

# Homework 7

## Labor Economics

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### 1 Setup

Wages are

$$\begin{aligned} y_0 &= \delta_0 + \beta_0 x + \theta + \epsilon_0 \\ y_1 &= \delta_1 + \beta_1 x + \alpha_1 \theta + \epsilon_1 \end{aligned}$$

Also define the utility shifter function  $C$  and an index function  $I$

$$\begin{aligned} C &= \gamma_0 + \gamma_2 z + \gamma_3 x + \alpha_C \theta \\ I &= E[y_1 - y_0 - C | \mathcal{F}] \\ &= \underbrace{(\delta_1 - \delta_0 - \gamma_0)}_{\tilde{\delta}} + \underbrace{(\beta_0 - \beta_1 - \gamma_3)}_{\tilde{\beta}} x_i - \gamma_2 z + \underbrace{(\alpha_1 - 1 - \alpha_C)}_{\tilde{\alpha}} \theta - \epsilon_c \end{aligned}$$

The distribution of shocks is

$$\begin{pmatrix} \epsilon_{i,0} \\ \epsilon_{i,1} \\ \epsilon_{i,C} \end{pmatrix} \bigg|_{x_i, z_i, \theta_i} \sim N \left[ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_0^2 & 0 & 0 \\ 0 & \sigma_1^2 & 0 \\ 0 & 0 & \sigma_C^2 \end{pmatrix} \right]$$

The information set for the agent is  $\mathcal{F}$ . The preference shock  $\epsilon_C \in \mathcal{F}$ , but  $\{\epsilon_0, \epsilon_1\} \notin \mathcal{F}$ . The decision rule is

$$s = 1 \iff E[I \geq 0 | \mathcal{F}]$$

### 2 Q1

There is no unobserved heterogeneity in this model since we know  $\theta$ . Thus,

$$E \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \bigg| x_i, \theta_i, s = k = \begin{bmatrix} \delta_0 + x\beta_0 + \theta \\ \delta_1 + x\beta_1 + \alpha_1 \theta \end{bmatrix} + \underbrace{E \begin{bmatrix} \epsilon_0 \\ \epsilon_1 \end{bmatrix} \bigg| x_i, \theta_i, \epsilon_c : \text{big/small}}_0$$

This is straight-up OLS, which means we recover  $\{\delta, \beta, \alpha_1, \sigma_0^2, \sigma_1^2\}$ .

$$\begin{aligned} \Pr[S = 1 | \mathcal{F}] &= \Pr \left[ \epsilon_c \leq \tilde{\delta} + \tilde{\beta} x - \gamma_2 z + \tilde{\alpha} \theta \bigg| \mathcal{F} \right] \\ &= \Phi \left[ \frac{\overbrace{[(\delta_1 - \delta_0) + (\beta_1 - \beta_0)x + (\alpha_1 - 1)\theta]}^{\text{known number}} - \gamma_0 - \gamma_2 z - \gamma_3 x - \alpha_c \theta}{\sigma_c} \bigg| \mathcal{F} \right] \end{aligned}$$

Now we can get  $\{\gamma_0, \gamma_2, \gamma_3, \alpha_c, \sigma_c^2\}$

### 3 Q2

Now we don't know  $\theta$  but agents do. However, we do have two measurement equations  $m \in \{A, B\}$ :

$$\begin{aligned} M_{iA} &= x_i^M \beta_A^M + \theta_i + \epsilon_{iA}^M \\ M_{iB} &= x_i^M \beta_B^M + \alpha_B \theta_i + \epsilon_{iB}^M \end{aligned}$$

where  $\epsilon_m^M \sim N(0, \sigma_m^{M2})$  are i.i.d.

#### 3.1 Heckman two-step

We can write

$$\begin{aligned} E[y_1|x, z, s = 1] &= \delta_1 + \beta_1 x + E[\epsilon_1 + \alpha_1 \theta | x, z, I \geq 0] \\ &= \delta_1 + \beta_1 x + \alpha_1 E[\theta | x, z, I \geq 0] \\ &= \delta_1 + \beta_1 x + \alpha_1 \sigma^* E \left[ \frac{\theta}{\sigma^*} \middle| 0 \leq \tilde{\delta} + \tilde{\beta}x - \gamma_2 z + \underbrace{(\alpha_1 - \alpha_0 - \alpha_c)\theta - \epsilon_c}_{\eta} \right] \\ &= \delta_1 + \beta_1 x + \alpha_1 \sigma^* E \left[ \frac{\theta}{\sigma^*} \middle| \eta \geq -(\tilde{\delta} + \tilde{\beta}x - \gamma_2 z) \right] \end{aligned}$$

Define  $\eta \equiv (\alpha_1 - \alpha_0 - \alpha_c)\theta - \epsilon_c$ . Then

$$\begin{pmatrix} \eta \\ \theta \end{pmatrix} \sim N \left[ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^{*2} & (\alpha_1 - \alpha_0 - \alpha_c)\sigma_\theta^2 \\ (\alpha_1 - \alpha_0 - \alpha_c)\sigma_\theta^2 & \sigma_\theta^2 \end{pmatrix} \right]$$

where  $\sigma^{*2} = (\alpha_1 - \alpha_0 - \alpha_c)^2 \sigma_\theta^2 + \sigma_c^2$ . We can project  $\theta$  onto  $\eta$ , which means

$$\theta = \frac{\text{Cov}(\eta, \theta)}{\text{Var } \eta} \eta + \nu = \frac{(\alpha_1 - \alpha_0 - \alpha_c)\sigma_\theta^2}{\sigma^{*2}} \eta + \nu$$

where

$$\nu \sim N(0, \sigma_\theta^2 (1 - \rho_{\eta\theta}^2)) \quad \text{and} \quad \rho_{\eta\theta} = \frac{(\alpha_1 - \alpha_0 - \alpha_c)\sigma_\theta^2}{\sigma^* \sigma_\theta}$$

Letting  $t \equiv -(\tilde{\delta} + \tilde{\beta}x - \gamma_2 z)/\sigma^*$ , we can now write

$$\begin{aligned} E[y_0|x, z, s = 1] &= \delta_0 + \beta_0 x + \alpha_0 \frac{(\alpha_1 - \alpha_0 - \alpha_c)\sigma_\theta^2}{\sigma^*} \overbrace{\frac{-\phi(t)}{\Phi(t)}}^{\lambda_0} \\ &= \delta_0 + \beta_0 x + \alpha_0 (\rho_{\eta\theta} \sigma_\theta) \lambda_{0i} \\ E[y_1|x, z, s = 1] &= \delta_1 + \beta_1 x + \alpha_1 \frac{(\alpha_1 - \alpha_0 - \alpha_c)\sigma_\theta^2}{\sigma^*} \underbrace{\frac{\phi(t)}{1 - \Phi(t)}}_{\lambda_1} \\ &= \delta_1 + \beta_1 x + \alpha_1 (\rho_{\eta\theta} \sigma_\theta) \lambda_{1i} \end{aligned}$$

A probit first-step has given us  $\{(\delta_1 - \delta_0 - \gamma_0)/\sigma_c, (\beta_1 - \beta_0 - \gamma_3)\sigma_c, \gamma_2/\sigma_c, (\alpha_1 - \alpha_0 - \alpha_c)\sigma_c\}$ . With the second step, we now get  $\{\delta_1, \delta_0, \beta_1, \beta_0\}$  and the ratio  $\alpha_1/\alpha_2$ . We can back out  $\{\sigma_c, \gamma_0, \gamma_2, \gamma_3\}$  from the original probit equations. We have 3  $\alpha$ s and only 2 equations for them, so those aren't identified. We can now turn to variances and covariances. Recall

$$\rho_{\eta\theta} = \frac{(\alpha_1 - \alpha_0 - \alpha_c)\sigma_\theta^2}{\sigma^*\sigma_\theta} \quad \text{and} \quad \sigma^{*2} = (\alpha_1 - \alpha_0 - \alpha_c)^2\sigma_\theta^2 + \sigma_c^2$$

It is clear to see that these are of little help since we have a bunch of parameters.

$$\begin{aligned} \text{Var}(Y_0|\eta < -t) &= \alpha_0^2 \text{Var}(\theta|\eta < t) + \sigma_0^2 \\ &= \alpha_0^2 (\rho_{\eta\theta}\sigma_\theta)^2 [1 - t\lambda_0 - \lambda_0^2] + \sigma_\theta^2 (1 - \rho_{\eta\theta}^2) + \sigma_0^2 \\ \text{Var}(Y_1|\eta \geq -t) &= \alpha_1^2 \text{Var}(\theta|\eta \geq t) + \sigma_1^2 \end{aligned}$$

### 3.2 MLE approach

The contribution to the likelihood of any given individual  $i$  is now the product of the likelihood of the wage and choice times the product of the likelihoods of the test equations.

$$\begin{aligned} L_i &= [f(y_{1i}|X, \theta, s_i = 1) \Pr(s_i = 1|X, Z, \theta)]^{s_i} \\ &\quad \times [f(y_{0i}|X, \theta, s_i = 0) \Pr(s_i = 0|X, Z, \theta)]^{1-s_i} \\ &\quad \times f(m_i^A|X_i^M, \theta) \\ &\quad \times f(m_i^B|X_i^M, \theta) \\ &\quad \times f(\theta) \end{aligned}$$

Define  $q_i \equiv 2s_i - 1$ . Since we only observe  $y_{1i}$  or  $y_{0i}$ , we simply use  $y_i$  in the likelihood equation. We can log everything and integrate w/ respect to  $\theta$ .

$$\begin{aligned} \mathcal{L}_i &= \int_{\theta} \log \left[ 1 - \Phi \left( q_i \times \frac{(\delta_1 - \delta_0 - \gamma_0) + (\beta_1 - \beta_0 - \gamma_3)X_i - \gamma_2 Z_i + (\alpha_1 - \alpha_0 - \alpha_c)\theta}{\sigma_c} \right) \right] \\ &\quad + s_i \log \left[ \phi \left( \frac{y_i - \delta_1 - \beta_1 x_i - \alpha_1 \theta}{\sigma_1} \right) \right] \\ &\quad + (1 - s_i) \log \left[ \phi \left( \frac{y_i - \delta_0 - \beta_0 x_i - \alpha_0 \theta}{\sigma_0} \right) \right] \\ &\quad + \log \left[ \phi \left( \frac{M_i^A - X_i^M \beta_A - \theta}{\sigma_A} \right) \right] \\ &\quad + \log \left[ \phi \left( \frac{M_i^B - X_i^M \beta_B - \alpha_B \theta}{\sigma_B} \right) \right] \\ &\quad + \log \left[ \phi \left( \frac{\theta}{\sigma_\theta} \right) \right] d\theta \end{aligned}$$

## 4 Histograms

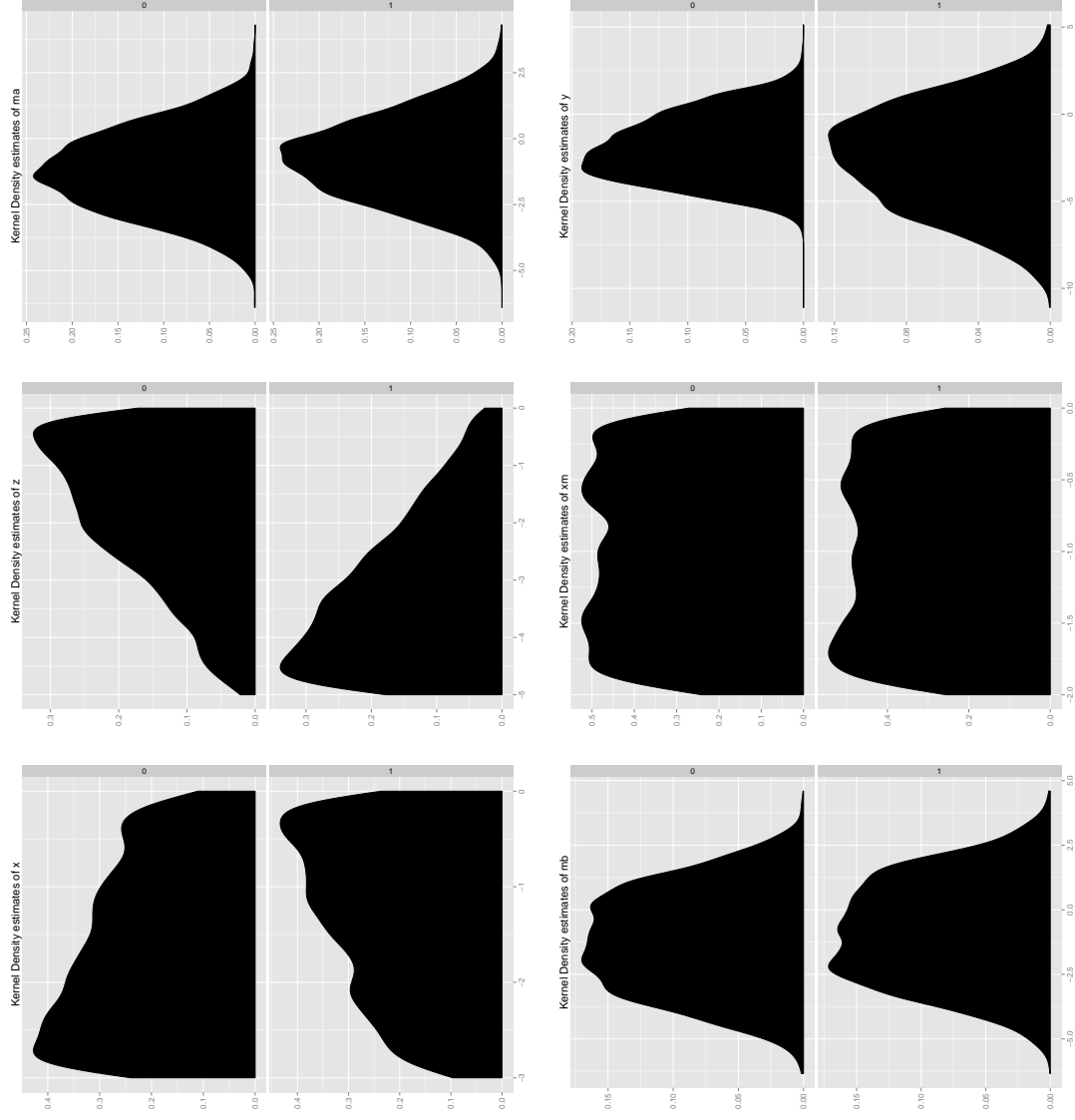


Figure 1: Plots