2, 1, "Sx", i, "Sy", i-1, "Sx", j, "Sy", j-1)
            add!(opO2, -1, "Sx", i, "Sy", i-1, "Sy", j, "Sx", j-1)
            add!(opO2, -1, "Sy", i, "Sx", i-1, "Sx", j, "Sy", j-1)
            add!(opO2, 1, "Sy", i, "Sx", i-1, "Sy", j, "Sx", j-1)
        end
    end
    return MPO(opO2,sites)
end

# Perform DMRG to find the ground state and calculate fluctuations

# Create the Hamiltonian and the sites and get the GS
 H, sites = tight_binding_chain(N)
 psi = randomMPS(sites)
 sweeps = Sweeps(10)
 maxdim!(sweeps, 10, 20, 100, 200)
```

```
 cutoff!(sweeps, 1e-10)
 energy, psi0 = dmrg(H, psi, sweeps)


 function calculate_fluctuations(L)

    # Calculate <O> = <theta>
    O1 = operator_O(L, sites)
    avgO = inner(psi0, O1, psi0)

    # Calculate <O^2> = <theta^2>
    O2 = operator_O2(L, sites)
    avgO2 = inner(psi0, O2, psi0)

    # Calculate fluctuations
    fluctuations = avgO2 - avgO^2

    println("system size =", L)

    return fluctuations
end

# Calculate the average value of O for different system sizes L
System_sizes = 5:5:200  # Change this range to explore more values of L
fluct = [calculate_fluctuations(L) for L in System_sizes]
fluct_re = pi^2*real(fluct)

# Open a file in write mode
open("flc_theta_U_$arg1.txt", "w") do file
    for i in 1:length(System_sizes)
        println(file, "$(System_sizes[i]),$(fluct_re[i])")
    end
end
```

I.3 Chiral fluctuations

```
    using ITensors, ITensorMPS

arg1 = 0.5 #parse(Float64, ARGS[1])

# Parameters of the system
t = 1
U = arg1
N = 2^10

# Get Hamiltonian and sites
function tight_binding_chain(L)
    sites = siteinds("S=1/2", L)
    h = OpSum()
    for i in 1:L-1
        add!(h, -2*t, "Sx", i, "Sx", i+1)
        add!(h, -2*t, "Sy", i, "Sy", i+1)
        add!(h, U, "Sz", i, "Sz", i+1)
    end
    add!(h, -2*t, "Sx", L, "Sx", 1) #PBC
    add!(h, -2*t, "Sy", L, "Sy", 1) #PBC
```

```
    add!(h, U, "Sz", L, "Sz", 1) #PBC
    return MPO(h, sites), sites
end


# Get operator theta-phi and theta+phi
function operator_theta_R(L, sites)
    opOR = OpSum()
    for i in 2:L
        add!(opOR, "Sx", i, "Sy", i-1)
        add!(opOR, -1, "Sy", i, "Sx", i-1)
        add!(opOR, -1, "Sz", i)
    end
    return MPO(opOR,sites)
end

function operator_theta_L(L, sites)
    opOL = OpSum()
    for i in 2:L
        add!(opOL, "Sx", i, "Sy", i-1)
        add!(opOL, -1, "Sy", i, "Sx", i-1)
        add!(opOL, 1, "Sz", i)
    end
    return MPO(opOL,sites)
end

# Get operator (theta-phi)^2 and (theta+phi)^2
function operator_theta_R2(L, sites)
    opOR2 = OpSum()
    for i in 2:L
        for j in 2:L
            add!(opOR2, 1, "Sx", i, "Sy", i-1, "Sx", j, "Sy", j-1)
            add!(opOR2, -1, "Sx", i, "Sy", i-1, "Sy", j, "Sx", j-1)
            add!(opOR2, -1, "Sy", i, "Sx", i-1, "Sx", j, "Sy", j-1)
            add!(opOR2, 1, "Sy", i, "Sx", i-1, "Sy", j, "Sx", j-1)
            add!(opOR2, -1, "Sz", i, "Sx", j, "Sy", j-1)
            add!(opOR2, 1,"Sz", i, "Sy", j, "Sx", j-1)
            add!(opOR2, 1, "Sz", i, "Sz", j)
        end
    end
    return MPO(opOR2,sites)
end

function operator_theta_L2(L, sites)
    opOL2 = OpSum()
    for i in 2:L
        for j in 2:L
            add!(opOL2, 1, "Sx", i, "Sy", i-1, "Sx", j, "Sy", j-1)
            add!(opOL2, -1, "Sx", i, "Sy", i-1, "Sy", j, "Sx", j-1)
            add!(opOL2, -1, "Sy", i, "Sx", i-1, "Sx", j, "Sy", j-1)
            add!(opOL2, 1, "Sy", i, "Sx", i-1, "Sy", j, "Sx", j-1)
            add!(opOL2, 1,"Sz", i, "Sx", j, "Sy", j-1)
            add!(opOL2, -1, "Sz", i, "Sy", j, "Sx", j-1)
            add!(opOL2, -1, "Sz", i, "Sz", j)
        end
    end
```

```
    return MPO(opOL2,sites)
end


# Perform DMRG to find the ground state and calculate fluctuations

# Create the Hamiltonian and the sites and get the GS
H, sites = tight_binding_chain(N)
psi = randomMPS(sites)
sweeps = Sweeps(10)
maxdim!(sweeps, 10, 20, 100, 200)
cutoff!(sweeps, 1e-10)
energy, psi0 = dmrg(H, psi, sweeps)

function calculate_fluctuations(L)

    # Calculate <theta_R>
    thetaR1 = operator_theta_R(L, sites)
    avgthetaR1 = inner(psi0, thetaR1, psi0)

    # Calculate <theta_R^2>
    thetaR2 = operator_theta_R2(L, sites)
    avgthetaR2 = inner(psi0, thetaR2, psi0)

    # Calculate fluctuations
    fluctuations = avgthetaR2 - avgthetaR1^2

    println("system size = ", L)

    return fluctuations
end

# Calculate the average value of O for different system sizes L
System_sizes = 5:5:200  # Change this range to explore more values of L
fluct = [calculate_fluctuations(L) for L in System_sizes]
fluct_re = pi^2*real(fluct)


# Open a file in write mode
open("flc_theta_R_U_$arg1.txt", "w") do file
    for i in 1:length(System_sizes)
        println(file, "$(System_sizes[i]),$(fluct_re[i])")
    end
end
```

References

[1] L. D. Landau. “The Theory of a Fermi Liquid”. In: *Zh. Eksp. Teor. Fiz.* 30.6 (1956), p. 1058.

[2] A A Abrikosov and I M Khalatnikov. “The theory of a fermi liquid (the properties of liquid 3He at low temperatures)”. In: *Reports on Progress in Physics* 22.1 (Jan. 1959), p. 329. DOI: 10.1088/0034-4885/22/1/310. URL: https://dx.doi.org/10.1088/0034-4885/22/1/310.

[3] D. Pines and P. Nozieres. *The Theory of Quantum Liquids: Normal Fermi liquids*. Advanced book classics series. W.A. Benjamin, 1966. URL: https://books.google.fr/books?id=rF59QgAACAAJ.

[4] Gerald D. Mahan. *Many Particles Physics*. Plenum New York, 1981.

[5] T. Giamarchi. *Quantum Physics in One Dimension*. International Series of Monographs on Physics. Clarendon Press, 2004. ISBN: 9780198525004. URL: https://books.google.fr/books?id=1MwTDAAAQBAJ.

[6] F D M Haldane. “’Luttinger liquid theory’ of one-dimensional quantum fluids. I. Properties of the Luttinger model and their extension to the general 1D interacting spinless Fermi gas”. In: *Journal of Physics C: Solid State Physics* 14.19 (July 1981), p. 2585. DOI: 10.1088/0022-3719/14/19/010. URL: https://dx.doi.org/10.1088/0022-3719/14/19/010.

[7] Sin-itiro Tomonaga. “Remarks on Bloch’s Method of Sound Waves applied to Many-Fermion Problems”. In: *Progress of Theoretical Physics* 5.4 (July 1950), pp. 544–569. ISSN: 0033-068X. DOI: 10.1143/ptp/5.4.544. eprint: https://academic.oup.com/ptp/article-pdf/5/4/544/5430161/5-4-544.pdf. URL: https://doi.org/10.1143/ptp/5.4.544.

[8] J. M. Luttinger. “An Exactly Soluble Model of a Many-Fermion System”. In: *Journal of Mathematical Physics* 4.9 (Sept. 1963), pp. 1154–1162. ISSN: 0022-2488. DOI: 10.1063/1.1704046. eprint: https://pubs.aip.org/aip/jmp/article-pdf/4/9/1154/19057386/1154_1_online.pdf. URL: https://doi.org/10.1063/1.1704046.

[9] Daniel C. Mattis and Elliott H. Lieb. “Exact Solution of a Many-Fermion System and Its Associated Boson Field”. In: *Bosonization*. 1994, pp. 98–106. DOI: 10.1142/9789812812650_0008. eprint: https://www.worldscientific.com/doi/pdf/10.1142/9789812812650_0008. URL: https://www.worldscientific.com/doi/abs/10.1142/9789812812650_0008.

[10] A. E. Hansen et al. “Mesoscopic decoherence in Aharonov-Bohm rings”. In: *Phys. Rev. B* 64 (4 July 2001), p. 045327. DOI: 10.1103/PhysRevB.64.045327. URL: https://link.aps.org/doi/10.1103/PhysRevB.64.045327.

[11] Karyn Le Hur. “Dephasing of Mesoscopic Interferences from Electron Fractionalization”. In: *Physical Review Letters* 95.7 (Aug. 2005). ISSN: 1079-7114. DOI: 10.1103/physrevlett.95.076801. URL: http://dx.doi.org/10.1103/PhysRevLett.95.076801.

[12] K.-V. Pham, M. Gabay, and P. Lederer. “Fractional excitations in the Luttinger liquid”. In: *Phys. Rev. B* 61 (24 June 2000), pp. 16397–16422. DOI: 10.1103/PhysRevB.61.16397. URL: https://link.aps.org/doi/10.1103/PhysRevB.61.16397.

[13] K.-V. Pham, M. Gabay, and P. Lederer. “Anyons in generalized Luttinger liquid models”. In: *Europhysics Letters* 51.2 (July 2000), p. 161. DOI: 10.1209/epl/i2000-00526-5. URL: https://dx.doi.org/10.1209/epl/i2000-00526-5.

[14] Georg Seelig and Markus Buttiker. “Charge-fluctuation-induced dephasing in a gated mesoscopic interferometer”. In: *Phys. Rev. B* 64 (24 Dec. 2001), p. 245313. DOI: 10.1103/PhysRevB.64.245313. URL: https://link.aps.org/doi/10.1103/PhysRevB.64.245313.

[15] Karyn Le Hur. “Electron fractionalization-induced dephasing in Luttinger liquids”. In: *Physical Review B* 65.23 (June 2002). ISSN: 1095-3795. DOI: 10.1103/physrevb.65.233314. URL: http://dx.doi.org/10.1103/PhysRevB.65.233314.

[16] Hadar Steinberg et al. “Charge fractionalization in quantum wires”. In: *Nature Physics* 4.2 (Dec. 2007), pp. 116–119. ISSN: 1745-2481. DOI: 10.1038/nphys810. URL: http://dx.doi.org/10.1038/nphys810.

[17] D. Senechal. *An introduction to bosonization*. 1999. arXiv: cond-mat/9908262 [cond-mat.str-el].

[18] Matthew P. A. Fisher and Leonid I. Glazman. *Transport in a one-dimensional Luttinger liquid*. 1996. arXiv: cond-mat/9610037 [cond-mat.mes-hall].

[19] A. R. Goni et al. "One-dimensional plasmon dispersion and dispersionless intersubband excitations in GaAs quantum wires". In: *Phys. Rev. Lett.* 67 (23 Dec. 1991), pp. 3298–3301. DOI: 10.1103/PhysRevLett.67.3298. URL: https://link.aps.org/doi/10.1103/PhysRevLett.67.3298.

[20] Henk van Houten and Carlo Beenakker. "Quantum Point Contacts". In: *Physics Today* 49.7 (July 1996), pp. 22–27. ISSN: 1945-0699. DOI: 10.1063/1.881503. URL: http://dx.doi.org/10.1063/1.881503.

[21] Inès Safi and H. J. Schulz. *Transport through a single-band wire connected to measuring leads.* 1997. arXiv: cond-mat/9605014 [cond-mat.mes-hall].

[22] Aleksandr V Eletskii. "Carbon nanotubes". In: *Physics-Uspekhi* 40.9 (Sept. 1997), p. 899. DOI: 10.1070/PU1997v040n09ABEH000282. URL: https://dx.doi.org/10.1070/PU1997v040n09ABEH000282.

[23] Marc Bockrath et al. "Luttinger-liquid behaviour in carbon nanotubes". In: *Nature* 397.6720 (1999), pp. 598–601. DOI: 10.1038/17569. URL: https://doi.org/10.1038/17569.

[24] Hiroyoshi Ishii et al. "Direct observation of Tomonaga–Luttinger-liquid state in carbon nanotubes at low temperatures". In: *Nature* 426.6966 (2003), pp. 540–544. DOI: 10.1038/nature02074. URL: https://doi.org/10.1038/nature02074.

[25] Herbert Levine, Stephen B. Libby, and Adrianus M. M. Pruisken. "Theory of the Quantized Hall Effect. 2." In: *Nucl. Phys. B* 240 (1984), pp. 49–70. DOI: 10.1016/0550-3213(84)90278-5.

[26] X. G. Wen. "Gapless boundary excitations in the quantum Hall states and in the chiral spin states". In: *Phys. Rev. B* 43 (13 May 1991), pp. 11025–11036. DOI: 10.1103/PhysRevB.43.11025. URL: https://link.aps.org/doi/10.1103/PhysRevB.43.11025.

[27] Xiao-Gang Wen. "Edge transport properties of the fractional quantum Hall states and weak-impurity scattering of a one-dimensional charge-density wave". In: *Phys. Rev. B* 44 (11 Sept. 1991), pp. 5708–5719. DOI: 10.1103/PhysRevB.44.5708. URL: https://link.aps.org/doi/10.1103/PhysRevB.44.5708.

[28] K.-V. Pham et al. "Tomonaga-Luttinger liquid with reservoirs in a multiterminal geometry". In: *Physical Review B* 68.20 (Nov. 2003). ISSN: 1095-3795. DOI: 10.1103/physrevb.68.205110. URL: http://dx.doi.org/10.1103/PhysRevB.68.205110.

[29] J. K. Jain. "Theory of the fractional quantum Hall effect". In: *Phys. Rev. B* 41 (11 Apr. 1990), pp. 7653–7665. DOI: 10.1103/PhysRevB.41.7653. URL: https://link.aps.org/doi/10.1103/PhysRevB.41.7653.

[30] Horst L. Stormer, Daniel C. Tsui, and Arthur C. Gossard. "The fractional quantum Hall effect". In: *Rev. Mod. Phys.* 71 (2 Mar. 1999), S298–S305. DOI: 10.1103/RevModPhys.71.S298. URL: https://link.aps.org/doi/10.1103/RevModPhys.71.S298.

[31] F D M Haldane. "General Relation of Correlation Exponents and Spectral Properties of One-Dimensional Fermi Systems: Application to the Anisotropic S=1/2 Heisenberg Chain". In: *Phys. Rev. Lett.* 45.16 (Oct. 1980), pp. 1358–1362. DOI: 10.1103/PhysRevLett.45.1358. URL: https://link.aps.org/doi/10.1103/PhysRevLett.45.1358.

[32] Karyn Le Hur, Bertrand I. Halperin, and Amir Yacoby. "Charge fractionalization in nonchiral Luttinger systems". In: *Annals of Physics* 323.12 (2008), pp. 3037–3058. ISSN: 0003-4916. DOI: https://doi.org/10.1016/j.aop.2008.04.006. URL: https://www.sciencedirect.com/science/article/pii/S0003491608000535.

[33] M. Kapfer et al. "A Josephson relation for fractionally charged anyons". In: *Science* 363.6429 (2019), pp. 846–849. DOI: 10.1126/science.aau3539. eprint: https://www.science.org/doi/pdf/10.1126/science.aau3539. URL: https://www.science.org/doi/abs/10.1126/science.aau3539.

[34] H. Francis Song, Stephan Rachel, and Karyn Le Hur. "General relation between entanglement and fluctuations in one dimension". In: *Physical Review B* 82.1 (July 2010). ISSN: 1550-235X. DOI: 10.1103/physrevb.82.012405. URL: http://dx.doi.org/10.1103/PhysRevB.82.012405.

[35] H. Francis Song et al. "Bipartite fluctuations as a probe of many-body entanglement". In: *Physical Review B* 85.3 (Jan. 2012). ISSN: 1550-235X. DOI: 10.1103/physrevb.85.035409. URL: http://dx.doi.org/10.1103/PhysRevB.85.035409.

[36] A. Crepieux et al. "Electron injection in a nanotube: Noise correlations and entanglement". In: *Phys. Rev. B* 67 (20 May 2003), p. 205408. DOI: 10.1103/PhysRevB.67.205408. URL: https://link.aps.org/doi/10.1103/PhysRevB.67.205408.

[37] Karyn Le Hur. "Electron lifetime in Luttinger liquids". In: *Physical Review B* 74.16 (Oct. 2006). ISSN: 1550-235X. DOI: 10.1103/physrevb.74.165104. URL: http://dx.doi.org/10.1103/PhysRevB.74.165104.

[38] Alexandru Petrescu et al. "Precursor of the Laughlin state of hard-core bosons on a two-leg ladder". In: *Physical Review B* 96.1 (July 2017). ISSN: 2469-9969. DOI: 10.1103/physrevb.96.014524. URL: http://dx.doi.org/10.1103/PhysRevB.96.014524.

[39] A. Luther and I. Peschel. "Calculation of critical exponents in two dimensions from quantum field theory in one dimension". In: *Phys. Rev. B* 12 (9 Nov. 1975), pp. 3908–3917. DOI: 10.1103/PhysRevB.12.3908. URL: https://link.aps.org/doi/10.1103/PhysRevB.12.3908.

[40] Satoshi Nishimoto. "Tomonaga-Luttinger-liquid criticality: Numerical entanglement entropy approach". In: *Physical Review B* 84.19 (Nov. 2011). ISSN: 1550-235X. DOI: 10.1103/physrevb.84.195108. URL: http://dx.doi.org/10.1103/PhysRevB.84.195108.

[41] P. I. Glicofridis et al. "Determination of the resistance across incompressible strips through imaging of charge motion". In: *Physical Review B* 65.12 (Mar. 2002). ISSN: 1095-3795. DOI: 10.1103/physrevb.65.121312. URL: http://dx.doi.org/10.1103/PhysRevB.65.121312.

[42] S. Ilani et al. "Measurement of the quantum capacitance of interacting electrons in carbon nanotubes". In: *Nature Physics* 2.10 (2006), pp. 687–691. DOI: 10.1038/nphys412. URL: https://doi.org/10.1038/nphys412.

[43] Steven R. White. "Density matrix formulation for quantum renormalization groups". In: *Phys. Rev. Lett.* 69 (19 Nov. 1992), pp. 2863–2866. DOI: 10.1103/PhysRevLett.69.2863. URL: https://link.aps.org/doi/10.1103/PhysRevLett.69.2863.

[44] Steven R. White. "Density-matrix algorithms for quantum renormalization groups". In: *Phys. Rev. B* 48 (14 Oct. 1993), pp. 10345–10356. DOI: 10.1103/PhysRevB.48.10345. URL: https://link.aps.org/doi/10.1103/PhysRevB.48.10345.

[45] G. Catarina and Bruno Murta. "Density-matrix renormalization group: a pedagogical introduction". In: *The European Physical Journal B* 96.8 (Aug. 2023). ISSN: 1434-6036. DOI: 10.1140/epjb/s10051-023-00575-2. URL: http://dx.doi.org/10.1140/epjb/s10051-023-00575-2.