1 10th of October 2018 — F. Poloni

This lecture has the goal of introducing the concept of linear combinations.

Definition 1.1 (Linear combination)**.** *In a very unformal way, we can define the goal of linear combination as the pursuit of obtaining a certain target vector $b \in \mathbb{R}^n$ using m (in principle $m \neq n$) vectors $a_1, a_2, \dots, a_m$ such that:*

$$a_1 x_1 + a_2 x_2 + \cdots + a_m x_m = b$$

where x_i are properly chosen.

The task of finding such vectors is called **solving a linear system** and it is formally written as $Ax = b$.

Theorem 1.1. *Let $A \in M(n, m)$ and let $b \in \mathbb{R}^n$. It holds that any linear system $Ax = b$ is solvable iff A is invertible.*

We are interested in finding approximate solutions of such systems, where the proximity to the target is expressed in terms as $\|Ax - b\|$ that should be close to zero. A geometric inutition is displayed in Figure 1.1.

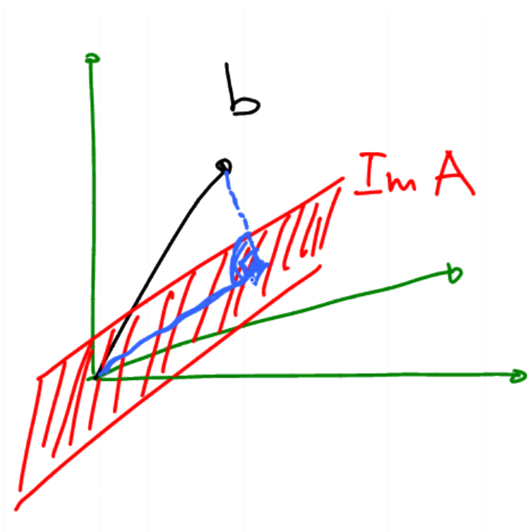

FIGURE 1.1: In this case the image of the matrix A (in red) does not contain b and the best one can do is to obtain a projection of b in the plane $Im(A)$ (drawn in blue).

Something on Matlab ...

Matlab provides syntactic sugar to solve linear systems.
Before introducing such syntax let we just notice the following `5\ 2` $(= 2/5) \neq$ `5/2`.
The syntax to solve $Ax = b$ is `A\ b`, where the algorithm used in Matlab is not inverting the matrix A and then performing the multiplication, but it is a more sophisticated and efficient one.

Definition 1.2 (Linearly square problem). *Let $A \in M(n, m, \mathbb{R})$ and let $b \in \mathbb{R}^n$, we term* ***linearly square problem*** *the task of computing* $\min\limits_{x\in\mathbb{R}^m} \|Ax - b\|_2$.

Something on Matlab ...

In Matlab the syntax `.func` means that function func should be performed entry by entry of the non-scalar variable.

An example of a practical least square problem may be predicting the salary of NBA players, assuming that the income is obtained as a linear combination of some features.

Definition 1.3 (Full rank matrix). *Let $A \in M(n, m, \mathbb{R})$ we say that A has* ***full column rank*** *if* $\ker A = \{0\}$.

Equivalently, $rk(A) = n$ or alternatively $\nexists z \in \mathbb{R}^n \backslash \{0\}$ such that $Az = 0$.

Fact 1.2. *Let $A \in M(n, m, \mathbb{R})$, the least square problem $\|Ax - b = 0\|$ has a unique solution iff A has full column rank.*

Theorem 1.3. *Let $A \in M(n, m, \mathbb{R})$. A has full column rank iff $A^T A$ is positive definite.*

Proof. A has full column rank $\iff \|Az\| \neq 0, \forall z \in \mathbb{R}^m \backslash \{0\} \iff \|Az\|^2 \neq 0, \forall z \in \mathbb{R}^m \backslash \{0\} \iff 0 = (Az)^T Az = z^T A^T Az$. □