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NOTE ON THE KITAEV SPIN CHAINS AND LADDERS —

1 Kitaev spin chain

We start from the $J_1 - J_2$ Kitaev spin chain (why do we choose this model ? this corresponds to weak or strong coupling ? Why spins are along x for the J_1 bonds and along y for the J_2 bond ?)

$$\mathcal{H} = \sum_{j=2m-1} J_1 \sigma_j^x \sigma_{j+1}^x + J_2 \sigma_{j+1}^y \sigma_{j+2}^y \tag{1.1}$$

Where if we choose J_1, $J_2 < 0$ this corresponds to FM Ising couplings.

1.1 Mapping to p-wave SC chain

Using the Jordan-Wigner transformation (see Appendix A.1), we can map spins to complex fermions f_j

$$\mathcal{H} = \sum_{j=2m-1} J_1 (f_j^\dagger - f_j)(f_{j+1}^\dagger + f_{j+1}) - J_2 (f_{j+1}^\dagger + f_{j+1})(f_{j+2}^\dagger - f_{j+2}) \tag{1.2}$$

Now it is convenient to define Majorana operators for odd and even sites $\gamma_{1,j}$ and $\gamma_{2,j}$

$$j = 2m-1 \ (odd), \begin{cases} \gamma_{1,j} = i(f_j^\dagger - f_j) \\ \gamma_{2,j} = f_j^\dagger + f_j \end{cases} \qquad j = 2m \ (even), \begin{cases} \gamma_{1,j} = f_j^\dagger + f_j \\ \gamma_{2,j} = i(f_j^\dagger - f_j) \end{cases} \tag{1.3}$$

we find

$$\mathcal{H} = -i \sum_{j=2m-1} J_1 \gamma_{1,j} \gamma_{1,j+1} - J_2 \gamma_{1,j+1} \gamma_{1,j+2} \tag{1.4}$$

We now introduce the bond fermions on the J_1 bonds (in between sites $j = 2m-1$ and $j+1 = 2m$) :

$$c_m = \frac{1}{2}(\gamma_{1,j} + i\gamma_{1,j+1}) \tag{1.5}$$

which leads to the p-wave superconductor

$$\mathcal{H} = \sum_{m=1}^{M} -2J_1 \left(c_m^\dagger c_m - \frac{1}{2} \right) - J_2 \left(c_m^\dagger c_{m+1} + h.c. \right) - J_2 \left(c_m^\dagger c_{m+1}^\dagger + h.c. \right) \tag{1.6}$$

We immediately identify

$$\mu = 2J_1, \quad t = -\Delta = J_2 \tag{1.7}$$

By Fourier transforming $c_m^\dagger = \frac{1}{\sqrt{M}} \sum_k e^{-ikm} c_k^\dagger$ and noting the spinors $\Psi_k^\dagger = (c_k^\dagger \ c_{-k})$ we arrive at

$$\mathcal{H} = \sum_k \Psi_k^\dagger H(k) \Psi_k, \quad H(k) = \begin{pmatrix} \xi_k & \Delta_k \\ \Delta_k^* & -\xi_k \end{pmatrix} \tag{1.8}$$

with

$$\xi_k = -J_1 - J_2 \cos(k) \tag{1.9a}$$

$$\Delta_k = -iJ_2 \sin(k) \tag{1.9b}$$

Fan has 2kl instead of k ...?

From the energy dispersion by diagonalizing the $H(k)$ we find

$$E_\pm(k) = \pm E(k) = \pm\sqrt{\xi_k^2 + |\Delta_k|^2} = \pm\sqrt{J_1^2 + J_2^2 + 2J_1J_2\cos(k)} \tag{1.10}$$

The gap closes at $J_1 = J_2, k^* = \pi$.

Fan : "at the gapless phase transition point, DMRG calculations reveal a central charge $c = \frac{1}{2}$" $\rightarrow$ how/why ?

1.2 Particle Chern number from relative polarization and correlations

It is also convenient to resolve the global topological invariant, particle Chern number of the Kitaev spin chain from the mapping to a Bloch sphere. Let's introduce the pseudospin vector $\vec{S}$ it's just semantics, but why is it called "pseudospin" ?

$$S^\nu(k) = \Psi_k^\dagger \sigma^\nu \Psi_k, \quad \nu = x, y, z \tag{1.11}$$

Such that the Hamiltonian rewrites into a form that manifests a spin in a magnetic fields

$$\mathcal{H} = -\vec{S}.\vec{d} \tag{1.12}$$

with

$$d_x = 0 \qquad d_y = -J_2\sin(k) = -|\Delta_k| \qquad d_z = J_1 + J_2\cos(k) = -\xi_k \qquad |\vec{d}| = E(k) \tag{1.13}$$

Equivalently, in terms of auxiliary angles (φ_k, θ_k)

$$\mathcal{H}(\varphi_k, \theta_k) = \begin{pmatrix} S^x & S^y & S^z \end{pmatrix} . \begin{pmatrix} \cos(\varphi_k)\sin(\theta_k) \\ \sin(\varphi_k)\sin(\theta_k) \\ \cos(\theta_k) \end{pmatrix} |\vec{d}| \tag{1.14}$$

with

$$\varphi_k = \pi/2 \qquad \sin(\theta_k) = \frac{-|\Delta_k|}{E(k)} \qquad \cos(\theta_k) = \frac{-\xi_k}{E(k)} \tag{1.15}$$

The $|BCS\rangle$ ground state can be built as a filled energy state according to $\eta_k^\dagger |BCS\rangle = 0$, where $\eta_k^\dagger$ creates a particle with the lowest eigenenergy from Bogoliubov-de Gennes (BdG) transformation (see Appendix A.2) :

$$\eta_k = \cos(\theta_k/2)c_k + i\sin(\theta_k/2)c_{-k}^\dagger \tag{1.16}$$

which diagonalizes the Hamiltonian into $\mathcal{H} = \sum_k -E(k)\eta_k^\dagger\eta_k$. The expectation values of S^ν as defined in (1.12) in the GS becomes (see Appendix A.2)

$$\langle S^x\rangle = 0 \quad \langle S^y\rangle = 2?\sin(\theta_k) \quad \langle S^z\rangle = 2?\cos(\theta_k) \tag{1.17}$$

How do we get these values directly ? Is there another way than "naively" rewriting the operators in the new BdG basis ?

The particle Chern number C can be resolved by counting magnetic charges inside a Bloch sphere, a quantity measurable from the relative polarization of a spin at its two, while the total polarization corresponds to the dual number $\overline{C}$

$$C = \frac{1}{2}\left(\langle S^z_{k=0}\rangle - \langle S^z_{k=\pi}\rangle\right) \tag{1.18a}$$

$$\overline{C} = \frac{1}{2}\left(\langle S^z_{k=0}\rangle + \langle S^z_{k=\pi}\rangle\right) \tag{1.18b}$$

Here we have from eq (1.17) that

$$\langle S^z_k\rangle = \frac{-\xi_k}{E(k)} = \frac{J_1 + J_2\cos(k)}{\sqrt{J_1^2+J_2^2+2J_1J_2\cos(k)}} \Rightarrow \begin{cases} \langle S^z_{k=0}\rangle = \frac{J_1+J_2}{|J_1+J_2|} = \text{sgn}(J_1+J_2) \text{ or } \text{sgn}(J_1) \text{ if } J_1 = J_2 \\ \langle S^z_{k=\pi}\rangle = \frac{J_1-J_2}{|J_1-J_2|} = \text{sgn}(J_1-J_2) \text{ or } -\text{sgn}(J_1) \text{ if } J_1 = J_2 \end{cases} \tag{1.19}$$

Therefore, in the gapped phases ($|J_1| \neq |J_2|$) if $|J_1| < |J_2|$, $|C| = 1$ and $|\overline{C}| = 0$ while if $|J_1| > |J_2|$, $|C| = 0$ and $|\overline{C}| = 1$. We have the following cases :

$$J_1 < J_2 \leq 0 \Rightarrow C = 0 \quad \overline{C} = -1 \tag{1.20a}$$
$$J_2 < J_1 \leq 0 \Rightarrow C = -1 \quad \overline{C} = 0 \tag{1.20b}$$
$$0 \leq J_1 < J_2 \Rightarrow C = +1 \quad \overline{C} = 0 \tag{1.20c}$$
$$0 \leq J_2 < J_1 \Rightarrow C = 0 \quad \overline{C} = +1 \tag{1.20d}$$

At the gapless phase transition point $|J_1| = |J_2|$ we identify $\cos(\theta_k) = -\cos(k/2)$ The minus sign here appear because Fan considered the FM case $J_1, J_2 < 0$, otherwise we get a $+$ and

$$C = \overline{C} = -\frac{1}{2} \tag{1.21}$$

Remarkably, the amplitude of particle Chern number is equal to the fractional central charge of the Kitaev spin chain

$$|C| = c = \frac{1}{2} \tag{1.22}$$

I still don't know how to get the value of c. It implies that in 1D, its central charge can be measured in the same way as particle Chern number. Equations (1.18) can also be rewritten in real space using the Fourier transforms (see Appendix A.3)

$$S^z_k = \frac{1}{\sqrt{M}}\sum_m e^{-ikm} S^z_m \tag{1.23a}$$
$$S^z_m = \frac{1}{\sqrt{M}}\sum_n c^\dagger_n c_{n+m} - c_n c^\dagger_{n+m} \tag{1.23b}$$

From (A.17a) we deduce

$$C = \frac{1}{\sqrt{M}}\sum_{m=1}^{M}\langle S_{2m-1}\rangle \qquad \overline{C} = \frac{1}{\sqrt{M}}\sum_{m=0}^{M}\langle S_{2m}\rangle \tag{1.24}$$

and

$$\langle S^z_m\rangle = \frac{1}{\sqrt{M}}\sum_k e^{ikm}\cos(\theta_k) = \frac{1}{\sqrt{M}}\sum_n \langle c^\dagger_n c_{n+m} + h.c. - \delta_{m,0}\rangle = \sqrt{M}(\langle\langle c^\dagger_n c_{n+m}\rangle + h.c. - \delta_{m,0}) \tag{1.25}$$

Why $\cos(2klm)$? Parity ?

In particular at $J_1 = J_2$ which renders $\cos(\theta_k) = -\cos(k/2)$ WHY ?? it can be verified that the nearest-neighbour (NN) correlator (m=1) is linked to the particle Chern number $C = -\frac{1}{2}$:

$$D_{NN} = \langle c^\dagger_n c_{n+1}\rangle + h.c. = \frac{1}{\sqrt{M}}\langle S^z_{m=1}\rangle = -\frac{1}{M}\sum_k e^{ik}\cos(k/2) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik}\cos(k/2)dk$$
$$= -\frac{1}{\pi}\int_0^\pi dk\cos(k)cos(k/2) = -\frac{2}{3\pi} = \frac{4}{3\pi}C \quad \text{Isn't it too easy to just say -1/2 = C ? Physical motivation ?} \tag{1.26}$$

From (A.17b) we can now also use the Jordan-Wigner transformation and rewrite (see Appendix A.3)

$$S^z_{m=0} = \frac{1}{\sqrt{M}} \sum_n (-\sigma^x_{2n-1}\sigma^x_{2n}) \tag{1.27}$$

$$S^z_{m=1} = \frac{1}{\sqrt{M}} \sum_n \frac{-1}{2} (\sigma^x_{2n-1}\sigma^z_{2n}\sigma^z_{2n+1}\sigma^x_{2n+2} + \sigma^y_{2n}\sigma^y_{2n+1}) \tag{1.28}$$

$$S^z_{m>1} = \frac{1}{\sqrt{M}} \sum_n \frac{-1}{2} (\sigma^x_{2n-1}\sigma^x_{2n+2m} O^{2n+2m-1}_{2n} + \sigma^y_{2n}\sigma^y_{2n+2m-1} O^{2n+2m-2}_{2n+1}) \tag{1.29}$$

Where $O^{x_f}_{x_i}$ denotes the Jordan-Wigner string that starts at x_i and ends at x_f

$$O^{x_f}_{x_i} = \prod_{x_i}^{x_f} -\sigma^z_j \tag{1.30}$$

Therefore, D_{NN} becomes

$$-\frac{1}{2}\left(\langle \sigma^x_{2n-1}\sigma^z_{2n}\sigma^z_{2n+1}\sigma^x_{2n+2}\rangle + \langle \sigma^y_{2n}\sigma^y_{2n+1}\rangle\right) = \frac{4}{3\pi} C \tag{1.31}$$

1.3 Local spin correlators

We can calculate these local spin correlators by rewriting them in the bond fermion picture c, $c^\dagger$ using the original $\gamma = \gamma_1$ formulation with the identification $\gamma_j = \gamma_{2m-1} = c_m + c^\dagger_m$ and $\gamma_{2m} = -i(c_m - c^\dagger_m)$

$$\begin{aligned} \sigma^x_{2n-1}\sigma^z_{2n}\sigma^z_{2n+1}\sigma^x_{2n+2} &= -i\gamma_{2n-1}\gamma_{2n+2} \\ &= -i(c_n + c^\dagger_n)(c_{n+1} - c^\dagger_{n+1}) \\ &= -(c^\dagger_{n+1}c_n + c^\dagger_n c_{n+1} + c_n c_{n+1} + c^\dagger_{n+1}c^\dagger_n) \end{aligned} \tag{1.32a}$$

$$\begin{aligned} \sigma^y_{2n}\sigma^y_{2n+1} &= i\gamma_{2n}\gamma_{2n+1} \\ &= (c_n - c^\dagger_n)(c_{n+1} + c^\dagger_{n+1}) \\ &= -(c^\dagger_{n+1}c_n + c^\dagger_n c_{n+1} - c_n c_{n+1} - c^\dagger_{n+1}c^\dagger_n) \end{aligned} \tag{1.32b}$$

Introducing the correlator that probes the p-wave superconducting (SU) pairing effect :

$$D_{SU} = \langle c_n c_{n+1}\rangle + h.c. \tag{1.33}$$

one arrives at

$$\langle \sigma^x_{2n-1}\sigma^z_{2n}\sigma^z_{2n+1}\sigma^x_{2n+2}\rangle = -(D_{NN} + D_{SU}) \tag{1.34a}$$

$$\langle \sigma^y_{2n}\sigma^y_{2n+1}\rangle = -(D_{NN} - D_{SU}) \tag{1.34b}$$

We can check that in momentum space

$$D_{NN} = \frac{1}{M}\sum_k e^{ik}\langle c^\dagger_k c_k\rangle + h.c. = \frac{1}{M}\sum_k \cos(k)\langle c^\dagger_k c_k + c^\dagger_{-k}c_{-k}\rangle = \frac{1}{M}\sum_k \cos(k)(1 + \langle c^\dagger_k c_k - c_{-k}c^\dagger_{-k}\rangle) \tag{1.35a}$$

$$D_{SU} = \frac{1}{M}\sum_k e^{-ik}\langle c_k c_{-k}\rangle + h.c. = \frac{1}{M}\sum_k \sin(k)\langle ic_{-k}c_k - ic^\dagger_k c^\dagger_{-k}\rangle \tag{1.35b}$$

Fan has a typo in the order of the † in D_{NN}

Therefore,

$$D_{NN} = \frac{1}{M}\sum_k \cos(k)\langle S^z(k)\rangle = \frac{1}{M}\sum_k \cos(k)\cos(\theta_k) \tag{1.36a}$$

$$D_{SU} = \frac{1}{M}\sum_k \cos(k)\langle S^y(k)\rangle = \frac{1}{M}\sum_k \sin(k)\sin(\theta_k) \tag{1.36b}$$

Recalling equations (1.15) and using results of Appendix A.4 we find the following results for $J_1, J_2 < 0$ (all signs are opposite for $J_1, J_2 > 0$)

	$\cos(\theta_k)$	$\sin(\theta_k)$	D_{NN}	D_{SU}
$J_2 < J_1 = 0$	$-\cos(k)$	$\sin(k)$	$-1/2$	$1/2$
$J_1 = J_2$	$-\cos(k/2)$	$\sin(k/2)$	$-2/3\pi$	$4/3\pi$
$J_1 < J_2 = 0$	-1	0	0	0

Table 1: Table of values of D_{NN} and D_{SU} for 3 limits of J_1/J_2 (0, 1, $+\infty$)

In Figures 1 and 2 respectively are shown the local spin correlators and their derivative over a range of J_1/J_2 which agree in all three limits with Table 1 calculated using DMRG. The numerical results are also in accordance with the exact calculation of the summations of eqs. (1.36) (Figure done by Fan).

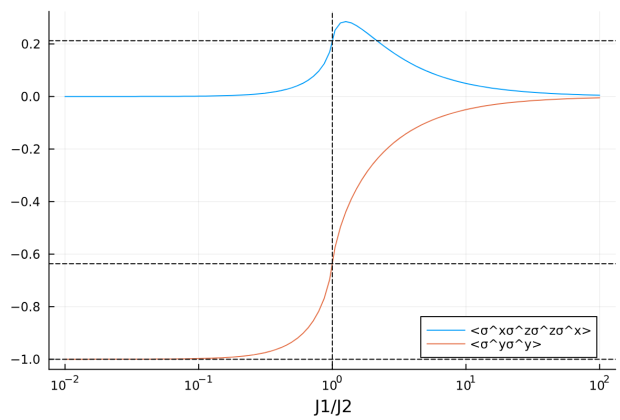

Figure 1: Local spin correlators as a function of $J1/J2$ with $J_1,\ J_2 > 0$!!! calculated with DMRG

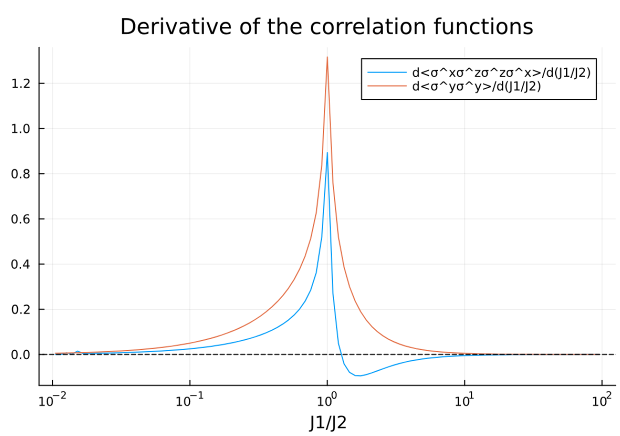

Figure 2: Derivative of the local spin correlators

1.4 Non-local spin correlations

On the other hand we can also examine the non-local still semantics but why is the $m = 0$ case called *non-local* ? spin correlation $\langle S^z_{m=0} \rangle$

$$\langle S^z_{m=0} \rangle = \frac{1}{\sqrt{M}} \sum_k \langle c^\dagger_k c_k - c_{-k} c^\dagger_{-k} \rangle = \frac{1}{\sqrt{M}} \sum_k \langle S^z(k) \rangle = \frac{1}{\sqrt{M}} \sum_k \cos(\theta_k) \tag{1.37}$$

Additionally, if we identify $c^\dagger_n c_n = \rho_n + \rho_0$ as the charge density operator (with $\rho_0 = 1/2$ at half-filling) we immediately see

$$\frac{\langle S^z_{m=0} \rangle}{\sqrt{M}} = \frac{2}{M} \sum_n \langle \rho_n \rangle = \frac{2}{M} \langle Q \rangle \tag{1.38}$$

Thus, at the phase transition $J_1 = J_2$, $\cos(\theta_k) = -\cos(k)$ (for $J_1, J_2 < 0$) so using (1.37) we arrive at

$$\frac{\langle S^z_{m=0} \rangle}{\sqrt{M}} = -\frac{2}{\pi} \tag{1.39}$$

In other regions, as $|J_1|$ increases, $\frac{\langle S^z_{m=0} \rangle}{\sqrt{M}}$ evolves from 0 to -1 ($J_1, J_2 < 0$).

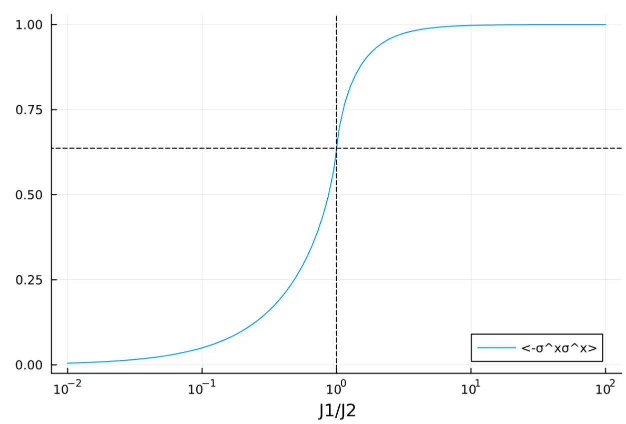

Figure 3: $\langle S^z_{m=0} \rangle / \sqrt{M} = -\langle \sigma^x_{2n-1} \sigma^x_{2n} \rangle$ for $J_1, J_2 > 0$ calculated with DMRG

1.5 Equivalence between QFI and BF at T=0

Starting from a density matrix $\rho = \sum_\lambda p_\lambda \left| \lambda \right\rangle \left\langle \lambda \right|$ and a global operator $\mathcal{O} = \sum_m o_m$, the Quantum Fisher Information (QFI) is defined as

$$f_{\mathcal{O}}(\rho) = 2 \sum_{\lambda, \lambda'} \frac{(p_\lambda - p_{\lambda'})^2}{p_\lambda - p_{\lambda'}} \left| \left\langle \lambda \right| \mathcal{O} \left| \lambda \right\rangle \right|^2, \; p_\lambda + P_{\lambda'} > 0. \tag{1.40}$$

At $T = 0$, the system stays in the ground state and becomes a pure state $\left| GS \right\rangle = \left| \lambda \right\rangle$, $p_\lambda = 1$. All others λ' have $p_{\lambda'} = 0$. Then, the QFI simplifies

$$f_{\mathcal{O}} = 4 \sum_{\lambda' \neq \lambda} \left| \left\langle \lambda' | \lambda \right\rangle \left\langle \lambda | \mathcal{O} | \lambda \right\rangle \right|^2 = 4 \left\{ \sum_{\lambda'} \left| \left\langle \lambda' | \mathcal{O} | \lambda \right\rangle \right|^2 - \left| \left\langle \lambda | \mathcal{O} | \lambda \right\rangle \right|^2 \right\} = 4 \left\{ \left\langle \lambda | \mathcal{O}^2 | \lambda \right\rangle - \left\langle \lambda | \mathcal{O} | \lambda \right\rangle^2 \right\} \tag{1.41}$$

Defining the *bipartite charge fluctuations* of the subsystem A with $Q_A = \sum_{n \in A} q_n$ and $q_n = \rho_n$ as

$$F_Q(A) = \langle Q_A^2 \rangle - \langle Q_A \rangle^2 \tag{1.42}$$

the link between the charge fluctuations and the QFI can be established by writing the density matrix of the subsystem A as a partial trace over the rest of the system $\rho_A = Tr_B \rho$:

$$f_Q(\rho_A) = 4F_Q(A) \tag{1.43}$$

Rewriting the charge operator in the Majorana basis, it becomes a valence bond correlator

$$q_n = c_n^\dagger c_n - \frac{1}{2} = \frac{1}{4}(\gamma_j - i\gamma_{j+1})(\gamma_j + i\gamma_{j+1}) - \frac{1}{2} = \frac{i}{2}\gamma_{2n-1}\gamma_{2n} = -\frac{1}{2}q_{VB},\ q_{VB} = \sigma^x_{2n-1}\sigma^x_{2n} \tag{1.44}$$

It leads to

$$F_{Q_{VB}}(A) = 4F_Q(A) = f_Q(\rho_A) \tag{1.45}$$

In agreement with existing results found in the literature, up to two leading orders.

2 Kitaev spin ladders

A Calculation details

A.1 Jordan-Wigner mapping

We define

$$\sigma_i^z = 2f_i^\dagger f_i - 1 \tag{A.1a}$$

$$\sigma_i^+ = f_i^\dagger e^{i\pi \sum_{k<i} f_k^\dagger f_k} \tag{A.1b}$$

$$\sigma_i^- = e^{-i\pi \sum_{k<i} f_k^\dagger f_k} f_i \tag{A.1c}$$

Using

$$\sigma_i^x = \sigma_i^+ + \sigma_i^- \qquad \sigma_i^y = -i(\sigma_i^+ - \sigma_i^-) \tag{A.2}$$

We can rewrite

$$\sigma_i^+\sigma_{i+1}^- = f_i^\dagger e^{-i\pi f_i^\dagger f_i} f_{i+1} = f_i^\dagger(1-2f_i^\dagger f_i)f_{i+1} = f_i^\dagger f_{i+1} \tag{A.3a}$$

$$\sigma_i^-\sigma_{i+1}^+ = f_i f_{i+1}^\dagger e^{i\pi f_i^\dagger f_i} = f_i f_{i+1}^\dagger(1-2f_i^\dagger f_i) = f_i f_{i+1}^\dagger + 2f_{i+1}^\dagger f_i = -f_i f_{i+1}^\dagger \tag{A.3b}$$

$$\sigma_i^+\sigma_{i+1}^+ = f_i^\dagger f_{i+1}^\dagger(1-2f_i^\dagger f_i) = f_i^\dagger f_{i+1}^\dagger \tag{A.3c}$$

$$\sigma_i^-\sigma_{i+1}^- = f_i f_{i+1}(1-2f_i^\dagger f_i) = -f_i f_{i+1} \tag{A.3d}$$

Therefore the Hamiltonian

$$\begin{aligned}\mathcal{H} &= \sum_{j=2m-1} J_1\sigma_j^x\sigma_{j+1}^x + J_2\sigma_{j+1}^y\sigma_{j+2}^y \\ &= \sum_{j=2m-1} J_1(\sigma_j^+ + \sigma_j^-)(\sigma_{j+1}^+ + \sigma_{j+1}^-) + J_2(\sigma_{j+1}^+ - \sigma_{j+1}^-)(\sigma_{j+2}^- - \sigma_{j+2}^+)\end{aligned} \tag{A.4}$$

Rewrites

$$\mathcal{H} = \sum_{j=2m-1} J_1(f_j^\dagger - f_j)(f_{j+1}^\dagger + f_{j+1}) - J_2(f_{j+1}^\dagger + f_{j+1})(f_{j+2}^\dagger - f_{j+2}) \tag{A.5}$$

A.2 Bogoliubov de Gennes transformation

The eigenvalues of the Hamiltonian are

$$E_{\pm}(k) = \pm\sqrt{\xi_k^2 + |\Delta_k|^2} \tag{A.6}$$

Or is equivalently parametrized by the angle θ_k such that

$$\tan(\theta_k) = \frac{\sin(\theta_k)}{\cos(\theta_k)} = \frac{|\Delta_k|}{\xi_k} \tag{A.7}$$

Where the angle θ_k is related to the "direction" of the vector formed by ξ_k and Δ_k. Specifically, θ_k is the angle that parameterizes the relative weights of these components in the eigenvectors.

Now to find the eigenvectors $u_k = (u_1 \quad u_2)^T$ we solve, for example for the negative eigenvalue

$$H(k)u_k = -E(k)u_k \tag{A.8}$$

which leads to

$$\xi_k u_1 + \Delta_k u_2 = -E(k)u_1 \Rightarrow \frac{u_2}{u_1} = -\frac{E(k)+\xi_k}{\Delta_k} \tag{A.9}$$

Since we want $|u_1|^2 + |u_2|^2$ we can write the vector $(u_1 \quad u_2) = (\cos(\alpha) \quad e^{i\phi}\sin(\alpha))$. Now from equation (A.9) and the half-angle formula

$$\tan(x/2) = \frac{1-\cos(x)}{\sin(x)} \tag{A.10}$$

We recognize that in fact

$$\frac{u_2}{u_1} = -\frac{E(k)+\xi_k}{\Delta_k} = e^{i\varphi_k}\tan(\theta_k/2) \tag{A.11}$$

therefore

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta_k/2) \\ -i\sin(\theta_k/2) \end{pmatrix} \tag{A.12}$$

and similarily the other eigenvector is

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} i\sin(\theta_k/2) \\ \cos(\theta_k/2) \end{pmatrix} \tag{A.13}$$

So the BdG transformation gives

$$\eta_k = \cos(\theta_k/2)c_k + i\sin(\theta_k/2)c_{-k}^\dagger \tag{A.14}$$

$$\eta_{-k}^\dagger = i\sin(\theta_k/2)c_k + \cos(\theta_k/2)c_{-k}^\dagger \tag{A.15}$$

$$\Gamma_k = \begin{pmatrix} \eta_k \\ \eta_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} \cos(\theta_k/2) & i\sin(\theta_k/2) \\ i\sin(\theta_k/2) & \cos(\theta_k/2) \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} \tag{A.16}$$

How do we calculate the excpectation values of $\vec{S}$??

A.3 Fourier and Jordan-Wigner transforms of the spin Chern numbers

$$S_k^z = \frac{1}{\sqrt{M}}\sum_m e^{-ikm} S_m^z \tag{A.17a}$$

$$S_m^z = \frac{1}{\sqrt{M}}\sum_k c_k^\dagger c_k - c_{-k}c_{-k}^\dagger \frac{1}{\sqrt{M}}\sum_n c_n^\dagger c_{n+m} - c_n c_{n+m}^\dagger \tag{A.17b}$$

(1.25) → (1.27) Fan

A.4 Calculation of the Chern number in the 4 limits of J_1/J_2

We have a BCS ground state

$$|\Psi_{GS}\rangle = \prod_{k<k_F} (u_k + v_k c_k^\dagger c_{-k}^\dagger) |0\rangle \tag{A.18}$$

With

$$u_k = \cos(\theta_k/2) \tag{A.19a}$$

$$v_k = i\sin(\theta_k/2) \tag{A.19b}$$

We have that the Chern number C is given by (1.24) as

$$C = \frac{1}{M} \sum_{m=2i-1} \sum_{n=1}^{M} \langle c_n^\dagger c_{n+m}\rangle + h.c. \tag{A.20}$$

Where the $\langle c_n^\dagger c_{n+m}\rangle$ is given by

$$\langle \Psi_{GS}|c_n^\dagger c_{n+m}|\Psi_{GS}\rangle = \frac{1}{\pi}\int_0^\pi \cos(km)|v_k|^2 dk = \frac{-1}{2\pi}\int_0^\pi \cos(km)\cos(\theta_k)dk \tag{A.21}$$

Since

$$|v_k|^2 = \sin^2(\theta_k/2) = \frac{1-\cos(\theta_k)}{2} \tag{A.22}$$

With

$$\cos(\theta_k) = \frac{J_1 + J_2\cos(k)}{\sqrt{J_1^2 + J_2^2 + 2J_1J_2\cos(k)}} \tag{A.23}$$

We can study 4 cases :

$$(1) : J_1 = 0, J_2 \neq 0 \Rightarrow \cos(\theta_k) = \cos(k) \tag{A.24a}$$

$$(2) : J_1 \neq 0, J_2 = 0 \Rightarrow \cos(\theta_k) = 1 \tag{A.24b}$$

$$(3) : J_1 = J_2 \Rightarrow \cos(\theta_k) = \cos(k/2) \tag{A.24c}$$

$$(4) : J_1 = -J_2 \Rightarrow \cos(\theta_k) = \sin(k/2) \tag{A.24d}$$

Then, the calculation of the correlation function from (A.21)

$$(1) : \tfrac{-1}{2\pi}\textstyle\int_0^\pi \cos(km)\cos(k)dk = \begin{cases} -\frac{1}{4} \text{ if } m = \pm 1 \\ \frac{m\sin(m\pi)}{2\pi(m^2-1)} = 0 \text{ otherwise} \end{cases} \tag{A.25a}$$

$$(2) : \frac{-1}{2\pi}\int_0^\pi \cos(km)dk = 0 \tag{A.25b}$$

$$(3) : \frac{-1}{2\pi}\int_0^\pi \cos(km)\cos(k/2)dk = \frac{1}{\pi(1-4m^2)} \tag{A.25c}$$

$$(4) : \frac{-1}{2\pi}\int_0^\pi \cos(km)\sin(k/2)dk = \frac{-1}{\pi(1-4m^2)} \tag{A.25d}$$

Plugging everything again in (A.21) and going into the continuous thermodynamic limit

$$C = \sum_{i=-\infty}^{i=+\infty} \langle c_n^\dagger c_{n+2i+1}\rangle + h.c. \tag{A.26}$$

we find

$$(1) : C = \langle c_n^\dagger c_{n+1} \rangle + h.c. + (+1) \leftrightarrow (-1) = 4 \times \frac{-1}{4} = -1 \tag{A.27a}$$

$$(2) : C = 0 \tag{A.27b}$$

$$(3) : C = \sum_{i=-\infty}^{i=+\infty} \frac{2}{\pi(1-4(2i+1)^2)} = -\frac{1}{2} \tag{A.27c}$$

$$(4) : C = \sum_{i=-\infty}^{i=+\infty} \frac{-2}{\pi(1-4(2i+1)^2)} = \frac{1}{2} \tag{A.27d}$$

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