Calculus Notes

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Contents

1 Prologue

1.1 Basic Properties of Numbers

All variables are assumed to be numbers.

Rule 1. $\forall(a,b,c): \quad a+(b+c)=(a+b)+c$

Rule 2. $\forall(a,b): \quad a+b=b+a$

Rule 3. $\forall a: \quad a+0=0+a=a$

Rule 4. $\forall a\ \exists(-a): \quad a+(-a)=(-a)+a=0$

Definition 1. $a-b \overset{def}{=} a+(-b)$

Theorem 1. $\forall a: \quad a+x=a \iff x=0$

Proof.

$$a+x=a \tag{1.1.0.1}$$

$$(-a)+(a+x)=(-a)+a \tag{1.1.0.2}$$

$$\overbrace{((-a)+a)}^{Rule\ 1}+x=\overset{Rule\ 4}{0} \tag{1.1.0.3}$$

$$\overset{Rule\ 4}{0}+x=0 \tag{1.1.0.4}$$

$$\overset{Rule\ 3}{x}=0 \tag{1.1.0.5}$$

□

Rule 5. $\forall(a,b,c): \quad a(bc)=(bc)a$

Rule 6. $\forall(a,b,c): \quad ab=ba$

Rule 7. $\forall a: \quad a\cdot 1=1\cdot a=a$

Rule 8. $1\neq 0$

Rule 9. $\forall a\neq 0\ \exists a^{-1}: \quad aa^{-1}=a^{-1}a=1$

Theorem 2. $\forall a: \quad ab=ac\wedge a\neq 0 \iff b=c$

Proof.

$$ab=ac \quad \wedge a\neq 0 \tag{1.1.0.6}$$

$$a^{-1}(ab)=a^{-1}(ac) \quad \wedge a\neq 0 \tag{1.1.0.7}$$

$$(\overbrace{a^{-1}a}^{Rule\ 5})b=(\overbrace{a^{-1}a}^{Rule\ 5})c \quad \wedge a\neq 0 \tag{1.1.0.8}$$

$$\overbrace{1}^{Rule\ 9}\cdot b=\overbrace{1}^{Rule\ 9}\cdot c \tag{1.1.0.9}$$

$$\overset{Rule\ 7}{b}=\overset{Rule\ 7}{c} \tag{1.1.0.10}$$

□

Rule 10. $\forall(a,b,c): \quad a(b+c) = ab + ac$

Theorem 3. $\forall a: \quad 0a = 0$

Proof.

$$0a + 0a = 0a + 0a \tag{1.1.0.11}$$

$$0a + 0a = \overset{Rule\ 10}{a(0+0)} \tag{1.1.0.12}$$

$$0a + 0a = \overset{Rule\ 3}{0a} \tag{1.1.0.13}$$

$$0a = 0 \tag{1.1.0.14}$$

□

Theorem 4. $ab = 0 \iff (a = 0 \vee b = 0)$

Proof.

$$a = 0 \wedge b = 0 \overset{Theorem\ 3}{\iff} ab = 0 \tag{1.1.0.15}$$

$$ab = 0 \quad \wedge a \neq 0 \tag{1.1.0.16}$$

$$a^{-1}(ab) = 0a^{-1} \quad \wedge a \neq 0 \tag{1.1.0.17}$$

$$(\overbrace{a^{-1}a}^{Rule\ 5})b = \overset{Theorem\ 3}{0} \quad \wedge a \neq 0 \tag{1.1.0.18}$$

$$\overbrace{1}^{Rule\ 9} b = 0 \tag{1.1.0.19}$$

$$\overset{Rule\ 7}{b} = 0 \tag{1.1.0.20}$$

□

Theorem 5. $a - b = b - a \iff a = b$

Proof.

$$a - b = b - a \tag{1.1.0.21}$$

$$(a - b) + b = (b - a) + b \tag{1.1.0.22}$$

$$a - \overbrace{((-b) + b)}^{Rule\ 1} = \overbrace{b + (b - a)}^{Rule\ 2} \tag{1.1.0.23}$$

$$a - \overset{Rule\ 4}{0} = (\overbrace{b + b}^{Rule\ 1}) - a \tag{1.1.0.24}$$

$$\overset{Rule\ 3}{a} = (b + b) - a \tag{1.1.0.25}$$

$$a + a = ((b + b) - a) + a \tag{1.1.0.26}$$

$$\overset{Rule\ 10}{a(1 + 1)} = \overset{Rule\ 10}{b(1 + 1)} + (\overbrace{(-a) + a}^{Rule\ 1}) \tag{1.1.0.27}$$

$$a(1 + 1) = b(1 + 1) + \overset{Rule\ 4}{0} \tag{1.1.0.28}$$

$$a(1 + 1) = b(1 + 1) \overset{Theorem\ 2}{\iff} a = b \tag{1.1.0.29}$$

□

Theorem 6. $\forall(a, b): \quad (-a)b = -(ab)$

Proof.

$$(-a)b + ab = (-a)b + ab \tag{1.1.0.30}$$

$$(-a)b + ab = \overset{Rule\ 10}{((-a) + a)b} \tag{1.1.0.31}$$

$$(-a)b + ab = \overbrace{0}^{Rule\ 4}\ b \tag{1.1.0.32}$$

$$(-a)b + ab = \overset{Theorem\ 3}{0} \tag{1.1.0.33}$$

$$((-a)b + ab) + (-(ab)) = 0 + (-(ab)) \tag{1.1.0.34}$$

$$(-a)b + (\overbrace{ab + (-(ab))}^{Rule\ 1}) = \overset{Rule\ 3}{-(ab)} \tag{1.1.0.35}$$

$$(-a)b + \overset{Rule\ 4}{0} = -(ab) \tag{1.1.0.36}$$

$$\overset{Rule\ 3}{(-a)b} = -(ab) \tag{1.1.0.37}$$

□

Theorem 7. $\forall(a, b): \quad (-a)(-b) = ab$

Proof.

$$(-a)(-b) + (-a)b = (-a)(-b) + (-a)b \tag{1.1.0.38}$$

$$(-a)(-b) + (\overset{Theorem\ 6}{-(ab)}) = \overset{Rule\ 10}{-a((-b) + b)} \tag{1.1.0.39}$$

$$(-a)(-b) - ab = -a(\overbrace{0}^{Rule\ 4}) \tag{1.1.0.40}$$

$$(-a)(-b) - ab = \overset{Theorem\ 3}{0} \tag{1.1.0.41}$$

$$((-a)(-b) - ab) + ab = 0 + ab \tag{1.1.0.42}$$

$$(-a)(-b) + (\overbrace{(-(ab)) + ab}^{Rule\ 1}) = \overset{Rule\ 3}{ab} \tag{1.1.0.43}$$

$$(-a)(-b) + \overset{Rule\ 4}{0} = ab \tag{1.1.0.44}$$

$$\overset{Rule\ 3}{(-a)(-b)} = ab \tag{1.1.0.45}$$

□