0 Assumptions

Currently, we assume the following assumptions (using Alex's paper notions):

- $\epsilon_{\mathcal{T}} = 0$ and $\tau = 1$
- f is balanced, so: $\mathbb{E}f = 1/2$
- $\beta = 0$ and $\gamma = \alpha$
- $h = f$
- $\epsilon_1 = \epsilon$ and $\lambda_1 = \lambda$

1 Extension of the FKN theorem (JOW's paper)

1.1 Theorem 5.1:

There exists some $k \in \{0, 1, \ldots .n\}$ such that:

$$a_k^2 \geq 1 - \frac{9 + \sqrt{17}}{2} \cdot \rho^2$$

Also,

$$\rho \leq d \leq \left(9 + \sqrt{17}\right)^{1/2} \cdot \rho \tag{1}$$

and

$$d \leq \left(\frac{9 + \sqrt{17}}{2}\right)^{1/2} \cdot \rho + o(\rho) \tag{2}$$

as $\rho \to 0$

1.2 Theorem 5.3:

1.2.1 the theorem as stated in the paper:

There exists a universal constant $L > 0$ with the following property: For $f : \{-1, 1\}^n \to \{-1, 1\}$, let:

$$\rho = \left(\sum_{A \subseteq [n]: |A| \geq 2} |\hat{f}(A)|^2 \right)^{1/2}$$

Then, there exists some $B \subseteq [n]$ with $|B| \leq 1$ such that

$$\sum_{A \subseteq [n]: |A| \leq 1, A \neq B} |\hat{f}(A)|^2 \leq L \cdot \rho^4 ln(2/\rho)$$

and $|\hat{f}(B)|^2 \geq 1 - \rho^2 - L \cdot \rho^4 ln(2/\rho)$

(part of the) Proof: Let $a_i = \hat{f}(\{i\})$

$$\sum_{i\in\{0,1,\ldots,n\}\setminus\{k\}} a_i^2 \leq 2d^4 log_2(2/d) \tag{3}$$

1.2.2 extracting value of the universal constant

If we substitute (1) in (3) we have:

$$\sum_{i\in\{0,1,\ldots,n\}\setminus\{k\}} a_i^2 \leq 2d^4 log_2(2/d) \leq 2\left(9+\sqrt{17}\right)^2 \cdot \rho^4 \cdot log_2(2/\rho) \tag{4}$$

2 On the entropy of a noisy function (Alex's paper)

2.1 Theorem 5.5

Depending on (4) we can restate theorem 5.5 to the following form:

Theorem 5.5: *For* $g : \{0,1\}^n \to \{-1,1\}$, *let* $\rho = \left(\sum_{A\subseteq[n]:|A|\geq 2} \hat{g}^2(A)\right)^{1/2}$. *Then there exists some* $B \subseteq [n]$ *with* $|B| \leq 1$ *such that:*

$$\sum_{A\subseteq[n]:|A|\leq 1, A\neq B} \hat{g}^2(A) \leq 2\left(9+\sqrt{17}\right)^2 \cdot \rho^4 \cdot log\left(\frac{2}{\rho}\right)$$

2.2 Equation (20)

It is true with $L = 2$:

$$\sum_{|A|\geq 2} \hat{f}^2(A) \leq \sum_{|A|\geq 2} \hat{g}^2(A) \leq 1 - \hat{g}^2(\{1\}) = 1 - \left(1+\gamma^2-2\gamma\right) = 2\gamma - \gamma^2 \leq 2\gamma \quad (5)$$

2.3 Equation (21)

Using (4), we have the following:

$$\sum_{k=2}^{n} \hat{f}^2\left(\{k\}\right) \leq \sum_{k=2}^{n} \hat{g}^2\left(\{k\}\right) \leq \left(9+\sqrt{17}\right)^2 \cdot \gamma^2 \cdot log\left(\frac{1}{\gamma}\right) \quad (6)$$

2.4 Proof of Lemma 5.2

The first inequality appears in proof of lemma 5.2, so, depending on (5) and (6) we have:

$$\begin{aligned}
\mathop{\mathbb{E}}_{T,1\notin T} Ent(g_T) &\leq \frac{2}{ln2} \cdot \frac{1}{\mathbb{E}f} \cdot \sum_{S,1\notin S} |S|\lambda^{|S|}\hat{f}^2(S) \\
&\leq \frac{4}{ln2} \cdot \left(\sum_{|S|\geq 2} |S|\lambda^{|S|}\hat{f}^2(S) + \sum_{k=2}^{n} \hat{f}^2\left(\{k\}\right)\cdot\lambda\right) \\
&\leq \frac{4}{ln2} \cdot \left(2\lambda^2 \cdot \sum_{|S|\geq 2} \hat{f}^2(S) + \lambda \cdot \sum_{k=2}^{n} \hat{f}^2\left(\{k\}\right)\right) \\
&\leq \frac{4}{ln2} \cdot \left(4(\lambda^2\gamma) + \left(9+\sqrt{17}\right)^2 \cdot \left(\lambda\gamma^2 log\left(\frac{1}{\gamma}\right)\right)\right)
\end{aligned} \quad (7)$$

2.5 Proof of Lemma 5.1

2.5.1 Lemma 5.1

For constant L, and for f that satisfies the conditions of theorem 1.14, we have:

$$\mathop{\mathbb{E}}_{T,1\in T}\left(Ent\left(f \mid T\right) - Ent\left(f \mid \{1\}\right)\right) \leq L \cdot \left(\lambda\gamma + \gamma^2 ln\left(\frac{1}{\gamma}\right)\right) \tag{8}$$

2.5.2 Equation 22

$$\mathop{\mathbb{E}}_{T,1\in T} Ent\left(g_T \mid x_1 = 0, x_2, \cdots, x_n\right) \leq L \cdot \lambda\gamma \tag{9}$$

Proof. Depending on (5) and (6), and in similar way as the proof of (7) we have:

$$\sum_{R,1\notin R} |R|\lambda^{|R|}\hat{f}^2(R) \leq 4(\lambda^2\gamma) + \left(9+\sqrt{17}\right)^2 \cdot \left(\lambda\gamma^2 log\left(\frac{1}{\gamma}\right)\right) \tag{10}$$

Also

$$\sum_{R,1\notin R} |R|\lambda^{|R|}\hat{f}^2(R\cup\{1\}) \leq 2\cdot\lambda\cdot\sum_{R,1\notin R}\hat{f}^2(R\cup\{1\}) \leq 2\cdot\lambda\cdot\sum_{|R|\geq 2}\hat{f}^2(R) \leq 4\cdot\lambda\gamma \tag{11}$$

If we fix a subset $T \subset [n]$ with $1 \in T$, then:

$$\mathbb{E}\left(g_T \mid x_1 = 0, x_2, \cdots, x_n\right) = \sum_{R\subseteq T\setminus\{1\}} \left(\hat{f}(R) + \hat{f}(R\cup\{1\})\right)\cdot W_R \tag{12}$$

And,

$$\mathbb{E}\left(g_T \mid x_1 = 0\right) = \hat{f}(0) + \hat{f}(\{1\}) = \mathbb{E}f\cdot(1+(1-\alpha)) = (2-\alpha)\cdot\mathbb{E}f \geq \mathbb{E}f \tag{13}$$

In particular, using (19) from the original article, we have:

$$\begin{aligned}
Ent\left(g_T \mid x_1 = 0, x_2, \cdots, x_n\right) &\leq \frac{2}{ln2}\cdot\frac{1}{\mathbb{E}\left(g_T \mid x_1 = 0\right)}\cdot\sum_{R\subseteq T\setminus\{1\}}|R|\cdot\left(\hat{f}(R) + \hat{f}(R\cup\{1\})\right)^2 \\
&\leq \frac{2}{ln2}\cdot\frac{1}{\mathbb{E}f}\cdot\sum_{R\subseteq T\setminus\{1\}}|R|\lambda^{|R|}\left(\hat{f}(R) + \hat{f}(R\cup\{1\})\right)^2 \\
&= \frac{4}{ln2}\cdot\sum_{R\subseteq T\setminus\{1\}}|R|\lambda^{|R|}\left(\hat{f}^2(R) + \hat{f}^2(R\cup\{1\}) + 2\cdot\hat{f}(R)\cdot\hat{f}(R\cup\{1\})\right) \\
&\leq \frac{8}{ln2}\cdot\sum_{R\subseteq T\setminus\{1\}}|R|\lambda^{|R|}\left(\hat{f}^2(R) + \hat{f}^2(R\cup\{1\})\right)
\end{aligned} \tag{14}$$

Averaging over T and depending on (10), (11) and (14) we have:

$$\begin{aligned}\underset{T,1\in T}{\mathbb{E}} Ent\left(g_T \mid x_1=0,x_2,\cdots,x_n\right) &\leq \frac{8}{ln2}\cdot\left(\left(4(\lambda^2\gamma)+\left(9+\sqrt{17}\right)^2\cdot\left(\lambda\gamma^2 log\left(\frac{1}{\gamma}\right)\right)\right)+4\cdot\lambda\gamma\right)\\ &\leq \frac{8}{ln2}\cdot(4\cdot\lambda\gamma+64\cdot\lambda\gamma+4\cdot\lambda\gamma)\\ &= \frac{576}{ln2}\cdot\lambda\gamma<831\cdot\lambda\gamma\end{aligned} \tag{15}$$

□

2.5.3 Equation 23

$$\underset{T,1\in T}{\mathbb{E}} Ent\left(g_T \mid x_1=1,x_2,\cdots,x_n\right)\leq L\cdot\left(\lambda\gamma+\gamma^2 ln\left(\frac{1}{\gamma}\right)\right) \tag{16}$$

Proof. Similarly to what we have in 12, we have:

$$\mathbb{E}\left(g_T \mid x_1=1,x_2,\cdots,x_n\right)=\sum_{R\subset T\setminus\{1\}}\left(\hat{f}(R)-\hat{f}(R\cup\{1\})\right)\cdot W_R \tag{17}$$

So, depending on the definition of α and λ (in our case, we always have $\alpha\geq\lambda$):

$$\mathbb{E}\left(g_T \mid x_1=1\right)=\hat{f}(0)-\hat{f}(\{1\})=\mathbb{E}f\cdot(1-(1-\alpha))\geq\lambda\cdot\mathbb{E}f \tag{18}$$

Let $g=\mathbb{E}\left(f \mid x_1=1,x_2,\cdots,x_n\right)$.
Applying (19) from the original paper, and averaging over T, we have:

$$\begin{aligned}\underset{T,1\in T}{\mathbb{E}} Ent\left(g_T \mid x_1=1,x_2,\cdots,x_n\right) &\leq \frac{2}{ln2}\cdot\frac{1}{\mathbb{E}\left(g_T \mid x_1=1\right)}\cdot\sum_{R\subseteq T\setminus\{1\}}|R|\lambda^{|R|}\left(\hat{f}(R)-\hat{f}(R\cup\{1\})\right)^2\\ &\leq \frac{4}{\lambda\cdot ln2}\cdot\sum_{R\subseteq T\setminus\{1\}}|R|\lambda^{|R|}\left(\hat{f}(R)-\hat{f}(R\cup\{1\})\right)^2\\ &= \frac{4}{\lambda\cdot ln2}\cdot\sum_{R\subseteq T\setminus\{1\}}|R|\lambda^{|R|}\hat{g}^2(R)\end{aligned} \tag{19}$$

g is a boolean function whose expectation equals to: $\hat{f}(0)-\hat{f}(\{1\})=\alpha\cdot\mathbb{E}f\leq\alpha\leq\gamma$, similarly: $\mathbb{E}g^2=\mathbb{E}g\leq\alpha\leq\gamma$ (we depended on $\alpha\leq\gamma$. actually, in our case we have $\alpha=\gamma$)

Inequality of Talagrand *For a boolean function* $g:\{0,1\}^m\to\{0,1\}$ *with expectation* $\mu\leq 1/2$ *holds* $\sum_{k=1}^m \hat{g}^2(\{k\})\leq\left(2+\sqrt{2}\right)^2\cdot\mu^2\cdot ln(1/\mu)$.

So, in our case we have:

$$\sum_{k=2}^{n} \hat{g}^2(\{k\}) \leq \left(2+\sqrt{2}\right)^2 \cdot \mathbb{E}^2 f \cdot ln\left(\frac{1}{\mathbb{E}f}\right) \leq \left(2+\sqrt{2}\right)^2 \cdot \alpha^2 \cdot ln\left(\frac{1}{\alpha}\right)$$
$$\leq \left(2+\sqrt{2}\right)^2 \cdot \gamma^2 \cdot ln\left(\frac{1}{\gamma}\right) \tag{20}$$

Depending on (5) and (20) we can continue (19) by:

$$\begin{aligned}
\mathop{\mathbb{E}}_{T,1\in T} Ent\left(g_T \mid x_1=1, x_2, \cdots, x_n\right) &\leq \frac{4}{\lambda \cdot ln2} \cdot \left(\lambda \cdot \sum_{k=2}^{n} \hat{g}^2(\{k\}) + 2\lambda^2 \sum_{|A|\geq 2} \hat{g}^2\right) \\
&\leq \frac{4}{\lambda \cdot ln2} \cdot \left(\left(2+\sqrt{2}\right)^2 \lambda\gamma^2 ln\left(\frac{1}{\gamma}\right) + 4\lambda^2\gamma\right) \\
&= \frac{16}{ln2} \cdot \left(\left(1+\frac{1}{\sqrt{2}}\right)^2 \gamma^2 ln\left(\frac{1}{\gamma}\right) + \lambda\gamma\right) \\
&\leq \frac{16}{ln2} \cdot \left(3 \cdot \gamma^2 ln\left(\frac{1}{\gamma}\right) + \lambda\gamma\right) \\
&\leq \frac{48}{ln2} \cdot \left(\gamma^2 ln\left(\frac{1}{\gamma}\right) + \lambda\gamma\right)
\end{aligned} \tag{21}$$

□

2.5.4 Proof of the lemma

Let $g_T = \mathbb{E}\left(f \mid T\right)$ for a subset $T \subseteq [n]$. Note that if $1 \in T$ then $\mathbb{E}\left(g_T \mid \{1\}\right) = \mathbb{E}\left(f \mid \{1\}\right)$.

Hence:

$$\mathop{\mathbb{E}}_{T,1\in T}\left(Ent\left(f \mid T\right) - Ent\left(f \mid \{1\}\right)\right) = \mathop{\mathbb{E}}_{T,1\in T}\left(Ent\left(g_T \mid T\right) - Ent\left(g_T \mid \{1\}\right)\right) =$$
$$\frac{1}{2} \cdot \mathop{\mathbb{E}}_{T,1\in T} Ent\left(g_T \mid x_1=0, x_2, \cdots, x_n\right) + \frac{1}{2} \mathop{\mathbb{E}}_{T,1\in T} Ent\left(g_T \mid x_1=1, x_2, \cdots, x_n\right)$$
$$\leq \frac{288}{ln2} \cdot \lambda\gamma + \frac{8}{ln2} \cdot \left(3 \cdot \gamma^2 ln\left(\frac{1}{\gamma}\right) + \lambda\gamma\right) = \frac{296}{ln2} \cdot \lambda\gamma + \frac{24}{ln2} \cdot \gamma^2 ln\left(\frac{1}{\gamma}\right) \tag{22}$$