Ising transverse

$$\begin{cases} \sigma_i^x = 1 - 2a_i^+ a_i \\ \sigma_i^z \sigma_{i+1}^z = (a_i^+ - a_i)(a_{i+1}^+ + a_{i+1}) \end{cases}$$

$$H = \sum_i \Delta \sigma_i^x + J \sigma_i^z \sigma_{i+1}^z$$

↖ here must be all sites, not just odd/even.

$$= \sum_i \underbrace{\Delta(1 - 2a_i^+ a_i)}_{-2\Delta(a_i^+ a_i - \frac{1}{2})} + J(a_i^+ - a_i)(a_{i+1}^+ + a_{i+1})$$

$$-2\Delta(a_i^+ a_i - \tfrac{1}{2}) + J\left\{a_i^+ a_{i+1}^+ + a_i^+ a_{i+1} \underbrace{- a_i a_{i+1}^+}_{+a_{i+1}^+ a_i} \underbrace{- a_i a_{i+1}}_{+a_{i+1} a_i}\right\}$$

$$\Rightarrow \mu = 2\Delta \quad t = -\Delta_{pair} = -J$$

$$\longrightarrow \boxed{H \cong \sum_i -\mu(c_i^+ c_i - \tfrac{1}{2}) - t(c_i^+ c_{i+1} + hc) + \Delta_{pair}(c_i^+ c_{i+1}^+ + hc)} \quad (\star)$$

mapping p-wave

In general for a Kitaev chain w/ Hamiltonian $(\star)$ we have

$$\cos \theta_k = \frac{-\mu - 2t \cos k}{E(k)} \qquad \sin \theta_k = \frac{2\Delta_{pair} \sin k}{E(k)}$$

$$E(k) = \sqrt{(\mu + 2t \cos k)^2 + 4\Delta_{pair}^2 \sin^2 k}$$

$\Rightarrow$ here we have
$$\begin{cases} E(k) = 2\sqrt{\Delta^2 + J^2 - 2\Delta J \cos k} \\ \cos \theta_k = \dfrac{-2\Delta + 2J \cos k}{E(k)} \\ \sin \theta_k = 2\dfrac{J \sin k}{E(k)} \end{cases}$$

$\boxed{(1)\ \Delta = 0,\ J \neq 0}$ $\rightarrow$
$$\begin{cases} E(k) = 2|J| \\ \cos \theta_k = (\mathrm{sgn}(J)) \cos k \\ \sin \theta_k = (\mathrm{sgn}(J)) \sin k \end{cases}$$

$\boxed{(2)\ J = 0,\ \Delta \neq 0}$ $\rightarrow$
$$\begin{cases} E(k) = 2|\Delta| \\ \cos \theta_k = -(\mathrm{sgn}(\Delta)) \\ \sin \theta_k = 0 \end{cases}$$

By default, assume $\Delta, J > 0$

$$\begin{cases} \langle g_j^+ g_{j+m} \rangle = \frac{1}{\pi}\int_0^\pi \cos(km)|v_k|^2 dk \quad \text{with } |v_k|^2 = \sin^2\frac{\theta_k}{2} = \dfrac{1 - \cos\theta_k}{2} \\ \langle g_j^+ g_{j+m}^+ \rangle = \frac{1}{\pi}\int_0^\pi \sin(km)|v_k^* u_k| dk \quad \text{with } |v_k u_k| = \sin\frac{\theta_k}{2}\cos\frac{\theta_k}{2} = \dfrac{\sin\theta_k}{2} \end{cases}$$

$\boxed{(1)}$ $\langle g_j^+ g_{j+m}\rangle = \frac{1}{2\pi}\int_0^\pi \cos(km)\,(1-\mathrm{sgn}(J)\cos k) = \frac{-\mathrm{sgn}(J)}{2\pi}\int_0^\pi \cos(km)\cos k$

$= \mathrm{sgn}(J)\times \begin{cases} -1/4 & \text{if } m=\pm 1 \\ 0 & \text{otherwise} \end{cases}$

$\langle g_j^+ g_{j+m}^+\rangle = \frac{1}{2\pi}\int_0^\pi \sin(km)\,\mathrm{sgn}(J)\sin k = \mathrm{sgn}(J)\begin{cases} \pm\frac{1}{4} & \text{if } m=\pm 1 \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow \langle g^+ g_{+1}\rangle = -\langle g^+ g_{+1}^+\rangle = -1/4$

$\boxed{(2)}$ $\langle g^+ g_{+m}\rangle = \langle g^+ g_{+m}^+\rangle = 0$

$\boxed{(3)\ \Delta = J}$ $\Rightarrow \begin{cases} E = 2\sqrt{2}\,\Delta\sqrt{1-\cos k} \\ \cos\theta_k = -\sin k/2 \\ \sin\theta_k = \cos k/2 \end{cases}$

$\begin{cases} \langle g^+ g_{+m}\rangle = \frac{1}{\pi(1-4m^2)} \\ \langle g^+ g_{+m}^+\rangle = \frac{-2m}{\pi(1-4m^2)} \end{cases}$

$\Rightarrow \begin{cases} \langle g^+ g_{+1}\rangle = \frac{-1}{3\pi} \\ \langle g^+ g_{+1}^+\rangle = \frac{2}{3\pi} = -2\langle g^+ g_{+1}\rangle \end{cases}$

$\boxed{(4)\ \Delta = -J}$ $\Rightarrow \begin{cases} E = 2\sqrt{2}\,\Delta\sqrt{1+\cos k} \\ \cos\theta_k = -\cos k/2 \\ \sin\theta_k = -\sin k/2 \end{cases}$

$\begin{cases} \langle g_j^+ g_{j+m}\rangle = \frac{(-1)^m}{\pi(1-4m^2)} \\ \langle g^+ g_{+m}^+\rangle = \frac{-2m(-1)^m}{\pi(1-4m^2)} \end{cases}$

$\Rightarrow \begin{cases} \langle g^+ g_{+1}\rangle = \frac{+1}{3\pi} \\ \langle g^+ g_{+1}^+\rangle = \frac{-2}{3\pi} = -2\langle g^+ g_{+1}^+\rangle \end{cases}$

$\Rightarrow$

	D_{NN}	D_{SU}	$D_{NN}-D_{SU}$ $=$ $\langle\sigma_i^z\sigma_{i+1}^z\rangle$	$-(D_{NN}+D_{SU})$ $=$ $\langle\sigma_i^y\sigma_{i+1}^y\rangle$
(1)	$-1/2$	$1/2$	-1	0
(2)	0	0	0	0
(3)	$-2/3\pi$	$4/3\pi$	$-2/\pi$	$-2/3\pi$
(4)	$2/3\pi$	$-4/3\pi$	$2/\pi$	$2/3\pi$

$\langle\sigma_i^x\sigma_{i+1}^x\rangle = \langle(1-2a_i^+a_i)(1-2a_{i+1}^+a_{i+1})\rangle = 1-2\{\langle n_i\rangle+\langle n_{i+1}\rangle-2\langle n_i n_{i+1}\rangle\}$

$\langle\sigma_i^y\sigma_{i+1}^y\rangle = \langle(a_i^++a_i)(a_{i+1}^+-a_{i+1})\rangle = -(\langle a_{i+1}^+a_i^+\rangle+\langle a_i^+a_{i+1}\rangle+\langle a_{i+1}^+a_i\rangle+\langle a_i a_{i+1}\rangle)$
$= -(D_{NN}+D_{SU})$ $= \langle\sigma^x\sigma^z\sigma^z\sigma^x\rangle$ of J_1-J_2 chain

$\langle\sigma_i^z\sigma_{i+1}^z\rangle = \langle(a_i^+-a_i)(a_{i+1}+a_{i+1}^+)\rangle = \langle a_i^+a_{i+1}\rangle + hc - \langle a_i a_{i+1}\rangle - hc = D_{NN}-D_{SU}$ $= -\langle\sigma^y\sigma^y\rangle$ of J_1-J_2 chain