Homework 8

Labor Economics

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1 Setup

Individual knows $\Omega_a = (x_a, y, \epsilon_a)$. His wage is

$$\log w_a^* = \alpha_1 + \alpha_2 x_a + \epsilon_a$$

We observe

$$\log w_a = \log w_a^* + \nu_a$$

Individual gets per-period payoff

$$u(\Omega_a) = \begin{cases} y + \gamma_1 + \gamma_2 y & \text{if } p_a = 0 \text{ (eg, no work)} \\ y + \exp\{f(x_a) + \epsilon_a\} & \text{if } p_a = 1 \text{ (eg, work)} \end{cases}$$

Assume iid shocks:

$$\begin{pmatrix} \epsilon_a \\ \nu_a \end{pmatrix} \sim N\left(0, \begin{bmatrix} \sigma_\epsilon^2 & \\ 0 & \sigma_\nu^2 \end{bmatrix}\right)$$

Objective is

$$\max_{\{p_a\}_{a=1}^A} \beta^{a-1} E\left[u(\Omega)|\Omega_a\right]$$

2 Recursive formulation

Write problem recursively for lazy:

$$V_a^0(x, y, \epsilon_a) = \gamma_1 + (1 + \gamma_2)y + \beta E[V_{a+1}(x + 1, y, \epsilon_{a+1})]$$

and working:

$$V_a^1(x, y, \epsilon_a) = \exp\{f(x_a) + \epsilon_a\} + y + \beta E[V_{a+1}(x + 1, y, \epsilon_{a+1})]$$

Value is

$$V_a(x, y, \epsilon_a) = \max\left\{V_a^0(x, y, \epsilon_a), V_a^1(x, y, \epsilon_a)\right\}$$

We normalize the value of afterlife to 0 after assuming earthly actions can't affect it

$$V_{A+1}(x, y, \epsilon) = 0$$

Let $\mathcal{W}_a$ be the event that we we work, which is

$$p_a = 1 \quad \Leftrightarrow \quad \epsilon_a \geq \underbrace{\log\left(\gamma_1 + \gamma_2 y + E[V^0_{a+1}(x,y)] - E[V^1_{a+1}(x,y)]\right) - \alpha_1 - \alpha_2 x_a}_{g(x,y,a)}$$

Then

$$\Pr(\mathcal{W}_a) = 1 - \Phi(g(x,y,a)/\sigma_\epsilon)$$

3 Backward induction

3.1 Last period

Last period's value is

$$V_A(x,y,\epsilon_A) = \max\{\gamma_1 + \gamma_2 y, \quad \exp(\alpha_1 + \alpha_2 x + \epsilon_A)\} + y$$

Now

$$g(x,y,A) = \log(\gamma_1 + \gamma_2 y) - (\alpha_1 + \alpha_2 x)$$

so

$$\Pr(\mathcal{W}_A) = 1 - \Phi\left(\frac{g(x,y,A)}{\sigma_\epsilon}\right) = \pi(x,y,A)$$

Expected terminal value is

$$E[V_A(x,y)] = y + [1 - \pi(x,y,A)]\,(\gamma_1 + \gamma_2 y) + \pi(x,y,A)\left[\exp\{\alpha_1 + \alpha_2 x\}\underbrace{\frac{1 - \Phi\left(\frac{g(x,y,A) - \sigma^2_\epsilon}{\sigma_\epsilon}\right)}{\pi(x,y,A)}\exp\{\frac{1}{2}\sigma^2_\epsilon\}}_{E[e^\epsilon_A | \mathcal{W}^A]}\right]$$

This can be written as

$$E[V_A(x,y)] = y + [1 - \pi(x,y,A)]\,[\gamma_1 + \gamma_2 y] + \exp\left\{\alpha_1 + \alpha_2 x + \frac{\sigma^2_\epsilon}{2}\right\}\left[1 - \Phi\left(\frac{g(x,y,A) - \sigma^2_\epsilon}{\sigma_\epsilon}\right)\right]$$

3.2 Other periods

This means

$$g(x,y,a) = \log\left(\gamma_1 + \gamma_2 y + \beta \overbrace{E[V^0_{a+1}(x,y)] - E[V^1_{a+1}(x,y)]}^{\Delta EV(x,y,a)}\right) - \alpha_1 - \alpha_2 x_a$$

and

$$\mathcal{W}_a = \{\epsilon_a \geq g(x,y,a)\}$$

so

$$\Pr(\mathcal{W}_a) = 1 - \Phi\left(\frac{g(x,y,a)}{\sigma_\epsilon}\right) = \pi(x,y,a)$$

and

$$E[V_a(x,y)] = y + [1-\pi(x,y,a)]\{\gamma_1 + \gamma_2 y + E[V_{a+1}(x,y)]\}$$
$$+ \pi(x,y,a)E[V_{a+1}(x+1,y)] + \exp\left\{\alpha_1 + \alpha_2 x + \frac{\sigma_\epsilon^2}{2}\right\}\left[1 - \Phi\left(\frac{g(x,y,a) - \sigma_\epsilon^2}{\sigma_\epsilon}\right)\right]$$

Note that we could simply use the general definition for V_a and $g(x,y,a)$ and specify $V_{A+1} = 0$. This would be a bit neater (ie, for each agent, have $A+1$ periods and just say $V_{A+1} = 0$... then start recursion at $a = A$.

4 Estimation

We have states $\Omega_{ia} = (x,y,a,\epsilon)_{ia}$ and control $p_{ia} \in \{0,1\}$.
Immediately we can get parameters governing distribution of non-labor income from a kernel density estimation of observed y_i values. Or, since we know if we know the underlying distribution we just need μ_y and σ_y which can be estimated by $N^{-1}\sum_i y_i$ and $\widehat{SE(y_i)}$. Number of periods is irrelevant and are consistent as $N \to \infty$. .
Remaining parameters are

$$\theta = \{\alpha_1, \alpha_2, \gamma_1, \gamma_2, \sigma_\epsilon^2, \sigma_\nu^2\}$$

We'll need functions(?) or matrices of(?)

$$V_a^0(x,y) \quad V_a^1(x,y) \quad g(x,y,a) \quad \pi(x,y,a) \quad w(x,\epsilon;\alpha_1,\alpha_2)$$

Note that $g(x,y,a)$ is a function of a because it has $\Delta E[V_{a+1}(x,y)]$

4.1 Plan

1. Given θ, calculate functions I think we are just starting at $t = A$ here and so $\Delta E[V_{a+1}(x,y)] = 0$ and we do not need to fix a θ. Can recover g without.

2. Run probit for working in A

3. Estimate wages as

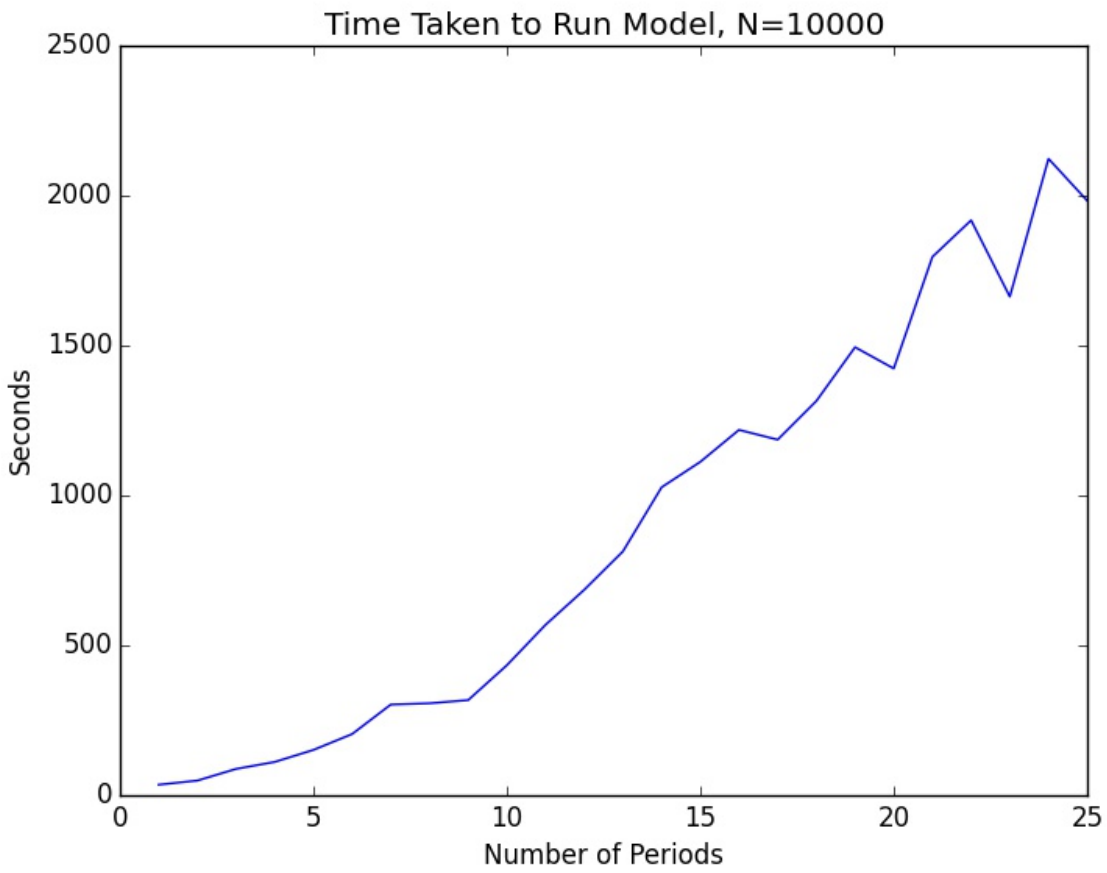
$$\log w = \underbrace{\alpha_1 + \alpha_2 x + \sigma_\epsilon \lambda(y,x,a)}_{E[\log w_a^* | \mathcal{W}_a]} + \nu$$

   where

$$\lambda(x,y,a) = E\left[\frac{\epsilon}{\sigma}\middle|\frac{\epsilon}{\sigma} \underbrace{\geq}_{\text{notes were} \leq} \frac{g(x,y,a)}{\sigma_\epsilon}\right] = \frac{\phi(g/\sigma)}{1-\Phi(g/\sigma)}$$

4. Compute $\Delta E[V_A(x,y)]$

5. Run probit for $A-1$...



5.1 Sample Output

Listing 1: Functions used

```
    ############################
    # Specify Model Parameters
    ############################
    # A    = 11          # Number of Periods
    γ_1  = 0.300        # Leisure Coefficient
    γ_2  = 0.500        # Consumption-Leisure Interaction Coefficient
    δ    = 0.1          # Discount Rate
    N    = 35000        # Number of Individuals
    σ_e  = 1.000        # Standard Error of Wage Shock
    σ_v  = 0.100        # Standard Error of Measurement Error
    α_1  = 5.000        # Wage Function Parameter
    α_2  = 0.500        # Wage Function Parameter
    α_3  = -0.100       # wage function parameter
    μ_y  = 0.0          # Mean of non-labor income
    σ_y  = 2.0          # std dev of non-labor income
    # yL = 0          # Minimum Non-Labor Income
    # yH = 1000          # Top Non-Labor Income

    θ_real = [γ_1; γ_2; α_1; α_2; α_3; σ_e]
    β_real = [α_1; α_2; α_3; σ_e]
```

Listing 2: Functions used

```
Eval 3: value = 3938.49124
Eval 4: value = 3393.73034
Eval 5: value = 2335.27842
```

```
Eval 10: value = 2823.92179
Eval 20: value = 2381.45825
Eval 30: value = 2366.84353
Eval 40: value = 1.7165000000000002e54
Eval 50: value = 2335.01217
Eval 75: value = 2334.88533
Eval 100: value = 2334.88218
Eval 125: value = 2334.88033
Eval 150: value = 2334.88032
Eval 175: value = 2334.88031
Eval 200: value = 2334.88031
Eval 250: value = 2334.88031
Eval 300: value = 3018.2841
Eval 350: value = 2334.89949
Eval 400: value = 2334.88031
Eval 450: value = 2334.88031
Eval 500: value = 2343.89512
Eval 600: value = 2334.88031
Eval 700: value = 2335.3637
Eval 800: value = 2334.88031
Eval 900: value = 2334.88488
Eval 1000: value = 2334.88031
Results of Optimization Algorithm
 * Algorithm: Nelder-Mead
 * Starting Point:
     [0.616155778922203,0.508501689628141,5.0377041439308625,1.2182100632738138,1.198399291904949,1.0062741405941

 * Minimum:
     [0.6161618008355569,0.5085065629459316,5.037713726302649,1.5515365722879093,1.5317258009190446,1.00627413594

 * Value of Function at Minimum: 2334.880305
 * Iterations: 91
 * Convergence: true
   * |x - x'| < NaN: false
   * |f(x) - f(x')| / |f(x)| < 1.0e-12: true
   * |g(x)| < NaN: false
   * Exceeded Maximum Number of Iterations: false
 * Objective Function Calls: 182
 * Gradient Call: 0elapsed time: 3731.772961383 seconds
There where 35000 workings and 11 periods
It took 3731.773 seconds to run

 Percentage that worked in period 1:
[0.92,0.92,0.93,0.93,0.93,0.93,0.92,0.89,0.84,0.75,0.57]


 LLN value at θ true: 2334.88



 MLE Parameters:
         TRUE    ESTIMATED

γ_1      0.3     0.616

γ_2      0.5     0.509

α_1      5.0     5.038

α_2      0.5     1.552

α_3      -0.1    1.532

σ_e      1.0     1.006

 OLS Parameters:
         TRUE    ESTIMATED
```

```
α_1      5.0     5.399

α_2      0.5     NaN

α_3      -0.1    NaN

σ_e      1.0     1.027
There where 35000 workings and 11 periods
It took 3731.773 seconds to run

 Percentage that worked in period 2:
[0.92,0.92,0.93,0.93,0.93,0.93,0.92,0.89,0.84,0.75,0.57]


 LLN value at θ true: 1902.961



 MLE Parameters:
         TRUE    ESTIMATED

γ_1      0.3     1.129

γ_2      0.5     0.514

α_1      5.0     5.009

α_2      0.5     0.517

α_3      -0.1    -0.092

σ_e      1.0     0.984

 OLS Parameters:
         TRUE    ESTIMATED

α_1      5.0     5.604

α_2      0.5     NaN

α_3      -0.1    NaN

σ_e      1.0     0.95
There where 35000 workings and 11 periods
It took 3731.773 seconds to run

 Percentage that worked in period 3:
[0.92,0.92,0.93,0.93,0.93,0.93,0.92,0.89,0.84,0.75,0.57]


 LLN value at θ true: 1687.306



 MLE Parameters:
         TRUE    ESTIMATED

γ_1      0.3     -0.0

γ_2      0.5     0.503

α_1      5.0     4.983

α_2      0.5     0.608

α_3      -0.1    -0.141

σ_e      1.0     0.996
```

```
 OLS Parameters:
         TRUE    ESTIMATED

α_1      5.0     5.589

α_2      0.5     -0.004

α_3      -0.1    0.004

σ_e      1.0     1.02
There where 35000 workings and 11 periods
It took 3731.773 seconds to run

 Percentage that worked in period 4:
[0.92,0.92,0.93,0.93,0.93,0.93,0.92,0.89,0.84,0.75,0.57]


 LLN value at θ true: 1732.686



 MLE Parameters:
         TRUE    ESTIMATED

γ_1      0.3     -0.0

γ_2      0.5     0.512

α_1      5.0     5.014

α_2      0.5     0.49

α_3      -0.1    -0.102

σ_e      1.0     0.993

 OLS Parameters:
         TRUE    ESTIMATED

α_1      5.0     5.464

α_2      0.5     -0.09

α_3      -0.1    0.022

σ_e      1.0     0.971
There where 35000 workings and 11 periods
It took 3731.773 seconds to run

 Percentage that worked in period 5:
[0.92,0.92,0.93,0.93,0.93,0.93,0.92,0.89,0.84,0.75,0.57]


 LLN value at θ true: 1853.577



 MLE Parameters:
         TRUE    ESTIMATED

γ_1      0.3     2.266

γ_2      0.5     0.531

α_1      5.0     5.11

α_2      0.5     0.457
```

```
α_3      -0.1    -0.095

σ_e      1.0     0.963

 OLS Parameters:
         TRUE    ESTIMATED

α_1      5.0     4.993

α_2      0.5     -0.03

α_3      -0.1    0.009

σ_e      1.0     0.98
There where 35000 workings and 11 periods
It took 3731.773 seconds to run

 Percentage that worked in period 6:
[0.92,0.92,0.93,0.93,0.93,0.93,0.92,0.89,0.84,0.75,0.57]


 LLN value at θ true: 2222.365



 MLE Parameters:
         TRUE    ESTIMATED

γ_1      0.3     0.146

γ_2      0.5     0.157

α_1      5.0     3.811

α_2      0.5     0.46

α_3      -0.1    -0.09

σ_e      1.0     1.002

 OLS Parameters:
         TRUE    ESTIMATED

α_1      5.0     4.312

α_2      0.5     0.043

α_3      -0.1    -0.005

σ_e      1.0     1.014
There where 35000 workings and 11 periods
It took 3731.773 seconds to run

 Percentage that worked in period 7:
[0.92,0.92,0.93,0.93,0.93,0.93,0.92,0.89,0.84,0.75,0.57]

 LLN value at θ true: 2950.578



 MLE Parameters:
         TRUE    ESTIMATED

γ_1      0.3     0.677

γ_2      0.5     0.499
```

```
α_1       5.0      5.004

α_2       0.5      0.521

α_3       -0.1     -0.103

σ_e       1.0      0.977

 OLS Parameters:
          TRUE     ESTIMATED

α_1       5.0      3.698

α_2       0.5      -0.026

α_3       -0.1     0.002

σ_e       1.0      0.963
There where 35000 workings and 11 periods
It took 3731.773 seconds to run

 Percentage that worked in period 8:
[0.92,0.92,0.93,0.93,0.93,0.93,0.92,0.89,0.84,0.75,0.57]


 LLN value at θ true: 4320.6



 MLE Parameters:
          TRUE     ESTIMATED

γ_1       0.3      -0.0

γ_2       0.5      0.435

α_1       5.0      4.817

α_2       0.5      0.532

α_3       -0.1     -0.104

σ_e       1.0      1.004

 OLS Parameters:
          TRUE     ESTIMATED

α_1       5.0      2.576

α_2       0.5      0.046

α_3       -0.1     -0.006

σ_e       1.0      0.966
There where 35000 workings and 11 periods
It took 3731.773 seconds to run

 Percentage that worked in period 9:
[0.92,0.92,0.93,0.93,0.93,0.93,0.92,0.89,0.84,0.75,0.57]

 LLN value at θ true: 6327.785

 MLE Parameters:
          TRUE     ESTIMATED

γ_1       0.3      0.243

γ_2       0.5      0.507
```

```
α_1      5.0     5.038

α_2      0.5     0.496

α_3      -0.1    -0.1

σ_e      1.0     1.009

 OLS Parameters:
         TRUE    ESTIMATED

α_1      5.0     1.452

α_2      0.5     -0.03

α_3      -0.1    0.003

σ_e      1.0     1.0
There where 35000 workings and 11 periods
It took 3731.773 seconds to run

 Percentage that worked in period 10:
[0.92,0.92,0.93,0.93,0.93,0.93,0.92,0.89,0.84,0.75,0.57]

 LLN value at θ true: 9500.981

 MLE Parameters:
         TRUE    ESTIMATED

γ_1      0.3     0.094

γ_2      0.5     0.148

α_1      5.0     3.97

α_2      0.5     0.444

α_3      -0.1    -0.096

σ_e      1.0     0.979

 OLS Parameters:
         TRUE    ESTIMATED

α_1      5.0     0.168

α_2      0.5     -0.03

α_3      -0.1    0.001

σ_e      1.0     0.981
There where 35000 workings and 11 periods
It took 3731.773 seconds to run

 Percentage that worked in period 11:
[0.92,0.92,0.93,0.93,0.93,0.93,0.92,0.89,0.84,0.75,0.57]

 LLN value at θ true: 16365.872

 MLE Parameters:
         TRUE    ESTIMATED

γ_1      0.3     0.373

γ_2      0.5     0.587

α_1      5.0     5.142
```

```
α_2      0.5     0.491

α_3      -0.1    -0.099

σ_e      1.0     0.985

 OLS Parameters:
         TRUE    ESTIMATED

α_1      5.0     -1.598

α_2      0.5     -0.015

α_3      -0.1    0.001

σ_e      1.0     1.023
NumPeriods: Time Taken:
1                0.0
NumPeriods: Time Taken:
2                0.0
NumPeriods: Time Taken:
3                0.0
NumPeriods: Time Taken:
4                0.0
NumPeriods: Time Taken:
5                0.0
NumPeriods: Time Taken:
6                0.0
NumPeriods: Time Taken:
7                0.0
NumPeriods: Time Taken:
8                0.0
NumPeriods: Time Taken:
9                0.0
NumPeriods: Time Taken:
10               0.0
NumPeriods: Time Taken:
11               3731.772961383
```