0 Assumptions

Currently, we assume the following assumptions (using Alex's paper notions):

- $\epsilon_\tau = 0$ and $\tau = 1$
- f is balanced, so: $\mathbb{E}f = 1/2$
- $\beta = 0$ and $\gamma = \alpha$
- $h = f$
- $\epsilon_1 = \epsilon$ and $\lambda_1 = \lambda$

1 Extension of the FKN theorem (JOW's paper)

1.1 Theorem 5.1:

There exists some $k \in \{0, 1, \ldots .n\}$ such that:

$$a_k^2 \geq 1 - \frac{9 + \sqrt{17}}{2} \cdot \rho^2$$

Also,

$$\rho \leq d \leq \left(9 + \sqrt{17}\right)^{1/2} \cdot \rho \tag{1}$$

and

$$d \leq \left(\frac{9 + \sqrt{17}}{2}\right)^{1/2} \cdot \rho + o(\rho) \tag{2}$$

as $\rho \to 0$

1.2 Theorem 5.3:

1.2.1 the theorem as stated in the paper:

There exists a universal constant $L > 0$ with the following property: For $f : \{-1,1\}^n \to \{-1,1\}$, let:

$$\rho = \left(\sum_{A \subseteq [n]:|A| \geq 2} |\hat{f}(A)|^2\right)^{1/2}$$

Then, there exists some $B \subseteq [n]$ with $|B| \leq 1$ such that

$$\sum_{A \subseteq [n]:|A| \leq 1, A \neq B} |\hat{f}(A)|^2 \leq L \cdot \rho^4 ln(2/\rho)$$

and $|\hat{f}(B)|^2 \geq 1 - \rho^2 - L \cdot \rho^4 ln(2/\rho)$

(part of the) Proof: Let $a_i = \hat{f}(\{i\})$

$$\sum_{i\in\{0,1,\ldots,n\}\setminus\{k\}} a_i^2 \leq 2d^4 log_2(2/d) \tag{3}$$

1.2.2 extracting value of the universal constant

If we substitute (1) in (3) we have:

$$\sum_{i\in\{0,1,\ldots,n\}\setminus\{k\}} a_i^2 \leq 2d^4 log_2(2/d) \leq 2\cdot\left(9+\sqrt{17}\right)^2\cdot\rho^4\cdot log_2(2/\rho) \tag{4}$$

2 On the entropy of a noisy function (Alex's paper)

2.1 Theorem 5.5

Depending on (4) we can restate theorem 5.5 to the following form:

Theorem 5.5: *For $g:\{0,1\}^n \to \{-1,1\}$, let $\rho = \left(\sum_{A\subseteq[n]:|A|\geq 2}\hat{g}^2(A)\right)^{1/2}$. Then there exists some $B\subseteq[n]$ with $|B|\leq 1$ such that:*

$$\sum_{A\subseteq[n]:|A|\leq 1, A\neq B} \hat{g}^2(A) \leq 2\cdot\left(9+\sqrt{17}\right)^2\cdot\rho^4\cdot log\left(\frac{2}{\rho}\right)$$

2.2 Equation (20)

It is true with $L=2$:

$$\sum_{|A|\geq 2}\hat{f}^2(A) \leq \sum_{|A|\geq 2}\hat{g}^2(A) \leq 1-\hat{g}^2(\{1\}) = 1-\left(1+\gamma^2-2\cdot\gamma\right) = 2\cdot\gamma-\gamma^2 \leq 2\cdot\gamma \tag{5}$$

2.3 Equation (21)

Using (4), we have the following:

$$\sum_{k=2}^{n}\hat{f}^2\left(\{k\}\right) \leq \sum_{k=2}^{n}\hat{g}^2\left(\{k\}\right) \leq \left(9+\sqrt{17}\right)^2\cdot\gamma\cdot log\left(\frac{1}{\gamma}\right) \tag{6}$$

2.4 Proof of Lemma 5.2

The first inequality appears in proof of lemma 5.2, so, depending on (5) and (6) we have:

$$
\begin{aligned}
\mathbb{E}_{T,1\notin T} Ent(g_T) &\leq \frac{2}{ln2} \cdot \frac{1}{\mathbb{E}f} \cdot \sum_{S,1\notin S} |S|\lambda^{|S|}\hat{f}^2(S) \\
&\leq \frac{4}{ln2} \cdot \left(\sum_{|S|\geq 2} |S|\lambda^{|S|}\hat{f}^2(S) + \sum_{k=2}^{n} \hat{f}^2\left(\{k\}\right) \cdot \lambda \right) \\
&\leq \frac{4}{ln2} \cdot \left(2\lambda^2 \cdot \sum_{|S|\geq 2} \hat{f}^2(S) + \lambda \cdot \sum_{k=2}^{n} \hat{f}^2\left(\{k\}\right) \right) \\
&\leq \frac{4}{ln2} \cdot \left(4(\lambda^2 \cdot \gamma) + \left(9+\sqrt{17}\right)^2 \cdot \left(\lambda \cdot \gamma^2 log\left(\frac{1}{\gamma}\right)\right)\right)
\end{aligned} \tag{7}
$$