Homework 7

Labor Economics

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1 Setup

Wages are

$$y_0 = \delta_0 + \beta_0 x + \theta + \epsilon_0$$
$$y_1 = \delta_1 + \beta_1 x + \alpha_1 \theta + \epsilon_1$$

Also define the utility shifter function C and an index function I

$$\begin{aligned} C &= \gamma_0 + \gamma_2 z + \gamma_3 x + \alpha_C \theta \\ I &= E[y_1 - y_0 - C |\, \mathcal{F}] \\ &= \underbrace{(\delta_1 - \delta_0 - \gamma_0)}_{\widetilde{\delta}} + \underbrace{(\beta_0 - \beta_1 - \gamma_3)}_{\widetilde{\beta}} x_i - \gamma_2 z + \underbrace{(\alpha_1 - 1 - \alpha_C)}_{\widetilde{\alpha}} \theta - \epsilon_c \end{aligned}$$

The distribution of shocks is

$$\left.\begin{pmatrix} \epsilon_{i,0} \\ \epsilon_{i,1} \\ \epsilon_{i,C} \end{pmatrix}\right|_{x_i, z_i, \theta_i} \sim N\left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_0^2 & 0 & 0 \\ 0 & \sigma_1^2 & 0 \\ 0 & 0 & \sigma_C^2 \end{pmatrix}\right]$$

The information set for the agent is $\mathcal{F}$. The preference shock $\epsilon_C \in \mathcal{F}$, but $\{\epsilon_0, \epsilon_1\} \notin \mathcal{F}$. The decision rule is

$$s = 1 \iff E[I \geq 0 |\, \mathcal{F}]$$

2 Q1

There is no unobserved heterogeneity in this model since we know θ. Thus,

$$E\left[\begin{matrix} y_0 \\ y_1 \end{matrix}\middle| x_i, \theta_i, s = k\right] = \begin{bmatrix} \delta_0 + x\beta_0 + \theta \\ \delta_1 + x\beta_1 + \alpha_1\theta \end{bmatrix} + \underbrace{E\left[\begin{matrix} \epsilon_0 \\ \epsilon_1 \end{matrix}\middle| x_i, \theta_i, \epsilon_c : big/small\right]}_{0}$$

This is straight-up OLS, which means we recover $\{\delta, \beta, \alpha_1, \sigma_0^2, \sigma_1^2\}$.

$$\begin{aligned} \Pr[S = 1 |\, \mathcal{F}] &= \Pr\left[\epsilon_c \leq \widetilde{\delta} + \widetilde{\beta} x - \gamma_2 z + \widetilde{\alpha}\theta \middle| \mathcal{F}\right] \\ &= \Phi\left[\frac{\overbrace{[(\delta_1 - \delta_0) + (\beta_1 - \beta_0)x + (\alpha_1 - 1)\theta]}^{\text{known number}} - \gamma_0 - \gamma_2 z - \gamma_3 x - \alpha_c \theta}{\sigma_c}\middle| \mathcal{F}\right] \end{aligned}$$

Now we can get $\left\{\gamma_0, \gamma_2, \gamma_3, \alpha_c, \sigma_c^2\right\}$

3 Q2

Now we don't know θ but agents do. However, we do have two measurement equations $m \in \{A, B\}$:

$$\begin{aligned}
M_{iA} &= x_i^M \beta_A^M + \theta_i + \epsilon_{iA}^M \\
M_{iB} &= x_i^M \beta_B^M + \alpha_B \theta_i + \epsilon_{iB}^M
\end{aligned}$$

where $\epsilon_m^M \sim N(0, \sigma_m^{M2})$ are i.i.d.

3.1 Heckman two-step

We can write

$$\begin{aligned}
E[y_1|x, z, s=1] &= \delta_1 + \beta_1 x + E[\epsilon_1 + \alpha_1 \theta | x, z, I \geq 0] \\
&= \delta_1 + \beta_1 x + \alpha_1 E[\theta | x, z, I \geq 0] \\
&= \delta_1 + \beta_1 x + \alpha_1 \sigma^* E\left[\frac{\theta}{\sigma^*}\middle| 0 \leq \widetilde{\delta} + \widetilde{\beta} x - \gamma_2 z + \underbrace{(\alpha_1 - \alpha_0 - \alpha_c)\theta - \epsilon_c}_{\eta}\right] \\
&= \delta_1 + \beta_1 x + \alpha_1 \sigma^* E\left[\frac{\theta}{\sigma^*}\middle| \eta \geq -\left(\widetilde{\delta} + \widetilde{\beta} x - \gamma_2 z\right)\right]
\end{aligned}$$

Define $\eta \equiv (\alpha_1 - \alpha_0 - \alpha_c)\theta - \epsilon_c$. Then

$$\begin{pmatrix} \eta \\ \theta \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^{*2} & (\alpha_1 - \alpha_0 - \alpha_c)\sigma_\theta^2 \\ (\alpha_1 - \alpha_0 - \alpha_c)\sigma_\theta^2 & \sigma_\theta^2 \end{pmatrix}\right]$$

where $\sigma^{*2} = (\alpha_1 - \alpha_0 - \alpha_c)^2 \sigma_\theta^2 + \sigma_c^2$. We can project θ onto η, which means

$$\theta = \frac{\operatorname{Cov}(\eta, \theta)}{\operatorname{Var} \eta} \eta + \nu = \frac{(\alpha_1 - \alpha_0 - \alpha_c)\sigma_\theta^2}{\sigma^{*2}} \eta + \nu$$

where

$$\nu \sim N\left(0, \sigma_\theta^2\left(1 - \rho_{\eta\theta}^2\right)\right) \qquad \text{and} \qquad \rho_{\eta\theta} = \frac{(\alpha_1 - \alpha_0 - \alpha_c)\sigma_\theta^2}{\sigma^* \sigma_\theta}$$

Letting $t \equiv -(\widetilde{\delta} + \widetilde{\beta} x - \gamma_2 z)/\sigma^*$, we can now write

$$\begin{aligned}
E[y_0|x, z, s=1] &= \delta_0 + \beta_0 x + \alpha_0 \frac{(\alpha_1 - \alpha_0 - \alpha_c)\sigma_\theta^2}{\sigma^*} \overbrace{\frac{-\phi(t)}{\Phi(t)}}^{\lambda_0} \\
&= \delta_0 + \beta_0 x + \alpha_0 \left(\rho_{\eta\theta} \sigma_\theta\right) \lambda_{0i} \\
E[y_1|x, z, s=1] &= \delta_1 + \beta_1 x + \alpha_1 \frac{(\alpha_1 - \alpha_0 - \alpha_c)\sigma_\theta^2}{\sigma^*} \underbrace{\frac{\phi(t)}{1 - \Phi(t)}}_{\lambda_1} \\
&= \delta_1 + \beta_1 x + \alpha_1 \left(\rho_{\eta\theta} \sigma_\theta\right) \lambda_{1i}
\end{aligned}$$

A probit first-step has given us $\{(\delta_1 - \delta_0 - \gamma_0)/\sigma_c, (\beta_1 - \beta_0 - \gamma_3)\sigma_c), \gamma_2/\sigma_c\}$. With the second step, we now get $\{\delta_1, \delta_0, \beta_1, \beta_0\}$ and the ratio α_1/α_0. We can back out $\{\sigma_c, \gamma_0, \gamma_2, \gamma_3\}$ from the original probit equations. We also get the quantity $(\alpha_1 - \alpha_0 - \alpha_c)\sigma_\theta^2$ since we have σ_c^2. However, we have 3 αs and only 2 equations for them, so those aren't identified. We can now turn to variances and covariances. Recall

$$\rho_{\eta\theta}\sigma_\theta = \frac{(\alpha_1 - \alpha_0 - \alpha_c)\sigma_\theta^2}{(\alpha_1 - \alpha_0 - \alpha_c)^2\sigma_\theta^2 + \sigma_c^2}$$

It is clear to see that these are of little help since we have a bunch of parameters in the euqations for the variances:

$$\begin{aligned}
\mathrm{Var}(Y_0|\eta < -t) &= \alpha_0^2 \,\mathrm{Var}\,(\theta|\eta < t) + \sigma_0^2 \\
&= \alpha_0^2 \left(\rho_{\eta\theta}\sigma_\theta\right)^2 \left[1 - t\lambda_0 - \lambda_0^2\right] + \sigma_\theta^2 \left(1 - \rho_{\eta\theta}^2\right) + \sigma_0^2 \\
\mathrm{Var}(Y_1|\eta \geq -t) &= \alpha_1^2 \,\mathrm{Var}\,(\theta|\eta \geq t) + \sigma_1^2
\end{aligned}$$

Fortunately, with the measurement equations, we can say things. Recall $I = E[Y_1 - Y_0 - C|X, Z, \theta]$. If we had an estimate of I, we would be in business... and when we do EM/MLE, we do get an estimate of I (right). **Previously: "I have no idea what to do with the last 2 eqns b/c how do we compute I w/ out θ This is maybe why we need MLE and EM??"**

$$\begin{aligned}
\mathrm{Cov}(Y_0 - \beta_0 X, M^A - X^M\beta_A) &= \alpha_0\sigma_\theta^2 && (1)\\
\mathrm{Cov}(Y_0 - \beta_0 X, M^B - X^M\beta_B) &= \alpha_0\alpha_B\sigma_\theta^2 && (2)\\
\mathrm{Cov}(Y_1 - \beta_1 X, M^A - X^M\beta_A) &= \alpha_1\sigma_\theta^2 && (3)\\
\mathrm{Cov}(Y_1 - \beta_1 X, M^B - X^M\beta_B) &= \alpha_1\alpha_B\sigma_\theta^2 && (4)\\
\mathrm{Cov}\left[I - \widetilde{\delta} - \widetilde{\beta}x - \gamma_2 z,\ \left(M^A - X^M\beta_A\right)\right] &= (\alpha_1 - \alpha_0 - \alpha_c)\sigma_\theta^2 && (5)\\
\mathrm{Cov}\left[I - \widetilde{\delta} - \widetilde{\beta}x - \gamma_2 z,\ \left(M^B - X^M\beta_B\right)\right] &= (\alpha_1 - \alpha_0 - \alpha_c)\alpha_B\sigma_\theta^2 && (6)\\
\mathrm{Cov}\left[I - \widetilde{\delta} - \widetilde{\beta}x - \gamma_2 z,\ (Y_0 - X\beta_0)\right] &= (\alpha_1 - \alpha_0 - \alpha_c)\alpha_0\sigma_\theta^2 && (7)\\
\mathrm{Cov}\left[I - \widetilde{\delta} - \widetilde{\beta}x - \gamma_2 z,\ (Y_1 - X\beta_1)\right] &= (\alpha_1 - \alpha_0 - \alpha_c)\alpha_1\sigma_\theta^2 && (8)
\end{aligned}$$

The top four equations give us two measurements for α_B. The bottom four plus knowledge of $(\alpha_1 - \alpha_0 - \alpha_c)\sigma_\theta^2$ from the two-step gives us α_0 and α_1 (just divide them). With α_ks in hand plus α_B give us multiple measurements for σ_θ^2. We plug these in to the variances for $\mathrm{Var}(Y_k|X, Z, s = k)$ and get σ_k^2. Done.

3.2 MLE approach

The contribution to the likelihood of any given individual i is now the product of the likelihood of the wage and choice times the product of the likelihoods of the test equations.

$$\begin{aligned}
L_i = &\left[f(y_{1i}|X, \theta, s_i = 1) \Pr(s_i = 1|X, Z, \theta)\right]^{s_i} \\
&\times \left[f(y_{0i}|X, \theta, s_i = 0) \Pr(s_i = 0|X, Z, \theta)\right]^{1-s_i} \\
&\times f(m_i^A|X_i^M, \theta) \\
&\times f(m_i^B|X_i^M, \theta) \\
&\times f(\theta)
\end{aligned}$$

Define $q_i \equiv 2s_i - 1$. Since we only observe y_{1i} or y_{i0}, we simply use y_i in the likelihood equation. We can log everything and integrate w/ respect to θ.

$$
\begin{aligned}
\mathcal{L}_i = \int_\theta & \log\left[1 - \Phi\left(q_i \times \frac{(\delta_1 - \delta_0 - \gamma_0) + (\beta_1 - \beta_0 - \gamma_3)X_i - \gamma_2 Z_i + (\alpha_1 - \alpha_0 - \alpha_c)\theta}{\sigma_c}\right)\right] \\
& + s_i \log\left[\phi\left(\frac{y_i - \delta_1 - \beta_1 x_i - \alpha_1\theta}{\sigma_1}\right)\right] \\
& + (1 - s_i)\log\left[\phi\left(\frac{y_i - \delta_0 - \beta_0 x_i - \alpha_0\theta}{\sigma_0}\right)\right] \\
& + \log\left[\phi\left(\frac{M_i^A - X_i^M\beta_A - \theta}{\sigma_A}\right)\right] \\
& + \log\left[\phi\left(\frac{M_i^B - X_i^M\beta_B - \alpha_B\theta}{\sigma_B}\right)\right] \\
& + \log\left[\phi\left(\frac{\theta}{\sigma_\theta}\right)\right] d\theta
\end{aligned}
$$

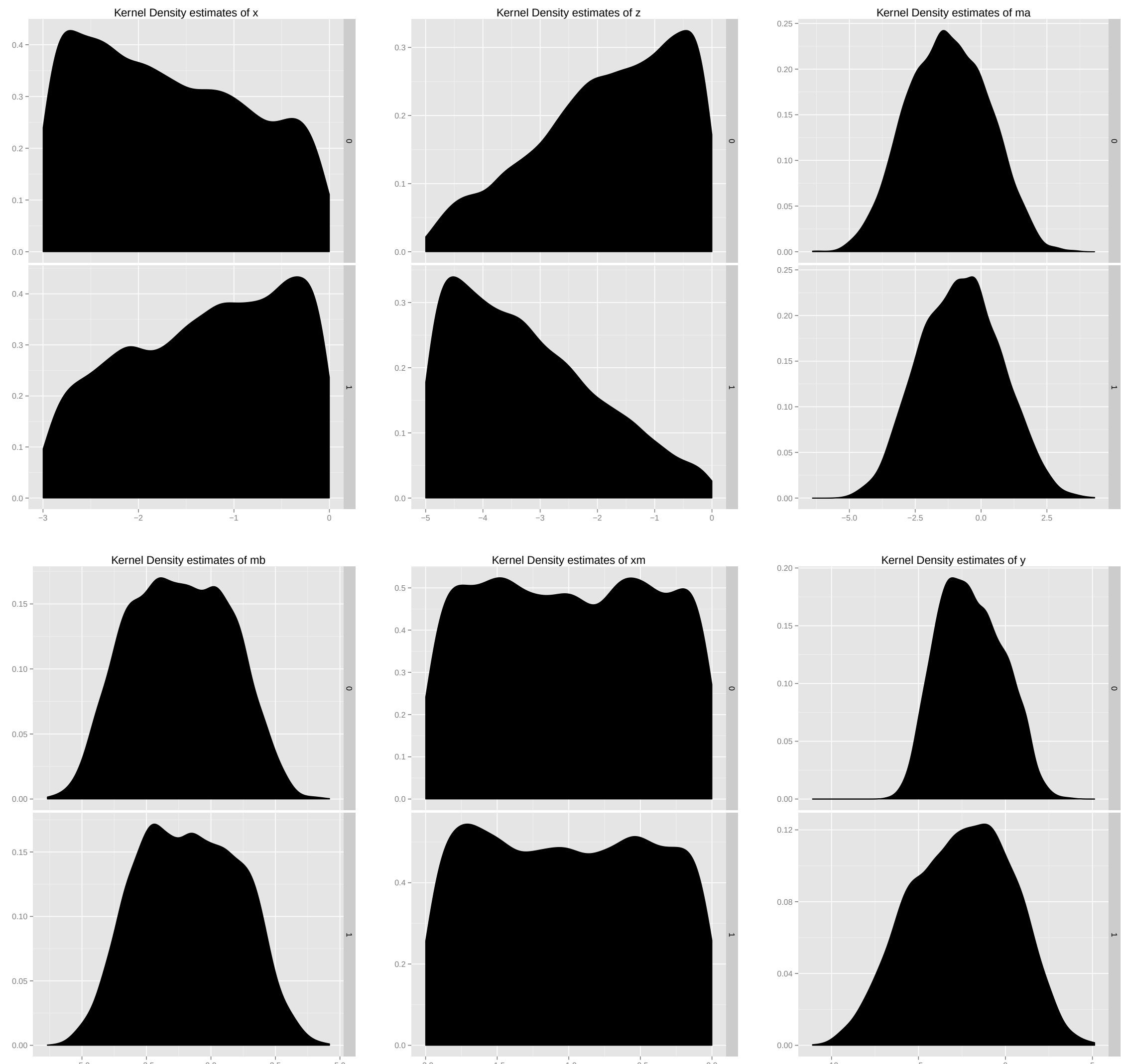

Figure 1: Histograms