

0 Assumptions

Currently, we assume the following assumptions (using Alex's paper notions):

- $\epsilon_\tau = 0$ and $\tau = 1$
- f is balanced, so: $\mathbb{E}f = 1/2$
- $\beta = 0$ and $\gamma = \alpha$
- $h = f$
- $\epsilon_1 = \epsilon$ and $\lambda_1 = \lambda$

1 Extension of the FKN theorem (JOW's paper)

1.1 Theorem 5.1:

There exists some $k \in \{0, 1, \dots, n\}$ such that:

$$a_k^2 \geq 1 - \frac{9 + \sqrt{17}}{2} \cdot \rho^2$$

Also,

$$\rho \leq d \leq \left(9 + \sqrt{17}\right)^{1/2} \cdot \rho \quad (1)$$

and

$$d \leq \left(\frac{9 + \sqrt{17}}{2}\right)^{1/2} \cdot \rho + o(\rho) \quad (2)$$

as $\rho \rightarrow 0$

1.2 Theorem 5.3:

1.2.1 the theorem as stated in the paper:

There exists a universal constant $L > 0$ with the following property: For $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$, let:

$$\rho = \left(\sum_{A \subseteq [n]: |A| \geq 2} |\hat{f}(A)|^2 \right)^{1/2}$$

Then, there exists some $B \subseteq [n]$ with $|B| \leq 1$ such that

$$\sum_{A \subseteq [n]: |A| \leq 1, A \neq B} |\hat{f}(A)|^2 \leq L \cdot \rho^4 \ln(2/\rho)$$

and $|\hat{f}(B)|^2 \geq 1 - \rho^2 - L \cdot \rho^4 \ln(2/\rho)$

(part of the) Proof: Let $a_i = \hat{f}(\{i\})$

$$\sum_{i \in \{0,1,\dots,n\} \setminus \{k\}} a_i^2 \leq 2d^4 \log_2(2/d) \quad (3)$$

1.2.2 extracting value of the universal constant

If we substitute (1) in (3) we have:

$$\sum_{i \in \{0,1,\dots,n\} \setminus \{k\}} a_i^2 \leq 2d^4 \log_2(2/d) \leq 2 \left(9 + \sqrt{17}\right)^2 \cdot \rho^4 \log_2(2/\rho) \quad (4)$$

2 On the entropy of a noisy function (Alex's paper)

2.1 Theorem 5.5

Depending on (4) we can restate theorem 5.5 to the following form:

Theorem 5.5: For $g : \{0,1\}^n \rightarrow \{-1,1\}$, let $\rho = \left(\sum_{A \subseteq [n]: |A| \geq 2} \hat{g}^2(A)\right)^{1/2}$. Then there exists some $B \subseteq [n]$ with $|B| \leq 1$ such that:

$$\sum_{A \subseteq [n]: |A| \leq 1, A \neq B} \hat{g}^2(A) \leq 2 \left(9 + \sqrt{17}\right)^2 \cdot \rho^4 \log\left(\frac{2}{\rho}\right)$$

2.2 Equation (20)

It is true with $L = 2$:

$$\sum_{|A| \geq 2} \hat{f}^2(A) \leq \sum_{|A| \geq 2} \hat{g}^2(A) \leq 1 - \hat{g}^2(\{1\}) = 1 - (1 + \gamma^2 - 2\gamma) = 2\gamma - \gamma^2 \leq 2\gamma \quad (5)$$

2.3 Equation (21)

Using (4), we have the following:

$$\sum_{k=2}^n \hat{f}^2(\{k\}) \leq \sum_{k=2}^n \hat{g}^2(\{k\}) \leq \left(9 + \sqrt{17}\right)^2 \cdot \gamma \cdot \log\left(\frac{1}{\gamma}\right) \quad (6)$$

2.4 Proof of Lemma 5.2

The first inequality appears in proof of lemma 5.2, so, depending on (5) and (6) we have:

$$\begin{aligned}
\mathbb{E}_{T,1 \notin T} Ent(g_T) &\leq \frac{2}{\ln 2} \cdot \frac{1}{\mathbb{E}f} \cdot \sum_{S, 1 \notin S} |S| \lambda^{|S|} \hat{f}^2(S) \\
&\leq \frac{4}{\ln 2} \cdot \left(\sum_{|S| \geq 2} |S| \lambda^{|S|} \hat{f}^2(S) + \sum_{k=2}^n \hat{f}^2(\{k\}) \cdot \lambda \right) \\
&\leq \frac{4}{\ln 2} \cdot \left(2\lambda^2 \cdot \sum_{|S| \geq 2} \hat{f}^2(S) + \lambda \cdot \sum_{k=2}^n \hat{f}^2(\{k\}) \right) \\
&\leq \frac{4}{\ln 2} \cdot \left(4(\lambda^2 \gamma) + (9 + \sqrt{17})^2 \cdot \left(\lambda \gamma^2 \log \left(\frac{1}{\gamma} \right) \right) \right)
\end{aligned} \tag{7}$$

I think we require here (in the 3rd line) that $\lambda \leq 2/3$.

2.5 Proof of Lemma 5.1

2.5.1 Equation 22

$$\mathbb{E}_{T,1 \in T} Ent(g_T | x_1 = 0, x_2, \dots, x_n) \leq L \cdot \lambda \gamma \tag{8}$$

Proof. Depending on (5) and (6), and in similar way as the proof of (7) we have:

$$\sum_{R, 1 \notin R} |R| \lambda^{|R|} \hat{f}^2(R) \leq 4(\lambda^2 \gamma) + (9 + \sqrt{17})^2 \cdot \left(\lambda \gamma^2 \log \left(\frac{1}{\gamma} \right) \right) \tag{9}$$

Also

$$\sum_{R, 1 \notin R} |R| \lambda^{|R|} \hat{f}^2(R \cup \{1\}) \leq 2 \cdot \lambda \cdot \sum_{R, 1 \notin R} \hat{f}^2(R \cup \{1\}) \leq 2 \cdot \lambda \cdot \sum_{|R| \geq 2} \hat{f}^2(R) \leq 4 \cdot \lambda \gamma \tag{10}$$

In addition, we have:

$$\begin{aligned}
Ent(g_T | x_1 = 0, x_2, \dots, x_n) &\leq \frac{2}{\ln 2} \cdot \frac{1}{\mathbb{E}f} \cdot \sum_{R \subseteq T \setminus \{1\}} |R| \cdot \left(\hat{f}(R) + \hat{f}(R \cup \{1\}) \right)^2 \\
&= \frac{4}{\ln 2} \cdot \sum_{R \subseteq T \setminus \{1\}} |R| \cdot \left(\hat{f}^2(R) + \hat{f}^2(R \cup \{1\}) + 2 \cdot \hat{f}(R) \cdot \hat{f}(R \cup \{1\}) \right) \\
&\leq \frac{8}{\ln 2} \cdot \sum_{R \subseteq T \setminus \{1\}} |R| \cdot \left(\hat{f}^2(R) + \hat{f}^2(R \cup \{1\}) \right)
\end{aligned} \tag{11}$$

Averaging over T and depending on (9), (10) and (11) we have:

$$\begin{aligned}
\mathbb{E}_{T, 1 \notin T} Ent(g_T | x_1 = 0, x_2, \dots, x_n) &\leq \frac{8}{\ln 2} \cdot \left(\left(4(\lambda^2 \gamma) + (9 + \sqrt{17})^2 \cdot \left(\lambda \gamma^2 \log \left(\frac{1}{\gamma} \right) \right) \right) + 4 \cdot \lambda \gamma \right) \\
&\leq \frac{8}{\ln 2} \cdot (4 \cdot \lambda \gamma + 64 \cdot \lambda \gamma + 4 \cdot \lambda \gamma) \\
&= \frac{576}{\ln 2} \cdot \lambda \gamma < 831 \cdot \lambda \gamma
\end{aligned}$$

(12)

□